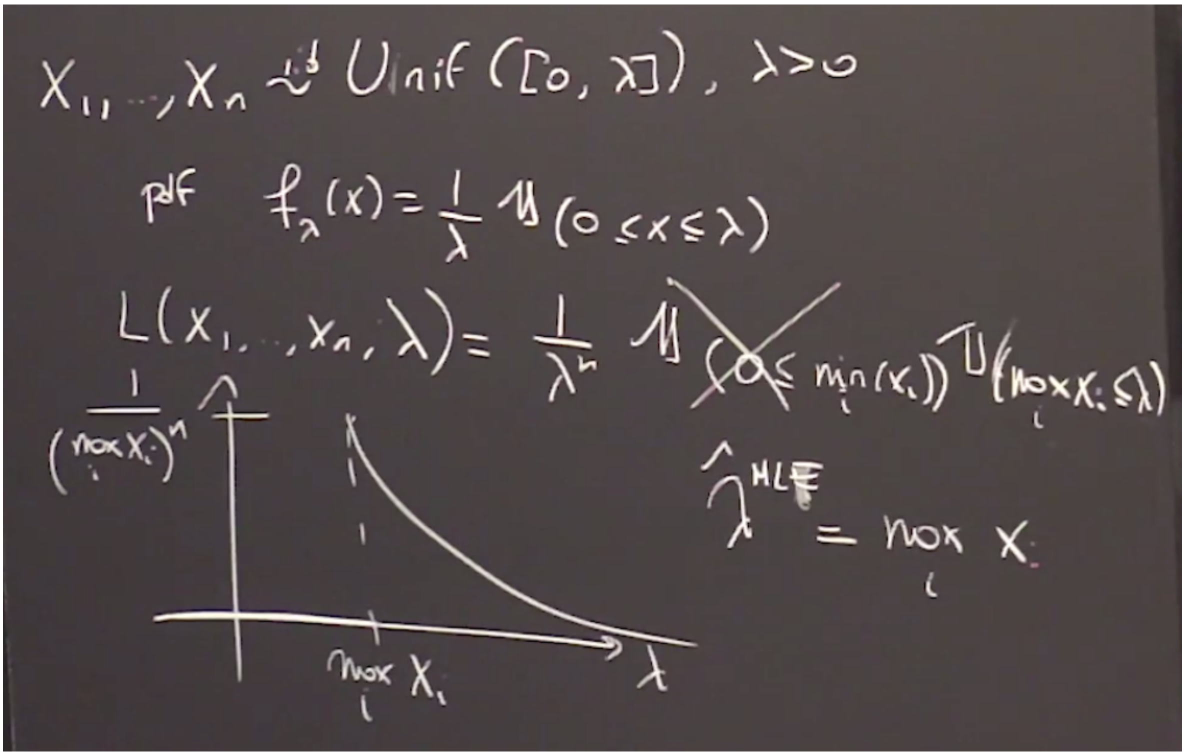
# Unit 3 cont’d Methods of Estimation

once lambda is less than the maximum of x_i then it is 0




this model does not satisfy the regulatory conditions - i.e. the derivative is not defined throughout so we can’t set it to 0 to find the MLE
(can’t take derivative below max x_i)

## Concept Check: Maximum Likelihood Estimator for a Uniform Statistical Model

1/1 point (graded)

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Unif}\,[0, \theta^*]$ where $\theta^*$ is an unknown parameter. We constructed the associated statistical model $(\mathbb{R}_{\geq 0}, \{\text{Unif}\,[0, \theta]\}_{\theta>0})$ (where $\mathbb{R}_{\geq 0}$ denotes the nonnegative reals).

For any $\theta > 0$, the density of $\text{Unif}\,[0, \theta]$ is given by $f(x) = \frac{1}{\theta}\mathbf{1}(x \in [0, \theta])$. Recall that

$$\mathbf{1}(x \in [0, \theta]) = \begin{cases} 1 & \text{if } x \in [0, \theta] \\ 0 & \text{otherwise.} \end{cases}$$

Hence we can use the product formula and compute the likelihood to be

$$L_n(x_1, \ldots, x_n, \theta) = \prod_{i=1}^{n} \left(\frac{1}{\theta}\mathbf{1}(x_i \in [0, \theta])\right) = \frac{1}{\theta^n}\mathbf{1}(x_i \in [0, \theta] \;\; \forall\, 1 \leq i \leq n).$$

For the fixed values $(1, 3, 2, 2.5, 5, 0.1)$ (think of these as observations of random variables $X_1, \ldots, X_6$), what value of $\theta$ maximizes $L_6(1, 3, 2, 2.5, 5, 0.1, \theta)$?

