CS F351 Theory of Computation
Tutorial-1

Problem 1 ***(In tutorial)*** *Given a string x over an alphabet Σ, x^R is the string obtained by reversing the string x. Example: Let $\Sigma = \{0,1\}$ and $x = 1010$ then $x^R = 0101$. Prove the following: for any two strings x and y over the same alphabet Σ, $(xy)^R = y^R x^R$.*

The formal definition of x^R is as follows: $\epsilon^R = \epsilon$ and $(xa)^R = ax^R$ where $a \in \Sigma$.

Hint: use mathematical induction.

Proof.
By induction on $|y|$ (i.e., the length of y).
Base: For $|y| = 0 \implies$, $y = \epsilon \implies y = \epsilon = y^R$.

$$(xy)^R = (x\epsilon)^R = x^R = \epsilon x^R = y^R x^R.$$

Inductive hypothesis: Assume that the statement is true for all strings y, over Σ, of length k. i.e., for all $y \in \Sigma^k$, $(xy)^R = y^R x^R$.

Induction step: We now show that the statement holds for all strings y of length $k+1$.

Let $y \in \Sigma^{k+1}$ be a string, of length $k+1$, such that $y = va$, where $v \in \Sigma^k$ and $a \in \Sigma$.

$$\begin{aligned}
(xy)^R = (x(va))^R &= ((xv)a)^R \quad (\textit{Associativity of concatenation}) \\
&= a(xv)^R \quad (\textit{Definition}) \\
&= a(v^R x^R) \quad (\textit{Inductive hypothesis} \implies (xv)^R = v^R x^R) \\
&= (av^R)x^R \quad (\textit{Associativity of concatenation}) \\
&= (a^R v^R)x^R \quad (\textit{Since } a^R = a) \\
&= (va)^R x^R \quad (\textit{Inductive hypothesis} \implies a^R v^R = (va)^R) \\
&= y^R x^R
\end{aligned}$$

$\therefore$ By the principle of the mathematical induction it holds.

Problem 2 ***(In tutorial)*** *Consider the following definitions of palindrome.*

Definition 2.1 *A palindrome is a string that reads the same from forward and backward.*

Definition 2.2

1. ϵ is a palindrome

2. If a is any symbol then the string a is a palindrome.

3. If a is any symbol and x is a palindrome then the string axa is a palindrome.

4. Nothing is palindrome unless it follows from (a) to (c)

Prove by induction that the two definitions are equivalent.

Proof.

For ϵ, it is part of **def2** (clause 1) while it trivially satisfies **def1**. Similar argument holds for strings of unit length (clause 2 in **def2**). For length 2 palindromes, they satisfy **def2** being of the type aa with $x = \epsilon$ (clause 3). Strings of type aa also satisfy **def1** being the same symbol repeated twice. Now let us assume both definitions to be equivalent upto strings of length $n > 2$ in Σ^*.

Consider a string σ with $|\sigma| = n + 1$ which is palindrome as per **def1**. That implies $\sigma = \sigma^R$ (applying **def1**). Hence it must be the case that σ starts and ends with same symbol. Hence $\exists \sigma' \in \Sigma^*$, $a \in \Sigma$ such that $\sigma = a\sigma'a$. Also, $\sigma = \sigma^R \implies a\sigma'a = (a\sigma'a)^R \implies a\sigma'a = a\sigma'^Ra \implies \sigma' = \sigma'^R$. Thus σ' is palindrome as per **def1**. Since $|\sigma'| = n - 1$ and **def2**, **def1** are equivalent for string length upto n, we have σ' as palindrome also for **def2**. Now, applying clause 3 of **def2**, we have $\sigma = a\sigma'a$ as palindrome (as per **def2**).

Consider a string σ with $|\sigma| = n + 1$ which is palindrome as per **def2**. Since $n > 2$, we must have a palindrome x such that $|x| = n - 1$ and $axa = \sigma$ for some symbol a. x should satisfy **def1** and hence $x = x^R$. So, $\sigma^R = (axa)^R = ax^Ra = axa = \sigma$. This σ is also palindrome as per **def1**.

Problem 3 *Prove that the following definitions for the string of balanced parentheses are equivalent.*

Definition 3.1 *A string w over alphabet $\{(,)\}$ is balanced iff*

1. w has an equal number of ('s as)'s.

2. any prefix of w has at least as many ('s as)'s.

Definition 3.2 *A string w over alphabet $\{(,)\}$ is balanced iff*

1. ϵ is balanced.

2. If w is balanced string then (w) is balanced.

3. If w and x are balanced strings the wx is balanced.

4. Nothing else is balanced string

Solution : Let P_Σ^n be the set of balanced parenthesis upto length $2n$. $P_\Sigma^0 = \{\epsilon\}$ which is trivially satisfying def1 and def2. Let def1 and def2 agree upto P_Σ^n. Now $P_\Sigma^{n+1} = P_\Sigma^n \cup X$ with X being the set of balanced parenthesis of length exactly $2n+2$.

