The possible outcomes of an introduction of an invasive rat species; a report for the Australian Government Department of Environment and Energy

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Contents

1 Executive Summary

This report provides an analysis and evaluation of the possible outcomes that may occur when an invasive species of rat is introduced to an island that is home to a vulnerable bird population, as requested by the Australian Government, Department of Environment and Energy.

Insular island birds are vulnerable to predation from foreign, introduced species due to having no behavioural adaptions that allow them to coexist with these animals. Rats are hypothesised to have a particularly large impact on an islands ecosystem due to both preying on natural inhabitants and competing for other necessary resources, such as nests, in some cases resulting in the extinction of small and vulnerable birds.

This report will endeavour to investigate this hypothesis by using Monte Carlo analysis of a continuous-time Markov Chain to simulate the population of rats and birds over time periods ranging from 5 to 100 years. It was determined that when considering rats and birds to be the only species on the island, extinction of birds was a certainty, but when accounting for a species of owl which preys on both species, a harmonious outcome was rare but possible. However, interventions which reduced the rat population at times when bird extinction appeared likely allowed the species to live in harmony for 100 years with a greater probability. While this model carries several limitations as a result of the assumptions and parameter choices made, assessment of the potential errors generated showed that these errors likely have little effect on the conclusions made.

Thus, the hypothesis was proven to be true and, as a result of the high chance of bird extinction, we do not recommend the introduction of this invasive rat species. Should the rat population be introduced, it is recommended that the Department of Environment and Energy monitors the animal population annually and makes use of common population-control tools, such as a potent rodenticide solution, in order to prevent the extinction of the native bird species.

2 Introduction

It is known that an endangered native bird lives on a small island off the Australian mainland in far north Queensland. The purpose of this report is to investigate the potential outcomes following the introduction of an invasive species of rat, which is known to prey on birds. As a result of their endangered status, this report will outline the circumstances in which birds have a chance of dying out and corresponding preventative solutions which can be undertaken by the Department of Environment and Energy to reduce the likelihood of extinction.

Basic information about the birds is supplied to us, but information which is not and information about the invasive species of rat is obtained via further research. We know that this bird is known to have a maximum population size of 1000, a lifespan of 2-5 years with females producing an average clutch size of 2.2 eggs per year [2]. At the time at which our study begins, there are 500 birds living on the island. Introduced rats are known to prey on birds as their main food source and compete for nests. An initial number of rats are introduced to the island in a controlled manner, and we let this number range from 20 to 140. These rats have an average birth rate of 2.8 infants per year per female rat, and natural lifespans of around 6 months.

2.1 Mathematical Background

Markov Chains are a common and conceptually intuitive way to model the stochastic nature of the spread of an invasive species. This study seeks to model how the introduction of a species of rat to the island affects the birds' natural cycle. The mathematical process used to model the effect of this rat on the birds will be addressed and explained to the extent that one might require to observe and understand the results. Any assumptions used to simplify and clarify the modelling process will also be addressed.

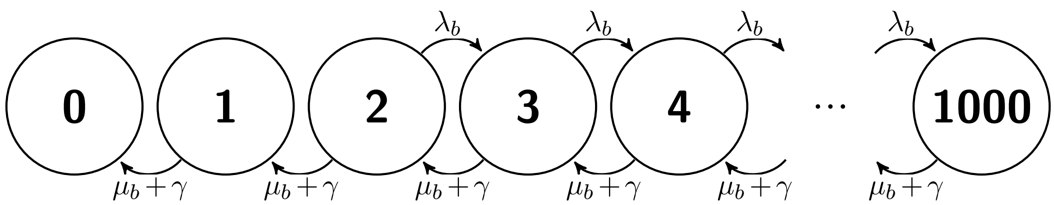
The basic building block of a Markov Chain is the random variable. A *random variable* is a variable for which the outcome is dependent on random phenomena and a *state* of a random variable is a particular value of that random variable. A *stochastic process* is a sequence of random variables. A Markov chain, denoted $X(t)$, is a model of some random process that happens over time, denoted by t. Markov chains follow the Markov property, which says that future states only depend on the present state, or are 'memoryless'.

Markov chains can be discrete or continuous, where Discrete-Time Markov Chains are split up into discrete time steps and Continuous-Time Markov Chains (CTMC) are chains where the amount of time the chain stays in a certain state is randomly picked from an exponential distribution, which means there's an average time a chain will stay in some state, plus or minus some random variation. As a result, a CTMC is used for the modelling of the rat introduction onto the island studied in this report. Furthermore, a Markov chain has states, and the collection of all states is called the state space, which can be finite or infinite depending on the model. If finite, the term transition capacity denotes the maximum state that the process can step in. The behaviour of the process can be represented using transition state diagrams or rate matrices. In transition state diagrams, we denote each stage as a node and the transitions into and out of a state with arrows representing each respective rate. In the rate matrix, the $i - j^{th}$ element represents the rate of which the process steps into the j^{th} state from the i^{th} state.

Fundamental properties of the Markov Chain were used to analyse the effects of the rats on the island, including the stationary distribution and the absorbing states. The stationary distribution of a Markov chain is a probability distribution that remains unchanged as time progresses, which for this problem represents the rats and birds living in harmony. The absorption state is the state in which once a process enters that state, it cannot leave. In this model, there are absorbing states for both the bird and rat population when their population size hits 0. This can either be calculated analytically or using numerical methods, and in this report will be found using Monte Carlo simulations. Monte Carlo simulation is a mathematical technique that generates random variables for modelling risk or uncertainty of a certain system, and is the most tenable method used when a model has uncertain parameters or a dynamic complex system needs to be analysed. It is a probabilistic method for modelling risk in a system, which means that it can be applied to this problem as there is minimal information known about the animals studied.

Our model can be described mathematically in the following way. We let $X(t)$ be the number of birds in the population living on this small island at time t. At the time at which our model begins, there are 500 birds currently living on the island. The state-space for $X(t)$ is $S = \{0, 1, ..., 1000\}$ with $X(0) = 500$. Hence we have a transition capacity of 1000 birds. The natural lifespan for adult birds gives a death rate which ranges from $\mu_{b,adult} = 0.2$ deaths per year to $\mu_{b,adult} = 0.5$ deaths per year, while infant birds have a death rate of $\mu_{b,infant} = 0.3$ deaths per year. As a result of our clutch size, the birth rate is $\lambda_b = 2.2$ births per year. The rats have an average birth rate of 2.8 infants per year per female rat, meaning that $\lambda_r = 2.8$ births per year. It is also known that adult rats have a typical death rate of $\mu_{r,adult} = 1.8$ deaths per year, while infant rats have a typical death rate of $\mu_{r,infant} = 1.2$ deaths per year. Rats prey on birds for food and also compete for nests, which birds require for egg incubation. Adult and infant birds are killed by rats with rates $\gamma_{adult} = 1.7$ adult birds killed by a rat per year and $\gamma_{infant} = 0.3$ infant birds killed by a rat per year.

The transition rate diagram for the population of birds is below. Here we denote μ_b to be equal to $\mu_{b,adult} + \mu_{b,infant}$ and γ to be equal to $\gamma_{adult} + \gamma_{infant}$.



The accompanying transition rate matrix is shown in the Appendix in Section 6.1.

As an extension to our model, we take into account the presence of owls which hunt both owls and rats. As owls and birds have lived in harmony for years preceding the beginning of this study, the effects of these owls on the birds' lifecycle is included within the death rates listed above. When this extension is explained, additional rates for the death of rats will be explained further. As the owls will have no further effect on the birds, the transition rate diagram and rate matrix remain the same.

The remainder of the report will contain the Report Body, where we will list the methodology, describe how the problem is analysed and present the results of our analysis. The interpretation of these results is found in the Discussion, where we will put the results in context and discuss any limitations or possible errors of our model, as well as suggested extensions. We will give our recommendation to the Australian Government Department of Environment and Energy in the conclusion, and all referenced code and other resources are listed in the Appendix.

3 Report Body

3.1 Methodology

This problem was broken up into five successive steps, listed below. This process intended to ensure that the model accurately reflected life on the island at each step and make it possible to easily compare previous versions of the model. All stages of this model were built in Matlab, with all code listed in Section 6.2.

