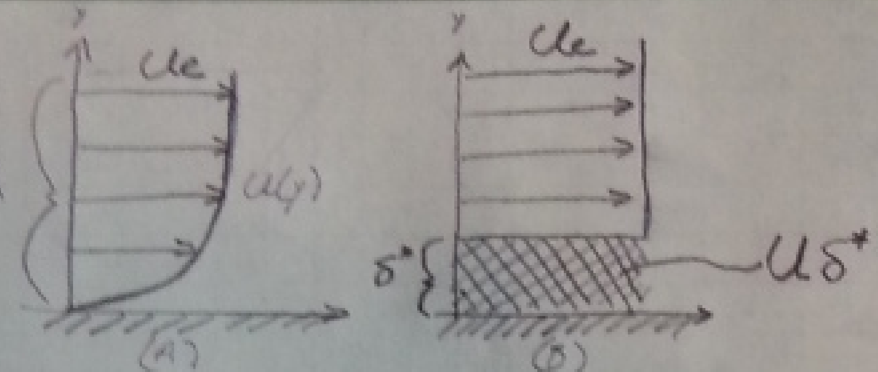
DISPLACEMENT THICKNESS | TICKET #3010 | 25 Nov 2013 | ①



DOWLING, 5TH ED
FIGURE 9.3

(p367) the displacement thickness is the distance by which the wall would have to be displaced outward in a hypothetical frictionless flow to maintain the same mass flux as that in the actual flow.

ASSUME DENSITY CONSTANT, ρ_e [illegible]

$$J_A = \int_0^\infty \rho_e u(y)\,dy = \rho_e \int_0^\infty u(y)\,dy = \rho_e \int_0^h u(y)\,dy$$

$$J_B = \int_{\delta^*}^h \rho_e U_e = \rho_e U_e h - \rho_e U_e \delta^*$$

EQUATING,

$$\rho_e \int_0^h u(y)\,dy = \rho_e U_e h - \rho_e U_e \delta^*$$

$$\rho_e U_e \delta^* = \rho_e U_e h - \rho_e \int_0^h u(y)\,dy$$

$$= \rho_e \int_0^h U_e - u(y)\,dy \qquad (\dagger)$$

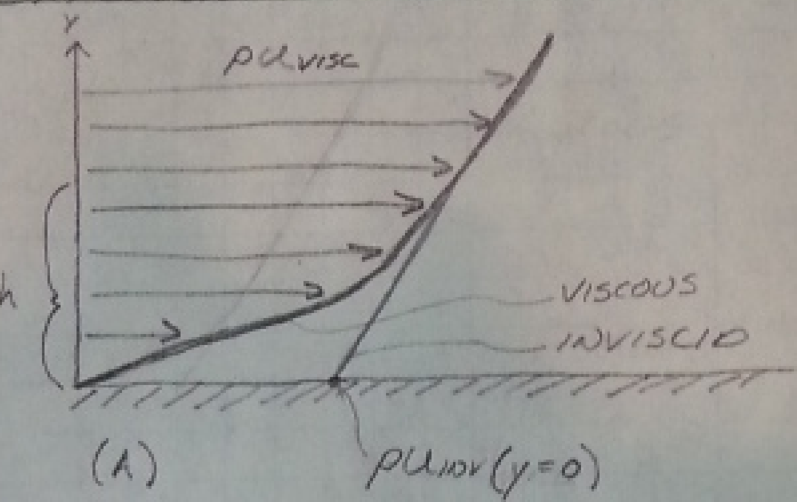
SO $$\delta^* = \frac{\rho_e \int_0^h U_e - u(y)\,dy}{\rho_e U_e}$$

$$= \int_0^h \frac{U_e}{U_e} - \frac{u(y)}{U_e}\,dy$$

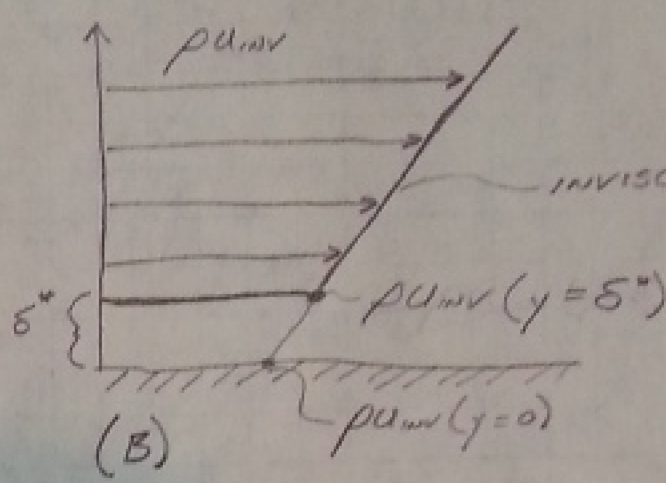
$$= \int_0^h 1 - \frac{u(y)}{U_e}\,dy$$

WHERE $$\delta^* = \int_0^\infty 1 - \frac{u(y)}{U_e}\,dy$$

ARISES BECAUSE INTEGRATING THE VELOCITY DEFECT OUTSIDE THE BOUNDARY LAYER HAS NEGLIGIBLE IMPACT ON THE INTEGRAL BUT MAKES IT INSENSITIVE TO THE DETERMINATION OF δ.



