Mathematical Modeling: Assignment 2

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A local daily newspaper has recently been acquired by a large media conglomerate. The paper currently sells for $1.50/week and has a circulation of 80,000 subscribers. Advertising sells for $250/page, and the paper currently sells 350 pages/week (50 pages/day). The new management is looking for ways to increase profits. It is estimated that an increase of ten cents/week in the subscription price will cause a drop in circulation of 5,000 subscribers. Increasing the price of advertising by $100/page will cause the paper to lose approximately 50 pages of advertising per week.The loss of advertising will also affect circulation, since one of the reasons people buy the paper is for the advertisements. It is estimated that a loss of 50 pages of advertisements per week will reduce circulation by 1,000 subscriptions.

(a) Find the weekly subscription price and the advertising price that will maximize profit. Use the five-step method, and model as an unconstrained optimization problem.

Step 1: Variables and Assumptions

$k-$ number of times the subscription price increased

$j-$ number of times the advertising price increased

$s-$ subscription price in dollars

$a-$ advertising price per page

$P-$ profit per week

$1.50/week-$ current subscription price in dollars

$80,000-$ current circulation at 1.50 dollars subscription price

$250/page-$ current advertising price in dollars

$350 pages/week-$ number of pages sold at 250 dollars advertising price

$350 - 50j-$ amount of pages sold with an increase in advertising price

$80,000 - 5,000k - 1000j-$ circulation with increase in prices

$s = (1.50 + 0.10k)$

$a = (250 + 100j)$

$P = a(350 - 50j) + s(80,000 - 5,000k - 1000j)$

where k, j, a, and $s \geq 0$

Step 2: Modeling Approach

"We are given a function $y = f(x_1, ..., x_n)$ on a subset S of the n-dimensional space R^n. We wish to find the maximum and/or minimum values of f on the set S. There is a theorem that states that if f attains its maximum or minimum at an interior point $(x_1, ..., x_n)$ in S, then gradient of f is 0 at that point, assuming that f is differentiable at that point. In other words, at the extreme point

$$\frac{df}{dx_1}(x_1, ..., x_n) = 0$$

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$$\frac{df}{dx_n}(x_1, ..., x_n) = 0$$

The theorem allows us to rule out as a candidate for max-min any point in the interior of S for which any of the partial derivatives of f do not equal zero. Thus, to find the max-min points we must solve simultaneously the n equations in n unknowns defined by the equations. Then we must also check any points on the boundary of S, as well as any points where one or more of the partial derivatives is undefined." - *Meerschaert, M. Mathematical Modeling 4th Edition.*

Step 3: Formulation of the Model

Let

$P = s(80,000 - 5000k - 1000j) + a(350 - 50j)$

$P = (1.50 + 0.1k)(80,000 - 5000k - 1000j) + (250 + 100j)(350 - 50j)$

let $y = P$, the variable to be maximized, and let $x_1 = k$ and $x_2 = j$ be the decision variables. Thus,

$y = f(x_1, x_2)$

$y = (1.50 + 0.1x_1)(80,000 - 5000x_1 - 1000x_2) + (250 + 100x_2)(350 - 50x_2)$

over the set

$S = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0$

Step 4: Solve the Model

$y = 207,500 + 500x_1 + 21,000x_2 - 100x_1x_2 - 500x_1^2 - 5000x_2^2$

$\frac{dy}{dx_1} = 500 - 100x_2 - 1000x_1 = 0$

$\frac{dy}{dx_2} = 21,000 - 100x_1 - 10,000x_2 = 0$

Solving for x_1 and x_2,

$x_2 = \frac{2095}{999} = 2.0971$

$x_1 = \frac{290}{999} = 0.29029$

Therefore, the profit is \$229,592.09 when the company gives $1.5 + .1 * 0.29 = \$1.53$ of subscription price and an advertising price of $250 + 100 * 2.0971 = \$459.71$.

Step 5: Answer the Question

With the results above, the company should leave the subscription price at \$1.50 per week and increase the advertising price at \$450. This will give a profit of \$229,500.

(b) Examine the sensitivity of your conclusions in part (a) to the assumption of 5,000 lost sales when the price of the paper increases by ten cents.

