17.12.2024

$$X\begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I \subseteq \mathbb{N}}$$

expected value

$$E(X) = \sum_{i \in I} x_i \cdot p_i$$

e.g. X = no. of a die

$$E(X) = \frac{1 + \dots + 6}{6} = 3.5$$

On avg., if we call ind. X „many times":

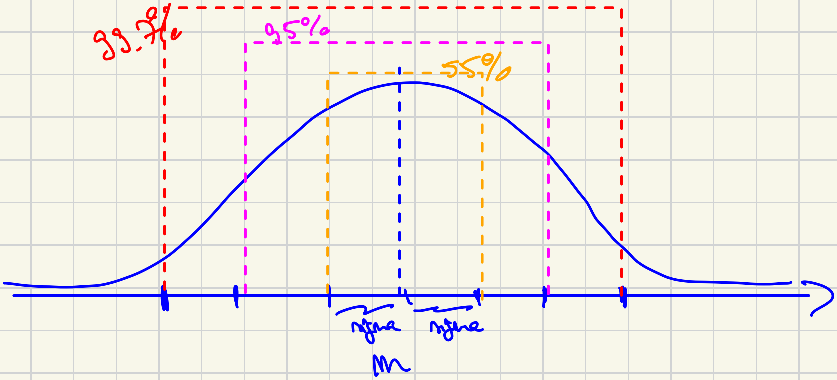
$$\frac{N_1 \cdot 1 + \dots + N_6 \cdot 6}{N} = \frac{N_1}{N} \cdot 1 + \dots + \frac{N_6}{N} \cdot 6 \approx 3.5$$

$$f_X : \mathbb{R} \to \mathbb{R}$$

$$F_X(x) = \int_{-\infty}^{x} f_X = P(X \leq x) \quad x \in \mathbb{R}$$

$$\boxed{\begin{aligned} V(X) &= E\left((X - E(X))^2\right) \\ &= E(X^2) - E^2(X) \end{aligned}}$$

$$\sigma(X) = \sqrt{V(X)}$$



1. Every day, the number of network blackouts has the following pdf

$$X\begin{pmatrix} 0 & 1 & 2 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}.$$

A small internet trading company estimates that each network blackout costs them \$500.
a) How much money can the company expect to lose each day because of network blackouts?
b) What is the standard deviation of the company's daily loss due to blackouts?

a) $E(X) = 0 \cdot 0.7 \cdot 500 + 1 \cdot 0.2 \cdot 500 + 2 \cdot 0.1 \cdot 500 =$

$= 100 + 100 = 200$

b) $X = 500$

$V(X) = E(X^2) - E^2(X) = E(250000) - 400 =$

$= 100.000 - 400 = 99.600$

$\sigma(X) = \sqrt{V(X)} = \sqrt{99.600} = 315,59$

a) Y = lost money

$Y = 500 \cdot X$

$E(Y) = E(X \cdot 500) = 500 \cdot E(X) = 200\,\$$

$E(X) = 0.4$

b) $X^2 \begin{pmatrix} 0 & 1 & 4 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}$

$E(X^2) = 0.6$

$V(X) = 0.6 - (0.4)^2 = 0.44$

$\sigma(Y) = \sqrt{V(Y)} = \sqrt{500^2 \cdot V(X)} = 500 \cdot \sqrt{0.44} \simeq 330\,\$$

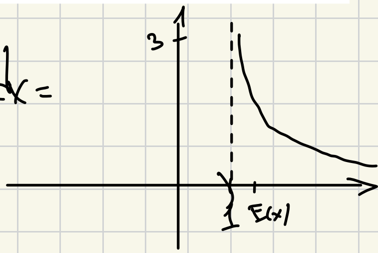
3. (Refer to Problem 1 in Sem. 5) The lifetime, in years, of some electronic component is a random variable with density

$$f(x) = \begin{cases} \dfrac{3}{x^4}, & \text{for} \quad x \geq 1 \\ 0, & \text{for} \quad x < 1. \end{cases}$$

How many years, on the average, can we expect that electronic equipment to last?

