Unit 4 Discrete Random Variables

Probability Mass Functions and Expectations

A random variable is loosely speaking, a numerical quantity whose value is determined by the outcome of a probabilistic experiment e.g. the weight of a randomly selected person

Discrete: takes values in finite or countable set
Random variable examples:
Bernoulli
Uniform
Binomial
Geometric

Expected value of a random variable aka Expectation (mean)
weighted average of the values of the random variable
weighted on their probabilities

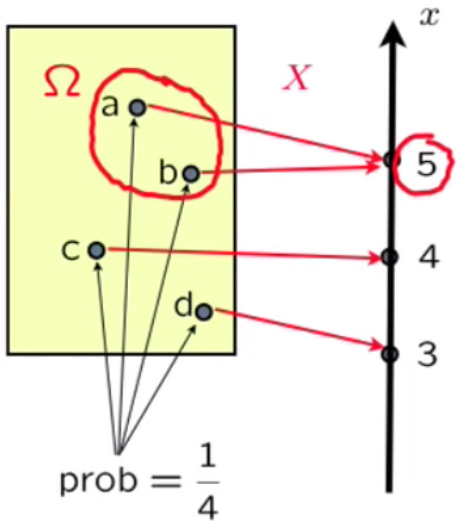
Random variables mathematically: A function from the sample space omega to the real numbers

Notation: random variable X numerical value x

A function of one or more random variables is also a random variable

Probability Mass Function (PMF) aka probability law or probability distribution of X

Probability space of getting different values
P(5) = 1/2 a and b



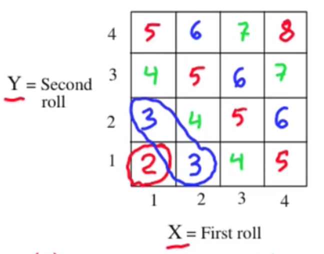
Notation:

$$p_X(x) = \mathbf{P}(X = x) = \mathbf{P}\left(\{\omega \in \Omega \text{ s.t. } X(\omega) = x\}\right)$$

- **Properties:** $p_X(x) \geq 0$ $\qquad \sum_x p_X(x) = 1$

PMF calculation

- Two rolls of a tetrahedral die
- Let every possible outcome have probability 1/16



$Z = X + Y$ $\qquad$ Find $p_Z(z)$ for all z

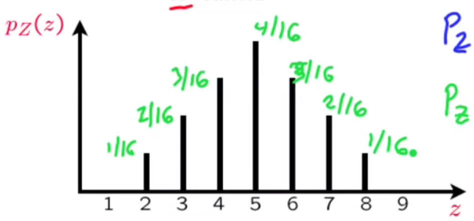
- repeat for all z:
  - collect all possible outcomes for which Z is equal to z
  - add their probabilities

$P_Z(2) = P(Z = 2) = 1/16$

$P_Z(3) = P(Z = 3) = 2/16$

$P_Z(4) = P(Z = 4) = 3/16$

⋮



Exercise: Random variables versus numbers

1/2 points (graded)

Let X be a random variable that takes integer values, with PMF $p_X(x)$. Let Y be another integer-valued random variable and let y be a number.

a) Is $p_X(y)$ a random variable or a number?

Random variable ✖ **Answer:** Number

b) Is $p_X(Y)$ a random variable or a number?

Random variable ✔ **Answer:** Random variable

Solution:

a) Recall that $p_X(\cdot)$ is a function that maps real numbers to real numbers. So, when we give it a numerical argument, y, we obtain a number.

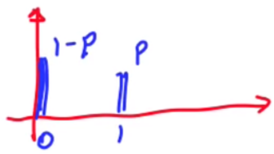
b) In this case, we are dealing with a function, the function being $p_X(\cdot)$, of a random variable Y. And a function of a random variable is a random variable. Intuitively, the "random" value of $p_X(Y)$ is generated as follows: we observe the realized value y of the random variable Y, and then look up the numerical value $p_X(y)$.

The simplest random variable: Bernoulli with parameter $p \in [0,1]$

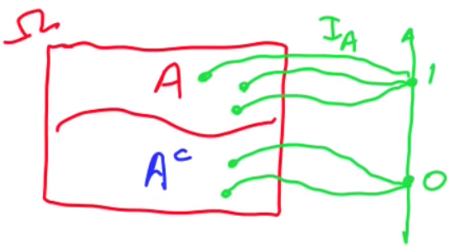
$$X = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1-p \end{cases}$$

$p_X(0) = 1-p$

$p_X(1) = p$



- Models a trial that results in success/failure, Heads/Tails, etc.
- Indicator r.v. of an event A: $I_A = 1$ iff A occurs



$p_{I_A}(1) = P(I_A = 1) = \underbrace{P(A)}_{p}$

Exercise: Indicator variables

2/2 points (graded)

Let A and B be two events (subsets of the same sample space Ω), with nonempty intersection. Let I_A and I_B be the associated indicator random variables.

For each of the two cases below, select one statement that is true.

a) $I_A + I_B$:

is not the indicator random variable of any event ✔

Answer: is not the indicator random variable of any event

b) $I_A \cdot I_B$:

is the indicator variable of the event $A \cap B$ ✔

Answer: is the indicator variable of the event $A \cap B$

(*Bug warning:* In some browsers, the mathematical content in each choice in the dropdown menu may appear duplicated, e.g. $A \cup B$ may show up twice as $A \cup BA \cup B$.)

Solution:

a) If the outcome of the experiment lies in the intersection of the events A and B, then $I_A + I_B$ takes the value of 2. But indicator random variables can take only the values 0 or 1. Therefore, $I_A + I_B$ is not an indicator random variable.

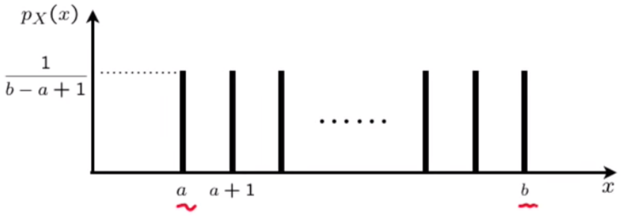
b) Note that $I_A \cdot I_B$ can take only the values 0 or 1. It is equal to 1 if and only if $I_A = 1$ (i.e., event A occurs) and $I_B = 1$ (i.e., event B occurs). Thus, $I_A \cdot I_B$ takes the value of 1 if and only if both A and B occur, and so it is the indicator random variable of the event $A \cap B$.

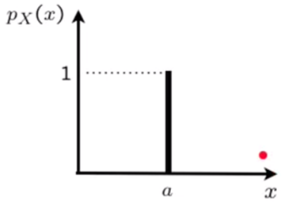
Discrete uniform random variable; parameters $a,\ b$

- **Parameters:** integers $a,\ b$; $\quad a \leq b$
- **Experiment:** Pick one of $a, a+1, \ldots, b$ at random; all equally likely
- **Sample space:** $\{a, a+1, \ldots, b\}$ $\quad$ $b - a + 1$ possible values
- **Random variable** X: $X(\omega) = \omega$ $\quad$ 11:52:26 $\quad \{0, 1, \ldots, 59\}$
- **Model of:** complete ignorance

Special case: $a = b$

constant/deterministic r.v.



