4. 12. 2023

4.1 Write the antimodels of U_1:

$U_1: (p \vee q) \wedge \neg r \longrightarrow p \wedge q \wedge r$

$\neg U_1: \neg((p \vee q) \wedge \neg r \longrightarrow p \wedge q \wedge r)$ (1) ✓

| $\alpha(1)$

$(p \vee q) \wedge \neg r$ (2) ✓

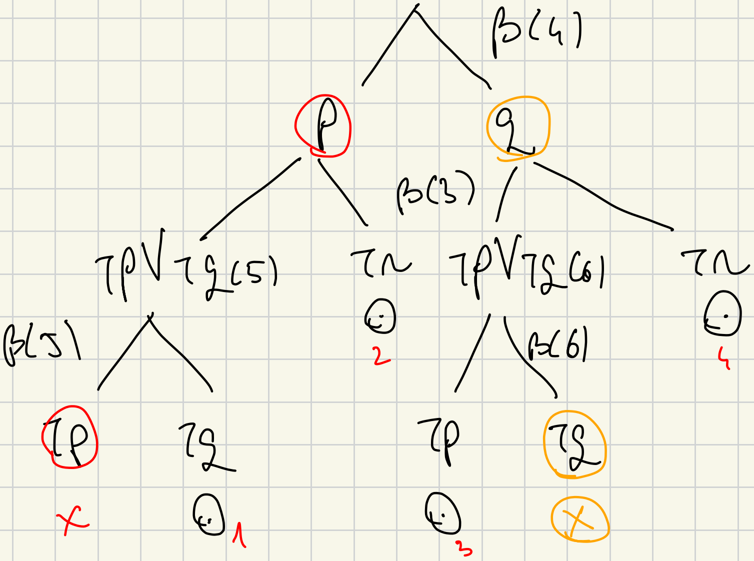
$\neg p \vee \neg q \vee \neg r \equiv \neg(p \wedge q \wedge r)$ (3) ✓

| $\alpha(2)$

$p \vee q$ (4) ✓

|

$\neg r$



$$DNF(\mathcal{U}_1) = (\underbrace{\neg q \wedge p \wedge \neg r}_{1}) \vee (\underbrace{\neg r \wedge p}_{2}) \vee (\underbrace{\neg p \wedge q \wedge \neg r}_{3}) \vee$$

$$\vee (\underbrace{\neg r \wedge q}_{4}) \equiv$$

$$\overset{abs.}{\underset{law}{\equiv}} (\neg r \wedge p) \vee (\neg r \wedge q)$$

- Cube $p \wedge \neg r \equiv T$ provides 2 models

$i_{1,2} : \{p, q, r\} \to \{T, F\}$

$i_1(p) = T$, $i_1(q) = T$, $i_1(r) = F$

$i_1(p) = T$, $i_2(q)$ F , $i_1(r) = F$

- Cube $q \wedge \neg r$ provides 2 models

$i_{3,4} : \{p, q, r\} \to \{T, F\}$

$i_3(p) = T$, $i_3(q) = T$, $i_3(r) = F$

$i_3(p) = F$, $i_3(q) = T$, $i_3(r) = F$

$i_1 = i_3$

$\neg\mathcal{U}_1$ has 3 models $i_1(\neg\mathcal{U}_1) = i_2(\neg\mathcal{U}_1) = i_4(\neg\mathcal{U}_1) = T$
which are the anti-models of $\mathcal{U}_1$:

$i_1(\mathcal{U}_1) = i_2(\mathcal{U}_1) = i_4(\mathcal{U}_1) = F$

7.1) $F(\exists x)(A(x) \wedge B(x)) \longrightarrow (\exists x)A(x) \wedge (\exists x)B(x)$

$\neg\mathcal{U}_1 = \neg((\exists x)(A(x) \wedge B(x)) \rightarrow (\exists x)A(x) \wedge (\exists x)B(x))$ (1) ✓

$\mid$ $\alpha(1)$

$(\exists x)(A(x) \wedge B(x))$ (2) ✓

$\mid$

$\neg((\exists x)A(x) \wedge (\exists x)B(x))$ (3) ✓

$\mid$ $\delta(2)$ (a new constant)

$A(a) \wedge B(a)$ (4) ✓

$\mid$ $\alpha(4)$

$\boxed{A(a)}$

$\mid$

$\boxed{B(a)}$

$\beta(3)$

$\neg((\exists x)A(x)) \equiv$

$\equiv (\forall x)\neg A(x)$ (5)

$\mid$ $\gamma(5)$

$\boxed{\neg A(a)}$

$\neg((\exists x)B(x)) \equiv$

$\equiv (\forall x)\neg B(x)$ (6)

