$s = \frac{1}{2}$ $\qquad$ $m = \pm\frac{1}{2}$

$$S^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle$$

$$S^z |s, m\rangle = \hbar m \, |s, m\rangle \qquad m = -s, s$$

$$-\frac{1}{2}, \frac{1}{2}$$

$$|s = \tfrac{1}{2}, m = +\tfrac{1}{2}\rangle = |\uparrow\rangle = |0\rangle$$

$$|s = \tfrac{1}{2}, m = -\tfrac{1}{2}\rangle = |\downarrow\rangle = |1\rangle$$

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \leftarrow$$

$\mathcal{H} \quad 2$

$$|\psi\rangle = \sum_{i=1}^{N} c_i |i\rangle$$

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad |a|^2 + |b|^2 = 1$$

$$|\uparrow\rangle \; (\equiv |0\rangle) \qquad \begin{vmatrix} 1 \\ 0 \end{vmatrix} \leftarrow$$

$$|\downarrow\rangle \; (\equiv |1\rangle) \qquad \begin{vmatrix} 0 \\ 1 \end{vmatrix} \leftarrow$$

$$\begin{vmatrix} a \\ b \end{vmatrix}$$

$$1 = \langle\psi|\psi\rangle = \begin{vmatrix} a^* & b^* \end{vmatrix} \begin{vmatrix} a \\ b \end{vmatrix} = a^* a + b^* b$$
$$= |a|^2 + |b|^2$$

$\hat{S}^z \qquad \hat{\sigma}^z$

$$\rightarrow \hat{\sigma}^z |\uparrow\rangle = 1 \, |\uparrow\rangle \qquad \frac{1}{2}\hbar$$

$$\to \underline{\sigma^z |\downarrow\rangle = -1|\downarrow\rangle} \qquad \cancel{-\tfrac{1}{2}\hbar}$$

$$\sigma^z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = 1 \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \end{vmatrix} = -1 \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

$$\hat{\sigma}^z \to \underset{\underset{Z}{\parallel}}{\sigma^z} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \qquad |\uparrow\rangle \begin{vmatrix} 1 \\ 0 \end{vmatrix} |0\rangle \qquad |\downarrow\rangle \begin{vmatrix} 0 \\ 1 \end{vmatrix} |1\rangle$$

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$
$$= a|0\rangle + b|1\rangle$$

$$\begin{vmatrix} a \\ b \end{vmatrix}$$

$$\hat{\sigma}^x \qquad \hat{\sigma}^y \qquad \hat{\sigma}^z$$

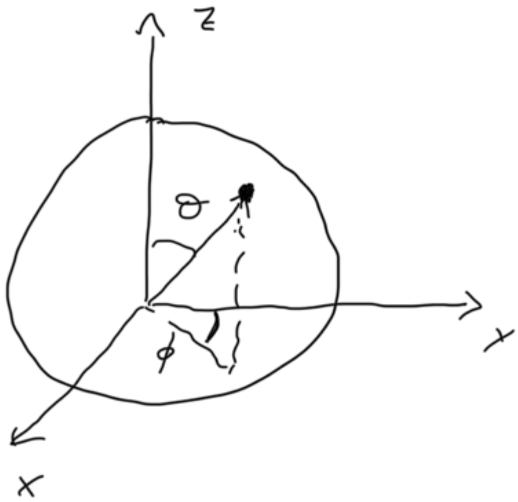
$$\sigma^x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$\sigma^x\sigma^y - \sigma^y\sigma^x$$

$$|\psi\rangle = a|\overset{0}{\uparrow}\rangle + b|\overset{1}{\downarrow}\rangle$$

