Unit 9 Bernoulli and Poisson Processes

Stochastic/ random processes
Model arrivals over time

- discrete time: Bernoulli
- continuous time: Poisson

Bernoulli process

- Memorylessness - past arrivals do not affect future ones
- Simplest stochastic process
- Like flipping a coin at each time interval t, and counting number of heads/ successes/ arrivals

The Bernoulli process

- A sequence of independent Bernoulli trials, X_i
- At each trial, i:

$$\begin{aligned} \mathbf{P}(X_i = 1) &= \mathbf{P}(\text{success at the } i\text{th trial}) = p \\ \mathbf{P}(X_i = 0) &= \mathbf{P}(\text{failure at the } i\text{th trial}) = 1 - p \end{aligned}$$

$0 < p < 1$

- Key assumptions:
  - Independence
  - Time-homogeneity
- Model of:
  - Sequence of lottery wins/losses
  - Arrivals (each second) to a bank
  - Arrivals (at each time slot) to server
  - ...

- Jacob Bernoulli (1655–1705)

Exercise: The Bernoulli process

3/4 points (graded)

Let $X_1, X_2, \ldots$ be a Bernoulli process. We will define some new sequences of random variables and inquire whether they form a Bernoulli process.

1. Let $Y_n = X_{2n}$. Is the sequence Y_n a Bernoulli process?

   No ✗ **Answer:** Yes

2. Let $U_n = X_{n+1}$. Is the sequence U_n a Bernoulli process?

   Yes ✓ **Answer:** Yes

3. Let $V_n = X_n + X_{n+1}$. Is the sequence V_n a Bernoulli process?

   No ✓ **Answer:** No

4. Let $W_n = (-1)^n X_n$. Is the sequence W_n a Bernoulli process?

   No ✓ **Answer:** No

Solution:

1. Yes, because the random variables X_{2n} are independent Bernoulli random variables with the same parameter.
2. Yes, for the same reason.
3. No, because, for example $V_1 = X_1 + X_2$ and $V_2 = X_2 + X_3$ are both affected by X_2 and are therefore dependent. In addition, each V_n can take value 2 and is therefore not Bernoulli.
4. No, because W_1 can take value -1 and therefore is not a Bernoulli random variable.

Stochastic processes

- First view: sequence of infinite random variables $X_1, X_2, \ldots$

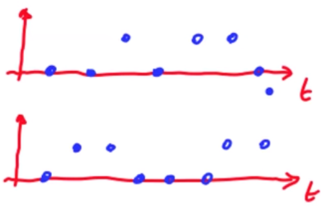
  Interested in: $\mathbf{E}[X_i] = p$ $\quad \text{var}(X_i) = p(1-p)$ $\quad p_{X_i}(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$

  $p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = p_{X_1}(x_1)\cdots p_{X_n}(x_n)$

  for all n

- Second view — sample space:

  $\Omega =$ set of infinite sequences of 0's and 1's



- Example (for Bernoulli process):

  $\mathbf{P}(X_i = 1 \text{ for all } i) = 0 \qquad (p<1)$

  $\leq P(X_1 = 1, \ldots, X_n = 1) = p^n$, for all n

Probability of all 1s for an infinite sequence becomes 0

Number of successes/arrivals S in n time slots

- $S = X_1 + \cdots + X_n$

- $\mathbf{P}(S = k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad k = 0, \ldots, n$

- $\mathbf{E}[S] = np$

- $\text{var}(S) = np(1-p)$

Time until the first success/arrival

- $T_1 = \min\{i : X_i = 1\}$

- $\mathbf{P}(T_1 = k) = P(\underbrace{0\,0\cdots0}_{k-1}\,1) = (1-p)^{k-1} p$ $\quad k = 1, 2, \ldots$

- $\mathbf{E}[T_1] = \dfrac{1}{p}$

- $\text{var}(T_1) = \dfrac{1-p}{p^2}$

6. Exercise: Time until the first failure

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Exercise: Time until the first failure

1/1 point (graded)

