Homework 3
Hypothesis testing

1. True or False

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(a)

2 points possible (graded)

Suppose the following: According to a fixed statistical model $\{\mathbf{P}_\theta\}_{\theta\in\Theta}$, a pair of hypotheses $H_0 : \theta \in \Theta_0$ and $H_1 : \theta \in \Theta_1$, and a 0.05-level test $\psi_{0.05}$ of the form

$$\psi_\alpha = \mathbf{1}(T_n > c_\alpha), \quad T_n = T_n(X_1, \ldots, X_n), \quad c_\alpha \in \mathbb{R},$$

we observe the sample $x_1, \ldots, x_n$ and compute the p-value to be 0.01. Here T_n is a test statistic, and c_α is a threshold constant.

For each of the following groups of statements, select the one that is necessarily true. If there is none, select "None of the above."

Which of the following is necessarily true?

- Any test ψ_α that rejects H_0 for this observation will have a Type 1 error of at most 0.05.
- Any test ψ_α that rejects H_0 for this observation will have a Type 2 error of at most 0.05.
- Any test ψ_α that does not reject H_0 for this observation will have a Type 1 error of at most 0.01. ✔
- Any test ψ_α that does not reject H_0 for this observation will have a Type 2 error of at most 0.01.
- None of the above.

Which of the following is necessarily true?

- ◯ There is exactly a 0.99 chance for the null hypothesis to be true.
- ◉ There is exactly a 0.99 chance for the null hypothesis to be false.
- ◯ There is exactly a 0.01 chance for the alternative hypothesis to be true.
- ◯ There is exactly a 0.01 chance for the alternative hypothesis to be false.
- ◯ None of the above ✔

Solution:

First, note that $\psi_{0.05}\left(x_1, \ldots, x_n\right) = 1$ since the p-value 0.01 of this sample is lower than 0.05, the level of the test. Thus, $T_n\left(x_1, \ldots, x_n\right) > c_{0.05}$. In fact, $T_n\left(x_1, \ldots, x_n\right) = c_{0.01}$ by the definition of p-value. If another test of the form $\psi_\alpha = \mathbf{1}\left(T_n > C_\alpha\right)$ does not reject H_0 at the given sample $x_1, \ldots, x_n$, then its threshold must satisfy

$$c_\alpha > T_n\left(x_1, \ldots, x_n\right) = c_{0.01}.$$

Since ψ_α has a higher threshold then $\psi_{0.01}$, we always have

$$\{T_n > c_\alpha\} \subset \{T_n > c_{0.01}\} \quad \text{and} \quad \mathbf{P}_\theta\{T_n > c_\alpha\} \leq \mathbf{P}_\theta\{T_n > c_{0.01}\} \quad \forall \theta \in \Theta_0.$$

Thus, its level must be $\alpha < 0.01$.

On the other hand, any test of this form that reject H_0 for this sample will have a type 1 error rate of at least 0.01.

This problem does not provide any information about the Type 2 errors.

(b)

1 point possible (graded)

Consider a statistical experiment $X_1, \ldots, X_n \stackrel{iid}{\sim} P_{\theta^*}$ with an associated statistical model $(E, \{P_\theta\}_{\theta \in \Theta})$. You perform a hypothesis test on the true parameter θ^* via a statistical test ψ.

Which of the following is true about the p-value associated to this statistical experiment?
(Choose all that apply.)

- ☐ The set of all possible values that the p-value can take varies depending on the distribution P_θ. For example, one distribution may have p-values in $(0, \infty)$, while another may be constrained to a discrete set like $\mathbb{Z}_{\geq 0}$.
- ☑ Regardless of the distribution of $X_1, \ldots, X_n$, the p-value lies in the interval $[0, 1]$. ✔
- ☑ The p-value will vary from one statistical experiment to another (*i.e.*, it varies depending on the particular sample), but it will always take values between 0 and 1. ✔

Solution:

We first examine the correct responses.

- The second and third choice are correct. Recall that the p-value is the smallest value α such that a test ψ of level α will reject the null hypothesis for the given sample $X_1, \ldots, X_n$. Recall that the level of a test is some α such that the type 1 error is uniformly bounded:

$$\alpha_\psi(\theta) = P_\theta(\psi = 1) \leq \alpha, \qquad \text{for all } \theta \in \Theta_0,$$

  and since it upper bounds a probability, the level α is in the interval $(0, 1)$. Thus, so is the p-value.

- The first choice is incorrect. The previous bullet explains how p-values are *always* on the scale $(0, 1)$, regardless of the particular distribution P_{θ^*}.

Remark: Part of the usefulness of p-values is that they put all distributions 'on the same scale'. Regardless of whether P_θ is Gaussian, Exponential, Poisson, or uniform, the p-value which lives in $(0, 1)$, so we just have to look at this number (and not particular properties of the distribution) to assess whether or not H_0 should be rejected.

2. Concept Check: Hypothesis Test Using a Single Observation

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Setup:

Let X be a **single** (i.e. $n = 1$) Gaussian random variable with unknown mean μ and variance 1. Consider the following hypotheses:

$$H_0 : \mu = 0 \quad \text{vs} \quad H_1 : \mu \neq 0.$$

(a)

1 point possible (graded)

Define a test $\psi_\alpha : \mathbb{R} \to \{0, 1\}$ with level α that is of the form

$$\psi_\alpha = \mathbf{1}\{f_\alpha(X) > 0\},$$

for some function $f_\alpha : \mathbb{R} \to \mathbb{R}$.

We want our test ψ above to satisfy the following:

- **Symmetric** in the value of X about 0, so $f(X) = f(-X)$;
- Its "acceptance region" is an interval. (The **acceptance region** of a test is the region in which the null hypothesis is **not rejected**, i.e. the complement of its rejection region.)

Specify the function $f_\alpha(X)$ in terms of α below.

