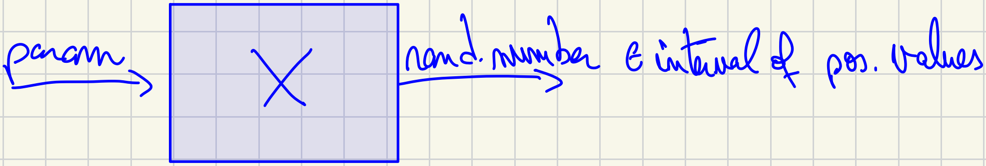
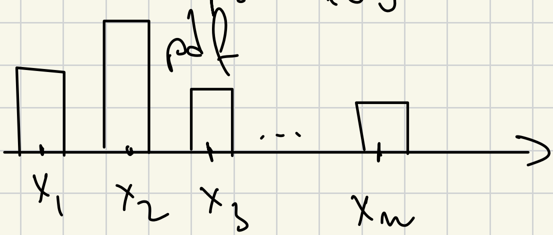
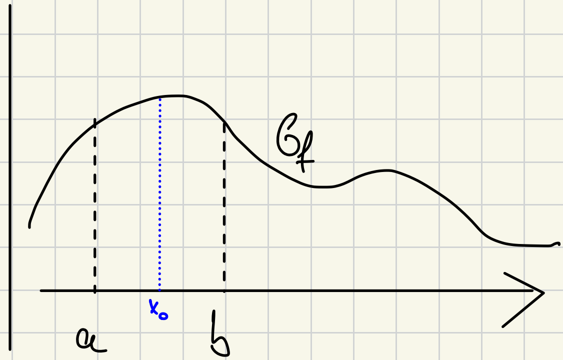
3.12.2024

Cont. random variable



$$X \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in J}$$

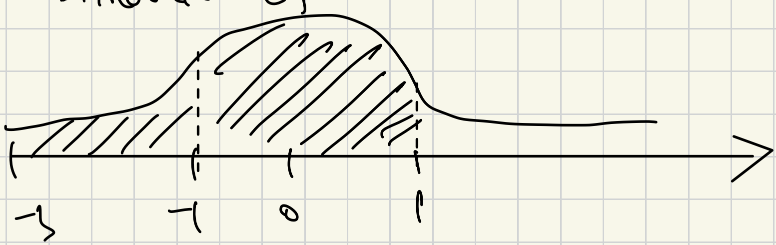




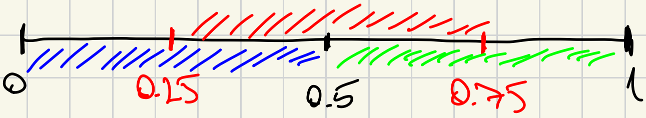
$$\int_a^b f(x)\,dx = P(X \in [a,b]),\ \forall -\infty \leq a < b \leq +\infty$$

$P(X = x_0) = 0,\ \forall x_0 \in \mathbb{R}$ (area of a line is infinitely small)

=? normrand()



→ rand() → uniform distrib. $P(X \in J) = \mathrm{len}(J)$



$\bullet = \bullet = \bullet = 0.5$

ex) Find the pdf. of $\underbrace{\text{rand}()}_{X}$.

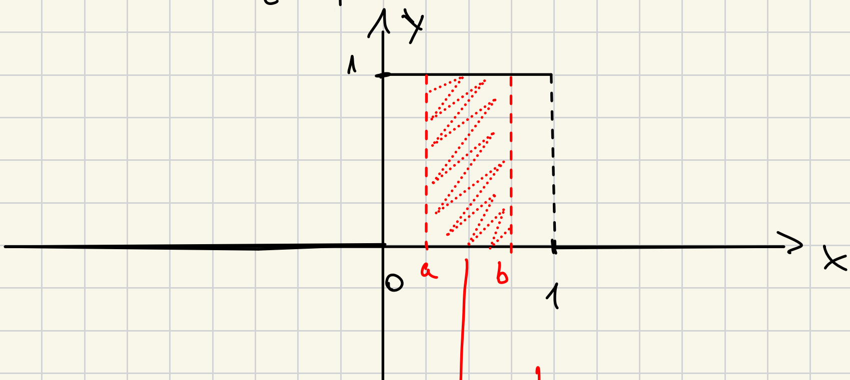
$$P(X \in \mathcal{I}) = \text{len}(\mathcal{I})$$

↳ interval.

