Spin systems

Heisenberg model:

spin vector: $\vec{s}_i = (s^x_i, s^y_i, s^z_i)$

$i = 1, \dots, N$ $\quad$ N number of spins in the system

Hamiltonian:
$$H = -\sum_{\langle i,j \rangle} J_{ij} \vec{s}_i \vec{s}_j$$
$$= -\sum_{\langle i,j \rangle} J_{ij} \left(s^x_i s^x_j + s^y_i s^y_j + s^z_i s^z_j \right)$$

J_{ij} exchange coupling constant between spin i and j

$\langle i,j \rangle$ means that we restrict the sum to run over nearest neighbour sites. However the model can of course be generalized to include interactions beyond nearest neighbour spins.

It can also be generalized to the anisotro= pic Heisenberg model where the interaction is different for x, y, and z spin compo= nents.

Ising model

Hamiltonian: $H = -\sum_{\langle i,j \rangle} J_{ij} \, S^z_i S^z_j$ (for spin ½ systems)

often rewritten as

$$H = -\sum_{\langle i,j \rangle} J_{ij} \, \sigma^z_i \sigma^z_j$$

($S^z_i = \frac{\hbar}{2} \sigma^z_i$) "absorbing" $\frac{\hbar^2}{4}$ inside J_{ij}

we can also add an external magnetic field h along z (x, y)

$$H = -\sum_{\langle i,j \rangle} J_{ij} \, \sigma^z_i \sigma^z_j - \sum_i h \, \sigma_i^{z\,(x,y)}$$

The Hilbert space for one spin $\frac{1}{2}$ is two-dimensional.

We typically take the eigenvectors of σ^z, i.e. $|\uparrow\rangle$ and $|\downarrow\rangle$, as basis states.
(in other words, we chose the spin-z quantization axis)

$$\sigma^z|\uparrow\rangle = |\uparrow\rangle$$
$$\sigma^z|\downarrow\rangle = -|\downarrow\rangle$$

These states can be represented in vector notation as

$$|\uparrow\rangle \rightarrow \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$
$$|\downarrow\rangle \rightarrow \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

A generic state expanded over this basis states is

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle,$$

or in vector-representation,

$$\begin{vmatrix} \alpha \\ \beta \end{vmatrix}$$

where α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$

The state $|\psi\rangle$ is an example of qubit with $|0\rangle \equiv |\uparrow\rangle$
$|1\rangle \equiv |\downarrow\rangle$

If $\alpha = 1$ $(\beta = 0)$ $|\psi\rangle = |0\rangle$ the qubit is in the state 0

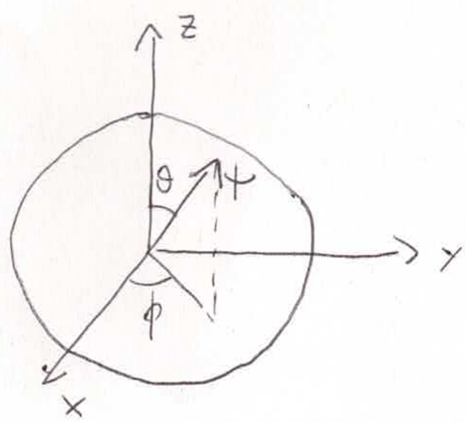
If $\beta = 1$ $(\alpha = 0)$ $|\psi\rangle = |1\rangle$ the qubit is in the state 1

If $\alpha \neq 0$ $\beta \neq 0$ $(|\alpha|^2 + |\beta|^2 = 1)$ $|\psi\rangle$ is a superposition of the state 0 and 1.

$|\psi\rangle$ can also have a representation as the direction of a ray on sphere, called Bloch sphere, with the polar angle representation

$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$$

$\theta \in [0, \pi]$
$\phi \in [0, 2\pi[$



Accordingly the state $|\uparrow\rangle$ lays on the z axis $(\theta = 0, \phi = 0)$, while the state $|\downarrow\rangle$

~~lays~~ along the $-z$ axis $(\theta = \pi, \phi = 0)$.
Complex combinations of the two states $|\uparrow\rangle$ and $|\downarrow\rangle$ lay somewhere on the surface of the sphere.

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} \otimes \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \\ \\ 0 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 \\ 1 \end{vmatrix} \otimes \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \begin{vmatrix} 1 \\ 0 \end{vmatrix} \\ \\ 1 \begin{vmatrix} 1 \\ 0 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 \\ 1 \end{vmatrix} \otimes \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \\ \\ 1 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

(remember: the tensor product of two 2d-vectors is

$$\begin{vmatrix} a_1 \\ a_2 \end{vmatrix} \otimes \begin{vmatrix} b_1 \\ b_2 \end{vmatrix} = \begin{vmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{vmatrix}$$
)

Now, we can write any state in the 2-spin Hilbert space as an expansion over these basis states

$$|\psi\rangle = \sum_{\sigma_1, \sigma_2 = \uparrow\downarrow} c_{\sigma_1 \sigma_2} |\sigma_1 \sigma_2\rangle$$

where $c_{\sigma_1 \sigma_2}$ are complex coefficients.