Sensitivity analysis on the decrease on circulation

Assume that the decrease in circulation is not 5,000. Instead, we replace it with n. Then,

$y = (1.50 + 0.1x_1)(80,000 - nx_1 - 1000x_2) + (250 + 100x_2)(350 - 50x_2)$

$y = 207,500 + 21,000x_2 - 1.5nx_1 + 8,000x_1 - 0.1nx_1^2 - 100x_1x_2 - 5,000x_2^2$

$y = 207,500 + 21,000x_2 - 1.5nx_1 + 8,000x_1 - 0.1nx_1^2 - 100x_1x_2 - 5,000x_2^2$

$\frac{dy}{dx_1} = -1.5n + 8,000 - 0.2nx_1 - 100x_2 = 0$

$\frac{dy}{dx_2} = 21,000 - 100x_1 - 10,000x_2 = 0$

$$x_1 = \frac{779,000 - 150n}{20n - 100}$$

$$x_2 = \frac{7790000 - 1500n}{-10,000(2n - 10)} + \frac{21}{10}$$

This implies that,

$s = 1.5 + 0.1x_1$

$s = 1.5 + \frac{77,900 - 15n}{20n - 100}$

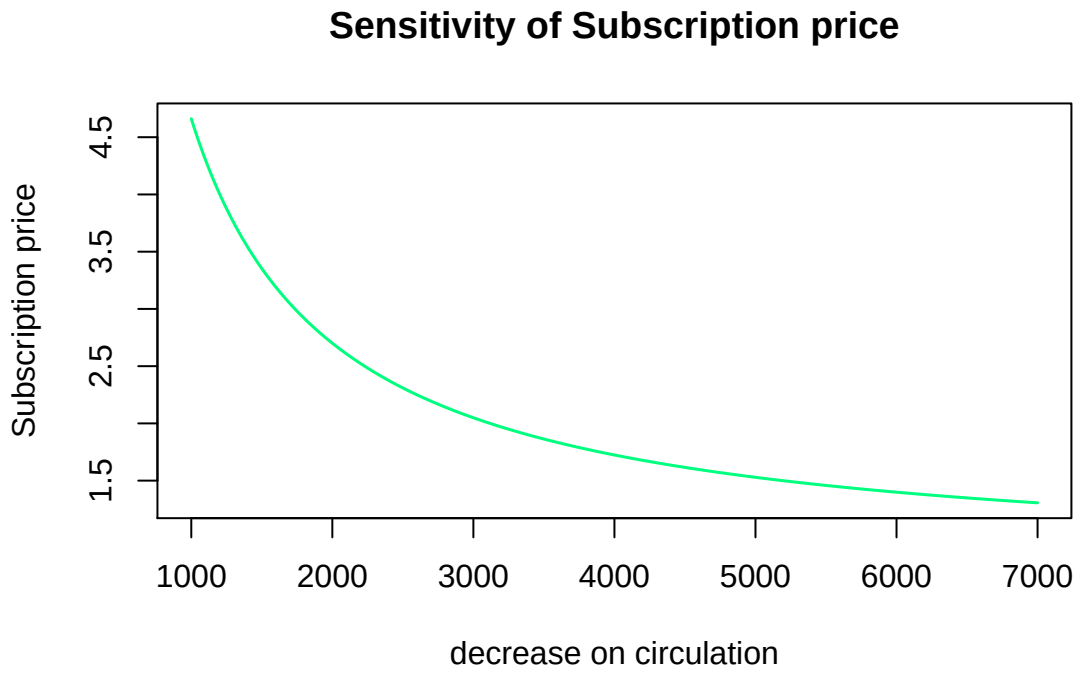
$$\frac{ds}{dn} = \frac{-1,556,500}{(20n - 100)^2}$$

At n = 5,000 and s = 1.5, the sensitivity of the subscription price to the amount of decrease in circulation is the following:

$$S(s,n) = \frac{-1,556,500}{(20n - 100)^2} * \frac{n}{s}$$

$$S(s,n) = \frac{-1,556,500}{9,980,010,000} * \frac{5,000}{1.5}$$

$$S(s,n) = -0.51987$$



Hence, the subscription price is very sensitive to the number of decrease in circulation with a 10% increase in subscription price will lead to a 5.2% decrease in number of subscription.

