

Unit 3 Recitations

Multivariate Gaussian

The Multivariate Gaussian

• X, Y follow a bivariate Gaussian w/ parameters $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho$ if

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right] \right]$$

• In the Multivariate Gaussian case, \vec{x} follows $MN(\vec{\mu}, \Sigma)$ if

$$f(\vec{x}) = \frac{1}{\sqrt{2\pi^d} |\det(\Sigma)|^{1/2}} \exp\left(-\frac{1}{2} (\vec{x}-\vec{\mu})^T \Sigma^{-1} (\vec{x}-\vec{\mu})\right)$$

- $f(x, y)$ is joint pdf
- multivariate has vector multiplication
-

Going to cover

1) Calculate marginal distr. of X, Y

2) Calculate $\text{Cov}(X, Y)$

3) Z_1, Z_2 are ind. $N(0, 1)$

Then, $X = \sigma_x Z_1 + \mu_x$,

$$Y = \sigma_y [\rho Z_1 + \sqrt{1-\rho^2} Z_2] + \mu_y$$

follow the bivariate Gaussian with parameters $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho$

4) Show how the MVG reduces to the bivariate case when $d=2$.

1. Calculate marginal distribution

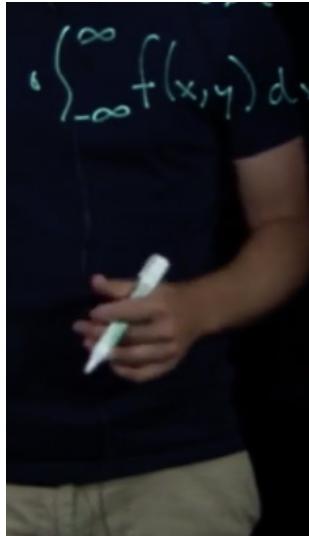
$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Let $a(y) = \mu_x + \rho \frac{\sigma_x}{\sigma_y} (y - \mu_y)$

Notice that

$$\frac{1}{(1-\rho^2)} \left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} \right) =$$
$$\left(\frac{(x-a(y))^2}{\sigma_x^2(1-\rho^2)} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right)$$

- integrate out x from the joint pdf
- trying to extract all the x's from the function to make it easier to integrate and then complete the square
- simplify the equation by using a trick to set $a = \dots$



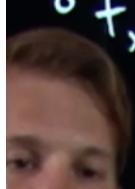
$$\begin{aligned} \int_{-\infty}^{\infty} f(x,y) dx &= \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right] \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2} \frac{(x-\alpha(y))^2}{\sigma_x^2(1-\rho^2)}\right] dx}_{N(\alpha(y), \sigma_x^2(1-\rho^2))} \\ &= \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right] \\ &\downarrow N(\mu_y, \sigma_y^2) \end{aligned}$$

- pulled out the terms that depend on just y and not x
- the second part is now the pdf of a normal exponential which integrates to 1

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right] \\ &\downarrow Y \sim N(\mu_y, \sigma_y^2) \\ &\quad X \sim N(\mu_x, \sigma_x^2) \end{aligned}$$

- so in a bivariate case we have two normal distributions

2. Calculate covariance



$$\begin{aligned} 2) \text{Cov}(X, Y) &= E(XY) - E(X) \cdot E(Y) \\ &\circ f_x(x|Y=y) \propto \exp\left[-\frac{1}{2} \frac{(x-\alpha(y))^2}{\sigma_x^2(1-\rho^2)}\right] \\ &\quad X|Y \sim N(\alpha(Y), \sigma_x^2(1-\rho^2)) \end{aligned}$$

- what is the distribution of X at a fixed Y ($Y=y$) proportional to

$$\begin{aligned}
 \mathbb{E}_{x,y} X \cdot Y &= \mathbb{E}_Y \left[\mathbb{E}_X X | Y \right] \\
 &= \mathbb{E}_Y \left[Y \cdot \mathbb{E}_X X | Y \right] \\
 &= \mathbb{E}_Y \left[Y \cdot \left(\mu_X + \rho \frac{\sigma_X}{\sigma_Y} Y - \rho \frac{\sigma_X}{\sigma_Y} \cdot \mu_Y \right) \right] \\
 &= \mu_Y \cdot \mu_X + \rho \frac{\sigma_X}{\sigma_Y} (\sigma_Y^2 + \mu_Y^2) - \rho \frac{\sigma_X}{\sigma_Y} \cdot \mu_Y^2 \\
 &= \mu_Y \cdot \mu_X + \rho \sigma_X \sigma_Y
 \end{aligned}
 \quad \left| \begin{array}{l} \text{Var}(Y) = \\ \mathbb{E}Y^2 - (\mathbb{E}Y)^2 \end{array} \right.$$

- use law of total expectations
- $\mathbb{E}(X|Y) = a(Y)$ (take from last section)
- $\sigma_y^2 + \mu_y^2$ for $\mathbb{E}[Y^2]$, from $\text{Var}(Y)$ equation on side

$$\begin{aligned}
 \text{Cov}(X, Y) &= \mu_Y \cdot \mu_X + \rho \sigma_X \sigma_Y - \mu_X \mu_Y \\
 &= \rho \sigma_X \sigma_Y \\
 \text{Corr}(X, Y) &= \rho
 \end{aligned}$$

now we have all the variables defined:

- μ_x = mean of x
- μ_y = mean of y
- σ_y^2 = marginal variance of y
- σ_x^2 = marginal variance of x
- ρ = correlation between x and y