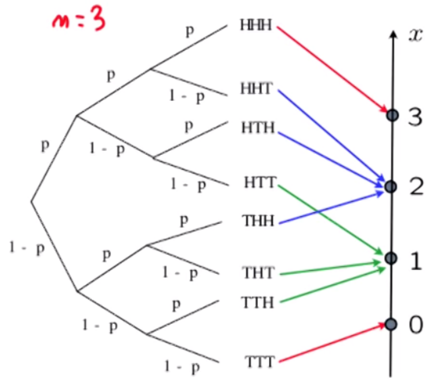


e.g. imagine looking at a clock at time 11:52:26 but only looking at the seconds the probability of getting a number 0 to 59 is equally likely

Special case where a = b is still a random variable in the mathematical sense, it just so happens that it can only be one value in this case

Binomial random variable; **parameters:** positive integer n; $p \in [0, 1]$

- **Experiment:** n independent tosses of a coin with $\mathbf{P}(\text{Heads}) = p$
- **Sample space:** Set of sequences of H and T, of length n
- **Random variable** X**:** number of Heads observed
- **Model of:** number of successes in a given number of independent trials



$$p_X(2) = P(X=2)$$
$$= P(HHT) + P(HTH) + P(THH)$$
$$= 3p^2(1-p) = \binom{3}{2} p^2 (1-p)$$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k = 0, 1, \ldots, n$$

Same formula as earlier but with slightly different notation

Exercise: The binomial PMF

2/2 points (graded)

You roll a fair six-sided die (all 6 of the possible results of a die roll are equally likely) 5 times, independently. Let X be the number of times that the roll results in 2 or 3. Find the numerical values of the following.

a) $p_X(2.5) =$ 0 ✔ **Answer:** 0

b) $p_X(1) =$ 0.3292 ✔ **Answer:** 0.32922

Solution:

a) A value of 2.5 is not possible for X since the number of rolls must be an integer, and therefore $p_X(2.5) = 0$.

b) For each die roll, there is a probability $2/6 = 1/3$ of obtaining a 2 or a 3. Hence, the random variable X is binomial with parameters $n = 5$ and $p = 1/3$, so that $p_X(1) = \binom{5}{1} \cdot (1/3) \cdot (2/3)^4 \approx 0.32922$.

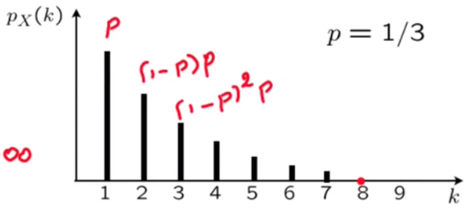
Geometric random variable; parameter p: $0 < p \leq 1$

- **Experiment:** infinitely many independent tosses of a coin; $\mathbf{P}(\text{Heads}) = p$
- **Sample space:** Set of infinite sequences of H and T $\quad \underbrace{TTTTH}HT\cdots$
- **Random variable** X**:** number of tosses until the first Heads $\quad X = 5$
- **Model of:** waiting times; number of trials until a success

$$p_X(k) = \mathbf{P}(X=k) = \mathbf{P}(\underbrace{T\cdots T}_{k-1}H) = (1-p)^{k-1}p \qquad k = 1, 2, 3, \ldots$$

$\mathbf{P}(\text{no Heads ever})$ $\leq \mathbf{P}(\underbrace{T\cdots T}_{k}) = (1-p)^k \xrightarrow{k\to\infty} 0$

$(TTT\cdots)$ "$X = \infty$" $= 0$



P(Tails forever) becomes 0 as k goes to infinity

Exercise: Geometric random variables

0/2 points (graded)

Let X be a geometric random variable with parameter p. Find the probability that $X \geq 10$. Express your answer in terms of p using standard notation (click on the "STANDARD NOTATION" button below.)

$\mathbf{P}(X \geq 10) =$ p/(1-((1-p)^9)) ✖ **Answer:** (1-p)^9

$$\frac{p}{1-((1-p)^9)}$$

STANDARD NOTATION

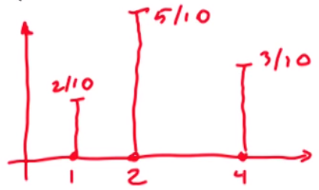
Solution:

We can calculate the desired probability by adding the probabilities of the events $\{X = 10\}$, $\{X = 11\}$, $\{X = 12\}$, etc., and using the formula for the sum of a geometric series. However, we can get the answer in an easier way, using the interpretation of geometric random variables as the number of trials until the first success. The event $\{X \geq 10\}$ is the event that the the first 9 trials resulted in failure, and therefore its probability is $(1-p)^9$.

Expectation/mean of a random variable

- **Motivation:** Play a game 1000 times. Random gain at each play described by:

$$X = \begin{cases} 1, & \text{w.p. } 2/10 \quad \sim 200 \\ 2, & \text{w.p. } 5/10 \quad \sim 500 \\ 4, & \text{w.p. } 3/10 \quad \sim 300 \end{cases}$$



- "Average" gain:

$$\frac{1\cdot 200 + 2\cdot 500 + 4\cdot 300}{1000}$$

$$= 1\cdot\frac{2}{10} + 2\cdot\frac{5}{10} + 4\cdot\frac{3}{10}$$

- **Definition:** $\mathrm{E}[X] = \sum_x x p_X(x)$

- **Interpretation:** Average in large number of independent repetitions of the experiment

- Caution: If we have an infinite sum, it needs to be well-defined. We assume $\sum_x |x|\, p_X(x) < \infty$

Expectation of a Bernoulli r.v.

$$X = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1-p \end{cases}$$

$$E[X] = 1\cdot p + 0\cdot(1-p) = p$$

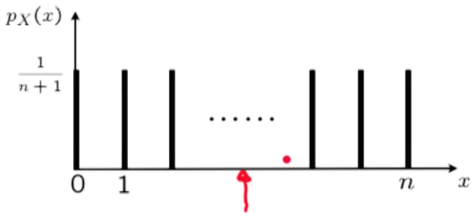
If X is the indicator of an event A, $X = I_A$:

$X = 1$ iff A occurs $\qquad p = P(A)$

$$E[I_A] = P(A)$$

Expectation of a uniform r.v.

- Uniform on $0, 1, \ldots, n$



- **Definition:** $\mathbf{E}[X] = \sum_x x p_X(x)$

$$\mathbf{E}[X] = 0\cdot\frac{1}{n+1} + 1\cdot\frac{1}{n+1} + \cdots + n\frac{1}{n+1}$$

$$= \frac{1}{n+1}(0+1+\cdots+n) = \frac{1}{n+1}\cdot\frac{n(n+1)}{2} = \frac{n}{2}$$

Elementary properties of expectations

- **Definition:** $\underbrace{\mathbf{E}[X]}_{\geq 0} = \sum_x \underbrace{x}_{\geq 0}\,\underbrace{p_X(x)}_{\geq 0}$

- If $X \geq 0$, then $\mathbf{E}[X] \geq 0$

for all ω: $X(\omega) \geq 0$

- If $a \leq X \leq b$, then $a \leq \mathbf{E}[X] \leq b$

for all ω: $a \leq X(\omega) \leq b$

$$E[X] = \sum_x x\,p_X(x) \geq \sum_x a\,p_X(x)$$

$$= a \sum_x p_X(x) = a \cdot 1 = a$$

- If c is a constant, $\mathbf{E}[c] = c$

$$E[c] = c \cdot p(c) = c$$

Exercise: Random variables with bounded range

3/3 points (graded)

Suppose a random variable X can take any value in the interval $[-1,2]$ and a random variable Y can take any value in the interval $[-2,3]$.

a) The random variable $X-Y$ can take any value in an interval $[a,b]$. Find the values of a and b:

$a =$ -4 ✔

$b =$ 4 ✔

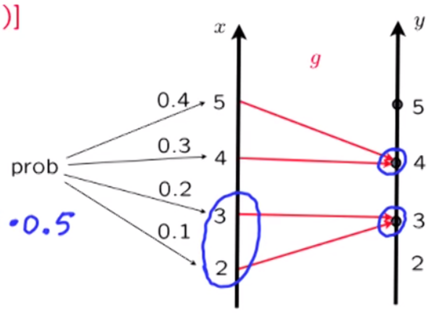
b) Can the expected value of $X+Y$ be equal to 6?