$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$$

Bloch sphere




$\theta \in [0, \pi]$

$\phi \in [0, 2\pi[$

$$\underline{U} = \begin{vmatrix} \cos\frac{\theta}{2} & e^{i\phi}\sin\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{vmatrix}$$

$$U^{\dagger} U = \mathbb{1}$$

$$U^{\dagger} = (U^{T})^{*}$$

$$\left( \begin{vmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{vmatrix} \right)^{*}$$

$$U^{\dagger} = \begin{vmatrix} \cos\frac{\theta}{2} & e^{i\phi}\sin\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{vmatrix} = U$$

$$U^{\dagger} U = U U = 1$$

$$\underline{\langle \psi | \psi \rangle} = \underline{\langle \psi' | U^{\dagger} U | \psi' \rangle}$$

$$|\psi'\rangle = U|\psi\rangle$$

$$\cos\theta \qquad e^{i\phi}\sin\theta$$

$U|0\rangle$ $\qquad U\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} \cdots & \cdots \\ e^{-i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{vmatrix}\begin{vmatrix} 1 \\ 0 \end{vmatrix}$

$$= \begin{vmatrix} \cos\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} \end{vmatrix}$$

$U|1\rangle$ $\qquad U\begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} \cos\frac{\theta}{2} & e^{i\phi}\sin\frac{\theta}{2} \\ e^{-i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{vmatrix}\begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$$= \begin{vmatrix} e^{i\phi}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{vmatrix}$$

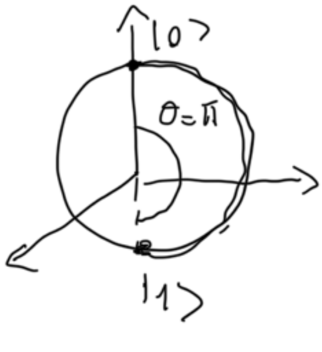
$\theta = \pi$ $\qquad \phi = 0$

$U|0\rangle$ $\qquad U\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{\sigma^x}\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$|\uparrow\rangle$

$X$

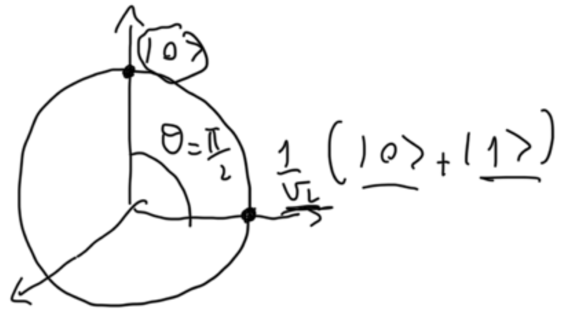
$$U|0\rangle = |1\rangle$$
$$U|\uparrow\rangle = |\downarrow\rangle$$




$\theta = \frac{\pi}{2}$ $\qquad \phi = 0$

$U|0\rangle$ $\qquad U\begin{vmatrix} 1 \\ \cdots \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \cdots & \cdots \end{vmatrix}\begin{vmatrix} 1 \\ \cdots \end{vmatrix} =$

$$| 0 | \quad \begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} \begin{vmatrix} 0 \end{vmatrix}$$



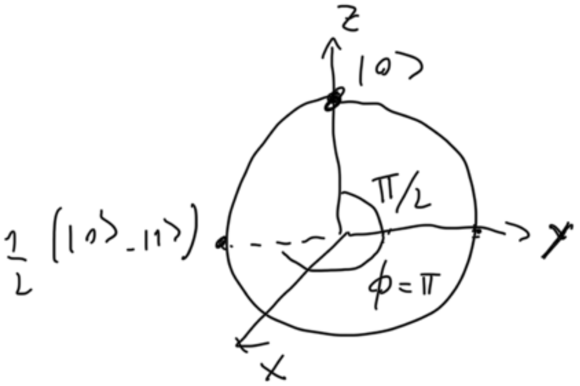

$$= \begin{vmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

Hadamard

$\theta = \frac{\pi}{2}$ $\qquad \phi = \pi$ $\qquad e^{i\pi}$

$$U \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{2}}{2} & \overset{\nearrow 1}{e^{i\pi}} \frac{\sqrt{2}}{2} \\ \underset{\substack{\downarrow \\ -1}}{e^{-i\pi}} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \end{vmatrix}$$

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two - qubit

$\mathcal{H}_1$ $\qquad \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}$

$\mathcal{H}_2$ $\qquad\qquad\qquad 2^2 = 4$

$\mathcal{H}_1 \quad |\uparrow\rangle_1, \ |\downarrow\rangle_1$

$\mathcal{H}_2 \quad |\uparrow\rangle_2 \quad |\downarrow\rangle_2$

$\rightarrow |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \qquad |\uparrow\uparrow\rangle \qquad |0\,0\rangle$