(Type **alpha** for α. If applicable, enter **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0, 1)$, and **q(alpha)** for q_α, the $1-\alpha$-quantile of a standard normal distribution, e.g. enter **q(0.01)** for $q_{0.01}$.)

$f(X) =$ **Answer:** abs(X) - q(alpha/2)

STANDARD NOTATION

Solution:

Since our test should be symmetric about zero and its "acceptance region" an interval, it must be of the form

$$\psi = \mathbf{1}\{|X| - q_{\alpha/2} > 0\}.$$

(b)

3 points possible (graded)

Assume you observe $X = 1.32$, and What is the value of your test ψ_α with level $\alpha = 0.05$?

$\psi(X) =$ **Answer:** 0

What is the p-value of your test (keeping in mind the symmetry and interval requirements)?
(If applicable, enter **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0,1)$, and **q(alpha)** for q_α, the $1-\alpha$-quantile of a standard normal distribution, e.g. enter **q(0.01)** for $q_{0.01}$.)

p-value $=$ **Answer:** 2*(1-Phi(1.32))

What is the conclusion of the test?

- Accept H_0
- Do not reject H_0 ✔
- Accept H_1
- Do not reject H_1

Solution:

First, determine q_α for our test ψ_α when $\alpha = 0.05$:

$$\begin{aligned} \mathbf{P}_{\mu=0}(|X| > q) = &\quad 0.05 \\ \iff 2(1-\Phi(q)) = &\quad 0.05 \\ \iff \Phi(q) = &\quad 0.975 \\ \iff q = q_{0.025} \approx &\quad 1.96. \end{aligned}$$

Hence, the test $\psi_{0.05}(X) = \mathbf{1}\{f_{0.05}(X) > 0\}$ where

$$f_{0.05}(X) = |X| - 1.96.$$

Since $|1.32| < 1.96$, $\psi_{0.05}(1.32) = 0$.

Next, under the requirements for the test, the p-value is defined as

$$\inf\{\alpha : \psi_\alpha(X) = 1\},$$

where

$$\psi_\alpha(X) = \mathbf{1}\{|X| > q(\alpha)\}.$$

In other words, the p-value is the smallest value so that we could still reject H_0 given the observation, when picking our hypothesis test from a family of hypothesis tests indexed by α . In this case, by the requirement of ψ_α having confidence level α ,

$$\begin{aligned} \mathbf{P}_{\mu=0}\left(\psi_\alpha\left(X\right) > q\left(\alpha\right)\right) = &\quad 2\left(1 - \Phi\left(q\right)\right) = \alpha \\ \iff \Phi\left(q\right) = &\quad 1 - \frac{\alpha}{2} \end{aligned}$$

and hence

$$q\left(\alpha\right) = q_{\alpha/2},$$

the $1 - \alpha/2$ quantile of a Normal variable. Now, by the form of the test ψ_α , we see that we get the infimum of α if $|X| = q_{\alpha/2}$, i.e., if

$$\alpha = 2\left(1 - \Phi\left(|X|\right)\right) = 2 - 2\Phi\left(1.32\right) \approx 0.19.$$

We do not reject H_0 because there is not enough evidence for doing so. That does not necessarily mean that we think H_0 true, so we should not “accept" it.

3. Simple Testing

Bookmark this page

Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}\left(\theta, 1\right)$. Consider testing

$$H_0 : \theta = 0 \quad \text{v.s.} \quad H_1 : \theta = 1.$$

(a)

2 points possible (graded)

What would a Type 1 error be in this test?

- (●) Rejecting H_0 when $\theta = 0$ ✔
- ( ) Not Rejecting H_0 when $\theta = 0$
- ( ) Rejecting H_0 when $\theta = 1$
- ( ) Not rejecting H_0 when $\theta = 1$

What would a Type 2 error be in this test?

- ( ) Rejecting H_0 when $\theta = 0$
- ( ) Not Rejecting H_0 when $\theta = 0$
- ( ) Rejecting H_0 when $\theta = 1$
- (●) Not rejecting H_0 when $\theta = 1$ ✔

Solution:

By definition of the type 1 and type 2 errors. The other choices are not errors.

(b)

1 point possible (graded)

Suppose that the rejection region of a test ψ has the form $R = \{\overline{X}_n : \overline{X}_n > c\}$. Find the smallest c such that ψ has level α.

(If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0,1)$, and **q(alpha)** for q_α, the $1-\alpha$ quantile of a standard normal variable.)

$c \geq$ **Answer:** q(alpha)/sqrt(n)

STANDARD NOTATION

Solution:

Since X_i are Gaussian,

$$\sqrt{n}\overline{X}_n \sim \mathcal{N}(0,1).$$

Given the rejection region $R = \{\overline{X}_n : \overline{X}_n > c\}$, the corresponding test $\psi_{n,\alpha} = \mathbf{1}\left(\overline{X}_n \in R\right)$ has level α for any c such that

$$\mathbf{P}_0\left(\overline{X}_n > c\right) = \mathbf{P}_0\left(\sqrt{n}\overline{X}_n > \sqrt{n}c\right) \leq \alpha.$$

Hence, the smallest such c is $c = \frac{q_\alpha}{\sqrt{n}}$.

(c)

2 points possible (graded)

Suppose that the test ψ has level $\alpha = 0.05$. What is the power of ψ?

(If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0,1)$, and **q(alpha)** for q_α, the $1-\alpha$ quantile of a standard normal variable, e.g. enter **q(0.01)** for $q_{0.01}$.)

Power of ψ: **Answer:** 1-Phi(q(0.05)-sqrt(n))

What does the power of ψ approach as $n \to \infty$?

