2 asset case

$$E(R_p) = w_1 R_1 + w_2 R_2 \quad \text{—— (I)}$$

$$\sigma^2(R_p) = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho_{12} \sigma_1 \sigma_2 \quad \text{—— (II)}$$

$$w_1 + w_2 = 1 \quad \text{—— (III)}$$

Equation I can be written as :-

$$E(R_p) = w_1 R_1 + (1 - w_1) R_2$$

Case 1 :- $\rho_{12} = +1$ Perfectly positively correlated.

$$E(R_p) = w_1 R_1 + (1 - w_1) R_2$$

$$\sigma^2(R_p) = w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2 w_1 (1 - w_1) \sigma_1 \sigma_2$$

or $\sigma^2(R_p) = [w_1 \sigma_1 + (1 - w_1) \sigma_2]^2$

or

$$\boxed{\begin{aligned} \sigma(R_p) &= +[w_1 \sigma_1 + (1 - w_1) \sigma_2] \\ E(R_p) &= w_1 R_1 + (1 - w_1) R_2 \end{aligned}} \quad \text{—— (IV)}$$

From equation (IV) we have :-

$$\sigma(R_p) = \omega_1 \sigma_1 + (1-\omega_1)\sigma_2$$

or $$\omega_1[\sigma_1 - \sigma_2] + \sigma_2 = \sigma(R_p)$$

or $$\boxed{\omega_1 = \frac{\sigma(R_p) - \sigma_2}{\sigma_1 - \sigma_2}} \quad \text{— (V)}$$

Putting value of ω_1 from equation (V) to equation (1) we have :-

$$E(R_p) = \left(\frac{\sigma(R_p) - \sigma_2}{\sigma_1 - \sigma_2}\right) R_1 + \left[1 - \left(\frac{\sigma(R_p) - \sigma_2}{\sigma_1 - \sigma_2}\right)\right] \times R_2$$

$$\boxed{E(R_p) = \left(\frac{R_1 - R_2}{\sigma_1 - \sigma_2}\right)\sigma_p + \left[R_2 - \frac{(R_1 - R_2)\sigma_2}{(\sigma_1 - \sigma_2)}\right]}$$

Straight line equation in Return-Risk space.

Case 2 Perfect Negative correlation ($\rho_{12} = -1$)

$$\sigma_p^2 = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2$$

$$= \left[w_1\sigma_1 - (1-w_1)\sigma_2\right]^2$$

$$\boxed{\sigma_p = \underset{\text{or}}{\overset{\text{either}}{\pm}}\left(w_1\sigma_1 - (1-w_1)\sigma_2\right)}$$

What is the weight invested in either of the security to have a risk free portfolio?

$$\sigma_p = 0$$

$$w_1\sigma_1 - (1-w_1)\sigma_2 = 0$$

or $$w_1\left[\sigma_1 + \sigma_2\right] = \sigma_2$$

or $$\boxed{w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}}$$ & $w_2 = 1 - w_1$

Zero-risk portfolio

$$\boxed{R_p = w_1R_1 + (1-w_1)R_2}$$

Case 3 $\rho_{12} = 0$. No correlation

$$E(R_P) = w_1 R_1 + (1-w_1) R_2$$

$$\sigma_P = \left[w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2\right]^{1/2}$$

What is the Minimum variance portfolio ?

$$\frac{d\sigma_P}{dw_1} = 0,$$

$$\frac{1}{2}\left[w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2\right]^{-1/2} \cdot \left[2w_1\sigma_1^2 - 2(1-w_1)\sigma_2^2\right]$$

$$= 0.$$

or $2w_1\sigma_1^2 - 2(1-w_1)\sigma_2^2 = 0$

or $w_1\sigma_1^2 - (1-w_1)\sigma_2^2 = 0$

or $w_1\left[\sigma_1^2 + \sigma_2^2\right] = \sigma_2^2$

or $\boxed{w_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$ MVP

Case 4 $\quad \rho_{12} = 0.5$

$$E(R_p) = w_1 R_1 + (1-w_1) R_2$$

$$\sigma^2(R_p) = w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2w_1(1-w_1)\rho_{12}\sigma_1\sigma_2$$

$$\sigma(R_p) = \left[w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\rho_{12}\sigma_1\sigma_2\right]^{1/2}$$

$$\frac{d\sigma(R_p)}{dw_1} = 0.$$

$$\frac{1}{2}\left[w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\rho_{12}\sigma_1\sigma_2\right]^{-1/2}.$$
$$\left[2w_1\sigma_1^2 - 2(1-w_1)\sigma_2^2 + 2(1-w_1)\rho_{12}\sigma_1\sigma_2 - 2w_1\rho_{12}\sigma_1\sigma_2\right] = 0$$

or $\quad w_1\sigma_1^2 - (1-w_1)\sigma_2^2 + (1-w_1)\rho_{12}\sigma_1\sigma_2 - w_1\rho_{12}\sigma_1\sigma_2 = 0$

or $\omega_1\sigma_1^2 - (1-\omega_1)\sigma_2^2 + (1-\omega_1)\rho_{12}\sigma_1\sigma_2$
$-\omega_1\rho_{12}\sigma_1\sigma_2 = 0$

or $\omega_1\left[\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2\right] = \sigma_2^2 - \rho_{12}\sigma_1\sigma_2$

or $$\boxed{\omega_1 = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}}$$

Minimum Variance Portfolio.

Q:1/ Suppose there are 2 stocks, A and B. Stock A has an expected return of 2% and a standard deviation of return of 2%. The corresponding statistics for Stock B are 9% and 7%, respectively. The correlation coefficient between the returns of stocks A and B is -0.5. An investor wants to achieve a standard deviation of 5% in his portfolio. What is the optimal portfolio for the investor and what is the expected return of this portfolio?

a) The optimal portfolio has 74.8% invested in Stock A and 25.2% invested in Stock B. The expected return of the portfolio is 3.8%.
b) The optimal portfolio has 25.2% invested in Stock A and 74.8% invested in Stock B. The expected return of the portfolio is 7.2%.
c) The optimal portfolio has 100% invested in Stock B. The expected return of the portfolio is 9.0%.
d) The optimal portfolio has 50% invested in Stock A and 50% invested in Stock B. The expected return of the portfolio is 5.5%.
e) None of the above

Q:2/ Suppose that there are only 2 stocks in the economy, C and D. Stock C has an expected return of 4% and a standard deviation of return of 8%. The corresponding statistics for Stock D are 6% and 12%, respectively. Suppose that the correlation coefficient between the returns of stocks C and D is -1.0. What is the only one possible value of the risk free rate from such combination?

a) 1.8%

b) 2.8%

c) 3.8%

d) 4.8%

e) None of the above

Q:3/ Suppose there are 2 stocks, A and B. Stock A has an expected return of 12%, a standard deviation of return of 30%, and current per share price of $100. The corresponding statistics for Stock B are 9%, 20%, and $75, respectively. The correlation coefficient between the returns of stocks A and B is 0.8. Suppose you buy 50 shares of Stock A and 50 shares of Stock B. What is the expected return of this portfolio and what is its standard deviation?

a) Expected Return=8.7%, Standard deviation=23.5%
b) Expected Return=9.7%, Standard deviation=24.5%.
c) Expected Return=10.7%, Standard deviation=24.5%
d) Expected Return=10.7%, Standard deviation=23.5%
e) None of the above

Q:4/ Assume two stocks have the following characteristics: -

	Expected Return	Standard deviation
C	14%	6%
S	8%	3%

The coefficient of correlation between the returns of the two stocks is 0.
What is the % investment in the two stocks in order to have a minimum variance portfolio? Investor is fully invested and there is no short selling.

a) 30% in C, 70% in S
b) 80% in C, 20% in S
c) 100% in S
d) 20% in C, 80% in S
e) None of the above

To develop the Efficient Frontier for the below Case

Case : 1 Risky asset
1 Risk-less or riskfree asset } Portfolio

Given : Risky asset X : Returns $= E(R_x) = R_x$
Riskfree asset F : Returns $= E(R_f) = R_f$

Standard deviation of returns of asset X
$\sigma(R_x) = \sigma_x$

The standard deviation of returns of risk-free asset $= 0$

The correlation between returns of risky asset X and risk-free asset $= 0$

Let w_x be proportion invested in X
w_f be proportion invested in risk-free asset.

We assume: Fully invested investor

$$w_x + w_f = 1$$
or
$$w_f = 1 - w_x$$

Portfolio Returns & Variance

$$E(R_p) = w_x R_x + (1 - w_x) R_f \quad \text{——} \quad (1)$$

$$\sigma_p^2 = w_x^2 \sigma_x^2 + 0 + 0 \quad \text{——} \quad (2)$$

From equation (2)

$$w_x = \left(\frac{\sigma_p}{\sigma_x} \right) \quad \text{——} \quad (3)$$

Putting the value of w_x from (3) in (1) we have :-

$$E(R_p) = \left(\frac{\sigma_p}{\sigma_x} \right) R_x + \left(1 - \frac{\sigma_p}{\sigma_x} \right) R_f$$

or $$E(R_p) = (R_x - R_f)\left(\frac{\sigma_p}{\sigma_x} \right) + R_f$$

or $$\boxed{E(R_p) = R_f + \left(\frac{R_x - R_f}{\sigma_x} \right) \sigma_p} \quad \text{——} \quad (4)$$

Straight line equation
Capital Allocation line

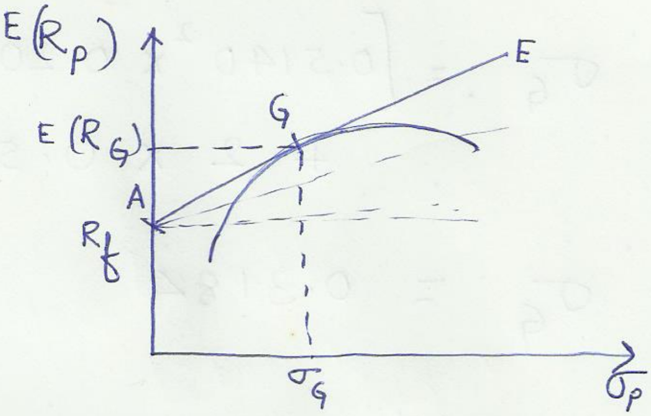
equation (4) is written in the slide of Lecture 8 as

$$E(R_c) = R_f + \left(\frac{E(R_A) - R_f}{\sigma_A} \right) \sigma_c$$

$$\sigma(R_G) = w_1^2\sigma_1^2 + w_2\sigma_2^2 + 2w_1w_2\,cov(R_1, R_2) \quad \text{—— (2)}$$

$$w_1 + w_2 = 1 \quad \text{—— (3)}$$

$$E(R_G) = w_1 E(R_1) + w_2 E(R_2) \quad \text{—— (1)}$$



In order to find point G.
We maximize the slope of the line
s.t. (1), (2) & (3)

$$\text{Max } S_p = \frac{E(R_G) - r_f}{\sigma_G} \quad \text{(Highest slope of CAL)}$$

s.t. equations (1), (2) & (3).

$$\frac{dS_p}{dw_1} = 0$$

We get

$$w_1 = \frac{[E(R_1) - r_f]\sigma_2^2 - [E(R_2) - r_f]\,cov(R_1, R_2)}{[E(R_1) - r_f]\sigma_2^2 + [E(R_2) - r_f]\sigma_1^2 - [E(R_1) - r_f + E(R_2) - r_f]\,cov(R_1, R_2)}$$

Putting the values we get

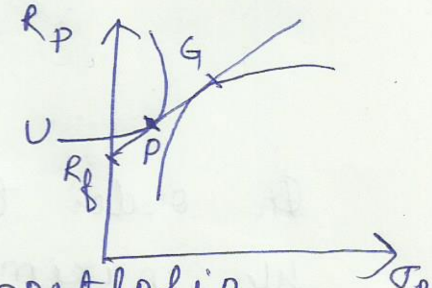
$w_1 = 0.5140 = 51.40\%$

$w_2 = 1 - w_1 = 48.60\%$

$$E(R_G) = 0.5140 \times 0.2 + 0.4860 \times 0.15$$
$$= 0.1757$$
$$= 17.57\ \%$$

$$\sigma_G = \Big[0.5140^2 \times 0.2025 + 0.4860^2 \times 0.1024 + 2 \times 0.5140 \times 0.4860 \times 0.0475\Big]^{1/2}$$

$$\sigma_G = 0.3184$$



Now, $U = E(R_P) - 0.5 A \sigma_P^2$

where P is a combination portfolio of riskless asset and portfolio G.

Let w_1' be invested in portfolio G

& $1 - w_1' = w_2'$ be invested in riskless asset with R_f as risk-free rate.

The equation of CAL is derived from following:-

$$E(R_P) = (1 - w_1') R_f + w_1' E(R_G) \quad \text{——— } (4)$$

$$\sigma_P^2 = w_1'^2 \sigma_G^2 \quad \text{——— } (5)$$

We Maximize U s.t. (4) & (5)

$$\frac{dU}{dw_1'} = 0.$$

$$\frac{dU}{dw_1'} = 0 \quad \text{where } U = w_1' E(R_G) + (1-w_1') R_f - 0.5 A w_1'^2 \sigma_G^2$$

$$\frac{d}{dw_1'}\left(w_1' E(R_G) + (1-w_1') R_f - 0.5 A w_1'^2 \sigma_G^2\right) = 0$$

$$E(R_G) - R_f - 0.5 \times 2 A w_1' \sigma_G^2 = 0$$

or $$w_1' = \frac{E(R_G) - R_f}{1 * A \sigma_G^2}$$

or $$w_1' = \frac{0.1757 - 0.08}{4 \times 0.3185^2} = 0.2358$$

The investor puts 23.58% of her wealth in Portfolio G and 76.42% (i.e. 1 - 23.58%) in riskless asset.