$\rightarrow |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \qquad |\uparrow\downarrow\rangle \qquad |0\,1\rangle$

$|\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \qquad |\downarrow\uparrow\rangle \qquad |1\,0\rangle$

$|\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \qquad |\downarrow\uparrow\rangle \qquad |1\,1\rangle$

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} \otimes \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \begin{vmatrix} 1 \\ 0 \end{vmatrix} \\ 0 \begin{vmatrix} 1 \\ 0 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} \otimes \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \\ 0 \begin{vmatrix} 0 \\ 1 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} \qquad \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} a_1 \\ a_2 \end{vmatrix} \otimes \begin{vmatrix} b_1 \\ b_2 \end{vmatrix} = \begin{vmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{vmatrix}$$

$$|\psi\rangle = \underline{\sum_{\alpha=\uparrow,\downarrow} \sum_{\beta=\uparrow,\downarrow} c_{\alpha\beta} |\alpha\,\beta\rangle}$$

$c_{\uparrow\uparrow} = w \qquad c_{\uparrow\downarrow} = x \qquad c_{\downarrow\uparrow} = y \qquad c_{\downarrow\downarrow} = z$

$$|\psi\rangle = w\,|\uparrow\uparrow\rangle + x\,|\uparrow\downarrow\rangle + y\,|\downarrow\uparrow\rangle + z\,|\downarrow\downarrow\rangle \leftarrow$$

$$\langle\psi|\psi\rangle = 1$$

$$|\psi\rangle \rightarrow \begin{vmatrix} w \\ x \\ y \\ z \end{vmatrix}$$

$$\begin{vmatrix} w^* & x^* & y^* & z^* \end{vmatrix} \begin{vmatrix} w \\ x \\ y \\ z \end{vmatrix} =$$

$$= w^* w + x^* x + y^* y + z^* z =$$

$$= |w|^2 + |x|^2 + |y|^2 + |z|^2 = 1$$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \leftarrow$$

$$\rightarrow \quad = \left(a\,|\uparrow\rangle_1 + b\,|\downarrow\rangle_1\right) \otimes \left(c\,|\uparrow\rangle_2 + d\,|\downarrow\rangle_2\right)$$

product states

Entangled

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle\right)$$

$$\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right)$$

Bell states

$\frac{1}{\sqrt{2}} \left( |\downarrow\downarrow\rangle + \ldots \right)$

---

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) =$$

$$\frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right) \leftarrow$$

$$\left( a|0\rangle + b|1\rangle \right) \otimes \left( c|0\rangle + d|1\rangle \right) =$$

$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

$$ac = \frac{1}{\sqrt{2}}$$

$$ad = 0 \rightarrow a = 0 \quad d = 0$$

$$bc = 0 \rightarrow b = 0 \quad c = 0$$

$$bd = \frac{1}{\sqrt{2}}$$

---

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$\left( a|0\rangle + b|1\rangle \right) \otimes \left( c|0\rangle + d|1\rangle \right) =$$

$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

$$ac = \frac{1}{2} \qquad a = \frac{1}{\sqrt{2}} \qquad c = \frac{1}{\sqrt{2}}$$

$$ad = \frac{i}{2} \qquad d = \frac{i}{\sqrt{2}}$$

b

$ac = -\frac{1}{2}$ $\qquad\qquad$ $b = -\frac{1}{\sqrt{2}}$

$bd = -\frac{i}{2}$

$$|\psi\rangle = \left(\frac{1}{2}|0\rangle_1 - \frac{1}{\sqrt{2}}|1\rangle_1\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle_2 + \frac{i}{\sqrt{2}}|1\rangle_2\right)$$

