Unit 9 Bernoulli and Poisson Processes

Stochastic/ random processes
Model arrivals over time
- discrete time: Bernoulli
- continuous time: Poisson

Bernoulli process
- Memorylessness - past arrivals do not affect future ones
- Simplest stochastic process
- Like flipping a coin at each time interval t, and counting number of heads/ successes/ arrivals

The Bernoulli process

- A sequence of independent Bernoulli trials, X_i
- At each trial, i:

$$\begin{aligned}\mathbf{P}(X_i = 1) &= \mathbf{P}(\text{success at the } i\text{th trial}) = p \\ \mathbf{P}(X_i = 0) &= \mathbf{P}(\text{failure at the } i\text{th trial}) = 1 - p\end{aligned}$$

$0 < p < 1$

- Key assumptions:
  - Independence
  - Time-homogeneity
- Model of:
  - Sequence of lottery wins/losses
  - Arrivals (each second) to a bank
  - Arrivals (at each time slot) to server
  - ...

- Jacob Bernoulli (1655–1705)

Exercise: The Bernoulli process

3/4 points (graded)

Let $X_1, X_2, \ldots$ be a Bernoulli process. We will define some new sequences of random variables and inquire whether they form a Bernoulli process.

1. Let $Y_n = X_{2n}$. Is the sequence Y_n a Bernoulli process?

   No ✖ **Answer:** Yes

2. Let $U_n = X_{n+1}$. Is the sequence U_n a Bernoulli process?

   Yes ✔ **Answer:** Yes

3. Let $V_n = X_n + X_{n+1}$. Is the sequence V_n a Bernoulli process?

   No ✔ **Answer:** No

4. Let $W_n = (-1)^n X_n$. Is the sequence W_n a Bernoulli process?

   No ✔ **Answer:** No

Solution:

1. Yes, because the random variables X_{2n} are independent Bernoulli random variables with the same parameter.
2. Yes, for the same reason.
3. No, because, for example $V_1 = X_1 + X_2$ and $V_2 = X_2 + X_3$ are both affected by X_2 and are therefore dependent. In addition, each V_n can take value 2 and is therefore not Bernoulli.
4. No, because W_1 can take value -1 and therefore is not a Bernoulli random variable.

Stochastic processes

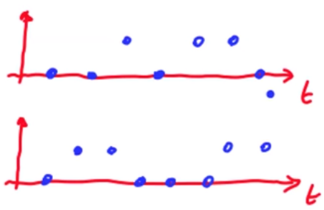
- First view: infinite sequence of random variables $X_1, X_2, \ldots$

  Interested in: $\mathbf{E}[X_i] = p$ $\quad$ $\text{var}(X_i) = p(1-p)$ $\quad$ $p_{X_i}(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$

  $p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = p_{X_1}(x_1)\cdots p_{X_n}(x_n)$ for all n

- Second view – sample space:

  $\Omega =$ set of infinite sequences of 0's and 1's



- Example (for Bernoulli process):

  $\mathbf{P}(X_i = 1 \text{ for all } i) = 0$ $\quad (p<1)$

  $\leq \mathbf{P}(X_1 = 1, \ldots, X_n = 1) = p^n$, for all n

Probability of all 1s for an infinite sequence becomes 0

Number of successes/arrivals S in n time slots

- $S = X_1 + \dots + X_n$

- $\mathbf{P}(S = k) = \binom{n}{k} p^k (1-p)^{n-k}$ $k = 0, \dots, n$

- $\mathbf{E}[S] = np$

- $\mathrm{var}(S) = np(1-p)$

Time until the first success/arrival

- $T_1 = \min\{i : X_i = 1\}$

- $\mathbf{P}(T_1 = k) = \mathbf{P}(\underbrace{0\,0 \dots 0}_{k-1}\ 1) = (1-p)^{k-1} p$ $k = 1, 2, \dots$

- $\mathbf{E}[T_1] = \dfrac{1}{p}$

- $\mathrm{var}(T_1) = \dfrac{1-p}{p^2}$

6. Exercise: Time until the first failure

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Exercise: Time until the first failure

1/1 point (graded)

Let the sequence X_n, $n = 1, 2, 3, \ldots$, be a Bernoulli process with parameter $\mathbf{P}(X_n = 1) = p$ for all $n \geq 1$. Let U be the time when a value of 0 is first observed: $U = \min\{n : X_n = 0\}$ Then, the random variable U is:

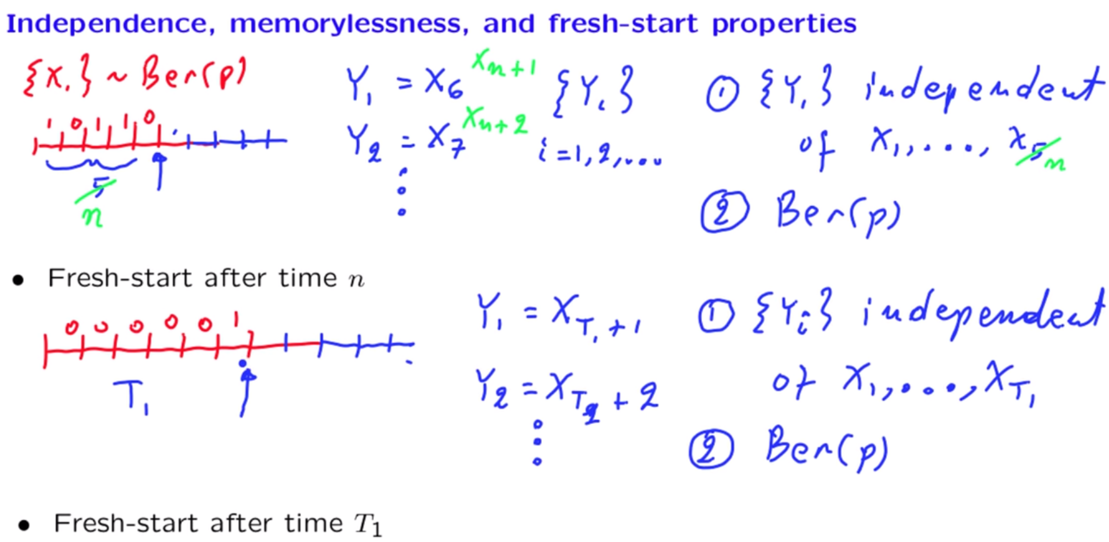
- ○ Geometric with parameter p
- ◉ Geometric with parameter $1 - p$
- ○ None of the above

