

# Recitation: inference of Gaussian distribution

## Recitation problem statement

Consider a sample of  $n$  i.i.d. Gaussian random variables  $X_1, \dots, X_n$ , with unknown parameters  $\mu \in \mathbb{R}$  and  $\sigma^2 > 0$ .

1. Let  $\hat{\sigma}^2$  be the sample variance of  $X_1, \dots, X_n$ . Recall an expression of  $\hat{\sigma}^2$ .
2. Prove that  $\hat{\sigma}^2$  is a consistent estimator of  $\sigma^2$ .
3. Using the central limit theorem, prove that  $\left( \frac{1}{n} \sum_{i=1}^n X_i, \frac{1}{n} \sum_{i=1}^n X_i^2 \right)'$  is asymptotically normal.
4. Conclude that  $\hat{\sigma}^2$  is asymptotically normal and compute its asymptotic variance.
5. Using the previous questions, find a confidence interval for  $\sigma^2$  with asymptotic level 95% (use two methods).

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$  iid.

Goal: Find asymptotic confidence interval for  $\sigma^2$

① Find estimator  $\hat{\sigma}^2$  for  $\sigma^2$ .

② Determine the asymptotic distribution of  $\hat{\sigma}^2$

③ Construct conf. interval based on  $\hat{\sigma}^2$

Getting a consistent estimator of the variance

$$\begin{aligned} \textcircled{1} \quad \text{Var}(X_1) &= \mathbb{E}[(X_1 - \mathbb{E}[X_1])^2] \\ &= \mathbb{E}[X_1^2] - (\mathbb{E}[X_1])^2 \\ \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n X_i^2 - \left( \frac{1}{n} \sum_{i=1}^n X_i \right)^2 \end{aligned}$$

By LLN:  $\frac{1}{n} \sum_{i=1}^n X_i^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathbb{E}[X_1^2]$

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n X_i &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mathbb{E}[X_1] \\ \Rightarrow \hat{\sigma}^2 &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} \text{Var}(X_1) \end{aligned}$$

- this estimator should go to 0 since the mean of the distribution is 0
- 

Applying the multivariate central limit theorem

②  $\begin{pmatrix} Y_1 \\ W_1 \end{pmatrix}, \dots, \begin{pmatrix} Y_n \\ W_n \end{pmatrix}$  iid.

CLT:  $\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n Y_i - \mathbb{E}[Y_i] \right) \xrightarrow{D} N(0, \text{Var}(Y_i))$

~~zD CLT:~~  $\sqrt{n} \left( \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n Y_i \\ \frac{1}{n} \sum_{i=1}^n W_i \end{pmatrix} - \begin{pmatrix} \mathbb{E}[Y_i] \\ \mathbb{E}[W_i] \end{pmatrix} \right)$

$\boxed{\begin{array}{l} Y_i = X_i^2 \\ W_i = X_i \end{array}}$

$\xrightarrow{D} N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \text{Var}(Y_i) & \text{Cov}(Y_i, W_i) \\ \text{Cov}(Y_i, W_i) & \text{Var}(W_i) \end{pmatrix} \right)$

$\boxed{\begin{array}{l} Y_i = X_i^2 \\ W_i = X_i \end{array}}$

$$\begin{aligned} \text{Var}(W_i) &= \sigma^2 && (\text{Cov}(Y_i, W_i) = 0) \\ \text{Var}(Y_i) &= \mathbb{E}[(X_i^2)^2] - (\mathbb{E}[X_i^2])^2 = 3\sigma^4 - (\sigma^2)^2 = 2\sigma^4. \\ \text{Cov}(Y_i, W_i) &= \mathbb{E}[X_i^2 \cdot X_i] - \mathbb{E}[X_i^2] \cdot \mathbb{E}[X_i] \\ &= 0 - \sigma^2 \cdot 0 = 0 \end{aligned}$$

- easily goes to 0 here because we are dealing with a centric distribution around 0