---

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$$

$2^3 = 8$

$|000\rangle \qquad |001\rangle \qquad |010\rangle \qquad |011\rangle$

$|100\rangle \qquad |101\rangle \qquad |110\rangle \qquad |111\rangle$

$|\psi\rangle$

GHZ : $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$

W : $\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$

---

$|\psi\rangle = a|0\rangle + b|1\rangle$

$|a|^2 + |b|^2 = 1$

$|a|^2 \qquad |b|^2$

$|0\rangle \qquad |1\rangle$

---

$$|\psi\rangle = \sum_{\alpha=0}^{1}\sum_{\beta=0}^{1} \langle\alpha\beta|\alpha\beta\rangle$$

$$= w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

$$|w|^2 \qquad |x|^2 \qquad |y|^2 \qquad |z|^2$$

$$\frac{3}{5}|00\rangle + \frac{4i}{5}|11\rangle$$

$$\frac{9}{25} \qquad \frac{16}{25}$$

---

$$\rightarrow |\psi\rangle = \sum_{\alpha\beta} c_{\alpha\beta} |\alpha\beta\rangle$$

$$\alpha \qquad \sum_{\beta=0,1} |c_{\alpha\beta}|^2$$

$$\rightarrow 0 \qquad \sum_{\beta=0,1} |c_{0\beta}|^2$$

$$\rightarrow |0\rangle \otimes \frac{|\phi\rangle}{\sqrt{\langle\phi|\phi\rangle}}$$

$$|\phi\rangle = \sum_\beta c_{0\beta}|\beta\rangle$$

$$|\psi\rangle = \sum_\alpha |\alpha\rangle \otimes \underbrace{\left(\sum_\beta c_{\alpha\beta}|\beta\rangle\right)}_{|\phi\rangle}$$

$$\alpha = 0$$

$$|\psi_\alpha\rangle = |\alpha = 0\rangle \otimes |\phi\rangle$$

$$\langle\phi|\phi\rangle = \sum_\beta |c_{\alpha\beta}|^2$$

$$|\psi_\alpha\rangle = |0\rangle \otimes \frac{|\phi\rangle}{\sqrt{\langle\phi|\phi\rangle}}$$

---

$$\to |\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

$$= |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle\right) + |1\rangle\left(\frac{i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle\right)$$

$$\frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \quad 66\%$$

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3} \quad 33\%$$

$$|0\rangle \otimes \frac{\overbrace{\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle\right)}^{|\phi\rangle}}{\underbrace{\sqrt{\frac{1}{2}+\frac{1}{6}}}_{\to \sqrt{\langle\phi|\phi\rangle}} = \sqrt{\frac{2}{3}}} =$$

$$|0\rangle \otimes \left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$$

$$|1\rangle \otimes \frac{\overbrace{\left(\frac{i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle\right)}^{|\phi\rangle}}{\sqrt{\langle\phi|\phi\rangle} = \sqrt{\frac{1}{3}}} =$$

$$= |1\rangle \otimes \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right)$$

---

$W$ : $\frac{1}{\sqrt{3}} |001\rangle + \frac{1}{\sqrt{3}} |010\rangle + \frac{1}{\sqrt{3}} |100\rangle$

$$|0\rangle \times \left( \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{3}} |10\rangle \right) + |1\rangle \otimes \left( \frac{1}{\sqrt{3}} |00\rangle \right)$$

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3} \quad \sim 66\%$$

$$\frac{1}{3} \quad \sim 33\%$$

$$|0\rangle \otimes \boxed{\left( \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{3}} |10\rangle \right)} \; |\phi\rangle$$

$$\sqrt{\langle\phi|\phi\rangle} = \sqrt{\frac{2}{3}}$$

$$= |0\rangle \otimes \left( \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle \right)$$

$$\frac{|1\rangle \otimes \frac{1}{\sqrt{3}} |00\rangle}{1/\sqrt{3}} = |1\rangle \otimes |00\rangle = |100\rangle$$