Method process

1. Build a model which only contains the lifecycle of the birds
2. Introduce the rats to the island, taking into account their own lifecycle as well as how they interact with the bird population
3. Take into account that adult animals are more capable than infant animals, in regards to predation, being prey and hunting food
4. Consider the outcomes when there is another animal that preys on both the rats and birds
5. Build scenarios in which there is further intervention from the Department of Environment and Energy, in the form of rodenticide to control the rat population

Several assumptions were made when constructing our model. While it is important to limit the magnitude of the effects these assumptions can have on the results, and hence the possible errors generated, the model cannot be built without the following assumptions.

Assumptions

1. Half the population is male and half are female and so in the case where 2 animals remain, reproduction is possible.
2. One nest is shared by two birds, but rats live independently in one nest each.
3. While rats prefer to feed on birds, they are able to survive off other animals and plants that exist on the island in the case where no birds remain.
4. We have a constant proportion of adult and infant animals. These ratios are 0.83-0.17 adults to infants for rats, and 0.95-0.05 adults to infants for birds [1, 4].

5. The predation rate of rats on birds increases if 80% of the nests are occupied.

6. The rate of rats killing adult and infant birds increases by 0.0001 and 0.0003, respectively [3].

7. The reproduction rate of both rats and birds increases by 0.0001 if less than 80% of the nests are occupied [3].

8. Owls do not live in the same nests as rats and birds and so they do not compete for nests.

9. Before this study began, owls were preying on birds at a constant rate, and as such, this is taken into account in their natural death rate.

10. Rodenticide only harms rat species.

3.2 Analysis of the problem

The problem was analysed in several phases, listed below. The accompanying code referenced is shown in Section 6.2.

Simple model
The first stage of of our model was simply accounting for the natural lifecycle of the birds on the island, before introduction of rats. This included their birth rate, death rate due to lifespan and the limiting factor of the number of nests available on the island. This simple model can be seen in our Matlab code *montecarlo.m*.

After doing this, we introduced the rats. We chose to initially have an introduction of 20 rats to the island, as we found that this initial number does not substantially change the predicted population trajectories, an example of which we will show in section 3.3. The birth rates, death rates, and predation rates of both adult and infant rats were included, as well as the death rates of adult and infant birds due to rat predation.

Introduction of predator owls
An extension to this model is to account for the possibility of another animal living on the island which preys on both rats and birds. As this predator owl coexisted with the birds prior to the introduction of the rats, we can assume that the death rates of the birds already established take these owls into account. However, once rats are introduced to the island, the owls have another source of food and the rats die at a higher

rate than seen in the simple model.

Rodenticide application

Another extension to this model is the introduction of rodenticide, which kills a certain percentage of the current rat population at the time applied, depending on the dose applied. This rodenticide can be applied at any time throughout the study and there are no restrictions on the frequency of application. This allowed us to test varying time points at which rodenticide was applied to rat nesting areas and observe the effects on what killing certain proportions of the rat populations had on the bird population. Previous studies have shown that rodenticide application usually kills between 20% - 80% of the population, values which can be seen in our Matlab code *montecarlo.m*. We applied the poison at varying time points, the resultant plots of which can be seen in the following section.

Rodenticide was also added using a systematic and automatic approach. It was determined whether rodenticide should be applied to the rat population at the beginning of each year using the calculation

$$|X_b - X_r| < 100.$$

That is, if the population size of birds and rats at each new year is within 100 animals, the rat population is deemed to be too close to the size of the bird population and the number of rats is decreased. This addition can be seen in the code *montecarlo.m*, and can be commented out in instances where rodenticide is not applied.

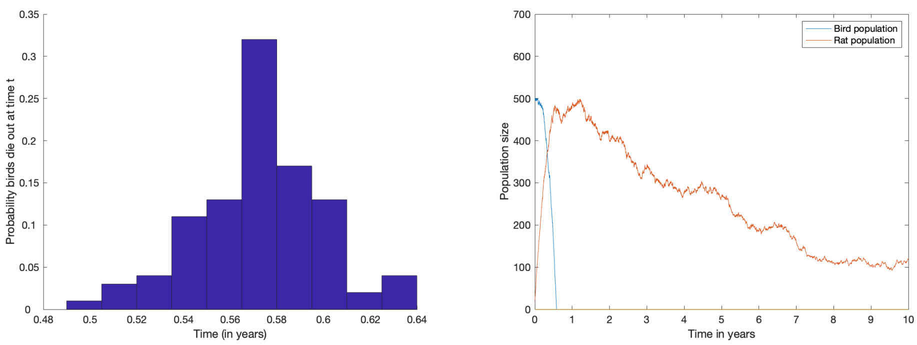
3.3 Results

Simple model

We ran the model 100 times in order to determine the probability mass function of the time at which the bird population goes extinct, or the size of the bird population reaches 0. We also plot a Monte Carlo simulation showing the population size over a total time of 10 years. To demonstrate our simple model, we show output for varying parameters, specifically the intial size of the rat population and the predation rates of rats on birds.

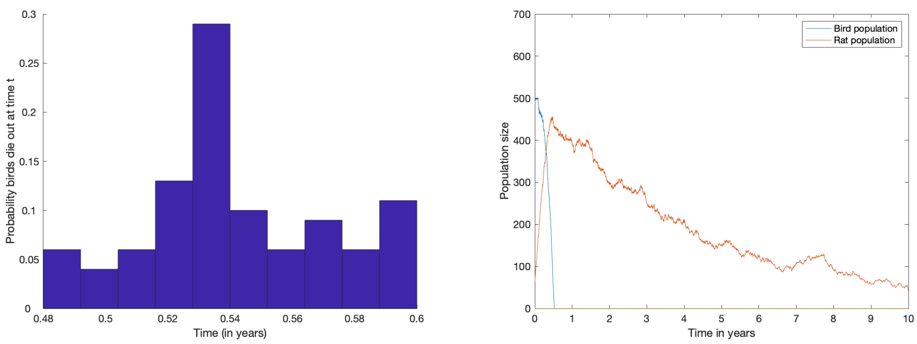
Varying the initial rat population size

We keep the other parameters constant and equal to the values listed in the introduction. As the death rate of adult birds can take a range of values, we choose to use $\mu_{b,adult} = 0.2$ deaths per year.



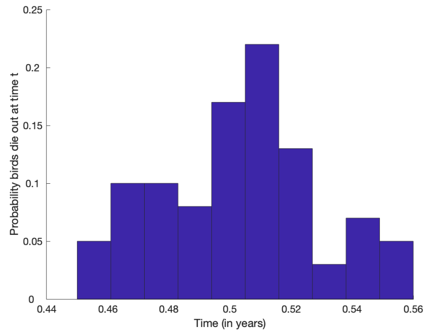
(a) Probability mass function (b) Population over time