On the other hand,

$a = 250 + 100x_2$

$a = 460 + \frac{7{,}790{,}000 - 1{,}500n}{-100(2n-10)}$

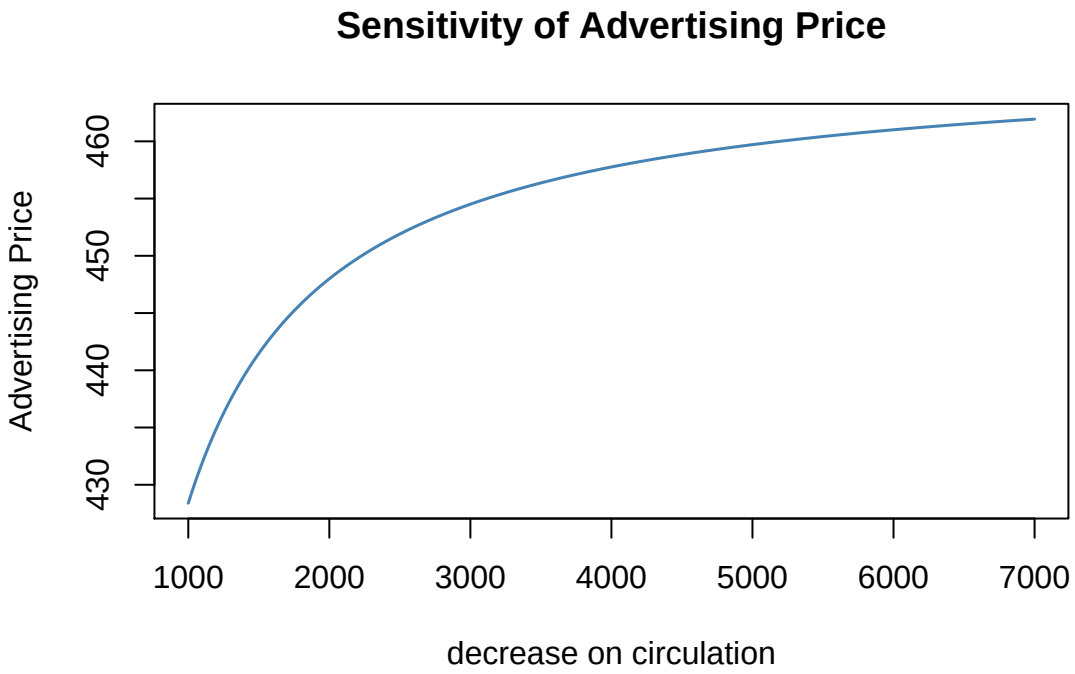
$$\frac{da}{dn} = \frac{1,556,500,000}{(1000 - 200n)^2}$$

This follows that the sensitivity of advertising price to the amount of decrease of circulation is as follows:

