1) I) $\begin{cases} x' = -y \\ y' = 5x \end{cases} \Rightarrow A = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$

a) type and stability of $(0,0)$

b) does it have global first integral?

c) find first integral

d) phase portrait

$\det(A - \lambda I_2) = 0$

$$\begin{vmatrix} -\lambda & -1 \\ 5 & -\lambda \end{vmatrix} = \lambda^2 + 5 = 0 \Rightarrow \lambda^2 = -5$$

$\Rightarrow \lambda = \pm i\sqrt{5} \Rightarrow$ CENTER $\Rightarrow$ saddle

b) $$\frac{dy}{dx} = \frac{5x}{-y} = 5x\,dx = -y\,dy \Leftrightarrow$$

$$\Leftrightarrow \int 5x\,dx = -\int y\,dy \Leftrightarrow 5\frac{x^2}{2} = -\frac{y^2}{2} + C$$

$\Leftrightarrow 5\frac{x^2}{2} + \frac{y^2}{2} = c \mid \cdot 2 \Leftrightarrow 5x^2 + y^2 = c_1$

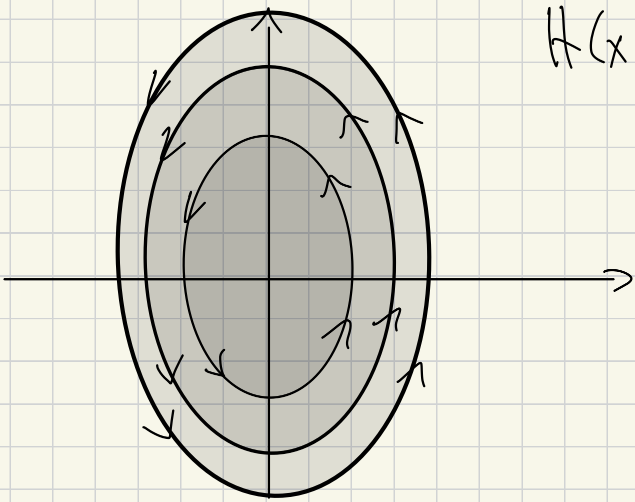
$\Rightarrow H : \mathbb{R}^2 \to \mathbb{R}, \; H(x,y) = 5x^2 + y^2$

c) $\frac{\partial H}{\partial x} f_1 + \frac{\partial H}{\partial y} f_2 \overset{?}{=} 0$ in $\mathbb{R}^2$

$10x \cdot (-y) + 2y \cdot 5x = 0 \Leftrightarrow 0 = 0$ true

d)

$H(x,y) = 5x^2 + y^2$



$x' = -y < 0$ when $y > 0$

II) $x' = -x$ $\quad A = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

$y' = 5y$

$\lambda_1 = -1$, $\lambda_2 = 5 \Rightarrow$ saddle, unstable

$$\frac{dy}{dx} = \frac{5y}{-x} \Leftrightarrow -x\,dy = 5y\,dx \Rightarrow$$

$$\Rightarrow \frac{dy}{y} = -5\frac{dx}{x} \Rightarrow \int \frac{dy}{y} = -5\int \frac{dx}{x}$$

$$\Rightarrow \ln|y| + 5\ln|x| = c$$

$$\Rightarrow \ln(y + x^5) = c$$

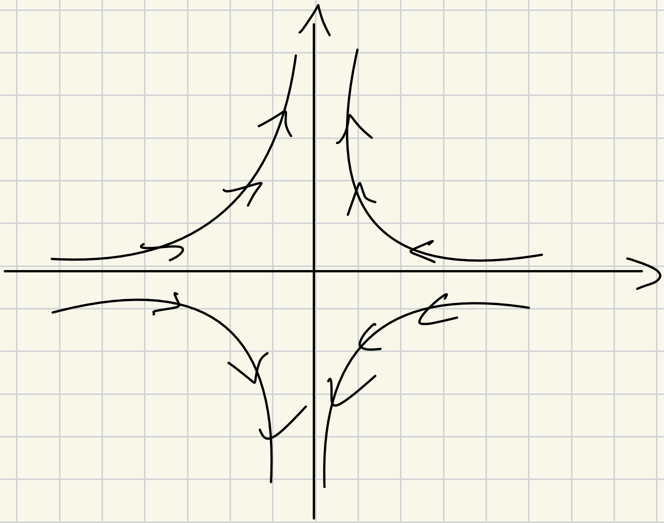
$$\Rightarrow y + x^5 = c_1$$

$H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = y \cdot x^5$

Since H not constant, we check if it is a global first integral

$$\frac{\partial H}{\partial x} f_1 + \frac{\partial H}{\partial y} f_2 = 0 \Leftrightarrow 5x^4 y \cdot (-x) + x^5 \cdot 5y = 0$$

$\Leftrightarrow 0 = 0$ true $\Rightarrow$ first integral (global)



$y = c \cdot \frac{1}{x^5}$

$x' = -x < 0$

$y' = 5y > 0$

2^{nd} quadrant : $x' = -x > 0$

$y' = 5y > 0$

3^{rd} quadrant : $x' = -x > 0$

$y' = 5y < 0$

4^{th} quadrant : $x' = -x > 0$

$y' = 5y < 0$

III) $\begin{cases} \dot{x} = -3x \\ \dot{y} = -2y \end{cases} \rightarrow \underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}}_{\dot{X}} = \underbrace{\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

