Unit 3 Counting

The counting principle
How many different combinations are there of 4 shirts, 3 ties and 2 jackets?

4 x 3 x 2 = 24
r = number of stages, 3
n[i] = number of options at each stage, 4,3,2

Examples
Permutations:
number of ways of ordering n elements

First element has n choices
Second element has n-1 choices
Third …. n-2 choices
Final …. 1 choice

n.(n-1)(n-2)….1 = n! factorial

Number of subsets of {1,….,n}:
Each stage has 2 choices, put element in subset or not
2 , 2 , 2 , 2… etc up to n times
= 2^n

When n=1 {1}, 2^1 = 2   which is subset of {1} and {} empty

Die roll example
P(six rolls of a 6sided die all give different numbers):

P(A) = # in A / # possible outcomes

possible outcomes = 6^6
how many times we can order 1-6 = 6!

P(A) = 6!/ 6^6

Exercise:
You are given the set of letters
{A, B, C, D, E}
. What is the probability that in a random five-letter string (in which each letter appears exactly once, and with all such strings equally likely) the letters A and B are next to each other? The answer to a previous exercise may also be useful here. (In this and subsequent questions, your answer should be a number. Do

not enter '!' or combinations in your answer.)

# of total combinations of 5 letters = 5!
# of ways A and B are next to each other = 48

Combinations

Definition "n choose k"

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

number of k element subsets
of a given n element set

Two ways of constructing an **ordered** sequence of k **distinct** items:
Method 1
choose each element in turn up to k
n(n-1)(n-2)...(n-k+1) = n! / (n-k)!

Method 2
First choose k items out of n (n choose k)
Then how many ways of ordering k elements (k!)
(n choose k).k!

Therefore
n! / (n-k)! = (n choose k).k!
Giving above definition

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = \#\text{ all subsets } = 2^n$$

Exercise: Counting committees

0.0/2.0 points (graded)

We start with a pool of n people. A chaired committee consists of $k \geq 1$ members, out of whom one member is designated as the chairperson. The expression $k\binom{n}{k}$ can be interpreted as the number of possible chaired committees with k members. This is because we have $\binom{n}{k}$ choices for the k members, and once the members are chosen, there are then k choices for the chairperson. Thus,

$$c = \sum_{k=1}^{n} k \binom{n}{k}$$

is the total number of possible chaired committees of any size.

Find the value of c (as a function of n) by thinking about a different way of forming a chaired committee: first choose the chairperson, then choose the other members of the committee. The answer is of the form

$$c = (\alpha + n^{\beta})\, 2^{\gamma n + \delta}.$$

What are the values of α, β, γ, and δ?

What are the values of α, β, γ, and δ?

$\alpha =$ 0

$\beta =$ 1

$\gamma =$ 1

$\delta =$ -1

Binomial Probabilities

Binomial coefficient $\underline{\binom{n}{k}}$ $\longrightarrow$ Binomial probabilities

- $n \geq 1$ independent coin tosses; $\mathbf{P}(H) = p$

$$\boxed{\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}}$$

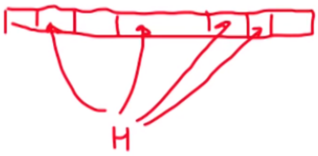
$n=6$

- $\mathbf{P}(HTTHHH) = p(1-p)(1-p)ppp = p^4(1-p)^2$

- $\mathbf{P}(\text{particular sequence}) = p^{\#\text{heads}}(1-p)^{\#\text{tails}}$

- $\mathbf{P}(\text{particular } k\text{-head sequence}) = p^k(1-p)^{n-k}$

$$\mathbf{P}(k \text{ heads}) = p^k(1-p)^{n-k} \cdot (\#\ k\text{-head sequences})$$



$$\binom{n}{k}$$

Exercise: Binomial probabilities

2/2 points (graded)

Recall that the probability of obtaining k Heads in n independent coin tosses is $\binom{n}{k}p^k(1-p)^{n-k}$, where p is the probability of Heads for any given coin toss.

