13.11.2023

8.1) Write all anti-models of U_1 using the appropriate normal form.

$$U_1 = (q \wedge r \xrightarrow{1} p) \xrightarrow{2} (p \xrightarrow{3} r) \wedge q$$

$$U \to V \equiv \neg U \vee V$$

- replace 1, 3

$$U_1 \equiv (\neg(q \wedge r) \vee p) \xrightarrow{2} (\neg p \vee r) \wedge q$$

- replace 2

$$U_1 \equiv \neg(\neg(q \wedge r) \vee p) \vee (\neg p \vee r) \wedge q$$

- De Morgan

$$U_1 \equiv (q \wedge r \wedge \neg p) \vee (\neg p \vee r) \wedge q$$

- distributivity

$$U_1 \equiv (q \wedge r \wedge \neg p) \vee (\neg p \wedge q) \vee (r \wedge q)$$

$\to$ DNF with 3 cubes

$$U_1 \equiv (q \vee \neg p \vee r) \wedge (q \vee \neg p \vee q) \wedge (q \vee q \vee r) \wedge (q \vee q \vee q)$$
$$\wedge (r \vee \neg p \vee r) \wedge (r \vee \neg p \vee q) \wedge (r \vee q \vee r) \wedge (r \vee q \vee q)$$
$$\wedge (\neg p \vee \neg p \vee r) \wedge (\neg p \vee \neg p \vee q) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee r)$$

$$U_1 \equiv (q \vee \neg p \vee r) \wedge (q \vee \neg p) \wedge (q \vee r) \wedge q \wedge$$
$$\wedge (r \vee \neg p) \wedge (r \vee \neg p \vee q) \wedge (r \vee q) \wedge (r \vee q) \wedge$$
$$\wedge (\neg p \vee r) \wedge (\neg p \vee q) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee r)$$

$$U_1 \equiv \underbrace{(\neg p \vee q \vee r)}_{\text{clause I}} \wedge \underbrace{(\neg p \vee q)}_{\text{clause II}} \wedge \underbrace{(q \vee r)}_{\text{clause III}} \wedge \underbrace{q}_{\text{clause IV}} \wedge \underbrace{(\neg p \vee r)}_{\text{clause V}}$$

$$U_1 \equiv (\neg p \wedge q) \wedge q \wedge (r \vee \neg p) \wedge (q \vee r)$$

$$U_1 \equiv q \wedge (r \vee \neg p) \quad : \text{CNF with 2 clauses.}$$

9.1) Using the def. of deduction, proove the following deductions:

$$p \to q,\ r \to t,\ p \vee r,\ \neg q \vdash t$$

$f_1 : p \to q$

$f_2 : r \to t$

$f_3 : p \vee r \equiv \neg p \to r$

$f_4 : \neg q$

$(f_1, \dots, f_m)$ is the deduction of t from hypotheses (and axioms)

Modus Tollens: $U \to V \vdash \neg V \to \neg U$ infer that

Modus Ponens: $U, U \to V \vdash V$

$$f_5 : (\overbrace{p \to q}^{f_1}) \xrightarrow{MT} (\neg q \to \neg p)$$

$f_1, f_5 \vdash_{mp} \neg q \to \neg p$

$f_6 : \neg q \to \neg p$

$f_4, f_6 \vdash_{mp} \neg p$

$f_7 : \neg p$

$f_3, f_7 \vdash_{mp} r$

$f_8 : r$

$f_2, f_8 \vdash_{mp} t$

$f_9 : t$

$(f_1, \dots, f_9)$ is the deduction of t from hypotheses (and axioms)

10.1) Prove the following theorems using the theorem of deduction and its reverse.

STEP 1: Apply reverse of theorem of deduction

$\vdash p \vee (q \to r) \to ((p \vee q) \to (p \vee r))$

then $p \vee (q \to r) \vdash (p \vee q) \to (p \vee r)$

then $p \vee (q \to r), (p \vee q) \vdash (p \vee r)$

then $p \vee (q \to r), (p \vee q) \vdash \neg p \to r$

then $p \vee (q \to r), (p \vee q), \neg p \vdash r$

STEP 2: hence deduction

$f_1 : p \vee (q \to r) \equiv \neg p \to (q \to r)$

$f_2 : p \vee q \equiv \neg p \to q$

$f_3 : \neg p$

MP: $U, U \to V \vdash V$

$f_3, f_1 \vdash_{mp} q \to r = f_4$

$f_3, f_2 \vdash_{mp} q$

$f_5, f_4 \vdash_{mp} r$

STEP 3: Apply theorem of deduction to get to the initial result

If $p \vee (q \to r), p \vee q, \neg p \vdash r$

then $p \vee (q \to r), p \vee q \vdash \neg p \to r \equiv p \vee r$

then $p \vee (q \to r) \vdash (p \vee q) \to (p \vee r)$

12.1) H_1: It is not sunny and it is cold

H_2: We will go swimming only if it is sunny

H_3: If we do not go swimming we will take a canoe trip

H_4: If we take canoe trip, home by sunset.

C: We will be home by sunset.

p = It is sunny

q = We go swimming

r = We will take a canoe trip

s = We will be home by sunset.

t = It is cold

MT: $U \to V \vdash_{mt} \neg V \to \neg U$

MP: $U, U \to V \vdash_{mp} V$

SIMPLIFICATION

$U \wedge V \vdash U$

$U \wedge V \vdash V$

$H_1 = \neg p \wedge t$

$H_2 = q \to p$

$H_3 = \neg q \to r$

$H_4 = r \to s$

$C = s$

$H_1 \vdash_{simplif} \neg p = H_5$

$H_2 \vdash_{mt} \neg p \to \neg q = H_6$

$H_5, H_6 \vdash_{mp} \neg q = H_7$

$H_7, H_3 \vdash_{mp} r = H_8$

$H_8, H_4 \vdash s = C$

Predicate Logic

1.1) Transform the following sentences from natural language into predicate formulas.

- variables
- constants
- functions : ex $square(x) = x^2$
- predicates :

Nat. language : In a plane if a line x is $\perp$ to a constant line d then all the lines $\parallel$ to x are $\perp$ to d.

$$(\forall x)(\forall y)\left(P(x,d) \land Q(y,x) \to P(y,d)\right)$$

Predicates

$$x \perp d \land y \parallel x \to y \perp d$$

$D = \mathcal{L}$

$P : D \times D \to \{T, F\}$, $P(x,y)$: "$x \perp y$"

$Q : D \times D \to \{T, F\}$, $Q(x,y)$: "$x \parallel y$"

Variables : x, y

Constants : d

$x \perp x$ False

From math: REFLEXIVITY $\to x \parallel x$ True

SYMETRY $\to x \perp y \to y \perp x$; $x \parallel y \to y \parallel x$

TRANSITIVITY $\to x \perp y , y \perp z \Rightarrow x \perp z$ False

$\to x \parallel z , y \parallel z \Rightarrow x \parallel z$

R: $(\forall x)\ Q(x,x)$

$$S: \begin{cases} (\forall x)(\forall y)(P(x,y) \to P(y,x)) \\ (\forall x)(\forall y)(Q(x,y) \to Q(y,x)) \end{cases}$$

T: $(\forall x)(\forall y)(\forall z)(Q(x,y) \wedge Q(y,z) \to Q(x,z))$