(A)



(B)

$$J_A = \int_0^\infty \rho u_{visc}(y)\,dy = \int_0^h \rho u_{visc}(y)\,dy + \int_h^\infty \rho u_{inv}(y)\,dy$$

$$J_B = \int_{\delta^*}^\infty \rho u_{inv}(y)\,dy = \int_{\delta^*}^h \rho u_{inv}(y)\,dy + \int_h^\infty \rho u_{inv}(y)\,dy$$

EQUATING AND CANCELING $\int_h^\infty \rho u_{inv}(y)\,dy$,

$$\int_0^h \rho u_{visc}(y)\,dy = \int_{\delta^*}^h \rho u_{inv}(y)\,dy$$

$$\begin{aligned} 0 &= \int_{\delta^*}^h \rho u_{inv}(y)\,dy - \int_0^h \rho u_{visc}(y)\,dy \\ &= \int_{\delta^*}^h \left[\rho u_{inv}(y) - \rho u_{visc}(y)\right] dy - \int_0^{\delta^*} \rho u_{visc}(y)\,dy \\ &= \int_{\delta^*}^\infty \left[\rho u_{inv}(y) - \rho u_{visc}(y)\right] dy - \int_0^{\delta^*} \rho u_{visc}(y)\,dy \end{aligned}$$

IMPLYING δ^* IS THE LENGTHSCALE SUCH THAT

$$\int_{\delta^*}^\infty \left[\rho u_{inv}(y) - \rho u_{visc}(y)\right] dy = \int_0^{\delta^*} \rho u_{visc}(y)\,dy \qquad (*)$$

WHICH IS NUMERICALLY NICE AS IT DEPENDS ON NEITHER h NOR IS SENSITIVE TO A DOMAIN TRUNCATION. IT IS SLIGHTLY MORE COMPLICATED TO COMPUTE

FOR SPECIAL CASE WHEN $\rho u_{inv} = \rho_e u_e$, $\rho u_{visc} = \rho_e u_{visc}(y)$,

$$\rho_e \int_{\delta^*}^{\cancel{\infty}\,h} u_e - u_{visc}(y)\,dy = \rho_e \int_0^{\delta^*} u_{visc}(y)\,dy$$

$$\rho_e \int_{\delta^*}^h u_e = \rho_e \int_0^h u_{visc}(y)\,dy$$

$$\rho_e u_e h - \rho_e u_e \delta^* = \rho_e \int_0^h u_{visc}(y)\,dy$$

$$\rho_e u_e \delta^* = \rho_e \int_0^h u_e - u_{visc}(y)\,dy$$

WHICH IS (†) ON PAGE 1. THIS MORE GENERAL STATEMENT DEGENERATES AS DESIRED.

RATHER THAN COMPUTE (*) TO DETERMINE Re_{δ^*}, WHY NOT AVOID THE NEED TO ISOLATE THE LENGTHSCALE?

$$Re_{\delta^*} = \frac{\rho_e u_e \delta^*}{\mu}$$

$$= \mu^{-1}\left[\rho_e \int_0^h u_e - u(y)\,dy\right] \quad \text{BY } (\dagger) \text{ ON PAGE ①}$$

$$= \mu^{-1}\left[\int_0^\infty \rho_e u_e - \rho_e u(y)\,dy\right] \quad \text{USING } \rho_e \text{ CONST, DEFECT} \to 0 \text{ AS } h \to \infty$$

$$= \mu^{-1}\int_0^\infty \left[\rho u_{inv}(y) - \rho u_{visc}(y)\right] dy$$

ADVANTAGES:

① UNAMBIGUOUS

② CHEAP + EASY TO COMPUTE

③ CLASSICAL INTERPRETATION IN Z.P.G. FLOW

DISADVANTAGES:

① PROVIDES NO LENGTHSCALE
(BUT CAN GET ONE FROM (*) ON PAGE ②)

DOWLING 5TH ED EXERCISE 9.6 DERIVES Θ VIA THE BALANCE LAWS. SCHLICTING 8TH ED JUST GIVES δ_2 IN §6.5.

