Ex_Mod1: Preliminaries

Riccardo Marega

March 2025

Indice

1 Ex_1: slide 39

Given $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ with $x_i, y_i \in \mathbb{R}$ for $i = 1, 2$, we verify the vector space axioms:

1. **Closure under addition**:
   $x + y = (x_1 + y_1, x_2 + y_2)$. Since $\mathbb{R}$ is closed under addition, $x_1 + y_1, x_2 + y_2 \in \mathbb{R}$, implying $x + y \in \mathbb{R}^2$.

2. **Associativity of addition**:
   Given also $z = (z_1, z_2) \in \mathbb{R}^2$, $(x + y) + z = ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2) = (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)) = x + (y + z)$. This follows since addition in $\mathbb{R}$ is associative.

3. **Existence of the additive identity**:
   The additive identity in $\mathbb{R}^2$ is $0 = (0, 0)$, since for all $x = (x_1, x_2)$: $x + 0 = (x_1 + 0, x_2 + 0) = (x_1, x_2) = x$.

4. **Existence of additive inverses**:
   For each $x = (x_1, x_2)$, its additive inverse is $-x = (-x_1, -x_2)$, since: $x + (-x) = (x_1 + (-x_1), x_2 + (-x_2)) = (0, 0)$.

5. **Commutativity of addition**:
   Since addition in $\mathbb{R}$ is commutative, $x + y = (x_1 + y_1, x_2 + y_2) = (y_1 + x_1, y_2 + x_2) = y + x$.

6. **Associativity of scalar multiplication**:
   $\alpha(\beta x) = \alpha(\beta x_1, \beta x_2) = (\alpha(\beta x_1), \alpha(\beta x_2)) = ((\alpha\beta)x_1, (\alpha\beta)x_2) = (\alpha\beta)x$.

7. **Existence of a multiplicative identity**:
   Since $1 \cdot x = (1 \cdot x_1, 1 \cdot x_2) = (x_1, x_2) = x$, the multiplicative identity in $\mathbb{R}$ acts as the identity for scalar multiplication in $\mathbb{R}^2$.

8. **Distributivity of scalar multiplication over vector addition**:
   $\alpha \cdot (x + y) = \alpha \cdot (x_1 + y_1, x_2 + y_2) = (\alpha(x_1 + y_1), \alpha(x_2 + y_2)) = (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) = \alpha x + \alpha y$.

9. **Distributivity of scalar multiplication over field addition**:
   $(\alpha + \beta) \cdot x = ((\alpha + \beta)x_1, (\alpha + \beta)x_2) = (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) = \alpha x + \beta x$.

Thus, all vector space axioms hold, proving that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$. $\square$

Given $|| \bullet ||_2 \doteq \sqrt{x^T x} = \sqrt{x_1^2 + ... + x_n^2}$ with $x \in \mathbb{R}^n$:

1. Given the domain of the function, $|| \bullet ||_2 \geq 0 \; \forall x \in \mathbb{R}^n$ with $||x|| = 0 \iff x = 0$
2. $\forall \alpha \in R$, $||\alpha x||_2 = \sqrt{\alpha^2 x_1^2 + ... + \alpha x_n^2} = \sqrt{\alpha^2 (x_1^2 + ... + x_n^2)} = |a| \sqrt{x_1^2 + ... + x_n^2} = |\alpha| ||x||_2$
3. Given $x \in \mathbb{R}^n$ and remembering the Cauchy-Schwarz inequality, $||x+y||_2^2 =< x+y|x+y >=< x|x > + < y|y > +2 < x|y >\leq ||x||_2^2 + ||y||_2^2 + 2||x||_2||y||_2 = (||x||_2 + ||y||_2)^2$
   $\longrightarrow ||x + y||_2 \leq ||x||_2 + ||y||_2$.

$\square$

The function $f(x) = a + bx$ with $a \in \mathbb{R}/\{0\}$ and $b, x \in R$ in not linear in $\mathbb{R}$ because:
$f(\alpha x+\beta y) = a+b(\alpha x+\beta y) = a+b\alpha x+b\beta y \neq \alpha f(x)+\beta f(y)$ for any $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ $\square$.

2 Ex_2: slide 61

1.

$$
\begin{aligned}
A &= \begin{vmatrix} 2 & \frac{1}{3} & \sqrt{2} \\ -1 & 0 & 9 \\ 3 & 1 & \frac{2}{7} \end{vmatrix} \\
&= 2\begin{vmatrix} 0 & 9 \\ 1 & \frac{2}{7} \end{vmatrix} - \frac{1}{3}\begin{vmatrix} -1 & 9 \\ 3 & \frac{2}{7} \end{vmatrix} + \sqrt{2}\begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} \\
&= 2\left(0 \cdot \frac{2}{7} - 9 \cdot 1\right) - \frac{1}{3}\left(-1 \cdot \frac{2}{7} - 9 \cdot 3\right) + \sqrt{2}\,(-1 \cdot 1 - 0 \cdot 3) \\
&= -18 + \frac{27 + \frac{2}{7}}{3} - \sqrt{2} \\
&= -18 + \frac{189}{21} + \frac{2}{21} - \sqrt{2} \\
&= -18 + \frac{191}{21} - \sqrt{2}.
\end{aligned}
$$

2.

$$B = \begin{vmatrix} 2 & -1 \\ -3 & \frac{3}{2} \end{vmatrix} = 2 \cdot \frac{3}{2} - (-1 \cdot -3) = 3 - 3 = 0.$$

3.

$$C = \begin{vmatrix} 1 & 2c \\ -3 & -3c \end{vmatrix} = 1(-3c) - 2c(-3) = -3c + 6c = 3c.$$

$\square$

Given:

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

since there are 4 vectors in $\mathbb{R}^3$, they must be linearly dependent. $\square$

Given I_3, the identity matrix of order 3,

$$I_3 \cdot I_3 = I_3. \qquad \square$$

3 Ex_3: slide 85

Given, $A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$, and given that the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where the determinant is: $\det(A) = (3)(-9) - (-18)(2) = -27 + 36 = 9$
Since $\det(A) \neq 0$, the inverse exists: $A^{-1} = \frac{1}{9} \begin{bmatrix} -9 & 18 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -\frac{2}{9} & \frac{1}{3} \end{bmatrix}$.
Given $B = \begin{bmatrix} 2 & 2c \\ -3 & -3c \end{bmatrix}$, the determinant is: $\det(B) = (2)(-3c) - (2c)(-3) = -6c + 6c = 0$ Since the determinant is zero for any real c, the matrix is singular and does not have an inverse. □

The eigenvalues of a matrix M are found by solving:

$$\det(M - \lambda I) = 0$$

Given A:

$$\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (3-\lambda)(-9-\lambda) - (-18)(2) = 0.$$

Expanding:

$$(-27 - 3\lambda + 9\lambda + \lambda^2) + 36 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

$$\lambda = -3$$

So, the only eigenvalue is $\lambda = -3$ with multiplicity 2.
Given B,

$$\begin{vmatrix} 2-\lambda & 2c \\ -3 & -3c-\lambda \end{vmatrix} = (2-\lambda)(-3c-\lambda) - (2c)(-3) = 0$$

Expanding:

$$-6c - 2\lambda + 3c\lambda + \lambda^2 + 6c = 0$$

$$\lambda^2 + (3c-2)\lambda = 0$$

$$\lambda(\lambda + 3c - 2) = 0$$

So, the eigenvalues are:

$$\lambda_1 = 0, \quad \lambda_2 = 2 - 3c$$ □