Let the sequence X_n, $n = 1, 2, 3, \ldots$, be a Bernoulli process with parameter $\mathbf{P}(X_n = 1) = p$ for all $n \geq 1$. Let U be the time when a value of 0 is first observed: $U = \min\{n : X_n = 0\}$ Then, the random variable U is:

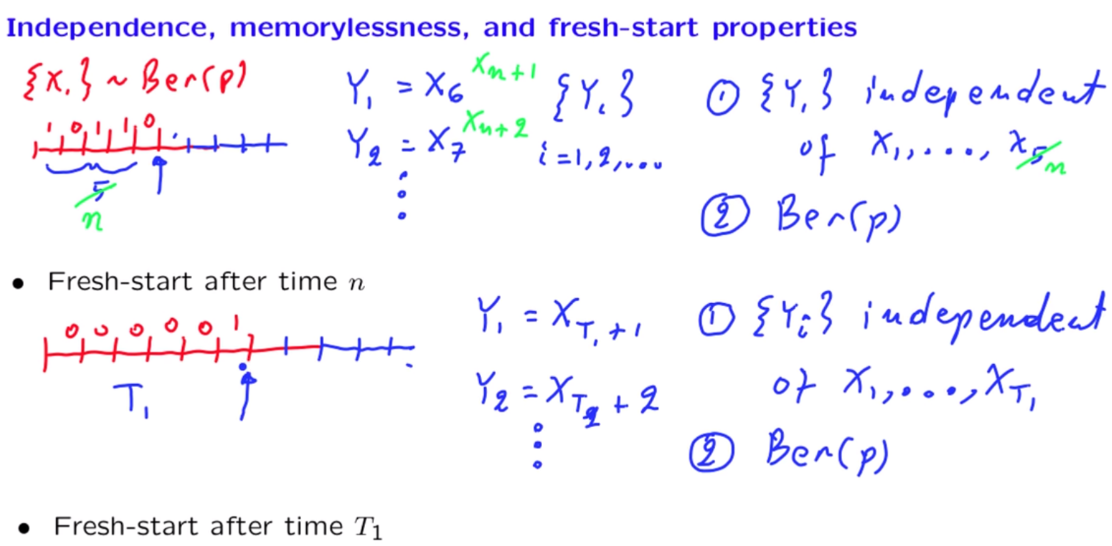
- ○ Geometric with parameter p
- ◉ Geometric with parameter $1 - p$
- ○ None of the above