---

$$\hat{\sigma}^z \; |\uparrow\rangle = 1 \, |\uparrow\rangle$$
$$|0\rangle = 1 \, |0\rangle$$

$$\hat{\sigma}^z \; |\downarrow\rangle = -1 \, |1\rangle$$
$$|1\rangle = -1 \, |1\rangle$$

$$\begin{aligned}
&\longrightarrow \hat{\sigma}_1^z\ |\uparrow\uparrow\rangle && |00\rangle && = 1\ |\uparrow\uparrow\rangle \\
&\longrightarrow \hat{\sigma}_1^z\ |\uparrow\downarrow\rangle && |01\rangle && = 1\ |\uparrow\downarrow\rangle \\
&\longrightarrow \hat{\sigma}_1^z\ |\downarrow\uparrow\rangle && |10\rangle && = -1\ |\downarrow\uparrow\rangle \\
&\longrightarrow \sigma_1^z\ |\downarrow\downarrow\rangle && |11\rangle && = -1\ |\downarrow\downarrow\rangle
\end{aligned}$$

$$\hat{\sigma}^x |\uparrow\rangle = |\downarrow\rangle$$

$$\hat{\sigma}^x |\downarrow\rangle = |\uparrow\rangle$$

$$\hat{\sigma}_1^x |\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle$$

$$\hat{\sigma}_1^x |\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle$$

$$\hat{\sigma}_2^z |\uparrow\downarrow\rangle = -1\ |\uparrow\downarrow\rangle$$

$$\sigma_1^z \otimes \mathbb{1} = \begin{vmatrix} 1\mathbb{1} & 0\mathbb{1} \\ 0\mathbb{1} & -1\mathbb{1} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$\hat{\sigma}_1^z\ |\uparrow\uparrow\rangle = 1\ |\uparrow\uparrow\rangle$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = 1 \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\hat{\sigma}_1^z\ |\downarrow\uparrow\rangle = -1\ |\downarrow\uparrow\rangle$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = - \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 & \omega & 1 \end{vmatrix}$$

$$\hat{\sigma}_2^z$$

$$\mathbb{1} \otimes \sigma^z = \begin{vmatrix} 1\sigma^z & 0\sigma^z \\ 0\sigma^z & 1\sigma^z \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$\underline{A} \otimes \underline{B} = \begin{vmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{vmatrix} =$$

$$= \begin{vmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{vmatrix}$$

$$\hat{\sigma}_1^z \hat{\sigma}_2^z$$

$$\hat{Z}_1 \hat{Z}_2 \quad \Rightarrow \quad \sigma_1^z \otimes \sigma_2^z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} =$$

$$= \begin{vmatrix} 1\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & 0\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \\ 0\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & -1\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

---

$$\hat{\sigma}_1^x \hat{\sigma}_2^x \qquad \sigma^x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\hat{X} \quad \hat{X}.$$

$$\sigma_1^x \otimes \sigma_2^x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$\hat\sigma_1^x \hat\sigma_2^x \,|\uparrow\uparrow\rangle = |\downarrow\downarrow\rangle \quad |11\rangle$$

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$\boxed{\hat\sigma_1^x \hat\sigma_2^x} \leftarrow$$

$$\sigma_1^x \otimes \sigma_2^x \otimes \mathbb{1}$$

---

1 quibit $\quad |\psi\rangle = a|0\rangle + b|1\rangle$

$$\langle \hat\sigma^z \rangle = \langle \psi | \hat\sigma^z | \psi \rangle =$$

$$\left( a^* \langle 0| + b^* \langle 1| \right) \hat\sigma^z \left( a|0\rangle + b|1\rangle \right) =$$

$$= \left(a^* \langle 0| + b^* \langle 1|\right)\left(a|0\rangle - b|1\rangle\right) =$$

$$= a^* a \underbrace{\langle 0|0\rangle}_{1} - a^* b \underbrace{\langle 0|1\rangle}_{0} + b^* a \underbrace{\langle 1|0\rangle}_{0} - b^* b \underbrace{\langle 1|1\rangle}_{1}$$

$$= a^* a - b^* b = |a|^2 - |b|^2$$

---

$\hat{O}$ $\quad \lambda_n \quad n = 1, \dots$

$|\lambda_n\rangle$

$$\hat{O}|\lambda_n\rangle = \lambda_n |\lambda_n\rangle$$

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \sum_n \lambda_n P(\lambda_n)$$

