27.05.2024

First and second order diff. eq. with const. coef.

→ Theory lecture 12.

2) a) Find solutions of the form $x_k = a \cdot 3^k$ of the d.e.

$\underbrace{x_{k+1} = 2 \cdot x_k + 3^k}$, $k \overset{\in \mathbb{N}}{\geq} 0$. Here we look for $a \in \mathbb{R}$.

first order linear non-homogeneous difference equation

$$x_k = a \cdot 3^k \Rightarrow x_{k+1} = 2 \cdot a \cdot 3^k + 3^k = 3^k(2a+1)$$

$$\Leftrightarrow \quad a \cdot 3 \cdot 3^k = 2 \cdot a \cdot 3^k + 3^k, \; k \geq 0 \;\Big|\; : 3^k > 0 \Rightarrow$$

$$\Rightarrow \quad 3a = 2a + 1 \Rightarrow a = 1 \Rightarrow x_k = 3^k \text{ is a sol.}$$

b) Find the general sol. of the equation $x_{k+1} = 2x_k + 3^k$

$$x_k = x_k^h + x_k^p$$

a) $\Rightarrow x_k^p = 3^k$

- find x_k^h:

$x_{k+1} = 2x_k$ LHDE

$x_{k+1} - 2x_k = 0$

The characteristic method $\Rightarrow r - 2 = 0 \Rightarrow r = 2$

We associate the sequence 2^k

$x_k^h = c \cdot 2^k, \ c \in \mathbb{R}.$

$x_k = x_k^h + x_k^p = c \cdot 2^k + 3^k, \ c \in \mathbb{R}$

c) Find the solution of the IVP

$x_{k+1} = 2x_k + 3^k, \ x_0 = 0$

$b \Rightarrow x_k = c \cdot 2^k + 3^k$

$x_0 = c + 1 = 0 \Rightarrow c = -1$

the unique solution of the IVP $= -2^k + 3^k.$

3) Find the solutions of the form $x_k = ak + b$ of the difference equation $x_{k+1} = -5x_k - k$; $a, b \in \mathbb{R}.$

$x_k = ak + b \Rightarrow x_{k+1} = ak + a + b$

$ak + a + b = -5(ak+b) - k \Leftrightarrow$

$\Leftrightarrow ak + a + b = -5ak - 5b - k \quad , k \geq 0$

$\Leftrightarrow ak + a + b = k(-5a-1) - 5b \quad , k \geq 0$

$\begin{cases} a = -5a-1 \Rightarrow 6a = -1 \Rightarrow a = -\frac{1}{6} \\ a+b = -5b \end{cases} \Rightarrow$

$\Rightarrow -\frac{1}{6} + 6b = 0 \Rightarrow b = \frac{1}{36}$

$x_k = -\frac{k}{6} + \frac{1}{36} \quad , k \geq 0$ is a solution

b) Find the general solution of $\underbrace{x_{k+1} = -5x_k - k}_{1^{st}\text{ order LinHDE}}$

difference

(not differential)

$x_k = x_k^h + x_k^p$

a) $\Rightarrow x_k^p = -\frac{k}{6} + \frac{1}{36}, \quad k \geq 0$

. find x_k^h

$x_{k+1} = -5x_k$ (it has a constant coef. so we use the charac. eq. method)

$x_{k+1} + 5x_k = 0 \iff \lambda + 5 = 0 \Rightarrow \lambda = -5 \Rightarrow$ sequence $(-5)^k$

$x_k^h = c \cdot (-5)^k, \quad c \in \mathbb{R}, k \geq 0.$

$x_k = x_k^h + x_k^p = c \cdot (-5)^k - \frac{k}{6} + \frac{1}{36}, \quad c \in \mathbb{R}, k \geq 0.$

c) Find the solution of the IVP

$x_{k+1} = -5x_k - k, \quad x_0 = -1$

$\underline{\underline{b)}} \Rightarrow$ use $\left. \begin{array}{l} x_k = c \cdot (-5)^k - \frac{k}{6} + \frac{1}{36} \\ x_0 = -1 \end{array} \right\} \Rightarrow$

$\Rightarrow \quad c + \frac{1}{36} = -1$

$\Rightarrow \quad c = -\frac{37}{36} \Rightarrow$ unique sol. of IVP is:

$x_{k+1} = -\frac{37}{36} \cdot (-5)^k - \frac{k}{6} + \frac{1}{36}, \quad k \geq 0.$

4) Find the general solution of:

a) $X_{k+2} - 6X_{k+1} + 9X_k = 0$

$\underbrace{\qquad\qquad\qquad\qquad}_{2^{nd}\text{ order LHDE with constant coef.}}$ $\Rightarrow$ apply the charac. eq. method.

