Unit 3 Counting

The counting principle
How many different combinations are there of 4 shirts, 3 ties and 2 jackets?

4 x 3 x 2 = 24
r = number of stages, 3
n[i] = number of options at each stage, 4,3,2

Examples
Permutations:
number of ways of ordering n elements

First element has n choices
Second element has n-1 choices
Third …. n-2 choices
Final …. 1 choice

n.(n-1)(n-2)….1 = n! factorial

Number of subsets of {1,….,n}:
Each stage has 2 choices, put element in subset or not
2 , 2 , 2 , 2… etc up to n times
= 2^n

When n=1 {1}, 2^1 = 2   which is subset of {1} and {} empty

Die roll example
P(six rolls of a 6sided die all give different numbers):

P(A) = # in A / # possible outcomes

possible outcomes = 6^6
how many times we can order 1-6 = 6!

P(A) = 6!/ 6^6

Exercise:
You are given the set of letters {A, B, C, D, E}. What is the probability that in a random five-letter string (in which each letter appears exactly once, and with all such strings equally likely) the letters A and B are next to each other? The answer to a previous exercise may also be useful here. (In this and subsequent questions, your answer should be a number. Do not enter '!' or combinations in your answer.)

# of total combinations of 5 letters = 5!
# of ways A and B are next to each other = 48

Combinations

Definition "n choose k"

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

number of k element subsets
of a given n element set

Two ways of constructing an **ordered** sequence of k **distinct** items:
Method 1
choose each element in turn up to k
n(n-1)(n-2)...(n-k+1) = n! / (n-k)!

Method 2
First choose k items out of n (n choose k)
Then how many ways of ordering k elements (k!)
(n choose k).k!

Therefore
n! / (n-k)! = (n choose k).k!
Giving above definition

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = \# \text{ all subsets } = 2^n$$

Exercise: Counting committees

0.0/2.0 points (graded)

We start with a pool of n people. A chaired committee consists of $k \geq 1$ members, out of whom one member is designated as the chairperson. The expression $k\binom{n}{k}$ can be interpreted as the number of possible chaired committees with k members. This is because we have $\binom{n}{k}$ choices for the k members, and once the members are chosen, there are then k choices for the chairperson. Thus,

$$c = \sum_{k=1}^{n} k\binom{n}{k}$$

is the total number of possible chaired committees of any size.

Find the value of c (as a function of n) by thinking about a different way of forming a chaired committee: first choose the chairperson, then choose the other members of the committee. The answer is of the form

$$c = (\alpha + n^{\beta})\, 2^{\gamma n+\delta}.$$

What are the values of α, β, γ, and δ?

What are the values of α, β, γ, and δ?

$\alpha =$ 0

$\beta =$ 1

$\gamma =$ 1

$\delta =$ -1

Binomial Probabilities

Binomial coefficient $\binom{n}{k}$ $\longrightarrow$ Binomial probabilities

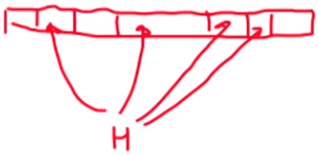
- $n \geq 1$ independent coin tosses; $\mathbf{P}(H) = p$

$$\boxed{\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}}$$

$n=6$

- $\mathbf{P}(HTTHHH) = p(1-p)(1-p)ppp = p^4(1-p)^2$
- $\mathbf{P}(\text{particular sequence}) = p^{\#\text{heads}} (1-p)^{\#\text{tails}}$
- $\mathbf{P}(\text{particular } k\text{-head sequence}) = p^k (1-p)^{n-k}$

$$\mathbf{P}(k \text{ heads}) = p^k (1-p)^{n-k} \cdot (\#\ k\text{-head sequences})$$



$\binom{n}{k}$

Exercise: Binomial probabilities

2/2 points (graded)

Recall that the probability of obtaining k Heads in n independent coin tosses is $\binom{n}{k} p^k (1-p)^{n-k}$, where p is the probability of Heads for any given coin toss.