No ✔

The expected value rule, for calculating $\mathbf{E}[g(X)]$

- Let X be a r.v. and let $Y = g(X)$
- Averaging over y: $\quad \mathbf{E}[Y] = \sum_y y p_Y(y)$

  $3\cdot(0.1+0.2) + 4\cdot(0.3+0.4)$
- Averaging over x: $3\cdot 0.1 + 3\cdot 0.2 + 4\cdot 0.3 + 4\cdot 0.5$



$$\boxed{\mathbf{E}[Y] = \mathbf{E}[g(X)] = \underline{\sum_x g(x) p_X(x)}}$$

Proof: $\displaystyle\sum_y \sum_{x:g(x)=y} g(x) p_X(x)$

$\displaystyle = \sum_y \sum_{x:g(x)=y} y\, p_X(x) = \sum_y y \underline{\sum_{x:g(x)=y} p_X(x)}$

$\displaystyle = \sum_y y\, p_Y(y) = E[Y]$

- $\mathbf{E}[X^2] = \displaystyle\sum_x x^2 p_X(x)$

  $g(x) = x^2$
- Caution: In general, $\mathbf{E}[g(X)] \neq g(\mathbf{E}[X])$

  $E[X^2] \neq (E[X])^2$

Linearity of expectation: $\mathbf{E}[aX+b] = a\mathbf{E}[X]+b$

X = salary $\quad E[X]$ = average salary

Y = new salary $= 2X+100 \quad E[Y] = E[2X+100] = 2E[X]+100$

- Intuitive
- **Derivation**, based on the expected value rule: $g(x) = ax+b$, $Y = g(X)$

$$E[Y] = \sum_x g(x) p_X(x)$$

$$= \sum_x (ax+b) p_X(x) = a\sum_x x\, p_X(x) + b\underbrace{\sum_x p_X(x)}_{1}$$

$$= aE[X] + b$$

$$E[g(X)] = g(E[X])\ ? \quad \text{exceptional } g$$

The blue equality is only true for linear functions

Exercise: Linearity of expectations

3/3 points (graded)

The random variable X is known to satisfy $\mathbf{E}[X] = 2$ and $\mathbf{E}[X^2] = 7$. Find the expected value of $8 - X$ and of $(X-3)(X+3)$.

a) $\mathbf{E}[8-X] =$ 6 ✔

b) $\mathbf{E}[(X-3)(X+3)] =$ -2 ✔

Variance; Conditioning on an event; Multiple random variables

Variance - quantifies the spread of a probability mass function (PMF)
We look at the distance from the mean
If we look at the average distance from the mean it is always = 0 so it not informative
However if we look at the average of the distance^2 it is informative

- **Definition of variance:** $\text{var}(X) = \mathbf{E}[(X-\mu)^2]$

Variance is always positive

$$\boxed{\textbf{Standard deviation:} \quad \sigma_X = \sqrt{\text{var}(X)}}$$

Variance is not in the same units as the original random variable but the square root of it is (standard deviation)

Properties of the variance

- Notation: $\mu = \mathbf{E}[X]$

$$\boxed{\text{var}(aX + b) = a^2\text{var}(X)}$$

$$\text{var}(3-4x) = (-4)^2\,\text{var}(x) = 16\ \text{var}(x)$$

- Let $Y = X + b$ $\quad \nu = E[Y] = \mu + b$

$$\text{var}(Y) = E[(Y-\nu)^2] = E[(X+\cancel{b} - (\mu+\cancel{b}))^2] = E[(X-\mu)^2] = \text{var}(X)$$

- Let $Y = aX$ $\quad \nu = E[Y] = a\mu$

$$\text{var}(Y) = E[(aX - a\mu)^2] = E[a^2(X-\mu)^2] = a^2 E[(X-\mu)^2] = a^2\,\text{var}(X)$$

$$\boxed{\text{A useful formula:} \quad \text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2}$$

$$\text{var}(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$

$$= E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - (E[X])^2$$

First calculation in red shows that adding a constant to a random variable does not change its variance (the b's cancel out)

Second calculation in red shows that multiplying a random variable by a constant would be multiplying the variance by a^2

Useful formula is a quick and easy way to calculate variance

Note: The derivation of the useful alternative formula for the variance, near the end of the video, uses a linearity property for multiple random variables (in this case, the variables X and X^2), which we have not yet discussed.

A derivation that does not rely on this linearity property, goes as follows: This expression is verified as follows:

$$\begin{aligned}
\operatorname{var}(X) &= \sum_x \left(x - \mathbf{E}[X]\right)^2 p_X(x) \\
&= \sum_x \left(x^2 - 2x\mathbf{E}[X] + \left(\mathbf{E}[X]\right)^2\right) p_X(x) \\
&= \sum_x x^2 p_X(x) - 2\mathbf{E}[X] \sum_x x p_X(x) + \left(\mathbf{E}[X]\right)^2 \sum_x p_X(x) \\
&= \mathbf{E}[X^2] - 2\left(\mathbf{E}[X]\right)^2 + \left(\mathbf{E}[X]\right)^2 \\
&= \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2.
\end{aligned}$$

(The sum of all x's multiplied by their probability is the expected value of x)

Exercise: Variance properties

0/1 point (graded)

Is it always true that $\mathbf{E}[X^2] \geq \left(\mathbf{E}[X]\right)^2$?

No ✖ **Answer:** Yes

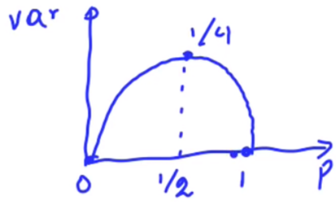
Solution:

We know that variances are always nonnegative and that $\mathsf{Var}(X) = \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2$. Therefore, $0 \leq \mathsf{Var}(X) = \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2$, or, equivalently, $\mathbf{E}[X^2] \geq \left(\mathbf{E}[X]\right)^2$.

Variance of the Bernoulli

$$X = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1-p \end{cases}$$

$E[X] = p$



$$\text{var}(X) = \sum_x (x - \mathbf{E}[X])^2 p_X(x) = (1-p)^2 p + (0-p)^2 \cdot (1-p)$$
$$= p - 2p^2 + \cancel{p^3} + p^2 - \cancel{p^3} = p - p^2 = p(1-p)$$

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = E[X] - (E[X])^2 = p - p^2 = \boxed{p(1-p)}$$

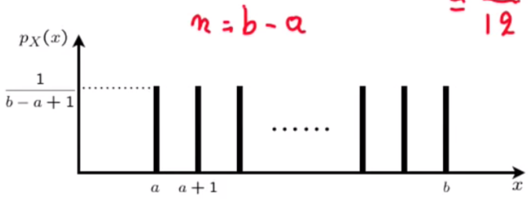
$X^2 = X$

X^2 = X here only because it is a Bernoulli random variable i.e. is a 0 or 1
Both methods show what the variance is
Plotting variance as a function of p we see that variance peaks when p is 0.5 i.e. when a coin toss is fair it is the most random

Variance of the uniform

$$\text{var}(X) = E[X^2] - (E[X])^2 = \frac{1}{n+1}\underbrace{(0^2 + 1^2 + 2^2 + \dots + n^2)}_{\frac{1}{6}n(n+1)(2n+1)} - \left(\frac{n}{2}\right)^2$$
$$= \frac{1}{12} n(n+2)$$

$n = b - a$



$$\text{Var}(X) = \frac{1}{12}(b-a)(b-a+2)$$

second is a more general form where it doesn't start from 0 necessarily
This shift is equivalent to adding a constant to the r.v. therefore the variance does not change
We just have to substitute b-a in for n

Exercise: Variance of the uniform

2/2 points (graded)

Suppose that the random variable X takes values in the set $\{0, 2, 4, 6, \ldots, 2n\}$ (the even integers between 0 and $2n$, inclusive), with each value having the same probability. What is the variance of X? *Hint:* Consider the random variable $Y = X/2$ and recall that the variance of a uniform random variable on the set $\{0, 1, \ldots, n\}$ is equal to $n\,(n+2)/12$.

