Midterm Exam

1.

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True or False

4 points possible (graded, results hidden)

Let A, B, and C be events associated with the same probabilistic model (i.e., subsets of a common sample space), and assume that $P(C) > 0$.

For each one of the following statements, decide whether the statement is True (always true), or False (not always true).

1. Suppose that $A \subset C$. Then, $P(A \mid C) \geq P(A)$.

   - ○ True
   - ● False

2. Suppose that $A \subset B$. Then, $P(A \mid C) \leq P(B \mid C)$.

   - ○ True
   - ● False

3. Suppose that $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.

   - ○ True
   - ● False

4. Suppose that $A \subset C$, $B \subset C$, and $P(A) \leq P(B)$. Then, $P(A \mid C) \leq P(B \mid C)$.

   - ○ True
   - ● False

2.

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A Drunk Person at the Theater

4 points possible (graded, results hidden)

There are n people in line, indexed by $i = 1, \ldots, n$, to enter a theater with n seats one by one. However, the first person ($i = 1$) in the line is drunk. This person has lost her ticket and decides to take a random seat instead of her assigned seat. That is, the drunk person decides to take any one of the seats 1 to n with equal probability. Every other person $i = 2, \ldots, n$ that enters afterwards is sober and will take his assigned seat (seat i) unless his seat i is already taken, in which case he will take a random seat chosen uniformly from the remaining seats.

Suppose that $n = 3$. What is the probability that person 2 takes seat 2?

(Enter a fraction or a decimal accurate to at least 3 decimal places.)

2/3

Suppose that $n = 5$. What is the probability that person 3 takes seat 3?

(Enter a fraction or a decimal accurate to at least 3 decimal places.)

0.4667

1. P(drunk person selecting correct) + P(2nd person selecting correct | drunk was incorrect) = 1/3 + 1/2 = 5/6
   or is it 1/3! ?
2. 1/5 +
3.

3.

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Expectation 1

1 point possible (graded, results hidden)

Compute $\mathbf{E}(X)$ for the following random variable X:

$$X = \text{Number of tosses until getting 4 (including the last toss) by tossing a fair 10-sided die.}$$

$\mathbf{E}(X) =$ 10

Expectation 2

2 points possible (graded, results hidden)

Compute $\mathbf{E}(X)$ for the following random variable X:

$$X = \text{ Number of tosses until all 10 numbers are seen (including the last toss) by tossing a fair 10-sided die.}$$

To answer this, we will use induction and follow the steps below:

Let $\mathbf{E}(i)$ be the expected number of additional tosses until all 10 numbers are seen (including the last toss) **given i distinct numbers have already been seen**.

1. Find $\mathbf{E}(10)$.

   $\mathbf{E}(10) =$ 1

2. Write down a relation between $\mathbf{E}(i)$ and $\mathbf{E}(i+1)$. Answer by finding the function $f(i)$ in the formula below.

   For $i = 0, 1, \ldots, 9$:

   $$\mathbf{E}(i) = \mathbf{E}(i+1) + f(i)$$

   where $f(i) =$ 10/i

   $\frac{10}{i}$

3. Finally, using the results above, find $\mathbf{E}[X]$.

   (Enter an answer accurate to at least 1 decimal place.)

   $\mathbf{E}[X] =$ 29.29

4.

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Conditional Independence 1

4 points possible (graded, results hidden)

Suppose that we have a box that contains two coins:

1. A fair coin: $\mathbf{P}(H) = \mathbf{P}(T) = 0.5$.
2. A two-headed coin: $\mathbf{P}(H) = 1$.

A coin is chosen at random from the box, i.e. either coin is chosen with probability $1/2$, and tossed twice. Conditioned on the identity of the coin, the two tosses are independent.

Define the following events:

- Event A: first coin toss is H.
- Event B: second coin toss is H.
- Event C: two coin tosses result in HH.
- Event D: the fair coin is chosen.

For the following statements, decide whether they are true or false.

1. A and B are independent.

   - (selected) True
   - False

2. A and C are independent.

- (selected) True
- False

3. A and B are independent given D.

- (selected) True
- False

4. A and C are independent given D.

- (selected) True
- False

Conditional Independence 2

2 points possible (graded, results hidden)

1. Suppose three random variables X, Y, Z have a joint distribution

$$\mathbf{P}_{X,Y,Z}(x,y,z) = \mathbf{P}_X(x)\,\mathbf{P}_{Z|X}(z \mid x)\,\mathbf{P}_{Y|Z}(y \mid z).$$

Then, X and Y are independent given Z.

- True
- (selected) False

2. Suppose random variables X and Y are independent given Z, then the joint distribution must be of the form

$$\mathbf{P}_{X,Y,Z}(x,y,z) = h(x,z)\,g(y,z),$$

where h, g are some functions.

- (selected) True
- False

5.

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Variance of Difference of Indicators

2 points possible (graded, results hidden)

Let A be an event, and let I_A be the associated indicator random variable (I_A is 1 if A occurs, and zero if A does not occur). Similarly, let I_B be the indicator of another event, B. Suppose that $P(A) = p$, $P(B) = q$, and $P(A \cap B) = r$.

Find the variance of $I_A - I_B$, in terms of p, q, r.

Var $(I_A - I_B) =$ (-p)^2 - (q^2) + (2*p*q) + p - q

$(-p)^2 - (q^2) + (2 \cdot p \cdot q) + p - q$

6.

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For all problems on this page, use the following setup:

Let N be a positive integer random variable with PMF of the form

$$p_N(n) = \frac{1}{2} \cdot n \cdot 2^{-n}, \qquad n = 1, 2, \ldots.$$

Once we see the numerical value of N, we then draw a random variable K whose (conditional) PMF is uniform on the set $\{1, 2, \ldots, 2n\}$.

Joint PMF

1 point possible (graded, results hidden)

Write down an expression for the joint PMF $p_{N,K}(n, k)$.

For $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2n$:

$p_{N,K}(n, k) =$ (1/4)*2^(-n)

$\left(\frac{1}{4}\right) \cdot 2^{-n}$

Since K has uniform PMF of 2n therefore probability of each one is 1/2n

Marginal Distribution

1.5 points possible (graded, results hidden)

Find the marginal PMF $p_K(k)$ as a function of k. For simplicity, provide the answer **only for the case when k is an even number**. (The formula for when k is odd would be slightly different, and you do not need to provide it).

Hint: You may find the following helpful: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$ for $0 < r < 1$.

For $k = 2, 4, 6, \ldots$:

$p_K(k) =$ 1/2^((k/2)+1)

$\frac{1}{2^{\left(\frac{k}{2}\right)+1}}$

Discrete PMFs

2 points possible (graded, results hidden)

Let A be the event that K is even. Find $P(A|N = n)$ and $P(A)$.

$P(A \mid N = n) =$

$P(A) =$ 1/2^((k/2)+1)

$\frac{1}{2^{\left(\frac{k}{2}\right)+1}}$

Independence 2

0.5 points possible (graded, results hidden)

Is the event A independent of N?

- ( ) yes
- (•) no
- ( ) not enough information to determine