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V, V' K-vector spaces

$f \in \mathrm{Hom}_K(V, V')$

$B = (v_1, \dots, v_n)$ basis of V

$B' = (v_1', \dots, v_m')$ basis of V'

$$[f]_{BB'} = ([f(v_1)]_{B'}, \dots, [f(v_n)]_{B'})$$

$f_1, f_2 \in \mathrm{Hom}_K(V, V')$

$g \in \mathrm{Hom}_K(V', V'')$

B, B', B'' bases of V, V', V''

$$[f_1 + f_2]_{B,B'} = [f_1]_{BB'} + [f_2]_{BB'}$$

$\forall \alpha \in K$:

$$[\alpha f_1]_{BB'} = \alpha \cdot [f_1]_{BB'}$$

$$[g \circ f_1]_{B,B''} = [g]_{B',B''} \cdot [f_1]_{B,B'}$$

$[f_1 g]$ does not make sense!

Base change

V, V' - K v.s.

B_1, B_2 bases of V

B_1', B_2' bases of V'

$f \in Hom(V, V')$

$$[f]_{\underset{1}{B_1'}, \underset{4}{B_2'}} = \underbrace{[id_{V'}]_{\underset{3}{B_2}, \underset{4}{B_2'}}}_{T_{B_2', B_2}} \cdot [f]_{\underset{2}{B_1}, \underset{3}{B_2}} \cdot \underbrace{[id_V]_{\underset{1}{B_1'}, \underset{2}{B_1}}}_{T_{B_1, B_1'}}$$

$\xleftarrow{\qquad\qquad\qquad\qquad\qquad}$

order of operations

$\forall\, v \in V$

$$[v]_{\underset{2}{B'}} = [id]_{\underset{1}{B}\underset{2}{B'}} \cdot [v]_{\underset{1}{B}}$$

$[id]_{B, B'} = T_{B', B} =$ = base-change matrix from B' to B

! $[id]_{B, B'} = [id]_{B', B}^{-1}$

NOT TRUE IN GENERAL

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

$$[2f]_B = 2[f]_B = 2\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B$$

$$[g]_B = [id]_{B'B} \cdot [g]_{B'} \cdot [id]_{BB'}$$

$$[id]_{B'B} = ([v_1']_B , [v_2']_B)$$

$$v_1' = av_1 + bv_2 = a(1,2) + b(1,3)$$
$$= (a+b, 2a+3b) = (1,0)$$

$$\begin{cases} a+b = 1 & \Rightarrow a = 1-b & \Rightarrow \boxed{a = 3} \\ 2a+3b = 0 & \Rightarrow 2-2b+3b = 0 & \Rightarrow \boxed{b = -2} \end{cases}$$

$$v_2' = av_1 + bv_2 = a(1,2) + b(1,3)$$
$$= (a+b, 2a+3b) = (2,1)$$

$$a+b = 2 \Rightarrow a = 2-b \Rightarrow \boxed{a=5}$$

$$2a+3b = 1 \Rightarrow 4-2b+3b = 1 \Rightarrow \boxed{b = -3}$$

$$[id]_{B'B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \Rightarrow [id]_{BB'} = [id]^{-1}_{B'B} =$$

$$\det([id]_{B'B}) = -9+10 = 1 \neq 0$$

$$[id]^t_{B'B} = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}$$

$$[id]^*_{B'B} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = [id]^{-1}_{B'B} = [id]_{BB'}$$

$$[g]_B = [id]_{B'B} \cdot [g]_{B'} \cdot [id]_{BB'}$$

$$= \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -5 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix} = [g]_B$$

$$\Rightarrow [f+g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix} =$$

$$= \begin{pmatrix} -19 & 30 \\ 12 & 19 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$[f]_{B'} = [id]_{BB'} \cdot [f]_B \cdot [id]_{B'B}$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

Eigenvectors and eigenvalues

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x, y) = (3x + 3y, 2x + 4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Step 1: pick a convenient basis and write f in that basis.

↳ usually the canonical basis.

We pick $E = (e_1, e_2)$

$$[f]_E = \left([f(e_1)]_E , [f(e_2)]_E\right) = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} = A.$$

$$f(e_1) = f(1,0) = (3,2)$$

$$f(e_2) = f(0,1) = (3,2)$$

Step 1: Write the characteristic polinomial of the matrix

$[f]_E := A$

$$p_A(x) = \det(A - X \cdot I_n)$$

$$p_A(x) = \begin{vmatrix} 3-x & 3 \\ 2 & 4-x \end{vmatrix} = (3-x)(4-x) - 6 =$$

$$= 12 - 3x - 4x + x^2 - 6$$

$$= x^2 - 7x + 6$$

Step 3: The eigen values of f are the distinct roots of the characteristic polinomial.

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49 - 24}}{2} = \frac{7 \pm 5}{2} = \begin{cases} \lambda_1 = 6 \\ \lambda_2 = 1 \end{cases}$$

Step 5: For an eigen value $\lambda \in \mathbb{R}$, we have the eigen-space $S_\lambda = \{ x \in V \mid \varphi(x) = \lambda x \}$ = the set of eigen vectors corresponding to λ, along with 0.

0 is NOT considered an eigen-vector.

$\forall$ λ_1, λ_2 eigen values. $S(\lambda_1) \cap S(\lambda_2) = 0$

$$S(\lambda) = \{ v \in V \mid [f]_E \cdot [v]_E = \lambda \cdot [f]_E \}$$

if $\lambda = \lambda_1 = 6$

$$\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 6 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left\{ \begin{array}{l} 3x + 3y = 6x \Rightarrow 3y = 3x \\ 2x + 4y = 6y \Rightarrow 2x = 2y \end{array} \right\} \Rightarrow x = y.$$

$$\Rightarrow S(\lambda_1) = \{ (x, x) \mid x \in \mathbb{R} \} = \langle (1, 1) \rangle$$

if $\lambda = \lambda_2 = 1$

$$\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{aligned} 3x + 3y &= x \Rightarrow 2x + 3y = 0 \\ 2x + 4y &= y \Rightarrow 2x + 3y = 0 \end{aligned}\right\} \Rightarrow y = \frac{-2x}{3}$$

$$S(\lambda_2) = \left\{\left(x, \frac{-2x}{3}\right) \mid x \in \mathbb{R}\right\} = \left\langle\left(1, -\frac{2}{3}\right)\right\rangle =$$

$$= \langle (3, -2) \rangle$$

basis of eigen vectors is: $((1,1), (3,-2))$

$$[f]_B = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

We can check that $[f]_B$ is $[id]_{E,B} [f]_E [id]_{B,E}$

$$[id]_{B,E} = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}$$

$$[id]_{E,B} = [id]_{B,E}^{-1} \dots$$

5) $A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$

charact polin. $P_A(x) = \begin{vmatrix} 3-x & 1 & 0 \\ -4 & -1-x & 0 \\ -4 & -8 & -2-x \end{vmatrix} =$

$= (-2-x) \begin{vmatrix} 3-x & 1 \\ -4 & -1-x \end{vmatrix} = \dots = -(x+2)(x-1)^2 \Rightarrow$

$\Rightarrow \begin{cases} \lambda_1 = -2 \\ \lambda_2 = 1 \end{cases}$ we have one eigenspace for each

$S(\lambda_1) = \{ v \in$