Homework 2

1. Confidence Intervals for Curved Gaussian Family

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(a)

0/1 point (graded)

Let $X_1, \ldots, X_n$ be i.i.d. random variables with distribution $\mathcal{N}(\theta, \theta)$, for some unknown parameter $\theta > 0$.

True or False: The sample average $\overline{X}_n$ follows a normal distribution for any integer $n \geq 1$.

- True ✔
- False (selected) ✖

Solution:

As a sum of independent normal variables, $\overline{X}_n$ again follows a normal distribution. This is a special property of normal variables.

(b)

2/2 points (graded)

What is the expectation and the variance of $\overline{X}_n$?

$\mathbb{E}[\overline{X}_n] =$ theta ✔ **Answer:** theta

θ

$\mathsf{Var}(\overline{X}_n) =$ theta/n ✔ **Answer:** theta/n

$\frac{\theta}{n}$

STANDARD NOTATION

Solution:

As a sum of independent normal variables, $\overline{X}_n$ again follows a normal distribution, that is in turn completely characterized by its expectation and variance,

$$\mathbb{E}[\overline{X}_n] = \theta, \quad \mathsf{Var}(\overline{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} \mathsf{Var}(X_i) = \frac{\theta}{n}.$$

Hence,

$$\sqrt{\frac{n}{\theta}}(\overline{X}_n - \theta) \sim \mathcal{N}(0, 1).$$

(c)

2/2 points (graded)

Find an interval $\mathcal{I}_\theta$ (that depends on θ) centered about $\overline{X}_n$ such that

$$\mathbf{P}\left(\mathcal{I}_\theta \ni \theta\right) = 0.9 \qquad \text{for all } n \text{(i.e, not only for large } n\text{)}.$$

(Write `barX_n` for $\overline{X}_n$. Use the estimate $q_{0.05} \approx 1.6448$ for best results.)

$\mathcal{I}_\theta = [A_\theta, B_\theta]$ for

$A_\theta =$ barX_n-(1.6448*sqrt(theta/n)) ✔ **Answer:** barX_n - 1.6448 * sqrt(theta)/sqrt(n)

$$\overline{X}_n - \left(1.6448 \cdot \sqrt{\frac{\theta}{n}}\right)$$

$B_\theta =$ barX_n+(1.6448*sqrt(theta/n)) ✔ **Answer:** barX_n + 1.6448 * sqrt(theta)/sqrt(n)

$$\overline{X}_n + \left(1.6448 \cdot \sqrt{\frac{\theta}{n}}\right)$$

Solution:

By parts (a) and (b),

$$\sqrt{\frac{n}{\theta}}\left(\overline{X}_n - \theta\right) \sim \mathcal{N}(0,1),$$

so together with looking up the quantile value for a symmetric 90% confidence interval for a Gaussian random variable $Z \sim \mathcal{N}(0,1)$,

$$\mathbf{P}\left(|Z| \le 1.6448\right) \approx 0.9,$$

we obtain

$$\mathbf{P}\left(\left|\sqrt{\frac{n}{\theta}}\left(\overline{X}_n - \theta\right)\right| \le 1.6448\right) = 0.9,$$

and hence can set

$$\mathcal{I}_1 = \left[\overline{X}_n - \frac{1.6448\sqrt{\theta}}{\sqrt{n}}, \overline{X}_n + \frac{1.6448\sqrt{\theta}}{\sqrt{n}}\right].$$

(d)

2/2 points (graded)

Again, use the estimate $q_{0.05} \approx 1.6448$ for best results.

Now, find a confidence interval $\mathcal{I}_{\text{plug-in}}$ with **asymptotic** confidence level 90% by plugging in $\overline{X}_n$ for all occurrences of θ in $\mathcal{I}_\theta$.