$f = ?$

$$P(X \in [a,b]) = \int_a^b f(x)\,dx = b-a \,, \quad \forall\, 0 \le a < b \le 1$$

$$f(x) = \begin{cases} 1 \,, & x \in [0,1] \\ 0 \,, & \text{otherwise} \end{cases}$$

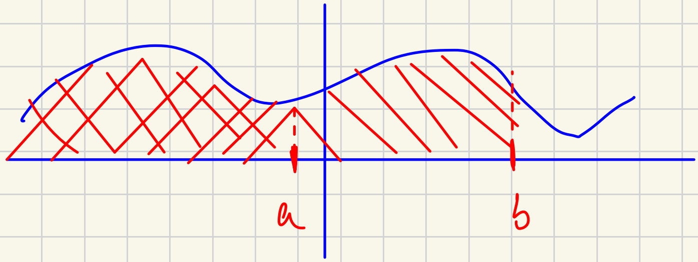


↳ $\int_a^b f(x)\,dx$

$$P(X < 0) = \int_{-\infty}^{0} f(x)\,dx = 0.$$

$F =$ cdf, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$, $x \in \mathbb{R}$

$$P(X \in [a,b]) = F(b) - F(a)$$



1. The lifetime, in years, of some electronic component is a random variable with density

$$f(x) = \begin{cases} \dfrac{k}{x^4}, & \text{for} \quad x \ge 1 \\ 0, & \text{for} \quad x < 1. \end{cases}$$

Find

a) the constant k;

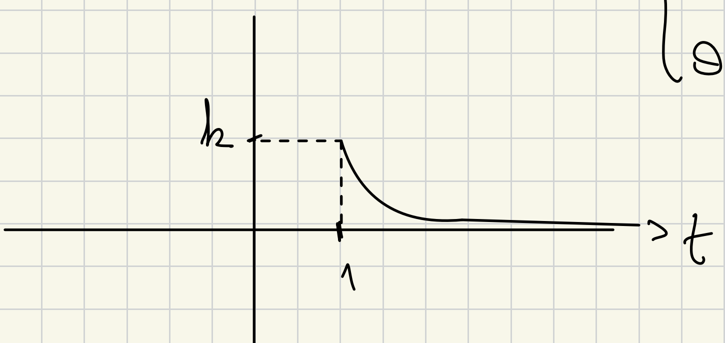
b) the corresponding cdf F;

c) the probability for the lifetime of the component to exceed 2 years.

$X =$ lifetime of a component in years

p.d.f. of X: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \dfrac{k}{x^4}, & x \ge 1 \\ 0, & x < 1 \end{cases}$

a)



$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} l(t)\, dt = 1$$

$$\int_{-\infty}^{\infty} l(t)\, dt = \int_{1}^{\infty} \frac{k}{x^4}\, dx = k \cdot \int_{1}^{\infty} \frac{1}{x^4}\, dx = \frac{k}{-3} \cdot \frac{1}{x^3}\Big|_{1}^{\infty} =$$

$$= \frac{k}{-3} \cdot \left(\lim_{x \to \infty} \frac{1}{x^3} - \frac{1}{1^3} \right) = \frac{k}{3}$$

$$\frac{k}{3} = 1 \Rightarrow k = 3$$

b) $$l(x) = \begin{cases} \frac{3}{x^4}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

