10.10.2023

$r = (A, B, R)$

$X \subseteq A$

$r\langle X \rangle = \{ y \in A \mid \exists x \in X : x \, r \, y \}$

$r\langle x \rangle = \{ y \in A \mid y \, r \, x \}$

$x \, r \, y \Leftrightarrow x \leq y$

$r\langle 1 \rangle = [1, +\infty)$

$r\langle [1,2] \rangle = [1, +\infty)$

$$r\langle X \rangle = \bigcup_{x \in X} r\langle x \rangle$$

Choose $x \in [1,2]$ $\quad r\langle x \rangle = [x, +\infty)$

$$r([1,2]) = \bigcup_{x \in [1,2]} r\langle x \rangle = \bigcup_{x \in [1,2]} [x, \infty) = [1, \infty)$$

! when in doubt, go back to the definitions

$r = (A, B, R)$ relation

domain codomain graph

$R = \underbrace{A \times B}$

cartesian product (set of $x\, r\, y$ where $x \in A$ and $y \in B$)

↳ indicates which $a \in A$ and $b \in B$ are so that $a\, r\, b$

if $A = B$ → homogeneous relation

- reflexivity : $\forall x \in A : x\, r\, x$
- symmetry : $\forall x, y \in A : [\text{if } x\, r\, y \Rightarrow y\, r\, x]$
- transitivity : $\forall x, y, z \in A : [\text{if } x\, r\, y \text{ and } y\, r\, z \Rightarrow x\, r\, z]$

→ r equivalence relation

1)

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2,3,4,5,6\}$ by

$$x\,r\,y \iff x < y$$

$$x\,s\,y \iff x|y$$

$$x\,t\,y \iff g.c.d.(x,y) = 1$$

$$x\,v\,y \iff x \equiv y \pmod 3.$$

Write the graphs R, S, T, V of the given relations.

r, s, t, v hom. relations

$M = \{2, 3, 4, 5, 6\}$

by: $x\,r\,y \Leftrightarrow x < y$

$x\,s\,y \Leftrightarrow x | y$

$x\,t\,y \Leftrightarrow g.c.d(x,y) = 1$

$x\,v\,y \Leftrightarrow x \equiv y \pmod 3$ $(x \% 3 = y \% 3)$

$R = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$

$S = \{(2,4), (2,6), (3,6), \underline{(2,2), (3,3),}$
$\underline{(4,4), (5,5), (6,6)}\}$!

$T = \{(2,3), (2,5), (3,4), (4,5), (5,6),$
$(3,2), (5,2), (4,3), (5,4), (6,5)\}$

$V = \{(2,5), (3,6)\}$

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

only symetry

$M = \{2, 3, 4, 5, 6\}$

$x \rho y \Leftrightarrow x \neq y$

$M = \{2, 3, 4, 5, 6\}$

$x \rho y \Leftrightarrow |x-y| = 1$

only transitivity

$M = \{2, 3, 4, 5, 6\}$

$x \rho y \Leftrightarrow x < y$

only reflexivity

$M = \{2, 3, 4, 5, 6\}$

$x \rho y \Leftrightarrow x = y$ or $x - y = 1$

$R = \Delta_M \cup \{(3,2), (4,3), (5,4), (6,5)\}$

r, ~~s~~, ~~t~~ → only reflexivity

$M = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$

obs: if we omit $(2,3)$, r will be transitive so we put it to BREAK it!!

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r, s, ~~t~~ → only symmetry

$R = \{(1,2), (2,1)\}$

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t, ~~s~~, ~~r~~

$R = \{(1,2), (2,3), (1,3)\}$

or

$R = \{(1,2)\}$:)

Th: M set. There is a bijection:

$$\underset{\text{set of equivalences on } M}{\underset{\|}{E(M)}} \xleftrightarrow{\sim} \underset{\text{set of partitions on } M}{\underset{\|}{P(M)}}$$

$$\sim \longmapsto \underbrace{M/\sim}_{\text{quotient set} \Leftrightarrow \text{associate partition}}$$

$$\sim_{\mathcal{P}} \longleftarrow\!\!\shortmid \ \mathcal{P}$$

$$M/\sim \ \overset{\text{set of classes}}{=} \{ \underbrace{\sim\langle x\rangle}_{=: \hat{x}} \mid x \in M \}$$