$\mathcal{I}_{\text{plug-in}} = [A_{\text{plug-in}}, B_{\text{plug-in}}]$ for

$A_{\text{plug-in}} =$ barX_n-(1.6448*sqrt(barX_n/n)) ✔ **Answer:** barX_n - 1.6448 * sqrt(barX_n)/sqrt(n)

$$\overline{X}_n - \left(1.6448 \cdot \sqrt{\frac{\overline{X}_n}{n}}\right)$$

$B_{\text{plug-in}} =$ barX_n+(1.6448*sqrt(barX_n/n)) ✔ **Answer:** barX_n + 1.6448 * sqrt(barX_n)/sqrt(n)

$$\overline{X}_n + \left(1.6448 \cdot \sqrt{\frac{\overline{X}_n}{n}}\right)$$

Solution:

By the Law of Large number, we have

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \theta.$$

Then CLT and Slutsky together imply that

$$\sqrt{\frac{n}{\bar{X}_n}}\left(\overline{X}_n - \theta\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1)\,.$$

Hence, we obtain an asymptotic confidence interval with level 90% by replacing θ by $\overline{X}_n$ in the expression for $\mathcal{I}_1$ and we can set

$$\mathcal{I}_2 = \left[\overline{X}_n - \frac{1.6448\sqrt{\overline{X}_n}}{\sqrt{n}}, \overline{X}_n + \frac{1.6448\sqrt{\overline{X}_n}}{\sqrt{n}}\right].$$

(e)

0/2 points (graded)

Finally, find a confidence interval $\mathcal{I}_{\text{solve}}$ for θ with **nonasymptotic** level 90% solving the bounds in $\mathcal{I}_\theta$ for θ .

$\mathcal{I}_{\text{solve}} = [A_{\text{solve}}, B_{\text{solve}}]$ for

$A_{\text{solve}} =$ barX_n^2 / (2*barX_n - 1) ✖

Answer: barX_n + 1.6448^2/(2*n) - 1/2*sqrt(1.6448^4/n^2 + 4*1.6448^2*barX_n/n)

$$\frac{\overline{X}_n^2}{2 \cdot \overline{X}_n - 1}$$

$B_{\text{solve}} =$ barX_n^2 / (2*barX_n + 1) ✖

Answer: barX_n + 1.6448^2/(2*n) + 1/2*sqrt(1.6448^4/n^2 + 4*1.6448^2*barX_n/n)

$$\frac{\overline{X}_n^2}{2 \cdot \overline{X}_n + 1}$$

Solution:

From part (c), we have

$$\mathbf{P}\left(\left|\sqrt{\frac{n}{\theta}}\left(\overline{X}_n-\theta\right)\right|\leq 1.65\right)=90\%.$$

With $t=1.6448$, the constraint on θ is equivalent to

$$\begin{aligned}
& \left|\sqrt{\frac{n}{\theta}}\left(\overline{X}_n-\theta\right)\right| && \leq t \\
\iff & \frac{n}{\theta}\left(\overline{X}_n-\theta\right)^2 && \leq t^2 \\
\iff & \theta^2-2\theta\overline{X}_n+\overline{X}_n^2 && \leq \frac{t^2\theta}{n} \\
\iff & \theta^2-\left(2\overline{X}_n+\frac{t^2}{n}\right)\theta+\overline{X}_n^2 && \leq 0 \\
\iff & \theta\in\left[\overline{X}_n+\frac{t^2}{2n}-\sqrt{\Delta},\ \overline{X}_n+\frac{t^2}{2n}+\sqrt{\Delta}\right], && \text{where } \Delta=\frac{t^4}{4n^2}+\frac{t^2\overline{X}_n}{n}
\end{aligned}$$

by the quadratic formula.Substituting $t=1.65$ gives

$$\mathcal{I}_{\text{solve}}=\left[\overline{X}_n+\frac{1.6448^2}{2n}-\sqrt{\frac{1.6448^4}{4n^2}+\frac{1.6448^2\overline{X}_n}{n}},\ \overline{X}_n+\frac{1.6448^2}{2n}+\sqrt{\frac{1.6448^4}{4n^2}+\frac{1.6448^2\overline{X}_n}{n}}\right].$$

2. Delta method and asymptotic variances

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(a) (Optional)

0 points possible (ungraded)