Applying the multivariate delta method

② Delta Method:  $\sqrt{n} \left( \bar{T}_n - \Theta \right) \xrightarrow[n \rightarrow \infty]{D} N(0, \Sigma)$   
 $g: \mathbb{R}^d \rightarrow \mathbb{R}$ , continuously differentiable at  $\Theta$

$$\Rightarrow \sqrt{n} (g(\bar{T}_n) - g(\Theta)) \rightarrow N(0, \nabla g(\Theta)^T \Sigma \nabla g(\Theta))$$

$$Y_i = X_i^2, \quad W_i = X_i, \quad \bar{T}_n = \frac{1}{n} \sum_{i=1}^n \left( \frac{X_i^2}{X_i} \right), \quad \hat{\sigma}^2 = g(\bar{T}_n),$$

$$g(y, w) = y - w^2, \quad \nabla g(y, w) = \begin{pmatrix} 1 \\ -2w \end{pmatrix}, \quad \Theta = \begin{pmatrix} \mathbb{E}[Y_i] \\ \mathbb{E}[W_i] \end{pmatrix} = \begin{pmatrix} \hat{\sigma}^2 \\ 0 \end{pmatrix}$$

$$\sqrt{n} \cdot (\hat{\sigma}^2 - \sigma^2) \xrightarrow[n \rightarrow \infty]{D} N(0, \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} 2\sigma^4 & 0 \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix})$$

$$= N(0, 2\sigma^4)$$

- delta method gives us the asymptotic distribution, because we know that it is continuously differentiable
- given a sequence of random variables  $T_n$ , such that they are centred by theta and re-scaled by  $\sqrt{n}$  - if it goes to a normal distribution with covariance matrix, and we have a continuously differentiable function  $g \Rightarrow$  (then) we know that the limiting distribution is as above (3rd line), and the covariance matrix is transformed by the gradient
- intuitively this means how sensitive is  $g$  to each of the variables
- $\nabla g$  is the partial derivatives with respect to  $y$  and  $w$
- inserting covariance matrix, terms were calculated in previous part
- 

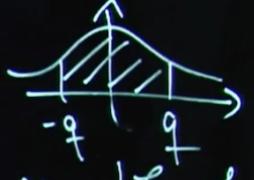
Confidence interval

$$\sqrt{n} \frac{\hat{\sigma}^2 - \sigma^2}{\sigma^2} \xrightarrow[n \rightarrow \infty]{D} N(0, 1)$$

using this formula

③ Two-sided confidence interval  $I = [\hat{\sigma}^2 - s, \hat{\sigma}^2 + s]$

$$\underbrace{P(\hat{\sigma}^2 \in I)}_{n \rightarrow \infty} \xrightarrow{} P(Z \in [-q, q]), Z \sim N(0, 1)$$

$$\Leftrightarrow \hat{\sigma}^2 \in [\hat{\sigma}^2 - s, \hat{\sigma}^2 + s] \quad \stackrel{!}{=} 1 - \alpha$$


$$\Leftrightarrow \hat{\sigma}^2 - \hat{\sigma}^2 \in [-s, s] \quad \Rightarrow q = q_{\alpha/2}$$

$$\Leftrightarrow \hat{\sigma}^2 - \hat{\sigma}^2 \in [-s, s] \quad 1 - \frac{\alpha}{2} \text{ quantile of } N(0, 1)$$

$$\Leftrightarrow \sqrt{\frac{n}{2}} \frac{\hat{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2} \in \left[ -\sqrt{\frac{n}{2}} \cdot \frac{s}{\hat{\sigma}^2}, \sqrt{\frac{n}{2}} \cdot \frac{s}{\hat{\sigma}^2} \right], \quad S = q_{\alpha/2} \cdot \frac{\hat{\sigma}^2 \cdot \sqrt{n}}{\sqrt{2}}$$

$$\Rightarrow I = \hat{\sigma}^2 + \left[ -\sqrt{\frac{n}{2}} \cdot q_{\alpha/2} \cdot \hat{\sigma}^2, \sqrt{\frac{n}{2}} \cdot q_{\alpha/2} \cdot \hat{\sigma}^2 \right]$$

- this confidence interval isn't great because it's dependent on the parameter we are trying to estimate ( $\sigma^2$ )
- we are interested in an asymptotic value so we can instead use  $\sigma_{\hat{\sigma}^2}$
- this is done using Slutsky's theorem

Slutsky's Thm:  $A_n \xrightarrow[n \rightarrow \infty]{D} A, B_n \xrightarrow[n \rightarrow \infty]{D} c \neq 0 \Rightarrow \frac{A_n}{B_n} \xrightarrow[n \rightarrow \infty]{D} \frac{A}{c}$

$$\Rightarrow \sqrt{\frac{n}{2}} \frac{\hat{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2} \xrightarrow[n \rightarrow \infty]{D} N(0, 1)$$

this allows us to turn all  $\sigma^2$  into  $\sigma_{\hat{\sigma}^2}$