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V, V' K-v.s.

$B = (v_1, v_2, \dots, v_n)$ basis of V

$B' = (v_1', v_2', \dots, v_m')$ basis of V'

$f : V \to V'$ linear map

column vector

$$[f]_{B,B'} = \begin{pmatrix} [f(v_1)]_{B'} & \dots & [f(v_n)]_{B'} \\ & & \\ & & \end{pmatrix}$$

$m \times n$ matrix

$w = a_1 v_1' + \dots + a_m v_m'$

$$[w]_{B'} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

$\forall$ linear map is a multiplication of a matrix

$$\boxed{\begin{array}{l} \forall v \in V \\ [f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B \end{array}}$$

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x,y,z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1,1,0),(0,1,1),(1,0,1))$ of $\mathbb{R}^3$, $B' = (v'_1, v'_2) = ((1,1),(1,-2))$ of $\mathbb{R}^2$ and let $E' = (e'_1, e'_2)$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

$e'_1 = (1,0)$, $e'_2 = (0,1)$

$$B = ((1,1,0),(0,1,1),(1,0,1))$$
$$B' = ((1,1),(1,-2))$$
$$E' = ((1,0),(0,1))$$

$$f(v_1) = f(1,1,0) = (1,-1)$$
$$f(v_2) = f(0,1,1) = (1,0)$$
$$f(v_3) = f(1,0,1) = (0,-1)$$

- $(1,-1) = a(1,0) + b(0,1)$

$$(1,-1) = (a,0) + (0,b)$$
$$(1,-1) = (a,b) \Rightarrow a = 1,\ b = -1$$

$$[f(v_1)]_{E'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- $(1,0) = a(1,0) + b(0,1)$

$$a = 1,\ b = 0 \Rightarrow [f(v_2)]_{E'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $(0,-1) = a(1,0) + b(0,1)$

  $a = 0,\ b = -1 \Rightarrow [f(v_3)]_{E'} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$\boxed{[f]_{BE'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}}$$

- $f(v_1) = (1,-1) = a(1,1) + b(1,-2)$

  $= (a+b,\ a-2b)$

$$\begin{cases} a+b=1 \\ a-2b=-1 \Rightarrow a = -1+2b \end{cases}$$

$-1+2b+b=1 \Rightarrow 3b = 2 \Rightarrow b = \frac{2}{3} \Rightarrow a = \frac{1}{3}$

$[f(v_1)]_{B'} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

- $f(v_2) = (1,0) \Rightarrow a(1,1) + b(1,-2) = (a+b, a-2b)$

$$\begin{cases} a+b=1 \\ a-2b=0 \Rightarrow a=2b \end{cases}$$

$3b = 1 \Rightarrow b = \frac{1}{3} \Rightarrow a = \frac{2}{3}$

$[f(v_2)]_{B'} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

- $l(v_3) = (0,-1) \Rightarrow a(1,1)+b(1,-2) =$
$= (a+b,\ a-2b)$

$$\begin{cases} a+b=0 \Rightarrow a=-b \\ a-2b=-1 \end{cases} \Rightarrow 3a=-1 \Rightarrow a=-\frac{1}{3} \Rightarrow$$

$\Rightarrow b=\frac{1}{3}$

$$[l(v_3)]_{B'} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$[l]_{B,B'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1,4,1,-1) \in Ker\, f$ and $v' = (2,-2,4,2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}$$

$E = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))$

i) Show that $v(1, 4, -1, 1) \in \mathrm{Ker} f$ and
$v'(2, -2, 4, 2) \in \mathrm{Im} f$.

ii) Det. a basis and the dim for $\mathrm{Ker} f$ and $\mathrm{Im} f$

iii) Define f, i.e. find $f(x, y, z, t)$

i) $v \in \mathrm{Ker} f \Leftrightarrow f(v) = 0 \Leftrightarrow [f(v)]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow [f]_E \cdot [v]_E = 0$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$v' \in \mathrm{Im} f \Rightarrow \exists u = (x, y, z, t),\ f(u) = v' \Leftrightarrow$