In this problem, you are going to compute the **asymptotic variance** of some estimators. Recall that the asymptotic variance of an estimator $\hat{\theta}$ for a parameter θ is defined as $V(\hat{\theta})$, if

$$\sqrt{n}(\hat{\theta}-\theta)\xrightarrow[n\to\infty]{(D)}\mathcal{N}(0,V(\hat{\theta})).$$

The arguments that we use to establish asymptotic normality are often the same in our setups, namely the Law of Large Numbers, the Central Limit Theorem, and the Delta Method. First, we review the assumptions and statements of those theorems:

Let $X_1,X_2,\ldots,$ be random variables. The (weak) Law of Large Numbers says that under suitable assumptions, with

$$\overline{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i,$$

we have

$$\overline{X}_n\xrightarrow{\mathbf{P}}\mathbb{E}[X_1].$$

What are the assumptions we need for the weak Law of Large Numbers? (Choose all that apply.)

- [x] $\mathbb{E}[|X_i|] < \infty$ for all i
- [x] $\mathsf{Var}(X_i) < \infty$ for all i
- [x] $X_1, X_2, \ldots$ independent
- [ ] There exists $M > 0$ such that $|X_i| \leq M$ for all i
- [ ] $|X_i| \geq |X_{i+1}|$ almost surely for all i
- [x] $X_1, X_2, \ldots$ identically distributed

✖

The Central Limit Theorem states that under some assumptions, there is a V such that

$$\sqrt{n}\,(\overline{X}_n - \mathbb{E}[X_1]) \xrightarrow{(D)} \mathcal{N}(0, V)\,.$$

What are the assumptions we need for the Central Limit Theorem? Pick all that apply.

- [x] $\mathbb{E}[|X_i|] < \infty$ for all i
- [x] $\mathsf{Var}(X_i) < \infty$ for all i
- [x] $X_1, X_2, \ldots$ independent
- [ ] There exists $M > 0$ such that $|X_i| \leq M$ for all i
- [ ] $|X_i| \geq |X_{i+1}|$ almost surely for all i
- [x] $X_1, X_2, \ldots$ identically distributed

✔

The Delta Method gives us a way to control the asymptotic variance of a transformation of a random variable. Let $\theta \in \mathbb{R}$ be a parameter and $Z_n \in \mathbb{R}$ be a sequence of random variables that satisfies

$$\sqrt{n}\,(Z_n - \theta) \xrightarrow[n \to \infty]{(\mathrm{D})} \mathcal{N}\,(0, V)$$

for some $V > 0$.

Given a function $g : \Omega \subseteq \mathbb{R} \to \mathbb{R}$,

$$\sqrt{n}\,(g\,(Z_n) - g\,(\theta)) \xrightarrow[n \to \infty]{(\mathrm{D})} \mathcal{N}\,(0, W)\,.$$

for some $W > 0$.

Pick the following assumptions and conditions that apply to the Delta Method as stated in class:

- [ ] g is monotonically increasing
- [x] g is continuously differentiable at θ
- [x] $W = g'(\theta)^2 V$
- [ ] $W = g(\theta)^2 V$
- [ ] $W = |g'\,(\theta)\,| V$

✔

Solution:

For the weak Law of Large Numbers to apply, we need that the X_i are independent and identically distributed (although there exist weaker versions of it). Moreover, the limit expectation needs to actually exist, i.e. $\mathbb{E}\,[|X_i|] < \infty$.

For the Central Limit Theorem, we have the same requirements, and on top of that, we need the variance to be finite, i.e. $\mathsf{Var}\,(X_i) < \infty$.

For the Delta Method, we need that g is continuously differentiable at θ and the correct asymptotic variance is given by $W = g'(\theta)^2 V$.

Note: There is also a multivariate version of the Delta Method, which we will discuss later in this course.

(b)

1/2 points (graded)

Argue that the proposed estimators $\widehat{\lambda}$ and $\widetilde{\lambda}$ below are both consistent and asymptotically normal. Then, give their **asymptotic variances** $V(\widehat{\lambda})$ and $V(\widetilde{\lambda})$, and decide if one of them is always bigger than the other.

