11.12.2023

Resolution in propositional logic

1.1) $U_1 = (A \to B \wedge C) \to (A \to B) \wedge (A \to C)$, $\overset{?}{\vdash} U_1$

$\neg U_1 = \neg [(A \overset{1}{\to} B \wedge C) \overset{2}{\to} (A \overset{3}{\to} B) \wedge (A \overset{4}{\to} C)]$,

we apply the norm. algorithm.

$\vdots$

$$\equiv \underbrace{(\neg A \vee B)}_{C_1} \wedge \underbrace{(\neg A \wedge C)}_{C_2} \wedge \underbrace{A}_{C_3} \wedge \underbrace{(\neg B \vee \neg C)}_{C_4}$$

$S = \{C_1, C_2, C_3, C_4\}$, $S \overset{?}{\vdash}_{Res} \square$

$C_5 = Res_A(C_1, C_3) = B$

$C_6 = Res_A(C_2, C_3) =$

$C_7 = Res_B(C_4, C_5) = \neg C$

$C_8 = Res_C(C_7, C_6) = \square$

$Res_l(f \vee l, g \vee \neg l) = f \vee g$

4.1) $S = \{ \overbrace{p \vee q \vee r}^{C_1}, \overbrace{\neg q \vee r}^{C_2}, \overbrace{\neg r}^{C_3}, \overbrace{\neg p \vee r}^{C_4} \}$

$S \vdash_{\text{Res}}^{\text{linear}} \square$?

top clause

$\boxed{C_1 = p \vee q \vee r}$ $\boxed{C_3 = \neg r}$ 1 C_2, C_4

conjunction

side clauses

central clauses

$\boxed{C_5 = p \vee q}$ $\boxed{C_2 = \neg q \vee r}$, C_4

$\boxed{C_6 = p \vee r}$ $\boxed{C_4 = \neg p \vee r}$, C_3

$\boxed{C_7 = r}$ $\boxed{C_3 = \neg r}$

$\boxed{C_8 = \square}$

$S \vdash_{\text{Res}}^{\text{lin}} \square$, so S is inconsistent

5.1) $S = \{ \overbrace{\neg q \vee \neg r}^{C_1}, \overbrace{p \vee r}^{C_2}, \overbrace{q \vee \neg p}^{C_3} \}$, prove consistency using linear resolution.

I top clause

$\boxed{C_1 = \neg q \vee \neg r}$ $\boxed{C_2 = p \vee r}$, C_3

$\boxed{C_4 = \neg \underline{q} \vee p}$ $\boxed{C_3 = \underline{q} \vee \neg p}$, C_3

$\boxed{C_5 = p \vee \neg p}$ $\equiv T \Rightarrow$ process is blocked $\Rightarrow$

$\Rightarrow$ backtracking

II top clause

$\boxed{C_1 = \neg q \vee \neg \underline{r}}$ $\boxed{C_2 = p \vee \underline{r}}$, C_3

$\boxed{C_4 = \neg q \vee \underline{p}}$ C_3, $\boxed{C_3 = q \vee \neg \underline{p}}$

$C_5 = \neg q \vee q$ $\equiv T \Rightarrow$ process is blocked $\Rightarrow$

$\Rightarrow$ backtracking

III top clause $\boxed{C_1 = \neg q \vee \neg r}$ $\quad C_2,$ $\boxed{C_3 = q \vee \neg p}$

$\boxed{C_4 = \neg r \vee \neg p}$ $\quad \boxed{C_2 = p \vee r},$ $\quad C_2$

$C_5 = \neg p \vee p \equiv \top$

IV top clause $\boxed{C_1 = \neg q \vee \neg r}$ $\quad C_2,$ $\boxed{C_3 = q \vee \neg p}$

$\boxed{C_4 = \neg r \vee \neg p}$ $\quad C_2,$ $\boxed{C_2 = p \vee r}$

$\boxed{C_5 = \neg r \vee r}$ $\equiv \top$

After a complete linear search using the backtracking strategy, without the derivation of the empty clause, we conclude that S is consistent.

6.1) Using lock resolution, prove inconsistency ($S \vdash^{lock}_{Res} \square$)

$$S^0 = S = \{ \overbrace{p \vee r}^{C_1}, \overbrace{\neg p \vee \neg q \vee r}^{C_2}, \overbrace{\neg p \vee q \vee r}^{C_3}, \overbrace{\neg r}^{C_4} \}$$

$C_1 = {}_{(1)}p \vee {}_{(2)} r$

$C_2 = {}_{(3)}\neg p \vee {}_{(4)}\neg q \vee {}_{(5)} r$

$C_3 = {}_{(6)}\neg p \vee {}_{(7)} q \vee {}_{(8)} r$

$C_4 = {}_{(9)}\neg r$

In this method we cannot resolve C_3 with C_4 bcs r does not have the lowest index in its literal.