$\lim_{n\to\infty} \text{Power} =$ **Answer:** 1

STANDARD NOTATION

Solution:

Since H_1 consists of a single point, the type 2 error of ψ is

$$\begin{aligned}\mathbf{P}_{\theta=1}(\psi = 0) = \mathbf{P}_{\theta=1}\left(\overline{X}_n \leq \frac{q_{0.05}}{\sqrt{n}}\right) &= \mathbf{P}_{\theta=1}\left(\sqrt{n}\left(\overline{X}_n - 1\right) \leq q_{0.05} - \sqrt{n}\right) \\ &= \Phi\left(q_{0.05} - \sqrt{n}\right).\end{aligned}$$

Hence, the power is $1 - \Phi\left(q_{0.05} - \sqrt{n}\right)$. As $n \to \infty$, this goes to 1.

4. Relating Hypothesis Tests and Confidence intervals

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(a)

2 points possible (graded)

Consider an i.i.d. sample $X_1, \ldots, X_n \sim \text{Poiss}(\lambda)$ for $\lambda > 0$.

Starting from the Central Limit Theorem, find a confidence interval $I = [A, B]$ with asymptotic level $1 - \alpha$ that is centered about $\overline{X}_n$ using the plug-in method.

(Write **barX_n** for $\overline{X}_n$. If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0, 1)$, and **q(alpha)** for q_α, the $1 - \alpha$ quantile of a standard normal variable.)

$A =$ **Answer:** barX_n - q(alpha/2)*sqrt(barX_n/n)

$B =$ **Answer:** barX_n + q(alpha/2)*sqrt(barX_n/n)

STANDARD NOTATION

Solution:

By the Central Limit Theorem,

$$\sqrt{n}\frac{\overline{X}_n - \lambda}{\sqrt{\lambda}} \xrightarrow[n\to\infty]{(D)} \mathcal{N}(0, 1).$$

Since

Since

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \lambda,$$

by Slutsky's Theorem, we get

$$\sqrt{n}\frac{\overline{X}_n - \lambda}{\sqrt{\overline{X}_n}} \xrightarrow[n\to\infty]{\text{(D)}} \mathcal{N}(0,1)\,.$$

That means for $q > 0$ that with

$$I = \left[\overline{X}_n - \frac{q\sqrt{\overline{X}_n}}{\sqrt{n}}, \overline{X}_n + \frac{q\sqrt{\overline{X}_n}}{\sqrt{n}}\right],$$

we have

$$\mathbf{P}_\lambda(\lambda \in I) \xrightarrow[n\to\infty]{} 1 - 2\Phi(q)\,.$$

If we want this quantity to be $1-\alpha$ to guarantee level $1-\alpha$ of the interval, that leads to

$$\Phi(q) = 1 - \frac{\alpha}{2} \iff q = q_{\alpha/2} = \Phi^{-1}(1 - \frac{\alpha}{2})\,.$$

(b)

2 points possible (graded)

Continuing the problem above, now consider the following hypothesis with a fixed number $\lambda_0 > 0$:

$$H_0 : \lambda = \lambda_0 \quad \text{vs} \quad H_1 : \lambda \neq \lambda_0.$$

Define a test for the above hypotheses with asymptotic level α, and rewrite it in the form

$$\psi = \mathbf{1}\{\lambda_0 \notin J\},$$

for some confidence interval $J = [C, D]$.

(Write **barX_n** for $\overline{X}_n$. If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0, 1)$, and **q(alpha)** for q_α, the $1 - \alpha$ quantile of a standard normal variable.)

$C =$

Answer: barX_n - q(alpha/2)*sqrt(barX_n)/sqrt(n)

$D =$

Answer: barX_n + q(alpha/2)*sqrt(barX_n)/sqrt(n)

Solution:

By setting

$$J = I = \left[\overline{X}_n - \frac{q_{\alpha/2}\sqrt{\overline{X}_n}}{\sqrt{n}}, \overline{X}_n + \frac{q_{\alpha/2}\sqrt{\overline{X}_n}}{\sqrt{n}} \right]$$

from part (a), the fact that I is a confidence interval with asymptotic level α means that

$$\mathbf{P}_\lambda (\lambda \in I) \xrightarrow[n\to\infty]{} 1-\alpha \quad \lambda > 0,$$

so

$$\mathbf{P}_\lambda (\lambda \notin I) \xrightarrow[n\to\infty]{} \alpha \quad \lambda > 0.$$

In particular, if we set

$$\psi = \mathbf{1}\{\lambda_0 \notin I\},$$

this means that

$$\mathbf{P}_{\lambda_0} (\psi = 1) = \mathbf{P}_{\lambda_0} (\lambda_0 \notin I) \xrightarrow[n\to\infty]{} \alpha,$$

yielding a hypothesis test with asymptotic level α .

5. P-Values Formulas

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In each of the following questions, you are given an i.i.d. sample and two hypotheses. For any $\alpha \in (0, 1)$, use the Central Limit Theorem to define a test with asymptotic level α, then give a formula for the asymptotic p-value of your test.

(a)

1 point possible (graded)

$X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Poiss (λ) for some unknown $\lambda > 0$;

$$H_0 : \lambda = \lambda_0 \quad \text{v.s.} \quad H_1 : \lambda \neq \lambda_0 \qquad \text{where } \lambda_0 > 0.$$

(Type **barX_n** for $\overline{X}_n$, **lambda_0** for λ_0. If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0, 1)$, and **q(alpha)** for q_α, the $1 - \alpha$ quantile of a standard normal variable, e.g. enter **q(0.01)** for $q_{0.01}$.)