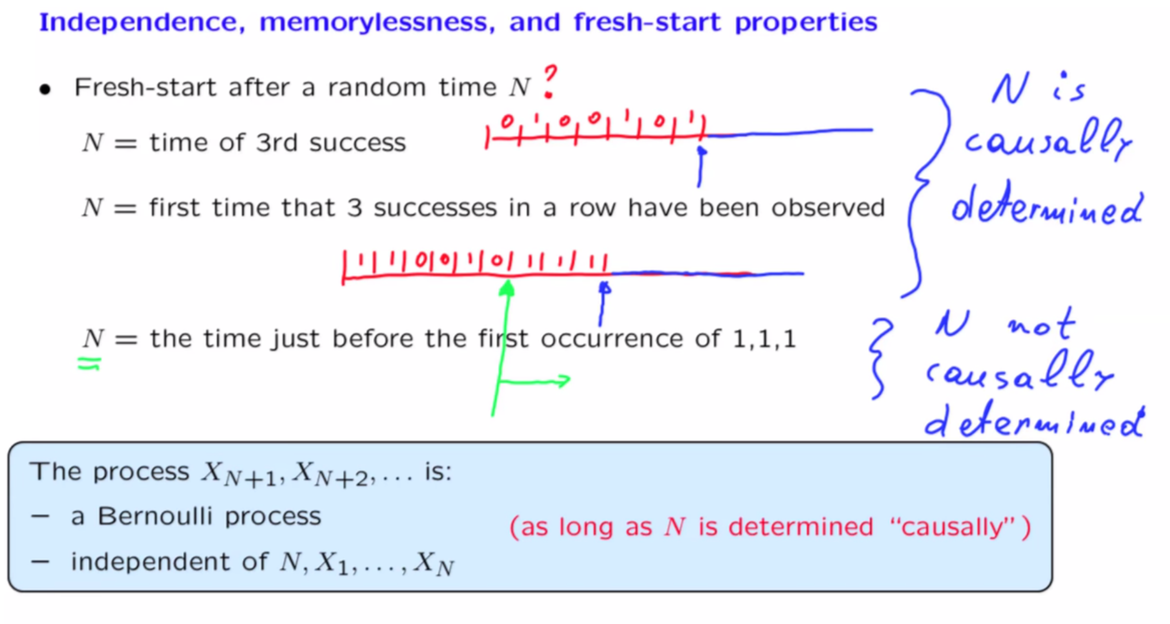
✔

Solution:

For $n \geq 1$, the event $\{U = n\}$ corresponds to $n - 1$ 1's followed by a 0. Its probability is $p^{n-1}(1 - p)$, which corresponds to a geometric PMF with parameter $1 - p$.

Properties





Exercise: Fresh start

3/3 points (graded)

Consider a Bernoulli process, with a “1" considered a success and a “0" considered a failure. Determine whether the process starts fresh right after each of the following random times:

1. The time of the kth failure

   Yes ✔ **Answer:** Yes

2. The first time that a failure follows a success

   Yes ✔ **Answer:** Yes

3. The first time at which we have a failure that will be followed by a success

   No ✔ **Answer:** No

Solution:

In the first two cases, the time of interest is determined causally, by past events, and we have the fresh-start property. In the last case, the time of interest is determined by something that is to happen in the future. In particular, we know that right after the time of interest, the next trial will result in a success.

Exercise: More on fresh start

1/1 point (graded)

Consider a Bernoulli process with parameter $p = 1/3$. Let T_1 be the time of the first success and let $T_1 + T_2$ be the time of the second success. We are told that the results of the two slots that follow the first success are failures, so that $X_{T_1+1} = X_{T_1+2} = 0$. What is the conditional expectation of the second interarrival time, T_2, given this information? (Recall that the expectation of a geometric random variable with parameter p is equal to $1/p$.)

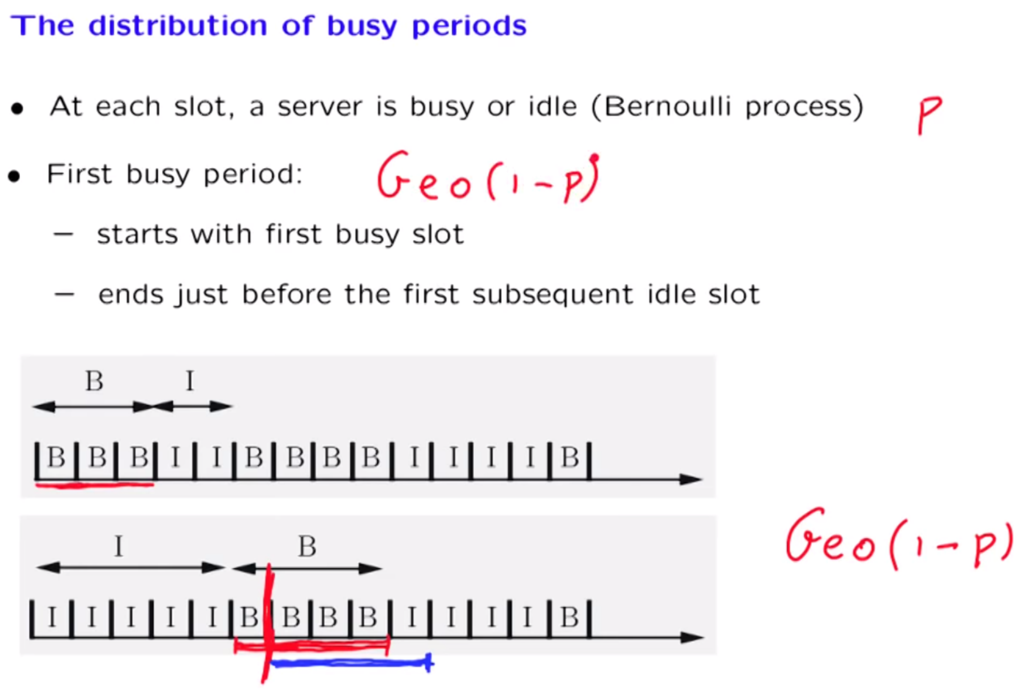
5

✔ **Answer:** 5

Solution:

After time T_1, we have two failures, and these are part of the interarrival time T_2. Given this information, the process starts fresh at time $T_1 + 3$ and the number of trials from time $T_1 + 3$ onwards until the next success is geometric with parameter $1/3$, and has an expected value of 3. Therefore, the conditional expectation of T_2, given the information we were given, is $2 + 3 = 5$.

Example



Exercise: Busy periods

1/1 point (graded)

Consider the same setting as in the last video. After the first busy period ends (with an idle slot), there will be a subsequent busy period, which starts with a busy slot, and lasts as long as the slots are busy. Is it true that the length of the second busy period is geometric?

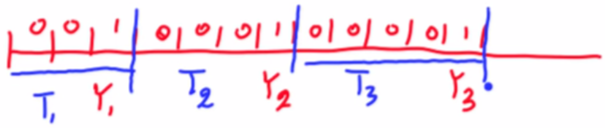
Yes

✔ **Answer:** Yes

Solution:

Yes, because the argument used for the first busy period applies without change.

Time of the kth success/arrival



- Y_k = time of kth arrival $\qquad Y_k = T_1 + \cdots + T_k$
- T_k = kth inter-arrival time $= Y_k - Y_{k-1} \quad (k \geq 2)$

- The process starts fresh after time T_1
- T_2 is independent of T_1; Geometric(p); etc.

