26.02.2024

Differential equations

1) Check that the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = 2e^{3t}$ is a solution of the Initial Value Problem (IVP)

$$\begin{cases} x' = 3x \to \text{is a } 1^{st} \text{ order scalar differential equation whose unknown is a function denoted by } x(t). \\ x(0) = 2 \to \text{initial value} \end{cases}$$

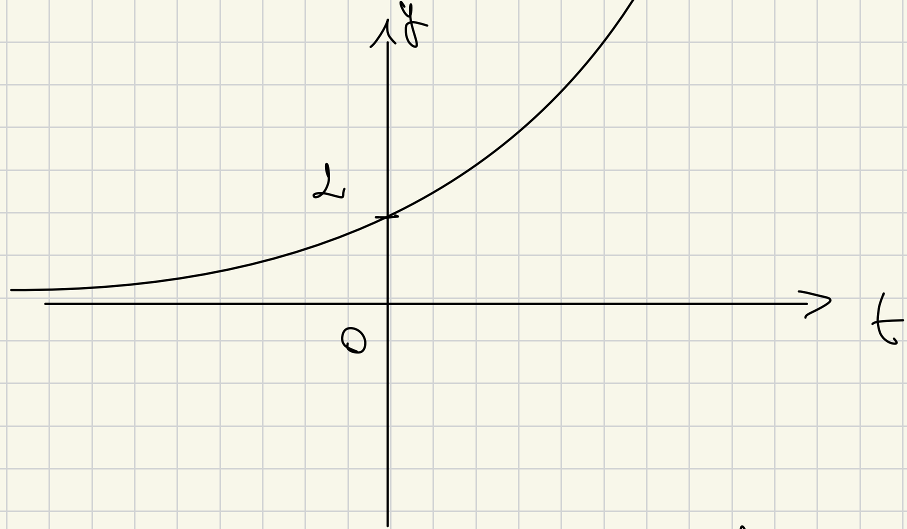
Represent the corresponding integral curve and solve its long-term behaviour.

$$x'\cancel{(t)} = 3 \cdot x\cancel{(t)}$$

Solution: We have to check the following relations:

$$\begin{cases} \varphi'(t) = 3\varphi(t), \forall t \in \mathbb{R} \\ \varphi(0) = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 6e^{3t} = 3 \cdot 2e^{3t}, \forall t \in \mathbb{R} \\ 2 \cdot e^0 = 2 \end{cases} \quad \text{TRUE}$$



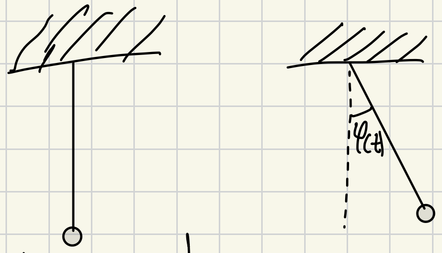
Long-term behaviour: - (exponentially) increasing in the future

- unbounded in the future ($\lim_{t\to\infty}\varphi(t)=\infty$)
- bounded in the past ($\lim_{t\to-\infty}\varphi(t)=0$)
- with positive values

2) Let $\varphi:\mathbb{R}\to\mathbb{R}$, $\varphi(t)=-2\sin t$, $\forall t\in\mathbb{R}$

Check that φ is a sol. $\begin{cases} x''+x=0 \\ x(0)=0 \\ x'(0)=-2 \end{cases}$

Represent the integral curve and describe its long-term behaviour. Describe the motion of the pendulum in this situation