$\Leftrightarrow \exists u = (x, y, z, t) : [f(u)]_E = [v']_E$

$\Leftrightarrow \exists u = (x, y, z, t)$ s.t. $\underbrace{[f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}}_{\exists x, y, z, t \Leftrightarrow \text{ we have a comp. system!}}$

system

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 = L_3 - 2L_1 \\ L_4 = L_4 - L_1}}{\overset{L_2 = L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 2 \end{array}\right)$$

$$\underset{L_4 = L_4 - \frac{1}{2}L_2}{\overset{L_3 = L_3 + \frac{1}{2}L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

row echelon form $\Rightarrow$ the syst. is compatible $\Rightarrow \exists u : f(u) = v' \Rightarrow$

$\Rightarrow v' \in \mathrm{Ker} f$

$$\mathrm{Ker} f = \{ v = (x, y, z, t) \mid f(v) = 0 \} \Leftrightarrow$$

$$\Leftrightarrow \mathrm{Ker} f = \{ v = (x, y, z, t) \mid [f]_E \cdot [v]_E = 0 \}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}} \left(\begin{array}{ccccc} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x + y - 3z + 2t = 0 \\ \quad 2y - 2z + 6t = 0 \end{cases}$$

$z = \alpha$

$t = \beta$

$x + y = 3\alpha - 2\beta$

$2y = 2\alpha - 6\beta \quad \Rightarrow y = \alpha - 3\beta \Rightarrow x = 2\alpha + \beta$

$\operatorname{Ker} f = \{(2\alpha + \beta, \alpha - 3\beta, \alpha, \beta), \ \alpha, \beta \in \mathbb{R}\}$

$= \{(2\alpha, \alpha, \alpha, 0) + (\beta, -3\beta, 0, \beta), \ \alpha, \beta \in \mathbb{R}\}$

$= \{\alpha(2, 1, 1, 0) + \beta(1, -3, 0, 1) \mid \alpha, \beta \in \mathbb{R}\}$

$= \langle (2, 1, 1, 0), (1, -3, 0, 1) \rangle$

Given that they are not proportional $\Rightarrow$ vect. are lin. indep. $\Rightarrow$

$\Rightarrow \mathcal{B} = \{(2, 1, 1, 0), (1, -3, 0, 1)\}$ basis of $\operatorname{Ker} f$

$\Rightarrow \dim \operatorname{Ker} f = 2.$

$\operatorname{Im} f = \{ w = (a,b,c,d) \mid \exists\, u = (x,y,z,t)$ s.t. $f(u) = w \}$

$\Leftrightarrow \quad w = (a,b,c,d) \mid$

$\exists\, u = (x,y,z,t)$ s.t. $[f]_E \cdot [u]_E = [w]_E$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{\substack{L_3 = L_3 - 2L_1 \\ L_4 = L_4 - L_1}}{\overset{L_2 = L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right)$$

$$\overset{L_2 \leftrightarrow L_3}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & 1 & -1 & 3 & d-a \end{array}\right) \underset{L_4 = L_4 + L_2}{\overset{L_3 = L_3 + 2L_2}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 0 & 0 & 0 & -3a+b+2c \\ 0 & 0 & 0 & 0 & -3a+c+d \end{array}\right)$$

In order for the syst. to be compatible $\Longrightarrow$

$$\Rightarrow \begin{cases} -3a + b + 2c = 0 \Rightarrow b = 3a - 2c \\ -3a + c + d = 0 \Rightarrow d = 3a - c \end{cases}$$

$$\operatorname{Im} f = \{ (a, 3a-2c, c, 3a-c) \mid a, c \in \mathbb{R} \}$$

$$= \langle (1, 3, 0, 3), (0, -2, 1, -1) \rangle$$

Since the vectors are not proportional $\Rightarrow$ basis. $\Rightarrow$

$\Rightarrow \dim \operatorname{Im} f = 2$

$$f(x, y, z, t) = ?$$

$$[f(x, y, z, t)]_E = [f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x + y - 3z + t \\ -x + y + z + 4t \\ 2x + y - 5z + t \\ x + 2y - 4z + 5t \end{pmatrix}$$