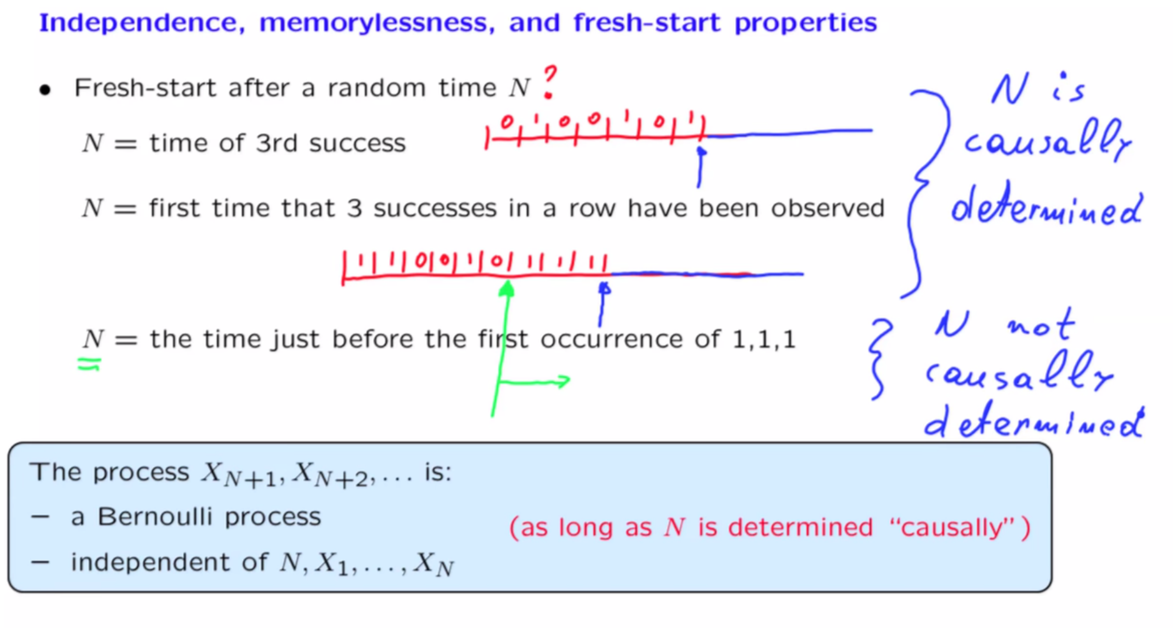
✔

Solution:

For $n \geq 1$, the event $\{U = n\}$ corresponds to $n - 1$ 1's followed by a 0. Its probability is $p^{n-1}(1 - p)$, which corresponds to a geometric PMF with parameter $1 - p$.

Properties





Exercise: Fresh start

3/3 points (graded)

Consider a Bernoulli process, with a “1" considered a success and a “0" considered a failure. Determine whether the process starts fresh right after each of the following random times:

1. The time of the kth failure

   Yes ✔ **Answer:** Yes

2. The first time that a failure follows a success

   Yes ✔ **Answer:** Yes

3. The first time at which we have a failure that will be followed by a success

   No ✔ **Answer:** No

Solution:

In the first two cases, the time of interest is determined causally, by past events, and we have the fresh-start property. In the last case, the time of interest is determined by something that is to happen in the future. In particular, we know that right after the time of interest, the next trial will result in a success.

Exercise: More on fresh start

1/1 point (graded)

Consider a Bernoulli process with parameter $p = 1/3$. Let T_1 be the time of the first success and let $T_1 + T_2$ be the time of the second success. We are told that the results of the two slots that follow the first success are failures, so that $X_{T_1+1} = X_{T_1+2} = 0$. What is the conditional expectation of the second interarrival time, T_2, given this information? (Recall that the expectation of a geometric random variable with parameter p is equal to $1/p$.)

5 ✔ **Answer:** 5

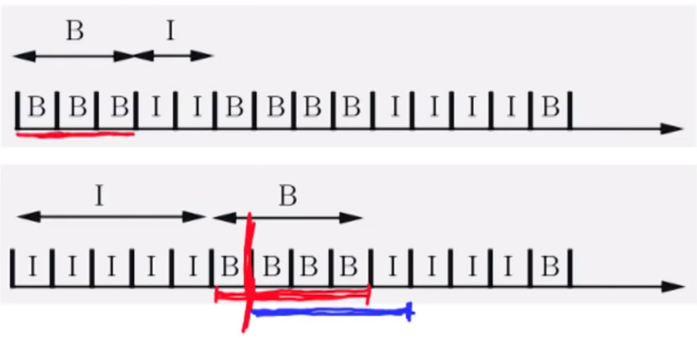
Solution:

After time T_1, we have two failures, and these are part of the interarrival time T_2. Given this information, the process starts fresh at time $T_1 + 3$ and the number of trials from time $T_1 + 3$ onwards until the next success is geometric with parameter $1/3$, and has an expected value of 3. Therefore, the conditional expectation of T_2, given the information we were given, is $2 + 3 = 5$.

Example

The distribution of busy periods

- At each slot, a server is busy or idle (Bernoulli process) p
- First busy period: Geo$(1-p)$
  - starts with first busy slot
  - ends just before the first subsequent idle slot



Geo$(1-p)$

Exercise: Busy periods

1/1 point (graded)

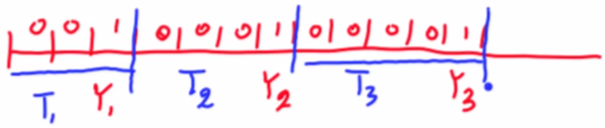
Consider the same setting as in the last video. After the first busy period ends (with an idle slot), there will be a subsequent busy period, which starts with a busy slot, and lasts as long as the slots are busy. Is it true that the length of the second busy period is geometric?