Asymptotic p-value =

Answer: 2*(1-Phi(sqrt(n)*abs(barX_n-lambda_0)/sqrt(lambda_0)))

Solution:

Since $X_i \sim$ Poiss (λ), $\mathbb{E}[X_i] = \lambda$ and $\sigma = \sqrt{\lambda}$. Hence, under $H_0 : \lambda = \lambda_0$, the Central Limit Theorem gives

$$T_{n,\lambda_0}(\overline{X}_n) = \sqrt{n}\left(\frac{\overline{X}_n - \lambda_0}{\sqrt{\lambda_0}}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, 1).$$

A test ψ with asymptotic level α is therefore

$$\psi_{n,\lambda_0,\alpha} = \mathbf{1}\left(\left|T_{n,\lambda_0}(\overline{X}_n)\right| > q_{\alpha/2}\right).$$

The asymptotic p-value is the smallest level α such that the test $\psi_{n,\lambda_0,\alpha}$ rejects the null hypothesis for a given sample (here, for a given realization of $\overline{X}_n$), hence:

$$\begin{aligned} p\text{-value} &= \mathbf{P}\left(|Z| > \left|T_{n,\lambda_0}(\overline{X}_n)\right|\right) \quad \text{where} \quad Z \sim \mathcal{N}(0, 1) \\ &= 2\left(1 - \Phi\left(\left|T_{n,\lambda_0}(\overline{X}_n)\right|\right)\right). \end{aligned}$$

Alternatively, define the test ψ and the p-value using

$$T_{n,\lambda_0}(\overline{X}_n) = \sqrt{n}\left|\frac{\overline{X}_n - \lambda_0}{\sqrt{\overline{X}_n}}\right|.$$

By Slutsky's theorem and the CLT, $T_{n,\lambda_0}(\overline{X}_n) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, 1)$.

(b)

1 point possible (graded)

$X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Poiss (λ) for some unknown $\lambda > 0$;

$$H_0 : \lambda \geq \lambda_0 \qquad \text{v.s.} \qquad H_1 : \lambda < \lambda_0 \qquad \text{where } \lambda_0 > 0.$$

(Type **barX_n** for $\overline{X}_n$, **lambda_0** for λ_0. . If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0, 1)$, and **q(alpha)** for q_α, the $1 - \alpha$ quantile of a standard normal variable.)

Asymptotic p-value =

Answer: Phi(sqrt(n)*(barX_n-lambda_0)/sqrt(lambda_0))

STANDARD NOTATION

Solution:

As in the previous problem, since $X_i \sim$ Poiss (λ), $\mathbb{E}[X_i] = \lambda$ and $\sigma = \sqrt{\lambda}$. Hence, assuming $\lambda = \lambda_0$, which is at the boundary of Θ_0 and Θ_1, the Central Limit Theorem gives again

$$T_{n,\lambda_0}(\overline{X}_n) = \sqrt{n}\left(\frac{\overline{X}_n - \lambda_0}{\sqrt{\lambda_0}}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1).$$

A candidate test ψ with asymptotic level α is therefore

$$\psi_{n,\lambda_0,\alpha} = \mathbf{1}\left(T_{n,\lambda_0}(\overline{X}_n) < -q_\alpha\right).$$

This is because

$$\mathbf{P}_\lambda\left(T_{n,\lambda_0}(\overline{X}_n) < -q_\alpha\right) \leq \mathbf{P}_\lambda\left(T_{n,\lambda_0}(\overline{X}_n) < -q_\alpha\right) \quad \text{for} \quad \lambda \geq \lambda_0$$

Recall that the (asymptotic) level α is an upper bound of the type 1 error. As argued in lecture and lecture exercises, the maximum of the type 1 error is achieved at the boundary of Θ_0 and Θ_1 for a one-sided tests, where the parameter space is 1-dimensional.

The asymptotic p-value is

$$\begin{aligned} p\text{–value} &= \mathbf{P}\left(Z < T_{n,\lambda_0}\left(\overline{X}_n\right)\right) \quad \text{where} \quad Z \sim \mathcal{N}(0,1) \\ &= \Phi\left(T_{n,\lambda_0}\left(\overline{X}_n\right)\right). \end{aligned}$$

Alternatively, again define the test ψ and the p-value using

$$T_{n,\lambda_0}(\overline{X}_n) = \sqrt{n}\left|\frac{\overline{X}_n - \lambda_0}{\sqrt{\overline{X}_n}}\right|.$$

By Slutsky's theorem and the CLT, $T_{n,\lambda_0}(\overline{X}_n) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1)$.

Solution:

As in the previous problem, since $X_i \sim$ Poiss (λ), $\mathbb{E}[X_i] = \lambda$ and $\sigma = \sqrt{\lambda}$. Hence, assuming $\lambda = \lambda_0$, which is at the boundary of Θ_0 and Θ_1, the Central Limit Theorem gives again

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A candidate test ψ with asymptotic level α is therefore

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This is because

$$\mathbf{P}_\lambda\left(T_{n,\lambda_0}(\overline{X}_n) < -q_\alpha\right) \le \mathbf{P}_\lambda\left(T_{n,\lambda_0}(\overline{X}_n) < -q_\alpha\right) \quad \text{for} \quad \lambda \ge \lambda_0$$

Recall that the (asymptotic) level α is an upper bound of the type 1 error. As argued in lecture and lecture exercises, the maximum of the type 1 error is achieved at the boundary of Θ_0 and Θ_1 for a one-sided tests, where the parameter space is 1-dimensional.

The asymptotic p-value is

$$\begin{aligned} p\text{-value} &= \mathbf{P}\left(Z < T_{n,\lambda_0}\left(\overline{X}_n\right)\right) \quad \text{where} \quad Z \sim \mathcal{N}(0,1) \\ &= \Phi\left(T_{n,\lambda_0}\left(\overline{X}_n\right)\right). \end{aligned}$$

Alternatively, again define the test ψ and the p-value using

$$T_{n,\lambda_0}(\overline{X}_n) = \sqrt{n}\left(\frac{\overline{X}_n - \lambda_0}{\sqrt{\overline{X}_n}}\right).$$

(c)

1 point possible (graded)

$X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Exp (λ) for some unknown $\lambda > 0$;

$$H_0 : \lambda = \lambda_0 \quad \text{v.s.} \quad H_1 : \lambda \neq \lambda_0 \quad \text{where} \;\; \lambda_0 > 0.$$

(Type **barX_n** for $\overline{X}_n$, **lambda_0** for λ_0. If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \le x)$ where $Z \sim \mathcal{N}(0,1)$, and **q(alpha)** for q_α, the $1-\alpha$ quantile of a standard normal variable.)