Figure 1: Population size of 20

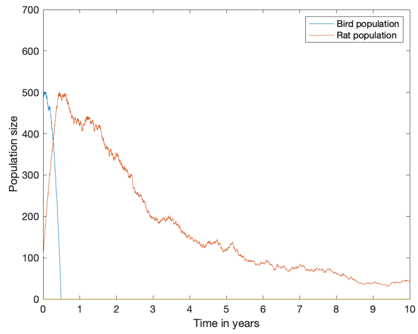


(a) Probability mass function (b) Population over time

Figure 2: Population size of 60

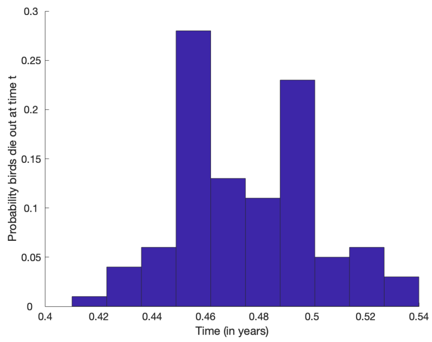


(a) Probability mass function

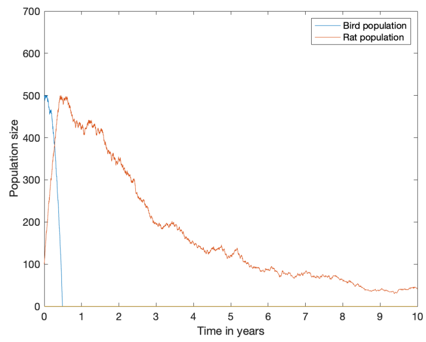


(b) Population over time

Figure 3: Population size of 100



(a) Probability mass function

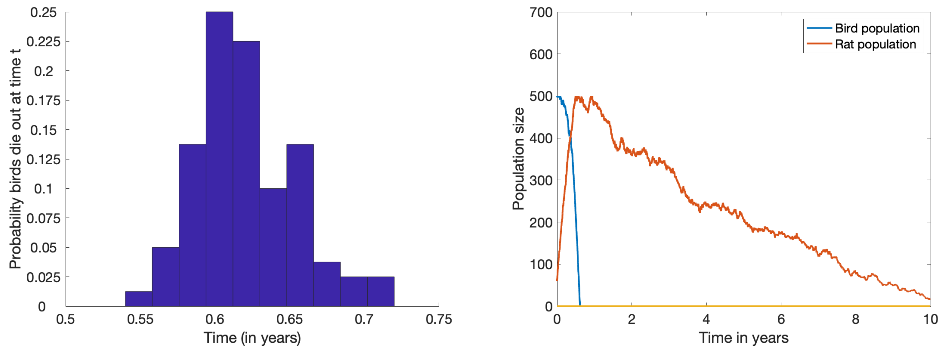


(b) Population over time

Figure 4: Population size of 140

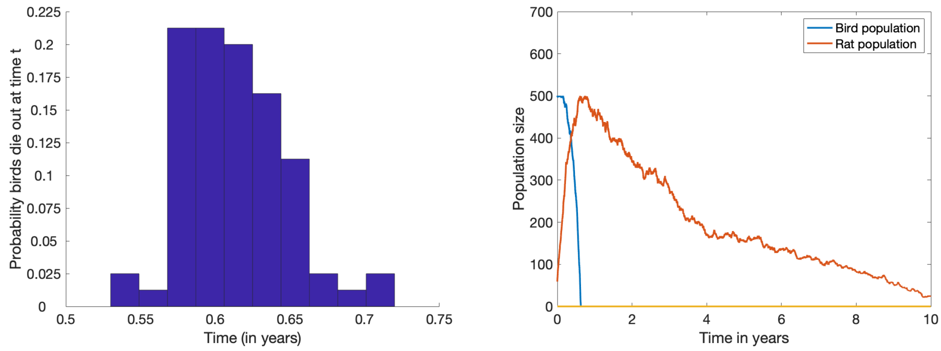
Varying predation rates of rats

Next, assuming that we keep the initial number of rats constant at 60, we can see the effects of changing the predation rates of both the infant and adult rats on the population size of birds. We set the natural death rates of adult birds to be $\mu_{b,adult} = 0.2$ deaths per year and use the rates as described in the introduction. Then we vary both the rates of adult and infant rats killing birds, ranging from $\gamma_{b,child} = \gamma_{b,adult} = 0.5$ birds killed by a rat per year to $\gamma_{b,child} = \gamma_{b,adult} = 1.5$ birds killed by a rat per year. We output the probability mass function of the time at which birds are expected to go extinct and a simulated run of the population size of birds and rats over 10 years.



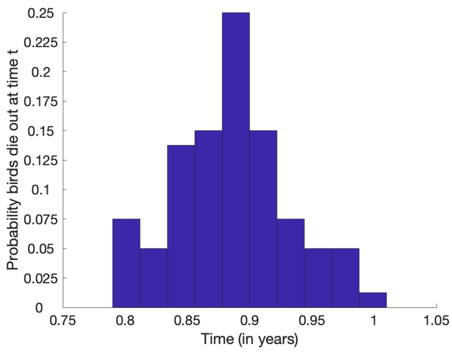
(a) Probability mass function (b) Population over time

Figure 5: $\gamma_{b,child} = 0.5, \gamma_{b,adult} = 1.5$

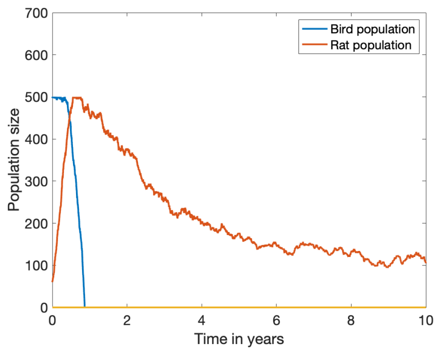


(a) Probability mass function (b) Population over time

Figure 6: $\gamma_{b,child} = 1, \gamma_{b,adult} = 1.5$

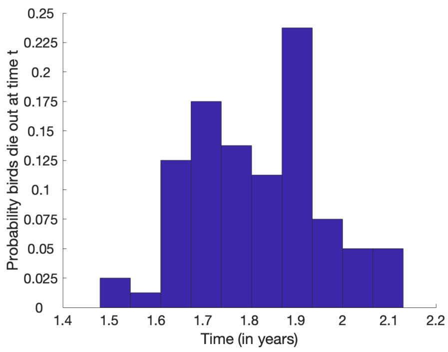


(a) Probability mass function

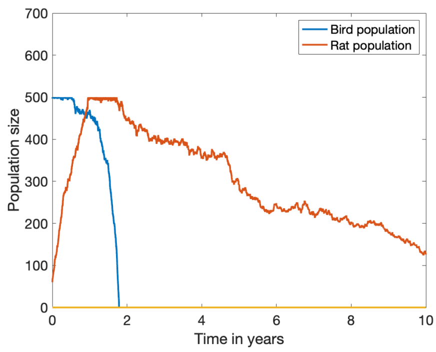


(b) Population over time

Figure 7: $\gamma_{b,child} = 1.5, \gamma_{b,adult} = 1$



(a) Probability mass function

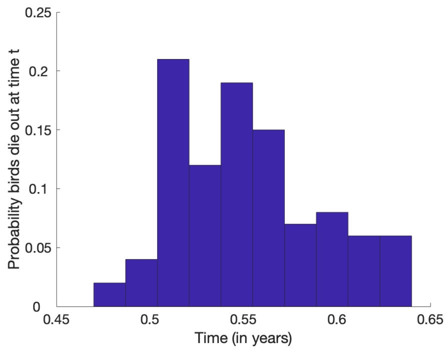


(b) Population over time

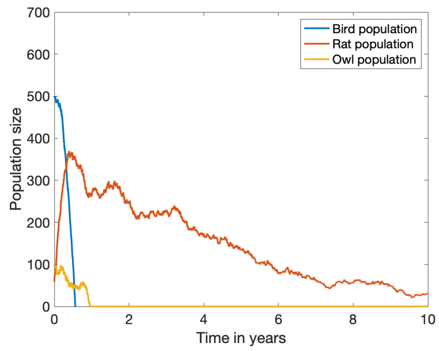
Figure 8: $\gamma_{b,child} = 1.5, \gamma_{b,adult} = 0.5$

Owls

Owls were introduced as a predator to both rats and birds, but it is assumed that owls have been hunting birds prior to the beginning of the study, so this is included in the birds natural death rate. For these owls, we have a natural death rate of $\mu_{owl} = 1.1$ deaths per year and a birth rate of $\lambda_{owl} = 2.2$ births per year. We can observe the effects on the rat population by varying the rate at which they kill rats, γ_{owl}. We choose to let γ_{owl} range from 0.5 to 2 rats killed by an owl per year. Once again, we output both the probability mass function of the time at which birds are expected to go extinct and a simulated run of the population size of birds, rats and owls over 10 years.