$\mid$ $\gamma(6)$

$\boxed{\neg B(a)}$

(5) copy of formula	(6) copy of formula
$\otimes$	$\otimes$

¬U₁ has a closed sem. table with two closed branches $(A(a);\ \neg A(a))$ and $(B(a),\ \neg B(a))$, so according to the theorem of s. and completeness $\Rightarrow\ \models U_1$ (tautology)

$$7.1)\ \not\models \overbrace{(\exists x)A(x) \wedge (\exists x)B(x)}^{T} \rightarrow \overbrace{(\exists x)(A(x) \wedge B(x))}^{F} : U_2$$

$$\neg U_2 = \neg((\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)(A(x) \wedge B(x)))\quad (1)\ ✓$$

| $\alpha(1)$

$$(\exists x)A(x) \wedge (\exists x)B(x)\quad (2)\ ✓$$

|

$$\neg((\exists x)A(x) \wedge B(x)) \equiv (\forall x)\neg(A(x) \wedge B(x))\quad (3)\ ✓$$

| $\alpha(2)$

$$(\exists x)A(x)\quad (4)\ ✓$$

|

$$(\exists x)B(x)\quad (5)\ ✓$$

|

! Resolution in predicate logic: 2.1 → prepare for next time

| $\delta(4)$ a const.

$A(a)$

| $\delta(5)$ b const.

$B(b)$

| $\gamma(3)$

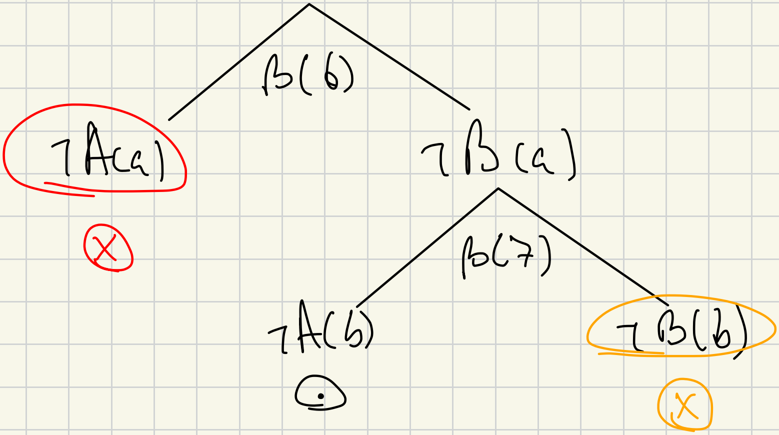
$\neg(A(a) \wedge B(a))$ $x \leftarrow a$ (6) ✓

|

$\neg(A(b) \wedge B(b))$ $x \leftarrow b$ (7) ✓

|

(3) copy



Conclusion: $\neg \mathcal{U}_2$ has an open sem. tabl. with two closed branches $(A(a), \neg A(a))$; $(B(b), \neg B(b))$ and one open branch, so $\neg \mathcal{U}_2$ is consistent and $\not\models \mathcal{U}_2$ (not a tautology). The open branch of $\neg \mathcal{U}_2$ provides a model of $\neg U_2$, which is an anti-model of U_2.

$$\underset{\text{interpretation}}{I} = \langle \overset{\text{domain}}{D}, \underset{\text{function}}{m} \rangle$$

$D = \{a, b\}$ (constants on the open branch)

$m(A)(a) = T$ $\quad\quad$ $m(A)(b) = F$

$m(B)(a) = F$ $\quad\quad$ $m(B)(b) = T$

"The predicate symbol A is satisfied for object a"

$v^I(\neg U_2) = T$ and $v^I(U_2) = F$

OBS: The open branch provides a generic model with a minimum number of constants!

- We build a complete model based on the generic model.

$I_1 = \langle D_1, m_1 \rangle$

$D_1 = \{2, 3\}$

$m_1(A)(x) =$ "x is even"

$m_1(B)(x) =$ "x is odd"

$v^{I_1}(\neg U_2) = T$ and $v^{I_1}(U_2) = F$

Resolution in propositional logic

3) $H_1 : L \wedge \neg G \to M \equiv \neg L \vee G \vee M : \overset{\text{clause}}{\overset{\uparrow}{C_1}}$

$H_2 : J \to L \equiv \neg J \vee L : C_2$

$H_3 : JT \to J \equiv \neg JT \vee J : C_3$

$H_4 : \neg G \wedge GS : C_4 \wedge C_5$

$H_5 : JT : C_6$

$C : M \; ; \neg C \equiv \neg M : C_7$

L - Lucy will go to the party
G - George will go to the party
M - Mary will go to the party
J - John will go to the party
JT - John is in town
GS - George is sick

$$H_1, H_2, H_3, H_4, H_5 \overset{?}{\vdash} C$$

Theorem of soundness and completeness

$$H_1, H_2, \dots, H_n \vdash C \text{ iff } CNF(H_1 \wedge \dots \wedge H_n \wedge \neg C) \vdash_{Res} \overset{\text{empty clause}}{\overset{\uparrow}{\square}}$$