At n = 5000 and a = 450

$$S(a,n) = \frac{da}{dn} * \frac{n}{a}$$

$$S(a,n) = 0.017$$

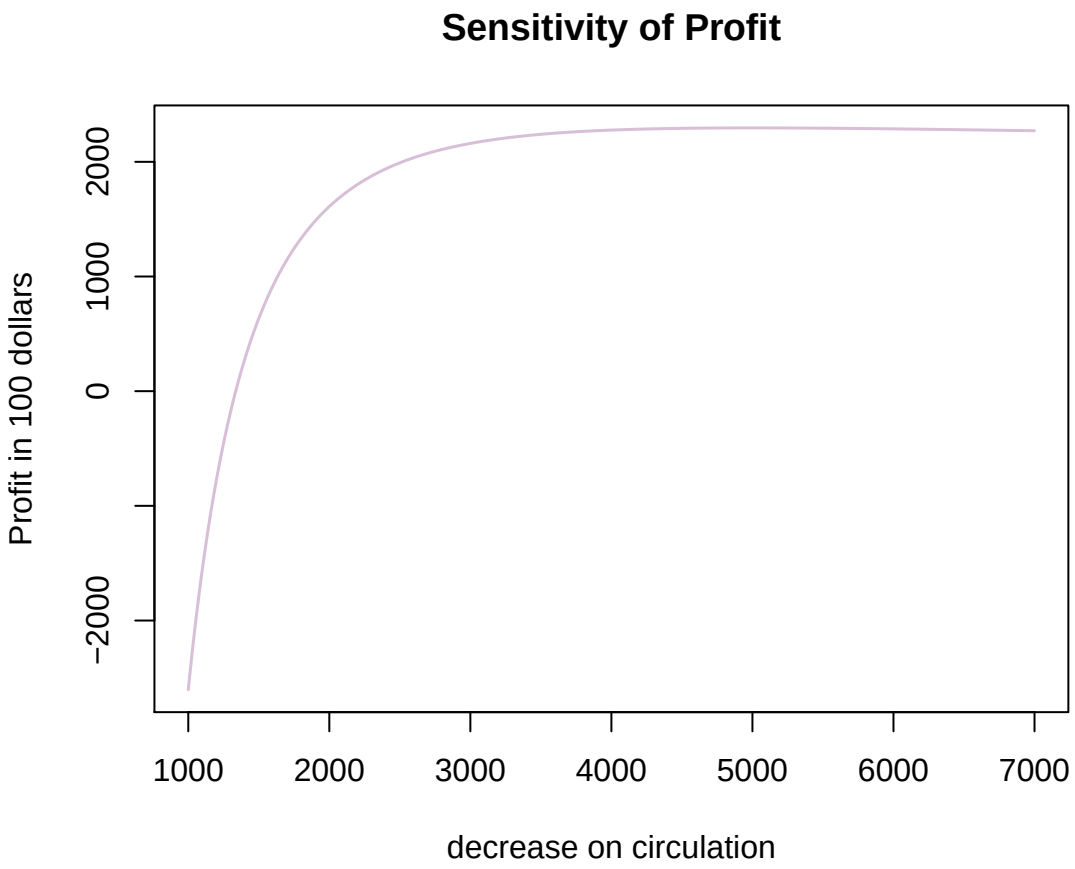


Thus, the advertising price is not that sensitive to the decrease in circulation.

Lastly, we need to see the sensitivity of the profit to the amount of decrease in circulation.

Since, $x_1 = \frac{77{,}900-15n}{2n-10}$ and $x_2 = \frac{77{,}900-15n}{1{,}000-200n} + \frac{21}{10}$, then,

$y = 229550 + 290\frac{77900-15n}{2n-10} - 499.5(\frac{77900-15n}{2n-10})^2$



circulation	profit
4800	229539.4
4900	229579.4
5000	229592.1
5100	229580.4
5200	229547.2

For the profit sensitivity, we will use the difference ratio and the table above.

$$\frac{dy}{dn} = \frac{229592.1 - 229580.4}{5000 - 5100} * \frac{5000}{229500}$$

$$S(y, n) = -0.00254902$$

This implies that the profit is not that sensitive to the amount of decrease in subscription.

(c) Examine the sensitivity of your conclusions in part (a) to the assumption of 50 pages/week of lost advertising sales when the price of advertising is increased by $100/page.

Now to test the sensitivity of the prices and profit to the amount of decrease in pages. Assume the decrease in number of pages is not 50 but p, then,

$y = (1.50 + 0.1x_1)(80,000 - 5,000x_1 - 1000x_2) + (250 + 100x_2)(350 - px_2)$

$y = 207,500 + 500x_1 + 33500x_2 - 500x_1^2 - 100x_1x_2 - 100px_2^2 - 250px_2$

$$\frac{dy}{dx_1} = 500 - 1000x_1 - 100x_2 = 0$$

$$\frac{dy}{dx_2} = 33500 - 100x_1 - 200px_2 - 250p = 0$$

Solving for x_1 and x_2,

$$x_1 = \frac{1}{2} - \frac{334500 - 2500p}{1000(20p - 1)}$$

$$x_2 = \frac{334500 - 2500p}{2000p - 100}$$

Thus,

$$a = 250 + 100\frac{334500 - 2500p}{2000p - 100}$$

$$s = 1.55 - \frac{334500 - 2500p}{10000(20p - 1)}$$

To test the sensitivity of advertising price to the amount of decrease in pages, we find the derivative of a. Thus follows,