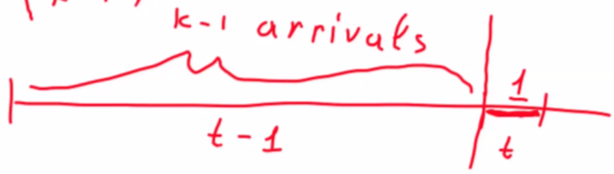
Time of the kth success/arrival

$P(Y_k = t)$

$= P(k-1 \text{ arrivals in time } t-1)$

$\cdot P(\text{arrival at time } t)$

$= \binom{t-1}{k-1} p^{k-1}(1-p)^{t-k} \cdot p$



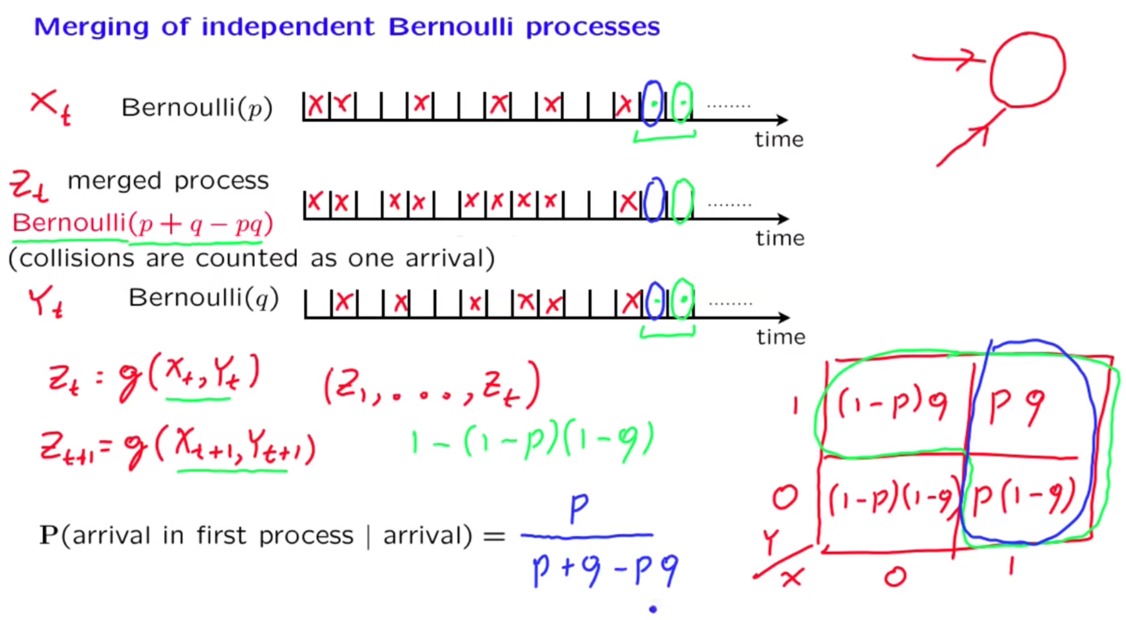
$Y_k = T_1 + \cdots + T_k$

the T_i are i.i.d., Geometric(p)

$\mathbf{E}[Y_k] = \frac{k}{p} \qquad \text{var}(Y_k) = \frac{k(1-p)}{p^2}$

$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k},$

$t = k, k+1, \ldots$



All sequences are independent from each other and in time
The conditional probability at the bottom is the blue event given the green event (box on right)

Exercise: A variation on merging

2/2 points (graded)

We start with two independent Bernoulli processes, X_n and Y_n, with parameters p and q, respectively. We form a new process Z_n by recording an arrival in a given time slot if and only if **both** of the original processes record an arrival in that same time slot. Mathematically, $Z_n = X_n Y_n$.

The new process Z_n is also Bernoulli with parameter p*q ✔ **Answer:** p*q

$p \cdot q$

(Enter an algebraic function of p and q using standard notation.)

Suppose that the two Bernoulli processes X_n and Y_n are dependent. We still assume, however, that the pairs (X_n, Y_n) are independent. E.g., (X_1, Y_1) is independent from (X_2, Y_2), etc. Is the process Z_n guaranteed to be Bernoulli?

No ✔ **Answer:** No

STANDARD NOTATION

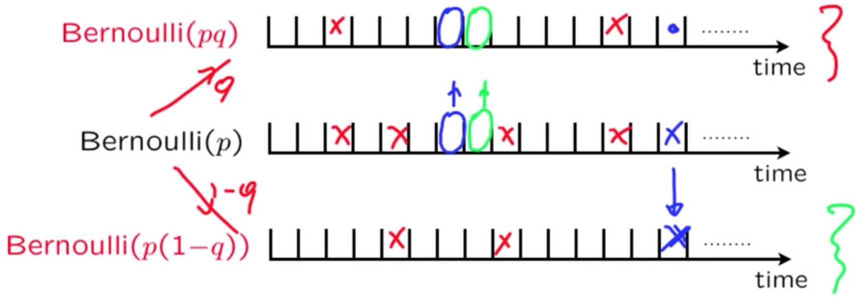
Solution:

The merged process records an arrival if and only if both of the original processes record an arrival, which happens with probability pq.

In the second case, since the pairs (X_n, Y_n) are independent, the random variables Z_n are also independent. However, there is nothing in the statement that would ensure that the Z_n are identically distributed. Thus, Z_n is not guaranteed to be a Bernoulli process. For example, consider the special case of $p = q$ and suppose that $Y_1 = X_1$ but Y_n is independent of X_n for $n > 1$. Then $\mathbf{P}(Z_1 = 1) = p$ while $\mathbf{P}(Z_n = 1) = p^2$ for $n > 1$, violating the time-homogeneity property of Bernoulli processes.

Splitting of a Bernoulli process

- Split successes into two streams, using independent flips of a coin with bias q
  - assume that coin flips are independent from the original Bernoulli process



- Are the two resulting streams independent? No

Not independent because presence of event in one stream tells us that there isn't one in the other

Exercise: Splitting

1/1 point (graded)

For each exam, Ariadne studies with probability $1/2$ and does not study with probability $1/2$, independently of any other exams. On any exam for which she has not studied, she still has a 0.20 probability of passing, independently of whatever happens on other exams. What is the expected number of total exams taken until she has had 3 exams for which she did not study but which she still passed?

30 ✔ **Answer:** 30

Solution:

The sequence of exams for which she does not study and passes can be modeled as follows. We look at the exams for which she has not studied (a Bernoulli process with parameter $1/2$) and "split" it according to whether she passes or not. This creates a new Bernoulli process for the exams for which she does not study and passes, with parameter $(1/2) \cdot 0.20 = 0.10$. The expected time until 3 successes in this process is $3/0.10 = 30$.