Find the value of $\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}$. (Your answer should be a number.)

1

✔ **Answer:** 1

Solution:

Note that the events “0 Heads", “1 Heads", ..., “n Heads" are disjoint, and their union is the entire sample space. The summation is adding up the probability of all of these events. Hence, the sum must be 1. In other words, each term in the summation gives the probability of obtaining k Heads out of n tosses. We then sum over all values of k, from 0 to n. Since the number of Heads must be one of $0, 1, \ldots, n$, these probabilities must sum up to 1.

A coin tossing problem

Assumptions:
- independence
- $\mathbf{P}(H) = p$

$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

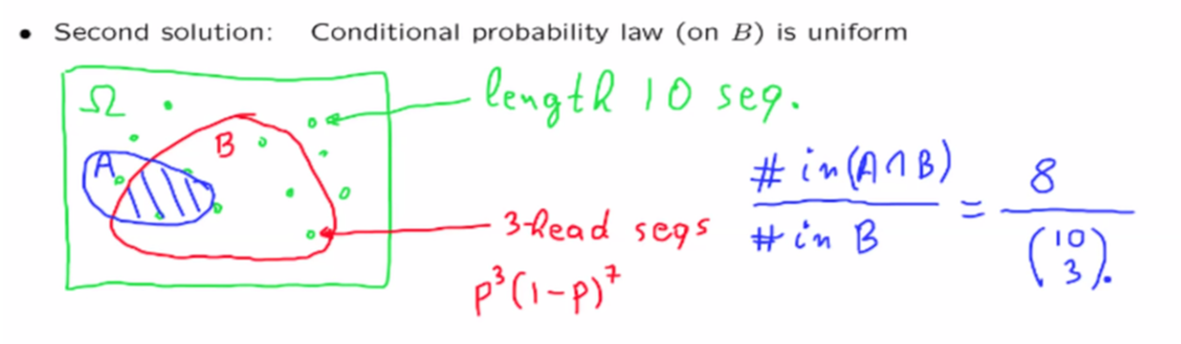
- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

- First solution:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{P(H_1 H_2 \text{ and one } H \text{ in tosses } 3, \ldots, 10)}{P(B)}$$

$$= \frac{p^2 \cdot \binom{8}{1} p^1 \cdot (1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{\binom{8}{1}}{\binom{10}{3}} = \frac{8}{\binom{10}{3}}$$

(8 comes from remaining tosses after toss 1 and 2)



Can apply this easier second solution because the condition probability law on B is uniform

Exercise: Coin tossing

2/2 points (graded)

Use the second method in the preceding segment to find the probability that the 6th toss out of a total of 10 tosses is Heads, given that there are exactly 2 Heads out of the 10 tosses. As in the preceding segment, continue to assume that all coin tosses are independent and that each coin toss has the same fixed probability of Heads. (In this and subsequent questions, your answer should be a number. Do not enter '!' or combinations in your answer.)

1/5 ✔

Submit You have used 1 of 3 attempts Save Show Answer

i.e. 9 / (10 choose 2) = 9/45 = 1/5

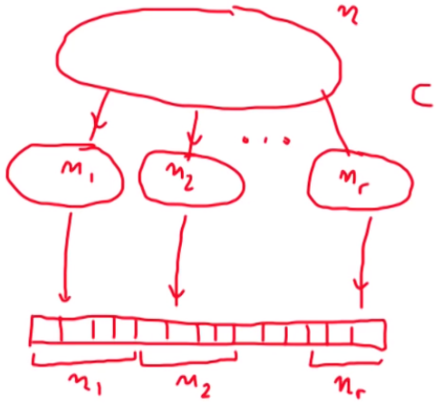
9 because toss 6 has to be a heads along with 1 of the 9 other options so there are 9 options
(10 choose 2) because 2 out of the 10 tosses are heads

Partitions

Partitions

- $n \geq 1$ distinct items; $r \geq 1$ persons give n_i items to person i
  - here $n_1, \ldots, n_r$ are given nonnegative integers
  - with $n_1 + \cdots + n_r = n$
- Ordering n items: $n!$
  - Deal n_i to each person i, and then order

$$c \, n_1! \, n_2! \cdots n_r! = n!$$



$r = 2 \quad n_1 = k \quad n_2 = n - k$

$$\text{number of partitions} = \frac{n!}{n_1!\, n_2! \cdots n_r!} \qquad \text{(multinomial coefficient)}$$

Exercise: Counting partitions

3/3 points (graded)

We have 9 distinct items and three persons. Alice is to get 2 items, Bob is to get 3 items, and Charlie is to get 4 items.

