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Linear independence

V K-vector space

$v_1, v_2, \dots, v_n \in V$

are linearly independent if

$$\left.\begin{array}{l}\forall\, \alpha_1, \alpha_2, \dots, \alpha_n \in K \\ \alpha_1 v_1 + \dots + \alpha_n v_n = 0\end{array}\right\} \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

$\Leftrightarrow$ no vector can be a comb. of the others.

Shortcut

If $V = K^m$:

the number of lin. indep. vectors in $(v_1, \dots, v_n) =$

$$= \operatorname{rank}\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_n \end{pmatrix} = \operatorname{rank}\left(v_1^t \mid v_2^t \mid \dots \mid v_n^t\right)$$

$$\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline v_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

ex) $v_1 = (1,2,3,4)$ | $v_3 = (9,10,11,12)$

$v_2 = (5,6,7,8)$

$v_1, v_2, \dots, v_n$ are linearly independent iff

$$\operatorname{rank}\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_n \end{pmatrix} = n$$

6.1) $v_1 = (1, -1, 0)$

$v_2 = (2, 1, 1)$

$v_3 = (1, 5, 2)$

$\in \mathbb{R}^3$

a) Show that v_1, v_2, v_3 are linearly dependent and find the dependence relationship.

b) Show that v_1 and v_2 are linearly independent.

a) $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\Leftrightarrow (\alpha_1, -\alpha_1, 0) + (2\alpha_2, \alpha_2, \alpha_2) + (\alpha_3, 5\alpha_3, 2\alpha_3) = 0$

$\Leftrightarrow (\alpha_1 + 2\alpha_2 + \alpha_3, -\alpha_1 + \alpha_2 + 5\alpha_3, \alpha_2 + 2\alpha_3) = (0, 0, 0)$

$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + 5\alpha_3 = 0 \\ \alpha_2 + 2\alpha_3 = 0 \end{cases}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 5 \\ 0 & 1 & 2 \end{vmatrix} = 2 + 0 - 1 - 0 - 5 + 4 = 0$$

$$d_p = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 + 2 = 3 \neq 0$$

$$d_c = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$$\alpha_3 = \beta \Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = -\beta \\ -\alpha_1 + \alpha_2 = -5\beta \end{cases}$$

$$\alpha_2 = -2\beta$$
$$\alpha_1 = 3\beta$$

$S = \{ 3\beta, -2\beta, \beta \} \Rightarrow 1$ sol.

$\Rightarrow$ v_1, v_2, v_3 linearly dependent

$\beta = 1 \Rightarrow 3v_1 - 2v_2 + v_3 = 0$

b) $v_1 = (1, -1, 0)$

$v_2 = (2, 1, 1)$

independent $\Leftrightarrow$ $\alpha_1 v_1 + \alpha_2 v_2 = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$

$\alpha_1 v_1 + \alpha_2 v_2 = 0$

$(\alpha_1, -\alpha_1, 0) + (2\alpha_2, \alpha_2, \alpha_2) = 0$

$(\alpha_1 + 2\alpha_2, -\alpha_1 + \alpha_2, \alpha_2) = 0$

$$\begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \\ 0 + \alpha_2 = 0 \Rightarrow \alpha_2 = 0 \end{cases}$$

$\alpha_1 + 2\alpha_2 = 0$

$\alpha_1 = 0$

$\Rightarrow v_1, v_2$ are linearly independent

6.3) $v_1 = (1, a, 0)$

$v_2 = (a, 1, 1)$

$v_3 = (1, 0, a)$

$v_1, v_2, v_3 \in \mathbb{R}^3$

$a \in \mathbb{R}$ s.t. v_1, v_2, v_3 are linearly independent $\Leftrightarrow$

$\Leftrightarrow \quad \operatorname{rank}\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline v_3 \end{pmatrix} = 3 \Rightarrow \det\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline v_3 \end{pmatrix} \neq 0$

$$\begin{vmatrix} 1 & a & 0 \\ a & 1 & 1 \\ 1 & 0 & a \end{vmatrix} = (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} a & 0 \\ 1 & 1 \end{vmatrix} + (-1)^{3+2} \cdot 0 \cdot \begin{vmatrix} 1 & 0 \\ a & 1 \end{vmatrix} +$$

$$+ (-1)^{3+3} \cdot a \cdot \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = a + a(1-a^2) =$$

$$= 2a - a^3 = a(2-a^2) \neq 0 \Rightarrow a \in \mathbb{R} \setminus \{-\sqrt{2}, 0, \sqrt{2}\}$$

for $a \in \mathbb{R} \setminus \{-\sqrt{2}, 0, \sqrt{2}\}$, v_1, v_2, v_3 are linearly independent

V $\mathbb{K}$- vector space

$\mathcal{B} = (v_1, v_2, v_3, \ldots, v_n) \to$ ORDER MATTERS

$\mathcal{B}$ is a basis $\Leftrightarrow$ i) $V = \langle v_1, \ldots, v_n \rangle$

($\mathcal{B}$ is a list of generators)

ii) $v_1, \ldots, v_n$ linearly independent

Short version: $\forall v \in V \ \exists! \ \alpha_1, \ldots, \alpha_n \in \mathbb{K}$ s.t.

$$v = \alpha_1 \cdot v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

if so, we will denote by $[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

! $\to$ was at the midterm / exam.