Find the value of $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$. (Your answer should be a number.)

1 ✔ **Answer:** 1

Solution:

Note that the events "0 Heads", "1 Heads", ..., "n Heads" are disjoint, and their union is the entire sample space. The summation is adding up the probability of all of these events. Hence, the sum must be 1. In other words, each term in the summation gives the probability of obtaining k Heads out of n tosses. We then sum over all values of k, from 0 to n. Since the number of Heads must be one of $0, 1, \ldots, n$, these probabilities must sum up to 1.

A coin tossing problem

- Given that there were 3 heads in 10 tosses, what is the probability that the first two tosses were heads?
  - event A: the first 2 tosses were heads
  - event B: 3 out of 10 tosses were heads

Assumptions:
- independence
- $\mathbf{P}(H) = p$

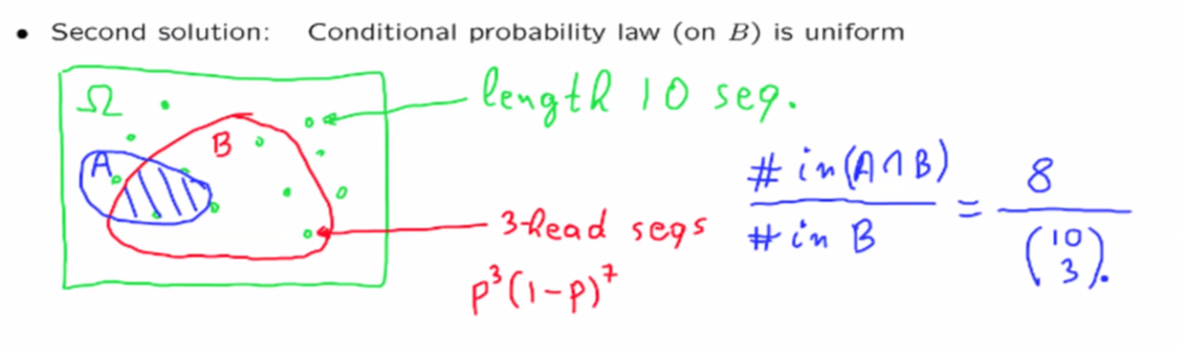
$$\mathbf{P}(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

- First solution:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{P(H_1 H_2 \text{ and one } H \text{ in tosses } 3, \ldots, 10)}{P(B)}$$

$$= \frac{p^2 \cdot \binom{8}{1} p^1 \cdot (1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{\binom{8}{1}}{\binom{10}{3}} = \frac{8}{\binom{10}{3}}$$

(8 comes from remaining tosses after toss 1 and 2)



Can apply this easier second solution because the condition probability law on B is uniform

Exercise: Coin tossing

2/2 points (graded)

Use the second method in the preceding segment to find the probability that the 6th toss out of a total of 10 tosses is Heads, given that there are exactly 2 Heads out of the 10 tosses. As in the preceding segment, continue to assume that all coin tosses are independent and that each coin toss has the same fixed probability of Heads. (In this and subsequent questions, your answer should be a number. Do not enter '!' or combinations in your answer.)

1/5 ✔

Submit You have used 1 of 3 attempts Save Show Answer

i.e. 9 / (10 choose 2) = 9/45 = 1/5

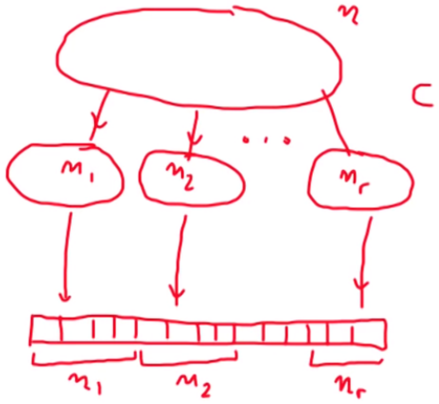
9 because toss 6 has to be a heads along with 1 of the 9 other options so there are 9 options
(10 choose 2) because 2 out of the 10 tosses are heads