Poisson approximation to binomial

- Interesting regime: large n , small p, moderate $\underline{\lambda = np}$ $\quad$ $n \to \infty$, $p \to 0$ $\quad$ $p = \frac{\lambda}{n}$

- Number of arrivals S in n slots: $\underline{p_S(k)} = \frac{n!}{(n-k)!\,k!} \cdot p^k (1-p)^{n-k}, \quad k = 0, \dots, n$

For fixed $k = 0, 1, \dots,$

$p_S(k) \to \frac{\lambda^k}{k!} e^{-\lambda},$

$$= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \cdot \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$\xrightarrow[n \to \infty]{} 1 \cdot 1 \cdots 1 \cdot \underbrace{\frac{\lambda^k}{k!} e^{-\lambda}}\cdot 1$$

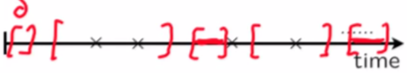
- Fact: $\lim_{n \to \infty} (1 - \lambda/n)^n = e^{-\lambda}$

Example of situation: number of earthquakes over 5 years split by hour

Definition of the Poisson process

Poisson

δ



$$\sum_{k=0}^{\infty} P(k, \tau) = \underline{1}$$

- Numbers of arrivals in disjoint time intervals are **independent**

$P(k, \tau) =$ Prob. of k arrivals in interval of duration τ

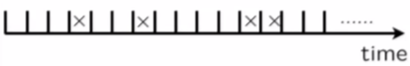
- **Small interval probabilities:**
  For VERY small δ:

$$P(k,\delta) \approx \begin{cases} 1 - \lambda\delta & \text{if } k = 0 \\ \lambda\delta & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \qquad P(k,\delta) = \begin{cases} 1 - \lambda\delta + O(\delta^2) & \text{if } k = 0 \\ \lambda\delta + O(\delta^2) & \text{if } k = 1 \\ 0 + O(\delta^2) & \text{if } k > 1 \end{cases}$$

$$\frac{O(\delta^2)}{\delta} \xrightarrow[\delta \to 0]{} 0$$

λ: "arrival rate"

Bernoulli



- Independence
- **Time homogeneity:**
  Constant p at each slot

Think of tau as a constant
Think of lambda as probability per unit time
Second order term of delta goes to 0 as delta goes to 0, becomes negligible for small delta

Exercise: Poisson process definition

0/1 point (graded)

Consider a Poisson process with rate $\lambda = 4$, and let $N(t)$ be the number of arrivals during the time interval $[0, t]$.

Suppose that you have recorded this process in a movie and that you play this movie at twice the speed. The process that you will be seeing in the sped-up movie satisfies the following (pick one of the answers):

- is a Poisson process with rate 2
- is a Poisson process with rate 4
- is a Poisson process with rate 8 ✔
- is not a Poisson process

✖

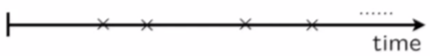
Solution:

Let $M(t)$ be the number of arrivals in the sped-up movie between times 0 and t. By time t, you have watched in the sped-up movie whatever happens in the original process from time 0 through time $2t$. Thus, $M(t) = N(2t)$. The independence and time-homogeneity properties of the original process can be seen to imply the same properties for the sped-up process. Furthermore,

$$\mathbf{P}\big(M(\delta) = 1\big) = \mathbf{P}\big(N(2\delta) = 1\big) \approx \lambda \cdot (2\delta) = (2\lambda)\,\delta,$$

which leads to the rather intuitive conclusion that the sped up process has a rate of $2\lambda = 8$.

Applications of the Poisson process



- Deaths from horse kicks in the Prussian army (1898)
- Particle emissions and radioactive decay
- Photon arrivals from a weak source
- Financial market shocks
- Placement of phone calls, service requests, etc.

Siméon Denis Poisson
(1781-1840)

Exercise: Poisson models

0/3 points (graded)

For each one of the following situations, state whether a Poisson model is a plausible model over the specified time frame.

1. The process of arrivals of passengers to the baggage claim section of an airport

   Yes ✖ **Answer:** No

2. The process of order arrivals at an online retailer between 3:00 and 3:15 pm

   No ✖ **Answer:** Yes

3. The process of order arrivals at a local pizza delivery shop over the course of a day

   Yes ✖ **Answer:** No

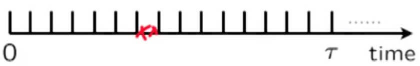
Solution:

1. Passengers go to the baggage claim area because their plane has just arrived. If I see that there were 20 arrivals to the baggage claim area over the last minute, I can infer that a plane just arrived, and I can expect a substantial number of arrivals over the next minute. Thus, the independence assumption does not hold.

2. Orders are generated from a large population of potential customers, and these are typically uncoordinated.

3. The rate of order arrivals should be much higher between during lunch and dinner meal hours and much lower at other times of the day, thus violating the time-homogeneity assumption.

The Poisson PMF for the number of arrivals



$P(k, \delta)$

- N_τ: arrivals in $[0, \tau]$ $\quad P(k,\tau) = \mathbf{P}(N_\tau = k)$

$n = \tau/\delta$ intervals/slots of length δ ← small

$\mathbf{P}$(some slot contains two or more arrivals)

$\le \sum_i P(\text{slot } i \text{ has} \ge 2 \text{ arrivals})$

$= \frac{\tau}{\delta} O(\delta^2) \xrightarrow[\delta \to 0]{} 0$

P(k arrivals in Poisson) $\approx P$(k slots have arrival)

$N_\tau \approx$ binomial $\quad p = \lambda\delta + O(\delta^2)$

$np = \lambda\tau + O(\delta) \approx \lambda\tau$

Bernoulli

$$p_S(k) = \frac{n!}{(n-k)!\,k!} \cdot p^k (1-p)^{n-k}, \quad k = 0, \ldots, n$$

$\lambda = np \quad n \to \infty \quad p \to 0$

For fixed $k = 0, 1, \ldots,$

$$p_S(k) \to \frac{\lambda^k}{k!} e^{-\lambda},$$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \quad k = 0, 1, \ldots$$

Poisson approximates Bernoulli as delta -> 0

Mean and variance of the number of arrivals

$$P(k,\tau) = \mathbf{P}(N_\tau = k) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$\mathrm{var}(N_\tau) = \lambda\tau$$

$$\mathbf{E}[N_\tau] = \sum_{k=0}^{\infty} k \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} = \cdots = \lambda\tau$$

$$\lambda_{\bullet} = \frac{E[N_\tau]}{\tau}$$

$N_\tau \approx$ Binomial(n, p)

$n = \tau/\delta, \quad p = \lambda\delta + O(\delta^2)$

$$E[N_\tau] \approx np \approx \lambda\tau$$

$$\mathrm{var}(N_\tau) \approx np(1-p) \approx \lambda\tau$$

Special property of the Poisson PMF is that expected value = variance

Example

- You get email according to a Poisson process, at a rate of $\lambda = 5$ messages per hour.