Now, 23.58% in Portfolio G means.

% investment in X = 0.5140 X 0.2358
= 0.1212 = 12.12%

% investment in Y = 0.4860 X 0.2358
= 0.1146 = 11.46%

Problem 1 solution

Given: $E(R_x) = 15\%$

$\sigma_x = 22\%$

$R_f = 7\%$

1 risky asset X and 1 riskfree asset

$$U = E(R_p) - \frac{1}{2} A \sigma_p^2$$

$$A = 4.$$

To find: Asset allocation in risky asset X and riskfree asset.

Soln.

Let w_x be invested in asset X

$1 - w_x$ is invested in riskfree asset

$$E(R_p) = w_x R_x + (1 - w_x) R_f$$

$$\sigma^2(R_p) = w_x^2 \sigma_x^2$$

$$U = w_x R_x + (1 - w_x) R_f - \frac{1}{2} \times A \times w_x^2 \sigma_x^2$$

Maximize Utility

$$\frac{\partial U}{\partial w_x} = 0 \text{ or } R_x - R_f - A w_x \sigma_x^2 = 0$$

$$\text{or } w_x = \frac{R_x - R_f}{A \sigma_x^2} = \frac{15\% - 7\%}{4 \times (22\%)^2} = 0.4132$$

$w_x = 41.32\%$

$w_{Riskfree} = 1 - w_x = 58.68\%$

The Investor would be investing 41.32% in risky asset and 58.68% in riskless asset.

Utility Functions

A procedure for ranking random wealth levels.

Utility function is defined on real numbers (representing possible wealth levels) and giving a real value.

Once, utility function is defined, all alternative random wealth levels are ranked by evaluating their expected utility values.

Larger the expected value better it is meaning larger values are preferred.

1st general restriction on utility function :-

Utility function is an increasing continuous function.

If x & y are nonrandom real values with

$x > y$, $U(x) > U(y)$.

$U'(x) > 0$.

Risk Aversion

The property here is
Utility function is concave.
$U''(x) < 0$

Risk Aversion Coefficients

Arrow Pratt absolute risk aversion coefficient,

$$a(x) = -\frac{U''(x)}{U'(x)}$$

Examples of utility functions :-

1. $U(x) = -e^{-ax}$
2. $U(x) = \ln(x)$
3. $U(x) = bx^b$
4. $U(x) = x - bx^2$

Name	Id no.

Question 1. [10 marks]

You are hired by Pension Plan Investment Board (PPIB) to manage $275 billion in investment assets for the Pension fund on behalf of 20 million pensioners. You have an asset allocation problem at hand. Each year the Pension fund was expected to deliver a 15% return on the investment portfolio in order to meet long-term actuarial projections and to be fully able to cover the promised retirement plan benefits.

PPIB suggests an overall investment plan that allocates the assets across four broad classes in order to archive diversification. There is no short selling allowed in the market.

1. A real estate fund
2. A small cap fund
3. A large cap fund
4. A socially responsible fund

You wanted to calculate the estimates of portfolio inputs to the optimization problem using historical data. You used a Market Index model to calculate the estimates. The Market Index has an annualized return of 11.2% and annualized standard deviation of returns being 14.26%. Following are the results of regression performed. The **monthly** returns of the fund (dependent variable; Ri) were regressed against the **monthly** returns of the market index (independent variable; Rm).

$$Ri = \alpha i + \beta i\, Rm + ei$$

The estimates of α's, β's and error variances for all four funds are provided in Table 1.

Table1

Funds	α	β	error variance
Real estate (RE)	0.98%	1.2	0.018%
Small cap (SC)	1.37%	1.4	0.42%
Large cap (LC)	0.79%	0.9	0.19%
Green and Renewable energy (GARE)	0.63%	1.62	0.86%

1.1 *(3 marks)* Calculate the annualized return for all four funds and write as a 4x1 matrix.

1.2 *(3 marks)* Calculate the annualized standard deviation of returns for all four funds.

1.3 *(2 marks + 2 marks)* Calculate the annualized covariance and correlation coefficients and present it as 4x4 matrix of variance-covariance. Comment on the correlation values.

Name	Id no.

Answer1

1.1 *(3 marks)* Calculate the annualized return for all four funds and write as a 4x1 matrix.

$E(R) = \alpha * 12 + \beta * Rm$ (1 mark)

Funds	Expected Return	Marks
Real estate (RE)	25.20%	0.5
Small cap (SC)	32.12%	0.5
Large cap (LC)	19.56%	0.5
Green and Renewable energy (GARE)	25.70%	0.5

1.2(3 marks) Calculate the annualized standard deviation of returns for all four funds

$\sigma = (\beta\text{^}2 * \sigma m\text{^}2 + \text{error variance} * 12)\text{^}1/2$ (1 mark)

Funds	stdev of returns	Marks
Real estate (RE)	17.73%	0.5
Small cap (SC)	30.04%	0.5
Large cap (LC)	19.82%	0.5
Green and Renewable energy (GARE)	39.57%	0.5

1.3 (2 marks + 2 marks) Calculate the annualized covariance and correlation coefficients and present it as 4x4 matrix of variance-covariance. Comment on the correlation values.

*Covariance (i,j) = βi * βj *σm*σm*

covariance of returns	Real estate (RE)	Small cap (SC)	Large cap (LC)	Green and Renewable energy (GARE)
Real estate (RE)	3.14%	3.42%	2.20%	3.95%
Small cap (SC)	**3.42%**	9.03%	2.56%	4.61%
Large cap (LC)	**2.20%**	2.56%	3.93%	2.96%
Green and Renewable energy (GARE)	**3.95%**	4.61%	2.96%	15.66%

*Correlation (i,j) = covariance (i,j)/(σi*σj)*

correlation of returns	Real estate (RE)	Small cap (SC)	Large cap (LC)	Green and Renewable energy (GARE)
Real estate (RE)	1.00	0.64	0.62	0.56
Small cap (SC)	0.64	1.00	0.43	0.39
Large cap (LC)	0.62	0.43	1.00	0.38
Green and Renewable energy (GARE)	0.56	0.39	0.38	1.00

Comment: Since correlation between funds is less than 1, with some being less than 0.5, we can expect that some level of diversification will be achieved. (2 mark)

Derivation of CAPM

⇒ The capital asset pricing model (CAPM):
If the market portfolio M is efficient, the expected return $\bar{r}_i$ of any asset i satisfies

$$\bar{r}_i - r_f = \beta_i (\bar{r}_m - r_f)$$

where $$\beta_i = \frac{\sigma_{im}}{\sigma_M^2}$$

Proof: For any α consider the portfolio consisting of a portion α invested in asset i and a portion $1-\alpha$ invested in the market portfolio M. (We allow $\alpha < 0$).

The expected rate of return of this portfolio is :-

$$\bar{r}_\alpha = \alpha \bar{r}_i + (1-\alpha) \bar{r}_m$$

and standard deviation of returns of this portfolio

$$\sigma_\alpha = \left[\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{i,M} + (1-\alpha)^2 \sigma_M^2\right]^{1/2}$$

As α varies, these values trace out a curve in $\bar{r}-\sigma$ diagram.
If $\alpha = 0$, 100% invested in M.

This curve cannot cross the CML.
As α passes through zero, the curve must be tangent to the CML at M.

This tangency condition can be translated into the condition that the slope of the curve is equal to the slope of the CML at point M.

Derivatives to be calculated are :-

$$\frac{d\bar{r}}{d\alpha} = \bar{r}_i - \bar{r}_m$$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}$$

We have.

$$\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\substack{at \\ \alpha=0}} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$$

$$\frac{d\bar{r}_\alpha}{d\sigma_\alpha} = \frac{d\bar{r}_\alpha/d\alpha}{d\sigma_\alpha/d\alpha}$$

to obtain

$$\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{(\bar{r}_i - \bar{r}_m)\sigma_M}{(\sigma_{iM} - \sigma_M^2)}$$

: ——— Slope of the curve at M

This slope must equal the slope of the CML line.

We have,

$$\frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

Solving for $\bar{r}_i$;

$$\bar{r}_i = \left(\frac{\bar{r}_M - r_f}{\sigma_M}\right)\left(\frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}\right) + \bar{r}_M$$

$$= \frac{(r_f - \bar{r}_M)}{\sigma_M^2}\left(-\sigma_{iM} + \sigma_M^2\right) + \bar{r}_M$$

$$= -r_f\frac{\sigma_{iM}}{\sigma_M^2} - \cancel{\frac{\bar{r}_M \sigma_M^2}{\sigma_M^2}} + \cancel{\bar{r}_M}$$

$$+ r_f\frac{\sigma_M^2}{\sigma_M^2} + \bar{r}_M\frac{\sigma_{iM}}{\sigma_M^2}$$

$$\bar{r}_i = r_f + (\bar{r}_M - r_f)\frac{\sigma_{iM}}{\sigma_M^2}$$

or $\boxed{\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f)}$

CAPM as a Pricing formula

$$\underbrace{\frac{\bar{Q} - P}{P}}_{\text{Return}} = r_f + \beta(\bar{r}_M - r_f)$$

$$\boxed{P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)}}$$

Pricing form of the CAPM.

The price P of an asset with payoff Q is

$$P = \frac{Q}{1 + r_f + \beta(\bar{r}_M - r_f)}$$

where β is the β (or beta) of the asset.

Question I:

.(10 MARKS) You are hired by National Pension Scheme Investment Board (NPSIB) to manage $275 billion in investment assets for the Pension fund on behalf of 20 million pensioners. You have an asset allocation problem at hand. Each year the Pension fund was expected to deliver a 20% return on the investment portfolio in order to meet long-term actuarial projections and to be fully able to cover the promised retirement plan benefits.
NPSIB suggests an overall investment plan that allocates the assets across four broad classes in order to achieve diversification. There is no short selling allowed in the market.

1. An emerging market fund (EM)
2. A global green fund (GG)
3. A resources & energy fund (RE)
4. A blue-chip equity fund (BE)

You wanted to calculate the estimates of portfolio inputs to the optimization problem using a Macroeconomic based multi-factor model like the Arbitrage Pricing theory model. You used a 2-factor model to calculate the estimates by running a multivariate regression.

$$Ri = \alpha i + \boldsymbol{\beta} i1\, \boldsymbol{F}1 + \boldsymbol{\beta} i2\, \boldsymbol{F}2 + ei$$

Following are the general equations related to the 2-factor model
Var(*Ri*) = ($\boldsymbol{\beta} i1$)^2 var(*F1*) + ($\boldsymbol{\beta} i2$)^2 var(*F2*) + 2 $\boldsymbol{\beta} i1 * \boldsymbol{\beta} i2 * cov(F1, F2)$ + var(*ei*)
E(Ri) = $\alpha i + \boldsymbol{\beta} i1\, E(\boldsymbol{F}1) + \boldsymbol{\beta} i2\, E(\boldsymbol{F}2) + E(ei)$
Cov(*Ri*,*Rj*) = $\boldsymbol{\beta} i1\, \boldsymbol{\beta} j1 var(F1) + \boldsymbol{\beta} i2\, \boldsymbol{\beta} j2 var(F2) + (\boldsymbol{\beta} i1\, \boldsymbol{\beta} j2 + \boldsymbol{\beta} i2\, \boldsymbol{\beta} j1) cov(F1, F2)$

Where Ri returns to asset i, αi *is the intercept term in the equation for asset i,* $\boldsymbol{\beta} iz$ *is sensitivity of return to asset i to factor z, Fz is the factor returns, ei is error term* var denotes variance, cov denotes covariance.

The 2 factors you had considered were unanticipated changes in food security (*denoted as Food security; F1*) and unanticipated changes in the climate (*denoted as Climate; F2*).
The annualized return related to *food security* was 10% and *climate* was 12%. The annualized standard deviation of returns for *food security* was 15%, and the same for *climate* was 20%. All index returns and index standard deviation are APR rate monthly compounding.
The two factor *food security* and *climate were correlated* with a correlation coefficient of 0.30.
Following are the results of the regression performed. The **monthly** returns of the fund (dependent variable; Ri) were regressed against the **monthly** returns of the food security index and **monthly** returns of climate index.

$$Ri = \alpha i + \boldsymbol{\beta} i1 * \textbf{Food security} + \boldsymbol{\beta} i2\, \textbf{Climate} + ei$$

The estimates of α's, β's and error standard deviation for all four funds are provided in Table 1 from the regression output ran with monthly returns.
Table1

Funds (i)	α_i	$\beta_i 1$	$\beta_i 2$	error standard deviation$_i$
EM	0.4%	1.2	1.6	1%
GG	1.4%	0.8	0.1	4%
RE	0.9%	0.7	1.2	3%
BE	0.6%	-0.2	-0.5	5%

1.1 *(2 marks)* Calculate the annualized return for all four funds and write as a 4x1 matrix.

1.2 *(2 marks)* Calculate the annualized standard deviation of returns for all four funds as 4x1 matrix

1.3 *(2 marks + 2 marks)* Calculate the annualized covariance and correlation coefficients and present it as 4x4 matrix of variance-covariance matrix and correlation matrix. Comment on the correlation values.

1.4 (*2 marks*) How will you solve the asset allocation problem? Write your optimization method equations (objective function) and constraints.

BITS Pilani presentation

BITS Pilani
Hyderabad Campus

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Department of Economics & Finance

Security Analysis & Portfolio Management (SAPM)

Agenda

Asset Pricing Models

CAPM Problem 1

a). You expect an RFR of 10% and the market return (Rm) of 14%. Compute the expected return for the following stocks and plot them on the SML graph.

Stock	Beta	E(Ri)
U	0.85	
N	1.25	
D	-0.2	

b). You ask the stockbroker what the firm's research department expects for these three stocks. The broker responds with the following information:

Stock	Current Price	Expected Price	Expected Dividend
U	22	24	0.75
N	48	51	2
D	37	40	1.25

Problem 1 contd.