Partitions

Partitions

- $n \geq 1$ distinct items; $r \geq 1$ persons
  give n_i items to person i
  - here $n_1, \ldots, n_r$ are given nonnegative integers
  - with $n_1 + \cdots + n_r = n$
- Ordering n items: $n!$
  - Deal n_i to each person i, and then order

$c \; n_1! \, n_2! \cdots n_r! = n!$



$r=2 \quad n_1 = k \quad n_2 = n-k$

$$\text{number of partitions} = \frac{n!}{n_1!\, n_2! \cdots n_r!} \qquad \text{(multinomial coefficient)}$$

Exercise: Counting partitions

3/3 points (graded)

We have 9 distinct items and three persons. Alice is to get 2 items, Bob is to get 3 items, and Charlie is to get 4 items.

1. As just discussed, this can be done in $\frac{a!}{b!\,3!\,4!}$ ways. Find a and b.

$a =$ 9 ✔

$b =$ 2 ✔

2. A different way of generating the desired partition is as follows. We first choose 2 items to give to Alice. This can be done in $\binom{c}{d}$ different ways. Find c and d. (There are 2 possible values of d that are correct. Enter the smaller value.)

$c =$ 9 ✔

$d =$ 2 ✔

3. Having given 2 items to the Alice, we now give 3 items to Bob. This can be done in $\binom{e}{f}$ ways. Find e and f. (There are 2 possible values of f that are correct. Enter the smaller value.)

$e =$ 7 ✔

$f =$ 3 ✔

Each person gets an ace example problem

Example: 52-card deck, dealt (fairly) to four players.
Find $\mathbf{P}$(each player gets an ace)

- Outcomes are: partition equally likely
  - number of outcomes: $\dfrac{52!}{13!\,13!\,13!\,13!}$
- Constructing an outcome with one ace for each person:
  - distribute the aces $4 \cdot 3 \cdot 2 \cdot 1$
  - distribute the remaining 48 cards $\dfrac{48!}{12!\,12!\,12!\,12!}$
- Answer: $$\frac{4 \cdot 3 \cdot 2 \cdot \dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}$$

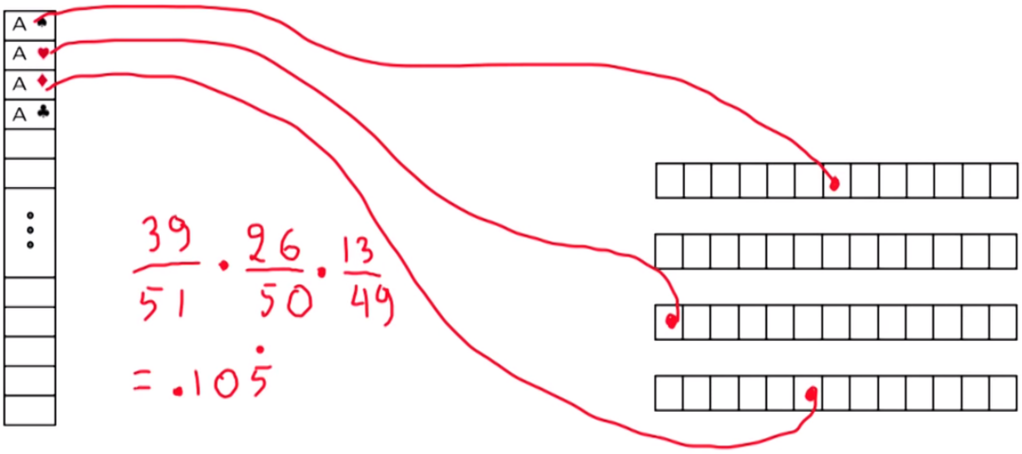
A smart solution

Distribute 4 aces randomly, what’s the chance each ace gets put into a different stack of 13 cards (4 x 13 = 52)
First ace doesn’t matter
Second ace 39 cards that are in a different stack out of 51 remaining cards
Third ace 26 cards that are in a different stack out of 50 remaining cards
Fourth ace 13 cards that are in a different stack out of 49 remaining cards