Asymptotic p-value =

Answer: 2*(1-Phi(sqrt(n)*abs(1/barX_n-lambda_0)/lambda_0))

Solution:

Since $X_i \sim \text{Exp}(\lambda)$, $\mathbb{E}[X_i] = \sigma = \frac{1}{\lambda}$. Hence, assuming $H_0 : \lambda = \lambda_0$, the central limit theorem and the delta method gives:

$$
\begin{aligned}
T_{n,\lambda_0}\left(\overline{X}_n\right) &= \sqrt{n}\left(\frac{g\left(\overline{X}_n\right) - g\left(1/\lambda_0\right)}{\left|g'\left(1/\lambda_0\right)\right|\left(1/\lambda_0\right)}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1) \qquad \text{where } g(x) := 1/x. \\
\iff \quad & \sqrt{n}\left(\frac{1/\overline{X}_n - \lambda_0}{\lambda_0}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1) \qquad \text{since } g'(1/\lambda) = -\lambda^2.
\end{aligned}
$$

As in Part (a), a test ψ with asymptotic level α is therefore

$$
\psi_{n,\lambda_0,\alpha} = \mathbf{1}\left(\left|T_{n,\lambda_0}\left(\overline{X}_n\right)\right| > q_{\alpha/2}\right).
$$

with asymptotic p-value:

$$
\begin{aligned}
p\text{-value} &= \mathbf{P}\left(|Z| > \left|T_{n,\lambda_0}\left(\overline{X}_n\right)\right|\right) \qquad \text{where } Z \sim \mathcal{N}(0,1) \\
&= 2\left(1 - \Phi\left(\left|T_{n,\lambda_0}\left(\overline{X}_n\right)\right|\right)\right).
\end{aligned}
$$

Alternatively, define the test ψ and the p-value using

$$
T_{n,\lambda_0}\left(\overline{X}_n\right) = \sqrt{n}\left(\frac{1/\overline{X}_n - \lambda_0}{1/\overline{X}_n}\right).
$$

where we plug-in the estimator $1/\overline{X}_n$ for λ_0.

(a)

3 points possible (graded)

The National Assessment of Educational Progress tested a simple random sample of 1000 thirteen year old students in both 2004 and 2008 and recorded each student's score. The average and standard deviation in 2004 were 257 and 39, respectively. In 2008, the average and standard deviation were 260 and 38, respectively.

Your goal as a statistician is to assess whether or not there were statistically significant changes in the average test scores of students from 2004 to 2008. To do so, you make the following modeling assumptions regarding the test scores:

- $X_1, \ldots, X_{1000}$ represent the scores in 2004.
- $X_1, \ldots, X_{1000}$ are iid Gaussians with standard deviation 39.
- $\mathbb{E}[X_1] = \mu_1$, which is an unknown parameter.
- $Y_1, \ldots, Y_{1000}$ represent the scores in 2008.
- $Y_1, \ldots, Y_{1000}$ are iid Gaussians with standard deviation 38.
- $\mathbb{E}[Y_1] = \mu_2$, which is an unknown parameter.
- $X_1, \ldots, X_n$ are independent of $Y_1, \ldots, Y_n$.

You define your hypothesis test in terms of the null $H_0 : \mu_1 = \mu_2$ (signifying that there were not significant changes in test scores) and $H_1 : \mu_1 \neq \mu_2$. You design the test

$$
\psi = \mathbf{1}\left(\sqrt{n}\left|\frac{\overline{X}_n - \overline{Y}_n}{\sqrt{38^2 + 39^2}}\right| \geq q_{\eta/2}\right).
$$

where q_η represents the $1 - \eta$ quantile of a standard Gaussian.

Hint: Under $H_0 : \mu_1 = \mu_2$, the test statistic is distributed as a standard Gaussian:

$$\sqrt{n}\frac{\overline{X}_n - \overline{Y}_n}{\sqrt{38^2 + 39^2}} \sim N(0, 1)$$

You are encouraged to check this.(Compute the mean and variance and recall that the sum of iid Gaussians is again Gaussian.)

What is the largest possible value of η so that ψ has level 10%?

$\eta =$ 0.05 **Answer:** 0.1

If ψ is designed to have level 10%, would you **reject** or **fail to reject** the null hypothesis given these data?

- ○ Reject ✔
- ◉ Fail to reject

What is the p-value for this data set?

0.01 **Answer:** 0.0815

Solution:

Recall the definition of quantiles a standard Gaussian: q_η is the number such that

$$\eta = P(Z \geq q_\eta)$$

where $Z \sim N(0, 1)$. Observe that under the null hypothesis,

$$\sqrt{n}\left|\frac{\overline{X}_n - \overline{Y}_n}{\sqrt{38^2 + 39^2}}\right| \sim N(0, 1).$$

By symmetry,

$$P(|Z| \geq q_{\eta/2}) = 2P(Z \geq q_{\eta/2}).$$

Thus our goal is to choose the smallest η such that $P(|Z| \geq q_{\eta/2}) \leq 0.1\%$. We get

$$2P(Z \geq q_{\eta/2}) = 0.1 \Rightarrow \eta = 0.1.$$

To determine if we should reject or accept the null based on the 2008 data, we need to compute $q_{0.1/2} = q_{0.05}$. Using computational software or a table, we find that $q_{0.05} \approx 1.64$. Now we evaluate our test statistic on the 2008 data:

$$\sqrt{n}\left|\frac{\overline{X}_n - \overline{Y}_n}{\sqrt{38^2 + 39^2}}\right| = \sqrt{1000}\left|\frac{260 - 257}{\sqrt{38^2 + 39^2}}\right| \approx 1.7422$$

Hence, $\psi = 1$, and we would **reject** the hypothesis that there were no changes in test scores between 2004 and 2008.