Plot your estimated returns on the graph from Part b and indicate what actions you would take with regard to these stocks. Explain your decisions.

CAPM Problem 2

The following information describes the expected return and risk relationship for the stocks of two of WAH’s competitors.

Using only data shown,

a) Draw and label a graph showing the security market line and position Stocks X and Y relative to it.

b) Compute the alphas both for Stock X and Y.

c) Assume RFR increases to 7%, select the stock with the higher expected risk adjusted return and justify your selection.

	Expected Return	**Standard deviation**	**Beta**
Stock X	12%	20%	1.3
Stock Y	9%	15	0.7
Market index	10%	12%	1
Risk free rate	5%		

CAPM Problem 3

As an equity analyst, you have developed the following return forecasts and risk estimates for two different stock mutual funds (Mutual fund T and U).

a) If risk free rate is 3.9% and expected market risk premium is 6.1%, calculate expected return for each mutual fund according to CAPM.
b) Using the estimated expected returns from Part a along with your own return forecasts, demonstrate whether Fund T and Fund U are currently priced to fall directly on SML, above SML or below SML.
c) According to your analysis, are funds T and U overvalued, undervalued or properly valued?

	Forecasted Return	CAPM Beta
Fund T	9%	1.2
Stock Y	10%	0.8

Thanks

1. $E(R_i) = RFR + \beta_i(R_M - RFR)$

$= .10 + \beta_i(.14 - .10)$

$= .10 + .04\beta_i$

a.

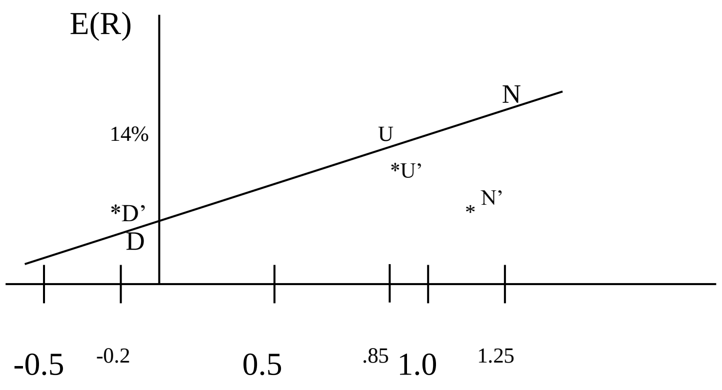
Stock	Beta	(Required Return) $E(R_i) = .10 + .04\beta_i$
U	85	.10 + .04(.85) = .10 + .034 = .134
N	1.25	.10 + .04(1.25)= .10 + .05 = .150
D	-.20	.10 + .04(-.20) = .10 - .008 = .092

b.

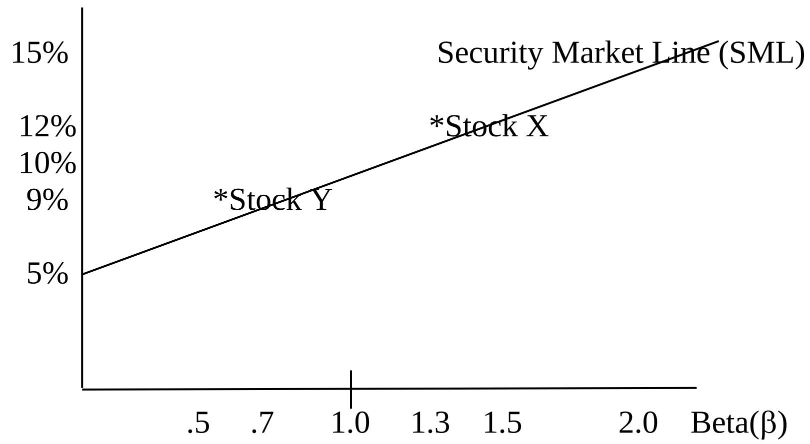
Stock	Current Price	Expected Price	Expected Dividend	Estimated Return
U	22	24	0.75	$\frac{24-22+0.75}{22} = .1250$
N	48	51	2.00	$\frac{51-48+2.00}{48} = .1042$
D	37	40	1.25	$\frac{40-37+1.25}{37} = .1149$

Stock	Beta	Required	Estimated	Evaluation
U	.85	.134	.1250	Overvalued
N	1.25	.150	.1042	Overvalued
D	-.20	.092	.1149	Undervalued

If you believe the appropriateness of these estimated returns, you would buy stocks D and sell stocks U and N.



2. a) The security market line (SML) shows the required return for a given level of systematic risk. The SML is described by a line drawn from the risk-free rate: expected return is 5 percent, where beta equals 0 through the market return; expected return is 10 percent, where beta equal 1.0. Using the SML:
Stock X expected return = 5% + 1.3 (10% - 5%) =11.50%
Stock Y expected return = 5% + 0.7(10% - 5%) = 8.50%



b). The expected risk-return relationship of individual securities may deviate from that suggested by the SML, and that difference is the asset's alpha. Alpha is the difference between the expected (estimated) rate of return for a stock and its required rate of return based on its systematic risk Alpha is computed as

ALPHA (α) $= E(r_i) - [r_f + \beta(E(r_M) - r_f)]$

where

$E(r_i)$ = expected return on Security i
r_f = risk-free rate
β_i = beta for Security i
$E(r_M)$ = expected return on the market

Calculation of alphas:

Stock X: = 12% - [5% + 1.3% (10% - 5%)] = 12% - 11.5% = 0.5%
Stock Y: = 9% - [5% + 0.7%(10% - 5%)] = 9% - 8.5% = 0.5%

In this instance, the alphas are equal and both are positive, so one does not dominate the other.

(c). By increasing the risk-free rate from 5 percent to 7 percent and leaving all other factors unchanged, the slope of the SML [$E(R_{MKT})$ - RFR] flattens and now becomes (10% - 7%) or 3%. Using the formula for alpha, the alpha of Stock X increases to 1.1 percent and the alpha of Stock Y falls to -0.1 percent. In this situation, the expected return (12.0 percent) of Stock X exceeds its required return (10.9 percent) based on the CAPM. Therefore, Stock X's alpha (1.1 percent) is positive. For Stock Y, its expected return (9.0 percent) is below its required return (9.1 percent) based on the CAPM. Therefore, Stock Y's alpha (-0.1 percent) is negative. Stock X is preferable

to Stock Y under these circumstances.
Calculations of revised alphas:

Stock X = 12% - [7% + 1.3 (10% - 7%)]
= 12% - 10.90% = 1.1%

Stock Y = 9% - [7% + 0.7(10% - 7%)]
= 9% - 9.1% = -0.1%

3. Security Market Line

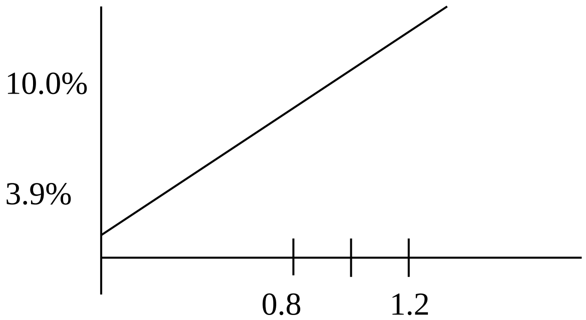
i. *Fair-value plot*. The following template shows, using the CAPM, the expected return, ER, of Fund T and Fund U on the SML. The points are consistent with the following equations:

ER on stock = Risk-free rate + Beta x (Market return – Risk-free rate)

ER for Fund T = 3.9% + 1.2(6.1%)
= 11.22%

ER for Fund U = 3.9% + 0.8(6.1%)
= 8.78%

ii. *Analyst estimate plot*. Using the analyst's estimates, Fund T plots below the SML and Fund U, above the SML.



(c). Over vs. Undervalue

Fund T is overvalued (a potential "sell" candidate) because it should provide a 11.22% return according to the CAPM whereas the analyst has estimated only a 9.0% return.

Fund U **is undervalued** (a potential "buy" candidate) because it should provide a 8.8% return according to the CAPM whereas the analyst has estimated a 10% return.

You are hired as an intern at *Bridge Associates* hedge fund (BAH). Your task is to figure out if there exist a riskless pure arbitrage opportunity in the market. There are no restrictions on short selling in the market. The Hedge fund BAH had allocations in the clean energy funds. Prior research by fund managers showed that returns were explained by a two factor model with the factors being the unanticipated changes in the energy policy (λ_1) and the unanticipated changes in industrial production (λ_2).

Following information on the sensitivities of returns (b_{i1} & b_{i2}) of the fund to the two factors (λ_1 and λ_2) are provided in addition to their expected returns: -

Fund	Expected Return E(R)	b_{i1}	b_{i2}
National Hydro fund (H)	12%	1	0.5
National Solar fund (S)	13.4%	3	0.2
National Wind fund (W)	12%	3	-0.5

1.Find the equation of the plane ($E(R) = \lambda_0 + b_{i1}\lambda_1 + b_{i2}\lambda_2$) that must describe equilibrium returns.

2. You had been following the Socially responsible fund (SRF). The sensitivity of SRF fund returns to the first factor was $b_{srf1} = 2$ and second factor was $b_{srf2} = 0$. The estimated return of SRF fund is 9%. Is there a riskless pure arbitrage opportunity existing, show how?

Find the equation of the plane ($E(R) = \lambda_0 + b_{i1}\lambda_1 + b_{i2}\lambda_2$) that must describe equilibrium returns.

$E(R) = \lambda_0 + b_{i1}\lambda_1 + b_{i2}\lambda_2$

1 mark for below 3 equations

$12\% = \lambda_0 + 1(\lambda_1) + 0.5\lambda_2$

$13.4\% = \lambda_0 + 3(\lambda_1) + 0.2\lambda_2$

$12\% = \lambda_0 + 3(\lambda_1) -0.5\lambda_2$

Solving above three equation we get values of factors

λ_0	10%
λ_1	1%
λ_2	2%

$E(Ri) = 10\% + b_{i1} * 1\% + b_{i2} * 2\%$

You had been following the Socially responsible fund (SRF). The sensitivity of SRF fund returns to the first factor was $b_{srf1} = 2$ and second factor was $b_{srf2} = 0$. The estimated return of SRF fund is 9%. Is there a riskless pure arbitrage opportunity existing? Illustrate.

As per the APT model, the E(R of SRF) = 10% + 2*1% + 0*3.5% = 12%

The estimated return is 10%. Hence it is overpriced and be short sold.

We create a long portfolio with similar factor sensitivities as SRF

$b_{srf1} = 2$, $b_{srf2} = 0$

$w_h + w_s + w_w = 1$

$w_h * 1 + w_s * 3 + w_w * 3 = 2$

$w_h * 0.5 + w_s * 0.2 + w_w * (-0.5) = 0$

We get 50% of Investment in H and 50% Investment in W, 0% in S

With above long portfolio with 50% in H and 50% in W, the sensitivities to the factors are the same as SRF (for factor 1 = 2, for factor 2 = 0)

For this long portfolio, E(R) = 50%*12% + 50%*12% = 12%

We go short on SRF , wsrf = -1 and wh = 0.5 and ww = 0.5 to make Net Investment as 0.

The arbitrage profit = 12% - 9% =3% for every $ invested in long portfolio

Security Valuation using APT

Given: $E(R_A) = 0.8\lambda_1 + 0.9\lambda_2$

$E(R_B) = -0.2\lambda_1 + 1.3\lambda_2$

$E(R_C) = 1.8\lambda_1 + 0.5\lambda_2$

$\lambda_1 = 4\%$

$\lambda_2 = 5\%$

Analyst forecast of prices one year from now

$P_{1A} = \$37.2$ / share

$P_{1B} = \$37.8$ / share

$P_{1C} = \$38.5$ / share

Also, $P_{0A} = P_{0B} = P_{0C} = \$35$/share are prices of securities A, B & C today.

Soln: From APT model,

$E(R_A) = 0.8 \times 4\% + 0.9 \times 5\% = 7.7\%$

$E(R_B) = -0.2 \times 4\% + 1.3 \times 5\% = 5.7\%$

$E(R_C) = 1.8 \times 4\% + 0.5 \times 5\% = 9.7\%$

From analyst forecast,
estimated returns are

	Expected as per APT
$R_A = \frac{37.2-35}{35} = 6.286\%$	7.7%
$R_B = \frac{37.8-35}{35} = 8\%$	5.7%
$R_C = \frac{38.5-35}{35} = 10\%$	9.7%

Hence, A is Overpriced & B&C are Underpriced.

Q/ Is there a pure arbitrage profit meaning a risk free zero investment arbitrage profit from long short strategy.

	β_1	β_2	w
A	0.8	0.9	w_A
B	-0.2	1.3	w_B
C	1.8	0.5	w_C

A/ Problem can be stated as
To solve: Can a portfolio be created by going long in B&C to have the same β's as A.

$$-0.2w_B + 1.8w_C = 0.8$$

$$1.3w_B + 0.5w_C = 0.9$$

We get, $w_B = 0.5$
$w_C = 0.5$.

For Net Investment to be zero;

we have

$w_A = -1$

$w_B = +0.5$

$w_C = +0.5$.

Above means that for every share of B & C held long, 2 shares of A is held short.

As 1 share of B; value = \$35 so $w_B = 0.5$ $\left(=\frac{35}{70}\right)$

1 share of C; value = \$35 $w_C = 0.5$

2 shares of A; value = \$70 $w_A = -1$
short short.

