CS F351 Theory of Computation
Tutorial-12

Problem 1 Show that $E_{CFG} = \{< G > | G \text{ is a CFG and } L(G) = \emptyset\}$ is decidable.

Solution: The TM R for E_{CFG} is:

On input $< G >$, where $G = (V, \Sigma, R, S)$ is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variables get marked:
   (a) Mark any variable A where G contains a rule $A \rightarrow U_1 \cdots U_k$ and each symbol $U_1, \cdots, U_k$ has already been marked.
3. If the start symbol is not marked, accept; otherwise, reject

Problem 2 Show that $E_{TM} = \{< M >| \ M \text{ is a TM and } L(M) = \emptyset\}$ is undecidable.

Solution: For the sake of contradiction, assume that E_{TM} is decidable. Then, there exists a TM H that decides E_{TM}.

We now show that the membership problem is decidable.
(recall that the membership problem is $A_{TM} = \{< M, w >| \ M \text{ is a TM and } M \text{ accepts } w\}$)

TM R to decide A_{TM} (see Figure 1): On input $< M, w >$:

1. Construct a new language $L(M_w) = L(M) \cap \{w\}$.
2. Run H on $L(M_w)$:
   (a) if H accepts $L(M_w)$, then *reject*.
   (b) if H rejects $L(M_w)$, then *accept*.

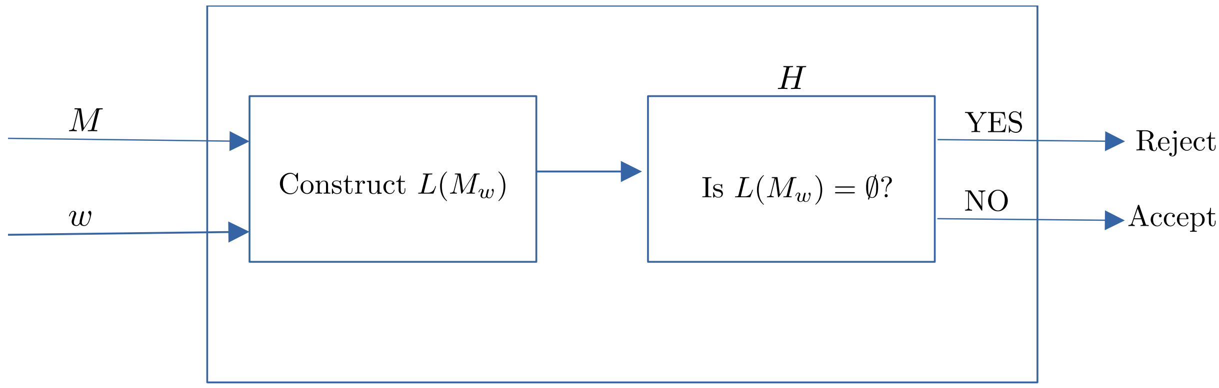

Figure 1: TM R that decides A_{TM}

But, we know that A_{TM} (membership problem) is undecidable. Hence, E_{TM} is also undecidable.

Problem 3 Show that $EQ_{TM} = \{< M_1, M_2 >| M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$ is undecidable.

Solution: For the sake of contradiction, assume that EQ_{TM} is decidable. Then, there exists a TM H that decides EQ_{TM}.

We now show that E_{TM} is decidable.

TM S that decides E_{TM} (see Figure 2):

On input $< M >$ (where M is a TM):

1. Run H on $< M, M_1 >$ where $L(M_1) = \emptyset$.
2. If H accepts, *accept*; if H rejects, *reject*.

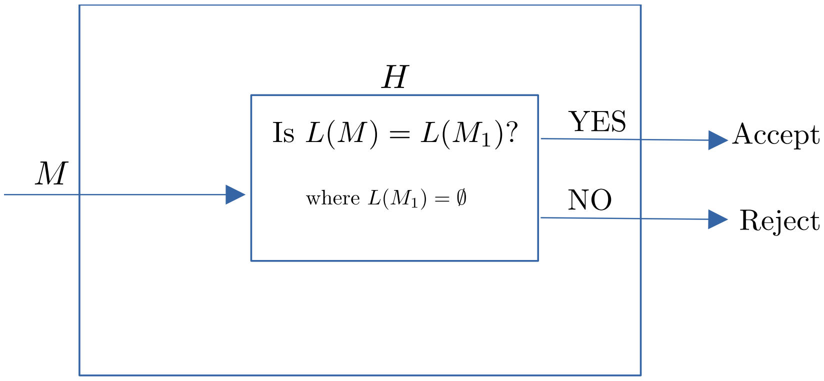

Figure 2: TM S that decides E_{TM}

Therefore, if H decides EQ_{TM}, then S decides E_{TM}. But E_{TM} is undecidable, so EQ_{TM} is undecidable.

Problem 4 Language A is *mapping reducible* to language B, written $A \leq_m B$, if there exists a computable function $f : \Sigma^* \rightarrow \Sigma^*$, where for every w, $w \in A \iff f(w) \in B$.

The function f is called the reduction from A to B.

Prove the following:

1. If $A \leq_m B$ and B is decidable, then A is decidable.

   Solution: Let M be the decider for B and f be the reduction from A to B. We describe a decider N for A as follows:

   $N =$ "On input w (see Figure 3):

   (a) Compute $f(w)$.

   (b) Run M on input $f(w)$ and output whatever M outputs."
   Clearly, if $w \in A$, then $f(w) \in B$ because f is a reduction from A to B. Thus, M accepts $f(w)$ whenever $w \in A$. Therefore, N works as desired.

2. If $A \leq_m B$ and A is undecidable, then B is undecidable.

   Solution: If B is decidable, from the above result, A is also decidable. Hence, B must be undecidable.

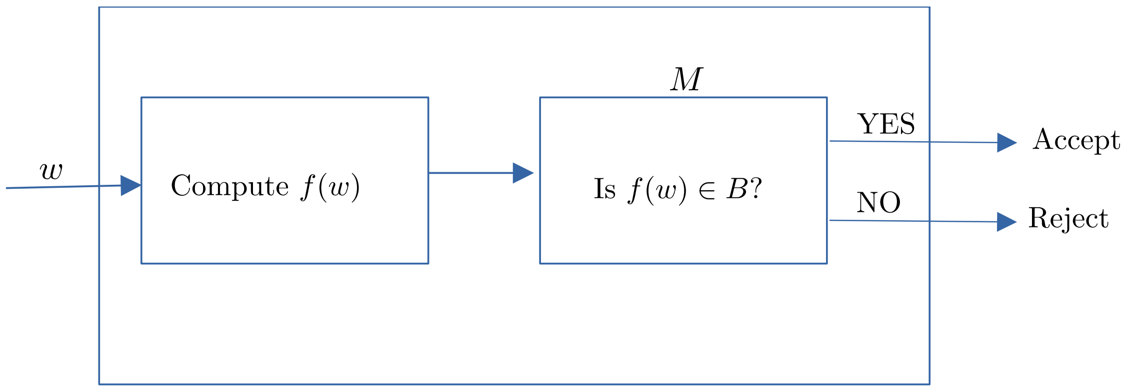

Figure 3: TM N that decides A

Problem 5 Find a match in the following instance of the Post Correspondence Problem instances

$$\left\{ \left[\frac{ab}{abab}\right], \left[\frac{b}{a}\right], \left[\frac{aba}{b}\right], \left[\frac{aa}{a}\right] \right\}$$

Solution:

$$\left[\frac{aa}{a}\right] \left[\frac{aa}{a}\right] \left[\frac{b}{a}\right] \left[\frac{ab}{abab}\right]$$