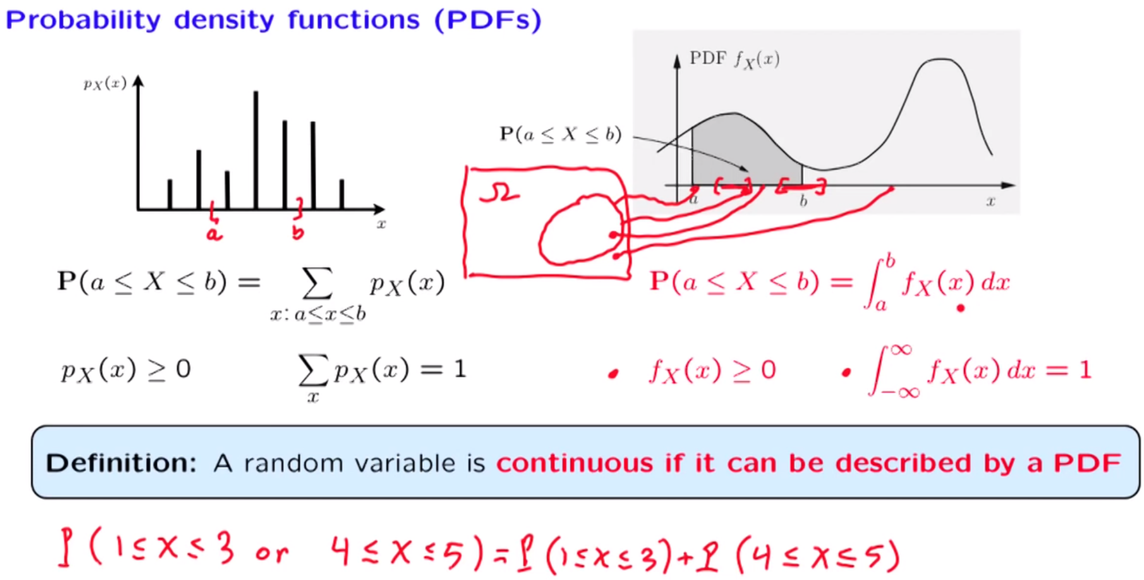
Unit 5 Continuous Random Variables

Very much a repetition of unit 4 but with continuous variables instead of discrete variables

Will use more calculus rather than counting/sums

Probability Density Functions (PMFs)
The mass of probability is spread out over a line, calculating the density of this gives us a probability of the interval



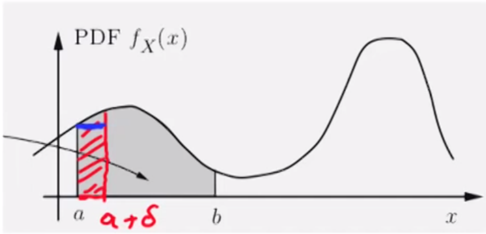
Any particular point has 0 probability but a group of points have a positive probability

Probability density functions (PDFs)

$\delta > 0$, small

$P(a \le X \le a+\delta)$

$\approx f_X(a) \cdot \delta$



$$\boxed{P(a \le X \le a+\delta) \approx f_X(a) \cdot \delta}$$

$$\boxed{P(X = a) = 0}$$

$$P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

$$f_X(x) \ge 0$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$P(a \le X \le b) = P(X=a) + P(X=b) + P(a < X < b)$

Exercise: PDFs

4/4 points (graded)

Let X be a continuous random variable with a PDF of the form

$$f_X(x) = \begin{cases} c(1-x), & \text{if } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Find the following values.

1. $c =$ 2 ✔
2. $\mathbf{P}(X = 1/2) =$ 0 ✔
3. $\mathbf{P}\big(X \in \{1/k : k \text{ integer}, k \ge 2\}\big) =$ 0 ✔
4. $\mathbf{P}(X \le 1/2) =$ 0.75 ✔

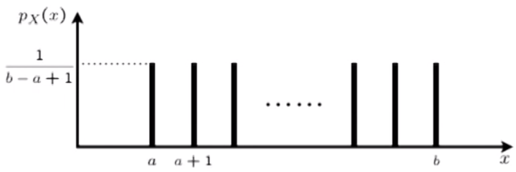
1. We have $1 = \int_{-\infty}^{\infty} f_X(x)\ dx = \int_0^1 c(1-x) = c(x - x^2/2)\,\Big|_1^0 = c/2$, and therefore, $c = 2$.

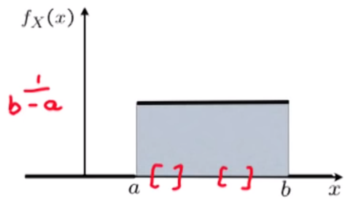
2. Individual points have zero probability.