Express your answer in terms of n using standard notation. Remember to write '*' for all multiplications and to include parentheses where necessary.

Var $(X) =$ (n*(n+2))/3 ✔

$$\frac{n \cdot (n+2)}{3}$$

Conditioning

Creating a new PMF based on information that an event has happened such as A
This is just another PMF and follows the same rules as ordinary PMFs except probabilities are replaced by conditional probabilities

Conditional PMF and expectation, given an event

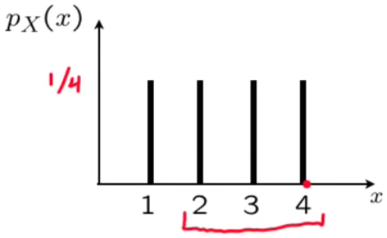
- Condition on an event $A \Rightarrow$ use conditional probabilities

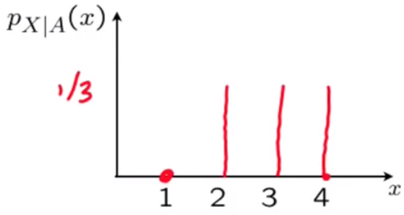
assume
$P(A) > 0$

$p_X(x) = \mathbf{P}(X = x)$	$\underline{p_{X\|A}}(x) = \underline{\mathbf{P}(X = x \mid A)}$
$\sum_x p_X(x) = 1$	$\sum_x p_{X\|A}(x) = 1$
$\mathbf{E}[X] = \sum_x x p_X(x)$	$\mathbf{E}[X \mid A] = \sum_x x p_{X\|A}(x)$
$\mathbf{E}[g(X)] = \sum_x g(x)\, p_X(x)$	$\mathbf{E}[g(X) \mid A] = \sum_x g(x)\, p_{X\|A}(x)$

Example of conditioning

- Let $A = \{X \geq 2\}$





$\mathbf{E}[X] = 2.5$

$\text{var}(X) = \frac{1}{12}(b-a)(b-a+2)$

$= \frac{1}{12}\, 3 \cdot 5 = \frac{5}{4}$

$\mathbf{E}[X \mid A] = 3$

$\text{var}(X \mid A) = \frac{1}{3}(4-3)^2 + \frac{1}{3}(3-3)^2$

$+ \frac{1}{3}(2-3)^2 = \frac{2}{3}$

In the last example, we saw that the conditional distribution of X, which was a uniform over a smaller range (and in some sense, less uncertain), had a smaller variance, i.e., $\mathsf{Var}\,(X \mid A) \leq \mathsf{Var}\,(X)$. Here is an example where this is not true. Let Y be uniform on $\{0, 1, 2\}$ and let B be the event that Y belongs to $\{0, 2\}$.

a) What is the variance of Y?

$\mathsf{Var}\,(Y) =$ 8/12 ✔

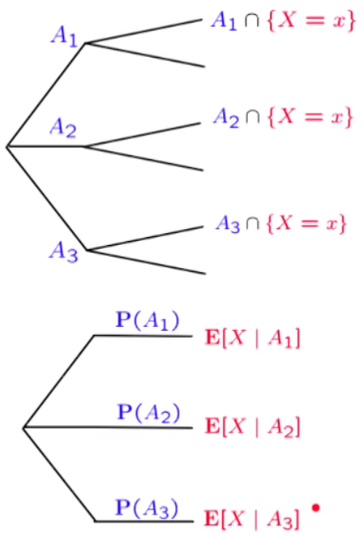
b) What is the conditional variance $\mathsf{Var}\,(Y \mid B)$?

$\mathsf{Var}\,(Y \mid B) =$ 1 ✔

Conditional probabilities allow us to divide and conquer

Reminder of Total Probability Theorem
translated to total expectation theorem

Total expectation theorem



$$\mathbf{P}(B) = \mathbf{P}(A_1)\,\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\,\mathbf{P}(B \mid A_n)$$

$B = \{X = x\}$

$$p_X(x) = \mathbf{P}(A_1)\,p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\,p_{X|A_n}(x)$$

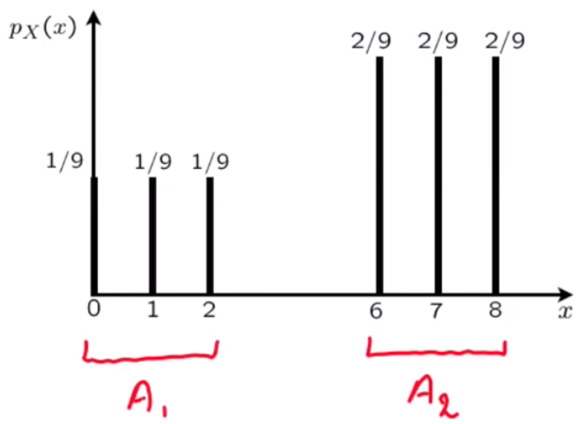
for all x

$$\underbrace{\sum_x x\,p_X(x)}_{E[X]} = P(A_1)\underbrace{\sum_x x\,p_{X|A_1}(x)}_{E[X|A_1]} + \cdots$$

$$\boxed{\mathbf{E}[X] = \mathbf{P}(A_1)\,\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\,\mathbf{E}[X \mid A_n]}$$

Each event is weighted by its probability

Total expectation example



$P(A_1) = \frac{1}{3}$

$P(A_2) = \frac{2}{3}$

$E[X|A_1] = 1$

$E[X|A_2] = 7$

$E[X] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 7$

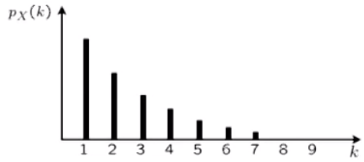
Geometric PMG, memorylessness and expectation

memorylessness: coin tosses have no ‘memory’, one coin toss isn’t affected by the last

Conditioning a geometric random variable

- X: number of independent coin tosses until first head; $\mathbf{P}(H) = p$

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$



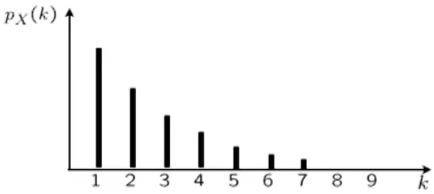
Memorylessness:
Number of **remaining** coin tosses, conditioned on Tails in the first toss, is **Geometric**, with parameter p

Conditioned on $X > n$, $X - n$ is geometric with parameter p

$$p_{X-1|X>1}(3) = P(X-1=3 \mid X>1) = P(T_2 T_3 H_4 \mid T_1) = P(T_2 T_3 H_4)$$
$$= (1-p)^2 p = p_X(3)$$
$$p_{X-1|X>1}(k) = p_X(k) = p_{X-n|X>n}(k).$$

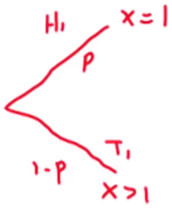
X>n is saying that the first n tosses were tails

The mean of the geometric



$$\mathbf{E}[X] = \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

$$\boxed{\mathbf{E}[X] = \frac{1}{p}.}$$



$$E[X] = 1 + E[X-1]$$
$$= 1 + p \cdot E[X-1 \mid X=1] + (1-p)E[X-1 \mid X>1]$$
$$= 1 + 0 + (1-p)E[X]$$

Exercise: Total expectation calculation

2/2 points (graded)

We have two coins, A and B. For each toss of coin A, we obtain Heads with probability $1/2$; for each toss of coin B, we obtain Heads with probability $1/3$. All tosses of the same coin are independent. We select a coin at random, where the probabilty of selecting coin A is $1/4$, and then toss it until Heads is obtained for the first time.