Hamiltonian for a two-spin system

Consider a spin operator, such as σ_1^z. This only acts on the first spin, for exampl

$$\sigma_1^z |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle$$

$$\sigma_1^z |\downarrow\uparrow\rangle = -|\downarrow\uparrow\rangle$$

$$\sigma_1^z |\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle$$

$$\sigma_1^z |\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle$$

Similarly a spin operator σ_2^z only acts on the second spin

$$\sigma_2^z |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle$$

$$\sigma_2^z |\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle$$

$$\sigma_2^z |\uparrow\downarrow\rangle = -|\uparrow\downarrow\rangle$$

$$\sigma_2^z |\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle$$

More rigorously, since the basis states are ~~built~~ constructed from tensor product $|\sigma_1 \sigma_2\rangle = |\sigma_1\rangle \otimes |\sigma_2\rangle$, an operator on these states need to be written as a tensor product. For instance

$$\sigma_1^z \otimes \mathbb{1}$$

$$\mathbb{1} \otimes \sigma_2^z$$

where $\mathbb{1}$ is the identity operator.

We now consider a two-spin ($\frac{1}{2}$) system (two-qubit)

Hilbert space of the first spin : $\mathcal{H}_1$

Hilbert space of the second spin : $\mathcal{H}_2$

The Hilbert space of the combined two-spin system is an expanded Hilbert space created by the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e.,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 .$$

This is a $2^2 = 4$-dimensional space.

Its basis states (assuming a z quatization axis) can be written as

$$|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \equiv |\uparrow\uparrow\rangle$$

$$|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \equiv |\uparrow\downarrow\rangle$$

$$|\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \equiv |\downarrow\uparrow\rangle$$

$$|\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \equiv |\downarrow\downarrow\rangle$$

(or, using a more compact notation)

These can also be expressed in a vector representation

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} \otimes \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \begin{vmatrix} 1 \\ 0 \end{vmatrix} \\ 0 \begin{vmatrix} 1 \\ 0 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Alternatively, renaming $c_{\uparrow\uparrow} \equiv w$, $c_{\uparrow\downarrow} \equiv x$

$c_{\downarrow\uparrow} \equiv y$, $c_{\downarrow\downarrow} \equiv z$

$$|\psi\rangle = w|\uparrow\uparrow\rangle + x|\uparrow\downarrow\rangle + y|\downarrow\uparrow\rangle + z|\uparrow\downarrow\rangle$$

with the normalization $\langle\psi|\psi\rangle = 1$ which gives

$$|w|^2 + |x|^2 + |y|^2 + |z|^2 = 1$$

A particular case for the state $|\psi\rangle$ is realized when the expansion coefficients can be written as

$w = ac$

$x = ad$ where a, b, c, d are complex

$y = bc$ numbers with $|a|^2 + |b|^2 = 1$

$z = bd$ $|c|^2 + |d|^2 = 1$

In such case, we can rewrite $|\psi\rangle$ as

$$|\psi\rangle = \left(a|\uparrow\rangle_1 + b|\downarrow\rangle_1\right) \otimes \left(c|\uparrow\rangle_2 + d|\downarrow\rangle_2\right)$$

$$= |\psi_1\rangle \otimes |\psi_2\rangle$$

where $|\psi_1\rangle = a|\uparrow_1\rangle + b|\downarrow\rangle_1$ is a spin state within $\mathcal{H}_1$, and $|\psi_2\rangle = c|\uparrow\rangle_2 + d|\downarrow\rangle_2$ is a spin state within $\mathcal{H}_2$.

$|\psi\rangle$ is therefore the tensor product of a state for spin 1 and a state for spin 2.

$|\psi\rangle$ is called separable or product state.

A state $|\psi\rangle$ that can <u>not</u> be written as the product of two single spin states is called "<u>entagled</u>".

Example of entangled states:

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right)$$

it is simple to see that they can not be written as the product $\left(a|\uparrow\rangle_1 + b|\downarrow\rangle_1\right)\otimes\left(c|\uparrow\rangle_2 + d\right.$

Note: a state $|\psi\rangle$ is defined by eight real numbers, that are the real and imaginary parts of w, x, y, z.

In case of a product state, these real numbers are reduced to four because of the normalization conditions

$$|a|^2 + |b|^2 = 1$$
$$|c|^2 + |d|^2 = 1$$

and because the overall phases of each $|\psi_1\rangle$ and $|\psi_2\rangle$ have no physical significance.

In case of an entangled state, we only have one normalization condition

$$|w|^2 + |x|^2 + |y|^2 + |z|^2 = 1$$

and only one overall phase to ignore. Therefore an entangled state is defined by six real parameters.

⇒ The parameter space of an entangled state is richer than that of a ~~product state~~ product of two states that can be prepared independently.

In matrix form

$$\sigma_1^z \otimes \mathbb{1} = \begin{vmatrix} 1\mathbb{1} & 0 \\ 0 & -1\mathbb{1} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$\mathbb{1} \otimes \sigma_2^z = \begin{vmatrix} 1\sigma_2^z & 0 \\ 0 & 1\sigma_2^z \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

(remember the tensor product of ~~m~~ two matrices, also called <u>Kronecker</u> product, is

$$A \otimes B = \begin{vmatrix} A_{11} B & A_{12} B \\ A_{21} B & A_{22} B \end{vmatrix} = \begin{vmatrix} A_{11} B_{11} & A_{11} B_{12} & A_{12} B_{11} & A_{12} B_{12} \\ A_{11} B_{21} & A_{11} B_{22} & A_{12} B_{21} & A_{12} B_{22} \\ A_{21} B_{11} & A_{21} B_{12} & A_{22} B_{11} & A_{22} B_{12} \\ A_{21} B_{21} & A_{21} B_{22} & A_{22} B_{21} & A_{22} B_{22} \end{vmatrix}$$

Similarly we can consider the operator $\sigma_1^z \sigma_2^z$ entering the Ising model.
They can be expressed as

$$\sigma_1^z \sigma_2^z \rightarrow \sigma_1^z \otimes \sigma_2^z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & 0 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \\ 0 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & -1 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$