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Poiss (λ), for some $\lambda > 0$. Let $\hat{\lambda} = \overline{X}_n$ and $\tilde{\lambda} = -\ln(\overline{Y}_n)$, where $Y_i = \mathbf{1}\{X_i = 0\}, i = 1, \ldots, n$.

$V(\widehat{\lambda}) =$ lambda

λ

✔ **Answer:** lambda

$V(\widetilde{\lambda}) =$ 1/lambda

$\frac{1}{\lambda}$

✖ **Answer:** exp(lambda) - 1

STANDARD NOTATION

Solution:

For $\widehat{\lambda}$, By the Law of Large Numbers,

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[X_1] = \lambda.$$

By the Central Limit Theorem,

$$\sqrt{n}(\overline{X}_n - \lambda) \sim \mathcal{N}(0, \mathsf{Var}(X_1)) = \mathcal{N}(0, \lambda),$$

hence

$$V(\hat{\lambda}) = \lambda.$$

For $\tilde{\lambda}$, first observe that by the Law of Large Numbers,

$$\overline{Y}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[Y_1] = \mathbf{P}(X_1 = 0) = \exp(-\lambda),$$

so with $g(t) = -\log(t)$

$$\tilde{\lambda} = g(\overline{Y}_n) \xrightarrow[n\to\infty]{\mathbf{P}} g(\exp(-\lambda)) = \lambda.$$

The Central Limit Theorem yields

$$\sqrt{n}(\overline{Y}_n - \mathbb{E}[Y_1]) \xrightarrow[n\to\infty]{(D)} \mathcal{N}(0, \mathsf{Var}(Y_1)) = \mathcal{N}(0, \exp(-\lambda)(1 - \exp(-\lambda)),$$

where we used the formula $\mathsf{Var}(Z) = p(1-p)$ if $Z \sim \mathsf{Be}(p)$. In order to apply the Delta Method for the above $g(t)$, we compute

$$g'(t) = -\frac{1}{t}, \quad g'(\exp(-\lambda)) = -\exp(\lambda),$$

which results in

$$\sqrt{n}(\tilde{\lambda} - \lambda) \xrightarrow[n\to\infty]{(D)} \mathcal{N}(0, \exp(\lambda) - 1).$$

Moreover, by the series expansion for the exponential,

$$\exp(\lambda) - 1 = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} > \lambda, \quad \text{for all } \lambda > 0,$$

so $V(\hat{\lambda}) < V(\tilde{\lambda})$ for all λ.

(c)

1/3 points (graded)

As above, argue that both proposed estimators $\hat{\lambda}$ and $\tilde{\lambda}$ are consistent and asymptotically normal. Then, give their **asymptotic** variances $V(\hat{\lambda})$ and $V(\tilde{\lambda})$, and decide if one of them is always bigger than the other.

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Exp (λ), for some $\lambda > 0$. Let $\hat{\lambda} = \frac{1}{\overline{X}_n}$ and $\tilde{\lambda} = -\ln(\overline{Y}_n)$, where $Y_i = \mathbf{1}\{X_i > 1\}, i = 1, \ldots, n$.

$V(\hat{\lambda}) =$ lambda^2 ✔ **Answer:** lambda^2

λ^2

$V(\tilde{\lambda}) =$ lambda ✖ **Answer:** exp(lambda) - 1

λ

- (●) $V(\hat{\lambda}) > V(\tilde{\lambda})$ for all λ.
- ( ) $V(\hat{\lambda}) < V(\tilde{\lambda})$ for all λ. ✔
- ( ) $V(\hat{\lambda}) = V(\tilde{\lambda})$ for all λ.
- ( ) There exists λ_1 such that $V(\hat{\lambda}) > V(\tilde{\lambda})$ and λ_2 such that $V(\hat{\lambda}) < V(\tilde{\lambda})$

✖

Solution:

For $\hat{\lambda}$, by the Law of Large Numbers,

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[X_1] = \frac{1}{\lambda}.$$

