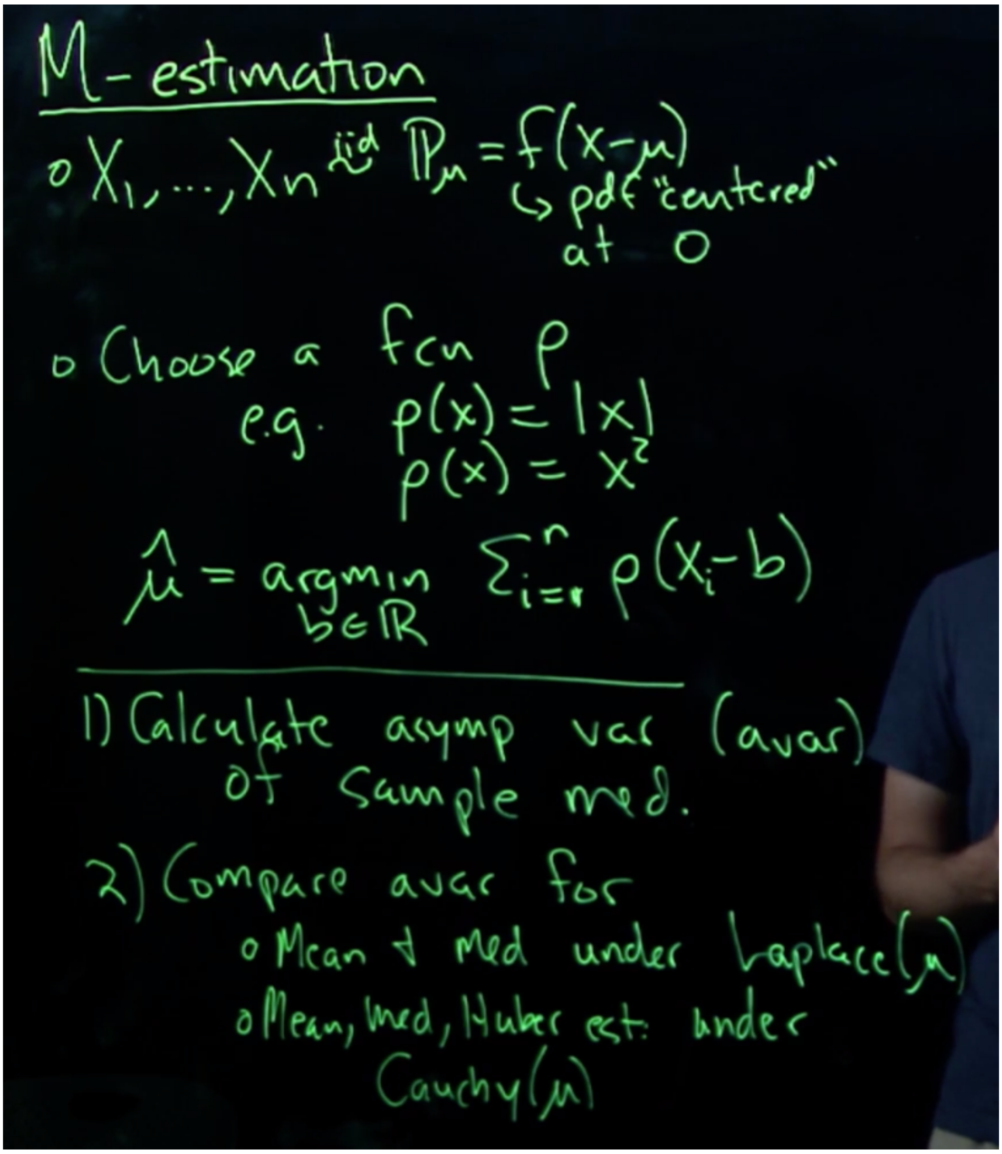
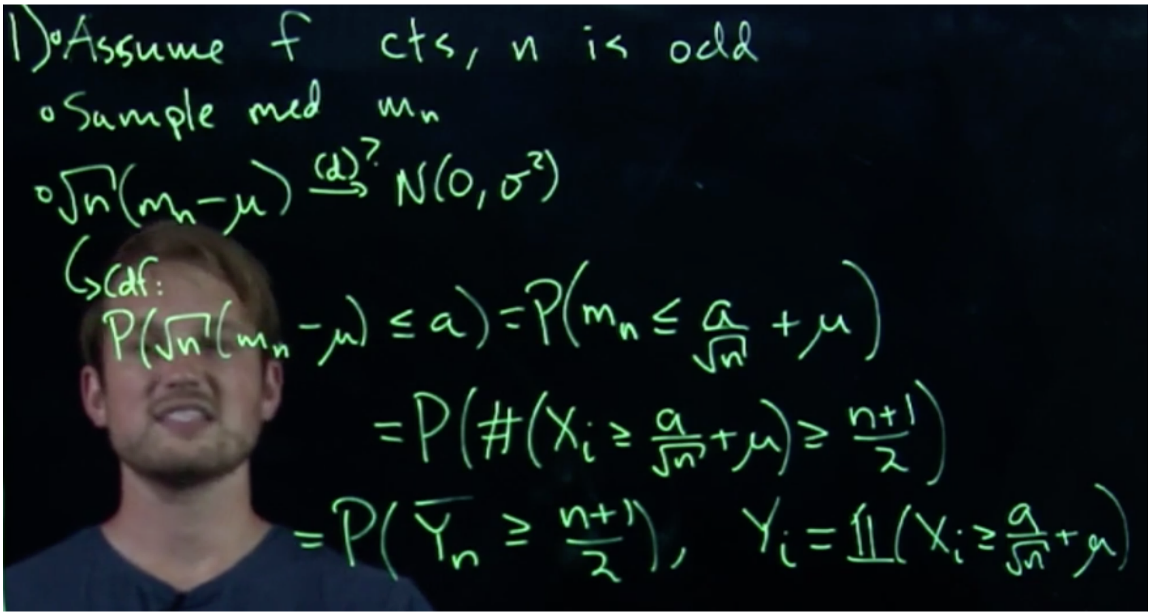
Recitation: M Estimation