To compute the p-value for this data set, we let $Z \sim N(0, 1)$ and compute using a table

$$P(|Z| > 1.7422) = 2P(Z > 1.7422) \approx 0.0815$$

7. Quiz: Composite Hypotheses for Bernoulli models

Bookmark this page

(a)

1 point possible (graded)

Let $X_1, \ldots, X_n$ be i.i.d. **Bernoulli** random variables with unknown parameter $p \in (0,1)$.

Find a function $T_{n,p}(\overline{X}_n)$, which depends on $\overline{X}_n, n,$ and p, such that

$$T_{n,p}\left(\overline{X}_n\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1),$$

by

- using the Central Limit Theorem on $\overline{X}_n$ and
- **substituting any occurrence of p in the variance by a plug-in estimator for p.**

Note: If $T_{n,p} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1)$, then so does $-T_{n,p}$. For this problem and the next part, use the expression for $T_{n,p}\left(\overline{X}_n\right)$ that is of the form $\left(\overline{X}_n - p\right) f\left(n, \overline{X}_n\right)$ where $f\left(n, \overline{X}_n\right)$ is always **positive**. (Or very loosely speaking, use $\left(\overline{X}_n - p\right)$ and not $\left(p - \overline{X}_n\right)$ where applicable.)

(Enter **barX_n** for $\overline{X}_n$).

$T_{n,p}(\overline{X}_n) =$ **Answer:** sqrt(n) * (barX_n - p)/sqrt(barX_n*(1-barX_n))

Solution:

By the Central Limit Theorem and plugging in the variance of a Bernoulli random variable,

$$\frac{\sqrt{n}}{\sqrt{p\,(1-p)}}(\overline{X}_n - p) \xrightarrow[n\to\infty]{(D)} \mathcal{N}\,(0,1)\,.$$

By the Law of Large Numbers,

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} p,$$

so by Slutsky's Theorem, we can replace $p\,(1-p)$ by $\overline{X}_n\,(1-\overline{X}_n)$ to obtain

$$\frac{\sqrt{n}}{\sqrt{\overline{X}_n\,(1-\overline{X}_n)}}(\overline{X}_n - p) \xrightarrow[n\to\infty]{(D)} \mathcal{N}\,(0,1)\,.$$

Hence, the function we are looking for is

$$T_{n,p}\,(\overline{X}_n) = \frac{\sqrt{n}}{\sqrt{\overline{X}_n\,(1-\overline{X}_n)}}(\overline{X}_n - p)$$

or

$$T_{n,p}\,(\overline{X}_n) = \frac{\sqrt{n}}{\sqrt{\overline{X}_n\,(1-\overline{X}_n)}}(p - \overline{X}_n)$$

(b)

3 points possible (graded)

(This is a quiz, hence only 1 attempt.)

Select a test with asymptotic level α , in terms of the function $T_{n,p}\left(\overline{X}_n\right)$, for each of the following pairs of hypotheses:

(Choose one for each column. Note the absolute values in the first 2 rows.)

$H_0 : p = 0.5 \quad \text{vs} \quad H_1 : p \neq 0.5 \quad :$	$H_0 : p \leq 0.5 \quad \text{vs} \quad H_1 : p > 0.5 \quad :$	$H_0 : p \geq 0.5 \quad \text{vs} \quad H_1 : p < 0.5 \quad :$
$\mathbf{1}\left(\left\lvert T_{n,0.5}\left(\overline{X}_n\right)\right\rvert > q_{\alpha/2}\right)$	$\mathbf{1}\left(\left\lvert T_{n,0.5}\left(\overline{X}_n\right)\right\rvert > q_{\alpha/2}\right)$	$\mathbf{1}\left(\left\lvert T_{n,0.5}\left(\overline{X}_n\right)\right\rvert > q_{\alpha/2}\right)$
$\mathbf{1}\left(\left\lvert T_{n,0.5}\left(\overline{X}_n\right)\right\rvert > q_{\alpha}\right)$	$\mathbf{1}\left(\left\lvert T_{n,0.5}\left(\overline{X}_n\right)\right\rvert > q_{\alpha}\right)$	$\mathbf{1}\left(\left\lvert T_{n,0.5}\left(\overline{X}_n\right)\right\rvert > q_{\alpha}\right)$
$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) > q_{\alpha/2}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) > q_{\alpha/2}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) > q_{\alpha/2}\right)$
$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) > q_{\alpha}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) > q_{\alpha}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) > q_{\alpha}\right)$
$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < -q_{\alpha/2}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < -q_{\alpha/2}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < -q_{\alpha/2}\right)$
$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < -q_{\alpha}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < -q_{\alpha}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < -q_{\alpha}\right)$
$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < q_{\alpha/2}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < q_{\alpha/2}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < q_{\alpha/2}\right)$
$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < q_{\alpha}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < q_{\alpha}\right)$	$\mathbf{1}\left(T_{n,0.5}\left(\overline{X}_n\right) < q_{\alpha}\right)$

Solution:

1. By part (a), with

$$T\left(X_1, \ldots, X_n\right) = \frac{\sqrt{n}}{\sqrt{\overline{X}_n\left(1 - \overline{X}_n\right)}}\left(\overline{X}_n - 0.5\right),$$

we know that

$$\mathbf{P}_{0.5}\left(|T| - q > 0\right) \xrightarrow[n \to \infty]{} 2\left(1 - \Phi\left(q\right)\right),$$

so to achieve asymptotic level $\alpha = 0.05$, set

$$q = q_{\alpha/2} \approx 1.96,$$

which means

$$\psi = \mathbf{1}\left\{\left|\frac{\sqrt{n}}{\sqrt{\overline{X}_n\left(1 - \overline{X}_n\right)}}\left(\overline{X}_n - 0.5\right)\right| - 1.96 > 0\right\}.$$

2. By part (a), with

$$T_{0.5}\left(X_1, \ldots, X_n\right) = \frac{\sqrt{n}}{\sqrt{\overline{X}_n\left(1-\overline{X}_n\right)}}\left(\overline{X}_n - 0.5\right),$$

we know that

$$\mathbf{P}_{0.5}\left(T - q > 0\right) \xrightarrow[n\to\infty]{} 1 - \Phi\left(q\right),$$

so to guarantee asymptotic confidence level $\alpha = 0.05$, we can set

$$q = q_\alpha \approx 1.65.$$

This gives us the required level for $p = 0.5$.

