31. 10. 2023

V K-vectorspace

$X \subseteq V$

↳ subset

$$\langle X \rangle = \bigcap_{\substack{S \leq_K V \\ S \supseteq X}} S$$ ($\forall \bigcap$ of subspaces is a subspace)

"angle brackets" // chevrons

↳ subspace generated by X (the span of X)

another notation $\mathrm{Span}_K(X)$

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} a_i x_i \;\middle|\; \begin{matrix} n \in \mathbb{N} \\ x_i \in X \\ a_i \in K \end{matrix} \right\}$$

If X is finite, $X = \{x_1, x_2, \dots, x_n\}$

then $\langle X \rangle = \left\{ \sum_{i=1}^{n} a_i x_i \;\middle|\; \begin{matrix} x_i \in X \\ a_i \in K \end{matrix} \right.$

5.1) Determine the following generated subspaces

i) $< 1, x, x^2 > \subseteq_{\mathbb{R}} \mathbb{R}[x]$

ii) $< \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} > \subseteq_{\mathbb{R}} M_2(\mathbb{R})$

iii) $< (2,0,1), (1,2,0) > \subseteq_{\mathbb{R}} \mathbb{R}^3$

i) $Y = \{ 1, x, x^2 \}$

$$< Y > = \{ \sum_{i=1}^{3} a_i \cdot y_i \mid a_i \in \mathbb{R}, y_i \in Y \}$$

$$= \{ a_1 \cdot 1 + a_2 \cdot x + a_3 x^2 \mid a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= \{ f \in \mathbb{R}[x] \mid \deg(f) = 2 \}$$

$$= \mathbb{R}_2[x]$$

ii) $Z = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$

$$< Z > = \{ \sum_{i=1}^{3} a_i \cdot z_i \mid a_i \in \mathbb{R}, z_i \in Z \}$$

$$\langle Z \rangle = \{ a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$a_1, a_2, a_3 \in \mathbb{R} \}$$

$$\langle Z \rangle = \{ \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}, \; a_1, a_2, a_3 \in \mathbb{R} \}$$

$\langle Z \rangle = T_2(\mathbb{R}) \rightarrow$ upper triangular matrices

iii) $A = \{ (2,0,1), (1,2,0) \}$

$$\langle A \rangle = \{ k_1(2,0,1) + k_2(1,2,0) \mid k_1, k_2 \in \mathbb{R} \}$$

$$\langle A \rangle = \{ (2k_1 + k_2, 2k_2, k_1) \mid k_1, k_2 \in \mathbb{R} \}$$

$$= \{ (x, y, z) \mid x = 2z + \tfrac{1}{2} y \}$$

$x = 2k_1 + k_2$, $y = 2k_2$, $z = k_1$

5.2) i) $A = \{ (x, y, z) \mid x = 0 \}$

ii) $B = \{ (x, y, z) \mid x + y + z = 0 \}$

iii) $C = \{ (x, y, z) \mid x = y = z \}$

iv) $D = \{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \mid x + t = 0, y = 0 \}$

v) $E = \{ a_0 + a_1 x + a_2 x^2 \in \mathbb{R}_2[x] \mid a_1 + 2a_2 = 0 \}$

Write A, B, C, D, E as generated subspaces with a minimal number of generators.

i) $A = \{ (x, y, z) \mid x = 0 \}$

$A = \{ (0, y, z) \in \mathbb{R}^3 \}$

$= \{ (0, y, 0) + (0, 0, z) \mid y, z \in \mathbb{R} \}$

$= \{ y(0,1,0) + z(0,0,1) \mid y, z \in \mathbb{R} \}$

$= \langle (0,1,0), (0,0,1) \rangle$

Because $(0,1,0) \notin \langle (0,0,1) \rangle$ because

if $(0,1,0) \in \langle (0,0,1) \rangle$ then $\exists \alpha \in \mathbb{R}$

s.t. $(0,1,0) = \alpha \cdot (0,0,1) \Rightarrow \alpha = 0 \Leftrightarrow$

$\Leftrightarrow (0,1,0) = (0,0,0)$ contradiction

Therefore, we have a MINIMAL set of generators.

ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

$B = \{(x, y, z) \in \mathbb{R}^3 \mid z = -x - y\}$

$B = \{(x, y, -x-y) \in \mathbb{R}^3\}$

$B = \{(x, 0, -x) + (0, y, -y) \mid x, y \in \mathbb{R}\}$
$= \{x(1, 0, -1) + y(0, 1, -1) \mid x, y \in \mathbb{R}\}$

$= \langle (1, 0, -1), (0, 1, -1) \rangle$

the set of generators is minimal because
$(1, 0, -1) \notin \langle (0, 1, -1) \rangle$

iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$

$C = \{(x, x, x) \in \mathbb{R}^3\}$

$C = \{x(1, 1, 1) \mid x \in \mathbb{R}\}$

$C = \langle (1, 1, 1) \rangle$

iv) $D = \{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \mid \begin{matrix} x+t=0 \\ y=0 \end{matrix} \}$

$D = \{ \begin{pmatrix} x & 0 \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \mid x=-t \}$

$D = \{ \begin{pmatrix} -t & 0 \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \}$

$D = \{ \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \mid t, z \in \mathbb{R} \}$

$D = \{ t \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mid t, z \in \mathbb{R} \}$

$D = < \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} >$ minimal bas.

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \notin < \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} >$

v) $E = \{ a_0 + a_1 x + a_2 x^2 \in \mathbb{R}_2[x] \mid a_1 + 2a_2 = 0 \}$

$E = \{ a_0 + (-2a_2) x + a_2 x^2 \in \mathbb{R}_2[x] \}$

$E = \{ a_0 + a_2(-2x + x^2) \in \mathbb{R}_2[x] \}$

$E = < 1 ; -2x + x^2 >$

V_1, V_2 K- vector spaces

$f: V_1 \to V_2$ is a K-homomorphism of vector spaces ($:=$ K-linear map)

K-linear map if:

1) $\forall\, v_1, v_2 \in V$, $f(v_1+v_2) = f(v_1) + f(v_2)$

2) $\forall\, \alpha \in K$, $\forall\, v \in V$, $f(\alpha \cdot v) = \alpha \cdot f(v)$

short version) $\forall\, \alpha_1, \alpha_2 \in K$, $\forall\, v_1, v_2 \in V$, $f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$

identity element of V_2

$\ker f = \{ v \in V_1 \mid f(v) = 0_{V_2} \} \leq_K V_1$

$\operatorname{Im} f = \{ f(v) \mid v \in V_1 \} \leq_K V_2$

5.6) $f, g : \mathbb{R}^2 \to \mathbb{R}^2$

$h : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x,y) = (x+y,\ x-y)$$

$$g(x,y) = (2x-y,\ 4x-2y)$$

$$h(x,y,z) = (x-y,\ y-z,\ z-x)$$

show that f, g are endomorphism of $\mathbb{R}^2$

— " — h — " — $\mathbb{R}^3$

5.5) Find generators for the kernels and images of f, g, h

<u>Solution</u>

<u>Function f:</u>

$$f(x,y) = (x+y,\ x-y)$$

$$\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$$

$$f(x_1+x_2, y_1+y_2) \overset{?}{=} f(x_1,y_1) + f(x_2,y_2)$$

$$(x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2) \overset{?}{=} (x_1+y_1, x_1-y_1) + (x_2+y_2, x_2-y_2)$$

$$\boxed{\begin{array}{l}(x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2) \overset{?}{=} (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2), \\ \forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2\end{array}}$$

$$f(\alpha x_1, \alpha y_1) \overset{?}{=} \alpha f(x_1,y_1)$$

$$(\alpha x_1+\alpha y_1, \alpha x_1-\alpha y_1) \overset{?}{=} \alpha(x_1+y_1, x_1-y_1)$$

$$\boxed{\begin{array}{l}\alpha(x_1+y_1, x_1-y_1) \overset{?}{=} \alpha(x_1+y_1, x_1-y_1), \\ \forall (x_1,y_1) \in \mathbb{R}^2 \text{ and } \forall \alpha \in \mathbb{R}.\end{array}}$$

$$\Longrightarrow \boxed{f \text{ is a } K\text{-linear map.}} \qquad f \in End_{\mathbb{R}}(\mathbb{R}^2)$$

$$\mathrm{Ker} f = \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) = (0,0) \}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid (x+y, x-y) = (0,0)\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid \begin{array}{l} x+y=0 \\ x-y=0 \end{array}\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid x=y=0\} = \{(0,0)\} =$$

$$= \langle \emptyset \rangle$$

$$\operatorname{Im} f = \{ f_{(x,y)} \mid x, y \in \mathbb{R}\}$$

$$= \{(x+y, x-y) \mid x, y \in \mathbb{R}\}$$

$$= \{x(1,1) + y(1,-1) \mid x, y \in \mathbb{R}\}$$

$$= \langle (1,1), (1,-1) \rangle$$