M estimation is process of estimating a parameter for a distribution using some kind of minimisation

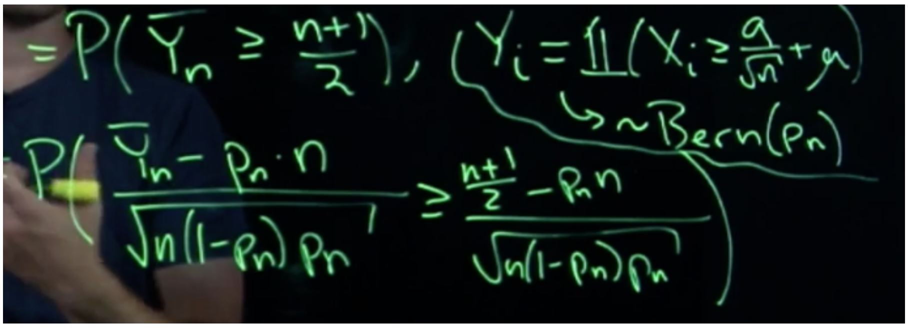


- trying to find the mu that is the centre of the distribution
- try to find a b that minimises the distance between mu and mu_hat
- roe are different loss functions
- if one estimator has a lower asymptotic variance than another then it is deviating from our value less

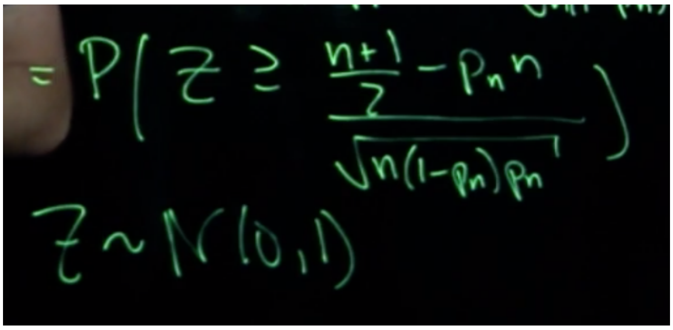
Calculate asymptotic variance for sample median



