Ch. 7.

1, 2, 4, 6, 7, 8.

$Q: \quad a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$

$Q \xrightarrow[\text{(rotation \& translation)}]{\text{Isometry (preserves length)}}$ $\lambda_1 x^2 + \lambda_2 y^2 = k$

OR

$\lambda_1 x^2 + \lambda_2 y = k$

$\downarrow$ rescaling

$k, \lambda_1, \lambda_2 = \pm 1$

r = rank Q	$(p, r-p)$	equation	name
2	(0,2) or (2,0)	$x^2+y^2+1=0$	imaginary ellipse
2	(0,2) or (2,0)	$x^2+y^2-1=0$	circle (ellipse)
2	(1,1)	$x^2-y^2-1=0$	hyperbola
2	(0,2) or (2,0)	$x^2+y^2=0$	two complex lines (a point)
2	(1,1)	$x^2-y^2=0$	two real lines
1	(0,1) or (1,0)	$x^2+1=0$	two complex lines
1	(1,0)	$x^2-1=0$	two real lines
1	(1,0)	$x^2=0$	a real, double line.
1	(0,1) or (1,0)	$x^2-y=0$	parabola

$$M_Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

2. Write down a quadratic eq with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A.

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$

$$P_A(\lambda) = \det(A - \lambda I_2)$$

$$= \begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)(9-\lambda) - 4 = 0$$

$$54 - 15\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 15\lambda + 50 = 0$$

$$\Delta = 225 - 200 = 25$$

$$\Rightarrow \lambda_{1,2} = \frac{15 \pm 5}{2} \begin{cases} 10 \\ 5 \end{cases}$$

$\lambda_1 = 10, \lambda_2 = 5$

$$S(\lambda) = \left\{ (x,y) \mid A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

$$= \left\{ (x,y) \mid (A - \lambda I_2) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$S(\lambda_1) = \left\{ (x,y) \mid \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$\begin{cases} -4x + 2y = 0 \\ 2x - y = 0 \end{cases} \Rightarrow S(\lambda_1) = \{ (x, 2x) \mid x \in \mathbb{R} \} =$

$= \langle (1,2) \rangle$

Choose $v_1 = \frac{1}{\sqrt{5}} \cdot (1,2)$

$S(\lambda_2) = \{ (-2y, y) \mid y \in \mathbb{R} \} = \langle (-2,1) \rangle$

Choose $v_2 = \frac{1}{\sqrt{5}} \cdot (2,-1)$

The base-change matrix is $M = M_{B'B} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$

$\det M_{B'B} = -1$, but we want it to be 1.

$\Rightarrow$ flip one of the vectors you chose

$\Rightarrow v_2 = \frac{1}{\sqrt{5}} (-2,1)$

$\Rightarrow M = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$\underbrace{M^{-1}}_{= M^T} \cdot A \cdot M = M^T \cdot A \cdot M$

(bc. M is orthogonal ($M \in O(2)$))

$= \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$$= \frac{1}{5}\begin{pmatrix} 10 & 20 \\ -10 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} =$$

$$= \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$Q: (x, y) \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + (x\ y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$$

$$Q: (6x + 2y \quad 2x + 9y) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + x + 2y - 1 = 0$$

$$Q: 6x^2 + 2xy + 2xy + 9y^2 + x + 2y - 1 = 0$$

$$Q: 6x^2 + 4xy + 9y^2 + x + 2y - 1 = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x\ y = \begin{pmatrix} x \\ y \end{pmatrix}^T = \left(M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}\right)^T = (x'\ y') \cdot M^T$$

$$Q: (x'\ y') \cdot \underbrace{M^T \cdot A \cdot M} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + (x'\ y')\, M^T \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$$

$$Q: (x'\ y') \cdot \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + (x'\ y') \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$$

$$Q: 10x'^2 + 5y'^2 + (x'y') \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 5 \\ 0 \end{pmatrix} - 1 = 0$$