$$P(\lambda_n) = |\langle \lambda_n | \psi \rangle|^2$$

$\hat{\sigma}^z |0\rangle = 1 |0\rangle$

$\sigma^z |1\rangle = -1 |1\rangle$

$|\psi\rangle = a|0\rangle + b|1\rangle$

$|\langle 0|\psi\rangle|^2 = |a|^2$

$|\langle 1|\psi\rangle|^2 = |b|^2$

$$\langle \hat{\sigma}^z \rangle = 1|a|^2 - 1|b|^2$$

---

2-qubit

$$|\psi\rangle = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

$\hat{\sigma}^z_1$

$\langle \psi | \hat{\sigma}^z_1 | \psi \rangle$

$\langle \psi | = w^* \langle 00| + x^* \langle 01| + y^* \langle 10|$
$+ z^* \langle 11|$

$$\sigma^z \otimes \mathbb{1} = \begin{pmatrix} 1\,\mathbb{1} & 0\,\mathbb{1} \\ 0\,\mathbb{1} & -1\,\mathbb{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sigma^z_1 |00\rangle = 1\,|00\rangle$$

$$\sigma^z_1 |01\rangle = 1\,|01\rangle$$

$$\sigma^z_1 |10\rangle = -1\,|10\rangle$$

$$\sigma^z_1 |11\rangle = -1\,|11\rangle$$

$$\langle \sigma^z_1 \rangle = |w|^2 + |x|^2 - |y|^2 - |z|^2$$

$$|w|^2 = \frac{\text{count}_{00}}{N_{\text{SHOTS}}}$$

$$|x|^2 = \frac{\text{count}_{01}}{N_{\text{SHOTS}}}$$

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$$\langle \hat{\sigma}^z_1 \hat{\sigma}^z_2 \rangle$$

$$|\psi\rangle = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

$$\langle \psi | \underbrace{\hat{\sigma}^z_1 \hat{\sigma}^z_2 \left( w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle \right)}_{|\psi\rangle}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

$$\hat\sigma_1^z \otimes \hat\sigma_2^z = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} \quad \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

$$\hat\sigma_1^z \hat\sigma_2^z |\underset{\uparrow}{0}\,\underset{\uparrow}{0}\rangle = 1\,|00\rangle$$

$$\hat\sigma_1^z \hat\sigma_2^z |\underset{\uparrow}{0}\,\underset{\downarrow}{1}\rangle = -1\,|01\rangle$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \\ 0 \\ 0 \end{vmatrix} = -1 \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$\hat\sigma_1^z \hat\sigma_2^z |\underset{\downarrow}{1}\,\underset{\uparrow}{0}\rangle = -1\,|10\rangle$$

$$\hat\sigma_1^z \hat\sigma_2^z |\underset{\downarrow}{1}\,\underset{\downarrow}{1}\rangle = \underbrace{(-1)(-1)}_{1}\,|11\rangle$$

$$\overbrace{(w^* \langle 00| + x^* \langle 01| + y^* \langle 10| + z^* \langle 11|)}^{\langle\psi|}\,(w|00\rangle - x|01\rangle - y|10\rangle + z|11\rangle)$$

$$|w^* \quad x^* \quad y^* \quad z^*| \begin{vmatrix} w \\ -x \\ -y \\ z \end{vmatrix} =$$

$$= |w|^2 - |x|^2 - |y|^2 + |z|^2$$

$\hat\sigma^x$ $\qquad$ $|\psi\rangle = a|0\rangle + b|1\rangle$

$\hat{X}$

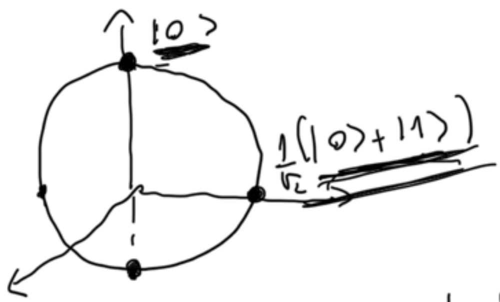
$\hat{\sigma}^z$

$$[\hat{\sigma}^x, \hat{\sigma}^z] \propto \sigma^y$$

$\langle \psi | \hat{\sigma}^x | \psi \rangle$

$U|\psi\rangle$

$$U = \begin{vmatrix} \cos\frac{\theta}{2} & e^{i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{vmatrix}$$