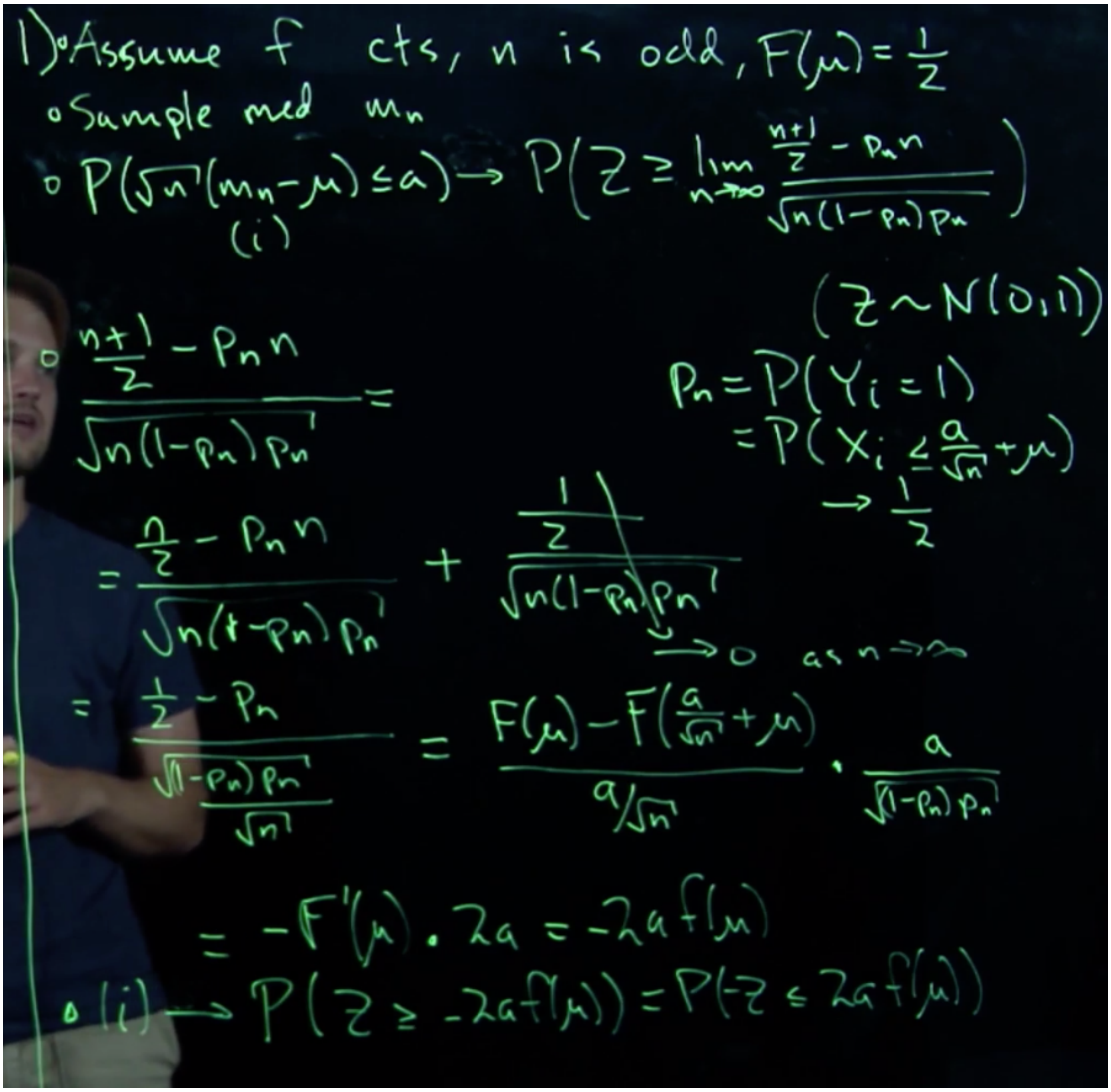
- the number of values greater than the median has to be at least half of the values
- Y_i is now Bernoulli, but it is dependent on n



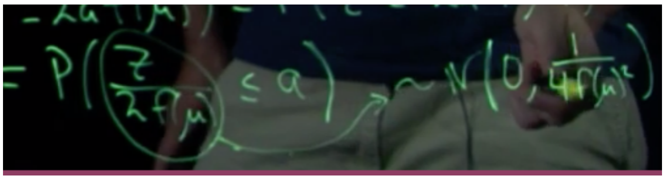
- p_n is the median (and parameter of the Bernoulli)
- as n-> infinity p_n goes to 1/2



