Beam Deflection Formula Derivations

A breakdown of the formulas used in the beam simulators.

Where do these formulas even come from?

Alright, let's break it down. All this stuff about beams bending comes from one main idea called the **Euler-Bernoulli beam equation**. It's basically the OG formula for figuring out how beams behave. The main equation looks like this:

$$M = EI\frac{d^2y}{dx^2}$$

So, what's all that junk mean? Lemme break it down for ya:

- **M** is the **Bending Moment**. Tbh, it's just how much the beam is trying to bend and fold up on the inside at any spot.
- **E** is **Young's Modulus**. Super fancy name, but it's just how stiff the material is. Like, steel is way stiffer than a pool noodle, right? That's its E value.
- **I** is the **Moment of Inertia**. Also sounds complicated, but it's literally just about the beam's shape. A tall, skinny beam is gonna bend way different than a short, fat one. That's I.
- **y** is just how much the beam is sagging down at any given point. Simple.
- And that crazy $\frac{d^2y}{dx^2}$ thingy? That's just math-speak for how curvy the beam gets when it bends.

To figure out the actual sag (y), we gotta do some calculus magic and integrate that moment equation two times. Don't worry 'bout the deets, lol. Let's just see how it works for our two beams.

1 The Cantilever Beam Breakdown

The setup: You've got a beam of length L stuck into a wall at one end ($x = 0$) and just hanging out in the air at the other. Then you drop a load P on it at some spot a.

The derivation requires us to look at two parts of the beam separately: before the load ($0 \leq x \leq a$) and after the load ($x > a$).

1.1 Derivation for $0 \leq x \leq a$

First, we find the internal bending moment $M(x)$ in this section. At the wall, there's a reaction force $R = P$ and a reaction moment $M_{\text{wall}} = P \cdot a$. This gives us:

$$M(x) = Pa - Px$$

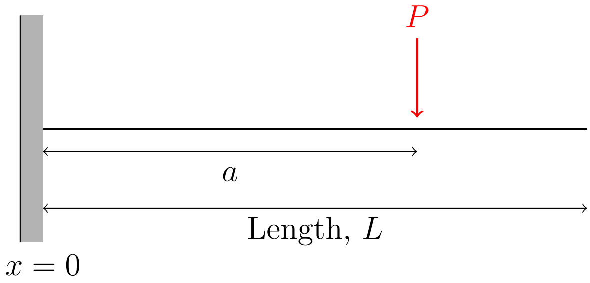

Figure 1: Cantilever beam with a point load.

Plugging this into the main equation (and saying downward movement is positive y) gives $EI\frac{d^2y}{dx^2} = Px - Pa$. We integrate this twice to get the deflection equation:

$$EIy(x) = \frac{Px^3}{6} - \frac{Pax^2}{2} + C_1x + C_2$$

At the wall ($x = 0$), the beam is fixed flat, so its sag ($y(0)$) and slope ($\frac{dy}{dx}(0)$) are both zero. Using this, we find that both constants C_1 and C_2 are zero. So, the final formula is:

$$y(x) = \frac{Px^2}{6EI}(3a - x)$$

1.2 Derivation for $x > a$

In this section, there's no force or moment acting on the beam. So, $M(x) = 0$. This means:

$$EI\frac{d^2y}{dx^2} = 0$$

Integrating this twice gives us a straight line for the deflected shape, which makes sense:

$$EI\frac{dy}{dx} = C_3$$

$$EIy(x) = C_3x + C_4$$

To find the new constants C_3 and C_4, we use a rule called **continuity**. This means the slope and the sag must be the same for both equations right at the point where they meet ($x = a$).

1. **Match the slopes at $x = a$:** We take the slope equation from the first section, $EI\frac{dy}{dx} = \frac{Px^2}{2} - Pax$, and plug in $x = a$. This must equal C_3.

$$C_3 = \frac{Pa^2}{2} - Pa(a) = -\frac{Pa^2}{2}$$

2. **Match the deflections at $x = a$:** We take the deflection equation from the first section, $EIy(x) = \frac{Px^3}{6} - \frac{Pax^2}{2}$, and plug in $x = a$. This must equal our new equation with the C_3 we just found.

$$\frac{Pa^3}{6} - \frac{Pa(a^2)}{2} = C_3(a) + C_4$$

$$-\frac{Pa^3}{3} = (-\frac{Pa^2}{2})a + C_4 \implies C_4 = \frac{Pa^3}{6}$$

Now we put C_3 and C_4 back into our equation for this section:

$$EIy(x) = (-\frac{Pa^2}{2})x + \frac{Pa^3}{6}$$

Cleaning that up gives us the final formula:

$$y(x) = \frac{Pa^2}{6EI}(3x - a)$$

2 The Simply Supported Beam Breakdown

The setup: This time the beam is just resting on two supports, one at each end ($x = 0$ and $x = L$). We put a load P somewhere in the middle at position a. The rest of the length is b.

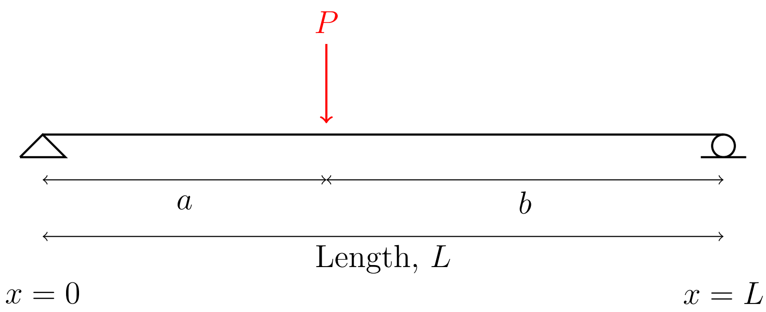

Figure 2: Simply supported beam with a point load.

2.1 Derivation for $0 \le x \le a$

First, how much is each support holding up? The left one (R_L) holds up $R_L = \frac{Pb}{L}$. So for any spot x before the load, the bending moment is:

$$M(x) = R_L \cdot x = \frac{Pbx}{L}$$

We plug that into the main equation and integrate twice, which gives us:

$$EIy(x) = \frac{Pbx^3}{6L} + C_1 x + C_2$$

We know the sag at the left support is zero ($y(0) = 0$), which tells us $C_2 = 0$. The other constant, C_1, is a bit more work to find, but by applying the other boundary and continuity conditions, we find it to be:

$$C_1 = -\frac{Pb}{6L}(L^2 - b^2)$$

Sticking that back in and cleaning it up gives the final formula:

$$y(x) = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2)$$

2.2 Derivation for $x > a$

For the second half of the beam, the easiest way to write the moment equation is by looking from the right support. The right support (R_R) holds up a force of $R_R = \frac{Pa}{L}$. The moment at a point x is:

$$M(x) = R_R \cdot (L - x) = \frac{Pa(L - x)}{L}$$

We pop this into the main equation and integrate it twice to get a general formula for this section:

$$EI\frac{d^2y}{dx^2} = \frac{Pa(L - x)}{L}$$

$$\implies EIy(x) = \frac{Pa}{L}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + D_1 x + D_2$$

Just like before, we use boundary and continuity conditions to find the constants D_1 and D_2. We know the sag at the right support is zero ($y(L) = 0$), and the slope and sag must match the first equation at $x = a$. After a fair bit of algebra to solve for the constants, the equation simplifies to:

$$y(x) = \frac{Pa(L - x)}{6LEI}(2Lx - x^2 - a^2)$$