The expected number of tosses until the first Heads is: 2.75 ✔

For A E[h] = 1 / (1/2) x P(selecting A) = 1/4 = 0.5
For B E[h] = 1 / (1/3) x P(selecting B) = 3/4 = 2.25
0.5 + 2.25 = 2.75

Exercise: Memorylessness of the geometric

2/2 points (graded)

Let X be a geometric random variable, and assume that $\textsf{Var}\,(X) = 5$.

a) What is the conditional variance $\textsf{Var}\,(X-4 \mid X>4)$?

$\textsf{Var}\,(X-4 \mid X>4) =$ 5 ✔ **Answer:** 5

b) What is the conditional variance $\textsf{Var}\,(X-8 \mid X>4)$?

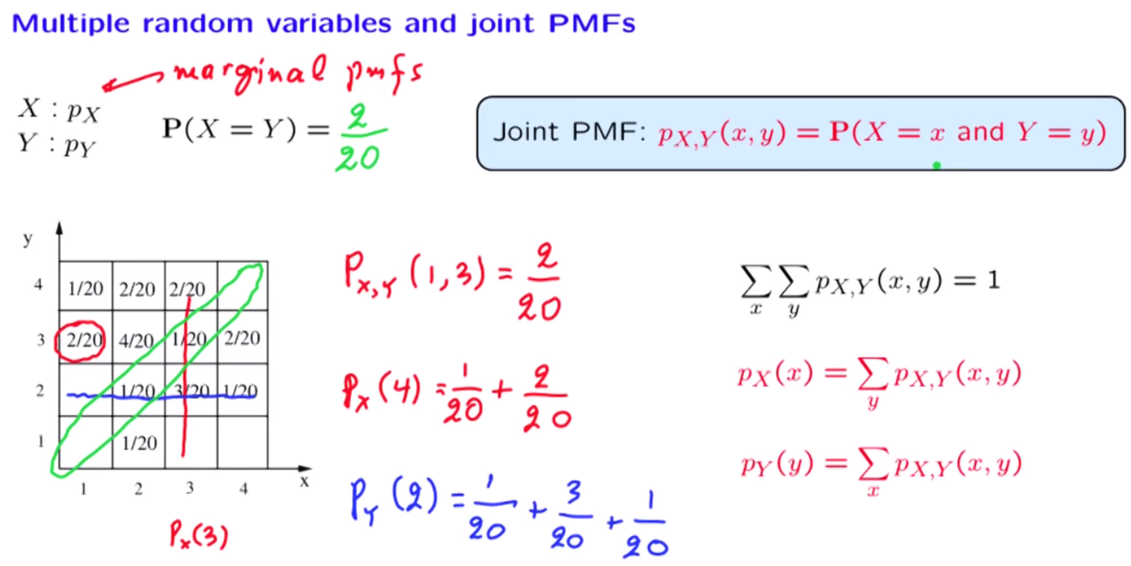
$\textsf{Var}\,(X-8 \mid X>4) =$ 5 ✔ **Answer:** 5

Solution:

a) The conditional distribution of $X-4$ given $X>4$ is the same geometric PMF that describes the distribution of X. Hence $\textsf{Var}\,(X-4 \mid X>4) = \textsf{Var}\,(X) = 5$.

b) In the conditional model (i.e., given that $X>4$), the random variables $X-4$ and $X-8$ differ by a constant. Hence they have the same variance and the answer is again 5.

Join PMFs and the expected value rule



Once we have a join PMF we can answer questions about the marginal PMFs (single)

More than two random variables

$$\boxed{p_{X,Y,Z}(x,y,z) = \mathbf{P}(X = x \text{ and } Y = y \text{ and } Z = z)}$$

$$\sum_x \sum_y \sum_z p_{X,Y,Z}(x,y,z) = 1$$

$$p_X(x) = \sum_y \sum_z p_{X,Y,Z}(x,y,z)$$

$$p_{X,Y}(x,y) = \sum_z p_{X,Y,Z}(x,y,z)$$

Functions of multiple random variables

$Z = g(X,Y)$

PMF: $p_Z(z) = \mathbf{P}(Z = z) = \mathbf{P}\big(g(X,Y) = z\big) = \sum_{(x,y):\, g(x,y) = z} p_{X,Y}(x,y)$

$$\boxed{\textbf{Expected value rule: } \mathbf{E}[g(X,Y)] = \sum_x \sum_y \underbrace{g(x,y)} \underline{p_{X,Y}(x,y)}}$$

$E[g(x)]$

Summing over x,y pairs to give an expected value of a random variable given by the function g(x,y)

Exercise: Joint PMF calculation

2/2 points (graded)

The random variable V takes values in the set $\{0, 1\}$ and the random variable W takes values in the set $\{0, 1, 2\}$. Their joint PMF is of the form

$$p_{V,W}(v, w) = c \cdot (v + w),$$

where c is some constant, for v and w in their respective ranges, and is zero everywhere else.

a) Find the value of c.

$c =$ 0.1111 ✔ **Answer:** 0.11111

b) Find $p_V(1)$.

$p_V(1) =$ 2/3 ✔ **Answer:** 0.66667

Solution:

a) The sum of the entries of the PMF is $c \cdot (0+0) + c \cdot (0+1) + c \cdot (0+2) + c \cdot (1+0) + \ldots = 9c$. Since this sum must be equal to 1, we have $c = 1/9$.

b)

$$p_V(1) = \sum_{w=0}^{2} p_{V,W}(1, w) = p_{V,W}(1, 0) + p_{V,W}(1, 1) + p_{V,W}(1, 2) = \frac{1}{9}(1 + 2 + 3) = \frac{6}{9}.$$

Let X and Y be discrete random variables. For each one of the formulas below, state whether it is true or false.

a) $\mathbf{E}[X^2] = \sum_x x p_X(x^2)$

False ✔ **Answer:** False

b) $\mathbf{E}[X^2] = \sum_x x^2 p_X(x)$

True ✔ **Answer:** True

c) $\mathbf{E}[X^2] = \sum_x x^2 p_{X,Y}(x)$

True ✖ **Answer:** False

d) $\mathbf{E}[X^2] = \sum_x x^2 p_{X,Y}(x, y)$

False ✔ **Answer:** False

e) $\mathbf{E}[X^2] = \sum_x \sum_y x^2 p_{X,Y}(x, y)$

False ✖ **Answer:** True

f) $\mathbf{E}[X^2] = \sum_z z p_{X^2}(z)$

True ✔ **Answer:** True

a) False. This does not follow from any of our formulas.

b) True. This is the expected value rule for a function of a single random variable.

c) False. This is syntactically wrong since the function $p_{X,Y}$ needs two arguments.

d) False. The left-hand side is a number whereas the right-hand side is actually a function of y.

e) True. This is the expected value rule

$$\mathbf{E}[g(X, Y)] = \sum_x \sum_y g(x, y)\, p_{X,Y}(x, y),$$

for the function $g(x, y) = x^2$.

f) True. This is just the definition of the expectation $\mathbf{E}[Z] = \sum_z z p_Z(z)$, where Z is the random variable X^2.