Conditions to hold for risk less arbitrage

1) $\sum w_i = 0$ [i.e. no net wealth invested]

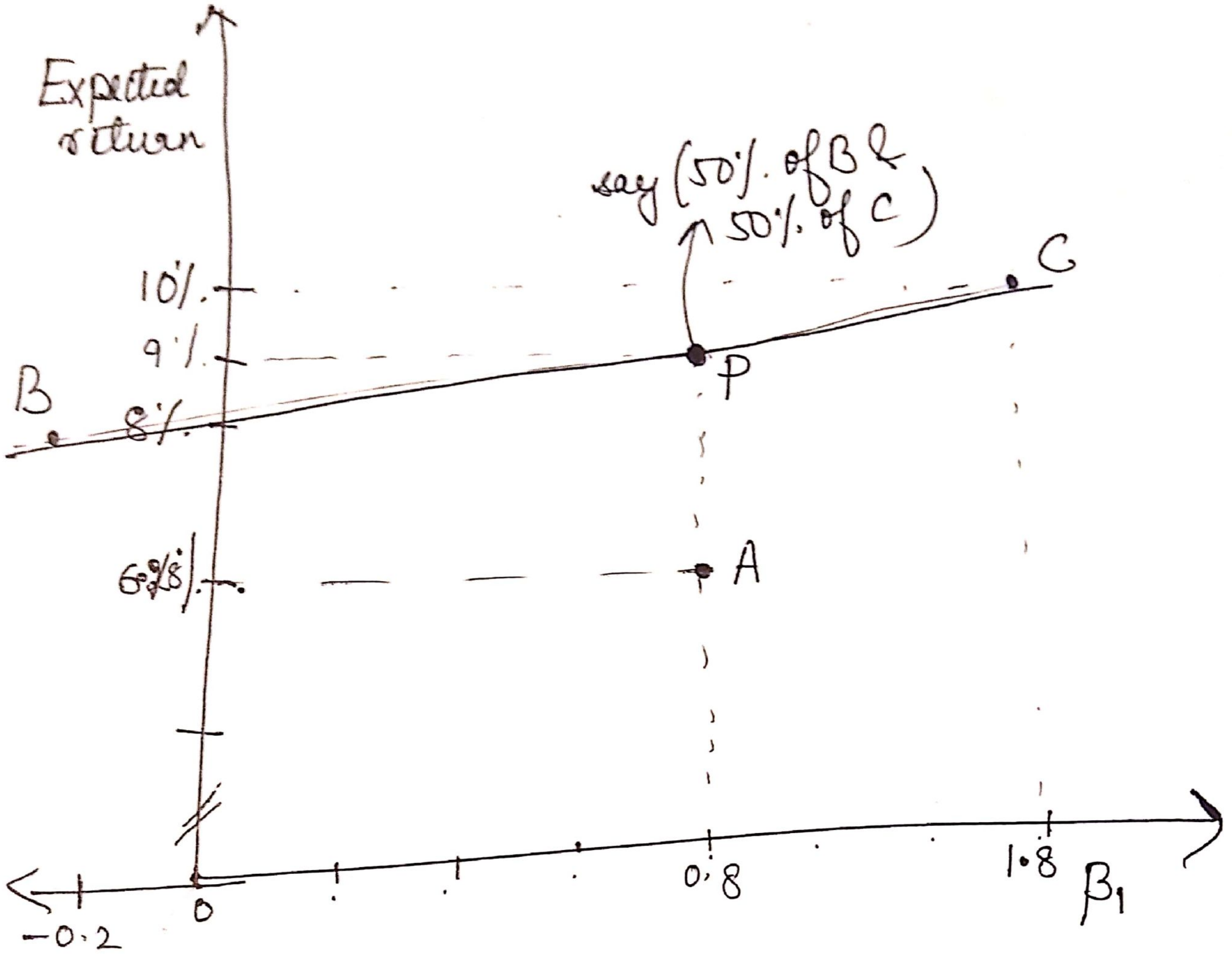
2) $\sum w_i b_{ij} = 0$ for all K factors
(no systematic risk]

3) $\sum w_i R_i > 0$ [Return of long-short portfolio is +ve]

Expected return (from estimated) of Long portfolio (with B & C)

$= 0.5 \times 8\% + 0.5 \times 10\% = 9\%$

Expected return from shorting A = (—)6.286%.



So short-selling A & going long in P formed by (50% of B & 50% of C) gives a positive payoff.

Arbitrage Profit = ~~8~~ 9% − 6.286% = 2.714% for every $ which we went long.

Problem 1.

Consider the following "portfolio choice" problem. The investor has initial wealth w and utility $u(x) = \ln(x)$. There is a *safe asset* (such as a US government bond) that has net real return of zero. There is also a *risky asset* with a random net return that has only two possible returns, R_1 with probability q and R_0 with probability $1-q$. Let A be the amount invested in the risky asset, so that $w - A$ is invested in the safe asset.

1. Find A as a function of w. Does the investor put more or less of his portfolio into the risky asset as his wealth increases?
2. Another investor has the utility function $u(x) = -e^{-x}$. How does her investment in the risky asset change with wealth?
3. Find the coefficients of absolute risk aversion $r(x) = -\frac{u''(x)}{u'(x)}$ for the two investors. How do they depend on wealth? How does this account for the qualitative difference in the answers you obtain in parts (1) and (2)?

Solution of Problem 1:

Firstly, let's set up the problem:

$$\max_{A\in[0,w]} \{q\, u\,((1+R_1)A + w - A) + (1-q)\, u\,((1+R_0)A + w - A)\}$$

Part 1: When we have a specific utility function $u(x) = \ln(x)$, we can get the first order condition as

$$\begin{aligned} q\frac{R_1}{R_1 A + w} + (1-q)\frac{R_0}{R_0 A + w} &= 0 \\ \Longrightarrow A &= -w\frac{qR_1 + (1-q)\,R_0}{R_0 R_1} \end{aligned}$$

Part 2: From the first order condition,

$$\begin{aligned} R_1 q e^{-(R_1 A + w)} + R_0(1-q)e^{-(R_0 A + w)} &= 0 \\ \Longrightarrow A &= \frac{1}{R_0 - R_1}\ln\left[-\frac{R_0(1-q)}{R_1 q}\right] \end{aligned}$$

Observe that $\frac{dA}{dw} = 0$; that is, her investment in the risky asset doesn't change with wealth.

Part 3: For $u(x) = \ln(x)$, we have $u'(x) = \frac{1}{x}$ and $u''(x) = -\frac{1}{x^2}$. So $r(x) = \frac{1}{x}$; i.e., as x gets bigger, $r(x)$ gets smaller, and so the wealthier the investor is, the less risk averse she is. Therefore, she will put more wealth into the risky asset.

For $u(x) = -e^{-x}$, we have $u'(x) = e^{-x}$ and $u''(x) = -e^{-x}$. So $r(x) = 1$. Therefore, the amount that the investor allocates to the risky asset is independent of her wealth.

Question on Event study methodology

“Cricket is considered as a religion in Indian sub-continent and people are crazy about Cricket, especially in this part of the world. The impact of sporting events on stock prices have already been captured in several recent research studies. A finding in these studies shows that stock prices react sharply to team performance in big sporting events, and when it comes to an event like the Cricket World Cup, the impact would be certainly significant, for sure. Many argue that a sporting event is a non-economic phenomenon and, as such, stock price will not be affected. However, behavioural finance theorists suggest that large sporting events affect the sentiments of viewers (who are/might be simultaneously investors) resulting in upwards or downwards “mood swings” in the stock market, which are subsequently reflected in stock prices.”

Reference-https://behavioralfinance.wordpress.com/2011/03/25/cricket-world-cup-and-indian-stock-market-is-there-any-relationship/

You are working as a research analyst and you wanted to check the impact of ICC Cricket World Cup (winning and losing) by Indian (Men) Cricket Team on Stock market performance of India (You use NIFTY50 Index as a proxy).

The data for Adjusted closing prices for NIFTY 50 is provided in the excel sheet along with the dates.
You realized that you can use event study methodology here to calculate the Cumulative Abnormal returns (CAR) during the event window. Plot a graph (2 graphs here, one for WIN one for LOSS) of event window CAR on Y-axis and Days relative to event on the X-axis. **For Expected returns calculation, use average of the daily returns during the estimation window as expected returns**. **For returns use Ln(Pt/Pt-1) using Adjusted closing prices.**

1.1 Event: ICC Cricket World Cup Win by Indian team 2011. Run the event study and plot the required graph for event window. Interpret your results.

Event date: April 2nd 2011. India winning the World cup (You can take the most recent next trading day as the event date – time 0).
Event window: -20 to +20 days to Event day
Estimation window: -220 to -21
Post event window: +21 to +40
Other dates for reference: 30th March 2011 (India won – Semi-final against Pakistan), 24th March 2011 (India won- Quarter final against Australia)

1.2 Event: ICC Cricket World Cup Loss by Indian team 2015. Run the event study and plot the required graph. Interpret your results.

Event date: March 26th 2015. India losing the World cup Semi-final (You can take the most recent next trading day as the event date – time 0).
Event window: -20 to +20 to Event day
Estimation window: -220 to -21
Post event window: +21 to +40

1.3 Overall what is your interpretation? Is there an impact of Cricket World Cup win or loss on the stock market performance?

Practice Problem on Event Study

You wanted to find the response of Oil & Gas companies to the comments of the MEA of the Government on the import of oil from a country X in war. The event day is 26th April 2022 when MEA had explained their stand on the import of oil from X.
You ran a market model to find the estimates of the sensitivity of returns for the estimation period (-220 to +20 days from event day).

$$Ri = \alpha i + \beta i\ Rm + ei$$

Following Table 6.1 are the results of the market model for two Oil & Gas companies.
Table 6.1 : Estimates of Market model

Company	Beta	alpha
Indi Petroleum (IP)	70% more volatile than market	Insignificant
Ril Petroleum (RP)	50% more volatile than market	Insignificant

You have the following information Table 6.2 on the Adjusted Closing Prices of the two Oil & Gas Companies and Adjusted Closing Prices for the Market Index.
Table 6.2: Adjusted Closing Prices

	Adjusted Closing Prices (INR)		
Date	Ril Petroleum	Indi Petroleum	Index
21-04-2022	2782	398	17392
22-04-2022	2758	393	17172
25-04-2022	2695	369	16954
26-04-2022	2775	373	17201
27-04-2022	2778	367	17038
28-04-2022	2819	369	17245
29-04-2022	2790	362	17102

6.1 (4 marks) Find the Cumulative Abnormal return for an event window of -1 to +1 for the two companies and Show results in Table format.*(Assume continuously compounding for returns calculation)*
6.2 (1 mark) Find the average Cumulative Abnormal return for -1 to +1 days relative to the event day. Interpret the response based on the results found. *(Assume continuously compounding for returns calculation)*

		Adjusted Closing Prices (INR)			Actual returns (RIL returns)	Actual returns (Indi returns)
	Date	Ril Petroleum	Indi Petrol	Index		RIL
	21-04-2022	2782	398	17392		
	22-04-2022	2758	393	17172	-0.87%	-1.26%
-1	25-04-2022	2695	369	16954	-2.31%	-6.30%
0	26-04-2022	2775	373	17201	2.93%	1.08%
1	27-04-2022	2778	367	17038	0.11%	-1.62%
	28-04-2022	2819	369	17245	1.47%	0.54%
	29-04-2022	2790	362	17102	-1.03%	-1.92%

beta of RIL	BETA OF Indi Petroleum
1.5	1.7

Actual returns (Index returns) Indi		Expected returns (rilreturns) RIL beta*Rm	Expected returns (Indi) Indi beta*Rm	Abnormal returns RIL	Abnormal returns Indi
-0.0127302		-0.0191	-0.02164	1.04%	0.90%
-0.0127764	-1	-0.01916	-0.02172	-0.39%	-4.13%
0.0144637	0	0.021696	0.024588	0.76%	-1.38%
-0.0095214	1	-0.01428	-0.01619	1.54%	0.00%
0.0120761		0.018114	0.020529	-0.35%	-1.51%
-0.0083268		-0.01249	-0.01416	0.21%	-0.50%

CAR	CAR				
RIL	**Indi**	**Average**			
		CAAR	**Event window**	AAR	**CAAR**
-0.39%	**-4.13%**	**-2.26%**	**-1**	-2.26%	-2.26%
0.36%	**-5.51%**	**-2.57%**	**0**	-0.31%	-2.57%
1.90%	**-5.51%**	**-1.81%**	**1**	0.77%	-1.81%

The response of Oil & Gas cos.
is NEGATIVE on the stand of
MEA of the Government on import of crude oil from X country

FAMA'S PERFORMANCE MEASURE

Let $R_a \rightarrow$ Return generated by the fund.

$R_f \rightarrow$ Risk-free rate

1). Overall Performance measure

The Overall performance measure is the return generated over a risk free rate.

$$\boxed{\text{Overall performance} = R_a - R_f}$$

2). Selectivity

The Overall performance measure can be divided into 2 components.

The first is Selectivity which measures the excess returns generated over and above the expected return due to systematic risk of the portfolio.

$$\boxed{\text{Selectivity} = R_a - R_x(\beta_a)}$$

where $R_x(\beta_a) = E(R_a) = R_f + \beta_a(R_m - R_f)$

3). Risk

The second component of the overall performance measure is the Risk component which measures the expected return generated only due to systematic risk (over and above a time value of money component).

$$\boxed{\text{Risk} = R_x(\beta a) - R_f}$$

Hence,

Overall Performance = Selectivity component + Risk component

$$\boxed{[R_a - R_f] = [R_a - R_x(\beta a)] + [R_x(\beta a) - R_f]}$$

4. <u>Diversification component</u>

The fund manager may have placed his bets (bought underpriced stocks) that he thought may be winners (provide much higher return). In doing so he may have compromised on the diversification by putting or buying more winning stocks which may be highly positively correlated with the other stocks in the portfolio.

The fund may be exposed to more risk i.e. unsystematic risk in addition to the systematic risk.