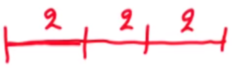
$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$\mathrm{var}(N_\tau) = \lambda\tau$$

- Mean and variance of mails received during a day = $5 \cdot 24$
- $\mathbf{P}$(one new message in the next hour) = $P(1,1) = 5e^{-5}$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- $\mathbf{P}$(exactly two messages during each of the next three hours) =



$$\left(P(2,1)\right)^3 = \left(\frac{5^2 e^{-5}}{2}\right)^3$$

Exercise: Poisson practice

1/1 point (graded)

Consider a Poisson arrival process with rate λ per hour. To simplify notation, we let $a = P(0,1)$, $b = P(1,1)$, and $c = P(2,1)$, where $P(k,1)$ is the probability of exactly k arrivals over an hour-long time interval.

What is the probability that we will have “at most one arrival between 10:00 and 11:00 and exactly two arrivals between 10:00 and 12:00"? Your answer should be an algebraic function of a, b, and c in standard notation.

(a*c)+b^2

✔ **Answer:** a*c+b^2

$(a \cdot c) + b^2$

STANDARD NOTATION

Solution:

The event of interest can happen in two ways:
(i) Zero arrivals during the first hour and two arrivals over the second hour; this has probability ac.
(ii) One arrival during each one of the two hours; this has probability b^2.
Thus, the answer is $ac + b^2$. (Note that for both scenarios, we have used independence to find the associated probabilities.)

The time T_1 until the first arrival

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- Find the CDF: $\mathbf{P}(T_1 \le t) =$

$$= 1 - P(T_1 > t) = 1 - P(0,t) = 1 - e^{-\lambda t}$$

$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad \text{for } t \ge 0$$

Exponential(λ)

Memorylessness: conditioned on $T_1 > t$, the PDF of $T_1 - t$ is again exponential

The time Y_k of the kth arrival

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \dots$$

- Can derive its PDF by first finding the CDF $P(Y_k \le y) = \sum_{m=k}^{\infty} P(m, y)$
- More intuitive argument:

$$\boxed{f_{Y_k}(y)\cancel{\delta}} \approx \mathbf{P}(y \le Y_k \le y + \delta) =$$

$$\approx \boxed{P(k-1, y)\,\lambda \cancel{\delta}}$$

$$+ P(k-2, y)\, \cancel{O(\delta^2)}$$

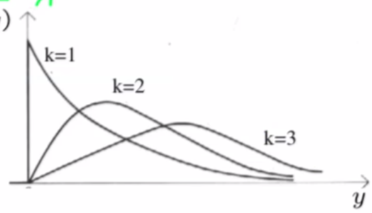
$$+ P(\cancel{k-3}, y)\, O(\delta^2)$$

0 ——— $k-1$ / $k-2$ ——— y $\quad y+\delta$

$$\frac{(\lambda y)^{k-1} e^{-\lambda y}}{(k-1)!}\lambda$$

Erlang distribution: $\quad f_{Y_k}(y) = \dfrac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \ge 0$

order k

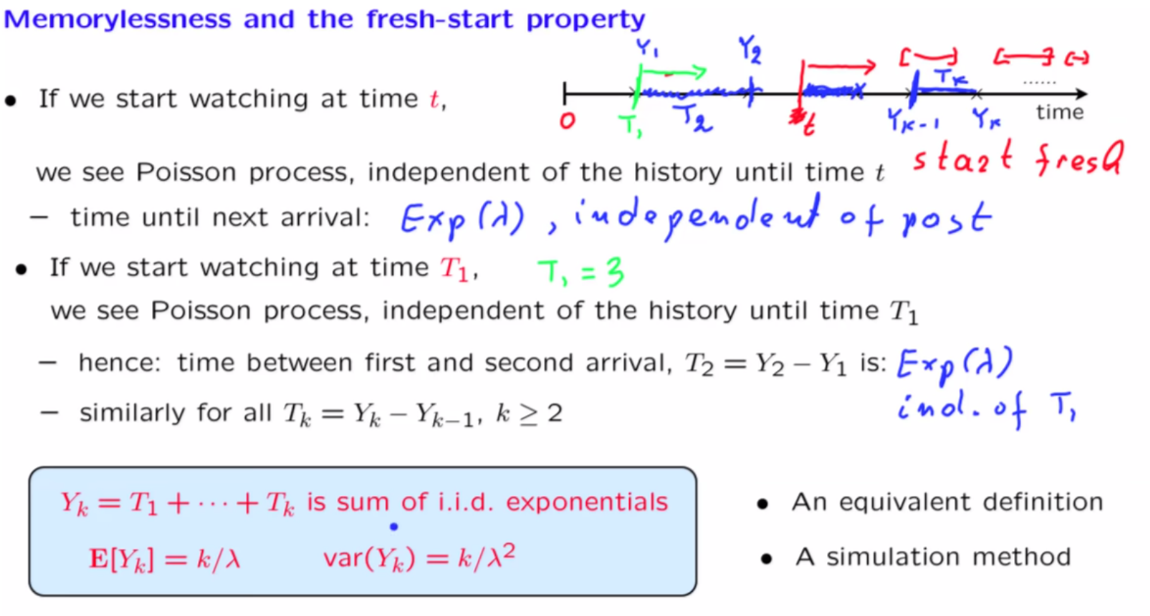


Have different Erlang distributions of order k, the distribution moves further to the right which makes sense as we have later arrivals

Important part of Poisson process

Memorylessness and the fresh-start property

- Analogous to the properties for the Bernoulli process
  - plausible, given the relation between the two processes
  - use intuitive reasoning
  - can be proved rigorously



t is a constant

Exercise: Describing events

4/4 points (graded)

Events related to the Poisson process can be often described in two equivalent ways: in terms of numbers of arrivals during certain intervals or in terms of arrival times. The first description involves discrete random variables, the second continuous random variables.

Let $N(t)$ be the number of arrivals during the time interval $[0, t]$ in a Poisson process. Let Y_k be the time of the kth arrival.

a) The event $\{N(5) > 1\}$ is equivalent to the event $\{Y_k \leq b\}$, for suitable b and k. Find b and k.

$b =$ 5 ✔ **Answer:** 5

$k =$ 2 ✔ **Answer:** 2

b) The event $\{2 < Y_3 \leq Y_4 \leq 5\}$ is equivalent to the event $\{N(2) \leq a \text{ and } N(5) \geq b\}$. Find a and b.

$a =$ 2 ✔ **Answer:** 2

$b =$ 4 ✔ **Answer:** 4

Solution:

a) We have $N(5) > 1$ if and only if we have had two or more arrivals by time 5, i.e., $T_2 \leq 5$. Thus, $b = 5$ and $k = 2$.

b) We have $2 < Y_3 \leq Y_4 \leq 5$ if and only if by time 2 we have not yet had 3 arrivals (i.e., $N(2) \leq 2$) and by time 5 we have had at least 4 arrivals (i.e., $N(5) \geq 4$). Thus, $a = 2$ and $b = 4$.