3. Using countable additivity and the fact that single points have zero probability, we have
$$\mathbf{P}\big(X \in \{1/2, 1/3, 1/4, 1/5, \ldots\}\big) = \sum_{n=2}^{\infty} \mathbf{P}(X = 1/n) = \sum_{n=2}^{\infty} 0 = 0.$$

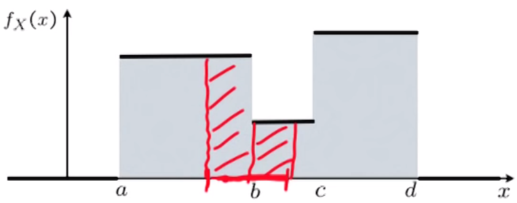
4. $\mathbf{P}(X \le 1/2) = \int_{-\infty}^{1/2} f_X(x)\ dx = \int_0^{1/2} 2(1-x)\ dx = 2(x - x^2/2)\,\Big|_{1/2}^{0} = \frac{3}{4}.$

Example: continuous uniform PDF





- Generalization: piecewise constant PDF



Exercise: Piecewise constant PDF

2/2 points (graded)

Consider a piecewise constant PDF of the form

$$f_X(x) = \begin{cases} 2c, & \text{if } 0 \le x \le 1, \\ c, & \text{if } 1 < x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the following values.

a) $c =$ 0.25 ✔ **Answer:** 0.25

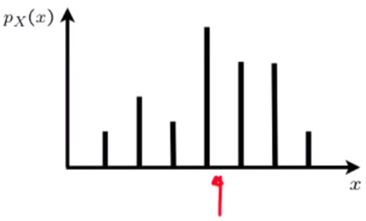
b) $\mathbf{P}(1/2 \le X \le 3/2) =$ 0.375 ✔ **Answer:** 0.375

Solution:

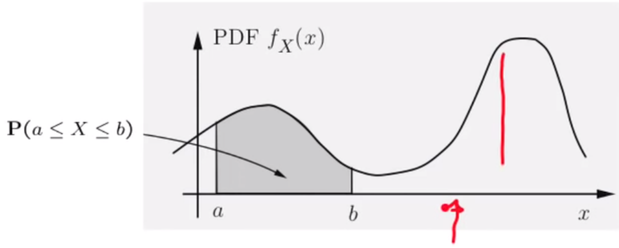
a) The total area under the PDF is the sum of the areas of two rectangles and is equal to $(2c) \cdot 1 + c \cdot 2 = 4c$. Therefore, $c = 1/4$.

b) The total area under the PDF over the interval of interest is the sum of the areas of two smaller rectangles and is equal to $(2c) \cdot (1/2) + c \cdot (1/2) = c \cdot (3/2) = 3/8$.

Expectation/mean of a continuous random variable



$$\mathbf{E}[X] = \sum_x x p_X(x)$$



$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- **Interpretation:** Average in large number of independent repetitions of the experiment

Fine print:
Assume $\int_{-\infty}^{\infty} |x| f_X(x)\, dx < \infty$

"Centre of gravity" = expectation

Properties of expectations

- If $X \ge 0$, then $\mathbf{E}[X] \ge 0$
- If $a \le X \le b$, then $a \le \mathbf{E}[X] \le b$
- Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

$$\boxed{\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx$$

- Linearity

$$\boxed{\mathbf{E}[aX + b] = a\mathbf{E}[X] + b}$$

Variance and its properties

- **Definition of variance:** $\text{var}(X) = \mathbf{E}[(X-\mu)^2]$

$\mu = E[X]$

- Calculation using the expected value rule, $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

$$\text{var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$g(x) = (x-\mu)^2$

Standard deviation: $\sigma_X = \sqrt{\text{var}(X)}$

✓ $\text{var}(aX+b) = a^2\text{var}(X)$

✓ A useful formula: $\text{var}(X) = \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2$

Exercise: Uniform PDF

3/3 points (graded)

Let X be uniform on the interval $[1,3]$. Suppose that $1 < a < b < 3$. Then,

(a) $\mathbf{P}(a \le X \le b) =$ (b-a)*0.5 ✔ **Answer:** (b-a)/2

$(b-a)\cdot 0.5$

(Your answer to part (a) should be an algebraic expression involving a and b.)