The expected return which one should be rewarded for holding total risk can be calculated using the CML (Capital Market Line).

For holding total risk σ_p, the expected return by CML line is given as follows:-

$$E(R_p) = R_f + \left(\frac{R_m - R_f}{\sigma_m}\right)\sigma_p .$$

This is denoted as

$$\boxed{E(\sigma(R_a)) = E(R_a) = R_f + \left(\frac{R_m - R_f}{\sigma_m}\right)\sigma_a}$$

or $R_x(\sigma(R_a))$

The expected return due to systematic risk is

$$\boxed{E(R_p) = R_x(\beta_a) = R_f + \beta_a(R_m - R_f)}$$

Hence, the expected return due to holding only unsystematic risk is

$$\boxed{E(\sigma(R_a)) - R_x(\beta_a)}$$

or $[R_x(\sigma(R_a)) - R_x(\beta_a)]$

The above is denoted as Diversification component which measures the additional expected return the manager must earn for holding unsystematic risk (i.e. for justifying buying winners by compromising on diversification).

Diversification component

$$= E(\sigma(R_a)) - R_x(\beta_a)$$

or $R_x(\sigma(R_a)) - R_x(\beta_a)$

5. <u>Net Selectivity</u>

Net Selectivity is measure the excess return generated by the fund over and above the expected return due to both systematic and unsystematic risk.

Net Selectivity = Selectivity − Diversification component

$$\boxed{\text{Net Selectivity} = R_a - \underbrace{R_x(\beta_a)}_{\substack{\text{E(R) due to}\\ \text{Systematic}\\ \text{risk}}} - \Big[\underbrace{R_x(\sigma(R_a)) - R_x(\beta_a)}_{\substack{\text{E(R) due to}\\ \text{Unsystematic}\\ \text{risk}}}\Big]}$$

You have prepared the following summary of the data, with the standard errors for each of the coefficients listed in parentheses.

	REGRESSION DATA			(R_{FUND} − RFR)	
Portfolio	α	β	R^2	Mean	σ
ABC	0.192 (0.11)	1.048 (0.10)	94.1%	1.022%	1.193%
DEF	−0.053 (0.19)	0.662 (0.09)	91.6	0.473	0.764
GHI	0.463 (0.19)	0.594 (0.07)	68.6	0.935	0.793
JKL	0.355 (0.22)	0.757 (0.08)	64.1	0.955	1.044
(MNO)	0.296 (0.14)	0.785 (0.12)	94.8	0.890	0.890

a. Which fund had the highest degree of diversification over the sample period? How is diversification measured in this statistical framework?

b. Rank these funds' performance according to the Sharpe, Treynor, and Jensen measures.

c. Since you know that according to the CAPM the intercept of these regressions (i.e., alpha) should be zero, this coefficient can be used as a measure of the value added provided by the investment manager. Which funds have statistically outperformed and underperformed the market using a two-sided 95 percent confidence interval? (Note: The relevant t-statistic using 60 observations is 2.00.)

4. You have just gathered the following performance data for three different money managers, based on a regression of their excess returns relative to those for the S&P 500 Index. Each manager's performance was measured over the same three-year period, but the return period for each was different.

Manager	Alpha	Beta	Std. Error of Regression	Return Period
A	0.058%	0.95	0.533%	Weekly
B	0.115	1.12	5.884	Biweekly
C	0.250	0.78	2.165	Monthly

a. Calculate the information ratio for each manager, ignoring the difference in return reporting periods.

b. Calculate the annualized information ratio for each manager.

c. Rank the managers' performance according to your answers in Parts a and b. Which manager performed the best? Explain.

5. Consider the following historical performance data for two different portfolios, the Standard and Poor's 500, and the 90-day T-bill.

Investment Vehicle	Average Rate of Return	Standard Deviation	Beta	R^2
Fund 1	26.40%	20.67%	1.351	0.751
Fund 2	13.22	14.20	0.905	0.713
S&P 500	15.71	13.25		
90-day T-bill	6.20	0.50		

a. Calculate the Fama overall performance measure for both funds.

b. What is the return to risk for both funds?

c. For both funds, compute the measures of (1) selectivity, (2) diversification, and (3) net selectivity.

d. Explain the meaning of the net selectivity measure and how it helps you evaluate investor performance. Which fund had the best performance?

6. You are evaluating the performance of two portfolio managers, and you have gathered annual return data for the past decade:

Year	Manager X Return (%)	Manager Y Return (%)
1	−1.5	−6.5
2	−1.5	−3.5
3	−1.5	−1.5
4	−1.0	3.5
5	0.0	4.5
6	4.5	6.5
7	6.5	7.5
8	8.5	8.5
9	13.5	12.5
10	17.5	13.5

a. For each manager, calculate (1) the average annual return, (2) the standard deviation of returns, and (3) the semi-deviation of returns.

b. Assuming that the average annual risk-free rate during the 10-year sample period was 1.5 percent, calculate the Sharpe ratio for each portfolio. Based on these computations, which manager appears to have performed the best?

c. Calculate the Sortino ratio for each portfolio, using the average risk-free rate as the minimum acceptable return threshold. Based on these computations, which manager appears to have performed the best?

d. When would you expect the Sharpe and Sortino measures to provide (1) the same performance ranking, or (2) different performance rankings? Explain.

7. Consider the following performance data for two portfolio managers (A and B) and a common benchmark portfolio:

	BENCHMARK		MANAGER A		MANAGER B	
	Weight	Return	Weight	Return	Weight	Return
Stock	0.6	−5.0%	0.5	−4.0%	0.3	−5.0%
Bonds	0.3	−3.5	0.2	−2.5	0.4	−3.5
Cash	0.1	0.3	0.3	0.3	0.3	0.3

a. Calculate (1) the overall return to the benchmark portfolio, (2) the overall return to Manager A's actual portfolio, and (3) the overall return to Manager B's actual portfolio. Briefly comment on whether these managers have under- or outperformed the benchmark fund.

b. Using attribution analysis, calculate (1) the *selection effect* for Manager A, and (3) the *allocation effect* for Manager B. Using these numbers in conjunction with your results from Part a, comment on whether these managers have added value through their selection skills, their allocation skills, or both.

8. A U.S. pension plan hired two offshore firms to manage the non-U.S. equity portion of its total portfolio. Each firm was free to own stocks in any country market included in Morgan Stanley/Capital International's Europe, Australia, and Far East Index (EAFE), and free to use

5(a). Overall performance (Fund 1) = 26.40% - 6.20% = 20.20%
Overall performance (Fund 2) = 13.22% - 6.20% = 7.02%

5(b). $E(R_i) = 6.20 + \beta(15.71 - 6.20)$
$= 6.20 + \beta\ (9.51)$
Total return (Fund 1) = 6.20 + (1.351)(9.51) = 6.20 + 12.85 = 19.05%
where 12.85% is the required return for risk
Total return (Fund 2) = 6.20 + (0.905)(9.51) = 6.20 + 8.61 = 14.81%
where 8.61% is the required return for risk

5(c)(i). Selectivity_1 = 20.2% - 12.85% = 7.35%

Selectivity_2 = 7.02% - 8.61% = -1.59%

5(c)(ii). Ratio of total $\text{risk}_1 = \sigma_1/\sigma_m = 20.67/13.25 = 1.56$

Ratio of total $\text{risk}_2 = \sigma_2/\sigma_m = 14.20/13.25 = 1.07$

$R_1 = 6.20 + 1.56\ (9.51) = 6.20 + 14.8356 = 21.04\%$

$R_2 = 6.20 + 1.07\ (9.51) = 6.20 + 10.1757 = 16.38\%$

Diversification_1 = 21.04% – 19.05% = 1.99%

Diversification_2 = 16.38% – 14.81% = 1.57%

5(c)(iii). Net Selectivity = Selectivity – Diversification

Net Selectivity_1 = 7.35% - 1.99% = 5.36%

Net Selectivity_2 = -1.59% - 1.57% = -3.16%

5(d). Even accounting for the added cost of incomplete diversification, Fund 1's performance was above the market line (best performance), while Fund 2 fall below the line.

Valuation Ratio	Definition	Measurement Issues	When to Use	Valuation Model
Price to earnings (P/E)	$\frac{\text{Price per Share}}{\text{Earnings per Share}}$ *Price per share* = Market price per share of common stock *Earnings per share* = Annual net income ÷ Shares outstanding	*Price per share*—Typically the most recent share price is used. However, if share price is very volatile, a normalized price (e.g., an average of beginning and ending prices for the month) may be used. *Earnings per share*—Although current-year or annualized current-quarter (trailing) earnings can be used, it is not uncommon to use analysts' expectations of future (leading) earnings. Moreover, earnings are typically measured before extraordinary items and may only include earnings from the firm's primary operations.	Firms with an established record of positive earnings that do not have significant noncash expenditures.	$\text{Estimated Stock Price} = (\text{EPS})_{\text{Firm}} \times \left(\text{P/E Ratio}\right)_{\text{Industry}}$
Price to earnings to growth (PEG)	$\frac{\text{Price per Share}}{\text{Earnings per Share (EPS)}}$ ÷ Growth rate in EPS *Growth rate in EPS* = Expected rate of growth in EPS over the next year	*Growth rate in EPS*—estimates of growth rates can be made from historical earnings estimates or can be obtained from analysts' estimates.	Firms that face stable prospects for future growth in EPS as well as similar capital structures and similar industry risk attributes.	$\text{Estimated Stock Price} = \left(\frac{\text{Growth Rate in EPS}}{\text{EPS}}\right)_{\text{Firm}} \times \left(\text{PEG Ratio}\right)_{\text{Industry}}$

Market to book value of equity	$\frac{\text{Market Value of Equity}}{\text{Book Value of Equity}}$ *Market value of equity* = Price per share × Shares outstanding *Book value of equity* = Total assets – Total liabilities	*Market value of equity*—The same issues that arise in selecting a price per share apply here. *Book value of equity*—Although the book value of equity is easy to pull from the firm's balance sheet, differences in the ages of firm assets (when acquired and rates of depreciation) as well as other differences in how various assets are accounted for (fair market value—mark to market—or cost), and the conservatism employed in accounting practices can lead to variation across firms.	Firms whose balance sheets are reasonable reflections of the market values of their assets. Financial institutions are the classic case.	$\text{Estimated Equity Value} = \left(\begin{array}{c}\text{Book Value}\\ \text{of Equity}\end{array}\right)_{\text{Firm}} \times \left(\begin{array}{c}\text{Market to}\\ \text{Book Ratio}\end{array}\right)_{\text{Industry}}$
Panel b. Enterprise Valuation Ratios				
Enterprise value (EV) to EBITDA	$\frac{\text{Enterprise Value}}{\text{EBITDA}}$ *Enterprise value* = Price per share × Shares outstanding plus Interest-bearing (short- and long-term) debt less cash *EBITDA* = Earnings before interest, taxes, depreciation, and amortization	*Enterprise value*—EV is typically estimated as the market value of the firm's equity (price per share times shares outstanding) plus the book value of the firm's interest-bearing debt. Consequently, the problems that arise in determining which stock price to use (most recent versus normalized) arise here, too. *EBITDA*—This earnings figure is easily accessed from the firm's income statement. However, cyclical variations in earnings and unusual variations in firm revenues may require some normalization to better reflect the firm's future earnings potential.	Firms that have significant noncash expenses (i.e., depreciation and amortization). Examples include industries with large investments in fixed assets, including health care, oil and gas equipment services, and telecommunications.	$\text{Estimated Enterprise Value} = (\text{EBITDA})_{\text{Firm}} \times \left(\begin{array}{c}\text{EV to EBITDA}\\ \text{Ratio}\end{array}\right)_{\text{Industry}}$

Valuation Ratio	Definition	Measurement Issues	When to Use	Valuation Model
Enterprise value (EV) to cash flow	$\frac{\text{Enterprise Value}}{\text{Cash Flow per Share}}$ *Enterprise value* = Price per share × Shares outstanding plus Interest-bearing debt *Cash flow* = Firm free cash flow (FFCF)	*Cash flow*—FFCF adjusts EBITDA to consider taxes and additional investment required in capital equipment (CAPEX). Consequently, the issues that arise with FFCF include those applicable to EBITDA, plus any normalization of CAPEX required to adjust for extraordinary circumstances in any given year.	Firms with stable growth and thus predictable capital expenditures. Examples include chemicals, paper products, forestry, and industrial metals.	$\text{Estimated Enterprise Value} = (\text{FFCF})_{\text{Firm}} \times \left(\text{EV to FFCF Ratio}\right)_{\text{Industry}}$
Enterprise value (EV) to sales	$\frac{\text{Enterprise Value}}{\text{Sales}}$ *Sales* = Firm annual revenues	*Sales*—Firm revenues sit at the top of the firm's income statement. However, they, too, may not be reflective of the firm's earning potential if they are abnormally large or depressed due to nonrecurring factors.	Young firms and start-ups that do not have an established history of earnings.	$\text{Estimated Enterprise Value} = (\text{Sales})_{\text{Firm}} \times \left(\text{EV to Sales Ratio}\right)_{\text{Industry}}$

Question IV ***(5 Marks):***

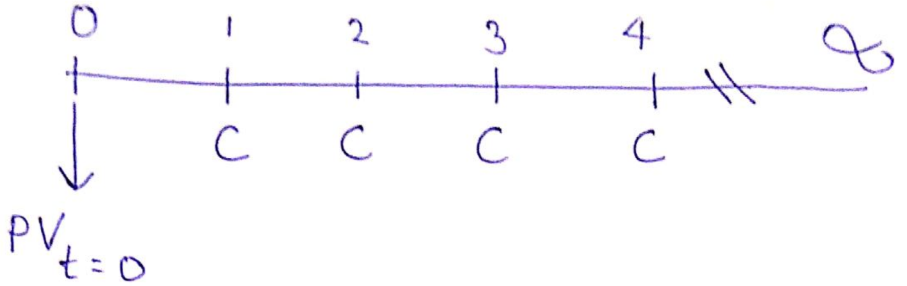
(5 marks) How much an investor should be willing to pay for growth? As a general rule, a PEG ratio of 1.0 or lower suggests a stock is fairly priced or undervalued respectively. A PEG ratio above 1.0 suggests a stock is overvalued.