$S = \{C_1, \dots, C_7\}$, $S \overset{?}{\underset{Res}{\vdash}} \square$

Resolution result

- $C_1 = \underset{\uparrow x}{p} \vee q \vee \neg r$

$C_2 = \boxed{\neg p} \vee t$

$R(C_1, C_2) = s \vee \neg r \vee t$

- $C_1 = p$
  $C_2 = \neg p$

$R(C_1, C_2) = \square$ → empty clause, symbolises inconsistency.

$C_8 = \mathrm{Res}_{JT}(C_3, C_6) = J$

$C_9 = \mathrm{Res}_{J}(C_8, C_2) = L$

$C_{10} = \mathrm{Res}_{L}(C_5, C_1) = G \vee M$

$C_{11} = \mathrm{Res}_{G}(C_{10}, C_4) = M$

$C_{12} = \mathrm{Res}_{M}(C_{12}, C_7) = \square$

1.) $U_1 = (A \to B \wedge C) \to (A \to B) \wedge (A \to C)$

$\neg U_1 = \neg((A \overset{1}{\to} B \wedge C) \overset{2}{\to} (A \overset{3}{\to} B) \wedge (A \overset{4}{\to} C))$

$\xlongequal[1,3,4]{\text{replace}} \neg((\neg A \vee (B \wedge C)) \overset{2}{\to} (\neg A \vee B) \wedge (\neg A \vee C))$

$\xlongequal[2]{\text{replace}} \neg(\neg(\neg A \vee (B \wedge C)) \vee ((\neg A \vee B) \wedge (\neg A \vee C)))$

$\xlongequal{\text{de Morgan}} (\neg A \vee (B \wedge C)) \wedge (\neg(\neg A \vee B) \vee \neg(\neg A \vee C))$

$\xlongequal{\text{distribute}} (\neg A \vee B) \wedge (\neg A \vee C) \wedge [(A \wedge \neg B) \vee (A \wedge \neg C)]$

$$\equiv \underbrace{(\neg A \vee B)}_{C_1} \wedge \underbrace{(\neg A \vee C)}_{C_2} \wedge \underbrace{A}_{C_3} \wedge \underbrace{(\neg B \vee \neg C)}_{C_4}$$

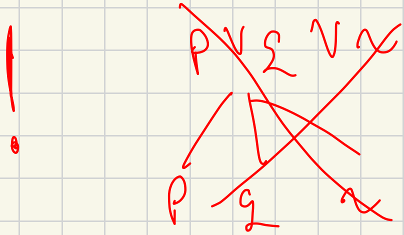
$$C_5 = \mathrm{Res}_A(C_1, C_3) = B$$

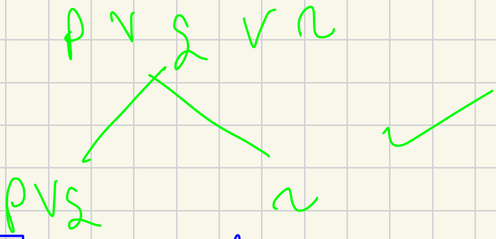
$$C_6 = \mathrm{Res}_A(C_2, C_3) = C$$

$$C_7 = \mathrm{Res}_B(C_4, C_5) = \neg C$$

$$C_8 = \mathrm{Res}_C(C_6, C_7) = \square$$

6.1) The open branches of sem. tabl. of $\neg U$ provide the models of $\neg U$, which are the anti-models of U.





Absorbtion law

$$U \vee (U \wedge V) = U$$

7.1) T.S.C. $\models U$ iff $\neg U$ has a closed sem. tabl.

α rules

$A \wedge B$ — A — B

$\neg(A \vee B)$ — $\neg A$ — $\neg B$

$\neg(A \to B)$ — A — $\neg B$

β rules

$A \vee B$ — A | B

$A \to B$ — $\neg A$ | B

$\neg(A \wedge B)$ — $\neg A$ | $\neg B$

δ (delta) rule

$(\exists x) U(x)$

$U(c)$

(new const.)

γ (gama) rule

$(\forall x) U(x)$

$U(c_1)$

⋮

$U(c_n)$

$c_1, \dots, c_n$ are all const. on that branch.

$(\forall x) U(x)$

Theorem

$\vdash U$ iff $CNF(\neg U) \vdash_{Res} \square$

a formula is a theorem