Continuing calculation

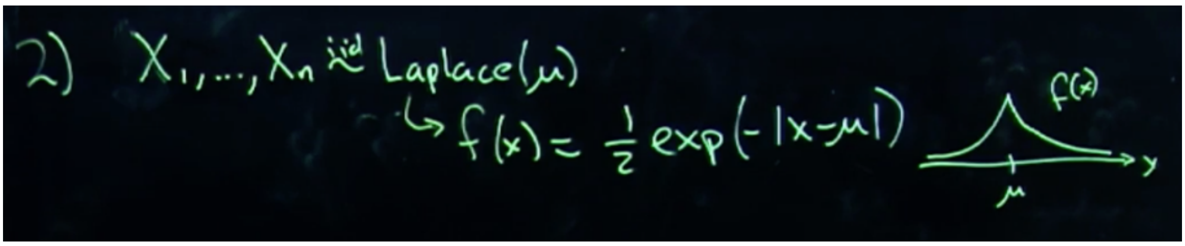


– showing that the initial Probability (i, above) converges in probability to :



– this gives a general formula for the asymptotic variance

Compare asymptotic variances for Laplace (mean and median)



– graph of probability density function

$$\sqrt{n}(\bar{X}-\mu) \xrightarrow{(d)} N(0, Var(X_1))$$

$$avar(\bar{X}) = var(X_1) = var(X_i') \quad , \quad X_i' \sim Laplace(0)$$

$$= E(X_i')^2 = 2$$

- asymptotic variance of the sample mean for a Laplace = 2

$$\sqrt{n}(m_n-\mu) \xrightarrow{(d)} N\left(0, \frac{1}{4f(\mu)^2}\right)$$

$$avar(m_n) = \frac{1}{4f(\mu)^2} = \frac{1}{4} \cdot \frac{1}{\left[\frac{1}{2}\exp(-|\mu-\mu|)\right]^2}$$

$$= 1$$

- asymptotic variance of the sample median for a Laplace (from previous section) = 1
- this means the median is a better estimator than the mean

Compare asymptotic variances for Cauchy (mean and median)

2) $X_1,\dots,X_n \sim Cauchy(\mu)$ if

pdf: $$f(x) = \frac{1}{\pi(1+(x-\mu)^2)}$$

cdf: $$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{x} \frac{1}{\pi(1+(t-\mu)^2)}\,dt$$

$$= \frac{\arctan(t-\mu)}{\pi}\Bigg|_{-\infty}^{x} = \frac{1}{2} + \frac{\arctan(x-\mu)}{\pi}$$

Note: $F(\mu) = \frac{1}{2}$

∘ Lets use $\bar{X}$ to estimate μ

$$E\bar{X} = EX_1 = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi(1+x^2)}\,dx \quad \text{diverges}$$

$$Var(\bar{X}) = \frac{1}{n} Var(X_1) : \text{diverges}$$

- has no variance or mean, they are both infinity so diverge
- same is true for asymptotic variance

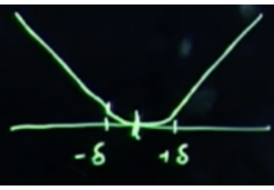
so try m_n instead (sample median)

$$\circ\ avar(m_n) = \frac{1}{4f(\mu)^2} = \frac{1}{4} \cdot \frac{1}{\left[\frac{1}{\pi}\right]^2} = \frac{\pi^2}{4}$$

- the median is a better estimator than the mean for Cauchy

Huber estimator for Cauchy

$$2)\quad \rho(x) = \begin{cases} \frac{x^2}{2} & |x| < \delta \\ \delta|x| - \frac{\delta^2}{2} & |x| > \delta \end{cases}$$



- Huber switches to a quadratic between -/+ delta to make it continuously differentiable
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$$\rho'(x) = \begin{cases} x & |x| < \delta \\ \delta\, \mathrm{sign}(x) & |x| \geq \delta \end{cases}$$

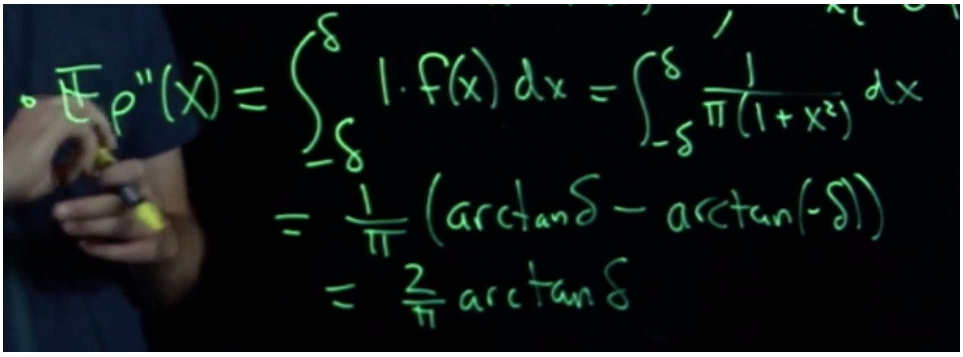
$$\rho''(x) = \begin{cases} 1 & |x| < \delta \\ 0 & |x| > \delta \end{cases}$$