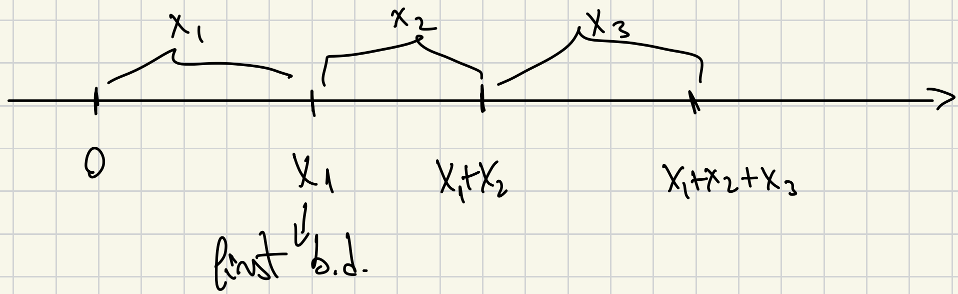
$$F(x) = \int_{-\infty}^{x} l(t)\, dt = \begin{cases} 0, & x < 1 \\ \int_{1}^{x} l(t)\, dt = -\frac{1}{t^3}\Big|_{1}^{x} = -\frac{1}{x^3} + 1, & x \geq 1 \end{cases}$$

c) $$\int_{2}^{\infty} \frac{3}{x^4}\, dx = 3 \cdot \int_{2}^{\infty} \frac{1}{x^4}\, dx = \frac{3}{-3} \cdot \frac{1}{x^3}\Big|_{2}^{\infty} =$$

$$= (-1) \cdot \left(\lim_{x \to \infty} \frac{1}{x^3} - \frac{1}{2^3} \right) = \frac{1}{8}.$$

3. On the average, a computer experiences breakdowns every 5 months. The time until the first breakdown and the times between any two consecutive breakdowns are independent Exponential random variables. After the third breakdown, a computer requires a special maintenance.
a) Find the probability that a special maintenance is required within the next 9 months;
b) Given that a special maintenance was not required during the first 12 months, what is the probability that it will not be required within the next 4 months?

average rate $= \frac{1}{5}$ break downs / months



X_1, X_2, X_3 independent $\in Exp(\lambda)$

a) $X = X_1 + X_2 + X_3$ = total time to get the third b.d.
$\in Gamma(3, \frac{1}{\lambda})$

$P(X \leq 9) = ?$ (octave online)

Exponential distribution $Exp(\lambda) = Gamma(1, 1/\lambda)$, $\lambda > 0$: pdf $f(x) = \lambda e^{-\lambda x}, x > 0$.
- Exponential distribution models *time*: waiting time, interarrival time, failure time, time between rare events, etc; the parameter λ represents the frequency of rare events, measured in time^{-1}.
- Gamma distribution models the *total* time of a multistage scheme.
- For $\alpha \in \mathbb{N}$, a $Gamma(\alpha, 1/\lambda)$ variable is the sum of α independent $Exp(\lambda)$ variables.

gamcdf $(9, 3, \frac{1}{\frac{1}{5}})$ = gamcdf $(9, 3, 5) = 0.2694$

b) $P(X \geq 12+4 \mid \underbrace{X \geq 12}) = \dfrac{P(X \geq 16 \cap X \geq 12)}{P(X \geq 12)} =$

"given that"

$$= \frac{P(X \geq 16)}{P(X \geq 12)} = \frac{1 - P(X < 16)}{1 - P(X < 12)} = \frac{1 - \text{gamcdf}(16, 3, 5)}{1 - \text{gamcdf}(12, 3, 5)}$$

5. Let X be a random variable with density $f_X(x) = \frac{1}{4} x e^{-\frac{x}{2}}$, $x \geq 0$ and let $Y = \frac{1}{2}X + 2$. Find f_Y.