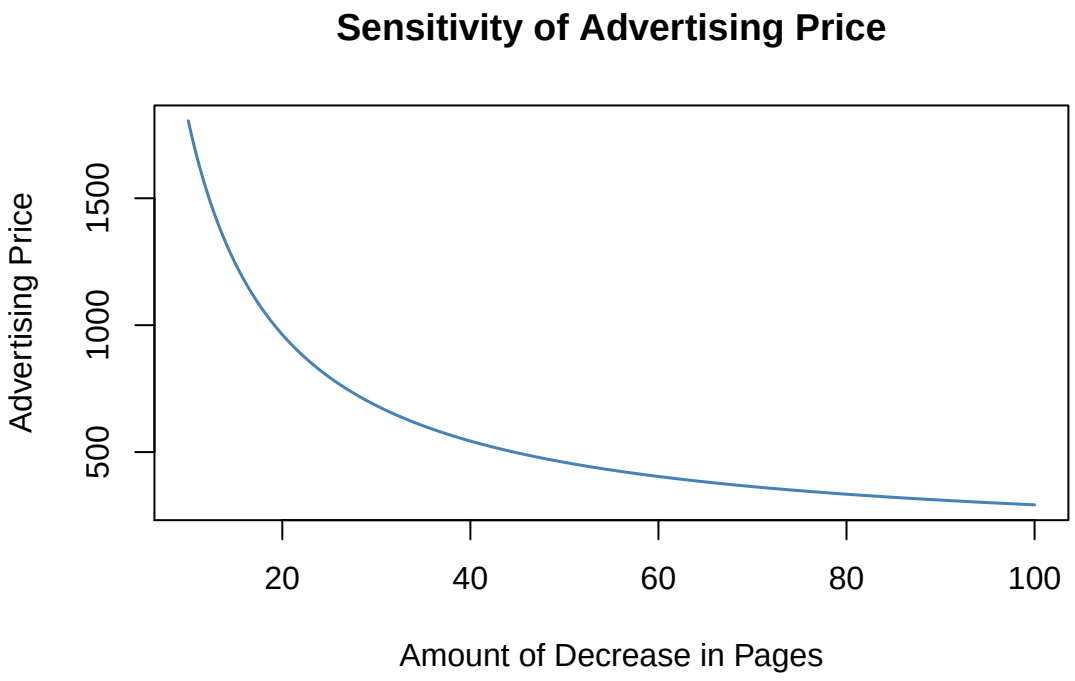
$$\frac{da}{dp} = 100\frac{-668750000}{(2000p - 100)^2}$$

$$S(a, p) = \frac{da}{dp} * \frac{p}{a}$$

At p = 50 and a = 450,

$S(a, p) = -0.7445$

This implies that the advertising price is sensitive to the amount of decrease in pages with a 100% increase in the amount of decrease will result to 74% decrease in the advertising price.



On the other hand, for the sensitivity of subscription price to the decrease on the number of pages can be calculated as follows:

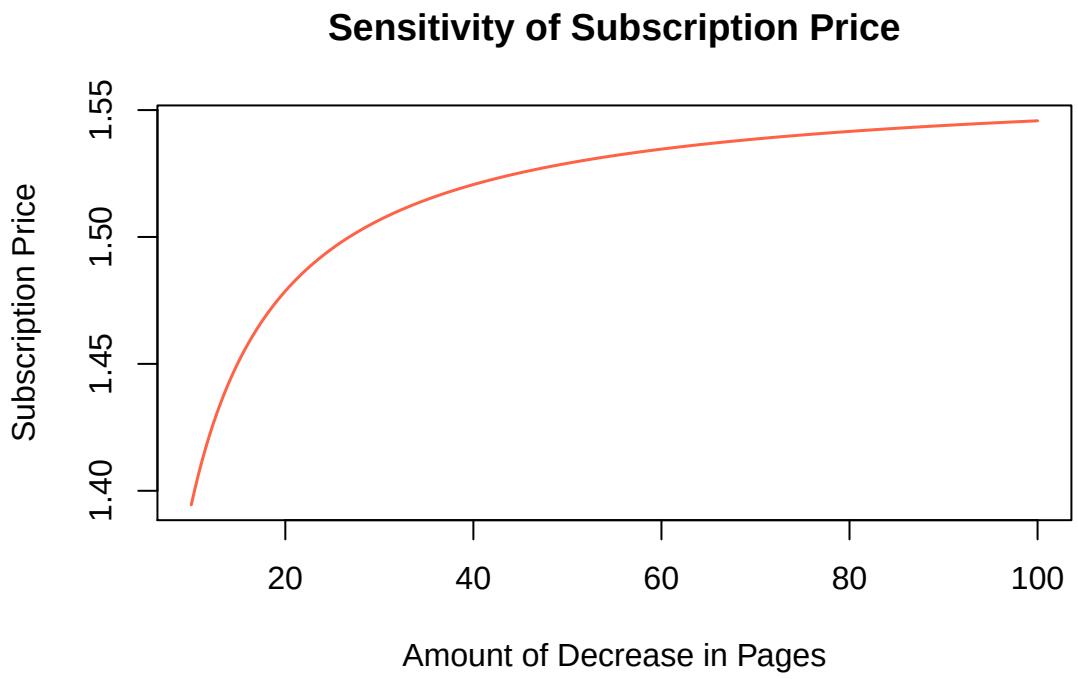
$$s = 1.55 - \frac{334500 - 2500p}{10000(20p - 1)}$$