Exercise: Erlang r.v.'s

1/1 point (graded)

Let X and Y be independent Erlang random variables with common parameter λ and of order m and n, respectively. Is the random variable $X+Y$ Erlang? If yes, enter below its order in terms of m and n using standard notation. If not, enter 0.

m+n

✔ **Answer:** m+n

$m+n$

STANDARD NOTATION

Solution:

The random variable X can be viewed as the sum of m i.i.d. exponential random variables. Similarly, Y can be viewed as the sum of n i.i.d. exponential random variables. Furthermore, since X and Y are independent, we take these two collections of random variables to be independent. Thus, $X+Y$ can be interpreted as the sum of $m+n$ i.i.d. exponentials, and is Erlang of order $m+n$.

Exercise: The time of the kth arrival

2/2 points (graded)

Let Y_k be the time of the kth arrival in a Poisson process with parameter $\lambda = 1$. In particular, $\mathbf{E}[Y_k] = k$.

Is it true that $\mathbf{P}(Y_k \geq k) = 1/2$ for any finite k?

No

✔ **Answer:** No

Is it true that $\lim_{k\to\infty} \mathbf{P}(Y_k \geq k) = 1/2$?

Yes

✔ **Answer:** Yes

Solution:

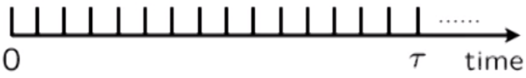
Consider the special case of $k = 1$. Then, $\mathbf{P}(Y_1 \geq 1) = e^{-1} \neq 1/2$.

When k is large, the central limit theorem applies because Y_k is the sum of k i.i.d. (exponential) random variables. Its (standardized) distribution is approximately normal, hence approximately symmetric around its mean. More formally, using the fact that the variance of an exponential with parameter 1 is 1, we have

$$\lim_{k\to\infty} \mathbf{P}(Y_k \geq k) = \lim_{k\to\infty} \mathbf{P}\left(\frac{Y_k - k}{\sqrt{k}} \geq 0\right) = \Phi(0) = \frac{1}{2},$$

where Φ is is the standard normal CDF.

Bernoulli/Poisson relation



$n = \tau/\delta$

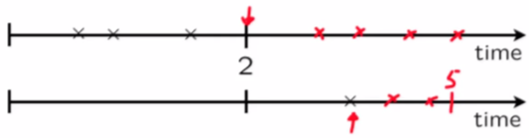
$p = \lambda\delta$

$np = \lambda\tau$

	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Example: Poisson fishing

- Fish are caught as a Poisson process, $\lambda = 0.6$/hour
  - fish for two hours;
  - if you caught at least one fish, stop
  - else continue until first fish is caught



$\mathbf{P}$(fish for more than two hours)= $P(0,2)$

$P(T_1 > 2) = \int_2^\infty f_{T_1}(t)\,dt$

$\mathbf{P}$(fish for more than two and less than five hours)=

$P(0,2)\,(1 - P(0,3))$

$P(2 < T_1 \le 5) = \int_2^5 f_{T_1}(t)\,dt$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

Example: Poisson fishing

- Fish are caught as a Poisson process, $\lambda = 0.6$/hour
  - fish for two hours;
  - if you caught at least one fish, stop
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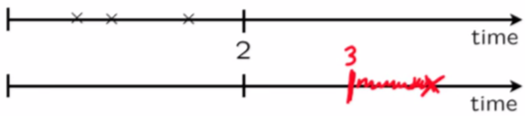


$\mathbf{P}$(catch at least two fish)=

$$\sum_{k=2}^{\infty} P(k,2) = 1 - P(0,2) - P(1,2)$$

$$P(Y_2 \le 2) = \int_0^2 f_{Y_2}(y)\,dy$$

$\mathbf{E}$[future fishing time | already fished for three hours]= $\frac{1}{\lambda}$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

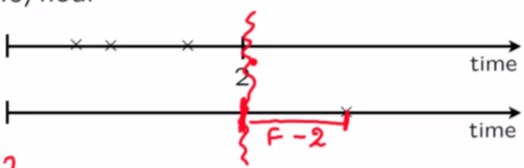
$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

1/lambda

Example: Poisson fishing

- Fish are caught as a Poisson process, $\lambda = 0.6$/hour
  - fish for two hours;
  - if you caught at least one fish, stop
  - else continue until first fish is caught



$\mathbf{E}$[total fishing time]= $E[F] = 2 + E[F-2]$

$= 2 + P(F=2)\cdot 0 + P(F>2)\, E[F-2 \mid F>2]$

$= 2 + P(0,2)\cdot 1/\lambda$

$\mathbf{E}$[number of fish]= $\underset{0.6 * 2}{\lambda\tau} + P(0,2)\cdot 1$

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

17. Exercise: Bank tellers

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Exercise: Bank tellers

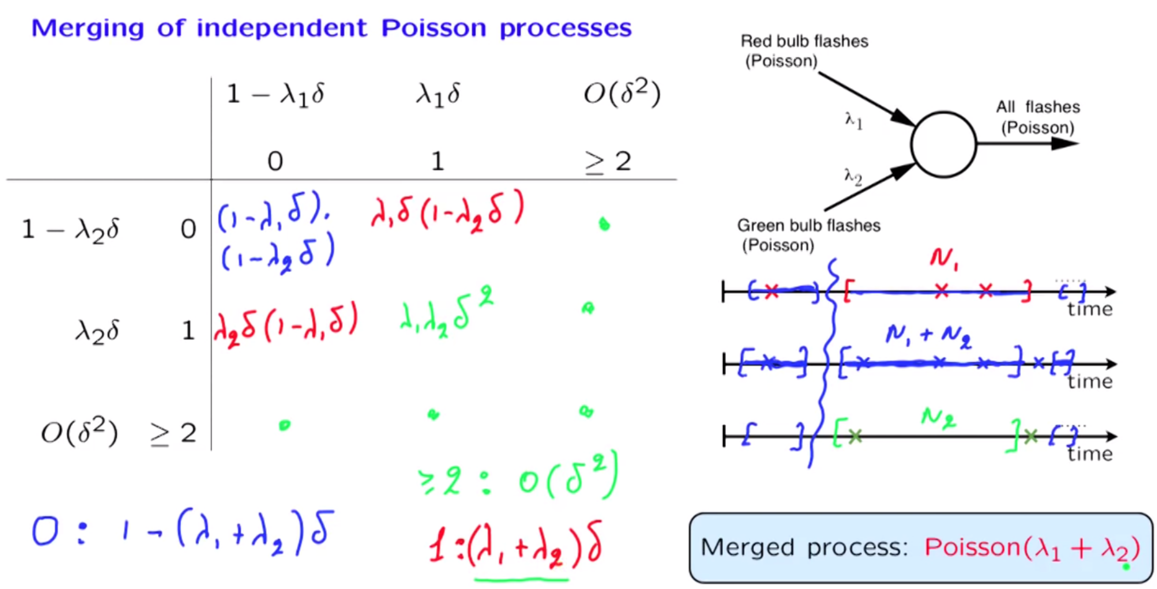
1/1 point (graded)

When you enter your bank, you find that there are only two tellers, both busy serving other customers, and that there are no other customers in line. Assume that the service times for you and for each of the customers being served are independent identically distributed exponential random variables. Also assume that after a service completion, the next customer in line immediately begins to be served. What is the probability that you will be the last to leave? *Hint:* Think of the situation at the time that you start getting served.