(p369) [The momentum thickness Θ] is defined such that $\rho U^2 \Theta$ is the momentum loss in the actual flow because of the presence of the boundary layer.

REFERRING BACK TO CASES (A) + (B) ON PAGE 1,

$$M_A = \int_0^h \rho u^2 \, dy = \rho_e \int_0^h u^2 \, dy$$

INTEGRATED MOMENTUM FLUX

$$M_B = \rho_e U_e^2 h - \rho_e U_e^2 (\delta^* + \Theta)$$
$$= \rho_e U_e^2 (h - (\delta^* + \Theta))$$

INTEGRATING OVER TWO RECTANGULAR REGIONS + TAKING THE DIFFERENCE.

EQUATING,

$$\rho_e U_e^2 (h - (\delta^* + \Theta)) = \rho_e \int_0^h u^2 \, dy$$
$$\rho_e U_e^2 \delta^* + \rho_e U^2 \Theta = \rho_e U_e^2 h - \rho_e \int_0^h u^2 \, dy$$
$$\rho_e U_e^2 \Theta = \rho_e \int_0^h U_e^2 \, dy - \int_0^h u^2 \, dy - U_e (\rho_e U_e \delta^*)$$
$$\rho_e U_e^2 \Theta = \rho_e \int_0^h [U_e^2 - u^2] \, dy - \rho_e U_e \int_0^h [U_e - u] \, dy$$
$$\Theta = \int_0^h \left[1 - \frac{u^2}{U_e^2}\right] dy - \int_0^h \left[1 - \frac{u}{U_e}\right] dy$$
$$= \int_0^h \left[\left(1 - \frac{u^2}{U_e^2}\right) - \left(1 - \frac{u}{U_e}\right)\right] dy$$
$$= \int_0^\infty \left[\left(1 - \frac{u^2}{U_e^2}\right) - \left(1 - \frac{u}{U_e}\right)\right] dy$$

TAKING $h \to \infty$ PERMISSIBLE AS DEFECTS VANISH

$$= \int_0^\infty \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy$$

AS EXPECTED. WHY IS THIS THICKNESS DEFINED RELATIVE TO δ^* (OR, WHY DOES THAT CAUSE THE ALGEBRA TO PUSH THROUGH)? SCHLICTING p161. SAYS "relative to that in the <u>inviscid outer flow</u>" WHERE PRESUMABLY THE DISPLACEMENT THICKNESS SERVES AS AN ADJUSTMENT TO "GET" TO THE OUTER FLOW. THIS CAN BE SEEN IN (B) ON PAGE 1.

IN NON-CONSTANT, INVISCID BASEFLOW...

$$M_A = \int_0^h \rho u_{visc}^2(y)\,dy$$

$$M_B = \int_{\delta^*+\Theta}^h \rho u_{inv}^2(y)\,dy$$

EQUATING, REARRANGING, AND TAKING $h \to \infty$

$$\int_{\delta^*+\Theta}^h \rho u_{inv}^2\,dy = \int_{\delta^*+\Theta}^h \rho u_{visc}^2\,dy + \int_0^{\delta^*+\Theta} \rho u_{visc}^2\,dy$$

$$\int_{\delta^*+\Theta}^\infty \left[\rho u_{inv}^2 - \rho u_{visc}^2\right] dy = \int_0^{\delta^*+\Theta} \rho u_{visc}^2\,dy \qquad (**)$$

WHICH RESEMBLES (*) ON p2 AND COULD BE SOLVED FOR Θ GIVEN δ^*

WOULD BE NICE TO ELIMINATE/SIMPLIFY APPEARANCE OF δ^* USING (*). NOTHING, HOWEVER, JUMPS OUT AT ME...

TURNING TO $Re_\Theta \equiv \dfrac{\rho_e u_e \Theta}{\mu} = \dfrac{\rho_e u_e^2 \Theta}{\mu u_e}$

SO

$$\mu Re_\Theta = \frac{1}{u_e}\left[\rho_e \int_0^h (u_e^2 - u^2)\,dy - \rho_e u_e \int_0^h (u_e - u)\,dy\right] \quad \text{by (***) on p4}$$

$$= \frac{\rho_e}{u_e}\int_0^h u_e^2 - u^2\,dy - \rho_e \int_0^h u_e - u\,dy$$

$$= \int_0^h \frac{\rho_e}{u_e}\left[u_e^2 - u^2\right] dy - \int_0^h \rho_e\left[u_e - u\right] dy$$