4.1 Calculate the PEG ratio for the firms below (present in Table format 5x1)
4.2 Based on the PEG ratio, suggest which stocks are overpriced, underpriced or fairly priced in a Table format (5X1)

Name of Company	**Market value of equity (INR crores)**	**Net Profit (INR crores)**	**Dividend pay-out ratio (%)**	**Shareholder's equity (INR crores)**
Techno Ltd. (TL)	5000	130	10%	500
Lesla Ltd. (LL)	2000	28	15%	55
UniverseX Ltd. (UL)	10000	400	20%	80
Kpharma Ltd. (KL)	500	15	25%	25
Risco Ltd. (RL)	7600	200	2%	55

Name of Company	P/E	EPS growth	PEG	Based on PE	Based on PEG
HCL	26.23	13.75%	1.90764	Underpriced	Overpriced
Tech Mahindra	26.87	9.10%	2.95275	Underpriced	Overpriced
Mindtree	47.49	87.37%	0.54355	Overpriced	Underpriced
WNS Holdings	38.19	-15.19%	-2.51415	Overpriced	Underpriced
Tata Consultancy services	37.06	3.15%	11.76508	Overpriced	Overpriced
Cognizant Tech solutions ADR	22.08	38.06%	0.58014	Underpriced	Underpriced
Genpact	25.11	7.41%	3.38866	Underpriced	Overpriced
EXL Services Holdings	41.69	74%	0.56338	Overpriced	Underpriced
Wipro	27.4	15.16%	1.80739	Underpriced	Overpriced
Infosys	36.96	17%	2.17412	Overpriced	Overpriced
Mphasis	50.84	3%	16.40000	Overpriced	Overpriced
Average (excl WNS Holdings)	**34.173**	**27%**	**4.20827**	Overpriced	Overpriced

Perpetuity or Constant perpetuity [Present Value]



$$PV_{t=0} = \sum_{t=1}^{\infty} \frac{C_t}{(1+r)^t}$$

$$= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \infty$$

$$= \frac{C}{1+r}\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots \infty\right)$$

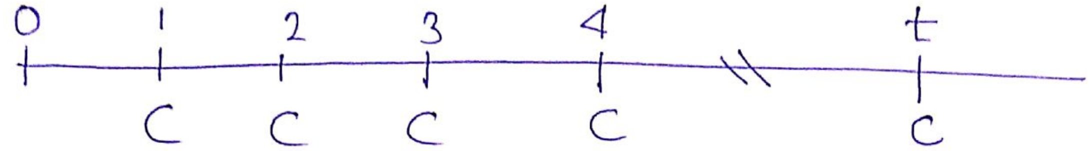
$$= \frac{C}{1+r}\left[\frac{1}{1-\left(\frac{1}{1+r}\right)}\right]$$

$$= \frac{C}{1+r}\left[\frac{1+r}{r}\right]$$

$$= \frac{C}{r} \; ; \quad r \neq 0$$

$PV_{t=0}$ for constant perpetuity $= \frac{C}{r}$

Present Value of Constant Annuity



$PV_{t=0}$

$$PV_{t=0} = \sum_{t=1}^{t} \frac{C_t}{(1+r)^t}$$

$$= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^t}$$

$$= \left(\frac{C}{1+r}\right)\left[1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{t-1}}\right]$$

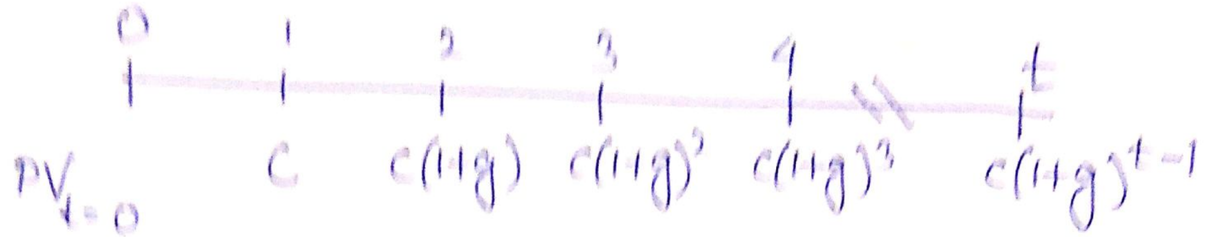
$$= \left(\frac{C}{1+r}\right)\left[\frac{1\left(1-\left(\frac{1}{1+r}\right)^t\right)}{1-\left(\frac{1}{1+r}\right)}\right]$$

$$= \left(\frac{C}{1+r}\right)\left(\frac{1+r}{r}\right)\left(1-\frac{1}{(1+r)^t}\right) \quad ; r \neq 0$$

$$\boxed{PV_{t=0} = \frac{C}{r}\left[1-\frac{1}{(1+r)^t}\right]}$$

of constant annuity

Present Value of Growing Annuity



$$PV_{t=0} = \sum_{t=1}^{t} \frac{C_t}{(1+r)^t}$$

$$= \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{t-1}}{(1+r)^t}$$

$$= \left(\frac{C}{1+r}\right)\left[1 + \left(\frac{1+g}{1+r}\right) + \cdots + \left(\frac{1+g}{1+r}\right)^{t-1}\right]$$

$$= \left(\frac{C}{1+r}\right)\left[\left(1 - \left(\frac{1+g}{1+r}\right)^{t}\right)\right] \times \frac{1}{1-\left(\frac{1+g}{1+r}\right)}$$

$$= \left(\frac{C}{1+r}\right)\left(\frac{1+r}{r-g}\right)\left[1 - \left(\frac{1+g}{1+r}\right)^{t}\right]$$

$$PV_{t=0} = \left(\frac{C}{r-g}\right)\left[1 - \left(\frac{1+g}{1+r}\right)^{t}\right] \quad ; \quad r \neq g$$

PV of growing annuity if
$r = g$

$$PV_{t=0} = \frac{C \cdot t}{(1+r)}$$

Why $r > g$ for a Growing Perpetuity?

PV formula to hold

Timeline of Cash Flows

	C	$C(1+g)$	$C(1+g)^2$	. . .	. . .	. . .		$C(1+g)^{n-1}$	
0	1	2	3	4	5	6	$\cdots$	n	∞

PV of growing perpetuity

$$= \lim_{n\to\infty}\left[\left(\frac{C}{1+r}\right) + \frac{C(1+g)}{(1+r)^2} + \cdots\cdots + \frac{C(1+g)^{n-1}}{(1+r)^n}\right]$$

$$= \lim_{n\to\infty}\left(\frac{C}{1+r}\right)\left[1 + \left(\frac{1+g}{1+r}\right) + \cdots\cdots + \left(\frac{1+g}{1+r}\right)^{n-1}\right]$$

$$= \lim_{n\to\infty}\left(\frac{C}{1+r}\right)\underbrace{\left[\frac{1.\left(1-\left(\frac{1+g}{1+r}\right)^n\right)}{1-\left(\frac{1+g}{1+r}\right)}\right]}_{\text{Sum of a finite Geometric series}}$$

$$= \lim_{n\to\infty}\left(\frac{C}{\cancel{1+r}}\right)\left[\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right].\cancel{(1+r)}$$

$$= \left(\frac{C}{r-g}\right)\left[1 - \lim_{n\to\infty}\left(\frac{1+g}{1+r}\right)^n\right]$$

Now $\left(\frac{1+g}{1+r}\right)^n \to 0$ as $n \to \infty$

only if $\left(\frac{1+g}{1+r}\right) < 1$

or $(1+g) < (1+r)$

or $r > g$

So PV of growing perpetuity $= \left(\frac{c}{r-g}\right)$

only if $r > g$.

If $g > r$,

The PV of growing perpetuity $\to \infty$

Why should r > g for the formula C/(r-g) to hold (Intuition is similar to the convergence explanation)? PV of Growing Perpetuity

A stream of cash flows growing at a rate of g each period and discounted at a constant interest rate r is worth

$$PV_0 = \frac{C_1}{r - g}$$

The first cash flow, C_1, occurs next period (time 1), the second cash flow of $C_2 = C_1 \cdot (1 + g)$ occurs in two periods, and so forth, *forever*. For the formula to work, g can be negative, but r must be greater than g.

You need to memorize the growing perpetuity formula!

IMPORTANT

Be careful to use the cash flow *next* year in the numerator. The subscript "1" is there to remind you. For example, if you want to use this formula on your firm, and it earned \$100 million this year, and you expect it to grow at a 5% rate forever, then the correct cash flow in the numerator is $C_1 = \$105$ million, not \$100 million!

Although a subscript on C makes this seem more painful, it is a good reminder here.

What would happen if the cash flows grew faster than the interest rate ($g > r$)? Wouldn't the formula indicate a negative PV? Yes, but this is because the entire scenario would be nonsense. The present value in the perpetuities formulas is only less than infinity, because *in today's dollars*, each term in the sum is a little less than the term in the previous period. If g were greater than r, however, the cash flow 1 period later would be worth more even in today's dollars. For example, take our earlier example with a discount rate of 10%, but make the growth rate of cash flows $g = 15\%$. The first cash flow would be $\$2 \cdot 1.15 = \$2.30$, which discounts to \$2.09 today. The second cash flow would be $\$2 \cdot 1.15^2 = \$2.645$, which discounts to \$2.186 today. The present value of each cash flow is higher than that preceding it. Taking a sum over an infinite number of such increasing terms would yield infinity as the value. A value of infinity is clearly not sensible, as nothing in this world is worth an infinite amount of money. Therefore, the growing perpetuity formula yields a nonsensical negative value if $g \geq r$—as it should!

The formula is nonsensical when $r < g$.

Note: Reference book Corporate Finance by Ivo Welch

Why $r > g$ for a Growing Perpetuity PV formula to hold?

Timeline of Cash Flows

	C	$C(1+g)$	$C(1+g)^2$	$\cdots$	$\cdots$	$\cdots$	$C(1+g)^{n-1}$	
0	1	2	3	4	5	6 $\cdots$	n	∞

PV of growing perpetuity

$$= \lim_{n\to\infty}\left[\left(\frac{C}{1+r}\right) + \frac{C(1+g)}{(1+r)^2} + \cdots\cdots + \frac{C(1+g)^{n-1}}{(1+r)^n}\right]$$

$$= \lim_{n\to\infty}\left(\frac{C}{1+r}\right)\left[1 + \left(\frac{1+g}{1+r}\right) + \cdots\cdots + \left(\frac{1+g}{1+r}\right)^{n-1}\right]$$

$$= \lim_{n\to\infty}\left(\frac{C}{1+r}\right)\left[\underbrace{\frac{1.\left(1-\left(\frac{1+g}{1+r}\right)^n\right)}{1-\left(\frac{1+g}{1+r}\right)}}_{\text{Sum of a finite Geometric series}}\right]$$

$$= \lim_{n\to\infty}\left(\frac{C}{\cancel{1+r}}\right)\left[\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right].\cancel{(1+r)}$$

$$= \left(\frac{C}{r-g}\right)\left[1 - \lim_{n\to\infty}\left(\frac{1+g}{1+r}\right)^n\right]$$

Now $\left(\frac{1+g}{1+r}\right)^n \to 0$ as $n \to \infty$

only if $\left(\frac{1+g}{1+r}\right) < 1$

or $(1+g) < (1+r)$

or $r > g$

So PV of growing perpetuity $= \left(\frac{C}{r-g}\right)$

only if $r > g$.

If $g > r$,

The PV of growing perpetuity $\to \infty$

Problem on Cash flow and Price of equity instrument

Manufacturing Ltd. has projected sales of $145 million next year. Costs including depreciation are expected to be $81 million and net investment (incl. working capital investment and capital spending less depreciation) is expected to be $15million. Each of these values is expected to grow at 14% the following year with the growth rate declining by 2% per year until the growth rate reaches 6% where it is expected to remain indefinitely. There are 5.5 million shares of stock outstanding and investors require a return of 13% return on co.'s stock. The corporate tax rate is 40%. There is no debt outstanding.

a. What is the estimate of the current stock price?
b. Suppose instead you estimate the terminal value of the company using a PE multiple. The industry PE multiple is 11. What is the new estimate of co.'s stock price?

Manufacturing Ltd. has projected sales of $145 million next year. Costs including depreciation are expected to be $81 million and net investment (incl. working capital investment and capital spending less depreciation) is expected to be $15million. Each of these values is expected to grow at 14% the following year with the growth rate declining by 2% per year until the growth rate reaches 6% where it is expected to remain indefinitely. There are 5.5 million shares of stock outstanding and investors require a return of 13% return on co.'s stock. The corporate tax rate is 40%. There is no debt outstanding.

a. What is the estimate of the current stock price?
b. Suppose instead you estimate the terminal value of the company using a PE multiple. The industry PE multiple is 11. What is the new estimate of co.'s stock price?