$$\frac{ds}{dp} = \frac{6.6875x10^{10}}{(10000(20p - 1))^2}$$

$$S(s, p) = \frac{ds}{dp} * \frac{p}{s}$$

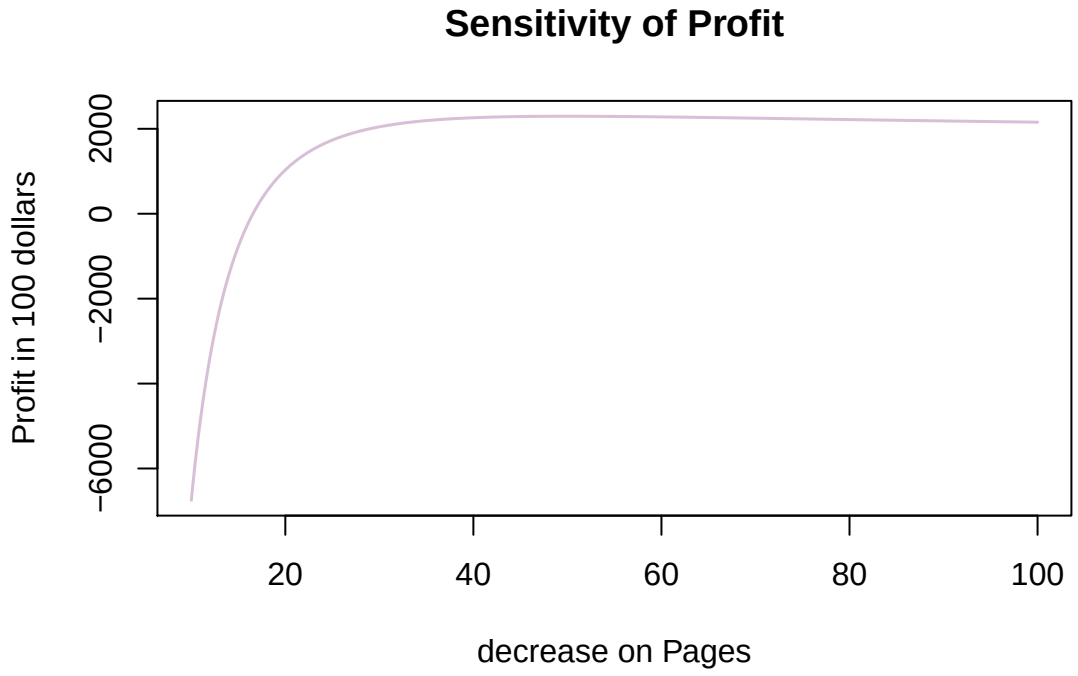
At p = 50 and s = 1.5,

$$S(s, p) = 0.022$$



Thus, the subscription price is not as sensitive to the amount of decrease in pages.

Lastly, the sensitivity of profit to the amount of decrease in pages using difference ratio.



pagesdecrease	profit
30	204638.2
40	226085.9
50	229592.1
60	228035.1
70	225017.5

Using the table above,

$$S(y,p) = \frac{229592.1 - 228035.1}{50 - 60} * \frac{50}{229592.1}$$

$$S(y,p) = -0.03390796$$

The optimal profit is not very sensitive to the amount of decrease in pages.

(d) Advertisers who currently place advertisements in the newspaper have the option of using direct mail to reach their customers. Direct mail would cost the equivalent of $500/page of newspaper advertising. How does this information alter your conclusions in part (a)?

The advertising price of the company should be lower than $450 to be competitive with the direct mail option since $450 is close to $500. This might prevent advertisers to choose the other option.