MOVING TO CASE w/ BASEFLOW AND VARIABLE DENSITY,

$$\mu Re_\Theta = \int_0^h \left[\rho u_{inv} - \frac{\rho u_{visc}^2}{u_{inv}}\right] dy - \int_0^h \left[\rho u_{inv} - \rho u_{visc}\right] dy$$

$$= \int_0^h \left[\rho u_{inv} - \frac{\rho u_{visc}^2}{u_{inv}} - \rho u_{inv} + \rho u_{visc}\right] dy$$

$$= \int_0^h \left[\rho u_{visc} - \frac{\rho u_{visc}^2}{u_{inv}}\right] dy$$

$$\text{so } Re_\Theta = \mu^{-1} \int_0^\infty \left[\rho u_{visc} - \frac{\rho u_{visc}^2}{u_{inv}}\right] dy$$

THOUGH $\dfrac{(\rho u_{visc})^2}{\rho u_{inv}}$ LOOKS PRETTIER

THE CLASSICAL ENERGY THICKNESS DERIVATION IS IDENTICAL TO THE MOMENTUM THICKNESS SAVE FOR THE POWER ON u:

$$E_A = \int_0^h \rho u^3 \, dy = \rho_e \int_0^h u^3 \, dy$$

$$E_B = \rho_e u_e^3 h - \rho_e u_e^3 (\delta^* + \delta_3)$$

$$= \rho_e u_e^3 (h - (\delta^* + \delta_3))$$

EQUATING,

$$\rho_e u_e^3 (h - (\delta^* + \delta_3)) = \rho_e \int_0^h u^3 \, dy$$

$$\rho_e u_e^3 \delta_3 = \rho_e \int_0^h [u_e^3 - u^3] \, dy - \rho_e u_e^2 \int_0^h [u_e - u] \, dy$$

$$\delta_3 = \int_0^h \left[1 - \frac{u^3}{u_e^3}\right] dy - \int_0^h \left[1 - \frac{u}{u_e}\right] dy$$

$$= \int_0^h \frac{u}{u_e}\left(1 - \frac{u^2}{u_e^2}\right) dy$$

AS EXPECTED. SEE SCHLICTING EQN (6.65)

BY ANALOGY w/ (**) ON PAGE 5,

$$\int_{\delta^* + \delta_3}^{\infty} \left[\rho u^3_{inv} - \rho u^3_{visc}\right] dy = \int_0^{\delta^* + \delta_3} \rho u^3_{visc} \, dy \qquad (***)$$

DEFINES δ_3 GIVEN δ^*.

SIMILARLY TO Re_θ, TO COMPUTE Re_{δ_3} SANS δ_3

$$Re_{\delta_3} = \frac{\rho_e u_e \delta_3}{\mu} = \frac{\rho_e u_e^3 \delta_3}{\mu \, u_e^2}$$

BY ( ), CAN BE TRANSFORMED INTO

$$Re_{\delta_3} = \mu^{-1} \int_0^{\infty} \left[\rho u_{visc} - \frac{\rho u^3_{visc}}{u^2_{inv}}\right] dy$$

THOUGH AGAIN $\dfrac{(\rho u_{visc})^3}{(\rho u_{inv})^2}$ LOOKS, AGAIN, PRETTIER.

CONSIDERING TOTAL ENTHALPY FLUX ($\rho H u$) IDENTICALLY TO HOW MOMENTUM FLUX ($\rho u u$) WAS BALANCED,

$$\rho_e H_e u_e \delta_H = \rho_e \int_0^h \left[H_e u_e - H(y)\,u(y)\right] dy - \rho_e H_e \int_0^h \left[u_e - u(y)\right] dy$$

IMPLYING

$$\delta_H = \int_0^\infty \frac{u(y)}{u_e}\left(1 - \frac{H(y)}{H_e}\right) dy$$

WHICH MATCHES SCHLICTING EQN (10.98) (IN SPIRIT) AS A DEFINITION OF ENTHALPY THICKNESS δ_H

FOR THE VARIABLE BASEFLOW CASE, SIMILAR TO (**) ON PAGE 5 ONE HAS

$$\int_{\delta^*+\delta_H}^{\infty} \left[\rho H u_{inv} - \rho H u_{visc}\right] dy = \int_0^{\delta^*+\delta_H} \rho H u_{visc}\, dy \qquad (****)$$

SIMILARLY TO Re_θ TO COMPUTE Re_{δ_H} SANS δ_H,

$$Re_{\delta_H} \equiv \frac{\rho_e u_e \delta_H}{\mu} = \frac{\rho_e H_e u_e \delta_H}{\mu H_e}$$

CAN BE REARRANGED TO GIVE

$$Re_{\delta_H} = \mu^{-1} \int_0^\infty \left[\rho u_{visc} - \frac{\rho H u_{visc}}{H_{inv}}\right] dy$$

PERMITTING DIRECT COMPUTATION W/O SOLVING (****)