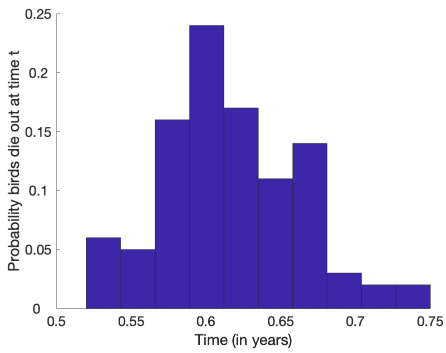


(a) Probability mass function

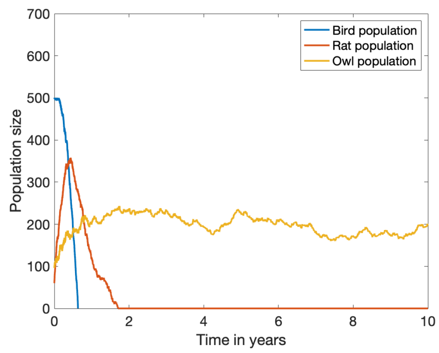


(b) Population over time

Figure 9: $\gamma_{owl} = 0.5$

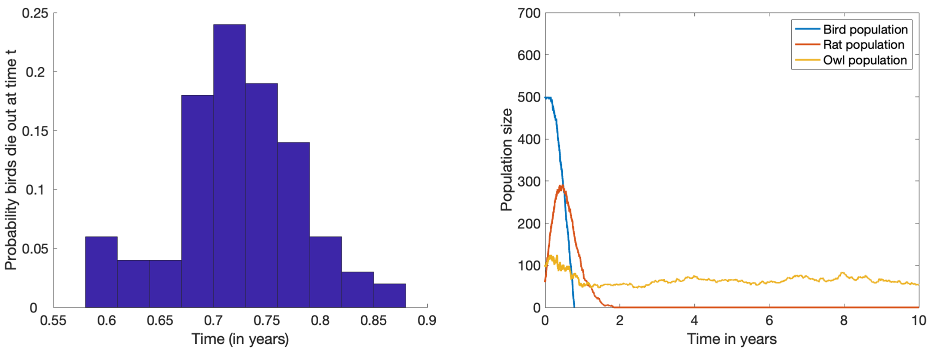


(a) Probability mass function



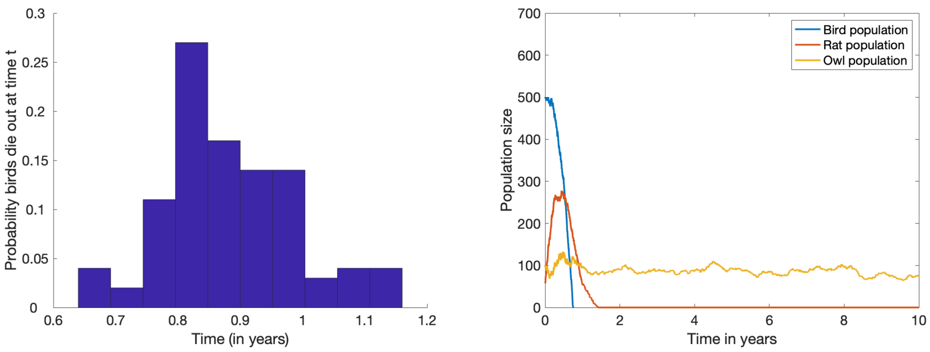
(b) Population over time

Figure 10: $\gamma_{owl} = 1$



(a) Probability mass function (b) Population over time

Figure 11: $\gamma_{owl} = 1.5$

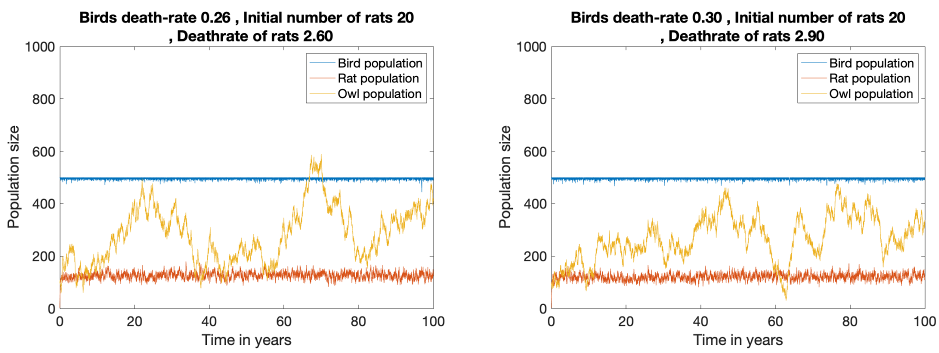


(a) Probability mass function (b) Population over time

Figure 12: $\gamma_{owl} = 2$

Harmonious living

For simulating animals live in harmony, the time constraint was increased to 100 years, as accumulating data over 100 years of the simulation is a better estimator of the long term outcome. The parameters of the initial size of the rat population is set to 20, birds death rate ranges from $\mu_{b,adult} = 0.2$ deaths per year to 0.5 with an increment of 0.06, infant rats death rate ranges from $\mu_{r,infant} = 1$ to 2 deaths per year with increment 0.1 and the remainder of parameters are as listed in the introduction. For some parameters the rat introduction is unsuccessful and the rats kill off all the birds which means that the animals are unable to live in harmony. For the following plots, however, the animals are able to survive for 100 years without either the rat or bird reaching extinction. Two pairs of parameters for which this is possible, $\mu_{b,adult} = 0.25$ deaths per year, $\mu_{r,infant} = 2.6$ deaths per year and $\mu_{b,adult} = 0.3$ deaths per year and $\mu_{r,infant} = 2.9$ deaths per year, are used to simulate the population over 100 years and graphed below.



(a) Population over time $\mu_b = 0.26, \mu_r = 2.6$

(b) Population over time $\mu_b = 0.3, \mu_r = 2.9$

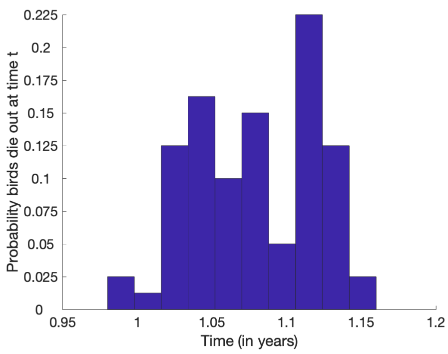
Figure 13: Harmonious living

When run 100 times, the number of simulations where the birds became extinct within a certain time range is shown below.

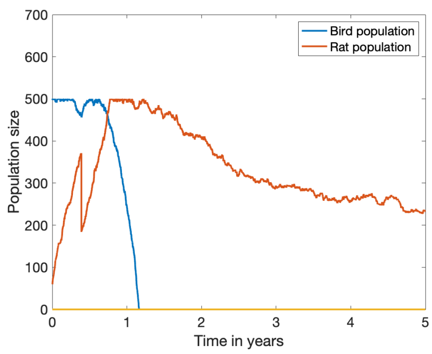
Time range (years)	Number of simulations
$0 \leq t < 10$	48
$10 \leq t < 20$	17
$20 \leq t < 30$	5
$30 \leq t < 40$	6
$40 \leq t < 50$	3
$50 \leq t < 60$	0
$60 \leq t < 70$	2
$70 \leq t < 80$	0
$80 \leq t < 90$	4
$90 \leq t < 100$	0
$t \geq 100$	15

Rodenticide

Rodenticide can be manually administered at specific time points and kills 20% - 80% of the current population of rats in attempt to prevent bird extinction or prolong their survival. In the following figures, rodenticide is applied at time points $t = 0.2$ and $t = 0.4$ and the population of rats and birds is observed over 5 years. The initial rat population size is 60, and $\mu_{b,adult} = 0.2$ deaths per year is chosen alongside the remainder of the parameters listed in the introduction.