0.5 ✔ **Answer:** 0.5

Solution:

The answer is $1/2$. To see this, focus at the moment when you start service with one of the tellers. Note that the probability that both customers currently being served have their service end at exactly the same time is zero, and so when you start service, there will be another customer still being served. Using the memorylessness property of the exponential, the remaining time of the other customer being served is exponential. The time until your own service will be completed has the same exponential distribution and is independent. By symmetry, you and the other customer have equal probability, $1/2$, of being the last to leave.



Ignoring terms of order delta squared

Merged process has a rate which is equal to the sum of the rates of the separate processes

Exercise: Processes in the park

2/2 points (graded)

As in an earlier exercise, busy people arrive at the park according to a Poisson process with rate $\lambda_1 = 3$/hour and stay in the park for exactly $1/6$ of an hour. Relaxed people arrive at the park according to a Poisson process with rate $\lambda_2 = 2$/hour and stay in the park for exactly half an hour. The arrivals of busy and relaxed people are independent processes. Assume that no other people arrive at the park.

Is the process of total arrivals at the park a Poisson process? If yes, enter the rate of that process in the answer box below. If it is not, enter 0.

5 ✔ **Answer:** 5

Whenever a relaxed person exits the park, he/she enters a nearby coffee shop. (Assume, for simplicity, that going from the park to the coffee shop takes zero time.)

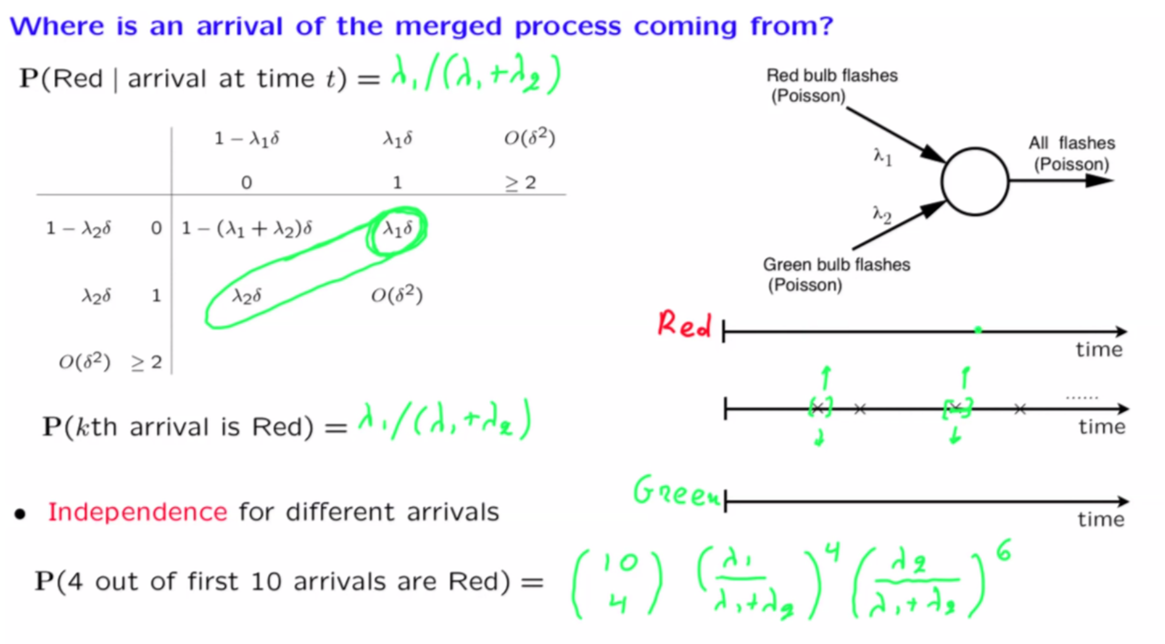
Is the process of arrivals of relaxed persons at the coffee shop a Poisson process? If yes, enter the rate of that process in the answer box below. If it is not, enter 0.

2 ✔ **Answer:** 2

Solution:

As discussed in the preceding video, it is a Poisson process whose rate is the sum, $3 + 2 = 5$, of the rates of the original processes.

The process of relaxed people arrivals at the coffee shop is identical to the process of relaxed people arrivals at the park, but delayed by half an hour. You can check that a Poisson process that is delayed by a constant amount has exactly the same statistical properties (independence, time-homogeneity, small time interval probabilities) and is therefore a Poisson process with the same rate, which is 2 in this case.



Exercise: What kind of people are they

1/1 point (graded)

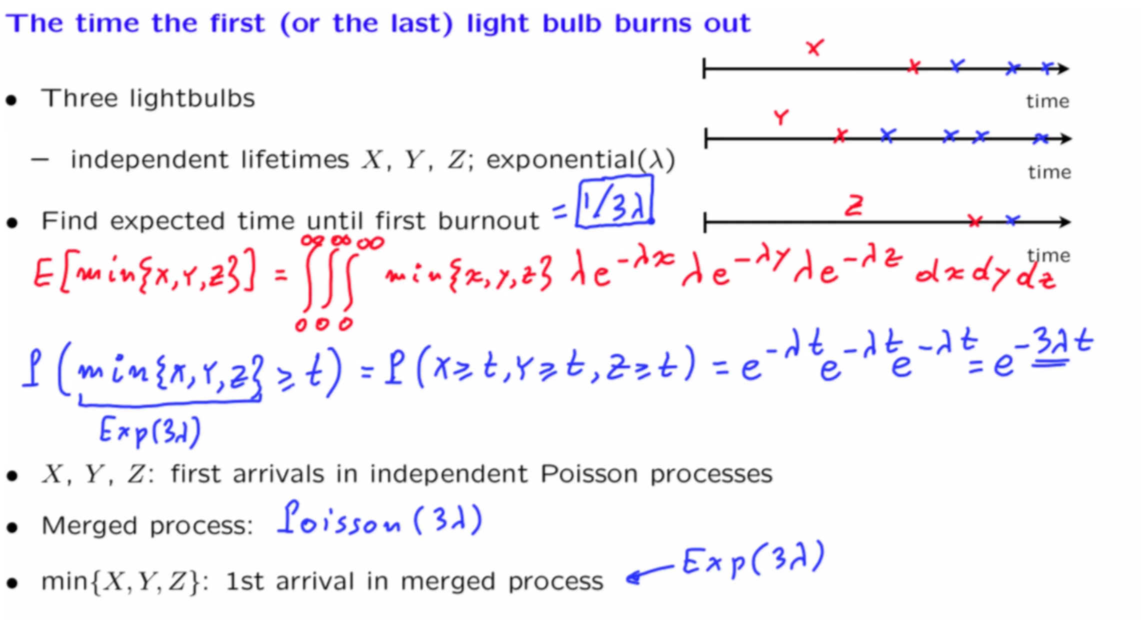
As in an earlier exercise, busy people arrive at the park according to a Poisson process with rate $\lambda_1 = 3$/hour. Relaxed people arrive at the park according to an independent Poisson process with rate $\lambda_2 = 2$/hour. Assume that no other people arrive at the park.