$\begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $\quad |00\rangle$

$\theta = \frac{\pi}{2} \quad \phi = 0$

$$H = U = \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\sigma^x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \to \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\sigma^x \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] = 1 \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]$$

$$\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \end{vmatrix}$$

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\frac{1}{\sqrt{2}}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} 1 \\ -1 \end{vmatrix} = \frac{1}{\sqrt{2}}\begin{vmatrix} -1 \\ 1 \end{vmatrix} = -1\left(\frac{1}{\sqrt{2}}\begin{vmatrix} 1 \\ -1 \end{vmatrix}\right)$$

$$\hat{\sigma}^x\left(\frac{1}{2}|0\rangle - |1\rangle\right) = -1\left(\frac{1}{2}|0\rangle - |1\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\langle\psi|\sigma^x|\psi\rangle = \langle\psi'|\underline{U^\dagger}\,\sigma^x\,U|\psi'\rangle$$

$$|\psi\rangle = U|\psi'\rangle \qquad U^\dagger \to U$$

$$= \langle\psi'|\underbrace{U\,\sigma^x\,U}_{\sigma^z}|\psi'\rangle$$

$$\sigma^z = U\sigma^x U =$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} =$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

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$$\langle\sigma^x_1\,\sigma^x_2\rangle$$

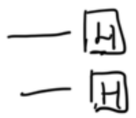
$$\underline{U} = U_1 \otimes U_2 = H_1 \otimes H_2$$

$$= \frac{1}{2}\begin{vmatrix} 1H & 1H \\ 1H & -1H \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

$\sigma_1^z \sigma_2^z$

$|00\rangle$
$|01\rangle$
$|10\rangle$
$|11\rangle$

$U^\dagger (\sigma_1^x \otimes \sigma_2^x) U$



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$\hat{O}$ $\quad |\lambda_n\rangle \qquad \hat{O}|\lambda_n\rangle = \lambda_n |\lambda_n\rangle$

$\cancel{\lambda}_n$

$\hat{P}_n = |\lambda_n\rangle\langle\lambda_n|$

$P_n|\psi\rangle = |\lambda_n\rangle\langle\lambda_n|\psi\rangle$

$\hat{O} = \sum_n \lambda_n |\lambda_n\rangle\langle\lambda_n|$

$\hat{O} = \sum_n \lambda_n \hat{P}_n$

$$\langle\psi|\hat{O}|\psi\rangle = \sum_n \lambda_n \langle\psi|\lambda_n\rangle\langle\lambda_n|\psi\rangle$$
$$= \sum_n \lambda_n |\langle\lambda_n|\psi\rangle|^2$$
$$= \sum_n \lambda_n P(\lambda_n)$$

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$|\psi\rangle = \sum_n c_n |n\rangle$

$O \quad G = \lambda G$

$\langle n|\hat{O}|m\rangle = O_{nm}$

$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad \sigma^z$$
$$z$$
$$|a|^2 \qquad |b|^2$$

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$$|0\rangle_1 \quad |0\rangle_2 \qquad |00\rangle$$
$$|1\rangle_1 \quad |1\rangle_2 \qquad |01\rangle$$
$$|10\rangle$$
$$|11\rangle$$

$$|\psi\rangle = w|00\rangle + x|01\rangle + y|10\rangle + z|11\rangle$$

$$|\psi\rangle = (a|0\rangle_1 + b|1\rangle_1) \otimes (c|0\rangle_2 + d|1\rangle_2)$$

$$ac = w$$
$$ad = x$$
$$bc = y$$
$$bd = z$$

$$|w|^2 \qquad |x|^2 \qquad |y|^2 \qquad |z|^2$$

$$\hat{A}_1 \quad \hat{B}_2$$

$$A_1 \otimes B_2$$ Kronecker

$$\langle \hat{\sigma}^z \rangle = \langle \psi | \sigma^z | \psi \rangle \leftarrow$$

$$\langle \hat{\sigma}_1^z \hat{\sigma}_2^z \rangle$$

$$\sigma^z \rightarrow \sigma^x$$

$\sigma^z \rightarrow \sigma^y$

$$\sigma^y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \qquad Y$$

$$U^\dagger \sigma^y U = \sigma^z$$

$$|\psi\rangle = U|\psi\rangle$$