However, for $p < 0.5$, we have that

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}_p} p,$$

which entails that

$$\frac{\sqrt{n}}{\sqrt{\overline{X}_n\left(1-\overline{X}_n\right)}}\left(\overline{X}_n - 0.5\right) \xrightarrow[n\to\infty]{\mathbf{P}_p} -\infty,$$

hence in the limit, for $p < 0.5$,

$$\mathbf{P}_p\left(T - q > 0\right) \to 0.$$

Overall, we get the desired test by setting

$$\psi = \mathbf{1}\left\{\frac{\sqrt{n}}{\sqrt{\overline{X}_n\left(1-\overline{X}_n\right)}}\left(\overline{X}_n - 0.5\right) - 1.65 > 0\right\}.$$

3. This is exactly analogous to the part above.

8. A Union-Intersection Test

Bookmark this page

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with unknown parameter $p \in (0,1)$. Suppose we want to test

$$H_0 : p \in [0.48, 0.51] \quad \text{vs} \quad H_1 : p \notin [0.48, 0.51]$$

We want to construct an asymptotic test ψ for these hypotheses using $\overline{X}_n$. For this problem, we specifically consider the family of tests ψ_{c_1,c_2} where we reject the null hypothesis if either $\overline{X}_n < c_1 \leq 0.48$ or $\overline{X}_n > c_2 \geq 0.51$ for some c_1 and c_2 that may depend on n, i.e.

$$\psi_{c_1,c_2} = \mathbf{1}\left((\overline{X}_n < c_1) \cup (\overline{X}_n > c_2)\right) \qquad \text{where } c_1 < 0.48 < 0.51 < c_2.$$

Throughout this problem, we will discuss possible choices for constants c_1 and c_2, and their impact to both the asymptotic and non-asymptotic level of the test.

(a)

1 point possible (graded)

Which expression represents the (smallest asymptotic) level α of this test? Recall the (smallest asymptotic) level equals the maximum Type 1 error rate.

- (selected, correct) $\alpha = \max_{p \in [0.48, 0.51]} \left(\mathbf{P}_p(\overline{X}_n < c_1) + \mathbf{P}_p(\overline{X}_n > c_2)\right)$ ✔
- $\alpha = \max_{p \in [0.48, 0.51]} \left(\max\left(\mathbf{P}_p(\overline{X}_n < c_1), \mathbf{P}_p(\overline{X}_n > c_2)\right)\right)$
- $\alpha = \max_{p \in [0.48, 0.51]} \mathbf{P}_p(\overline{X}_n < c_1)$
- $\alpha = \max_{p \in [0.48, 0.51]} \mathbf{P}_p(\overline{X}_n > c_2)$
- $\alpha = \max_{p \in [0.48, 0.51]} \left(\mathbf{P}_p(\overline{X}_n < c_1) \cdot \mathbf{P}_p(\overline{X}_n > c_2)\right)$

Solution:

A Type I error occurs when $\psi = 1$ but H_0 is true; hence the type 1 error rate is

$$\alpha_\psi(p) = \mathbf{P}_p\left((\overline{X}_n < c_1) \cup (\overline{X}_n > c_2)\right)$$

Since $c_1 < 0.48 < 0.51 < c_2$, we have

$$\mathbf{P}_p\left((\overline{X}_n < c_1) \cup (\overline{X}_n > c_2)\right) = \mathbf{P}_p\left(\overline{X}_n < c_1\right) + \mathbf{P}_p\left(\overline{X}_n > c_2)\right).$$

Maximizing over this over $p \in [0.48, 0.51]$, we get that the maximum Type 1 error rate of this test, i.e. the smallest le

$$\boxed{\alpha = \max_{p \in [0.48, 0.51]} \left(\mathbf{P}_p(\overline{X}_n < c_1) + \mathbf{P}_p(\overline{X}_n > c_2)\right)}$$

(b)

4 points possible (graded)

Use the central limit theorem and the approximation $\sqrt{p(1-p)} \approx \frac{1}{2}$ for $p \in [0.48, 0.51]$ to approximate $\mathbf{P}_p(\overline{X}_n < c_1)$ and $\mathbf{P}_p(\overline{X}_n > c_2)$ for large n. Express your answers as a formula in terms of c_1, c_2, n and p.

(Write **Phi** for the cdf of a Normal distribution, **c_1** for c_1, and **c_2** for c_2.)

$\mathbf{P}_p\left(\overline{X}_n < c_1\right) \approx$ **Answer:** Phi(2*(c_1 - p)*sqrt(n))

For what value of $p \in [0.48, 0.51]$ is the expression above for $\mathbf{P}_p\left(\overline{X}_n < c_1\right)$ maximized?

$\mathbf{P}_p\left(\overline{X}_n < c_1\right)$ is max at $p =$ **Answer:** 0.48

$\mathbf{P}_p\left(\overline{X}_n > c_2\right) \approx$ **Answer:** 1 - Phi(2*(c_2 - p)*sqrt(n))

For what value of $p \in [0.48, 0.51]$ is the expression above for $\mathbf{P}_p\left(\overline{X}_n > c_2\right)$ maximized?