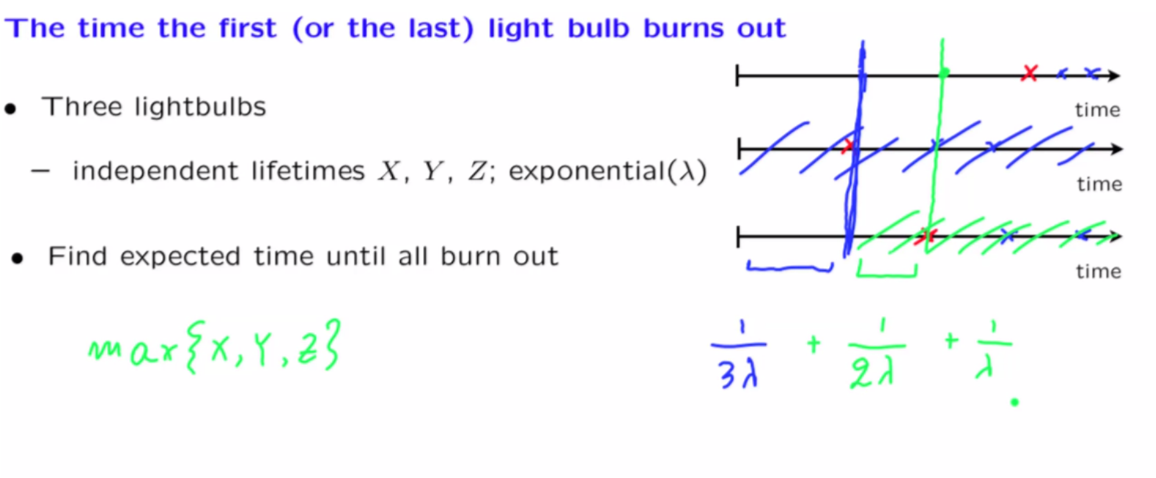
During the last 10 minutes, exactly two people arrived at the park. What is the probability that they are both relaxed?

4/25 ✔ **Answer:** 0.16

Solution:

As discussed in the preceding video, each arrival has probability $2/(3+2) = 2/5$ of being a relaxed person. Furthermore, the types (busy or relaxed) of the different arrivals are independent. Therefore, the probability that both arrivals are relaxed is $(2/5)^2 = 4/25$.





Exercise: Lightbulb burnouts

0/1 point (graded)

As in the preceding video, consider three lightbulbs each of which has a lifetime that is an independent exponential random variable with parameter $\lambda = 1$. The variance of the time until all three burn out is:

1.83333

✖ **Answer:** 1.36111

Recall that the variance of an exponential with parameter λ is $1/\lambda^2$.

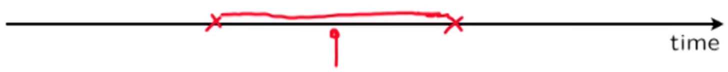
Solution:

As we discussed, the time until all three lighbulbs burn out is the sum of an exponential random variable with parameter 3λ, an exponential random variable with parameter 2λ, and an exponential random variable with parameter λ. Furthermore, because of the fresh-start property, we argued that these three random variables are independent. Therefore, since $\lambda = 1$, the variance is

$$\frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{1^2} = \frac{49}{36}.$$

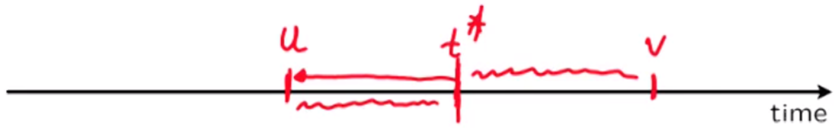
"Random incidence" in the Poisson process

- Poisson process that has been running forever



- Believe that $\lambda = 4$/hour, so that $\mathbf{E}[T_k] = \frac{1}{4}$ hrs = 15 mins
- Show up at some time and measure interarrival time
  - do it many times, average results, see something around 30 mins! Why?

"Random incidence" in the Poisson process — analysis



- Arrive at time t^*
- U: last arrival time
- V: next arrival time
- $V - U = \underbrace{(V - t^*)}_{Exp(\lambda)} + \underbrace{(t^* - U)}_{Exp(\lambda)}$

$\frac{1}{\lambda}$

- $\mathbf{E}[V - U] = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda}$
- $V - U$: interarrival time you see, versus kth interarrival time

A "backwards" running poisson process is still a poisson process
bias towards longer interarrival times as more likely to be placed in one compared to a shorter time

Exercise: Random incidence

1/1 point (graded)

Consider an arrival process whose interarrival times are independent exponential random variables with mean 2 (and consequently variance equal to 4), and consider the interarrival interval S seen by an observer who arrives at a fixed time t^*, as in the preceding video. What is the variance of S?

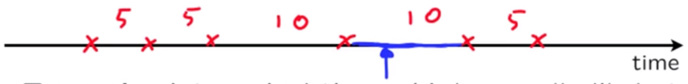
8

✔ **Answer:** 8

Solution:

As discussed in the preceding video, such an interval is the sum of two independent exponential random variables. Its variance is the sum of the variances of these two exponentials: $4 + 4 = 8$.

Random incidence "paradox" is not special to the Poisson process



- **Example:** interrarival times, i.i.d., equally likely to be 5 or 10 minutes

  expected value of kth interarrival time: $\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 10 = 7.5$

- you show up at a "random time"

  $\mathbf{P}$(arrive during a 5-minute interarrival interval) $= \frac{1}{3}$

  expected length of interarrival interval during which you arrive $= \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 10$

  ≈ 8.3

- Calculation generalizes to "renewal processes:"
  i.i.d. interarrival times, from some general distribution

- "Sampling method" matters

the 10 minute interval is longer so more likely to end up in one of those

Different sampling methods can give different results

- Average family size? $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 6$
  - look at a "random" family (uniformly chosen)
  - look at a "random" person's (uniformly chosen) family $\frac{3}{9} \cdot 1 + \frac{6}{9} \cdot 6$

- Average bus occupancy?

  - look at a "random" bus (uniformly chosen)
  - look at a "random" passenger's bus

- Average class size?