Linearity of expectations and the mean of the binomial

Derivation of linearity of expectations

Linearity of expectations

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$$

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$E[X+Y] = E[g(X,Y)]$$

$$(g(x,y) = x+y)$$

$$= \sum_x \sum_y (x+y)\, p_{X,Y}(x,y)$$

$$= \underbrace{\sum_x \sum_y x\, p_{X,Y}(x,y)} + \sum_x \sum_y y\, p_{X,Y}(x,y)$$

$$= \sum_x x \underbrace{\sum_y p_{X,Y}(x,y)} + \sim\sim\sim$$

$$= \sum_x x\, p_X(x) + \sum_y y\, p_Y(y) = E[X] + E[Y]$$

And for multiple variables

$$\boxed{\mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]}$$

The mean of the binomial

- X: binomial with parameters n, p
  - number of successes in n independent trials

$$\mathbf{E}[X] = \sum_{k=0}^{n} k \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{p_X(k)}$$

$$\boxed{\mathbf{E}[X] = np}$$

$X_i = 1$ if ith trial is a success; — p

$X_i = 0$ otherwise — $1-p$

(indicator variable)

$$X = X_1 + \cdots + X_n$$

$$E[X] = \underbrace{E[X_1]}_{p} + \cdots + \underbrace{E[X_n]}_{p} = np$$

Exercise: Linearity of expectations drill

1/1 point (graded)

Suppose that $\mathbf{E}[X_i] = i$ for every i. Then,

$\mathbf{E}[X_1 + 2X_2 - 3X_3] =$ -4

Exercise: Using linearity of expectations

2/2 points (graded)

We have two coins, A and B. For each toss of coin A, we obtain Heads with probability $1/2$; for each toss of coin B, we obtain Heads with probability $1/3$. All tosses of the same coin are independent.

We toss coin A until Heads is obtained for the first time. We then toss coin B until Heads is obtained for the first time with coin B.

The expected value of the total number of tosses is: 5 **Answer:** 5

Solution:

Let T_A and T_B be the number of tosses of coins A and B, respectively. We know that T_A is geometric with parameter $p = 1/2$, so that $\mathbf{E}[T_A] = 1/p = 1/(1/2) = 2$. Similarly, $\mathbf{E}[T_B] = 3$. The total number of coin tosses is $T_A + T_B$. Using linearity,

$$\mathbf{E}[T_A + T_B] = \mathbf{E}[T_A] + \mathbf{E}[T_B] = 2 + 3 = 5.$$

Conditioning on a random variable; Independence of random variables

Conditional PMFs

$A = \{Y = y\}$

$p_{X|A}(x \mid A) = \mathbf{P}(X = x \mid A)$ $\qquad$ $p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y) = \dfrac{P(X = x, Y = y)}{P(Y = y)}$

$$\boxed{p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \qquad \text{defined for } y \text{ such that } p_Y(y) > 0}$$

$$\sum_x p_{X|Y}(x \mid y) = 1$$

$Y = 2$ $\qquad$ $P_Y(2) = 5/20$

$P_{X|Y}(1|2) = 0$

$P_{X|Y}(2|2) = 1/5$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X,Y}(x,y) = p_Y(y)\, p_{X|Y}(x \mid y)$

$p_{X,Y}(x,y) = p_X(x)\, p_{Y|X}(y \mid x)$

Like scaling the original probabilities to a new 'world' i.e. 5 as the denominator

Conditional PMFs involving more than two r.v.'s

- Self-explanatory notation

$p_{X|Y,Z}(x \mid y, z) = P(X = x \mid Y = y, Z = z) = \dfrac{P(X = x, Y = y, Z = z)}{P(Y = y, Z = z)} = \dfrac{P_{X,Y,Z}(x,y,z)}{P_{Y,Z}(y,z)}$

$p_{X,Y|Z}(x, y \mid z) = P(X = x, Y = y \mid Z = z)$

- Multiplication rule

$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B \mid A)\,\mathbf{P}(C \mid A \cap B)$

$A = \{X = x\}$ $\quad$ $B = \{Y = y\}$ $\quad$ $C = \{Z = z\}$

$p_{X,Y,Z}(x,y,z) = p_X(x)\, p_{Y|X}(y \mid x)\, p_{Z|X,Y}(z \mid x, y)$

Multiplication rule at the bottom is the P(X=x) happening x P(Y=y | X=x) x P(Z=z | X=x and Y=y)

Note: The notation $p_{X|A}(x \mid A)$ should be $p_{X|A}(x)$.

For each of the formulas below, state whether it is true or false.

a) $p_{X,Y,Z}(x,y,z) = p_Y(y)\; p_{Z|Y}(z \mid y)\; p_{X|Y,Z}(x \mid y, z)$

True ✔ **Answer:** True

b) $p_{X,Y|Z}(x, y \mid z) = p_X(x)\ p_{Y|Z}(y \mid z)$

False ✔ **Answer:** False

c) $p_{X,Y|Z}(x, y \mid z) = p_{X|Z}(x \mid z)\ p_{Y|X,Z}(y \mid x, z)$

True ✔ **Answer:** True

d) $\sum_x p_{X,Y|Z}(x, y \mid z) = 1$

False ✔ **Answer:** False

e) $\sum_x \sum_y p_{X,Y|Z}(x, y \mid z) = 1$

True ✔ **Answer:** True

f) $p_{X,Y|Z}(x, y \mid z) = \dfrac{p_{X,Y,Z}(x, y, z)}{p_Z(z)}$

True ✔ **Answer:** True

g) $p_{X|Y,Z}(x \mid y, z) = \dfrac{p_{X,Y,Z}(x, y, z)}{p_{Y,Z}(y, z)}$

True ✔ **Answer:** True

Conditional expectation

$A = \{Y = y\}$

$\mathbf{E}[X] = \sum_x x p_X(x)$ $\qquad \mathbf{E}[X \mid A] = \sum_x x\, p_{X|A}(x)$ $\qquad \mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$

- Expected value rule

$\mathbf{E}[g(X)] = \sum_x g(x)\, p_X(x)$ $\qquad \mathbf{E}[g(X) \mid A] = \sum_x g(x)\, p_{X|A}(x)$

$\mathbf{E}[g(X) \mid Y = y] = \sum_x g(x)\, p_{X|Y}(x \mid y)$

Total probability and expectation theorems

- $A_1, \ldots, A_n$: partition of Ω $\qquad Y = \{y_1, \ldots, y_n\}$ $\qquad A_i = \{Y = y_i\}$

- $p_X(x) = \mathbf{P}(A_1)\, p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\, p_{X|A_n}(x)$

$$p_X(x) = \sum_y p_Y(y)\, p_{X|Y}(x \mid y)$$

- $\mathbf{E}[X] = \mathbf{P}(A_1)\, \mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\, \mathbf{E}[X \mid A_n]$

$$\boxed{\mathbf{E}[X] = \sum_y p_Y(y)\, \mathbf{E}[X \mid Y = y]}$$

- Fine print:
  Also valid when Y is a discrete r.v. that ranges over an infinite set,
  as long as $\mathbf{E}[|X|] < \infty$

For each of the formulas below, state whether it is true or false.

