S.1) How many solutions has the following problem

a) $x'' + t^2 x = 0, \quad x(0) = 0$

b) $x'' + t^2 x = 0, \quad x(0) = 0, \quad x'(0) = 0$

c) $x'' + t^2 x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1$

b) is the only iVP, but the others are not

<u>Remark</u>

The characteristic eq. method doesn't work for $x'' + t^2 x = 0$ (we don't have const. coef.)

<u>The existance and uniqueness theorem for an iVP</u> $\Rightarrow$
An iVP has a unique solution

$\Rightarrow$ b) has only one solution : $x(t) = 0, \ \forall t \in \mathbb{R}$.

$x'(0) = \eta, \ x(0) = 0, \ x'' + t^2 x = 0$ iVP $\Rightarrow$ one solution for $\forall \eta$ $\Rightarrow$ <u>infinitely many solutions for a)</u>

c) has no solutions

3.2) $\begin{cases} x' = -x + xy \\ y' = -2y + 3y^2 \end{cases}$

a) Equilibrium and stability

$\left.\begin{cases} -x + xy = 0 & \Leftrightarrow x(-1+y) = 0 \\ -2y + 3y^2 = 0 & \Leftrightarrow y(-2+3y) = 0 \end{cases}\right\} \Rightarrow$

$\Rightarrow (0,0)$, $(0, \frac{2}{3}) \to$ two eq. points

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -1+y & x \\ 0 & -2+6y \end{pmatrix}$$

The new system : $\dot{X} = J X$

$(0,0)$: $J = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \to \lambda_1 = -1, \lambda_2 = -2$

$Re(\lambda_1) < 0$, $Re(\lambda_2) < 0 \Rightarrow (0,0)$ attractor

$(0, \frac{2}{3})$: $J = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = -\frac{1}{3}, \lambda_2 = 2$

$\lambda_1 < 0$, $\lambda_2 > 0 \Rightarrow (0, \frac{2}{3})$ saddle point, unstable

b). $\varphi(t, \underline{0}, \underline{\tfrac{2}{3}})$ (flow) by def, this φ will be the unique sol of the ivp.

$$\begin{cases} x' = -x + xy \\ \underline{x(0) = 0} \\ y' = -2y + 3y^2 \\ \underline{y(0) = \frac{2}{3}} \end{cases}$$

$(0, \frac{2}{3})$ eq. point (from a)) $\Rightarrow \varphi(t, 0, \frac{2}{3}) =$

$= (0, \frac{2}{3})$ $\forall t \in \mathbb{R}$.

- $\varphi(t, 4, 0)$ is the unique sol of ivp

$$\begin{cases} x' = -x + xy \\ x(0) = 4 \\ y' = -2y + 3y^2 \end{cases}$$ ivp $\Rightarrow$ we can find unique y. $\Rightarrow y = 0$

$$\left\{ y(0) = 0 \right.$$

now replace y in the first equation

$$\begin{cases} x' = -x \\ x(0) = 4 \end{cases}$$

$x' + x = 0$

$\lambda + 1 = 0$

$\Rightarrow x = c \cdot e^{-t}, \quad c \in \mathbb{R}$

$\xRightarrow{x(0)=4} c = 4$

$\Rightarrow x = 4 \cdot e^{-t}$

$\Rightarrow \varphi(t, 4, 0) = (4 \cdot e^{-t}, 4, 0), \quad \forall t \in \mathbb{R}$

- $\varphi(t, 1, \frac{2}{3})$

$$\begin{cases} x' = -x + xy \\ x(0) = 1 \\ y' = -2y + 3y^2 \end{cases}$$

$\Rightarrow$ has 0 and $\frac{2}{3}$ eq. points $\Rightarrow$

$y(0) = \frac{2}{3}$

$\Rightarrow y = \frac{2}{3}$

$$\begin{cases} x' = -x + \frac{2}{3}x \\ x(0) = 1 \end{cases} \Leftrightarrow \begin{cases} x' = -\frac{1}{3}x \\ x(0) = 1 \end{cases}$$