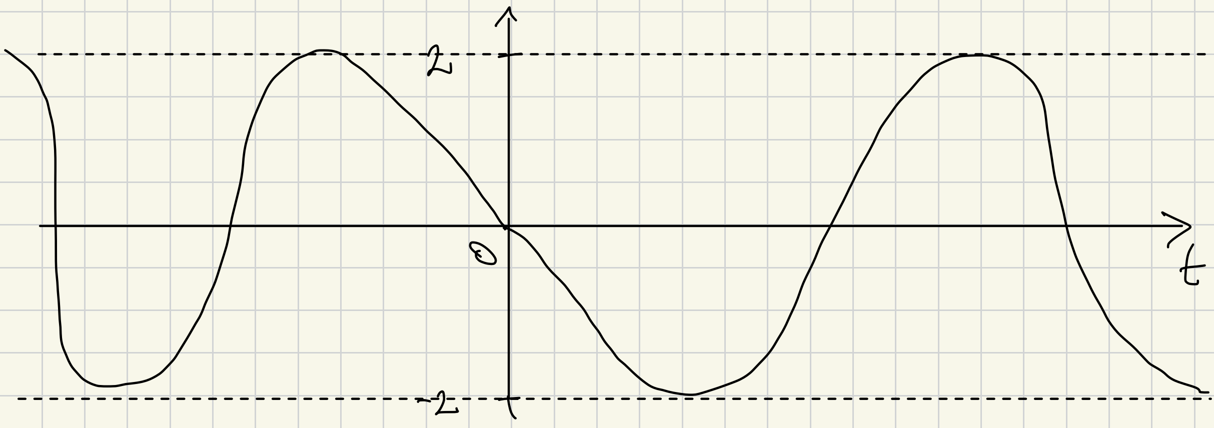


$\varphi(t)$ is the angle between the rod and the vertical at time t. (measured in radians in the trigonometric sense). We consider a simple pendulum (idealized)

2nd order ⇒ we have 2 initial values for x and x'

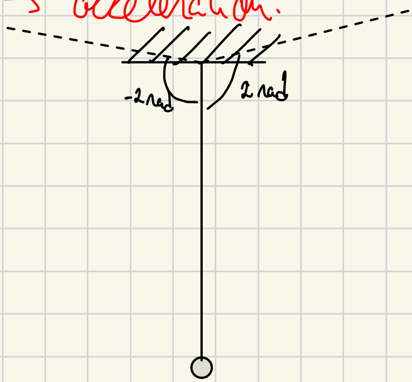
$$\begin{cases} \varphi''(t) + \varphi(t) = 0, \ \forall t \in \mathbb{R} \\ \varphi(0) = 0 \\ \varphi'(0) = -2 \end{cases} \iff$$

$$\iff \begin{cases} 2\sin(t) - 2\sin(t) = 0, \ \forall t \in \mathbb{R} \\ 2\sin(0) = 0 \\ -2\cos(0) = -2 \end{cases} \quad \text{TRUE}$$



$\varphi(t)$: – is 2π periodic
↳ minimum periodicity
– is bounded with values in the interval $[-2, 2]$.
– oscilates around 0

! obs: $\begin{cases} x' \to \text{velocity} \\ x'' \to \text{acceleration.} \end{cases}$



3) Same exercise but $\varphi(t) = e^{-2t} \cdot \cos(t)$

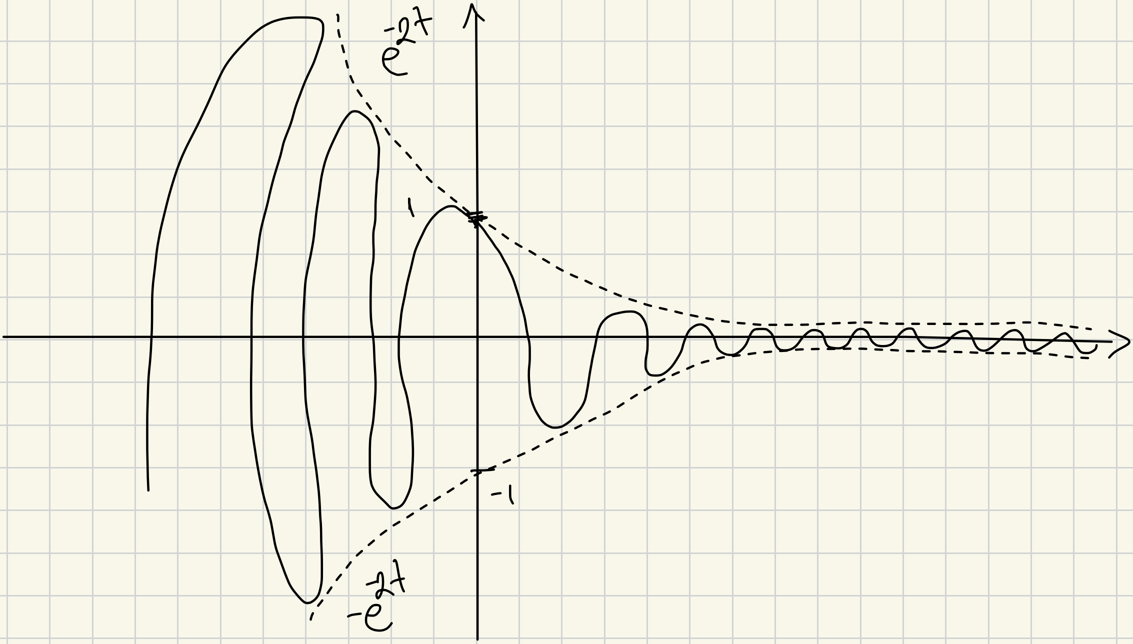
$$\begin{cases} x'' + 4x' + 5x = 0 \\ x(0) = 1 \\ x'(0) = -2 \end{cases}$$