Yes ✔ **Answer:** Yes

Solution:

Yes, because the argument used for the first busy period applies without change.

Time of the kth success/arrival



- Y_k = time of kth arrival

$Y_k = T_1 + \cdots + T_k$

- T_k = kth inter-arrival time $= Y_k - Y_{k-1}$ $\quad (k \geq 2)$

- The process starts fresh after time T_1
- T_2 is independent of T_1; Geometric(p); etc.

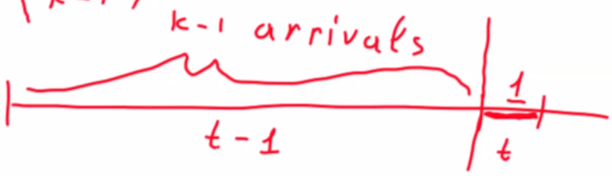
Time of the kth success/arrival

$$P(Y_k = t)$$

$$= P(k-1 \text{ arrivals in time } t-1)$$

$$\cdot P(\text{arrival at time } t)$$

$$= \binom{t-1}{k-1} p^{k-1} (1-p)^{t-k} \cdot p$$



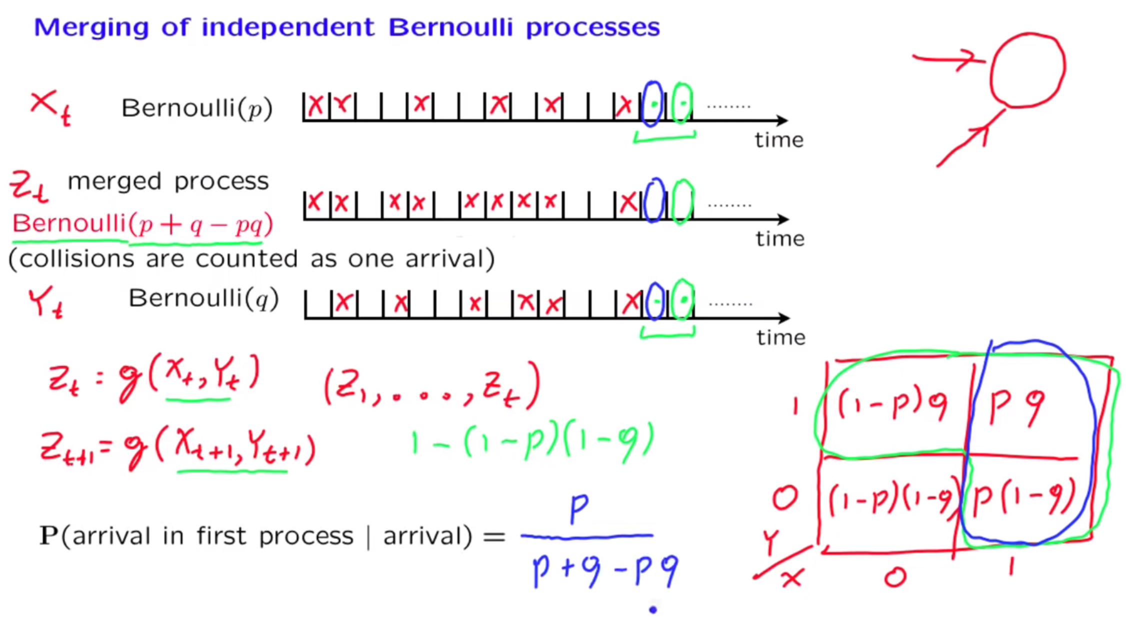
$Y_k = T_1 + \cdots + T_k$

the T_i are i.i.d., Geometric(p)

$$\mathbf{E}[Y_k] = \frac{k}{p} \qquad \text{var}(Y_k) = \frac{k(1-p)}{p^2}$$

$$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k},$$

$$t = k, k+1, \ldots$$



All sequences are independent from each other and in time
The conditional probability at the bottom is the blue event given the green event (box on right)

Exercise: A variation on merging

2/2 points (graded)

We start with two independent Bernoulli processes, X_n and Y_n, with parameters p and q, respectively. We form a new process Z_n by recording an arrival in a given time slot if and only if **both** of the original processes record an arrival in that same time slot. Mathematically, $Z_n = X_n Y_n$.