6.8) $\mathbb{R}_2[x] = \{ f \in \mathbb{R}[x] \mid \deg f \leq 2 \}$

Show that the lists: $E = (1, x, x^2)$, $\mathcal{B} = (1, x-a, (x-a)^2)$ — lists of vectors!

are bases of $\mathbb{R}^2[x]$ and determined for any

$f = a_0 + a_1 x + a_2 x^2 \in \mathbb{R}_2[x]$

$[f]_E, [f]_{\mathcal{B}} = ?$ (coordinates)

$f \in \mathbb{R}_2[x]$

$\Rightarrow f = a_0 + a_1 x + a_2 x^2 \quad , \; a_0, a_1, a_2 \in \mathbb{R}$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t. $f = \alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3 \cdot x^2$

$\Leftrightarrow \quad a_0 + a_1 x + a_2 x^2 = \alpha_1 + \alpha_2 x + \alpha_3 x^2$

$\Leftrightarrow (a_0 - \alpha_1) + x(a_1 - \alpha_2) + x^2(a_2 - \alpha_3) = 0$

$$\Leftrightarrow \begin{cases} a_0 - \alpha_1 = 0 \\ a_1 - \alpha_2 = 0 \\ a_2 - \alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = a_0 \\ \alpha_2 = a_1 \\ \alpha_3 = a_2 \end{cases}$$

unique solution $\Rightarrow E$ is a basis of $\mathbb{R}_2[x]$

$$[f]_E = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$B = (1, (x-a), (x-a)^2)$

$f \in \mathbb{R}_2[x] \Rightarrow f = a_0 + a_1 x + a_2 x^2$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t.

$$f = \alpha_1 \cdot 1 + \alpha_2(x-a) + \alpha_3(x-a)^2$$

$$a_0 + a_1 x + a_2 x^2 = \alpha_1 + \alpha_2(x-a) + \alpha_3(x-a)^2$$

$$a_0 - \alpha_1 + a\alpha_2 - a^2\alpha_3 + x(a_1 - \alpha_2 + 2a\alpha_3) + x^2(a_2 - \alpha_3) = 0$$

$$\begin{cases} a_0 - \alpha_1 + a\alpha_2 - a^2\alpha_3 = 0 \\ a_1 - \alpha_2 + 2a\alpha_3 = 0 \\ a_2 - \alpha_3 = 0 \Rightarrow \alpha_3 = a_2 \end{cases}$$

$$\begin{cases} \alpha_3 = a_2 \\ a_0 - \alpha_1 + a\alpha_2 - a^2 \cdot a_2 = 0 \\ a_1 - \alpha_2 + 2a a_2 = 0 \end{cases}$$

$$\begin{cases} \alpha_3 = a \\ \alpha_2 = a_1 + 2a a_2 \\ a_0 - \alpha_1 + a a_1 - a^2 a = 0 \end{cases}$$

$$\Rightarrow \alpha_1 = a_0 + a a_1 + a^2 a_2$$

$\Rightarrow$ unique sol.

$\Rightarrow B$ is a basis of $\mathbb{R}_2[x]$

$$[f]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a_0 + a a_1 + a^2 a_2 \\ a_1 + 2a a_2 \\ a_2 \end{pmatrix}$$

6.7) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Prove that the list $B = (A_1, A_2, A_3, A_4)$ is a basis of $M_2(\mathbb{R})$

Determine $\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_B$ (the coordinates of the matrix)

Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$

so that $f = \alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 + \alpha_3 \cdot A_3 + \alpha_4 \cdot A_4$

$$f = \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \alpha_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_3 & \alpha_3 \\ \alpha_3 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_4 & \alpha_4 \\ \alpha_4 & \alpha_4 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \alpha_2+\alpha_3+\alpha_4 \\ +\alpha_3+\alpha_4 & \alpha_1+\alpha_4 \end{pmatrix}$$

$$\begin{cases} a = \alpha_1+\alpha_2+\alpha_3+\alpha_4 \\ b = \alpha_2+\alpha_3+\alpha_4 \\ c = \alpha_3+\alpha_4 \\ d = \alpha_1+\alpha_4 \end{cases}$$

$$\begin{cases} \alpha_1 = a-b \\ \alpha_2 = b-c \\ \alpha_3 = c-\alpha_4 \\ \alpha_4 = d-a+b \end{cases}$$

$$\begin{cases} \alpha_1 = a-b \\ \alpha_2 = b-c \\ \alpha_3 = c-d+a-b \\ \alpha_4 = d-a+b \end{cases}$$

is unique so therefore B is a basis

$$\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_B = \begin{pmatrix} 2-1 \\ 1-1 \\ 1-0+2-1 \\ 0-2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$