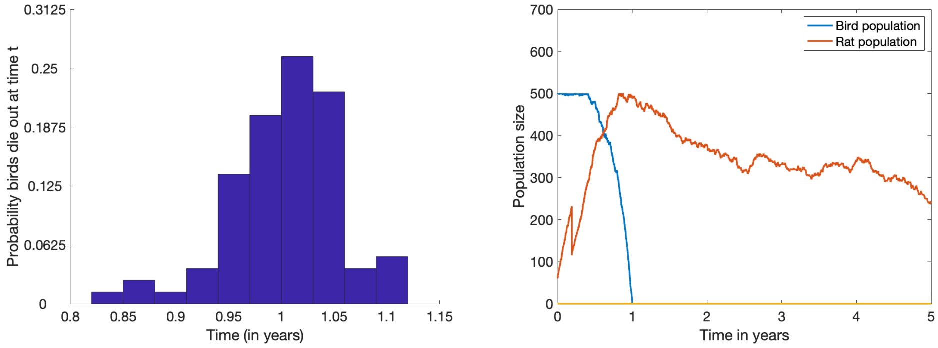


(a) Probability mass function



(b) Population over time

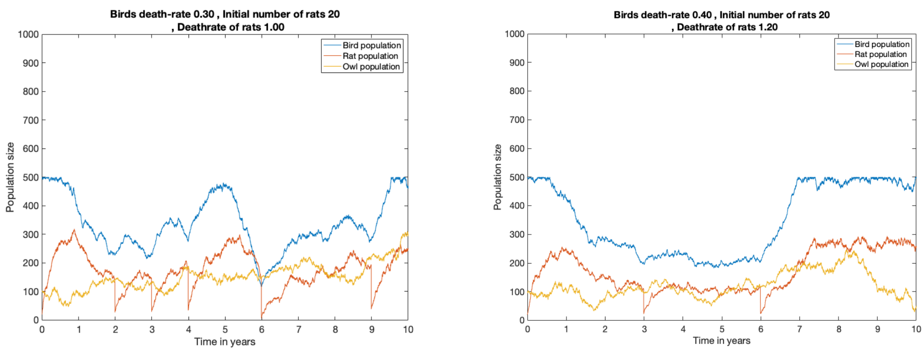
Figure 14: Rodenticide application at time $t = 0.4$



(a) Probability mass function (b) Population over time

Figure 15: Rodenticide application at time $t = 0.2$

Rodenticide can also be systematically applied at yearly time points if the rat and bird population are deemed too close together, say within 100 animals. This allows researchers to control the rat population while only having to visit and count the population numbers once a year. Below we have accounted for two different scenarios, both where there are initially 20 rats introduced to the island. We make parameter choices of $\mu_{b,adult} = 0.3$ deaths per year, $\mu_{r,infant} = 1$ deaths per year and $\mu_{b,adult} = 0.4$ deaths per year, $\mu_{r,infant} = 1.2$ deaths per year and keep all other parameters as described in the introduction. We plot the populations of birds, rats and owls simulated over a 10 year period below.



(a) $\mu_b = 0.3, \mu_{r,infant} = 1$ (b) $\mu_b = 0.4, \mu_{r,infant} = 1.2$

Figure 16: Rodenticide application at yearly time points

We can run the above simulation 100 times to assess the probability of extinction within a certain time period. We use the method of rodenticide application as described above over 10 years.

Time range (years)	Number of simulations
$0 \leq t < 10$	52
$t \geq 10$	48

When run 100 times over a time period of 100 years, the number of simulations where the birds lived until a certain time range with rodenticide application occurring as listed above is shown below.

Time range (years)	Number of simulations
$0 \leq t < 10$	51
$10 \leq t < 20$	14
$20 \leq t < 30$	11
$30 \leq t < 40$	2
$40 \leq t < 50$	3
$50 \leq t < 60$	1
$60 \leq t < 70$	0
$70 \leq t < 80$	0
$80 \leq t < 90$	0
$90 \leq t < 100$	0
$t \geq 100$	18

4 Discussion

4.1 Analysis of results

Simple model

We varied two key parameters when looking at the simple model; the initial rat population size and the rat predation rate.

We can see from Figures 1-4 that the initial rat population size does not have a substantial effect on the time at which the bird population dies out or the rate of decline. The probability mass functions for each parameter choice are centred around values which range from 0.57 for an initial rat size of 20 to 0.47 for an initial rat size of 140. From this, we can conclude that for larger initial rat sizes, the birds tend to die out more quickly, but this varies by around a tenth of a year.

The effect of the rat predation rate was tested using a combination of bird predation rates for both adults and infants. From Figures 5 and 6, we can see that when $\gamma_{b,adult} = 1.5$ adult birds killed by a rat per year, the expected time of bird extinction is centred around 0.6 years, no matter the value of $\gamma_{b,infant}$. When the value of $\gamma_{b,adult}$ is reduced to be equal to 1 and 0.5 in Figures 7 and 8, the expected time of extinction is increased from being centred around 0.9 to being centred around 1.8, despite the value of $\gamma_{b,child}$ remaining constant. This suggests that the rate at which the adult birds are killed is more important to the survival time of birds than the rate at which infant birds are killed.

Predator owls

The predation rate of owls affects the rat population which consequently affects the bird population. The predation rate tested ranged from $\gamma_{owl} = 0.5$ birds killed by an owl per year to 2 birds killed by an owl per year, and in Figures 9-12 it is clear that by increasing γ_{owl}, the expected time of bird survival increases from having a mean of around 0.55 when $\gamma_{owl} = 0.5$ rats killed by an owl per year to have a mean of around 0.9 when $\gamma_{owl} = 2$ rats killed by an owl per year. From this, we can see that the predation rate of owls on rats increasing benefits the bird population and results in a longer survival period. We can also see from Figure 9b that when $\gamma_{owl} = 0.5$ rats killed by an owl per year, the owl population quickly dies out and as a result, has little effect on either bird or rat population. For $\gamma_{owl} = 1$ rats killed by an owl per year and above, the owls live at a constant population size over the 10 years studied.

Harmonious living

The population sizes of rats, birds and owls were analysed over a long time period, 100 years. The parameter ranges chosen, $\mu_{b,adult} = 0.2$ to 0.5 deaths per year and $\mu_{r,infant} = 1$ to 7 deaths per year, resulted in harmonious conditions only when neither the rat or bird population declined substantially. This outcome occurred in 15% of the simulations ran, and is not possible without the presence of owls.

Rodenticide application

We analysed the effect of applying rodenticide both manually at specific time points and systematically at yearly time points depending on whether the rat and bird populations were deemed too close to one another.

As we used parameters previously studied, we knew the expected time at which the bird population would die. By applying rodenticide at time points $t = 0.4$ years and $t = 0.2$ years, we can compare the output to Figure 2, where we have used the same parameters without rodenticide application. In Figure 2b, we have a mean expected bird extinction time of around 0.54 years. In Figures 14 and 15, when rodenticide is applied, we have a mean expected bird extinction time of around 1.05 and 1 years, for application at time $t = 0.4$ years and $t = 0.2$ years respectively. We can see from this comparison that any rodenticide application before the point at which birds die is going to prolong the birds survival time, but applying it closer to $t = 1$ years, when the rat population is larger, is going to have a greater effect on slowing the decline of the bird population.

The circumstance in which rat and bird populations are close together reflects the rat population growing and the bird population decreasing, due to the respective initial values of each animal and parameters the model begins with. As such, when the populations become close, say each population only differs by 100 animals, the rat population must be substantially decreased in order to prevent the bird population from going extinct. The result of this is shown in Figure 16, where we showed that for $\mu_{b,adult} = 0.3$ birds die per year, $\mu_{r,infant} = 1$ rats die per year, the population required rodenticide application 5 times in 10 years to remain in an harmonious state, and for $\mu_{b,adult} = 0.4$ birds die per year, $\mu_{r,infant} = 1.2$ rats die per year, the population required two applications of rodenticide to remain in this state. This shows that if rodenticide is applied at these time points whenever the populations are within 100 animals difference, the bird and rat populations can live together on the island without either dying out 48% of the time over a 10 year span. If we look at a 100 year period, only 18% of the simulations will result in birds surviving for the entire time period. This is still an improvement over the results found without the use of rodenticide, but only an improvement of 3%. This method can be used at any time point, but spacing out the times of rodenticide application increases the chance that birds will die out before extinction can be prevented.

4.2 Possible errors

Every assumption listed in Section 2.1 carries a certain degree of error, the most influential of which are possible incorrect birth, death and predation parameters. By systematically changing these parameters and observing the effects on the resultant population size and estimated time of bird extinction, the error size can be estimated.

We start this analysis with the simple model, which does not include owls, rodenticide or any distinction between infant and adult animals. The death rate of the rats has the biggest effect on the birds extinction time. Most of the known and widely modelled death rates for different species of rat are covered within the parameter range selected and modelled, although we cannot be assured that this model will accurately predict how any species of rat would behave in a tropical setting. That could mean that the rats are not able to adapt to the new environment, which is unlikely, or that rats could have natural predators on the island, which would increase their death rate. The fact that the rats have the potential to change the whole ecosystem of the island, for example through being carriers of disease, was not considered in the model.