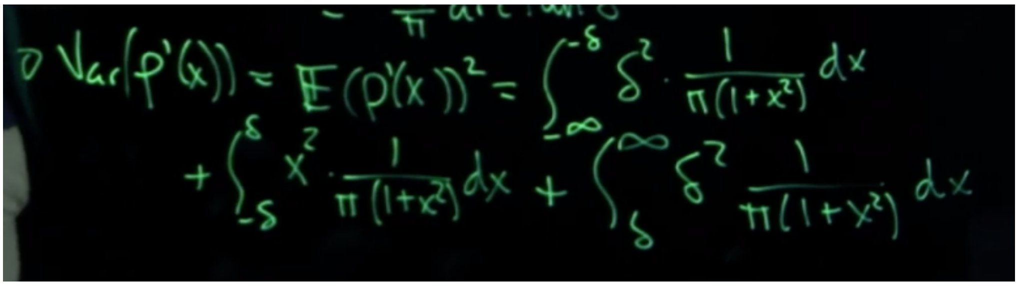
$$\circ\ \text{Huber est: } \hat{\mu}_{huber} = \underset{b \in \mathbb{R}}{\operatorname{argmin}} \sum_{i=1}^{n} \rho(X_i - b)$$

$$\circ\ \sqrt{n}(\hat{\mu}_{huber} - \mu) \xrightarrow{(d)} N\left(0, \frac{Var(\rho'(X))}{[\mathbb{E}\,\rho''(X)]^2}\right)$$

$$X \sim f(x), \quad X_i \sim f(x - \mu)$$

- calculate these quantities to get the asymptotic variance





- because E[roe’] is 0
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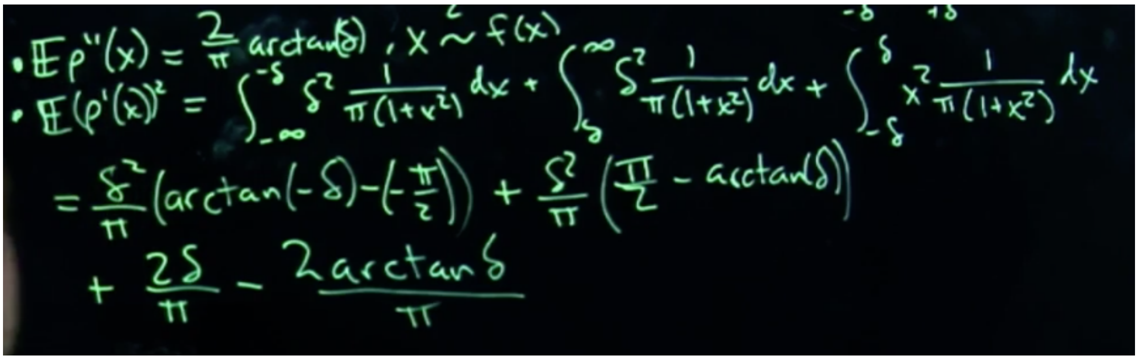
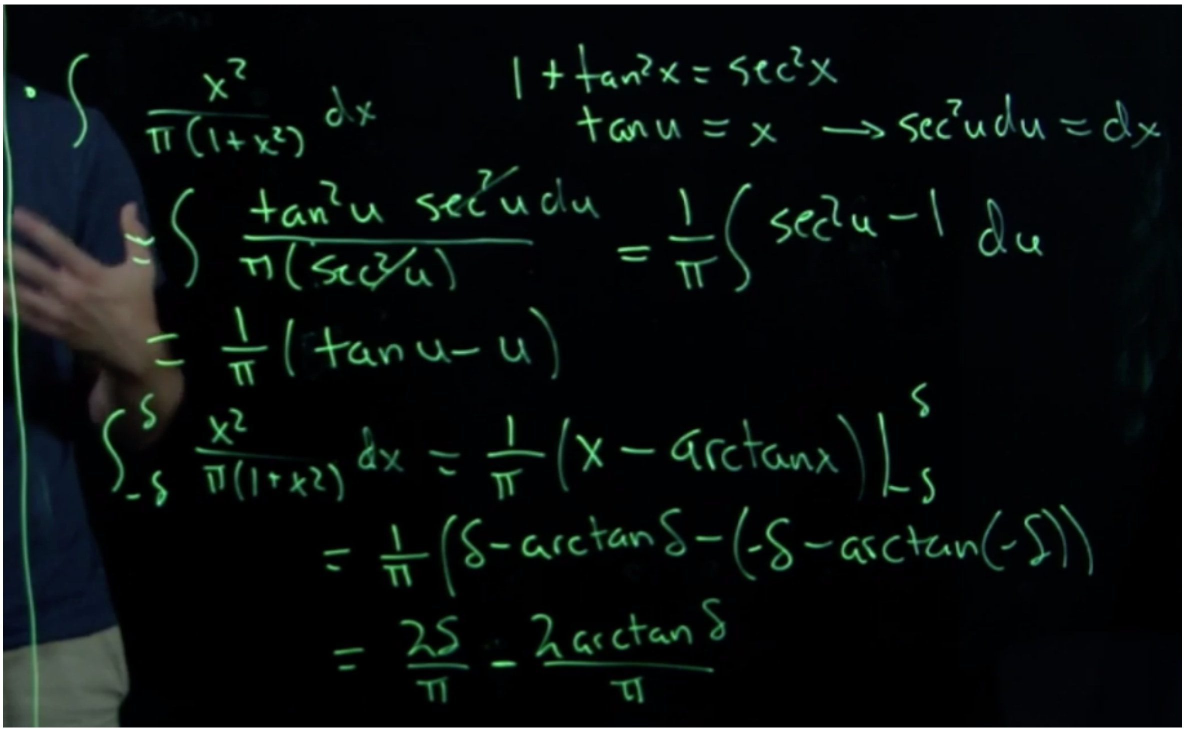
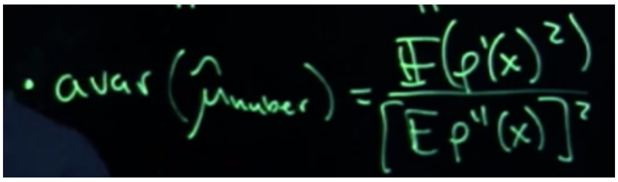
calculating integrals