Net Investment	15	$million
Costs	81	$million
Sales revenue1	145	$million
Shares outstanding	5.5	million
Cost of equity	13%	
Tax rate	40%	

t	0	1	2	3	4
growth rate from t to t+1		14%	12.00%	10.00%	8.00%
Sales revenue		145	165.3	185.136	203.6496
Costs		81	92.34	103.4208	113.7629
EBT		64	72.96	81.7152	89.88672
EBT(1-Tax rate)		38.4	43.776	49.02912	53.93203
Net Investment		15	17.1	19.152	21.0672
		1	2	3	4
FCFE		23.4	26.676	29.87712	32.86483
growth rate from t-1 to t			0.14	0.12	0.1

TV at t= 5

PV of TV	291.7230731	$million
PV of planning period CFs	₹ 101.73	$million
Total PV	₹ 393.45	$million
Price per share	₹ 71.54	$ per share

b

TV ar t=5

TV at t=0	347.7530973	$million
Total PV	₹ 449.48	$million
Price per share	₹ 81.72	$ per share

5	6	7
6.00%	6.00%	6.00%
219.9416	233.1381	247.1263
122.8639	130.2357	138.0499
97.07766	102.9023	109.0765
58.24659	61.74139	65.44587
22.75258	24.11773	25.56479
5	6	7
35.49402	37.62366	39.88108
0.08	0.06	0.06
537.4809		

640.7125

Immunization

Immunization techniques are strategies used by investors to shield their overall financial status from exposure to interest rate fluctuations.

Immunization requires that the duration of the portfolio of assets equal the duration of liability.

Q. An insurance co. must make a payment of \$19,487 in 7 years. The market interest rate is 10%, so the present value of the obligation is \$10,000. The co's portfolio manager wishes to fund the obligation using 3-year zero-coupon bonds and perpetuities paying annual-coupons. How can the manager immunize the obligation?

Given :

$\text{Payment}_7 = \$19,487$

$PV_0 = \$10,000$

$ytm_0 = 10\%$

Duration of zero coupon bond_0 = Maturity of bond_0 = 3 years

*

Duration of a perpetual bond $= \frac{1+ytm_0}{ytm_0}$

$= \frac{1.10}{0.10}$

$= 11$ years

Let the fraction of portfolio invested in zero-coupon bond be w.

Asset duration $= w \times 3 + (1-w) \times 11$

Asset duration = Liability duration

$w \times 3 + (1-w) \times 11 = 7$

or $w = 1/2$

50% invested in zero coupon bond & 50% invested in perpetual bond.

50% x \$10,000 = \$5000 invested in zero coupon bond & \$5000 invested in perpetual bond

Price risk : Price risk is the risk that the market price of a bond will fall usually due to a rise in the market interest rate.

Reinvestment risk :

Reinvestment risk is the risk that a bond is repaid early, and an investor has to find a new place to invest with risk of lower returns.

Duration Measures

Composite measure that considers both coupon and maturity.

Macaulay Duration

$$D = \frac{\sum_{t=1}^{n} \frac{C_t \times t}{(1+r)^t}}{\sum_{t=1}^{n} \frac{C_t}{(1+r)^t}}$$

$r \rightarrow$ ytm of bond

$t \rightarrow$ time of coupon or principal repayment

$C_t \rightarrow$ Coupon & Principal repayment.

$D \propto$ ttm.

$\propto \frac{1}{\text{ytm}}$

$\propto \frac{1}{\text{coupon}}$.

Modified Duration

An adjusted measure of duration used to approximate the interest rate sensitivity.

$$\text{Modified M.D.} = D_{mod} = \frac{D}{(1 + ytm_{\text{per period}})}$$

$$\boxed{\frac{\Delta P}{P} \times 100 = -D_{mod} \times \Delta ytm.}$$

Bond Convexity

$$\text{Convexity} = \frac{d^2P/dr^2}{P}$$

$$\frac{d^2P/dr^2}{P} = \frac{1}{(1+r)^2 P}\left[\sum_{t=1}^{n} \frac{CF_t}{(1+r)^t} \times (t^2+t)\right]$$

Price change due to convexity $= \frac{1}{2} \times \text{price} \times \text{convexity} \times (\Delta \text{ in yield})^2$

Appendix C: Duration and Convexity

This appendix discusses different ways of measuring the *interest rate risk* from holding bonds. Interest rate risk is the risk that bond prices will fall if market interest rates rise. It is the main form of market risk for bonds paying fixed coupons. There are two principal measures of interest-rate risk that we will consider: duration and convexity.

C.1 DURATION

The measure of interest rate risk typically used by bond analysts is called *duration*, which was invented by Macaulay (1938). Duration is defined as the weighted average maturity of a bond using the relative discounted cash flows in each period as weights. If we assume annual coupons, then (see Chua, 1984):

$$\begin{aligned}D &= \frac{d}{P_d} \times \sum_{t=1}^{T} \frac{t}{(1+rm)^t} + \frac{B}{P_d} \times \frac{T}{(1+rm)^T} \\ &= \frac{d}{P_d} \times \left[\frac{(1+rm)^{T+1} - (1+rm) - (rm)(T)}{(rm)^2(1+rm)^T} \right] + \frac{B}{P_d} \times \frac{T}{(1+rm)^T}\end{aligned} \tag{C.1}$$

where:

D = duration (measured in years)
d = annual coupon
B = par value of bond
P_d = dirty price of bond (i.e. clean price plus accrued interest)
t = time in years to tth cash flow
T = time in years to maturity
rm = yield to maturity.

In (C.1), $d/(1+rm)^t$ is the discounted value of the tth cash flow and so $d/P_d(1+rm)^t$ is the relative discounted value of the tth cash flow; similarly with the terminal value.

Example of duration. A 10% annual coupon bond is trading at par with three years to maturity, so $P_d = B = 100$, $d = 10$, $rm = 10\%$, $T = 3$ years. Therefore, duration is given by:

$$\begin{aligned} D &= \frac{10}{100}\left[\frac{1}{(1.1)} + \frac{2}{(1.1)^2} + \frac{3}{(1.1)^3}\right] + \frac{100}{100}\left[\frac{3}{(1.1)^3}\right] \\ &= \frac{10}{100}\left[\frac{(1.1)^4 - (1.1) - (0.1)(3)}{(0.1)^2(1.1)^3}\right] + \frac{100}{100}\left[\frac{3}{(1.1)^3}\right] \\ &= 2.74 \text{ years} \end{aligned}$$

which implies that the average time taken to receive the cash flows on this bond is 2.74 years (see Figure C.1).

We can examine some of the properties of duration. Duration is always less than (or equal to) maturity. This is because some weight is given to the cash flows in the early years of the bond's life and this helps to bring forward the average time at which cash flows are received. In the above example, the coupon element contributes 0.5 years to duration, while the principal element contributes 2.24 years. Duration also varies with coupon, yield and maturity.

For a zero-coupon bond, duration equals the term to maturity. This is obvious from the definition of a zero-coupon bond and also from (C.1) given that for a zero-coupon bond $d = 0$ and $P_d = B/(1 + rm)^T$. For a perpetual bond, duration is given by:

$$D = \frac{1 + rc}{rc} = \frac{1}{rc} + 1 \tag{C.2}$$

where $rc = (d/P_d)$ is the current yield. This follows from (C.1) as $T \to \infty$, recognising (from Equation (B.2) in Appendix B) that, for a perpetual bond, $rm = rc$.

Equation (C.2) provides the limiting value to duration. For bonds trading at or above par (so that $rm \leq rc$), duration increases with maturity and approaches this limit from below. For bonds trading at a discount to par (so that $rm > rc$), duration increases to a maximum at around 20 years and then declines towards the limit given by (C.2). So, in general, duration increases with maturity (see Figure C.2).

Duration increases as coupon and yield decrease, as shown in Figure C.3. As the coupon falls, more of the relative weight of the cash flows is transferred to the maturity date and this causes duration to rise.

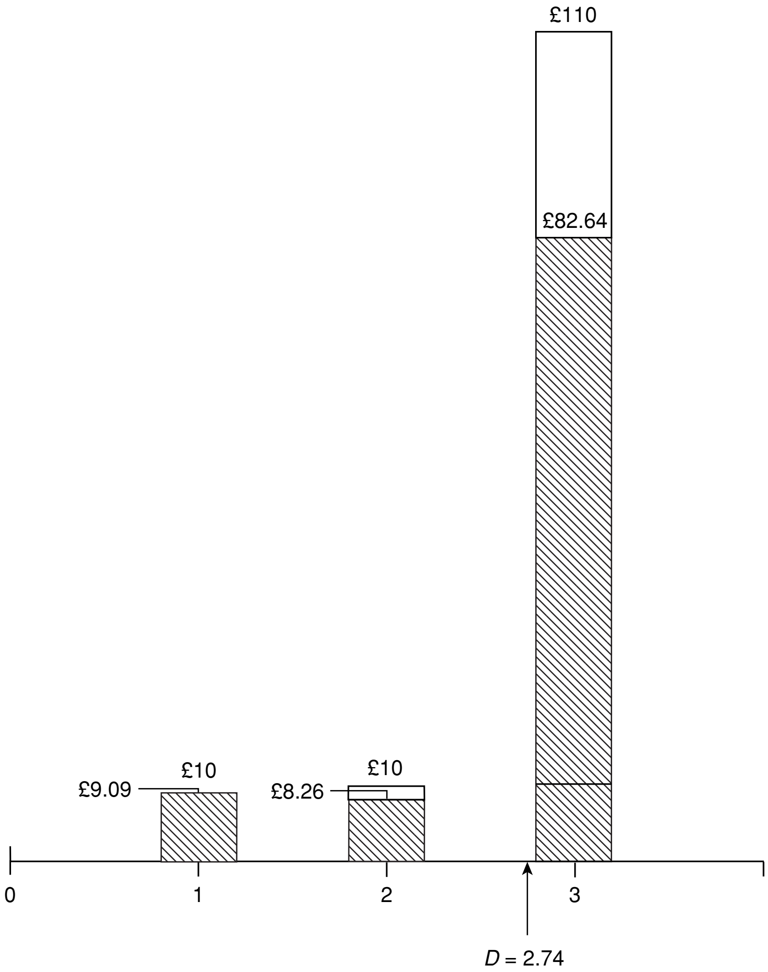


Figure C.1 Duration as the weighted average maturity of a bond

As yield increases, the present values of all future cash flows fall, but the present values of the more distant cash flows fall relatively more than those of the nearer cash flows. This has the effect of increasing the relative weight given to nearer cash flows and hence of reducing duration. Because the coupon on index-linked gilts is much lower than on conventional gilts, this means that the duration of index-linked gilts will be much higher than for conventional gilts with the same maturity.

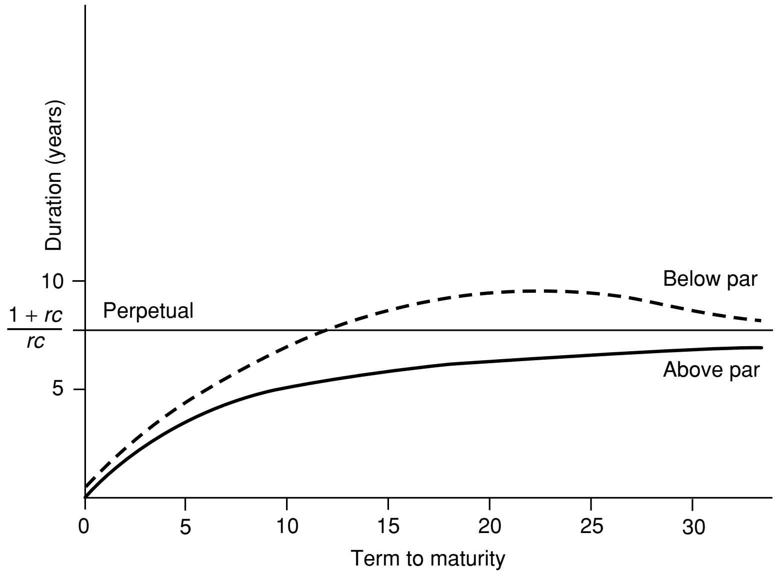


Figure C.2 Duration against maturity

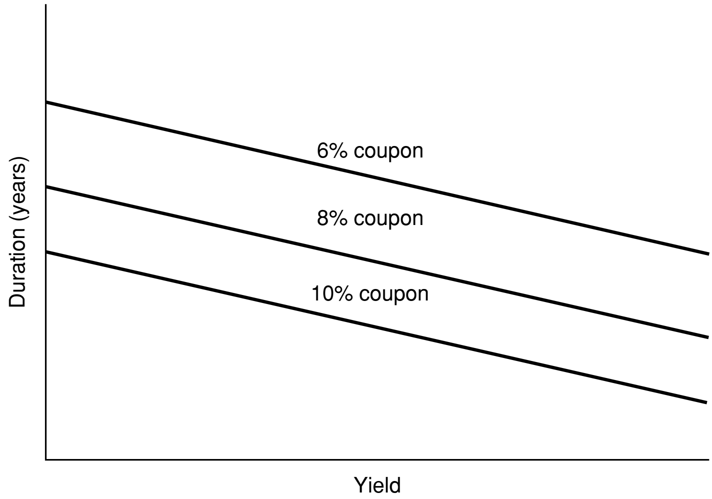


Figure C.3 Duration against coupon and yield

That duration is a measure of interest rate risk is demonstrated as follows. The present value equation for an annual coupon bond is given by:

$$P_d = \sum_{t=1}^{T} \frac{d}{(1+rm)^t} + \frac{B}{(1+rm)^T} \tag{C.3}$$

Differentiating this equation with respect to $(1+rm)$ gives:

$$\frac{\Delta P_d}{\Delta(1+rm)} = -d \times \sum_{t=1}^{T} \frac{t}{(1+rm)^{t+1}} - B \times \frac{T}{(1+rm)^{T+1}} \tag{C.4}$$

where Δ means 'a small change in'. Multiplying both sides of (C.4) by $(1+rm)/P_d$ gives:

$$\begin{aligned}\frac{\Delta P_d/P_d}{\Delta(1+rm)/(1+rm)} &= -\frac{d}{P_d} \times \sum_{t=1}^{T} \frac{t}{(1+rm)^t} - \frac{B}{P_d} \times \frac{T}{(1+rm)^T} \\ &= -D\end{aligned} \tag{C.5}$$

The LHS of (C.5) is the elasticity of the bond price with respect to (one *plus*) the yield to maturity, $\varepsilon[P_d, (1+rm)]$, where:

$$\begin{aligned}\varepsilon[P_d, (1+rm)] &= \frac{\Delta \ln P_d}{\Delta \ln(1+rm)} \\ &= \frac{\Delta P_d/P_d}{\Delta(1+rm)/(1+rm)}\end{aligned} \tag{C.6}$$

The RHS of (C.5) is (the negative of) duration. So, duration measures the interest-rate elasticity of the bond price, and is therefore a measure of interest-rate risk. The lower the duration, the less responsive is the bond's value to interest-rate fluctuations.