Another big assumption concerning only the simple model is the population size of the birds. As it was given to us, there was no further research done investigating whether or not this is true. In reality, this number may not be accurate, which would have a huge impact on the time when the birds die out. The birth, death and predation rates of both infant and adult rats was found using independent research, each of which carries a certain degree of uncertainty. However, according to the model it carries not much weight in the light of the outcome.
Extending the model with the possibility of rodenticide application can be used to maintain harmonious living between rats and birds. Rats may adapt to the use of the poison which makes them more tolerable to it and consequently, a smaller proportion of rats may die out with each application.

Another assumption which has a large effect on the outcome of the model is that the rodenticide only effects the rats, as you would expect that applying poison would carry further adverse effects than considered here. Furthemore, the model was again extended by taking into account the possibility of owls preying on both animals. We assumed that once the rats were introduced, the owls had not changed the rate at which they preyed on the birds. If this assumption turns out to be false, then the death rate of the birds will change which is a huge factor in predicting when the birds will die out.

To see which parameters carry a larger potential for error, sensitivity analysis was done. Changing the value of the each parameter by a small amount will show whether or not that parameter choice is crucial for the success of the model. The default settings used within the model, and those which are changed in our sensitivity analysis are the proportion of bird infants, $prob_b = 0.17$, the proportion of rat infants, $prob_r = 0.5$, the predation rate of rats on infant birds, $\gamma_{b,infant} = 1.5$, the death rate of infant birds, $\mu_{b,infant} = 0.7$, the predation rate of rats on adult birds, $\gamma_b = 1$, and the birth rate of birds, $\lambda_b = 2.2$.

If we run the simulations for a 100 times with an initial rat size of 20, birds death rate equal to $\mu_b = 0.2$ and $\mu_{r,infant} = 1.2$ on average, 50% of the simulations will result in bird extinction within 10 years. For a small change in a parameter it is expected that there will be a small change in the proportion of the birds that die out. We expect to see between 40-60% of the simulations resulting in extinction, but we perform sensitivity analysis on each of the parameters to assess the choices made. The number of simulations that resulted in bird extinction when changing the value of each parameter estimated is shown in the table below.

	Value used	+10%	+20%	+30%	+40%
Birds infants:adults	42	41	45	56	52
Rats infants:adults	50	45	44	43	57
$\gamma_{b,infant}$	49	37	51	43	56
$\mu_{b,infant}$	49	48	47	53	52
γ_b	50	53	55	46	63
λ_b	50	54	53	56	50

As we can see, the parameter choices for rats and birds and reasonably stable and small changes in their values do not substantially affect the number of simulations for which birds become extinct within 10 years.

4.3 Extensions of the model

Each subsection and extension of the simple model can be extended further to imitate reality more accurately. Firstly, a system can be added to keep track of the age of the animals and track the point at which they become mature, capable of reproduction, better predators or more challenging prey. This would be a more realistic depiction of the natural life-cycle of the animals.

It is safe to assume that over time birds would adapt to rat predation and develop preventative strategies to save themselves. This would change the predation rates of the birds, which are one of the two key parameters in predicting the time when the birds become extinct. As, by assumption, the predation rate would decrease by a small amount and the time when the birds terminate would increase by a small amount. It is not expected to substantially affect the result, but is it safe to expect to be able to predict the amount of rodenticide needed for a certain time frame.

After introducing the rats to the island, the death rate of the owls could increase due to the rats preying on their nests for eggs, which would shift the whole model by a small fraction. Yet again, this would not have an a big enough effect on the outcome to change the results, but would only allow for more accurate prediction of the final outcome.

Rats adapting to rodenticide would change the effectiveness of the poison. Over time a higher dose or different kind of rodenticide is advised to be used. This could be done by introducing a new parameter for rats that would measure the immunity for the animals to the poison at a certain time. Furthermore, the poison would likely affect the overall life-cycle of the island, which can be researched further and added into the model.

5 Conclusions

The existence of native birds on the island is dependent on their predators and the availability of the necessary resources on the island that are required for survival. Mathematical techniques were applied to this problem, primarily a Monte Carlo simulation of a continuous-time Markov chain, for which parameters used were both given to us and obtained through previous literature.

Various models were constructed, beginning with a simple model which considered only the interaction between adult birds and adult rats, and showed that rats quickly forced birds to extinction. Furthermore, other animals on the island were considered when an owl that preys on both birds and rats was introduced. This allowed for some control of the rat population, and as a result, harmonious outcomes where birds and rats were able to coexist were produced for a variety of birth, death and predatory parameters. As a final extension to this model, a population control method to prevent the rat population growing in an uncontrolled manner was introduced in the form of a rodenticide. With a single dose of rodenticide, the rat population was able to bounce back quickly, but with regular and systematic doses, there is potential that this strategy will reliably prevent bird extinction.

Utilisation of these mathematical techniques have shown that, assuming minimal parameter error, introducing rats to this island has risks that at worst result in the extinction of a species of birds and at best, further intervention can reduce the risk of extinction but it cannot be certain that this will remove the risk all together. Specifically, we have showed that the rat population is fast-growing and that, unless it is found that rats have predators on the island, the growth of their population will always result in a rapid decrease of the bird population. As a result, our recommendation to the Department of Environment and Energy is not to introduce this particular rat species as our models have shown that there is a high likelihood that doing so will be devastating for the population of birds, and if it is decided that this rat population should be introduced, regular monitoring of the population numbers should be undertaken in order to apply population control methods when applicable.

6 Appendix & Code

References

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6.1 Transition rate matrix

The transition rate matrix for the population size of birds is

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \mu_b + \gamma & -(\mu_b + \gamma) & 0 & 0 & 0 & \dots & 0 \\ 0 & \mu_b + \gamma & -(\mu_b + \gamma + \lambda_b) & \lambda_b & 0 & \dots & 0 \\ 0 & 0 & \mu_b + \gamma & -(\mu_b + \gamma + \lambda_b) & \lambda_b & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -(\mu_b + \gamma) \end{bmatrix}$$

