

Chapter 5 : 16, 18, 19, 20, 3, 22, 23, 24

$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ affine morphism

$$f(p) = A \cdot p + b, A \in M_m(\mathbb{R}), b \in \mathbb{R}^m$$

f isometry $\Leftrightarrow \forall x, y \in \mathbb{R}^n$

$$\text{dist}(f(x), f(y)) = \text{dist}(x, y)$$

$$(\lim f)(p) = A \cdot p$$

$$\Leftrightarrow A \in O(n) = \left\{ M \in M_m(\mathbb{R}) \mid M^{-1} = M^T \right\}$$

orthogonal matrices

$$\Leftrightarrow A \cdot A^T = I_n \Rightarrow \text{check if the distances are preserved}$$

$A \in O(n)$ (special orthogonal)

I $\det A = 1 \Rightarrow$ direct isometry

II $\det A = -1 \Rightarrow$ indirect isometry

$$m=2$$

- direct isometry:

- identity

- translation by a vector $\vec{v} \rightarrow T_{\vec{v}}$

- rotation around a point Q by an angle θ — $\text{Rot}_{Q,\theta}$

- indirect isometry:

- reflection wrt a line l — Ref_l

- glide reflection wrt a line l w/ a vector $\vec{v} \in \mathbb{A}(l)$ — $T_{\vec{v}} \circ \text{Ref}_l$

\underline{l}



$$\text{Ref}_l(P)$$

$$T_{\vec{v}} \circ \text{Ref}_l(P)$$

$$\text{Fix}(f) = \{ p \in \mathbb{R}^n \mid f(p) = p \}$$

J.16. f isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a translation by the vector $(-1, \sqrt{3})$

Let the inverse f^{-1}

$$\Rightarrow f = \text{Rot}_{-\frac{\pi}{3}} \circ T_{(-1, \sqrt{3})} \quad (f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

$$\begin{pmatrix} \cos -\frac{\pi}{3} & -\sin -\frac{\pi}{3} & 0 \\ \sin -\frac{\pi}{3} & \cos -\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos -\frac{\pi}{3} & -\sin -\frac{\pi}{3} & -2\cos -\frac{\pi}{3} + \sqrt{3}\sin -\frac{\pi}{3} \\ \sin -\frac{\pi}{3} & \cos -\frac{\pi}{3} & -2\sin -\frac{\pi}{3} + \sqrt{3}\cos -\frac{\pi}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & -1 + \frac{5\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{1}{2} + \sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow f^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{5\sqrt{3}}{2} & \frac{1}{2} - \sqrt{3} & 1 \end{pmatrix}$$

$$[R^{-1}] = \text{Rot}_{\frac{\pi}{3}} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T] = T_{(2, -\sqrt{3})} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$[T^{-1}] \cdot [R^{-1}] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$J.R. \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$f(P) = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \cdot P + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\text{Tr } A = 2 \cos \theta$$

- Show that f is a rotation (direct isometry)

- Find its center and the rotation angle

$$H \cdot H^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = I_2 \Rightarrow \text{direct isometry.}$$

$$\det A = \frac{9+16}{25} = 1 \Rightarrow A \in SO(2)$$

$\Rightarrow f$ direct isometry.

$$\text{Let } P \in \mathbb{R}^2, P \in \text{Fix}(f) \Leftrightarrow f(P) = P$$

$$\begin{aligned} P(x) \\ f(P) = AP + B = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3x - 4y}{5} + 1 \\ \frac{4x + 3y}{5} - 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$\Rightarrow x = \frac{1}{2}, y = 0 \Rightarrow P = \frac{1}{5} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow f \text{ is a rotation.}$$

$$\text{Tr}(f) = \frac{6}{5} = 2 \cos \theta \Rightarrow \frac{3}{5} = \cos \theta \Rightarrow \theta = \arccos \frac{3}{5}$$

$$n=3$$

direct:

- identity

- motion around an axis l by an angle θ $\text{Rot } l, \theta$

- glide rotation

$$T_{\vec{v}} \circ \text{Rot } l, \theta$$

indirect:

- reflection with respect to a plane $T \cap \text{Rel } \pi$

- glide reflection

$$T_{\vec{v}'} \circ \text{Rel } \pi$$
$$\vec{v}' \in \text{D}(\pi)$$

- rotational reflection

$$\text{Rot } l, \theta \circ \text{Rel } \pi \text{, where } l \perp \pi$$

If T rotation: $T_{\vec{v}} A = 2 \cos \theta + I$

FINAL EXAM 18, 19

Verify that the matrix:

$$A = \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \in \text{SO}(3)$$

det. the axis and the angle of rotation

$$\begin{pmatrix} -1 & 2 & 2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$\Rightarrow A \in O(3)$

$$\det A = \frac{1}{27} (4 + 4 + \dots) = \frac{27}{27} = 1 \quad A \in SO(3)$$

\Rightarrow f direct isometry

$$\text{Let } P \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{such that } D \cdot P = P.$$

$$\Rightarrow \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} -x + 2y - 2z \\ -2x - 2y - z \\ -2x + y + 2z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow -x + 2y - 2z = 0$$

$$-2x - 2y - z = 0$$

$$\underline{-2x + y + 2z = 0}$$

$$\Rightarrow -5y = y \Rightarrow y = 0$$

$$-4x - 2z = 0 \quad | \cdot \frac{1}{2}$$

$$2x + z = 0 \Rightarrow z = -2x$$

$$\Rightarrow S(\alpha, 0, -2\alpha)$$

$$\text{Fix}(A) = \{(x, 0, -2x) \mid x \in \mathbb{R}\}$$

$$D(e) = \langle(1, 0, -2)\rangle \Rightarrow A \text{ rotation}$$

$$\text{Tr } A = 2 \cos \theta + 1$$

$$\Rightarrow -\frac{1}{3} = 2 \cos \theta + \frac{2}{3}$$

$$-\frac{4}{3} = 2 \cos \theta \Rightarrow -\frac{2}{3} = \cos \theta$$

22. Using Euler-Rodriguez formula write a
rotation around the axis $\mathbf{v}/\|\mathbf{v}\|$ where $\mathbf{v}^T(1, 1, 0)$
and use this matrix for the parametrization of
a cylinder of axis $\mathbf{v}/\|\mathbf{v}\|$ and diameter s_2

Euler-Rodriguez.

$$\text{Rot}_{e, \theta}(\mathbf{P}) = \cos \theta \cdot \mathbf{P} - \sin \theta (\vec{\mathbf{v}} \times \mathbf{P}) + (1 - \cos \theta) \vec{\mathbf{v}} \mathbf{P} \vec{\mathbf{v}}$$

plug in \mathbf{P} as \mathbf{v} \Rightarrow write result as matrix =
matrix of rotation.

