8.04.2024

$\dot{x} = -k(x-21), \forall k > 0$

a) Find its flow $(\varphi(t, \eta))$

$\eta \in \mathbb{R}$ consider the IVP

$$\begin{cases} x = -k(x-21) \\ x(0) = \eta \end{cases}$$

$x' = kx + 21k$

linear, non-hom.

1) Solve the homogeneous part

$$\underbrace{x' + kx}_{\text{hom. part}} = \underbrace{21k}_{\text{non-hom. part}}$$

$x' + kx = 0$ 1^{st} order l.h with const. coef.

$r + k = 0 \Rightarrow r = -k \longmapsto e^{-kt}$

$x_h = c \cdot e^{-kt}$

$x_p = ?$ The non-h part is a const. function $\Rightarrow$

$\Rightarrow$ Look for a $\underline{\text{constant}}$ partial solution

$x = 21$

$$\boxed{x = x_h + x_p}$$

$$x = c_1 \cdot e^{-kt} + 21 , \quad \forall c_1 \in \mathbb{R}$$

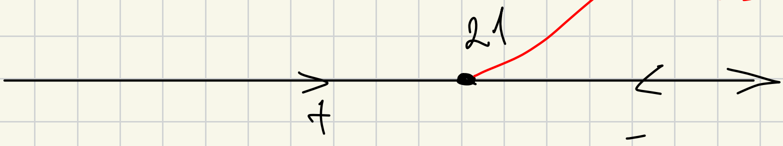
$$x(0) = c_1 + 21 = \eta \Rightarrow c_1 = \eta - 21$$

$$\varphi(t, \eta) = (\eta - 21) \cdot e^{-kt} + 21$$

flow

b) represent its phase portrait

equilibrium point (const. sol.)



derivative sign depending on x ⇒

⇒ we get the monotony

c) $x = -k(x-21)$

$x(0) = 43$

$x(10) = 37$

initial temp.

$\varphi(10', 49) = 37$ → temp after 10 mins

↳ time after 10 minutes

$(49-21)\cdot e^{-k\cdot 10} + 21 = 37$

$28 \cdot e^{-10k} = 16 \iff e^{-10k} = \frac{\overset{4}{16}}{28} = \frac{4}{7} \iff$

$\iff -10k = \ln\frac{4}{7} \iff k = -\frac{1}{10}\cdot \ln\frac{4}{7}$

$n = ?$ s.t. $\varphi(20', n) = 37$

$(n-21)\cdot e^{\frac{1}{\cancel{10}}\cdot \ln\left(\frac{4}{7}\right)\cdot \overset{t}{\cancel{20}}\,2} + 21 = 37$

$(n-21)\cdot e^{2\cdot \ln\frac{4}{7}} = 16$

$(n-21)\cdot \cancel{e}^{\cancel{\ln}\left(\frac{4}{7}\right)^2} = 16$

$(n-21)\cdot \frac{16}{49} = 16$

$n-21 = 49$

$n = 70$

The linearization method

Let $f \in C^1(\mathbb{R})$ and $x^* \in \mathbb{R}$ be s.t. $f(x^*) = 0$

If $f'(x^*) < 0$, then x^* is an attractor eq. p. of $\dot{x} = f(x)$

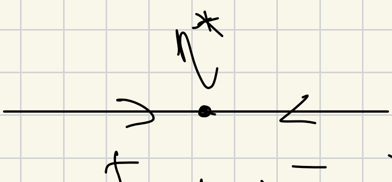
equilibrium point

If $f'(x^*) > 0$, then x^* is a repeller eq. p. of $\dot{x} = f(x)$

<u>Proof</u> (sketch)

Assume that $f'(x^*) < 0$

x	x^*
$f'(x^*)$	$-$ $-$ $-$ $-$
f	$+$ 0 $-$



the phase portrait $\Rightarrow$ x^* is an attractor.

2. Let $0 < c < 1$ be a parameter and consider the scalar dynamical system $\dot{x} = x(1-x) - cx$.

~~a)~~ Find its equilibria and study their stability using the linearization method.