1) $\mathbf{E}[g(X,Y) \mid Y=2] = \sum_x g(x,y)\, p_{X,Y}(x,y)$

False ✔ **Answer:** False

2) $\mathbf{E}[g(X,Y) \mid Y=2] = \sum_x g(x,y)\, p_{X,Y}(x,2)$

False ✔ **Answer:** False

3) $\mathbf{E}[g(X,Y) \mid Y=2] = \sum_x g(x,2)\, p_{X,Y}(x,2)$

False ✔ **Answer:** False

4) $\mathbf{E}[g(X,Y) \mid Y=2] = \sum_x g(x,2)\, p_{X|Y}(x \mid 2)$

True ✔ **Answer:** True

5) $\mathbf{E}[g(X,Y) \mid Y=2] = \sum_x g(x,2) \dfrac{p_{X,Y}(x,2)}{p_Y(2)}$

False ✖ **Answer:** True

6) $\mathbf{E}[g(X,Y) \mid Y=2] = \sum_x \sum_y g(x,y)\, p_{X,Y|Y}(x,y \mid 2)$

False ✖ **Answer:** True

1-3) There is no reason for any of the first three formulas to be true.

4) True. This is just the usual expected value rule, in a model in which the event $\{Y = 2\}$ is known to have occurred. Given the information that $Y = 2$, the function $g(x, y)$ is replaced by $g(x, 2)$, and we are dealing with a function $g(x, 2)$ of a single variable x. We apply the expected value rule for a function of a single variable, but since we are within a conditional model, we need to use the conditional PMF of X.

5) True. This is the same as the fourth statement, except that we have substituted in the definition of $p_{X|Y}(x \mid 2)$.

6) True. This is just the expected value rule for a function of two variables, applied within a conditional universe where the event $\{Y = 2\}$ is known to have occurred.

Notice that $p_{X,Y|Y}(x, y \mid 2)$ will be zero for any $y \neq 2$. And for $y = 2$,

$$p_{X,Y|Y}(x, 2 \mid 2) = \mathbf{P}(X = x, Y = 2 \mid Y = 2) = \mathbf{P}(X = x \mid Y = 2) = p_{X|Y}(x \mid 2),$$

so that the sixth formula agrees with the fourth one.

Independence of random variables

Independence

- of two events: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$ $\qquad$ $\mathbf{P}(A \mid B) = \mathbf{P}(A)$
- of a r.v. and an event: $\mathbf{P}(X = x \text{ and } A) = \mathbf{P}(X = x) \cdot \mathbf{P}(A)$, for all x

$p_{X|A}(x) = p_X(x)$, for all x $\qquad$ $\mathbf{P}(A \mid X = x) = \mathbf{P}(A)$, for all x

- of two r.v.'s: $\mathbf{P}(X = x \text{ and } Y = y) = \mathbf{P}(X = x) \cdot \mathbf{P}(Y = y)$, for all x, y

$p_{X|Y}(x|y) = p_X(x)$ $\qquad$ $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$, for all x, y

$p_{Y|X}(y|x) = p_Y(y)$

X, Y, Z are **independent** if:
$p_{X,Y,Z}(x, y, z) = p_X(x)\, p_Y(y)\, p_Z(z)$, for all x, y, z

The occurrence of one event happening does not affect our beliefs(probabilities) of other events happening

Exercise: Independence

3/5 points (graded)

Let X, Y, and Z be discrete random variables.

a) Suppose that Z is identically equal to 3, i.e., $\mathbf{P}(Z=3)=1$. Is X guaranteed to be independent of Z?

No ✖ **Answer:** Yes

b) Would either of the following be an appropriate definition of independence of the pair (X,Y) from Z?

- $p_{X,Y,Z}(x,y,z) = p_X(x)\; p_Y(y)\; p_Z(z)$, for all x, y, z

  No ✔ **Answer:** No

- $p_{X,Y,Z}(x,y,z) = p_{X,Y}(x,y)\; p_Z(z)$, for all x, y, z

  Yes ✔ **Answer:** Yes

c) Suppose that X, Y, Z are independent. Is it true that X and Y are independent?

Yes ✔ **Answer:** Yes

d) Suppose that X, Y, Z are independent. Is it true that (X,Y) is independent from Z?

No ✖ **Answer:** Yes

a) Since Z is deterministic, the value of Z does not provide any information, and so, intuitively, we have independence. For a formal argument, suppose that $z \neq 3$. Then, $p_{X,Z}(x,z) = 0 = p_X(x)\,p_Z(z)$. And for $z=3$, $p_{X,Z}(x,3) = \mathbf{P}(X=x, Z=3) = \mathbf{P}(X=x) = \mathbf{P}(X=x)\cdot 1 = p_X(x)\,p_Z(3)$, so that the definition of independence is satisfied.

b) The second definition is correct, because it says that events of the form $\{X=x \text{ and } Y=y\}$ are independent from events of the form $\{Z=z\}$. On the other hand, the first imposes the stronger requirement that X is also independent of Y.

c) Intuitively, since X, Y, Z are independent, none of the random variables provides information about the others. For a formal argument,

$$p_{X,Y}(x,y) = \sum_z p_{X,Y,Z}(x,y,z) = \sum_z p_X(x)\,p_Y(y)\,p_Z(z) = p_X(x)\,p_Y(y)\sum_z p_Z(z) = p_X(x)\,p_Y(y).$$

d) Intuitively, the value of the pair (X,Y) provides no information about the random variable Z. We will verify that the appropriate definition of independence of (X,Y) from Z from part (b) is satisfied. We first use independence of X, Y, Z, and then the fact, from part (c), that $p_X(x)\,p_Y(y) = p_{X,Y}(x,y)$, to obtain

$$p_{X,Y,Z}(x,y,z) = p_X(x)\,p_Y(y)\,p_Z(z) = p_{X,Y}(x,y)\,p_Z(z),$$

as desired.

Exercise: A criterion for independence

1/1 point (graded)

Suppose that the conditional PMF of X, given $Y = y$, is the same for every y for which $p_Y(y) > 0$. Is this enough to guarantee independence?

Yes ✔ **Answer:** Yes

Solution:

The condition given means that when I tell you the value of Y, the conditional PMF of X will be the same. Thus, the value of Y makes no difference, and, intuitively, we have independence.

For a formal argument, let $c(x) = p_{X|Y}(x \mid y)$; we can define $c(x)$ this way (without a dependence on y) since we are assuming that $p_{X|Y}(x \mid y)$ is the same for all y. Now,

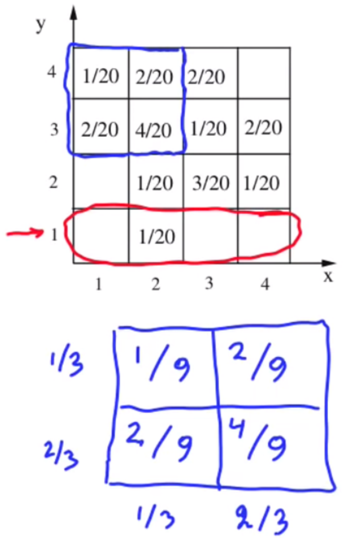
$$p_{X,Y}(x, y) = p_Y(y)\, p_{X|Y}(x \mid y) = p_Y(y)\, c(x) .$$

Summing over all y, we obtain

$$p_X(x) = \sum_y p_{X,Y}(x, y) = \sum_y p_Y(y)\, c(x) = c(x) .$$

Therefore, $c(x) = p_X(x)$. It follows that
$p_{X,Y}(x, y) = p_{X|Y}(x \mid y)\, p_Y(y) = c(x)\, p_Y(y) = p_X(x)\, p_Y(y)$, which establishes independence.

Example: independence and conditional independence



- Independent? No

$P_X(1) = 3/20$

$P_{X|Y}(1 \mid 1) = 0$

- What if we condition on $X \leq 2$ and $Y \geq 3$?