1. As just discussed, this can be done in $\frac{a!}{b!\,3!\,4!}$ ways. Find a and b.

   $a =$ 9 ✔

   $b =$ 2 ✔

2. A different way of generating the desired partition is as follows. We first choose 2 items to give to Alice. This can be done in $\binom{c}{d}$ different ways. Find c and d. (There are 2 possible values of d that are correct. Enter the smaller value.)

   $c =$ 9 ✔

   $d =$ 2 ✔

3. Having given 2 items to the Alice, we now give 3 items to Bob. This can be done in $\binom{e}{f}$ ways. Find e and f. (There are 2 possible values of f that are correct. Enter the smaller value.)

   $e =$ 7 ✔

   $f =$ 3 ✔

Each person gets an ace example problem

Example: 52-card deck, dealt (fairly) to four players.
Find $\mathbf{P}$(each player gets an ace)

- Outcomes are: partition equally likely
  - number of outcomes: $\frac{52!}{13!\,13!\,13!\,13!}$
- Constructing an outcome with one ace for each person:
  - distribute the aces $4 \cdot 3 \cdot 2 \cdot 1$
  - distribute the remaining 48 cards $\frac{48!}{12!\,12!\,12!\,12!}$
- Answer:

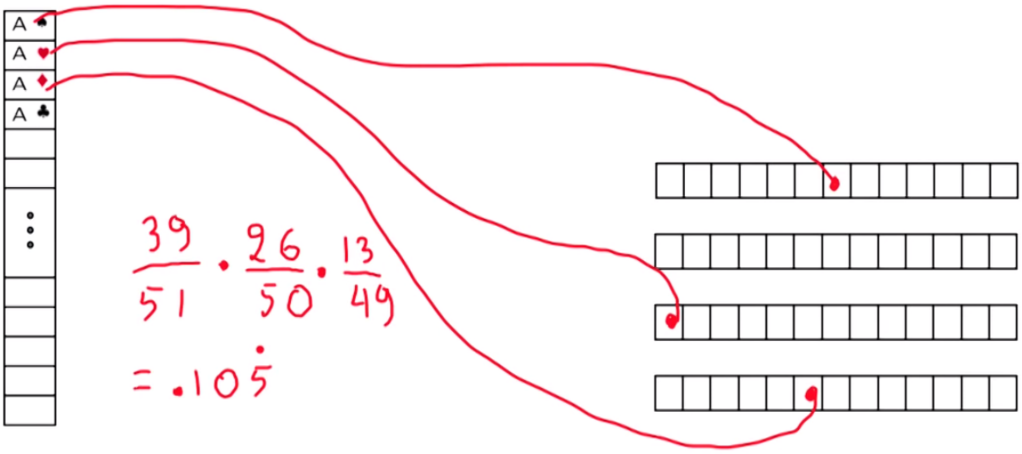
$$\frac{4 \cdot 3 \cdot 2 \cdot \dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$

A smart solution

Distribute 4 aces randomly, what's the chance each ace gets put into a different stack of 13 cards (4 x 13 = 52)
First ace doesn't matter
Second ace 39 cards that are in a different stack out of 51 remaining cards
Third ace 26 cards that are in a different stack out of 50 remaining cards
Fourth ace 13 cards that are in a different stack out of 49 remaining cards

Example: 52-card deck, dealt (fairly) to four players. A smart solution
Find $\mathbf{P}$(each player gets an ace)

Stack the deck, aces on top



$$\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}$$

$= .105$