Function $Y = g(X)$: X r.v., $g : \mathbb{R} \to \mathbb{R}$ differentiable with $g' \neq 0$, strictly monotone

$$f_Y(y) = \frac{f_X\left(g^{-1}(y)\right)}{\left|g'\left(g^{-1}(y)\right)\right|}, \ y \in g(\mathbb{R})$$

$$X \Rightarrow \text{pdf}, \quad f_X(x) = \begin{cases} \frac{1}{4} x e^{-\frac{x}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$Y = \frac{1}{2}X + 2, \ \text{find } f_Y(y) = ?; \quad y = \frac{1}{2}\left(\frac{1}{4} \cdot x \cdot e^{-\frac{x}{2}}\right),$$

$$f_Y(y) = \frac{\frac{1}{4} \cdot (2y - 4) \cdot e^{-y+2}}{\left|\frac{1}{2}\right|} = \frac{\cancel{2} \cdot (\cancel{2y - 4}) \cdot e^{-y+2}}{\cancel{4} \ \cancel{2}} =$$

$$= \begin{cases} (y - 2) \cdot e^{-y+2}, & y \geq 2 \\ 0, & y < 2 \end{cases}$$

1) $g(x) = \frac{1}{2}x + 2, \ x \in \mathbb{R}$

$g'(x) = \frac{1}{2}$

2) $g^{-1}(y) = 2(y - 2), \ y \in \mathbb{R}$

3) $f_Y(y) = \dots = \{$

4. The joint density for (X, Y) is $f_{(X,Y)}(x, y) = \frac{1}{16}x^3y^3$, $x, y \in [0, 2]$.
a) Find the marginal densities f_X, f_Y.
b) Are X and Y independent?
c) Find $P(X \leq 1)$.

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{16}x^3y^3, & x,y \in [0,2] \\ 0, & x \notin [0,2] \text{ or } y \notin [0,2] \end{cases}$$

a) $$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,dy \overset{x,y}{\underset{\in[0,2]}{=}} \int_0^2 \frac{1}{16}x^3 \cdot y^3\,dy =$$

$f_X(x) = \int\limits_{\mathbb{R}} f(x,y)dy, \ \forall x \in \mathbb{R}, \ f_Y(y) = \int\limits_{\mathbb{R}} f(x,y)dx, \ \forall y \in \mathbb{R}$ (marginal densities)

$$= \frac{x^3}{16} \cdot \int_0^2 y^3\,dy = \frac{x^3}{16} \cdot \frac{y^4}{4}\Big|_0^2 = \frac{x^3}{\cancel{16}} \cdot \cancel{4} = \frac{x^3}{4}.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)\,dx \overset{x,y\in}{\underset{[0,2]}{=}} \int_0^2 \frac{1}{16}x^3y^3\,dy = \dots =$$

$$= \frac{y^3}{4}, \quad x,y \in [0,2]; \quad f_Y(y) = \begin{cases} \frac{y^3}{4}, & y \in [0,2] \\ 0, & \text{otherwise} \end{cases}$$

b) $\Rightarrow f_{(X,Y)}(x,y) \overset{?}{=} f_X(x) \cdot f_Y(y)$

$$\frac{1}{16}x^3y^3 \overset{?}{=} \frac{1}{4}\cdot x^3 \cdot \frac{1}{4}y^3 \quad \text{„True"}$$

$$f_{(X,Y)}(x,y) \overset{?}{=} \begin{cases} f_X(x)\cdot f_Y(y) & , x,y \in [0,2] \\ 0\cdot 0 & , x \notin [0,2] \text{ or } y \notin [0,2] \end{cases}$$

"True"

$\Rightarrow X, Y$ indep.

c) $P(X \leq 1) \Rightarrow \underbrace{\int_0^1 f_X(x)\,dx}_{} = \frac{1}{4} = 0.25$

add branches $\int_{-\infty}^{0} + \int_0^1$

$\int_{-\infty}^{0} = 0$