Example: 52-card deck, dealt (fairly) to four players.
Find $\mathbf{P}$(each player gets an ace)

A smart solution

Stack the deck, aces on top



$$\frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49} = .105$$

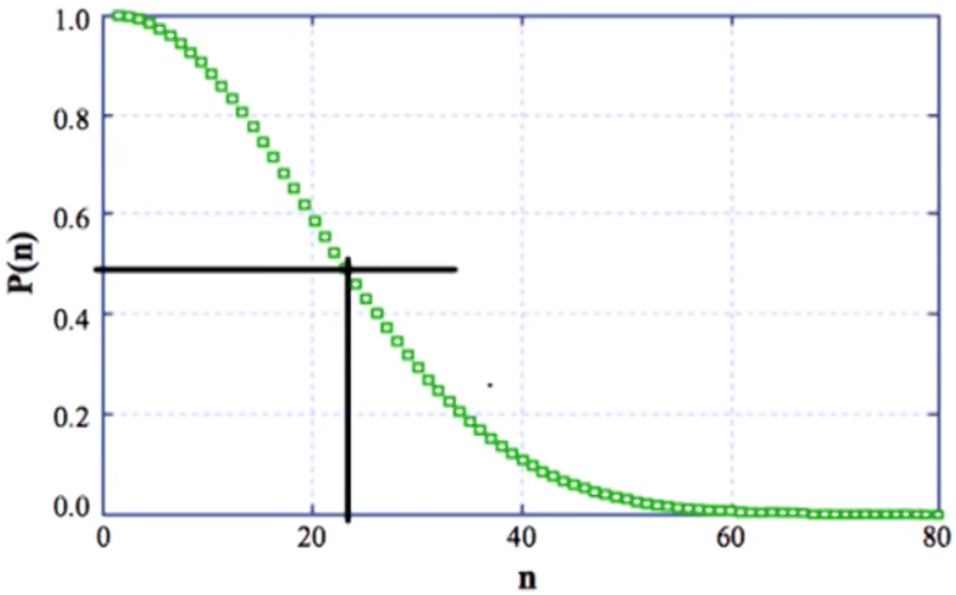
Solved Problems

The birthday problem
Consider n people who are attending a party. What is the probability that each person has a distinct birthday?

n . P(no 2 birthdays coincide)
omega = all birthday combinations = (365)^n
First person choice = 365 days
2nd = 364
3rd = 363
nth = 365-n+1

P(no 2 birthdays coincide) = $\frac{365.364.... (365-n+1)}{365^n}$

23 people gives ~50%
Only valid for n<=365 otherwise the "pigeon hole principle" applies



Rooks on a chessboard

Eight rooks are placed in distinct squares of an 8×8 chessboard, with all possible placements being equally likely. Find the probability that all the rooks are safe from one another, i.e., that there is no row or column with more than one rook.

Denominator is number of squares to power of rooks available
Numerator is first choice x second choice x 3rd .... etc
First rook = 64 choices, this eliminates a column and row (15 squares)
2nd rook = 64-15 = 49 choices
3rd = 49 - 13 = 36 (13 as 2 of the spots are already overlapping ones that aren't available)
4th = 36 - 11 = 25

5th = 25 - 9 = 16
6th = 16 - 7 = 9
7th = 9 - 5 = 4
8th = 4 - 3 =1

O	-	-	-	-	-	-	-
-	O	-	-	-	-	-	-
-	-	O	-	-	-	-	-
-	-	-	O	-	-	-	-
-	-	-	-	O	-	-	-
-	-	-	-	-	O	-	-
-	-	-	-	-	-	O	-
-	-	-	-	-	-	-	O

Total arrangements = $\frac{64!}{56!}$

56 is how many spaces remain for the 8th rook (not considering “safe” squares)

Safe arrangements = 64 . 49 . 36 . 25 . 16 . 9 . 4 . 1

P (8 safe rooks arranged) = $\dfrac{64 . 49 . 36 . 25 . 16 . 9 . 4 . 1}{\frac{64!}{56!}}$

Hypergeometric probabilities

An urn contains n balls, out of which exactly m are red. We select k of the balls at random, without replacement (i.e., selected balls are not put back into the urn before the next selection). What is the probability that i of the selected balls are red?