$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_1^{\infty} \frac{3x}{x^4}\, dx =$

$= 3 \int_1^{\infty} \frac{1}{x^3}\, dx = 3 \left. \frac{x^{-2}}{-2} \right|_1^{\infty} =$



$$= 3 \cdot \frac{1}{-2x^2}\Big|_1^{\infty} = 3\left(0 + \frac{1}{2}\right) = 1.5$$

$$V(X) = E(X^2) - E^2(X) =$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx =$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$$

$$= \int_1^{\infty} \frac{3x^2}{x^4}\,dx = 3\int_1^{\infty} \frac{1}{x^2}\,dx = 3 \cdot \frac{x^{-1}}{-1}\Big|_1^{\infty} =$$

$$= 3 \cdot \left(-\frac{1}{x}\right)\Big|_1^{\infty} = 3 \cdot (0+1) = 3$$

$$E^2(X) = (1.5)^2 = 2.25$$

$$\sigma(X) = \sqrt{0,75}$$

4. (Optimal portfolio) A businessman wants to invest \$600 and has two companies to choose from, company A, where shares cost \$20 each and company B, where shares cost \$30 per share. The market analysis shows that for company A the return per share is distributed as follows: lose \$1 with probability 0.2, win \$2 with probability 0.6, or win/lose nothing. For company B: lose \$1 with probability 0.3, win \$3 with probability 0.6, or win/lose nothing. The returns from the two companies are independent. In order to maximize the expected return and minimize the risk, which way is better to invest:

a) all money in company A; $\rightarrow 30A$

b) all money in company B; $\rightarrow 20B$

c) half the amount in each? $\rightarrow 15A + 10B$

600 \$

$A \begin{pmatrix} -1 & 0 & 2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$ $\quad B \begin{pmatrix} -1 & 0 & 3 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}$

20\$ / share $\quad$ 30\$ / share

X = total profit

$E(X)$

a) $X = 30A$

$E(30 \cdot A) = 30E(A) = 30 \cdot 1 = 30$

b) $X = 20B$

$E(20 \cdot B) = 20 \cdot E(B) = 20 \cdot 1.5 = 30$

c) $X = 15A + 10B$

$E(15A + 10B) = \ldots = 15 \cdot E(A) + 10 \cdot E(B) = 15 + 15 = 30$

a) $V(X) = E(X^2) - E^2(X) =$

$= 900\, E(A^2) - 900 = 900 \cdot 2.6 - 900 =$

$A^2 = \begin{pmatrix} 1 & 0 & 4 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$ $\qquad = 900 \cdot 1.6 = 1440$

b) $V(X) = 400 \cdot E(B^2) - 900 = 400 \cdot 5.7 - 900$

$B^2 = \begin{pmatrix} 1 & 0 & 9 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}$ $\qquad = 1380$

c) $V(X) = 15^2 \cdot V(A) + 10^2 \cdot V(B) =$

$= 15^2 \cdot (E(A^2) - E^2(A)) + 10^2 (E(B^2) - E^2(B))$

$= 15^2 \cdot (2.6 - 1) + 10^2 \cdot (5.7 - 2.25) =$

$= 15^2 \cdot 1{,}6 + 10^2 \cdot 3{,}45 =$

$= \underline{\underline{705}}$

6. The joint density function of the vector (X, Y) is $f(x, y) = x + y,\ (x, y) \in [0, 1] \times [0, 1]$. Find
a) the means and variances of X and Y;
b) the correlation coefficient $\rho(X, Y)$.

$$f(x,y) = \begin{cases} x+y, & (x,y) \in [0,1] \times [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$$f : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$$

a) $E(X) = E(Y) = \ldots$
$V(X) = V(Y) = \ldots$

b) $\rho(X,Y) = \dfrac{\operatorname{cov}(X,Y)}{\sqrt{V(X)} \cdot \sqrt{V(Y)}}$
$\in [-1, 1]$

↳ close to $0 \Rightarrow$ close to being independent
↳ close to $-1, 1 \Rightarrow X$ can be written as a linear comb of Y.

a) $\boxed{E(h(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\, dx\, dy.}$

$$\underset{h(x,y)=x}{E(X)} = \int_0^1 \left(\int_0^1 x^2 \cdot (x+y)\, dx \right) dy.$$

$$= \int_0^1 \int_0^1 x^3 + x y^2 \, dx \, dy$$

$$= \int_0^1 \left(\frac{x^4}{4} + y \cdot \frac{x^3}{3} \right) \Big|_0^1 \, dy$$

$$= \int_0^1 - \frac{1}{4} - \frac{y}{3} \, dy \; = \ldots = \frac{7}{12}$$

$$\operatorname{cov}(X, Y) = E\big((X - \underbrace{E(X)}_{\frac{7}{12}}) \cdot (Y - \underbrace{E(Y)}_{\frac{7}{12}})\big) =$$

$$\underline{\underline{h(x,y) = (x - \tfrac{7}{2})(y - \tfrac{7}{2})}} \quad \int_0^1 \int_0^1 (x - \tfrac{7}{2})(y - \tfrac{7}{2}) \cdot (x+y) \, dx \, dy =$$

$=$