Midterm Exam

1.

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True or False

4 points possible (graded, results hidden)

Let A, B, and C be events associated with the same probabilistic model (i.e., subsets of a common sample space), and assume that $P(C) > 0$.

For each one of the following statements, decide whether the statement is True (always true), or False (not always true).

1. Suppose that $A \subset C$. Then, $P(A \mid C) \geq P(A)$.

- ( ) True
- (•) False

2. Suppose that $A \subset B$. Then, $P(A \mid C) \leq P(B \mid C)$.

- ( ) True
- (•) False

3. Suppose that $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.

- ( ) True
- (•) False

4. Suppose that $A \subset C$, $B \subset C$, and $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.

- ( ) True
- (•) False

2.

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A Drunk Person at the Theater

4 points possible (graded, results hidden)

There are n people in line, indexed by $i = 1, \ldots, n$, to enter a theater with n seats one by one. However, the first person ($i = 1$) in the line is drunk. This person has lost her ticket and decides to take a random seat instead of her assigned seat. That is, the drunk person decides to take any one of the seats 1 to n with equal probability. Every other person $i = 2, \ldots, n$ that enters afterwards is sober and will take his assigned seat (seat i) unless his seat i is already taken, in which case he will take a random seat chosen uniformly from the remaining seats.

Suppose that $n = 3$. What is the probability that person 2 takes seat 2?

(Enter a fraction or a decimal accurate to at least 3 decimal places.)

2/3

Suppose that $n = 5$. What is the probability that person 3 takes seat 3?

(Enter a fraction or a decimal accurate to at least 3 decimal places.)

0.4667

1. P(drunk person selecting correct) + P(2nd person selecting correct | drunk was incorrect) = 1/3 + 1/2 = 5/6
 or is it 1/3! ?
2. 1/5 +
3.

3.

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Expectation 1

1 point possible (graded, results hidden)

Compute $\mathbf{E}(X)$ for the following random variable X:

$$X = \text{Number of tosses until getting 4 (including the last toss) by tossing a fair 10-sided die.}$$

$\mathbf{E}(X) =$ 10

Expectation 2

2 points possible (graded, results hidden)

Compute $\mathbf{E}(X)$ for the following random variable X:

$$X = \text{ Number of tosses until all 10 numbers are seen (including the last toss) by tossing a fair 10-sided die.}$$

To answer this, we will use induction and follow the steps below:

Let $\mathbf{E}(i)$ be the expected number of additional tosses until all 10 numbers are seen (including the last toss) **given i distinct numbers have already been seen**.

1. Find $\mathbf{E}(10)$.

   $\mathbf{E}(10) =$ 1

2. Write down a relation between $\mathbf{E}(i)$ and $\mathbf{E}(i+1)$. Answer by finding the function $f(i)$ in the formula below.

   For $i = 0, 1, \ldots, 9$:

   $$\mathbf{E}(i) = \mathbf{E}(i+1) + f(i)$$

   where $f(i) =$ 10/i

   $\frac{10}{i}$

3. Finally, using the results above, find $\mathbf{E}[X]$.

   (Enter an answer accurate to at least 1 decimal place.)

   $\mathbf{E}[X] =$ 29.29

4.

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Conditional Independence 1

4 points possible (graded, results hidden)

Suppose that we have a box that contains two coins:

1. A fair coin: $\mathbf{P}(H) = \mathbf{P}(T) = 0.5$.
2. A two-headed coin: $\mathbf{P}(H) = 1$.

A coin is chosen at random from the box, i.e. either coin is chosen with probability $1/2$, and tossed twice. Conditioned on the identity of the coin, the two tosses are independent.

Define the following events:

- Event A: first coin toss is H.
- Event B: second coin toss is H.
- Event C: two coin tosses result in HH.
- Event D: the fair coin is chosen.

For the following statements, decide whether they are true or false.

1. A and B are independent.

   - (selected) True
   - False

2. A and C are independent.

- [x] True
- [ ] False

3. A and B are independent given D.

- [x] True
- [ ] False

4. A and C are independent given D.

- [x] True
- [ ] False

Conditional Independence 2

2 points possible (graded, results hidden)

1. Suppose three random variables X, Y, Z have a joint distribution

$$\mathbf{P}_{X,Y,Z}(x, y, z) = \mathbf{P}_X(x)\, \mathbf{P}_{Z|X}(z \mid x)\, \mathbf{P}_{Y|Z}(y \mid z)\,.$$

Then, X and Y are independent given Z.

- [ ] True
- [x] False

2. Suppose random variables X and Y are independent given Z, then the joint distribution must be of the form

$$\mathbf{P}_{X,Y,Z}(x, y, z) = h(x, z)\, g(y, z)\,,$$

where h, g are some functions.

- [x] True
- [ ] False

5.