~~b)~~ Represent its phase portrait. ↳ attractor or repeller

c) When $x(t) > 0$ is considered to be the density of fish in a lake, and $0 < c < 1$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a) and b). ◇

$$\dot{x} = x(1-x) - cx$$

$$f(x) = x(1-x) - cx$$

$$f(x) = 0$$

$$x - x^2 - cx = 0$$

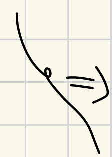
$$x(1 - x - c) = 0$$

$$x_1 = \eta_1^* = 0$$

$$\eta_2^* = 1 - c$$

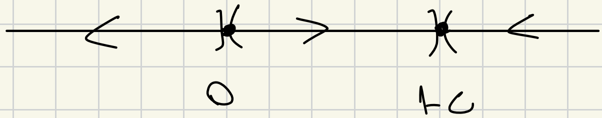
$$f'(x) = (x - x^2 - cx)' = 1 - 2x - c$$

$$f'(0) = 1 - c > 0$$



$$f'(1-c) = 1 - 2(1-c) - c) = 1 - 2 + 2c - c = -1 + c < 0$$

$$\Rightarrow \begin{cases} \eta_1^* \text{ is a repeller equilibrium point} \\ \eta_2^* \text{ is an attractor equilibrium point} \end{cases}$$



From the beginning

$l(x) = x(1 - x - c)$

x		0		$1-c$	
l	$- -$	0	$+ + +$	0	$- - -$

c) What happens when initially, we have a density $n > 1-c$?

Optimum a-la-long $\to$ start with a small ammount n and keep c small.

3. Represent the phase portrait of the scalar dynamical system $\dot{x} = x - x^3$. Find $\varphi(t,-1)$ and $\varphi(t,0)$ and justify. Specify the properties of the functions $\varphi(t,-2)$, $\varphi(t,3)$ and, respectively, $\varphi(t,-0.5)$.

$$\dot{x} = x - x^3$$

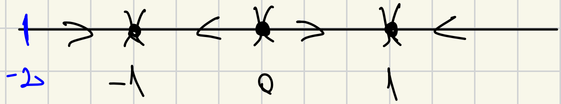
$$l(x) = x(1 - x^2) = 0$$

$$x_1 = 0 \; ; \; x_2 = -1 ; \; x_3 = 1 \quad (3 \text{ eq. points})$$

$f'(x) = -3x^2 + 1$

$f'(0) = 1 > 0 \Rightarrow$ repeller

$$\left.\begin{array}{l} f'(-1) = -2 < 0 \\ f'(1) = -2 < 0 \end{array}\right\} \text{attractor}$$



η^* is an eq. point $\Leftrightarrow \varphi(t, \eta^*) = \eta^*, \forall t \in \mathbb{R}$

$\varphi(t, \eta)$ is the unique sol of the ivp $\begin{cases} x' = x - x^3 \\ x(0) = \eta \end{cases}$

$\eta^* = -1 \Rightarrow \varphi(t, -1) = -1, \forall t \in \mathbb{R}$

$\eta^* = 0 \Rightarrow \varphi(t, 0) = 0, \forall t \in \mathbb{R}$

$\eta^* = -2 \Rightarrow$ has $\text{Im} f(t, -2) = (-\infty, -1)$ and it is strictly increasing

$\eta^* = 3 \Rightarrow$ has $\text{Im} f(t, 3) = (1, +\infty)$ and is strictly decreasing ; $\lim_{t \to \infty} \varphi(t, 3) = 1$

$\eta^* = -\frac{1}{2} \Rightarrow$ has Im $f(t, -\frac{1}{2}) = (-1, 0)$, strictly decreasing

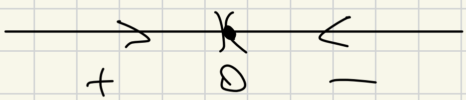
$$\lim_{t \to \infty} \varphi(t, -\tfrac{1}{2}) = -1$$

$$\lim_{t \to -\infty} \varphi(t, -\tfrac{1}{2}) = 0$$

Examples

$f'(\eta^*) = 0$? $\dot{x} = f(x)$

1) $\dot{x} = -x^3$

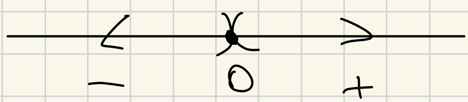


$f(x) = -x^3$

$f(x) = 0 \Rightarrow x = 0$, $f'(x) = -3x^2$

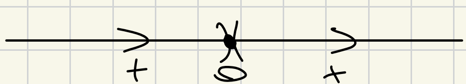
$f'(0) = 0 \rightarrow 0$ is attractor

2) $\dot{x} = x^3$



$f'(0) = 0 \rightarrow 0$ is repeller

3) $\dot{x} = x^2$ $f(x) = x^2$ $f(0) = 0$, $f'(0) = 0$



neither attractor or repeller

4) $\dot{x} = -x^2$