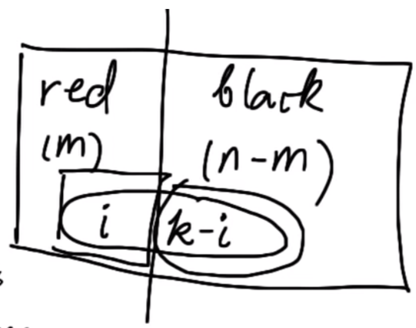
$|\Omega| = \binom{n}{k}$

C = # ways to get i red balls $\quad a = \binom{m}{i}$

$= a \cdot b \quad b = \binom{n-m}{k-i}$

a = # ways to get i out of m red balls

b = # ways to get $k-i$ out of $n-m$ black balls



$$P_r = \frac{C}{|\Omega|} = \frac{\binom{m}{i}\binom{n-m}{k-i}}{\binom{n}{k}}$$

Multinomial probabilities

An urn contains balls of r different colors. We draw n balls, with different draws being independent. For any given draw, there is a probability pi, i=1,...,r, of getting a ball of color i. Here, the pi's are nonnegative numbers that sum to 1. Let n1,...,nr be nonnegative integers that sum to n. What is the probability that we obtain exactly ni balls of color i, for each i=1,...,r?

The multinomial probabilities

$$n_1 + \cdots + n_r = n$$

$r = 3$

$n = 7$

1 1 3 1 2 2 1

type: $(4, 2, 1)$

Prob $= P_1 P_1 P_3 P_1 P_2 P_2 P_1 = P_1^4 P_2^2 P_3$

$$\mathbf{P}\big(\text{particular sequence of "type" } (n_1, n_2, \ldots, n_r)\big) = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

sequence of type $(n_1, n_2, \ldots, n_r)$ $\longleftrightarrow$ partition of $\{1, \ldots, n\}$ into subsets of sizes $n_1, n_2, \ldots, n_r$

$$\boxed{\mathbf{P}\big(\text{get type } (n_1, n_2, \ldots, n_r)\big) = \frac{n!}{n_1!\, n_2! \cdots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}}$$

Problem sets

Problem 1. Customers arriving at a restaurant

2/2 points (graded)

Six customers enter a three-floor restaurant. Each customer decides on which floor to have dinner. Assume that the decisions of different customers are independent, and that for each customer, each floor is equally likely. Find the probability that exactly one customer dines on the first floor.

0.2634 ✔

Using multinomial probabilities where

n = 6

p1 = 1/3

n1 = 1

p2 = 2/3

n2 = 5

Problem 2. A three-sided die, part 1

1/1 point (graded)

The newest invention of the 6.431x staff is a three-sided die. On any roll of this die, the result is 1 with probability 1/2, 2 with probability 1/4, and 3 with probability 1/4.

Consider a sequence of six independent rolls of this die.

Find the probability that exactly two of the rolls result in a 3.

- [x] $\binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4$

1. Given that exactly two of the six rolls resulted in a 1, find the probability that the first roll resulted in a 1.
   Note: Your answer should be a number. Do not enter "!" or combinations in your answer.

   1/3 ✔

2. We are told that exactly three of the rolls resulted in a 1 and exactly three rolls resulted in a 2. Given this information, find the probability that the six rolls resulted in the sequence $(1, 2, 1, 2, 1, 2)$.
   Note: Your answer should be a number. Do not enter "!" or combinations in your answer.