$x' + \frac{1}{3}x = 0$

$r + \frac{1}{3} = 0 \Rightarrow r = -\frac{1}{3}$

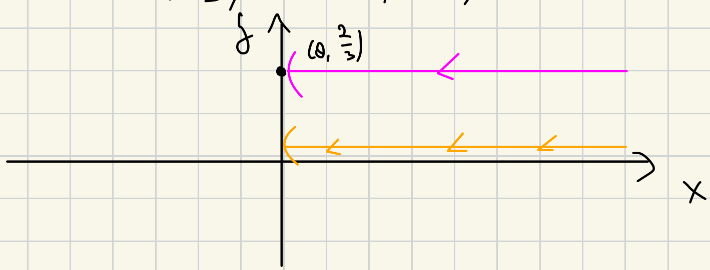
$x = c \cdot e^{-\frac{t}{3}}, c \in \mathbb{R} \xrightarrow{x(0)=1} c = 1 \Rightarrow$

$\Rightarrow x = e^{-\frac{t}{3}}$

$\varphi\left(t, 1, \frac{2}{3}\right) = \left(e^{-\frac{t}{3}}, \frac{2}{3}\right), \forall t \in \mathbb{R}.$

c) orbita for $\left(0, \frac{2}{3}\right), (1, 0), \left(1, \frac{2}{3}\right)$

$\varphi\left(t, 0, \frac{2}{3}\right) = \left(0, \frac{2}{3}\right)$



$\varphi(t, 4, 0) = (4e^{-t}, 0)$

$\varphi(t, 1, \frac{2}{3}) = (e^{-\frac{1}{3}t}, \frac{2}{3})$

9.3) Small oscilations of a simple idealised pendulum

a) Find IVP

b) Describe the oscillations of the pendulum

$T = ?$ -> the time the pendulum will return to its $t=0$

c) represent the phase portrait, specify its type.

justify why $\phi(t) = 0$ stable, not attractor $\forall\, t \in \mathbb{R}$

$\omega > 0$ => (LH 2nd order DE with const. coef.)

$$\left\{\begin{aligned} &\phi'' + \omega^2 \phi = 0 \\ &\phi(0) = \frac{\pi}{6} \\ &\phi'(0) = 0 \end{aligned}\right\} \overset{\text{IVP}}{\Rightarrow} \text{unique sol.}$$

a) $\phi'' + \omega^2 \phi = 0$

$r^2 + w^2 = 0 \Rightarrow r^2 = -w^2 \Leftrightarrow r = \pm\sqrt{-w^2} \Leftrightarrow$

$\Leftrightarrow r = \pm i w \rightarrow \cos(t), \sin(t)$

$$\phi(t) = c_1 \cdot \cos(wt) + c_2 \cdot \sin(wt)$$

$$\left.\begin{aligned} &\phi(0) = \underbrace{c_1 \cdot \cos 0}_{=c_1} + \underbrace{c_2 \cdot \sin 0}_{=0} = c_1 \\ &\phi(0) = \frac{\pi}{6} \end{aligned}\right\} \Rightarrow c_1 = \frac{\pi}{6}$$

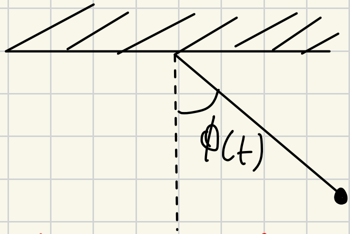
$$\phi'(t) = -c_1 \cdot w \sin(wt) + c_2 \cdot w \cdot \cos(wt)$$

$$\left.\begin{aligned} &\phi'(0) = \underbrace{-c_1 \cdot w \cdot \sin 0}_{=0} + \underbrace{c_2 \cdot w \cdot \cos 0}_{=c_2 w} \\ &\phi'(0) = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \quad c_2 \cdot w = 0 \quad , \forall w > 0 \Rightarrow c_2 = 0$$