With $g(t) = 1/t$, we have that

$$\hat{\lambda} \xrightarrow[n\to\infty]{\mathbf{P}} \frac{1}{\mathbb{E}[X_1]} = \lambda.$$

By the Central Limit Theorem,

$$\sqrt{n}(\overline{X}_n - \frac{1}{\lambda}) \sim \mathcal{N}(0, \mathsf{Var}(X_1)) = \mathcal{N}\left(0, \frac{1}{\lambda^2}\right).$$

The fact that

$$g'(t) = -\frac{1}{t^2}$$

together with the Delta Method then yields

$$V(\hat{\lambda}) = \lambda^2.$$

For $\tilde{\lambda}$, first observe that it is the average of Bernoulli variables, and by the Law of Large Numbers,

$$\overline{Y}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[Y_1] = \mathbf{P}(X_1 > 1) = \exp(-\lambda),$$

so with $\tilde{g}(t) = -\log(t)$

$$\tilde{\lambda} = \tilde{g}(\overline{Y}_n) \xrightarrow[n\to\infty]{\mathbf{P}} g(\exp(-\lambda)) = \lambda.$$

The Central Limit Theorem yields

$$\sqrt{n}(\overline{Y}_n - \mathbb{E}[Y_1]) \xrightarrow[n\to\infty]{(D)} \mathcal{N}(0, \mathsf{Var}(Y_1)) = \mathcal{N}(0, \exp(-\lambda)(1 - \exp(-\lambda)).$$

In order to apply the Delta Method for the above $\tilde{g}(t)$, we compute

$$\tilde{g}'(t) = -\frac{1}{t}, \quad \tilde{g}'(\exp(-\lambda)) = -\exp(\lambda),$$

which results in

$$\sqrt{n}(\tilde{\lambda} - \lambda) \xrightarrow[n\to\infty]{(D)} \mathcal{N}(0, \exp(\lambda) - 1).$$

In order to compare these two asymptotic variances, first observe that similar to part (b),

$$\exp(\lambda) - 1 \geq \lambda, \quad \text{for all } \lambda > 0,$$

and since $\lambda^2 < \lambda$ for $\lambda \in (0, 1)$, we have

$$\exp(\lambda) - 1 \geq \lambda^2, \quad \text{for } \lambda \in (0, 1).$$

Moreover,

$$\exp(1) - 1 = e - 1 > 1 = 1^2,$$

and

$$\frac{d}{d\lambda}(\exp(\lambda) - 1) = \exp(\lambda), \quad \frac{d}{d\lambda}\lambda^2 = 2\lambda,$$

so that

$$\frac{d}{d\lambda}(\exp(\lambda) - 1) = \exp(\lambda) \geq 1 + \lambda + \frac{\lambda^2}{2} > 2\lambda = \frac{d}{d\lambda}\lambda^2, \quad \text{for all } \lambda > 0,$$

which can be checked by the quadratic formula. This means that for $\lambda \geq 1$,

$$\exp(\lambda) - 1 = e + \int_1^\lambda \exp(t)\ dt - 1 > 1 + \int_1^\lambda 2t\, dt = \lambda^2.$$

Hence, the asymptotic variance of $\hat{\lambda}$ is always lower than that of $\tilde{\lambda}$.

(e)

1/3 points (graded)

As above, argue that both proposed estimators $\widehat{p}, \widetilde{p},$ are consistent and asymptotically normal. Then, give their **asymptotic** variances $V(\widehat{p})$ and $V(\widetilde{p})$ and decide if one of them is always bigger than the other.

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Geom(p), for some $p \in (0,1)$. That means that

$$\mathbf{P}(X_1 = k) = p(1-p)^{k-1}, \quad \text{for } k = 1, 2, \ldots.$$

Let

$$\widehat{p} = \frac{1}{\overline{X}_n},$$

and $\widetilde{p}$ be the **number of ones in the sample divided by** n.