$$Q: 10x'^2 + 5y'^2 + \sqrt{5}\,x' - 1 = 0$$

$$Q: \left(10x'^2 + \sqrt{5}x'\right) + 5y'^2 - 1 = 0$$

$$10\left(x'^2 + \frac{2}{20}\sqrt{5}x' + \frac{5}{400}\right) + 5y'^2 - 1 - \frac{1}{8} = 0$$

$$Q: 10\left(x' + \frac{\sqrt{5}}{20}\right)^2 + 5y^2 - \frac{9}{8} = 0$$

$$\begin{cases} x'' = x' + \frac{\sqrt{5}}{20} \\ y'' = y' \end{cases}$$

$$10x''^2 + 5y''^2 = \frac{9}{8}$$

$$\Rightarrow \frac{x''^2}{\frac{9}{80}} + \frac{y''^2}{\frac{9}{40}} = 1$$

$$\left.\begin{cases} x''' = \frac{x''}{\frac{3}{\sqrt{80}}} \\ y''' = \frac{y''}{\frac{3}{\sqrt{40}}} \end{cases}\right\} \Rightarrow Q: x'''^2 + y'''^2 = 1$$

2^{nd} approach (Lagrange)

$$Q: 6x^2 + 4xy + 3y^2 + x + 2y - 1 = 0$$

$$\Rightarrow (6x^2 + 4xy) + 3y^2 + x + 2y - 1 = 0$$

$$\left(6x^2 + 2 \cdot \sqrt{6}x \cdot \frac{2}{\sqrt{6}}y + \frac{2}{3}y^2\right)$$

$$\overset{3/}{+}\,9y^2 - \frac{2}{3}y^2 + x + 2y - 1 = 0$$

$$= \left(\sqrt{6}x + \frac{2}{\sqrt{6}}y\right)^2 + \frac{25}{3}y^2 + x + 2y - 1 = 0$$

$$x' = \sqrt{6}x + \frac{2}{\sqrt{6}}y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sqrt{6} & \frac{2}{\sqrt{6}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = \frac{x' - \frac{2}{\sqrt{6}}y}{\sqrt{6}}$$

$$Q: \quad x'^2 + \frac{25}{3}y^2 + \frac{1}{6}x' - \frac{1}{3}y + 2y - 1 = 0$$

$$Q: \left(x'^2 + \frac{1}{6}x'\right) + \left(\frac{25}{3}y^2 + \frac{5}{3}y\right) - 1 = 0$$

$$Q: \left(x'^2 + 2 \cdot \frac{1}{12}x' + \frac{1}{144}\right) +$$

$$+ \left(\frac{25}{3}y^2 + 2 \cdot \frac{5}{\sqrt{3}}y \cdot \frac{\sqrt{3}}{6} + \frac{1}{12}\right)$$

$$- \frac{1}{144} - \frac{1}{12} - 1 = 0$$

$$Q: \left(x' + \frac{1}{12}\right)^2 + \left(\frac{5}{\sqrt{3}}y + \frac{\sqrt{3}}{6}\right)^2$$

$$= \frac{157}{144}$$

$x'' = x' + \frac{1}{12}$

$y'' = \frac{5}{\sqrt{3}} y' + \frac{\sqrt{3}}{6}$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{12} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$

$Q: \; x''^2 + y''^2 = \frac{157}{144}$

$x''' = x'' \cdot \sqrt{\frac{144}{157}}$

$y''' = y'' \cdot \sqrt{\frac{144}{157}}$

$Q: \; x'''^2 + y'''^2 = 1$

7.4. $Q: \; -x^2 + xy - y^2 = 0$

Bring Q to the canonical form

$(x^2 - xy) + y^2 = 0$

$\left(x^2 - 2 \cdot \frac{1}{2} \cdot x \cdot y + \frac{y^2}{4}\right) + \frac{3}{4} y^2 = 0$

$Q: \left(x - \frac{y}{2}\right)^2 + \frac{3}{4} y^2 = 0$

$$\begin{cases} x' = x - \frac{y}{2} \\ y' = y \end{cases}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Q: $x'^2 + \frac{3}{4} y' = 0$ Two complex lies (a point)