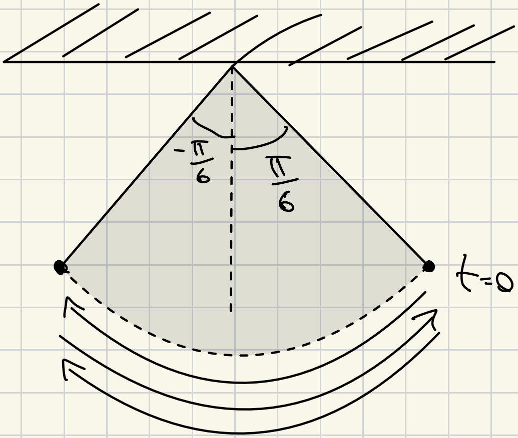
$$\phi(t) = c_1 \cdot \cos(wt) + c_2 \cdot \sin(wt) =$$

$$= \frac{\pi}{6} \cdot \cos(wt) \rightarrow \text{the unique solution}$$



b) $\underline{\phi(t)} = \frac{\pi}{6} \cdot \cos(wt)$

↳ represents the angle between the rod and the vertical



$\phi(t)$ periodic function $\Rightarrow$ pendullum won't stop

The pendulum will return to the initial pos. after one period $T = \frac{2\pi}{\omega}$.

c) Let $x = \phi$, $y = \phi'$

$$\phi'' + \omega^2\phi = y' + \omega^2 x = 0$$

$$y' = \phi''$$

$$\begin{cases} x' = y \\ y' = -\omega^2 x \end{cases}$$ $\Leftarrow$ planar system assoc. to a 2nd order DE.

$\mathring{X} = A \cdot X$ lin system

$$\begin{pmatrix}x'\\ y'\end{pmatrix} = \underbrace{\begin{pmatrix}0 & 1\\ -w^2 & 0\end{pmatrix}}_{A}\begin{pmatrix}x\\ y\end{pmatrix}$$

The eigenvalues of A:

$$\det(A-\lambda I_2)=0 \iff \begin{vmatrix}-\lambda & 1\\ -w^2 & -\lambda\end{vmatrix} = 0 \iff$$

$$\iff \lambda^2 + w^2 = 0 \iff \lambda^2 = -w^2 \iff$$

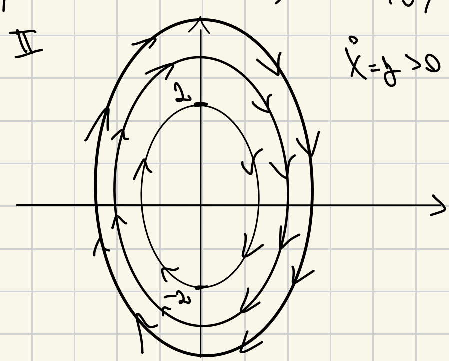
$$\iff \lambda = \sqrt{-w^2} \iff \lambda_{1,2} = \pm i w$$

$$\left.\begin{array}{l}\lambda_1, \lambda_2 \text{ complex conjugated}\\ Re(\lambda_1) = Re(\lambda_2) = 0\end{array}\right\} \Rightarrow \text{lin. system in a center} \Rightarrow$$

$\Rightarrow$ STABLE

1^{st} integral ..... (skipped computations, no time)

$$H(x,y) : \mathbb{R}^2 \to \mathbb{R}, \quad H(x,y) = \underbrace{w^2x^2 + y^2}_{\text{eq. of an ellipse}}$$



$w^2x^2 + y^2 = 4$

$x = 0 \Rightarrow y = \pm 2$

$y = 0 \Rightarrow x = \pm \frac{w}{2}, \ w > 0$

d) $\exists$ first integral $\Rightarrow$ not attractor.