Yes

Not independent but conditionally independent (blue)
This can be checked by checking that the product of the marginal PMFs (singular) are equal to the join PMF i.e. 2/3 x 1/3 = 2/9

Independence and expectations

- In general: $\mathbf{E}[g(X,Y)] \neq g\big(\mathbf{E}[X], \mathbf{E}[Y]\big)$

always true

- Exceptions: $\mathbf{E}[aX+b] = a\mathbf{E}[X]+b$ $\qquad$ $\mathbf{E}[X+Y+Z] = \mathbf{E}[X]+\mathbf{E}[Y]+\mathbf{E}[Z]$

> If X, Y are **independent:** $\quad \mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
>
> $g(X)$ and $h(Y)$ are also independent: $\quad \mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

$E[g(X,Y)] \qquad g(x,y) = xy$

$$= \sum_x \sum_y xy\, p_{X,Y}(x,y) = \sum_x \sum_y x y\, p_X(x)\, p_Y(y)$$

$$= \sum_x x\, p_X(x) \underbrace{\sum_y y\, p_Y(y)} = E[X]\, E[Y]$$

Exercise: Independence and expectations

2/2 points (graded)

Let X and Y be independent positive discrete random variables. For each of the following statements, determine whether it is true (that is, always true) or false (that is, not guaranteed to be always true).

1. $\mathbf{E}[X/Y] = \mathbf{E}[X]/\mathbf{E}[Y]$

   False ✔ **Answer:** False

2. $\mathbf{E}[X/Y] = \mathbf{E}[X]\ \mathbf{E}[1/Y]$

   True ✔ **Answer:** True

Solution:

1. There is no reason why this statement should be true, and it is easy to come up with examples where it fails.

2. True. Note that $X/Y = X \cdot (1/Y)$. Furthermore, since X and Y are independent, so are X and $1/Y$. The validity of the statement follows.

Independence and variances

- Always true: $\text{var}(aX) = a^2\text{var}(X)$ $\text{var}(X+a) = \text{var}(X)$
- In general: $\text{var}(X+Y) \neq \text{var}(X) + \text{var}(Y)$

If X, Y are independent: $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$

assume
$E[X] = E[Y] = 0$
$E[XY] = E[X]E[Y] = 0$

$\text{var}(X+Y) = E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$
$= E[X^2] + 2\underbrace{E[XY]} + E[Y^2] = \text{var}(X) + \text{var}(Y)$

- Examples:
  - If $X = Y$: $\text{var}(X+Y) =$ $\text{var}(2X) = 4\text{var}(X)$
  - If $X = -Y$: $\text{var}(X+Y) =$ $\text{var}(0) = 0$
  - If X, Y independent: $\text{var}(X - 3Y) =$ $\text{var}(X) + \text{var}(-3Y) = \text{var}(X) + 9\text{var}(Y)$

Variance of the binomial

- X: binomial with parameters n, p
  - number of successes in n independent trials

$X_i = 1$ if ith trial is a success;
$X_i = 0$ otherwise (indicator variable) independent

$X = X_1 + \cdots + X_n$

$\boxed{\text{var}(X)} = \text{var}(X_1) + \cdots + \text{var}(X_n)$
$= n \cdot \text{var}(X_1) = \boxed{np(1-p)}$

The pair of random variables (X, Y) is equally likely to take any of the four pairs of values $(0, 1)$, $(1, 0)$, $(-1, 0)$, $(0, -1)$. Note that X and Y each have zero mean.

a) Find $\mathbf{E}[XY]$.

$\mathbf{E}[XY] =$ 0 ✔ **Answer:** 0

b) For this pair of random variables (X, Y), is it true that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$?

No ✖ **Answer:** Yes

c) We know that if X and Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. Is the converse true? That is, does the condition $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ imply independence?

No ✔ **Answer:** No

Solution:

a) At each possible outcome, we have $XY = 0$, and therefore $\mathbf{E}[XY] = 0$.

b) Since the random variables have zero mean, $\mathbf{E}[X + Y] = 0$, $\text{Var}(X) = \mathbf{E}[X^2]$, and $\text{Var}(Y) = \mathbf{E}[Y^2]$. Combining this with the result from part (a), we conclude that

$$\begin{aligned}
\text{Var}(X + Y) &= \mathbf{E}[(X + Y)^2] - (\mathbf{E}[X + Y])^2 \\
&= \mathbf{E}[(X + Y)^2] \\
&= \mathbf{E}[X^2] + 2\mathbf{E}[XY] + \mathbf{E}[Y^2] \\
&= \mathbf{E}[X^2] + \mathbf{E}[Y^2] \\
&= \text{Var}(X) + \text{Var}(Y).
\end{aligned}$$

c) We have here an example of two random variables that satisfy the condition $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. But these random variables are not independent. For example, the information that $X = 1$ tells us that the value of Y must be zero.

The Hat Problem

The hat problem

- n people throw their hats in a box and then pick one at random
  - All permutations equally likely $1/n!$
  - Equivalent to picking one hat at a time

$\frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{3!}$

$p_X(k)$: hard!

$\sum_k k\, p_X(k)$

- X: number of people who get their own hat
  - Find $\boxed{\mathbf{E}[X]} = E[X_1] + \dots + E[X_n] = n \cdot \frac{1}{n} = \boxed{1}$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

- $\mathbf{E}[X_i] = E[X_1] = P(X_1 = 1) = \frac{1}{n}$

Equivalent to a Bernoulli random variable when X[i] is 1 or 0 (got own hat back or not)

The variance in the hat problem

- X: number of people who get their own hat
  - Find $\text{var}(X)$

$n = 2$

$X_1 = 1 \Rightarrow X_2 = 1$

$X_1 = 0 \Rightarrow X_2 = 0$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$X^2 = \overbrace{\sum_i X_i^2}^{n} + \overbrace{\sum_{i,j:i\neq j} X_i X_j}^{\substack{n(n-1) \\ n^2 - n}}$$

- $\boxed{\text{var}(X)} = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = 2 - 1 = \boxed{1}$
- $\mathbf{E}[X_i^2] = E[X_1^2] = E[X_1] = 1/n$

$E[X^2] = n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n} \cdot \frac{1}{n-1}$

- For $i \neq j$: $\mathbf{E}[X_i X_j] = E[X_1 X_2] = P(X_1 X_2 = 1) = P(X_1 = 1, X_2 = 1)$

$= P(X_1 = 1)\, P(X_2 = 1 \mid X_1 = 1) = \frac{1}{n} \cdot \frac{1}{n-1}$

In green the n’s cancel out to give a value of 2
and (E[X])^2 is also equal to 1 so 2 -1 = 1

Exercise: The hat problem

0/2 points (graded)

Consider the hat problem, with $n = 10$. What is the expected value of $X_3X_6X_7$?

$\mathbf{E}[X_3X_6X_7] =$ 1/3 ✖ **Answer:** 0.00139

Solution:

By symmetry, this is the same as $\mathbf{E}[X_1X_2X_3]$. Since the product $X_1X_2X_3$ is either zero or one, this is the same as

$$\mathbf{P}(X_1X_2X_3 = 1) = \mathbf{P}(X_1 = 1) \cdot \mathbf{P}(X_2 = 1 \mid X_1 = 1) \cdot \mathbf{P}(X_3 = 1 \mid X_1 = X_2 = 1).$$

By thinking in terms of the sequential description of the process, we have seen that $\mathbf{P}(X_1 = 1) = 1/10$ and $\mathbf{P}(X_2 = 1 \mid X_1 = 1) = 1/9$. By a similar argument, given that the first two people obtained their own hats, the third person is faced with 8 hats, one of which is his/her own, and has probability $\mathbf{P}(X_3 = 1 \mid X_1 = X_2 = 1) = 1/8$ of picking it. Thus, the final answer is $(1/10) \cdot (1/9) \cdot (1/8)$.