

Unit 3 Recitation

Method of moments

- gamma distribution
- $x > 0$
- want to find some moments to use them to find estimators of alpha and beta

Gamma Method of Moments

$X_1, \dots, X_n \stackrel{iid}{\sim} \text{Gamma}(\alpha, \beta)$

$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad x > 0$

Moment generating fcn:

$M_x(t) = \left(1 - \frac{t}{\beta}\right)^\alpha \quad (t < \beta)$

1) Find first 4 moments
of $X \sim \text{Gamma}(\alpha, \beta)$

2) Use MoM to find
estimators for α, β

3) Find asymptotic distribution
of $\hat{\alpha}_{\text{mom}}$

- use moment generating function by taking derivatives
- k'th derivative gives k'th moment

1. Find the first 4 moments

$$\text{Ex: } M_x^{(1)}(0) = E[X], \quad M_x^{(k)}(0) = E[X^k]$$

1st Moment: $M_x'(t) = \frac{d}{dt} \left(1 - \frac{t}{\beta}\right)^{-\alpha}$
 $= -\alpha \cdot \left(1 - \frac{t}{\beta}\right)^{-\alpha-1} \cdot -\frac{1}{\beta}$

$$E[X] = M_x'(0) = \alpha/\beta$$

- apply chain rule for derivative (derive outside then inside separately)

2nd Moment: $M_x''(t) = \frac{d^2}{dt^2} \left(1 - \frac{t}{\beta}\right)^{-\alpha}$
 $= \frac{d}{dt} \frac{\alpha}{\beta} \left(1 - \frac{t}{\beta}\right)^{-\alpha-1}$
 $= \frac{\alpha(\alpha+1)}{\beta^2} \left(1 - \frac{t}{\beta}\right)^{-\alpha-2}$

$$E[X^2] = M_x''(0) = \frac{\alpha(\alpha+1)}{\beta^2}$$

2nd M.

- just keep taking the next derivative

3rd Moment: $E[X^3] = M_x'''(0) = \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3}$

4th Moment: $E[X^4] = M_x''''(0) = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\beta^4}$

2. Use MoM to find estimators

$$E[X] = \frac{\alpha}{\beta}, \quad E[X^2] = \frac{\alpha(\alpha+1)}{\beta^2} = \frac{\alpha + \alpha}{\beta^2}$$

- starting with first 2 moments

Method of Moments estimators for $\alpha + \beta$.

$$m_1 = \frac{\alpha}{\beta}, \quad m_2 = \frac{\alpha^2 + \alpha}{\beta^2}$$

$$\alpha = g_1(m_1, m_2), \quad \beta = g_2(m_1, m_2)$$

- combining m_1 and m_2 to find alpha and beta

$$\alpha = g_1(m_1, m_2), \quad \beta = g_2(m_1, m_2)$$

$$\text{Simplify: } m_1^2 = \frac{\alpha^2}{\beta^2}, \quad m_2 - m_1^2 = \frac{\alpha}{\beta^2}$$

$$g_1(m_1, m_2) = \frac{m_1^2}{m_2 - m_1^2} = \frac{\alpha^2 / \beta^2}{\alpha / \beta^2} = \alpha$$

$$g_2(m_1, m_2) = \frac{m_1}{m_2 - m_1^2} = \frac{\alpha / \beta^2}{\alpha / \beta^2} = \beta$$

- find alpha and beta by trying different combinations of m_1 and m_2 such that terms cancel

$$\hat{\alpha}_{mom} = g_1(\bar{X}, \bar{X}^2) = \frac{\bar{X}^2}{\bar{X}^2 - \bar{X}^2}$$

$$\hat{\beta}_{mom} = g_2(\bar{X}, \bar{X}^2) = \frac{\bar{X}}{\bar{X}^2 - \bar{X}^2}$$

- then to get estimators for alpha and beta we plug in the sample means where:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad \bar{X}^2 = \frac{1}{n} \sum_{i=1}^n X_i^2$$

3. Find asymptotic distribution of $\hat{\alpha}$

$$\text{Asymptotic distribution of } \hat{\alpha} = \frac{\bar{X}^2}{\bar{X}^2 - \bar{X}^2}.$$

use multivariate central limit theorem

$$\sqrt{n} \left(\left(\frac{\bar{X}}{\bar{X}^2} \right) - \left(\frac{E(X_1)}{E(X_1^2)} \right) \right) \xrightarrow{(d)} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_1^2) \\ \text{Cov}(X_1, X_1^2) & \text{Var}(X_1^2) \end{pmatrix} \right)$$

$\hookrightarrow = \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} X_i \\ X_i^2 \end{pmatrix}$

- we calculated $E(X_1)$ and $E(X_1^2)$ earlier

$$\text{so } m_k = E(X_1^k)$$

$$\sqrt{n} \left(\left(\frac{\bar{X}}{\bar{X}^2} \right) - \left(\frac{m_1}{m_2} \right) \right) \xrightarrow{(d)} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} m_2 - m_1^2 & m_3 - m_1 m_2 \\ m_3 - m_1 m_2 & m_4 - m_2^2 \end{pmatrix} \right)$$

$\underbrace{\sum}_{\text{a}}$

- variance is $m_2 - (m_1)^2$
- using:

$$\text{Cov}(X_1, X_1^2) = E(X_1^3) - E(X_1)E(X_1^2)$$

$$g \left(\frac{\bar{X}}{\bar{X}^2} \right) = \frac{\bar{X}^2}{\bar{X}^2 - \bar{X}^2} \rightarrow \text{Use multivariate delta method.}$$

\sum^n

using the multivariate delta method

$$\sqrt{n} \left(g\left(\frac{\bar{X}}{\bar{X}^2}\right) - g\left(\frac{m_1}{m_2}\right) \right) \xrightarrow{(d)} N\left(0, \underbrace{\nabla g\left(\frac{m_1}{m_2}\right) \sum \nabla g\left(\frac{m_1}{m_2}\right)}_{\text{M}}\right)$$

$$\sqrt{n} \left(\frac{1}{2} m_2 m_1 - \alpha \right) \xrightarrow{(d)} N\left(0, \frac{1}{2} \right)$$

$$\nabla g\left(\frac{m_1}{m_2}\right) = \nabla \left(\frac{m_1}{m_2 - m_1^2} \right) = \left(\frac{(m_2 - m_1^2) \cdot 2m_1 - m_1^2 (-2m_1)}{(m_2 - m_1^2)^2} \right)$$

- gradient of g found by taking derivative wrt m_1 (quotient rule), then wrt to m_2