$V(\widehat{p}) =$ p/(1-p)^2

$\frac{p}{(1-p)^2}$

✖ **Answer:** p^2*(1-p)

$V(\widetilde{p}) =$ (1-p)^2/p

$\frac{(1-p)^2}{p}$

✖ **Answer:** p*(1-p)

- ○ $V(\widehat{p}) > V(\widetilde{p})$ for all p.
- ◉ $V(\widehat{p}) < V(\widetilde{p})$ for all p.
- ○ $V(\widehat{p}) = V(\widetilde{p})$ for all p.

Solution:

By the Law of Large Numbers,

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[X_1] = \frac{1}{p}.$$

Setting

$$g(t) = \frac{1}{t},$$

we obtain consistency of $\widehat{p} = g(\overline{X}_n)$, i.e.,

$$\widehat{p} = g(\overline{X}_n) \xrightarrow[n\to\infty]{\mathbf{P}} g(\mathbb{E}[X_1]) = p.$$

By the Central Limit Theorem,

$$\sqrt{n}(\overline{X}_n - \frac{1}{p}) \xrightarrow[n\to\infty]{(\mathrm{D})} \mathcal{N}(0, \mathsf{Var}(X_1)) = \mathcal{N}\left(0, \frac{1-p}{p^2}\right),$$

and hence by the Delta Method, together with

$$g'\left(\frac{1}{p}\right)^2 = \left(-p^2\right)^2 = p^4,$$

we end up with

$$\sqrt{n}\left(\widehat{p}-p\right)\xrightarrow[n\to\infty]{(\mathrm{D})}\mathcal{N}\left(0,p^2\left(1-p\right)\right),$$

so

$$V\left(\widehat{p}\right)=p^2\left(1-p\right).$$

For $\tilde{p}$, note that we can write it as

$$\tilde{p}=\overline{Y}_n,\qquad \text{where } Y_i=\mathbf{1}\{X_i=1\},$$

so it is again an average over Bernoulli variables. The Law of Large Numbers gives

$$\overline{Y}_n\xrightarrow[n\to\infty]{\mathbf{P}}\mathbb{E}\left[Y_1\right]=\mathbf{P}\left(X_1=1\right)=p,$$

while the Central Limit Theorem yields

$$\sqrt{n}\left(\overline{Y}_n-p\right)\xrightarrow[n\to\infty]{(\mathrm{D})}\mathcal{N}\left(0,\mathsf{Var}\left(Y_1\right)\right)=\mathcal{N}\left(0,p\left(1-p\right)\right).$$

Since $p^2<p$ for $p\in(0,1)$,

$$V\left(\widehat{p}\right)<V\left(\tilde{p}\right).$$

3. Application of Delta Method on Gamma Variables

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The **Gamma distribution** $\mathrm{Gamma}\left(\alpha,\beta\right)$ with paramters $\alpha>0$, and $\beta>0$ is defined by the density

$$f_{\alpha,\beta}\left(x\right)=\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}x^{\alpha-1}e^{-\beta x},\qquad \text{for all} x\geq 0.$$

The Γ function is defined by

$$\Gamma\left(s\right)=\int_0^\infty x^{s-1}e^{-x}dx.$$

As usual, the constant $\frac{\beta^\alpha}{\Gamma(\alpha)}$ is a normalization constant that gives $\int_0^\infty f_{\alpha,\beta}\left(x\right)dx=1$.

In this problem, let $X_1,\ldots,X_n$ be i.i.d. Gamma variables with

$$\beta=\frac{1}{\alpha}\text{for some } \alpha>0.$$

That is, $X_1,\ldots,X_n\sim\mathrm{Gamma}\left(\alpha,\frac{1}{\alpha}\right)$ random variables for some $\alpha>0$. The pdf for X_i is therefore

$$f_\alpha\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)\alpha^\alpha}x^{\alpha-1}e^{-x/\alpha},\qquad \text{for all } x\geq 0.$$

(a)

0/1 point (graded)

What is the limit, in probability, of the sample average $\overline{X}_n$ of the sample in terms of α?