$\mathbf{P}_p\left(\overline{X}_n > c_2\right)$ is max at $p =$ **Answer:** 0.51

Solution:

Consider a specific $p \in [0.48, 0.51]$. Then,

$$\mathbf{P}_p(\overline{X}_n < c_1) = \mathbf{P}_p\left(\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}\sqrt{n} < \frac{c_1 - p}{\sqrt{p(1-p)}}\sqrt{n}\right).$$

By the Central Limit Theorem and noting that the variance of X_1 is $\sqrt{p(1-p)}$, we see that $\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}\sqrt{n}$ has a standard Gaussian distribution, so

$$\mathbf{P}_p(\overline{X}_n < c_1) = \Phi\left(\frac{c_1 - p}{\sqrt{p(1-p)}}\sqrt{n}\right) \approx \Phi\left(2(c_1 - p)\sqrt{n}\right).$$

As $\Phi(x)$ is an increasing function, $\Phi(2(c_1 - p)\sqrt{n})$ is maximized at the minimum possible p in the range, which is $p = 0.48$. Hence, $\max_{p\in[0.48,0.51]} \mathbf{P}_p(\overline{X}_n < c_1) = \Phi(2(c_1 - 0.48)\sqrt{n})$.

Similarly for a specific $p \in [0.48, 0.51]$,

$$\mathbf{P}_p\left(\overline{X}_n > c_2\right) = \mathbf{P}_p\left(\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}\sqrt{n} > \frac{c_2 - p}{\sqrt{p(1-p)}}\sqrt{n}\right)$$

Applying the Central Limit Theorem as in the previous part and then the approximation $\sqrt{p(1-p)} \approx \frac{1}{2}$ gives

$$\mathbf{P}_p\left(\overline{X}_n > c_2\right) \approx 1 - \Phi\left(2\left(c_2 - p\right)\sqrt{n}\right).$$

As $\Phi(x)$ is an increasing function, $1 - \Phi\left(2\left(c_2 - p\right)\sqrt{n}\right)$ is maximized at the maximum possible p in the range, which is $p = 0.51$. Hence,

$$\max_{p\in[0.48,0.51]} \mathbf{P}_p\left(\overline{X}_n > c_1\right) = 1 - \Phi\left(2\left(c_2 - 0.51\right)\sqrt{n}\right)$$

(c)

1 point possible (graded)

Next, we combine the results from parts (a) and (b).

Apply the inequality $\max_x\left(f(x) + g(x)\right) \leq \max_x f(x) + \max_x g(x)$ to the expression for the (asymptotic) level α obtained in part (a) and use the results from part (b) to give an upper bound on α.

Express your answer as a formula in terms of c_1, c_2, and n.
(Write **Phi** for the cdf of a Normal distribution, **c_1** for c_1, and **c_2** for c_2.)

$\alpha \leq$ **Answer:** 1+Phi(2*(c_1-0.48)*sqrt(n))-Phi(2*(c_2-0.51)*sqrt(n))

(*Food for thought:* Is this upper bound tight? A bound is tight if equality may be achieved.)

Solution:

Recall that the (smallest) asymptotic level α of a test is equal to the maximum Type 1 error rate. Recalling from part (a) the expression for (smallest) asymptotic level α, applying the given inequality $\max_x\left(f(x) + g(x)\right) \leq \max_x f(x) + \max_x g(x)$, and using all the results from part (b), we have

$$\begin{aligned}\max_{p\in[0.48,0.51]}\left(\mathbf{P}_p\left(\overline{X}_n < c_1\right) + \mathbf{P}\left(\overline{X}_n > c_2\right)\right) &\leq \max_{p\in[0.48,0.51]} \mathbf{P}\left(\overline{X}_n < c_1\right) + \max_{p\in[0.48,0.51]} \mathbf{P}\left(\overline{X}_n > c_2\right)\\ &\approx \boxed{\Phi\left(2\left(c_1 - 0.48\right)\sqrt{n}\right) + 1 - \Phi\left(2\left(c_2 - 0.51\right)\sqrt{n}\right)}\end{aligned}$$

(This bound is not tight because the the maxima for the two summands are not obtained at the same p.)

(d)

2 points possible (graded)

Suppose that we wish to have a level $\alpha = 0.05$. What c_1 and c_2 will achieve $\alpha = 0.05$? Choose c_1 and c_2 by setting the expressions you obtained above for $\max_{p \in [0.48, 0.51]} \mathbf{P}_p\left(\overline{X}_n < c_1\right)$ and $\max_{p \in [0.48, 0.51]} \mathbf{P}_p\left(\overline{X}_n > c_2\right)$ to both be 0.025.

(If applicable, enter **q(alpha)** for q_α, the $1 - \alpha$ -quantile of a standard normal distribution, e.g. enter **q(0.01)** for $q_{0.01}$.)

$c_1 =$ **Answer:** -q(0.025)/(2* sqrt(n)) + 0.48

$c_2 =$ **Answer:** q(0.025)/(2* sqrt(n)) + 0.51

Solution:

To have a test of level 0.05 and equal left and right tails, according to the description above, we want to set

$$\Phi\left(2 * (c_1 - 0.48) * \sqrt{n}\right) = 0.025$$

and

$$\Phi\left(2 * (c_2 - 0.51) * \sqrt{n}\right) = 0.975.$$

Taking the inverse Phi function to both equations gives

$$2 * (c_1 - 0.48) * \sqrt{n} = -1.96$$

and

$$2 * (c_2 - 0.51) * \sqrt{n} = 1.96,$$

respectively. Solving the two equations (independently) gives

$$c_1 = \boxed{-\frac{0.98}{\sqrt{n}} + 0.48}$$

$$c_2 = \boxed{\frac{0.98}{\sqrt{n}} + 0.51}$$