6.2 Code

montecarlo.m

```
%calls the MonteCarlo simulation

% function input
% number of initial rats
% deathrate of birds (known to be betwenn 0.2-0.5)
% boolean to print a single plot

% function returns
% the time when the birds died out in time t

function [birdsout] = montecarlo(X_r_in,mu_b, printPlot,mu_r_child_in,version,T_rodenticide)
% hold all
prob_b=0.17; % percentage of rat infants
prob_r=0.5; % percentage of bird babies

% bird parameters
mu_b_child = 0.7; % death rate of baby birds
gamma_b_child = 1.5; % Rate of rats killing  baby birds
gamma_b = 1; % Rate of rats killing adult birds
lambda_b = 2.2; % birth rate
mu_b=mu_b;
counts=0;

% rat parameters
mu_r_child=mu_r_child_in; %deathrate of infant birds
mu_r = mu_r_child*1.3; % death rate of adult rats
lambda_r = 2.8; % birth rate

% owl parameters to be used when including owls
gamma_owl = 2; % rate of owls killing rats
lambda_owl = 2.2; % birth rate of owls
mu_owl = 1.1; % death rate of owls
N_owl = 500; % number of owls possible
X_owl = 100; % Initial number of owls
X_out_owl = X_owl;
a_owl = zeros(2, 1);

% % owl parameters to be used when NOT including owls
% gamma_owl = 0; % rate of owls killing rats
% lambda_owl = 0; % birth rate of owls
% mu_owl = 0; % death rate of owls
% N_owl = 0; % number of owls possible
% X_owl = 0; % Initial number of owls
% X_out_owl = X_owl;
% a_owl = zeros(2, 1);

N_nest=1000;
N_b = 1000; % number of birds possible.
N_r = 500; % number of rats possible
T = 100; % length of simulation(in years)
T_all1 = [1:T] - 0.01;
T_all2 = [1:T] + 0.01;
T_all = sort([T_all1, T_all2]);

% initial conditions for birds and rats
X_b = 500; % initial number of birds
X_r=X_r_in; % initial number of rats
t = 0;

% Set parameter values
a_b = zeros(3,1);
a_r = zeros(3,1);

X_out_b = X_b;
t_out = 0;

X_r=X_r_in;
X_out_r = X_r;



while true
    X_b(X_b<0) = 0; % Population of birds has to be non-negative
    X_r(X_r<0) = 0; % Population of rats has to be non-negative
```

```
X_owl(X_owl<0) = 0; % Population of owls has to be non-negative

% step 1. Calculate the rates of each event given the current state.
% Natural deaths
a_b(1) = mu_b*X_b*(1-prob_b)+prob_b*X_b*mu_b_child; % natural bird death
a_r(1) = mu_r*X_r*(1-prob_r)+prob_r*X_r*mu_r_child; % natural rat death
a_owl(1) = mu_owl*X_owl; % natural owl death

rat_rate_adult=X_r*(1-prob_r);
rat_rate_infant=prob_r*X_r;
bird_rate_adult=X_b*(1-prob_b);
bird_baby=prob_b*X_b;
number_of_adult_female_birds=(X_b/2)*(1-prob_b);
number_of_adult_female_rats=(X_r/2)*(1-prob_r);



% Bird death due to rats killing birds
if X_b + X_r > 0.8*N_nest % Fighting rate increases if more than 80% of nests are occupied
    gamma_b = gamma_b + 0.0001;
    gamma_b_child = gamma_b_child + 0.0003;
    a_b(3) = rat_rate_adult*gamma_b+rat_rate_infant*gamma_b_child; % bird death due to rat predation
else
    a_b(3) = rat_rate_adult*gamma_b+rat_rate_infant*gamma_b_child;
end

% Rat death due to owls killing rats
if X_owl > 0
    a_r(3) = gamma_owl*X_r;
else
    a_r(3) = 0;
end

% Bird births
if X_b<N_b/2   % as nests are shared by 2 birds
    if X_b*(1-prob_b) < 0.2*N_b
        lambda_b = lambda_b + 0.0001; % Birth rate increases if there are more than 80% the total nests
available
        a_b(2) = lambda_b*number_of_adult_female_birds;  % birth multiplied by number of female adult birds
    else
        a_b(2) = lambda_b*number_of_adult_female_birds;  % birth multiplied by number of female adult birds
    end
else
    a_b(2) = 0; %
end



% Rat births if there are birds and if there are not
if X_b>0
    if X_r<N_r % If less rats than nests
        if X_r < 0.2*N_r
            lambda_r = lambda_r + 0.0001; % Birth rate increases if there are more than 80% the total nests available
            a_r(2) = lambda_r*number_of_adult_female_rats + gamma_b*bird_rate_adult+gamma_b_child*bird_baby;  % birth
multiplied by number of female rats + number of birds to eat
        else
            a_r(2) = lambda_r*number_of_adult_female_rats + gamma_b*bird_rate_adult+gamma_b_child*bird_baby;  % birth
multiplied by number of female rats + number of birds to eat
        end
    else
        a_r(2) = 0;
    end
else
    if X_r<N_r    % Less rats than nests
        a_r(2) = lambda_r*(X_r/2);  % birth multiplied by number of female rats
    else
        a_r(2) = 0;
    end
end

% Owl births
if X_owl > 0
    a_owl(2) = lambda_owl*(X_owl/2);
else
    a_owl(2) = 0;
end

a0_b = a_b(1)+a_b(2) + a_b(3);
a0_r = a_r(1)+a_r(2) + a_r(3);
a0_owl = a_owl(1) + a_owl(2);



% step 2. Calculate the time to the next event.
```

```
    t = t - log(rand)/(a0_b + a0_r + a0_owl);

    if t > T
        break
    end



    % step 3. Update the state.
    if X_r >= 0 || X_b > 0 && X_owl > 0
        if rand*a0_b < a_b(1) + a_b(3)
            X_b = X_b - 1;
        elseif X_b > 1
            X_b = X_b + 1;
        end
        if rand*a0_r < a_r(1) + a_r(3)
            X_r = X_r - 1;
        elseif X_r > 1
            X_r = X_r + 1;
        end
        if rand*a0_owl < a_owl(1)
            X_owl = X_owl - 1;
        elseif X_owl > 1
            X_owl = X_owl + 1;
        end
    else
        if rand*a0_r >= a_r(1) && X_r < N_nest/2 && X_r >= 2
            X_r = X_r + 1;
        else
            X_r = X_r - 1;
        end
    end

    % Rodenticide introduction kills a proportion of the rats in the population
    % If no rodenticide, make T_rodenticide > T
    if t <= T_rodenticide + 0.01 && t >= T_rodenticide - 0.01 && counts == 0
        X_r = 0.1*X_r;
        counts = counts + 1;
    end

% comment it out for figure 16
% Rodenticide that applies rodenticide at T = 1, 2,... and kills 80% of
% rats if diff between X_b and X_r is less than 100
    for k = 1:2:2*T
        if t >= T_all(k) && t <= T_all(k+1) && abs(X_r - X_b) < 100
        X_r = 0.2*X_r;
        end
    end

    % record the time and state after each jump
    X_out_b = [X_out_b, X_b];
    t_out = [t_out, t];
    X_out_r = [X_out_r, X_r];
    X_out_owl = [X_out_owl, X_owl];

end

%time when birds died out
%comment in if you would like to see the errors tables
 birdsout = t_out(min(find(X_out_b==0)));



% % comment this on if intrested in the proportion of the simulations fails
% % comment in if you would like to see the errors tables

% birdsout=1;
% birdsout=min((find(X_out_b==0)));
%
% if birdsout>1
%     birdsout= 2;
% else
%     birdsout= 0;
% end



% creates a plot representing the trajectory of the simulation
% depending on what we are after
```

```
% version one represent the  stationary plots
%else the other plots where either or both dies out
if version ==1
    if(X_out_b~=0)
        if printPlot
            figure();
            plotSimulation(X_r_in, mu_b, X_out_r, X_out_b, X_out_owl, t_out,T,mu_r_child_in);
        end
    end
else
    if printPlot
        figure();
        plotSimulation(X_r_in, mu_b, X_out_r, X_out_b, X_out_owl, t_out,T,mu_r_child_in);
    end
end

% comment it out if you would like to see the possible error mesaure
plotSimulation(X_r_in, mu_b, X_out_r, X_out_b, X_out_owl, t_out,T,mu_r_child_in);
```

plotPmf.m

```
% creates a pmf bar-chart representing when the probability of birds dying at a certain timeframe

% function input
% buckets proportion
% timeframe

function [] = plotPmf(nRuns,time_of_birds_deaths)

 %creates an array of the possible number of steps the simulation can hit 0

%   creates the bar-chart with
%   buckets proportion
%   timeframe
  time_of_birds_deaths=round(time_of_birds_deaths,2);
  hist(time_of_birds_deaths);
  ytix = get(gca, 'YTick');
  set(gca, 'YTick',ytix, 'YTickLabel',ytix*1/nRuns)
  title(sprintf('(PMF-(Monte Carlo Simulation))\n Birds dying out by time t'));
  xlabel('Time (in years)')
  ylabel('Probability birds die out at time t');
end
```

plotSimulation.m

```
% creates a plot of the simulation with
% x axis time
% y axis number of birds
% two trajectory : number of birds, number of rats

% function input
% number of rats
% birds death rate
% vector holding rats state trajectory
% vector holding birds state trajectory
% vector holding rats time trajectory
% vector holding birds time trajectory

function [] =  plotSimulation(X_r_in, mu_b, X_out_r, X_out_b, X_out_owl, t_out,T,mu_r_child_in)
    plot(t_out, X_out_b, t_out, X_out_r, t_out, X_out_owl);
    title(sprintf('\n Birds death-rate %.2f , Initial number of rats %d \n  , Deathrate of 
rats %.2f',mu_b, X_r_in,mu_r_child_in), 'LineWidth', 2);
    ylim([0, 1000]);
    xlim([0, T]);
    ylabel('Population size');
    xlabel('Time in years');
    legend('Bird population', 'Rat population', 'Owl population');
    set(gca,'FontSize',16)
end
```