difference

$$r^2 - 6r + 9 = 0$$

$$\Delta = 36 - 4 \cdot 1 \cdot 9 = 0 \; ; \; \sqrt{\Delta} = 0$$

$$r_{1,2} = \frac{6 \pm 0}{2} = 3 \underset{m=2}{\mapsto} 3^k, \; k \cdot 3^k$$

• general sol: $X_k = C_1 3^k + C_2 \cdot k \cdot 3^k, \quad C_1, C_2 \in \mathbb{R}, \; k \geq 0$

b) $X_{k+2} - 2X_{k+1} + X_k = 0$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \underset{m=2}{\longmapsto} \underbrace{1^k, \; k \cdot 1^k}_{\Leftrightarrow 1, \, k}.$$

general sol: $X_k = C_1 \cdot 1 + C_2 \cdot k = C_1 + C_2 \cdot k \; ; \; C_1, C_2 \in \mathbb{R}$

$k \geq 0$

check that $X = C_1 + C_2 k$ is a general solution of

$X_{k+2} - 2X_{k+1} + X_k = 0$?

$$\left.\begin{array}{l} x_{k+2} = c_1 + c_2(k+2) \\ x_{k+1} = c_1 + c_2(k+1) \end{array}\right\} \Leftarrow$$

$$\Leftrightarrow \cancel{c_1} + \cancel{c_2 k} + \cancel{2c_2} - \cancel{2c_1} - \cancel{2c_2 k} - \cancel{2c_2} + \cancel{c_1} + \cancel{c_2 k} = 0$$

True

c) $x_{k+2} + x_{k+1} + x_k = 0$

$$r^2 + r + 1 = 0$$

$$\Delta = 1 - 4 = -3 < 0$$

$$r_{1,2} = \frac{-1 \mp i\sqrt{3}}{2} \in \mathbb{C} \setminus \mathbb{R} \longmapsto$$

$$\mapsto r_1^k = \cos\left(\frac{2\pi k}{3}\right) + i \cdot \sin\left(\frac{2\pi k}{3}\right)$$

$\Rightarrow$ the general sol $x_k = c_1 \cdot \cos\left(\frac{2\pi k}{3}\right) + i \cdot \sin\left(\frac{2\pi k}{3}\right)$,

$c_1, c_2 \in \mathbb{R}$

5) Find the expression of the Fibbonacci eq.

$$x_{k+2} = x_{k+1} + x \quad , x_0 = 0, \; x_1 = 1$$

We have second order LHDE with const coef.

$\Rightarrow$ We can apply the characteristic eq. method

$$r^2 - r - 1 = 0$$

$$r_{1,2} = \frac{1 \pm \sqrt{5}}{2} \in \mathbb{R}$$

$$r_1, r_2 \in \mathbb{R} \quad , r_1 \neq r_2 \longmapsto r_1^k \, , \, r_2^k$$

$$x_k = c_1 \cdot \left(\frac{1-\sqrt{5}}{2} \right)^k + c_2 \cdot \left(\frac{1+\sqrt{5}}{2} \right)^k \quad , c_1, c_2 \in \mathbb{R}, k \geq 0$$

$$\left. \begin{array}{l} x_0 = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \\ x_1 = c_1 \cdot \frac{1-\sqrt{5}}{2} + c_2 \frac{1+\sqrt{5}}{2} \end{array} \right\} \Longrightarrow$$

$$\Rightarrow c_1 \left(\frac{\cancel{1} - \sqrt{5} - \cancel{1} - \sqrt{5}}{2} \right) = 1 \Rightarrow$$

$$\Rightarrow c_1 \cdot \frac{-\cancel{2}\sqrt{5}}{\cancel{2}} = 1$$

$$\Rightarrow c_1 \cdot (-\sqrt{5}) = 1 \Rightarrow c_1 = \frac{-\sqrt{5}}{5} \Rightarrow c_2 = \frac{\sqrt{5}}{5}$$

unique sol: $X_k = \frac{-\sqrt{5}}{5} \cdot \left(\frac{1-\sqrt{5}}{2} \right)^k + \frac{\sqrt{5}}{5} \left(\frac{1+\sqrt{5}}{2} \right)$, $k \geq 0$.