a) $C_5 = Res_p^{lock}(C_1, C_2) = {}_{(2)} r \vee {}_{(4)} \neg q$

$C_6 = Res_p^{lock}(C_1, C_3) = {}_{(2)} r \vee {}_{(7)} q$

$C_7 = Res_r^{lock}(C_4, C_5) = {}_{(4)} \neg q$

$C_8 = Res_r^{lock}(C_4, C_6) = {}_{(7)} q$

$C_9 = Res_{q}^{lock}(C_7, C_8) = \square \Rightarrow$

$\Rightarrow$ S is inconsistent.

b) lock res + level saturation strategy (and diff indexing)

$C_1 = {}_{(2)}p \vee {}_{(1)}r$

$C_2 = {}_{(5)}\neg p \vee {}_{(4)}\neg q \vee {}_{(3)} r$

$C_3 = {}_{(8)}\neg p \vee {}_{(7)} q \vee {}_{(6)} r$

$C_4 = {}_{(5)}\neg r$

$S^0 = \{C_1, C_2, C_3, C_4\}$

$S^1 = \{Res^{lock}(C_i, C_j) \mid C_i, C_j \in S^0\}$

$C_5 = Res_{r}^{lock}(C_1, C_4) = {}_{(2)}p$

$C_6 = Res^{lock}(C_2, C_4) = {}_{(5)}\neg p \vee {}_{(4)}\neg q$

$C_7 = Res_{r}^{lock}(C_3, C_4) = {}_{(8)}\neg p \vee {}_{(7)} q$

$S^1 = \{C_5, C_6, C_7\}$

$S^2 = \{Res^{lock}(C_i, C_j) \mid C_i \in S^1, C_j \in S^0 \cup S^1\}$

$C_8 = Res_2^{lock}(C_6, C_7) \underset{(5)}{=} \neg p$

$S^2 = \{C_8\}$

we stop if one clause is empty on a certain level or the last level is empty!

Theory: if $\Box \in S^k$, then S is inconsistent

if $S^k = \emptyset$, then S is consistent

$S^3 = \{Res^{lock}(C_i, C_j) \mid C_i \in S^2 \mid C_j \in S^0 \cup S^1 \cup S^2\}$

$C_9 = Res_p^{lock}(C_8, C_5) = \Box \Rightarrow S$ is inconsistent

OBS: a) and b) are two different strategies paired with two diff indexings. In strat b) we care about indexing since we cannot resolve two literals if the indexes of the resolvent is not the smallest one in the literal in where it's from.

! Indexing is only relevant for the RELATIVE order!

7.1) Check the consistency using lock resolution.

a) $S = \{ \underbrace{p \vee q \vee r}_{C_1}, \underbrace{\neg q \vee r}_{C_2}, \underbrace{\neg r \vee \neg p}_{C_3} \}$

a) $C_1 = {}_{(1)}p \vee {}_{(2)}q \vee {}_{(3)}r$

$C_2 = {}_{(6)}\neg q \vee {}_{(7)}r$ $\qquad S^0 = S$

$C_3 = {}_{(4)}\neg r \vee {}_{(5)}\neg p$

$S^1 = \{ Res^{lock}(C_i, C_j) \mid C_i, C_j \in S^0 \}$

$S^1 = \emptyset \Rightarrow S$ is consistent

Obs: if we choose the appropriate indexing we can get to consistency in a faster way!

b) $C_1 = {}_{(2)}p \vee {}_{(1)}q \vee {}_{(3)}r$

$C_2 = {}_{(4)}\neg q \vee {}_{(5)}r$ $\qquad S^0 = S$

$C_3 = {}_{(7)}\neg r \vee {}_{(6)}\neg p$

$S^1 = \{ Res^{lock}(C_i, C_j) \mid C_i, C_j \in S^0 \}$

$C_4 = Res^{lock}_2(C_1, C_2) = {}_{(2)}p \vee {}_{(3)}r$

$S^2 = \{ Res^{lock}(C_i, C_j) \mid C_i \in S^0, C_j \in S^0 \cup S^1 \}$

$C_5 = Res_p^{lock}(C_4, C_3) = {}_{(2)}r \vee_{(7)} \neg r \equiv T$ in lock resolution we continue with tautologies.

$C_9 = {}_{(1)}p \vee_{(2)} s$

index(p) < index(s) => index(s) < index(p).

$C_{10} = {}_{(3)}\neg p \vee_{(4)} p$

$C_{11} = Res_p^{lock}(C_9, C_{10}) = {}_{(2)}s \vee_{(4)} p$

Here is an example that shows us that we can change the relative order using a tautology!

$S^3 = \{ Res^{lock}(C_i, C_j) \mid C_i \in S^2, C_j \in S^0 \cup S^1 \cup S^2 \}$

$S^3 = \emptyset \Rightarrow S$ is consistent.