$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}}$ alpha

α

✖ **Answer:** alpha^2

STANDARD NOTATION

Solution:

By the weak law of large numbers

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[X_i].$$

In general, the expectation for a Gamma variable with parameters α, β is $\frac{\alpha}{\beta}$, since

$$\begin{aligned}\int_0^\infty x f_{\alpha,\beta}(x)\,dx &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{x^\alpha e^{-\beta x}}{-\beta} \Big|_0^\infty - \int_0^\infty (\alpha x^{\alpha-1}) \left(\frac{e^{-\beta x}}{-\beta}\right) dx \right) = \frac{\alpha}{\beta}.\end{aligned}$$

Hence, for $X_i \sim \mathrm{Gamma}\left(\alpha, \frac{1}{\alpha}\right)$, we have

$$\mathbb{E}[X_i] = \frac{\alpha}{1/\alpha} = \alpha^2.$$

(b)

1 point possible (graded)

Use the result from the previous problem to give a consistent estimator $\hat{\alpha}$ of α in terms of $\overline{X}_n$.

(Enter `barX_n` for $\overline{X}_n$)

$\hat{\alpha} =$

Answer: sqrt(barX_n)

Solution:

From the previous problem, we know that $\overline{X}_n \xrightarrow[\mathbf{P}]{n\to\infty} \alpha^2$. By the continuous mapping theorem, $\hat{\alpha} = \sqrt{\overline{X}_n} \xrightarrow[\mathbf{P}]{n\to\infty} \sqrt{\alpha^2} = \alpha$ since $\alpha > 0$.

(c)

3 points possible (graded)

For the Delta method to apply, at what value of x does g need to be continuously differentiable? (Your answer should be in terms of α.)

$x =$ alpha **Answer:** alpha^2

α

What distribution does $\sqrt{n}\,(\hat{\alpha} - \alpha)$ converge to as $n \to \infty$?

- Gamma distribution
- Normal distribution ✔
- None of the above

What is its **asymptotic** variance of $\hat{\alpha}$?

$V(\hat{\alpha}) =$ alpha/beta^2 **Answer:** alpha/4

$\frac{\alpha}{\beta^2}$

Solution:

The Delta method would give

$$\sqrt{n}\,(\hat{\alpha} - \alpha) = \sqrt{n}\left(\sqrt{\overline{X}_n} - \alpha\right) \xrightarrow[d.]{n\to\infty} \mathcal{N}\left(0, \left(g'\left(\mathbb{E}\left[X_i\right]\right)\right)^2 \mathsf{Var}\left(X_i\right)\right) = \mathcal{N}\left(0, \left(g'\left(\alpha^2\right)\right)^2 \mathsf{Var}\left(X\right)\right) \qquad \textit{where } g(x) = \sqrt{x}$$

if g is continuously differentiable at α^2. Indeed, since $g'(x) = \frac{1}{2\sqrt{x}}$ exists and is continuous for all $x > 0$, g' is continuously differentiable at any α^2 value. Hence, the Delta method does apply.
To compute the asymptotic variance $\left(g'\left(\alpha^2\right)\right)^2 \mathsf{Var}\left(X_i\right)$, we need to compute $g'\left(\alpha^2\right)$ and $\mathsf{Var}\left(X_i\right)$.

$$g'\left(\alpha^2\right) = \frac{1}{2\sqrt{\alpha^2}} = \frac{1}{2\alpha}$$

In general, the variance for a Gamma variable X with parameters α, β is $\frac{\alpha}{\beta^2}$, since

$$\begin{aligned}
\mathbb{E}[X^2] = \int_0^\infty x^2 f_{\alpha,\beta}(x)\,dx &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\beta x} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{x^{\alpha+1} e^{-\beta x}}{-\beta}\Bigg|_0^\infty - \int_0^\infty \left((\alpha+1)\,x^\alpha\right)\left(\frac{e^{-\beta x}}{-\beta}\right) dx \right) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\alpha+1}{\beta} \int_0^\infty x^\alpha e^{-\beta x}\,dx \right) \\
&= \frac{\alpha+1}{\beta} (\mathbb{E}[X]) = \frac{\alpha+1}{\beta}\left(\frac{\alpha}{\beta}\right) \\
\mathsf{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \frac{\alpha+1}{\beta}\left(\frac{\alpha}{\beta}\right) - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2}
\end{aligned}$$