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Variance of Difference of Indicators

2 points possible (graded, results hidden)

Let A be an event, and let I_A be the associated indicator random variable (I_A is 1 if A occurs, and zero if A does not occur). Similarly, let I_B be the indicator of another event, B. Suppose that $P(A) = p$, $P(B) = q$, and $P(A \cap B) = r$.

Find the variance of $I_A - I_B$, in terms of p, q, r.

Var $(I_A - I_B) =$ (-p)^2 - (q^2) + (2*p*q) + p - q

$(-p)^2 - \left(q^2\right) + (2 \cdot p \cdot q) + p - q$

6.

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For all problems on this page, use the following setup:

Let N be a positive integer random variable with PMF of the form

$$p_N(n) = \frac{1}{2} \cdot n \cdot 2^{-n}, \qquad n = 1, 2, \ldots.$$

Once we see the numerical value of N, we then draw a random variable K whose (conditional) PMF is uniform on the set $\{1, 2, \ldots, 2n\}$.

Joint PMF

1 point possible (graded, results hidden)

Write down an expression for the joint PMF $p_{N,K}(n, k)$.

For $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2n$:

$p_{N,K}(n, k) =$ (1/4)*2^(-n)

$\left(\frac{1}{4}\right) \cdot 2^{-n}$

Since K has uniform PMF of 2n therefore probability of each one is 1/2n

Marginal Distribution

1.5 points possible (graded, results hidden)

Find the marginal PMF $p_K(k)$ as a function of k. For simplicity, provide the answer **only for the case when k is an even number**. (The formula for when k is odd would be slightly different, and you do not need to provide it).

Hint: You may find the following helpful: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $0 < r < 1$.

For $k = 2, 4, 6, \ldots$:

$p_K(k) =$ 1/2^((k/2)+1)

$\frac{1}{2^{\left(\frac{k}{2}\right)+1}}$

Discrete PMFs

2 points possible (graded, results hidden)

Let A be the event that K is even. Find $P(A|N = n)$ and $P(A)$.

$P(A \mid N = n) =$

$P(A) =$ 1/2^((k/2)+1)

$\frac{1}{2^{\left(\frac{k}{2}\right)+1}}$

Independence 2

0.5 points possible (graded, results hidden)

Is the event A independent of N?

- ( ) yes
- (•) no
- ( ) not enough information to determine

Solutions

Question 1

1. Suppose that $A \subset C$. Then, $P(A \mid C) \geq P(A)$. This is **TRUE**:

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)}{P(C)} \geq P(A), \tag{7.1}$$

since $P(C) \leq 1$.

2. Suppose that $A \subset B$. Then, $P(A \mid C) \leq P(B \mid C)$. This is **TRUE**.

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} \leq \frac{P(B \cap C)}{P(C)} = P(B \mid C)$$

where the inequality follows from $A \cap C \subset B \cap C$.

3. Suppose that $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.This is **FALSE**, with the following counter example: Suppose that A and B are disjoint events with positive probability and that $C = A$. Then, $P(A \mid C) = P(A) > 0$, whereas $P(B \mid C) = 0$.

4. Suppose that $A \subset C$, $B \subset C$, and $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$. This is **TRUE**: Since $A, B \subset C$, we have $P(A \mid C) = \dfrac{P(A)}{P(C)}$ and similarly $P(B \mid C) = \dfrac{P(B)}{P(C)}$. Then, $P(A) \leq P(B)$ implies $P(A \mid C) \leq P(B \mid C)$.

Question 2

1. $\mathbf{P}(\text{Person 2 takes seat 2}) = \mathbf{P}(\text{Person 1 takes seat 1 or 3}) = \frac{2}{3}$.

2.
$$\begin{aligned} & \mathbf{P}(\text{Person 3 takes seat 3}) \\ = & \mathbf{P}(\text{Person 1,2 does not take seat 3}) \\ = & \mathbf{P}(\text{Person 1 takes seat 1 or 4 or 5}) \cdot 1 + \mathbf{P}(\text{Person 1 takes seat 2}) \cdot \mathbf{P}(\text{Person 2 does not take seat 3}) \\ = & \frac{3}{5} \cdot 1 + \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{4}. \end{aligned}$$

Question 3

This is just the mean of a geometric random variable with parameter $1/10$. Hence, $\mathbf{E}(X) = 10$.

Recall $\mathbf{E}(i)$ is the expected number of additional tosses until all 10 numbers are seen (including the last toss) given i distinct numbers have already been seen.