3.4) $\omega > 0$

2^{nd} nLE : $\ddot{\theta} + \omega^2 \sin\theta = 0$

a) equilibria and using lin. method, study the stability

$\ddot{\theta} + \omega^2 \cdot \sin\theta = 0$ (we need to transform the 2^{nd} order nLE into a planar system) $\Rightarrow$

$\Rightarrow$ introduce 2 new variables x and y.

$$\begin{cases} x = \theta \\ y = \dot{\theta} \end{cases} \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 \cdot \sin x \end{cases}$$

$$f(x,y) = \begin{pmatrix} y \\ -\omega^2 \cdot \sin x \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y = 0 \\ -\omega^2 \cdot \sin x = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = k\pi, k \in \mathbb{Z} \end{cases}$$

$\eta_k^*(k\pi, 0), \forall k \in \mathbb{Z}$.

$$Jf(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 \cdot \cos x & 0 \end{pmatrix}$$

$$Jf(k2\pi, 0) = \begin{pmatrix} 0 & 1 \\ -w^2 \cos(k2\pi) & 0 \end{pmatrix} = A_k$$

Case I : k even $\Rightarrow \cos(k2\pi) = 1 \Rightarrow$

$$\Rightarrow A_k = \begin{pmatrix} 0 & 1 \\ -w^2 & 0 \end{pmatrix}$$

$$\det(A_k - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ -w^2 & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \lambda_{1,2} = \pm i w$$

From $Re(\lambda_1) = Re(\lambda_2) = 0 \Rightarrow \eta^*$ not hyperbolic $\Rightarrow$ we cannot apply the linearization method

Case 2: k odd $\Rightarrow \cos(k2\pi) = -1$

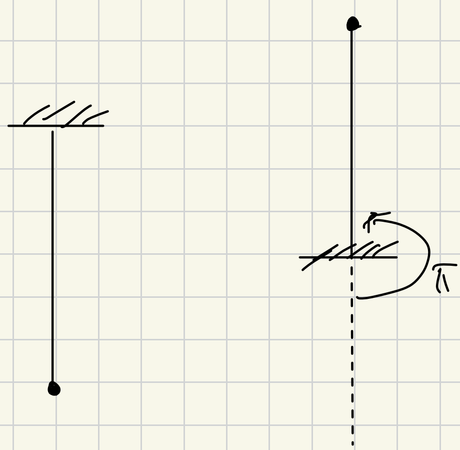
$$A_k = \begin{pmatrix} 0 & 1 \\ w^2 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \lambda_1 = w \\ \lambda_2 = -w \end{matrix}$$

$\left.\begin{array}{l} Re(\lambda_1) \neq 0 \\ Re(\lambda_2) \neq 0 \end{array}\right\} \Rightarrow m^*$ hyperbolic point $\Longrightarrow$ we can apply lin. method

$\left\{\begin{array}{l} \lambda_1, \lambda_2 \in \mathbb{R} \\ \lambda_1 > 0 \\ \lambda_2 < 0 \end{array}\right\} \Rightarrow$ linear system will be a saddle $\Rightarrow$

$\Rightarrow$ m^* unstable.

$k = 0$ $\qquad$ $k = 1$ $\qquad$ $k = 2$



Obs: Physically, we only have 2 equilibrium positions

if we move just a bit from an equilibrium point, the pendulum will not come back to the initial position.

Finding the 1st integral of a nLE

$$\frac{dy}{dx} = \frac{-w^2 \sin x}{y} \xLeftrightarrow{\text{separate var.}} y\,dy = -w^2 \sin x\,dx \xLeftrightarrow{\int}$$

$$\Leftrightarrow \int y\,dy = -w^2 \int \sin x\,dx$$

$$\Leftrightarrow \frac{1}{2} y^2 = w^2 \cdot \cos x + C \quad | \cdot 2$$

$$\Leftrightarrow y^2 = 2 \cdot w^2 \cos x + C_1$$

$$\Leftrightarrow y^2 - 2w^2 \cos x = C_1$$

$H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = y^2 - 2w^2 \cdot \cos x$

check equality to see if H is 1st integral

$$\frac{\partial H}{\partial x} \cdot f_1 + \frac{\partial H}{\partial y} \cdot f_2 \overset{?}{=} 0 \Rightarrow \text{will be true}$$

$\lambda_{1,2} = \pm i w$, the eigenvalues corresponding for $\eta^*(0,0)$ $\Big\} \Rightarrow$
$\exists$ a first integral defined on $\mathbb{R}^2$

$\Rightarrow (0,0)$ stable