$$\varphi'(t) = (-2)e^{-2t} \cdot \cos(t) + e^{-2t} \cdot (-\sin(t))$$

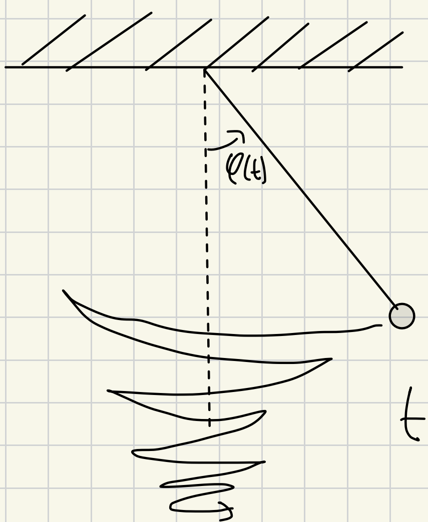
$$\varphi''(t) = 4e^{-2t}\cdot\cos(t) + 2e^{-2t}\cdot\sin(t) +$$
$$+ 2e^{-2t}\cdot\sin(t) + e^{-2t}\cdot(-\cos(t))$$
$$= 3e^{-2t}\cdot\cos t + 4e^{-2t}\cdot\sin(t)$$

$$\begin{cases} \varphi''(t) + 4\varphi'(t) + 5\varphi(t) = 0 \\ \varphi(0) = 0 \\ \varphi'(0) = -2 \end{cases} \iff$$

$$\iff \begin{cases} \cancel{3e^{-2t}\cdot\cos t} + \cancel{4e^{-2t}\cdot\sin(t)} + \cancel{(-8)e^{-2t}\cdot\cos(t)} + \\ \cancel{-4e^{-2t}(-\sin(t))} + \cancel{5\cdot e^{-2t}\cdot\cos(t)} = 0 \\ 1 = 1 \\ -2 = -2 \end{cases}$$



- oscilates around 0, with decreasing amplitude in the future.
- bounded in the future ($\lim_{t\to\infty} \varphi(t) = 0$)
- unbounded in the past ($\lim_{t\to-\infty} \varphi(t) \nexists$)



$t \to \infty$, the pendulum will stop.