runSimulations.m

```
% functions runs Nruns times the moncecarlo() function

% function input
% number of simulations runs
% number of initial birds
% death rate of birds

% function returns
%the time when the birds died out in time

function [time_of_birds_deaths] = runSimulations(nRuns,X_r_in,mu_b,mu_r_child_in,version,T_rodenticide)

%    vectors of the time when the birds died out in each simulation
    time_of_birds_deaths=zeros(1,nRuns);

    %populates the vector with the right component
    for i = 1:nRuns
        time_of_birds_deaths(i)=montecarlo(X_r_in,mu_b,false,mu_r_child_in,version,T_rodenticide);
    end

% use it for the 1st tables
%hist(time_of_birds_deaths);
%plots=find(time_of_birds_deaths==0);
%length(plots)

End
```

bigProject.m

```
%in the working progress 2 pair of plots attached
% 2 examples of the boundary cases of the deathrate of the birds
% but notice that the plots have multiple dimensions, therefore there are more boundary conditions

function [] = bigProject(nRuns,X_r_in,mu_b, mu_r_child_in,T_rodenticide)
    hold all
    time_of_birds_deaths=runSimulations(nRuns,X_r_in,mu_b,mu_r_child_in,2,T_rodenticide);
    plotPmf(nRuns,time_of_birds_deaths);
    montecarlo(X_r_in,mu_b,true,mu_r_child_in,2,T_rodenticide);
    hold off
end
```

stationaryLife.m

```
% function inputs

    % number of initial size of the birds
    % min deathrate of birds
    % rat infants deathrate
    % which version of the monte carlo simulation will be run
    % time when the rodenticide is implemented
% function runs and plots all the simulations with this variables

function [] = stationaryLife(X_r_in,mu_b,mu_r_child_in,version,T_rodenticide)

hold all

montecarlo(X_r_in,mu_b,true,mu_r_child_in,version,T_rodenticide);

hold off

end
```

typeOfBirds.m

```
% function inputs

    % number of initial size of the birds
    % min deathrate of birds
    % rat infants deathrate minimum
    % rat infants deathrate maximum
    % the number we divide the difference between max and minimum infants deathrate
    % maximum deathrate of birds
    % which version of the monte carlo simulation will be run
          %--version one represent the  stationary plots
          %--else the other plots where either or both dies out
    % time when the rodenticide is implemented

% function runs and plots all the simulations with this variables 5 times
% with the each run increased birds deathrate starting at min birds
% deathrate finishing at max birds deathrate

function [] = typeOfBirds(X_r_in,mu_b,mu_r_child_min,mu_r_child_max,divideBy,mu_b_max,version,T_rodenticide)

% calculates the difference between max and min birds deathrate
diff=mu_b_max-mu_b;

% divide the difference by 5
increment=diff/5;

% for loop to run for different birds deathrate values

    for i=1:5
         mu_b=increment+mu_b;
         typeOfRats(X_r_in,mu_b,mu_r_child_min,mu_r_child_max,divideBy,version,T_rodenticide);
    end

end
```

typeOfRats.m

```
% function inputs

    % number of initial size of the birds
    % min deathrate of birds
    % rat infants deathrate minimum
    % rat infants deathrate maximum
    % the number we divide the difference between max and minimum infants deathrate
    % which version of the monte carlo simulation will be run
          %--version one represent the  stationary plots
          %--else the other plots where either or both dies out
    % time when the rodenticide is implemented

% function runs and plots all the simulations with this variables divideBy times
% with the each run increased ratinfants deathrate starting at min birds
% deathrate finishing at max ratinfants deathrate



function [] = typeOfRats(X_r_in,mu_b,mu_r_child_min,mu_r_child_max,divideBy,version,T_rodenticide)

    %calculates the differnce between max and min rat infants deathrate
    diff=mu_r_child_max-mu_r_child_min;

    %divide the difference by divideBy
    increment=diff/divideBy;

    % for loop to run for different rat infats deathrate values
    for i=1:divideBy

        mu_r_child_min= mu_r_child_min+increment;
        stationaryLife(X_r_in,mu_b,mu_r_child_min,version,T_rodenticide);

    end
end
```

runMe.m

```
%% 16.10.2019
% semester2 Random Process
% Group G-5 Vivienne Esser , Ron Au Yeung, Richard Pinter
% main program that runs all the simulations and plots of version 2
% this is for calling the pmf for different parameters



%% Script to use for the project
% there will be commented parts, depending on the expected outcome
% follow the instructions and comment in and out parts for results



%% to see plots for the simple model for varying initial ratsizes
% figure 1
% bigProject(100,20,0.2,1,0);
% figure 2
% bigproject(100,60,0.2,1,0);
% figure 3
% bigproject(100,100,0.2,1,0);
% figure 4
% bigproject(100,140,0.2,1,0);

% % varying predation rates of rats
% go to montecarlo and change the parameters to one of the following then run :

%  figure 5
% gamma_b_child = 0.5; % Rate of rats killing  baby birds
% gamma_b = 1.5; % Rate of rats killing adult birds

% figure 6
% gamma_b_child = 1; % Rate of rats killing  baby birds
% gamma_b = 1.5; % Rate of rats killing adult birds

% figure 7
% gamma_b_child = 1.5; % Rate of rats killing  baby birds
% gamma_b = 1; % Rate of rats killing adult birds

% figure 8
% gamma_b_child = 1; % Rate of rats killing  baby birds
% gamma_b = 1.5; % Rate of rats killing adult birds

% After changed the parameters uncomment the next line  and run it in order to see the plots
% bigproject(100,60,0.2,1,0);



%% Next, the model was  extended with the owls, to see those plots go to monte carlo functon and
% follow the instructions, and change manually the parameter mu_owl  to

% figure 9
% mu_owl = 0.5; % death rate of owls

% figure 10
% mu_owl = 1; % death rate of owls

% figure 11
% mu_owl = 1.5; % death rate of owls

% figure 12
% mu_owl = 2; % death rate of owls

% then run :
% bigproject(100,60,0.2,1,0);



%% To see plots of harmonious living  go to the function montecarlo and change up the parameter T manually to :
% T = 100; % length of simulation(in years)

% Then run the following:
% figure 13
% stationaryLife(20,0.26,2.6,1,0)
% stationaryLife(20, 0.3, 2.9, 1, 0)

% run the following to see many plots where for some parameteres the birds
% survive for some they terminate
% typeOfBirds(20,0.2,2,4,10,0.5,1,0)

%  for plots of the application of rodenticide, first change the time T manually to:
% T = 5; % length of simulation(in years)
% rodenticide is added for 2 different parameters:

% figure 14
```

```
% t=0.4
% bigproject(100,60,0.2,2,0.4);

% figure 15
% t=0.2
% bigproject(100,60,0.2,2,0.2);

%  for plots of the application of rodenticide, first change the time T manually to:
% T = 10; % length of simulation(in years):

% figure 16
% comment out montecarlo
%  montecarlo(20,0.2,true,1,2,0.2)
%  montecarlo(20,0.3,true,1.3,2,0.2)

%% for the tables the time T was changed systematically:
% go to montecarlo, and follow the instructions from line 207 to see the
% errors
% T = 5; % length of simulation(in years)

% for possible errors

% prob_b=0.17;
% prob_b=0.21;
% prob_b=0.25;
% prob_b=0.29;
% prob_b=0.31;
% prob_b=0.37;

% bigproject(100,20,0.2,1,0);

% prob_r=0.5;
% prob_r=1;
% prob_r=1.5;
% prob_r=2;
% prob_r=2.5;

% bigproject(100,20,0.2,1,0);

% gamma_b_child = 1.5;
% gamma_b_child = 1.55;
% gamma_b_child = 1.65;
% gamma_b_child = 1.8;
% gamma_b_child = 1.95;

% bigproject(100,20,0.2,1,0);

% mu_b_child = 0.7;
% mu_b_child = 0.8;
% mu_b_child = 0.9;
% mu_b_child = 1;
% mu_b_child = 1.2;

% bigproject(100,20,0.2,1,0);

% gamma_b = 1;
% gamma_b = 1.1;
% gamma_b = 1.2;
% gamma_b = 1.3;
% gamma_b = 1.4;
% gamma_b = 1.5;

% bigproject(100,20,0.2,1,0);

% lambda_b = 2.2;
% lambda_b = 2.3;
% lambda_b = 2.4;
% lambda_b = 2.5;
% lambda_b = 2.6;
% lambda_b = 2.7;

% bigproject(100,20,0.2,1,0);
```