Figure C.4 shows the present-value profile for a bond. There is a negative-sloping and convex relationship between (the natural logarithm of) the price of the bond and (the natural logarithm of) one *plus* the yield to maturity. The slope of the present-value profile at the current bond price and yield to maturity is equal to the (negative of the) duration of the bond. The flatter the present-value profile, the lower the duration and the lower the interest-rate risk.

Example of first-order interest-rate risk. A 10% annual coupon bond is trading at par with a duration of 2.74 years. If yields rise from 10 to

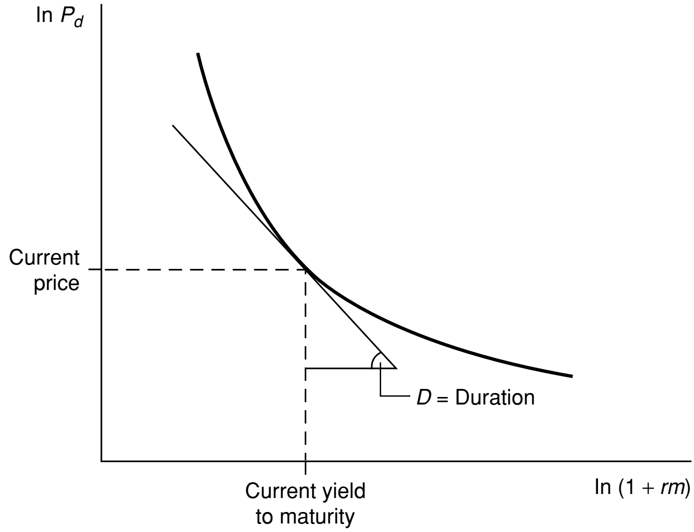


Figure C.4 Present-value profile and duration

10.5%, then the price of the bond will fall by:

$$\begin{aligned}\Delta P_d &= -D \times \frac{\Delta(rm)}{1+rm} \times P_d \\ &= -2.74 \times \frac{0.005}{1.1} \times 100 \\ &= -1.25\end{aligned}$$

to £98.75.

The UK markets use a concept called *modified duration* (also known as *volatility*), which is related to duration as follows:

$$MD = \frac{D}{1+rm} \tag{C.7}$$

where MD = modified duration in years. This means that the following relationship holds between modified duration and bond prices:

$$\Delta P_d = -MD \times \Delta rm \times P_d \tag{C.8}$$

For example, if $D = 2.74$ years and yields are 10% then:

$$\begin{aligned}MD &= \frac{2.74}{1.1} \\ &= 2.49 \text{ years}\end{aligned}$$

Market practitioners also use a concept called *basis point value* (BPV) (which is sometimes referred to as *risk*). It is related to modified duration as follows:

$$BPV = \frac{MD \times P_d}{10\,000} \tag{C.9}$$

While modified duration gives the percentage change in the price of a bond, BPV gives the money change in the price of a bond in response to a one-basis-point change in yield: from (C.8) it is clear that:

$$BPV = -\frac{\Delta P_d}{\Delta rm} \times \frac{1}{10\,000}$$

Some practitioners in the financial markets (an example is Bloomberg, suppliers of online financial information) calculate duration and modified duration using numerical approximations to the first-order derivative given in (C.5). For example, modified duration can be calculated using the following expression:

$$MD = \frac{10^4}{2}\left(\frac{\left|\Delta P_d''\right|}{P_d} + \frac{\left|\Delta P_d''\right|}{P_d}\right), \tag{C.10}$$

where:

$\left|\Delta P_d'\right|$ = absolute value of the change in dirty price if the yield increases by 1 basis point (0.01%); i.e. by one BPV.

$\left|\Delta P_d''\right|$ = absolute value of the change in dirty price if the yield falls by 1 basis point (0.01%); i.e. by one BPV.

The scaling factor 10^4 is explained by the fact that the price difference is calculated on the basis of a 100th of 1% change in yield, whereas the price level is based on 100% of par value.

To illustrate this we can use the same bond as in the above example of duration. If the yield increases from 10.00 to 10.01% then, using (C.3) above and Equation (A.18) in Appendix A, the price of the bond will fall to:

$$\begin{aligned} P_d' &= \frac{10}{(0.1001)}\left[1 - \frac{1}{(1.1001)^3}\right] + \frac{100}{(1.1001)^3} \\ &= 99.9751359 \end{aligned}$$

or by $\Delta P_d' = -0.0248641$. If the yield falls to 9.99%, the price of the bond will rise to:

$$P_d'' = \frac{10}{(0.0999)}\left[1 - \frac{1}{(1.0999)^3}\right] + \frac{100}{(1.0999)^3}$$
$$= 100.0248729$$

or by $\Delta P_d'' = 0.0248729$. Therefore:

$$MD = \frac{10^4}{2}\left(\frac{0.0248641}{100} + \frac{0.02489729}{100}\right)$$
$$= 2.49.$$

C.2 CONVEXITY

Duration can be regarded as a first-order measure of interest-rate risk: it measures the *slope* of the present-value profile. *Convexity*, on the other hand, can be regarded as a second-order measure of interest-rate risk: it measures the *curvature* of the present-value profile.

A second-order Taylor expansion of the present-value equation (C.3) gives:

$$\frac{\Delta P_d}{P_d} = \frac{1}{P_d} \times \frac{\Delta P_d}{\Delta rm} \times (\Delta rm) + \frac{1}{2P_d} \times \frac{\Delta^2 P_d}{\Delta rm^2} \times (\Delta rm)^2$$
$$= -MD \times (\Delta rm) + \frac{C}{2} \times (\Delta rm)^2 \qquad \text{(C.11)}$$

where:

MD = modified duration
C = convexity

Convexity is the rate at which price variation to yield changes with respect to yield and, as is clear from (C.11), it is found by taking the second derivative of Equation (C.3) with respect to rm and dividing the result by P_d. Blake and Orszag (1996) show that this expression for convexity can be simplified as follows:

$$C = \frac{1}{P_d} \times \frac{\Delta^2 P_d}{\Delta rm^2}$$
$$= \frac{d}{P_d} \times \sum_{t=1}^{T} \frac{t(t+1)}{(1+rm)^{t+2}} + \frac{B}{P_d} \times \frac{T(T+1)}{(1+rm)^{T+2}}$$

$$
\begin{aligned}
= -\frac{d}{P_d} \Bigg\{ & \frac{(T+1)(T+2)\left(\dfrac{1}{1+rm}\right)^{T+2}}{rm} \\
& + \frac{2\left[(T+2)\left(\dfrac{1}{1+rm}\right)^{T+2} - \left(\dfrac{1}{1+rm}\right)\right]}{rm^2} \\
& + \frac{2\left[\left(\dfrac{1}{1+rm}\right)^{T+2} - \left(\dfrac{1}{1+rm}\right)\right]}{rm^3} \Bigg\} + \frac{B}{P_d} \times \frac{T(T+1)}{(1+rm)^{T+2}}
\end{aligned}
\tag{C.12}
$$

Convexity can also be approximated by the following expression for the numerical second-order derivative:

$$
C = 10^8 \left(\frac{\Delta P'_d}{P_d} + \frac{\Delta P''_d}{P_d} \right) \tag{C.13}
$$

where:

$\Delta P'_d$ = change in dirty bond price if yield increases by 1 basis point (0.01%); i.e. by one *BPV*.

$\Delta P''_d$ = change in dirty bond price if yield decreases by 1 basis point (0.01%); i.e. by one *BPV*.

Example of convexity. A 10% annual coupon bond is trading at par with three years to maturity, so $P_d = B = 100$, $d = 10$, $rm = 10\%$, $T = 3$ years. Therefore, using the second line of (C.12), convexity is given by:

$$
\begin{aligned}
C &= \frac{10}{100}\left[\frac{2}{(1.1)^3} + \frac{6}{(1.1)^4} + \frac{12}{(1.1)^5}\right] + \frac{100}{100} \times \frac{12}{(1.1)^5} \\
&= 8.76
\end{aligned}
$$

We get the same answer if we use the numerical approximation to the second-order derivative (C.13). We know that if the yield increases from 10.00 to 10.01%, the price of the bond will fall by $\Delta P'_d = -0.0248641$, while if the yield falls to 9.99%, the price of the bond will rise by

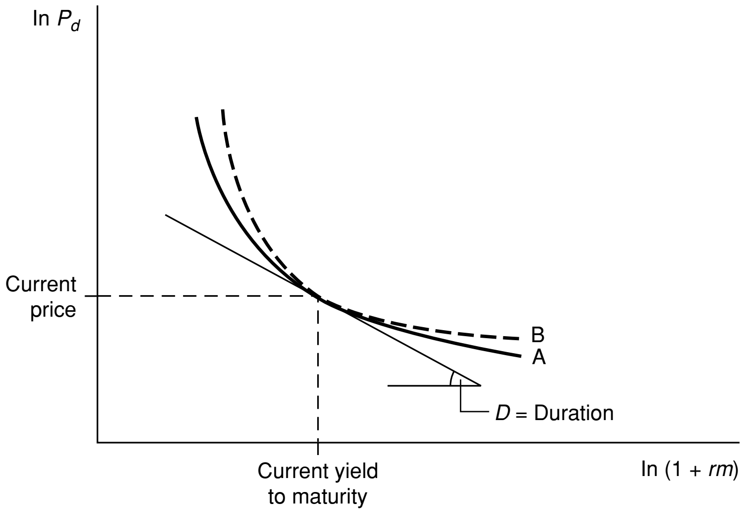


Figure C.5 Present-value profile and convexity

$\Delta P_d'' = 0.0248729$. Therefore:

$$C = 10^8 \left(\frac{-0.0248641}{100} + \frac{0.0248729}{100} \right)$$
$$= 8.76$$

It can be shown that convexity increases with the square of maturity. It decreases with both coupon and yield. Index-linked bonds are more convex than conventional bonds.

That convexity is a second-order measure of interest-rate risk is demonstrated in Figure C.5. This figure shows the present-value profiles for two bonds A and B, trading at the same price and yield to maturity, and having the same duration. Bond B, however, is more convex than bond A. B is clearly more desirable than A, since B will outperform A whatever happens to market interest rates. If yields rise, the price of B falls by less than the price of A, while if yields fall, the price of B rises by more than that of A. High convexity is therefore a desirable property for bonds to have.

When convexity is high, the first-order approximation for interest-rate risk, (C.5) or (C.6), will become increasingly less accurate as the change in yield becomes large. This is clear from Figure C.5. Duration is a linear approximation to the present-value profile. For small changes in yield, the linear approximation will be reasonably good; but for large jumps in yield, the linear approximation will become very poor if convexity is high. It will overestimate the risk (i.e. overestimate the price adjustment) when convexity is positive and underestimate risk and price adjustment when convexity is negative (which is the case, for example, with callable bonds near par). So, for large jumps in yield, the quadratic approximation to the present-value profile given by (C.12) or (C.13) is preferred.

Example of second-order interest-rate risk. A 10% annual coupon bond is trading at par with a modified duration of 2.49 and convexity of 8.76. If yields rise from 10 to 12%, the price of the bond will fall by:

$$\begin{aligned}
\Delta P_d &= -MD \times (\Delta rm) \times P_d + \frac{C}{2} \times (\Delta rm)^2 \times P_d \\
&= -(2.49) \times (0.02) \times 100 + \frac{8.76}{2} \times (0.02)^2 \times 100 \\
&= -4.98 + 0.18 \\
&= -4.80
\end{aligned}$$

to £95.20. The first-order approximation overestimates the fall by £0.18.

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You are hired as a fixed income portfolio manager. You have information about two corporate bonds which are issued recently in the market. One is a green bond which is issued to fund projects with positive environmental benefits. The other is the social bond which is issued to raise funds for projects with positive social outcomes. The two bonds issued were AAA-rated bonds. The par value of the bonds is ₹1000. The characteristics of the bonds are provided below: -

Bond	Time to maturity	Coupon rate p.a.	Payments
Green bond (G)	10 years	8%	Semi-Annual
Social bond (S)	15 years	10%	Semi-Annual

The market interest rate (yield to maturity) is 10% for Social bonds and 8% for Green bonds at the time of issuance. You plan to invest in either of the two. You have also got the analyst forecast for expected inflation in the market in the near future. The analyst has suggested that the expected inflation in the market is going to decrease in the future leading to a 10 basis points change in YTM.

1.1 Which bond will you invest in and why based on the analyst forecast of expected inflation? Explain your answer by calculating a relevant measure.

1.2 Calculate the price of the bonds if YTM decreases by 10 basis points (now) for both the bonds.

1.3 Calculate the % change in price for 10 basis points reduction in YTM for both the bonds and interpret.

1.4 Calculate the price of the bonds if YTM increases by 10 basis points (now) for both the bonds.

1.5 Calculate the % change in price for 10 basis points increase in YTM for both the bonds and interpret.

1.6 What will be the relevant measure if you expect much larger changes in YTM? If you expect the YTM to decrease by 350 basis points, which bond should you invest in?

1.7 If say the government announces that the coupon payments will be taxable in the hands of the investor, what do you think will be the impact on the market interest rate? Which bond should you invest in then?