In this problem, $\beta = 1/\alpha$, hence

$$\mathsf{Var}(X_i) = \alpha^3.$$

Putting these together, the asymptotic variance is

$$\left(g'(\alpha^2)\right)^2 \mathsf{Var}(X_i) = \frac{1}{4\alpha^2}\left(\alpha^3\right) = \frac{\alpha}{4}.$$

(d)

0.0/4.0 points (graded)

Using the previous part, find confidence intervals for α with asymptotic level 90% using both the "solving" and the "plug-in" methods. Use $n = 25$, and $\overline{X}_n = 4.5$.
(Enter your answers accurate to 2 decimal places. Use the Gaussian estimate $q_{0.05} \approx 1.6448$ for best results.)

$\mathcal{I}_{\text{solve}} = [$ **Answer:** 1.89 , **Answer:** 2.37 $]$

$\mathcal{I}_{\text{plug-in}} = [$ **Answer:** 1.88 , **Answer:** 2.36 $]$

STANDARD NOTATION

Solution:

Recall from the last part that

$$\sqrt{n}\left(\hat{\alpha} - \alpha\right) \xrightarrow[d.]{n\to\infty} \mathcal{N}\left(0, \tau^2\right) \qquad \text{where } \tau^2 = \frac{\alpha}{4}$$

This implies

$$\frac{\sqrt{n}}{\tau}(\hat{\alpha} - \alpha) \xrightarrow[d.]{n\to\infty} \mathcal{N}(0,1) \text{ where } \tau^2 = \frac{\alpha}{4}$$

Therefore, following the usual procedure for confidence intervals, for large n, approximately

$$\mathbf{P}\left(\hat{\alpha} - q_{0.05}\frac{\tau}{\sqrt{n}} < \alpha < \hat{\alpha} + q_{0.05}\frac{\tau}{\sqrt{n}}\right) = 0.9.$$

Plugging in the asymptotic variance $\tau = \sqrt{\alpha}/2$ gives

$$\mathbf{P}\left(\hat{\alpha} - q_{0.05}\frac{\sqrt{\alpha}}{2\sqrt{n}} < \alpha < \hat{\alpha} + q_{0.05}\frac{\sqrt{\alpha}}{2\sqrt{n}}\right) = 0.9.$$

We now go through the three methods of solving for the confidence interval:

1. Conservative bound: Since $\sqrt{\alpha}$ is not bounded, the conservative bound method does not give a confidence interval.

2. Solving for α: we need to solve the following for α:

$$\begin{aligned} & |\hat{\alpha} - \alpha| && < q_{0.05}\frac{\tau}{\sqrt{n}} = q_{0.05}\frac{\sqrt{\alpha}}{2\sqrt{n}} \\ \Longleftrightarrow\ & (\hat{\alpha} - \alpha)^2 && < q_{0.05}^2\frac{\alpha}{4n} \\ \Longleftrightarrow\ & \alpha^2 - \left(2\hat{\alpha} + \frac{q_{0.05}^2}{4n}\right)\alpha + \hat{\alpha}^2 && = 0 \end{aligned}$$

   where $\hat{\alpha}^2 = \overline{X}_n = 4.5$, and $q_{0.05} = 1.6448$. Using the quadratic formula or software, we get the confidence interval

$$\mathcal{I}_{\text{solve}} = [1.89, 2.37]$$

3. Plug-in: Since $\hat{\alpha}^2 = \overline{X}_n = 4.5$, the plug-in confidence interval is

$$\begin{aligned} \mathcal{I}_{\text{plug-in}} &= \left[\hat{\alpha} - q_{0.05}\frac{\sqrt{\hat{\alpha}}}{2\sqrt{n}}, \hat{\alpha} + q_{0.05}\frac{\sqrt{\hat{\alpha}}}{2\sqrt{n}}\right] \\ &= [1.88, 2.36] \end{aligned}$$