   1/64 ✖

3. The conditional probability that exactly k rolls resulted in a 3, given that at least one roll resulted in a 3, is of the form:

$$\frac{1}{1-(c_1/c_2)^{c_3}}\binom{c_3}{k}\left(\frac{1}{c_2}\right)^k\left(\frac{c_1}{c_2}\right)^{c_3-k}, \quad \text{for } k = 1, 2, \ldots, 6.$$

   Find the values of the constants c_1, c_2, and c_3:

   $c_1 =$ 3 ✔

   $c_2 =$ 4 ✔

   $c_3 =$ 6 ✔

1. We know there are two 1s, so how many different combinations are there that have exactly two 1s? = denominator 15 from (6 choose 2)
   Now how many combinations are there that have a 1 at the start? = numerator 5
2. Applying similar logic, if there are 6 numbers but two different is the total amount of combinations (6 choose 2) or 2^6?
   Then how many combinations of (1,2,1,2,1,2) are there? I thought 1 but potentially have to think about having different 1s/2s in different positions as well

Problem 3. Forming a committee

2/2 points (graded)

Out of five men and five women, we form a committee consisting of four different people. Assuming that each committee of size four is equally likely, find the probabilities of the following events:

1. The committee consists of two men and two women.

0.4762 ✔

2. The committee has more women than men.

0.2619 ✔

3. The committee has at least one man.

0.9762 ✔

For the remainder of the problem, assume that Alice and Bob are among the ten people being considered.

4. Both Alice and Bob are members of the committee.

0.1333 ✔

1. $\frac{(5 \text{ choose } 2).(5 \text{ choose } 2)}{(10 \text{ choose } 4)}$
2. (1 - above) / 2 because the only other possibility is more men or women and they have symmetric possibilities
3. Same as 1 - P(having all women) or can add up all the possibilities of 1 man, 2 men, 3 men, 4 men (with their corresponding number of women, using same formula as 1.)
4. $\frac{(2 \text{ choose } 2).(8 \text{ choose } 2)}{(10 \text{ choose } 4)}$ treat it as choosing both Bob and Alice out of the 2, and 2 from the other 8

Problem 4. Proving Binomial identities

Out of $2n$ people, we want to choose a committee of n people, one of whom will be its chair. In how many different ways can this be done?

$$n\binom{2n}{n} = 2n\binom{2n-1}{n-1}.$$

In a group of $2n$ people, consisting of n boys and n girls, we want to select a committee of n people. In how many ways can this be done?

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}.$$

How many subsets does a set with $2n$ elements have?

$$2^{2n} = \sum_{i=0}^{2n} \binom{2n}{i}.$$

Out of n people, we want to form a committee consisting of a chair and other members. We allow the committee size to be any integer in the range 1,2,...,n. How many choices do we have in selecting a committee-chair combination?

$$n2^{n-1} = \sum_{i=0}^{n} \binom{n}{i} i.$$

Problem 5. Hats in a box

5/5 points (graded)

Each one of n persons, indexed by $1, 2, \ldots, n$, has a clean hat and throws it into a box. The persons then pick hats from the box, at random. Every assignment of the hats to the persons is equally likely. In an equivalent model, each person picks a hat, one at a time, in the order of their index, with each one of the remaining hats being equally likely to be picked. Find the probability of the following events.

1. Every person gets his or her own hat back.

$$\frac{1}{n!}$$

2. Each one of persons 1,...,m gets his or her own hat back, where 1≤m≤n.

$$\frac{(n-m)!}{n!}$$

3. Each one of persons 1,...,m gets back a hat belonging to one of the last m persons (persons n−m+1,...,n), where 1≤m≤n.

$$\frac{1}{\binom{n}{m}}$$

Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). Find the probability that:

4. Persons 1,…,m will pick up clean hats.

$$(1-p)^m$$

5. Exactly m persons will pick up clean hats.

$$\binom{n}{m}(1-p)^m p^{n-m}$$