5

✔ **Answer:** 5

**Solution:**

Observe that

$$L_6(1, 3, 2, 2.5, 5, 0.1, \theta) = \frac{1}{\theta^6}\mathbf{1}(\{1, 3, 2, 2.5, 5, 0.1\} \subset [0, \theta]).$$

If $\theta < \max\{1, 3, 2, 2.5, 5, 0.1\}$, then we have $\{1, 3, 2, 2.5, 5, 0.1\} \not\subset [0, \theta]$. By the definition of the indicator function, this means $L_6(1, 3, 2, 2.5, 5, 0.1, \theta) = 0$ for $\theta < \max\{1, 3, 2, 2.5, 5, 0.1\} = 5$. Hence, when maximizing $L_6(1, 3, 2, 2.5, 5, 0.1, \theta)$, we need to consider $\theta \in [5, \infty)$. Restricted to this interval, we observe that

$$L_6(1, 3, 2, 2.5, 5, 0.1, \theta) = \frac{1}{\theta^n}.$$

The above is a decreasing function on $[5, \infty)$, so the maximum is attained when $\theta = \max\{1, 3, 2, 2.5, 5, 0.1\} = 5$.

**Remark**: In general, the maximum likelihood estimator for $\theta^*$ in this uniform statistical model is

$$\widehat{\theta_n}^{MLE} = \max_{1 \leq i \leq n} X_i.$$

## MLE for a Loaded Die: Likelihood

1/1 point (graded)

You have a loaded (i.e. possibly unfair) six-sided die with the probability that it shows a "3" equal to $\eta$ and the probability that it shows any other number equal to $(1-\eta)/5$.

Let $X$ be a random variable representing a roll of this die. You roll this die $n$ times, and record your data set, consisting of the values of the faces as $X_1, X_2, X_3, \ldots X_n$.

Let the outcome of a set of $n$ rolls of the die be modeled by the i.i.d. random variable sequence $(X_1, \ldots, X_n)$. We model the $i$'th roll as $X_i$ where $X_i = j$ if the top face of the die shows a "j".

You roll the die $n$ times and observe a sequence of outcomes $x_1, \ldots, x_n$ which contains exactly $k$ outcomes $x_i = 3$. What is the likelihood function $L_n(x_1, \ldots, x_n, \eta)$ for the entire sequence of outcomes?

(Enter **eta** for $\eta$.)

eta^k*((1-eta)/5)^(n-k)

**Answer:** eta^k*((1-eta)/5)^(n-k)

$\eta^k \cdot \left(\frac{1-\eta}{5}\right)^{n-k}$

**Solution:**

Denote by $p_\eta(x)$ the pmf of $X_i$. The probability that $X_i$ takes on a value $3$ is equal to $\eta$ and the probability that $X_i$ takes on any other value is $(1-\eta)/5$. Therefore, using the probability of observing a particular sequence of outcomes, we obtain the likelihood function as

$$\begin{aligned} L_n(x_1, \ldots, x_n, \eta) &= \prod_{i=1}^{n} p_\eta(x_i) \\ &= \eta^k \left(\frac{1-\eta}{5}\right)^{n-k}. \end{aligned}$$

Note that the above does not contain a combinatorial expression as we are interested in the probability of observing a particular sequence of die rolls and not in the probability of obtaining a certain number of $3$'s in $n$ rolls.

## MLE for a Loaded Die: MLE

1/1 point (graded)

Find the ML estimator $\widehat{\eta}_n^{\text{MLE}}$.

k/n

**Answer:** k/n

$\frac{k}{n}$

STANDARD NOTATION

**Solution:**

Since we are looking for the $\text{argmax}_{\eta\in[0,1]} L_n(x_1, \ldots, x_n, \eta)$, we can ignoring any scaling constant in $L_n(x_1, \ldots, x_n, \eta)$. Hence, we will maximize $\widetilde{L}_n(x_1, \ldots, x_n, \eta) = \eta^k (1-\eta)^{n-k}$.

Taking the derivative of $\widetilde{L}_n(x_1, \ldots, x_n, \eta)$ with respect to $\eta$ and setting it to $0$, we get

$$\begin{aligned} k(1-\eta) &= (n-k)\eta \\ \Longrightarrow \widehat{\eta}_n^{\text{MLE}} &= \frac{k}{n}. \end{aligned}$$

**Remark:** The function $\widetilde{L}_n(x_1, \ldots, x_n, \eta) = \eta^k (1-\eta)^{n-k}$ whose maximizer is $\widehat{\eta}_n^{\text{MLE}}$ is the same as the likelihood function for a Bernoulli experiment with parameter $\eta$, even though each roll of a die has 6 potential outcomes.