1. $\mathbf{E}(10) = 0$
2. The induction step is as follows. For $i = 1, \ldots 9$:

$$\begin{aligned} \mathbf{E}(i) &= (\mathbf{E}(i) + 1) \times \frac{i}{10} + (\mathbf{E}(i+1) + 1) \times \left(1 - \frac{i}{10}\right) \\ \iff \mathbf{E}(i) &= \mathbf{E}(i+1) + \frac{10}{10 - i}. \end{aligned}$$

Using $\mathbf{E}(10) = 0$ and the induction step, we have

$$\mathbf{E}(0) = \frac{10}{10} + \frac{10}{9} + \ldots + \frac{10}{2} + \frac{10}{1} + 0 \approx 29.28968.$$

Question 4

1. False. Since we do not know whether it is a fair coin or the two-headed one when the coin is being tossed, getting a Heads during one toss increases our belief the the coin is the two-headed one, so that also increases our belief that the other toss also results in a Heads. Or we can also verify by definition: $\mathbf{P}(A \cap B) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{5}{8} \neq \frac{9}{16} = \frac{3}{4} \cdot \frac{3}{4} = \mathbf{P}(A)\mathbf{P}(B)$.
2. False. $\mathbf{P}(A \cap C) = \mathbf{P}(C) \neq \mathbf{P}(A)\mathbf{P}(C)$.
3. True. Conditioned on D, A and B becomes the outcome of two independent fair coin tosses.
4. False. $\mathbf{P}(A \cap C|D) = \mathbf{P}(C|D) \neq \mathbf{P}(A|D)\mathbf{P}(C|D)$.

Question 5

Solution:

$$\begin{aligned} \mathrm{Var}(I_A - I_B) &= \mathbf{E}[(I_A - I_B)^2] - (\mathbf{E}[(I_A - I_B)])^2 \\ &= \mathbf{E}[I_A^2 - 2I_AI_B + I_B^2] - (\mathbf{E}[I_A] - \mathbf{E}[I_B])^2 \\ &= \mathbf{E}[I_A^2] - 2\mathbf{E}[I_AI_B] + \mathbf{E}[I_B^2] - (\mathbf{E}[I_A]) - (\mathbf{E}[I_B])^2 \\ &= \mathbf{E}[I_A] - 2\mathbf{E}[I_AI_B] + \mathbf{E}[I_B] - (\mathbf{E}[I_A]) - (\mathbf{E}[I_B])^2 \\ &= P(A) - 2P(A \cap B) + P(B) - (P(A) - P(B))^2 \\ &= p - 2r + q - (p - q)^2 \end{aligned}$$

Question 6

We are given that:

$$p_{K|N}(k \mid n) = \frac{1}{2n}, \qquad k = 1, 2, \dots, 2n.$$

By definition:

$$p_{N,K}(n,k) = p_{K|N}(k \mid n)\, p_N(n) = \frac{1}{2n}\frac{1}{2} \cdot n \cdot 2^{-n} = (\frac{1}{2})^{n+2}, \qquad n = 1, 2, \dots, \qquad k = 1, 2, \dots, 2n$$

Solution
Observe that in the infinite sum $p_K(k) = \sum_{n=1}^{\infty} p_{N,K}(n,k)$ only the terms from $n = k/2$ and above have non-zero probability. Indeed, $K = k = 4$ has probability 0 if $n < k/2 = 4/2 = 2$.
Hence:

$$\begin{aligned}
p_K(k) &= \sum_{n=k/2}^{\infty} p_{N,K}(n,k) = \sum_{n=k/2}^{\infty} (\frac{1}{2})^{n+2} \\
&= \sum_{n=k/2}^{\infty} (\frac{1}{2})^{n+2} = \frac{1}{4} \sum_{n=k/2}^{\infty} (\frac{1}{2})^{n} \\
&= \frac{1}{4}\Big[\sum_{n=0}^{\infty} (\frac{1}{2})^{n} - \sum_{0}^{k/2-1} (\frac{1}{2})^{n}\Big] \\
&= \frac{1}{4}\Big[\frac{1}{1-\frac{1}{2}} - \frac{1-(\frac{1}{2})^{k/2-1+1}}{1-\frac{1}{2}}\Big] \\
&= (\frac{1}{2})^{k/2+1} \qquad \text{for} k = 2, 4, \dots
\end{aligned}$$

Let A be then event that K is even. We need to check whether $P(A \mid N = n) = P(A)$ is true for the event A to be independent of N. Now because $p_{K|N}(k \mid n)$ is uniform over the $2n$-size set $\{1, 2, \dots, 2n\}$ and there are exactly n even numbers in this set, we have that:

$$P(A \mid N = n) = \frac{n}{2n} = \frac{1}{2}, \qquad n \geq 1. \tag{7.4}$$

Intuitively, knowledge of n does not affect the beliefs about A, and we have independence. A full, formal argument goes as follows:

$$\begin{aligned}
P(A) &= \sum_{n=1}^{\infty} P(A \mid N = n)\, P(N = n) \\
&= \frac{1}{2} \sum_{n=1}^{\infty} P(N = n) = \frac{1}{2},
\end{aligned}$$

where the last step follows because PMFs always sum to 1. So, $P(A \mid N = n) = P(A)$, for all n.
Equivalently, $P(A \text{ and } N = n) = P(A \mid N = n) \cdot P(N = n) = P(A) \cdot P(N = n)$, for all n, which is the defining property of independence.