4) find $r \in \mathbb{R}$ s.t. e^{rt} is a solution of the $\underbrace{\text{differential equation}}_{\text{d.e.}}$: a) $x'' - 5x' + 6x = 0$

b) $x'' - x = 0$

c) $x'' + x = 0$

$$(e^{rt})'_t = r \cdot e^{rt}$$

$$(e^{rt})''_t = r^2 \cdot e^{rt}$$

a) $r^2 \cdot e^{rt} - 5 \cdot r \cdot e^{rt} + 6 \cdot e^{rt} = 0 \quad \Big| \cdot \frac{1}{e^{rt}}, e^{rt} > 0$

$\Leftrightarrow r^2 - 5r + 6 = 0$

$\Leftrightarrow (r-2)(r-3) = 0$

$\begin{cases} r_1 = 2 \\ r_2 = 3 \end{cases} \qquad S = \{2, 3\}$

b) $r^2 \cdot e^{rt} - e^{rt} = 0 \quad \Big| \cdot \frac{1}{e^{rt}}, e^{rt} > 0$

$\Leftrightarrow r^2 - 1 = 0$

$\Leftrightarrow (r-1)(r+1) \qquad S = \{\pm 1\}$

$\Leftrightarrow \begin{cases} r_1 = 1 \\ r_2 = -1 \end{cases}$

c) $r^2 \cdot e^{rt} + e^{rt} = 0 \quad \Big| \cdot \frac{1}{e^{rt}}, \; e^{rt} > 0$

$\Leftrightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1$

$\nexists\, r \in \mathbb{R} \Rightarrow S = \emptyset$

$e^{it} = \cos t + i \sin t$

$\cos t$ and $\sin t$ are real value sol. of this ec.

5) Find all the solutions to the equation

$x' = x \Leftrightarrow x' - x = 0$

0 is a sol. (const function)

e^t is a sol.

$c \cdot e^t, \; \forall c \in \mathbb{R}$

$x' - x = 0 \quad \Big| \cdot e^{-t}$

$e^{-t} x' - e^{-t} \cdot x = 0 \Leftrightarrow (e^{-t} \cdot x)' \Leftrightarrow e^{-t} \cdot x = \underset{\text{const}}{c} \Leftrightarrow$

$\Leftrightarrow x = c \cdot e^{t}, \; c \in \mathbb{R}.$

↳ the general solution of $x' - x = 0$

6) Find the general sol. of $x' = ax$, where $a \in \mathbb{R}$ is a fixed param.

$$x' - ax = 0 \quad \Big| \cdot e^{-at} \iff$$

$$\iff x' \cdot e^{-at} - a \cdot x \cdot e^{-at} = 0$$

$$\iff \left(e^{-at} \cdot x\right)' = 0 \implies e^{-at} \cdot x = c \quad \Big| \cdot e^{at}$$

$$\iff x = c \cdot e^{at}$$ → general solution of the d.e.

$c \in \mathbb{R}$.

7) Let $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$, $\begin{cases} x_1(t) = \cos(t) \\ x_2(t) = \sin(t) \\ x_3(t) = e^t \end{cases}$

a) Prove that they are linearly independent in the vectorspace of continuous func ($C(\mathbb{R})$)

b) Find $a, b, c \in \mathbb{R}$ s.t.: $x(t) = a \cdot \cos(t) + b \sin(t) + c \cdot e^t$ is a sol of the equation $x' + x = -3\sin t + 2e^t$

b) $x' + x = -3\sin t + 2e^t \Leftrightarrow$

$$\Leftrightarrow \underbrace{-a\sin(t) + b\cdot\cos(t) + c\cdot e^t}_{x'} + a\cos(t) + b\sin(t) +$$

$$+ c\cdot e^t = -3\sin(t) + 2e^t$$

$$\Leftrightarrow \sin(t)(b-a) + \cos(t)(b+a) + 2c\cdot e^t = -3\sin(t) + 2e^t$$

$$\Leftrightarrow \begin{cases} b-a=-3 \\ b+a=0 \\ 2c=2 \Rightarrow c=1 \end{cases} \Rightarrow \begin{array}{l} b=-3+a \Leftrightarrow -3+2a=0 \Rightarrow a=\frac{3}{2} \Rightarrow \\ \Rightarrow b = -3+\frac{3}{2} = -\frac{3}{2} \end{array}$$

$x(t) = \frac{3}{2}\cos(t) - \frac{3}{2}\sin t + 2e^t$ sol. of d.e. given.

we can use that bcs. at a) we "proved" that $\sin(t)$, $\cos(t)$ and e^t are lin. indep $\Rightarrow$ we will have a unique combination for which we have a solution.