Let $x \in X$ be a balanced parenthesis satisfying def1. Consider the prefix of x of length 1. As per condition 2 of def1, it has to be '('. (If the first symbol is ')', condition 2 is not satisfied.) As per condition 1 of def1, x has $n+1$ '(' and $n+1$ ')'. We now argue that the last symbol of x has to be ')'. Otherwise, if $x = x_1($, then x_1 is a prefix with $n+1$ ')' and n '(' (violates condition 2). Hence, as per def1 $x = (w)$ (has to start and end with '(' and ')' respectively). There are two options now.

(a) w is a balanced string of length $2n$ as per def1. Then w is a balanced string also as per def2 (the definitions agree upto length 2n). Then x satisfies def2 (clause2).

(b) w is not a balanced string as per def1. This can only happen if clause 2 of def1 is violated by w (clause 1 is satisfied). Let w_1 be the smallest such prefix of w with less '(' than ')'. Note that '$(w_1$' manages to be a prefix with at least as many '(' as ')'. Hence, in w_1 there is exactly one '(' less. Thus '$(w_1$' satisfies both conditions of def1 (condition 2 is guaranteed by w_1 being the smallest possible violator). Thus $(w_1$ is a balanced string as per def1 (and hence def2). With $x = (w) = (w_1w_2)$, what about w_2 ? Since condition 1 and 2 of def1 are satisfied by both $(w_1$ and w, they are also satisfied by $w_2)$. Still let us show that. Surely, $w_2)$ has same number of '(' and ')' since both (w) and $(w_1$ are balanced. Consider any prefix σ of $w_2)$. Note, $(w_1\sigma$ has at least as many '(' as ')'. $(w_1$ has same number of '(' and ')'. Hence σ has at least as many '(' as ')'.

Both $(w_1$ and $w_2)$ are balanced as per both defs. Hence $x = (w)$ is balanced also as per def2 (clause3).

Assume def2 and prove the reverse now.

Problem 4 ***(In tutorial)*** *Prove that any equivalence relation R on a set S partitions S into disjoint equivalence classes.*

Proof. Let R be an equivalence relation on S, and suppose a and b are elements of S. Let C_a and C_b be the equivalence classes containing a and b respectively; that is, $C_a = \{c \mid aRc\}$ and $C_b = \{c \mid bRc\}$. We shall show that either $C_a = C_b$ or $C_a \cap C_b = \emptyset$. Suppose $C_a \cap C_b \neq \emptyset$; let d be in $C_a \cap C_b$. Now let e be an arbitrary member of C_a. Thus aRe. As d is in $C_a \cap C_b$ we have aRd and bRd. By symmetry dRa, By transitivity (twice), bRa and bRe. Thus e is in C_b and hence $C_a \subseteq C_b$. A similar proof shows that $C_b \subseteq C_a$, so $C_a = C_b$. Thus distinct equivalence classes are disjoint. To show that the classes form a partition, we have only to observe that by reflexivity, each a is in the equivalence class C_a , so the union of the equivalence classes is S.

Problem 5 *Show that the following are equivalent relation and give their classes.*

1. *R_1 on integers $\rightarrow iR_1j$ iff $i = j$*
2. *R_2 on people $\rightarrow pR_2q$ if p and q were born on the same hour of same day of some year*

Solution :

(a) R_1 is a relation on set of integers $\mathbb{Z}$ such that $iR_1j \implies i = j$

   i. **Proof of Reflexivity**: Let, $a \in \mathbb{Z}$
   As we can say $a = a \implies aR_1a$

   ii. **Proof of Symmetry**: Let, $a, b \in \mathbb{Z}$ such that $a = b \implies aR_1b$
   From $a = b$ we can say $b = a \implies bR_1a$

   iii. **Proof of Transitivity**: Let, $a, b, c \in \mathbb{Z}$ such that $a = b, b = c \implies aR_1b,\ bR_1c$
   As we can say $a = c \implies aR_1c$

Clearly this R_1 will divide $\mathbb{Z}$ into as many equivalent classes as many integers are present in Integer set.

(b) R_2 is a relation on set of people $\mathcal{P}$ such that $pR_2q \implies p$ and q were born on the same hour of same day of some year.

   i. **Proof of Reflexivity**: Let, $p \in \mathcal{P}$
   Then we can say p and p were born on the same hour of same day of some year. $\implies pR_1p$

   ii. **Proof of Symmetry**: Let, $p, q \in \mathcal{P}$ and pR_2q
   Then we can say q and p were born on the same hour of same day of some year. $\implies qR_1p$

   iii. **Proof of Transitivity**: Let, $p, q, r \in \mathcal{P}$ and $pR_2q,\ qR_2r$
   As $pR_2q \implies p$ and q were born on the same hour of same day of some year and $qR_2r \implies q$ and r were born on the same hour of same day of some year.then we can clearly say, p and r were born on the same hour of same day of some year. $\implies pR_1r$

Clearly this R_2 will divide $\mathcal{P}$ into 365*24 (366*24, for leap years) equivalence classes i.e. the number of hours in any year.