$f_1(x,y) = -3x + 0y$

$f_2(x,y) = 0x - 2y$

$\det(A - \lambda I_2) = 0 \;\; = 0 \Rightarrow \underbrace{\lambda_1 = -3,\ \lambda_2 = -2}_{<0} \Rightarrow$

$\Downarrow$

$\Rightarrow$ node global attractor $\Rightarrow$

theorem $\Rightarrow$ no global first integrals $\Rightarrow$ look for regional first integrals

iii) $\dfrac{dy}{dx} = \dfrac{f_2(x,y)}{f_1(x,y)} \Leftrightarrow \dfrac{dy}{dx} = \dfrac{-2y}{-3x} \Leftrightarrow$

$\Leftrightarrow \dfrac{dy}{-2y} = \dfrac{dx}{-3x} \overset{|\cdot(-6)}{\Longleftrightarrow} \dfrac{3\,dy}{y} = \dfrac{2\,dx}{x} \overset{\int}{\Longleftrightarrow}$

$\Leftrightarrow 3\displaystyle\int \frac{dy}{y} = 2\int \frac{dx}{x}$

$\Leftrightarrow 3\ln|y| = 2\ln|x| + c$

$\Leftrightarrow \ln\left|\dfrac{y^3}{x^2}\right| = c \Leftrightarrow \dfrac{y^3}{x^2} = c_1\ (e^c)$

$\mathbb{R}^2$ (that's why it's not global)

$$H : \mathbb{R}^* \times \cancel{\mathbb{R}^*} \to \mathbb{R}$$

$$H(x,y) = \frac{y^3}{x^2}$$

$$U_1 = \{ (x,y) \in \mathbb{R}^2 \mid x < 0 \}$$

$$U_2 = \{ (x,y) \in \mathbb{R}^2 \mid x > 0 \}$$

$$U = U_1 \cap U_2$$

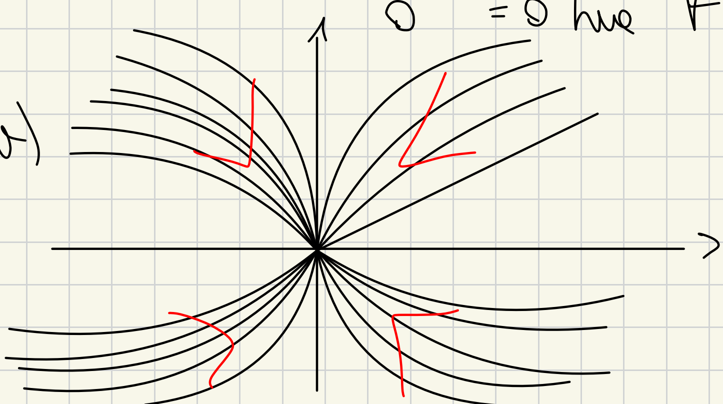
$$l_1(x,y) \frac{\partial H}{\partial x}(x,y) + l_2(x,y) \frac{\partial H}{\partial y} \overset{?}{=} 0, \ \forall (x,y) \in U$$

$$-3x \cdot y^3 \cdot \left(\frac{1}{x^2}\right)' - 2y \frac{(y^3)'}{x^2} \overset{?}{=} 0$$

$$6x^{-2} \cdot y^3 - 6y^3 \cdot x^{-2} = 0$$

$$0 = 0 \text{ true } \forall (x,y) \in U \Rightarrow$$

iv)



$$H(x,y) = \frac{y^3}{x^2}$$

$$\frac{y^3}{x^2} = c \Rightarrow y = \sqrt[3]{cx^2}$$

$\underline{\text{iv}}$ $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases} \rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\det(A - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow (1-\lambda)^2 + 1 = 0$$

$$\Leftrightarrow \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda_1 = 1 - i$$
$$\lambda_2 = 1 + i$$

$\left.\begin{array}{l} \mathrm{Re}(\lambda) \neq 0 \\ \mathrm{Im}(\lambda) \neq 0 \end{array}\right\} \Rightarrow$ focus point

$\boxed{\dot{X} = AX \Rightarrow \text{linear}}$

hom. syst.

$\mathrm{Re}(\lambda) > 0 \Rightarrow$ global repeller $\xrightarrow{\text{theorem}}$ no global first integral

3)(i) For what values of $a \in \mathbb{R}$

$$\begin{cases} \dot{x} = ax - 5y \\ \dot{y} = x - 2y \end{cases}$$ has a center?

Find general sol. of system

$$\dot{X} = AX \iff \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues of A

$$\det(A - \lambda I_2) = 0 \iff \det \begin{pmatrix} a-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)(-2-\lambda) + 5 = 0 \iff$$

$$\iff -2a - a\lambda + 2\lambda + \lambda^2 + 5 = 0$$

$$\iff \lambda^2 + \lambda(2-a) - 2a + 5 = 0$$

$$\lambda_{1,2} = \frac{-(2-a) \pm \sqrt{\Delta}}{2} =$$

Center iff. $\exists \beta \neq 0$ s.t $\lambda_{1,2} = \pm i\beta$

iff $\Delta < 0$ and $\frac{-(2-a)}{2} = 0$

$$\frac{-(2-a)}{2} = 0 \Rightarrow a = 2$$

$$\Delta = (2-a)^2 - 4 \cdot 1(-2a+5) = 0^2 - 4(-4+5) = -4 \overset{a=2}{\iff}$$

$$\iff \lambda_{1,2} = \pm \frac{2i}{2} = \pm i$$

ii) $a = ?$ s.t.