$=: \hat{x}$ (just like residue classes)

$$\sim\langle x\rangle = \{ y \in M \mid x \sim y \}$$

$$x \sim_{\mathcal{P}} y \Leftrightarrow \exists A \in \mathcal{P}: A \ni x, y$$

2. Let A and B be sets with n and m elements respectively ($m, n \in \mathbb{N}^*$). Determine the number of:

(i) relations having the domain A and the codomain B;

(ii) homogeneous relations on A.

$M = \{1, 2, 3, 4\}$, r_1, r_2 hom. relations on M and we have $\pi_1, \pi_2 \subseteq \mathcal{P}(M)$

$R_1 = \Delta_M \cup \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

$\{(1,1), (2,2), (3,3), (4,4)\}$: diagonal elements

Δ (= greek letter for Δ)

$R_2 = \Delta_M \cup \{(1,2), (1,3)\}$

$\pi_1 = \{\{1\}, \{2\}, \{3,4\}\}$

$\pi_2 = \{\{1\}, \{1,2\}, \{3,4\}\}$

i) Are r_1, r_2 equivalences on M? If so, write the corresponding partition

ii) Are π_1, π_2 partitions of M? If so, write the graph of the corresponding equivalence

i) Are r_1, r_2 equivalences on M? If so, write the corresponding partition

r_1 = eq. relation (has r, s, t)

reflexive because $\Delta_M \subset R_1$ ✓

symmetric ✓

transitive ✓

$$\boxed{\begin{array}{l} M/r_1 = \{r\langle x\rangle,\ x \in M\} = \{\{1,2,3\},\{4\}\} \\ r_1\langle 1\rangle = \{1, 2, 3\} \\ r_1\langle 4\rangle = \{4\} \end{array}}$$

r_2 is not symmetric because $(1,2) \in R$ but $(2,1) \notin R_2$

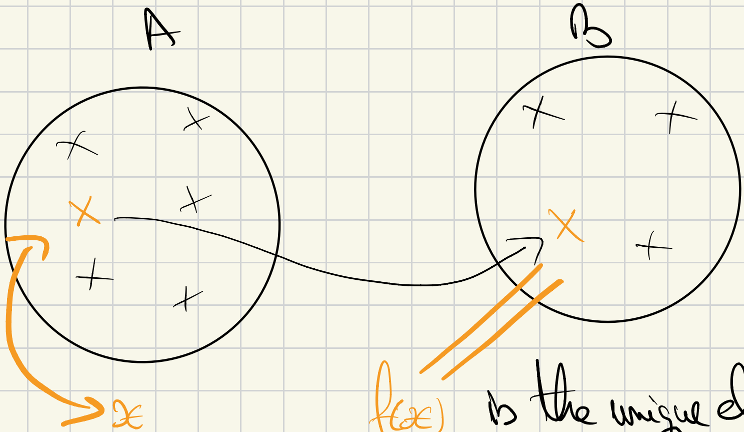
ii) $\left.\begin{array}{l} \{1\} \cup \{2\} \cup \{3,4\} = M \\ \forall A, B \in \pi_1,\ A \cap B = \emptyset \end{array}\right\} \Rightarrow \pi_1$ is a partition

$R_{\pi_1} = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$

π_2 is NOT a partition because $\{1\} \cap \{1,2\} \neq \emptyset$

$r = (A, B, R)$ relation

r function $\iff |r\langle x \rangle| = 1, \forall x \in A$



$f(x)$ is the unique element of $r\langle x \rangle$

9) $M = \{0, 1, 2, 3\}$, $h = (\mathbb{Z}, M, H)$ relation with

$H = \{(x,y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}$

is h a function?

$\forall x \in \mathbb{Z}, \exists y \in \{0,1,2,3\}$ so that $x \equiv y \pmod 4$

$\iff \forall x \in \mathbb{Z}, \exists y \in M : x - y = 4z$ ✓ $(\Leftrightarrow 4 \mid (x-y))$

$\Rightarrow h$ is a function