6) Find the linear homogeneous diff. eq. of minimal order that has the solution:

a) $X_k = \frac{7}{2^k} - \frac{2}{3^k} = 7 \cdot \left(\frac{1}{2}\right)^k - 2 \cdot \left(\frac{1}{3}\right)^k$ this is a linear combination r_1^k and r_2^k where $r_1 = \frac{1}{2}$ and $r_2 = \frac{1}{3}$. which are also the roots of the charac. eq.

$$\left(r - \frac{1}{2}\right)\left(r - \frac{1}{3}\right) = 0$$

$$r^2 - \frac{5}{6} r + \frac{1}{6} = 0 \Rightarrow X_{k+2} - \frac{5}{6} X_{k+1} + \frac{1}{6} X_k = 0.$$

b) $X_k = 7 \cdot Re(i^k) - 2\, Im(i^k)$ this is a linear combination of $Re(i^k)$ and $Im(i^k)$

$r_1 = \overline{r_2} \in \mathbb{R}\setminus\mathbb{C}$, $r_1 = i$, $r_2 = -i$ are roots of the charac. eq.

$$(r-i)(r+i) = 0$$

$$r^2 + \cancel{ir} - \cancel{ir} + 1 = 0 \Leftrightarrow r^2 + 1 = 0 \Rightarrow X_{k+2} + X_k = 0$$

$$\boxed{c_1 \cdot \mathrm{Re}(r_1^k) + c_2 \cdot \mathrm{Im}(r_2^k)}$$

*7) Find the solution of the ivp:

$$\begin{cases} x_{k+2} + 2x_{k+1} + 2x_k = 0 \\ x_0 = 1 \\ x_1 = 0 \end{cases}$$

First we find the general solution:

$x_{k+2} + 2x_{k+1} + 2x_k = 0$

We have 2^{nd} order LHDE with cost. coef $\Rightarrow$

$\Rightarrow$ we can apply the charac. method

$r^2 + 2r + 2 = 0$

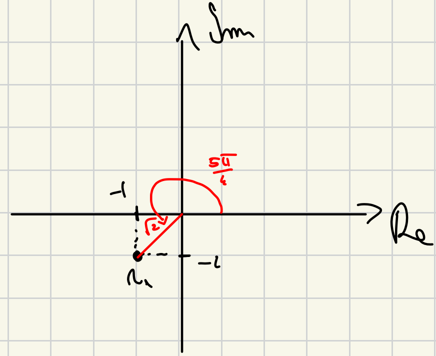
$\Delta = 4 - 4 \cdot 1 \cdot 2 = -4 < 0$

$$r_{1,2} = \frac{-2 \pm i\sqrt{4}}{2} = -1 \pm i$$

$r_1 = \overline{r_2} \in \mathbb{C} \setminus \mathbb{R} \longmapsto \mathrm{Re}(r_1^k), \mathrm{Im}(r_2^k)$

$r_1 = -1 - i$

$|r_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$



$$r_1 = \overset{|r_1|}{\sqrt{2}} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right) =$$

$$= \sqrt{2} \left(\cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right) \cdot i \right)$$

$$r_1^k = (\sqrt{2})^k \cdot \left(\cos\left(\frac{5\pi k}{4}\right) + i \cdot \sin\left(\frac{5\pi k}{4}\right) \right) =$$

$$= \underbrace{(\sqrt{2})^k \cdot \cos\left(\frac{5\pi k}{4}\right)}_{Re(r_1^k)} + \underbrace{(\sqrt{2})^k \cdot \sin\left(\frac{5\pi k}{4}\right)}_{Im(r_1^k)} \cdot i$$

$$x_k = c_1 \cdot (\sqrt{2})^k \cdot \cos\left(\frac{5\pi k}{4}\right) + c_2 \cdot (\sqrt{2})^k \cdot \sin\left(\frac{5\pi k}{4}\right),$$

$c_1, c_2 \in \mathbb{R}, \quad k \geq 0$

$x_0 = 1 \Leftrightarrow c_1 \cdot \cos 0 + c_2 \cdot \sin 0 = c_1 \Rightarrow c_1 = 1$

$$x_1 = 0 \Leftrightarrow c_1 \cdot \sqrt{2} \cdot \cos\frac{5\pi}{4} + c_2 \cdot \sqrt{2} \cdot \sin\frac{5\pi}{4} =$$

$$= c_1 \cdot \sqrt{2} \cdot \left(-\frac{1}{\sqrt{2}}\right) - c_2 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} = -c_1 - c_2 = -1 - c_2$$

$-1 - c_2 = 0 \Rightarrow c_2 = -1$

$$x_k = (\sqrt{2})^k \cdot \cos\left(\frac{5\pi k}{4}\right) - (\sqrt{2})^k \cdot \sin\left(\frac{5\pi k}{4}\right), \quad k \geq 0.$$