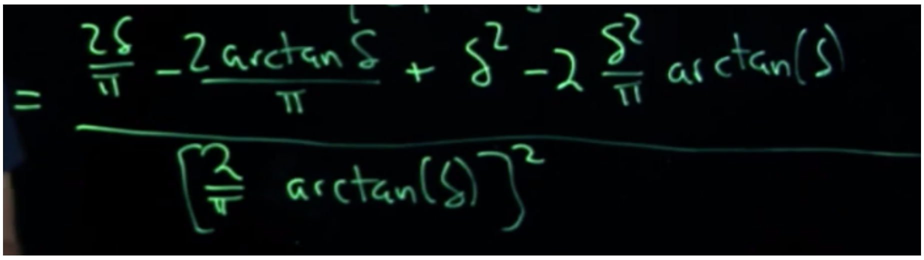


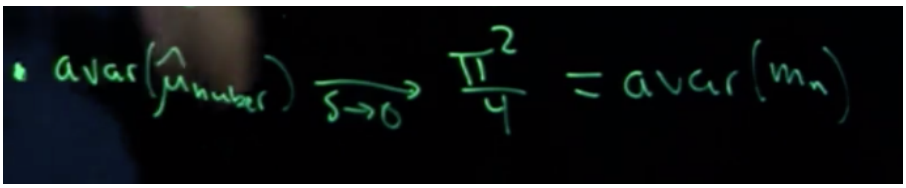
- calculation of 3rd term below, subbing in trigonometric identities for integrating



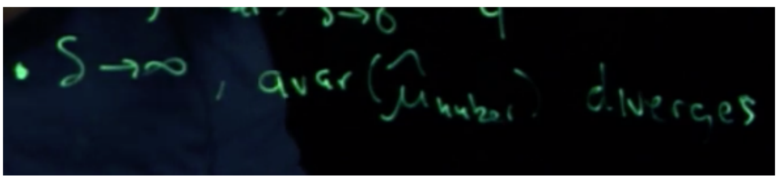
Asymptotic variance of Huber estimator for Cauchy distribution







- for delta -> 0, the asymptotic variance converges to median



- for delta -> infinity, the asymptotic variance diverges