The new process Z_n is also Bernoulli with parameter p*q ✔ **Answer:** p*q

$p \cdot q$

(Enter an algebraic function of p and q using standard notation.)

Suppose that the two Bernoulli processes X_n and Y_n are dependent. We still assume, however, that the pairs (X_n, Y_n) are independent. E.g., (X_1, Y_1) is independent from (X_2, Y_2), etc. Is the process Z_n guaranteed to be Bernoulli?

No ✔ **Answer:** No

STANDARD NOTATION

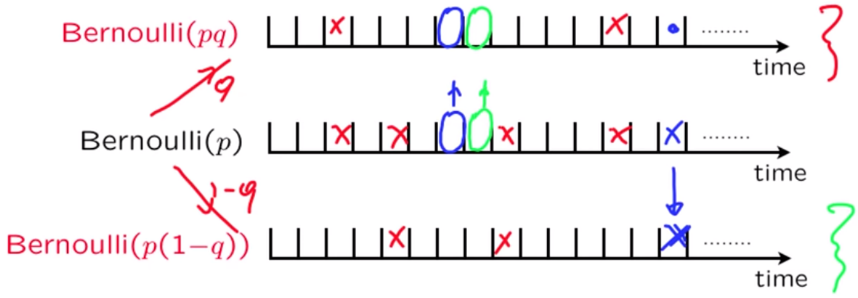
Solution:

The merged process records an arrival if and only if both of the original processes record an arrival, which happens with probability pq.

In the second case, since the pairs (X_n, Y_n) are independent, the random variables Z_n are also independent. However, there is nothing in the statement that would ensure that the Z_n are identically distributed. Thus, Z_n is not guaranteed to be a Bernoulli process. For example, consider the special case of $p = q$ and suppose that $Y_1 = X_1$ but Y_n is independent of X_n for $n > 1$. Then $\mathbf{P}(Z_1 = 1) = p$ while $\mathbf{P}(Z_n = 1) = p^2$ for $n > 1$, violating the time-homogeneity property of Bernoulli processes.

Splitting of a Bernoulli process

- Split successes into two streams, using independent flips of a coin with bias q
  - assume that coin flips are independent from the original Bernoulli process



- Are the two resulting streams independent? No

Not independent because presence of event in one stream tells us that there isn't one in the other

Exercise: Splitting

1/1 point (graded)

For each exam, Ariadne studies with probability $1/2$ and does not study with probability $1/2$, independently of any other exams. On any exam for which she has not studied, she still has a 0.20 probability of passing, independently of whatever happens on other exams. What is the expected number of total exams taken until she has had 3 exams for which she did not study but which she still passed?

30

✔ **Answer:** 30

Solution:

The sequence of exams for which she does not study and passes can be modeled as follows. We look at the exams for which she has not studied (a Bernoulli process with parameter $1/2$) and "split" it according to whether she passes or not. This creates a new Bernoulli process for the exams for which she does not study and passes, with parameter $(1/2) \cdot 0.20 = 0.10$. The expected time until 3 successes in this process is $3/0.10 = 30$.

Poisson approximation to binomial

- Interesting regime: large n, small p, moderate $\lambda = np$

$n \to \infty$ $\quad$ $p \to 0$ $\quad$ $p = \frac{\lambda}{n}$

- Number of arrivals S in n slots: $p_S(k) = \frac{n!}{(n-k)!\,k!} \cdot p^k (1-p)^{n-k}, \quad k = 0, \ldots, n$

$$= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} \cdot \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$\xrightarrow[n \to \infty]{} 1 \cdot 1 \cdots 1 \cdot \frac{\lambda^k}{k!} e^{-\lambda} \cdot 1$$

For fixed $k = 0, 1, \ldots,$

$p_S(k) \to \frac{\lambda^k}{k!} e^{-\lambda},$

- Fact: $\lim_{n \to \infty} (1 - \lambda/n)^n = e^{-\lambda}$

Example of situation: number of earthquakes over 5 years split by hour