(b) $\mathbf{E}[X] =$ 2 ✔ **Answer:** 2

(c) $\mathbf{E}[X^3] =$ 10 ✔ **Answer:** 10

Solution:

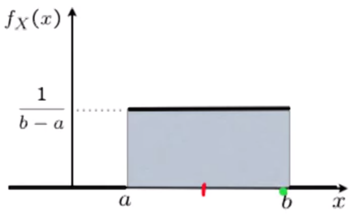
(a) The value of the PDF on the interval $[1,3]$ must be equal to 1/2, so that it integrates to 1. Thus,

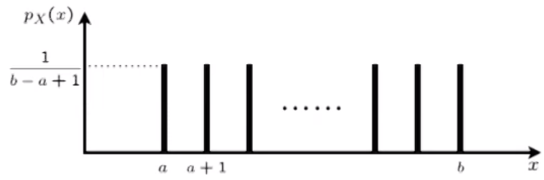
$$\mathbf{P}(a \le X \le b) = \int_a^b \frac{1}{2}\,dx = \frac{b-a}{2}.$$

(b) The expected value of a uniform is the midpoint of its range: $\mathbf{E}[X] = (1+3)/2 = 2$.

(c) Using the expected value rule, $\mathbf{E}[X^3] = \int_1^3 x^3 \cdot \frac{1}{2}\,dx = \frac{1}{2}\cdot\frac{1}{4}x^4\Big|_3^1 = \frac{1}{2}\cdot\frac{1}{4}\cdot(81-1) = 10.$

Continuous uniform random variable; parameters a, b





$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{a+b}{2}$$

$$\mathbf{E}[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)$$

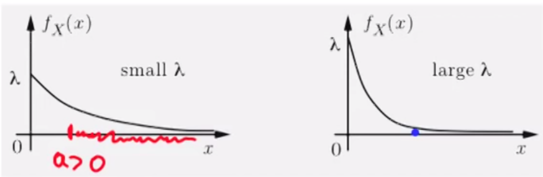
$$\text{var}(X) = E[X^2] - (E[X])^2 = \boxed{(b-a)^2/12} \qquad \sigma = \frac{b-a}{\sqrt{12}}$$

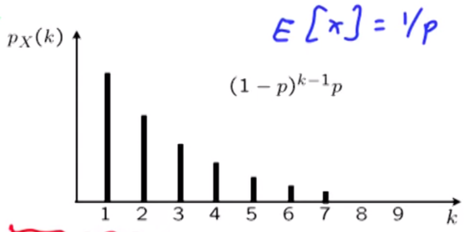
$$\mathbf{E}[X] = \frac{a+b}{2}$$

$$\text{var}(X) = \frac{1}{12}(b-a)(b-a+2)$$

Exponential random variable; parameter $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \int f_X(x)\,dx = 1$$





$$E[X] = 1/p$$

$$\mathbf{E}[X] = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx = 1/\lambda$$

$$\mathbf{E}[X^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx = 2/\lambda^2$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = 1/\lambda^2$$

$$\boxed{P(X \geq a)} = \int_a^{\infty} \lambda e^{-\lambda x}\,dx$$

$$\left[\int e^{ax}\,dx = \frac{1}{a} e^{ax} \quad a \leftrightarrow -\lambda \right]$$

$$= \lambda \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_a^{\infty}$$

$$= -e^{-\lambda \cdot \infty} + e^{-\lambda a} = \boxed{e^{-\lambda a}}$$

Shows that the probability decreases exponentially beyond a, and if a=0 then the probability equals 1 which is the entire space of probability

When x is zero we have f(x) = lambda since e^0 = 1
The exponential is similar to the discrete geometric and are used in conjunction with one another
The exponential is also used a lot to model the time for an event to happen such as the time for a customer to arrive, a machine to break down, a meteorite to hit a house etc.

Exercise: Exponential PDF

1/2 points (graded)

Let X be an exponential random variable with parameter $\lambda = 2$. Find the values of the following. Use 'e' for the base of the natural logarithm (e.g., enter `e^(-3)` for e^{-3}).

a) $\mathbf{E}\left[(3X+1)^2\right] =$ 3.5 ✖ **Answer:** 8.5

b) $\mathbf{P}(1 \le X \le 2) =$ 0.1170 ✔ **Answer:** 0.11702

Solution:

a) By expanding the quadratic, using linearity of expectations, and the facts that $\mathbf{E}[X] = 1/\lambda$ and $\mathbf{E}[X^2] = 2/\lambda^2$, we have

$$\mathbf{E}\left[(3X+1)^2\right] = 9\mathbf{E}\left[X^2\right] + 6\mathbf{E}[X] + 1 = 9 \cdot \frac{2}{2^2} + 6 \cdot \frac{1}{2} + 1 = \frac{17}{2}.$$

b) We have seen that for $a > 0$, we have $\mathbf{P}(X \ge a) = e^{-\lambda a}$, so that $\mathbf{P}(X \le a) = 1 - e^{-\lambda a}$. Therefore,

$$\mathbf{P}(1 \le X \le 2) = \mathbf{P}(X \le 2) - \mathbf{P}(X \le 1) = (1 - e^{-4}) - (1 - e^{-2}) = e^{-2} - e^{-4}.$$