3. Generate a bivariate normal distribution from 2 independent standard normals

3) We can take $X = \sigma_X Z_1 + \mu_X$,
 $Y = \sigma_Y [\rho Z_1 + \sqrt{1-\rho^2} Z_2] + \mu_Y$, where
 Z_1, Z_2 are independent $N(0, 1)$

3) o Know $X + Y$ have a bivariate Gaussian distribution.

$$o \mathbb{E} X = \mathbb{E}(\sigma_x Z_1 + \mu_x) = \mathbb{E}\sigma_x Z_1 + \mathbb{E}\mu_x = \mu_x$$

- by plugging in formula

$$o \mathbb{E} Y = \mathbb{E}(\sigma_y (\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_y) \\ = \mathbb{E}\sigma_y \rho Z_1 + \mathbb{E}\sigma_y \sqrt{1-\rho^2} Z_2 + \mathbb{E}\mu_y = \mu_y$$

- Z_1 and Z_2 have means of 0 because they are normal

$$o \text{Var}(X) = \text{Var}(\sigma_x Z_1 + \mu_x) = \text{Var}(\sigma_x Z_1) \\ = \sigma_x^2 \text{Var}(Z_1) = \sigma_x^2$$

$$o \text{Var}(Y) = \text{Var}(\sigma_y \rho Z_1 + \sigma_y \sqrt{1-\rho^2} Z_2) \\ = \text{Var}(\sigma_y \rho Z_1) + \text{Var}(\sigma_y \sqrt{1-\rho^2} Z_2) \\ = \sigma_y^2 \rho^2 + \sigma_y^2 (1-\rho^2) = \sigma_y^2$$

- can pull the constant out of the variance and square it (sigma because it's multiplying the random variable)
- didn't include or removed the mu constant because it's adding to the random variable so does not affect the variance

$$o \text{Cov}(X, Y) = \text{Cov}(\sigma_x Z_1 + \mu_x, \sigma_y (\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_y) \\ = \text{Cov}(\sigma_x Z_1, \sigma_y \rho Z_1 + \sigma_y \sqrt{1-\rho^2} Z_2) \\ = \text{Cov}(\sigma_x Z_1, \sigma_y \rho Z_1) + \text{Cov}(\sigma_x Z_1, \sigma_y \sqrt{1-\rho^2} Z_2) \\ = \sigma_x \sigma_y \rho (\text{Cov}(Z_1, Z_1) + \sigma_x \sigma_y \sqrt{1-\rho^2} \cancel{\text{Cov}(Z_1, Z_2)}) \\ = \sigma_x \sigma_y \rho$$

- removed μ_x and μ_y again as they do not affect the covariance since they're constants
- break up covariance of all of it into a sum of covariances
- last term is 0 because Z_1 and Z_2 are independent
- so we can correctly generate the bivariate x and y case from 2 independent standard normals and they'll have the correct marginal parameters and

covariance

4. Reduce multivariate case to bivariate case when dimensions = 2

$$4) \text{ When } d=2, \vec{x} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\circ \bar{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \Sigma = \begin{pmatrix} \text{Cov}(X, X) & \text{Cov}(X, Y) \\ \text{Cov}(Y, X) & \text{Cov}(Y, Y) \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_X^2 & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma_Y^2 \end{pmatrix}$$

- the order of the rv's in the covariance doesn't matter as they're the same

$$\circ \text{Recall, } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \det(A) = a_{11}a_{22} - a_{21} \cdot a_{12}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

trying to show that the 2 sections of the bivariate and multivariate pdf are equal to each other

- the $1/2\pi$ part and the exponential part

$$\circ \det \Sigma = \sigma_X^2 \sigma_Y^2 - \rho^2 \sigma_X^2 \sigma_Y^2 = (1-\rho^2) \sigma_X^2 \sigma_Y^2$$

\hookrightarrow normalization constants ✓

$$\circ \Sigma^{-1} = \frac{1}{(1-\rho^2) \sigma_X^2 \sigma_Y^2} \begin{pmatrix} \sigma_Y^2 & -\rho \sigma_X \sigma_Y \\ -\rho \sigma_X \sigma_Y & \sigma_X^2 \end{pmatrix}$$

$$\left(\begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} \right)^T \begin{pmatrix} \sigma_Y^2 & -\rho \sigma_X \sigma_Y \\ -\rho \sigma_X \sigma_Y & \sigma_X^2 \end{pmatrix} \cdot \left(\frac{1}{(1-\rho^2) \sigma_X^2 \sigma_Y^2} \right) \cdot \left(\left(\begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} \right) \right)$$

then pull the constant out and multiply the matrices from left to right

$$\begin{aligned}
 &= \frac{1}{(1-\rho^2) \sigma_x^2 \sigma_y^2} \begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}^\top \begin{pmatrix} \sigma_y^2(x - \mu_x) - \rho \sigma_x \sigma_y (y - \mu_y) \\ -\rho \sigma_x \sigma_y (x - \mu_x) + \sigma_x^2 (y - \mu_y) \end{pmatrix} \\
 &= \frac{1}{(1-\rho^2)} \left[\frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{(y - \mu_y)^2}{\sigma_y^2} - \frac{2\rho (x - \mu_x)(y - \mu_y)}{\sigma_x \sigma_y} \right] \\
 &\quad \text{Exponent matches.}
 \end{aligned}$$

when simplifying we end up with the same parameter