Sem 6 → Test (1 h) - 10 points
Lab 7 → Test (2 h) - 15 points

const. coef.

11.03.2024

LHDE's with CC. The characteristic equation method.

1) Find the general sol. of

a) $x' + 6x = 0$

b) $x'' + 5x' + 6x = 0$

c) $x'' + 4x' + 4x = 0$

d) $x''' = 0$

e) $x'' + x' + x = 0$

f) $x^{(IV)} - x = 0$

LHDE with CC ⇒ we can apply the charac. meth.

associate function

a) $r + 6 = 0 \Rightarrow r = -6 \longmapsto e^{-6t}$

$r = \alpha \in \mathbb{R} \longmapsto e^{\alpha t}$

b) $r^2 + 5r + 6 = 0 \Leftrightarrow (r+2)(r+3) = 0 \Rightarrow$

$$\Rightarrow \begin{cases} r_1 = -2 \longmapsto e^{-2t} \\ r_2 = -3 \longmapsto e^{-3t} \end{cases} \Rightarrow x = c_1 \cdot e^{-2t} + c_2 e^{-3t},$$

$c_1, c_2 \in \mathbb{R}$

c) $r^2 + 4x + 4 = 0 \Leftrightarrow (r+2)^2 = 0 \Rightarrow$

$\Rightarrow r_{1,2} = -2$ (double root) $\longmapsto e^{-2t}, t \cdot e^{-2t} \Rightarrow$

$$\Rightarrow x = c_1 \cdot e^{-2t} + c_2 \cdot t \cdot e^{-2t}, \quad c_1, c_2 \in \mathbb{R}$$

d) $r^3 = 0$

$r_{1,2,3} = 0$ (triple root) $\longmapsto \underbrace{e^{0 \cdot t}}_{1}, \underbrace{t \cdot e^{0 \cdot t}}_{t}, \underbrace{t^2 \cdot e^{0 \cdot t}}_{t^2}$

$$x = c_1 + c_2 t + c_3 t^2, \quad c_1, c_2, c_3 \in \mathbb{R}$$

OBS: We can solve this by integrating 3 times

$x'' = c_1, \quad x' = c_1 t + c_2, \quad x = c_1 \frac{t^2}{2} + c_2 t + c_3$

$c_1, c_2, c_3 \in \mathbb{R}$.

e) $x'' + x' + x = 0$

$r^2 + r + 1 = 0$

$\Delta = 1 - 4 \cdot 1 \cdot 1 = -3$

$r_{1,2} = \dfrac{-1 \pm i\sqrt{3}}{2} = -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$

$\alpha = -\dfrac{1}{2}, \ \beta = \dfrac{\sqrt{3}}{2}, \ m = 1 \longmapsto e^{\alpha t} \cdot \cos \beta t, \ e^{\alpha t} \cdot \sin \beta t$

$\longmapsto \underbrace{e^{-\frac{1}{2}t} \cdot \cos \frac{\sqrt{3}}{2} \cdot t}_{A}, \ \underbrace{e^{-\frac{1}{2}t} \cdot \sin \frac{\sqrt{3}}{2} \cdot t}_{B}$

$x = c_1 \cdot A + c_2 \cdot B, \ c_1, c_2 \in \mathbb{R}$

f) $x^{(IV)} - x = 0$

$r^4 - 1 = 0$

$(r^2 - 1)(r^2 + 1) = 0$

$(r-1)(r+1)(r-i)(r+i) = 0$

$r_1 = 1 \longmapsto e^{t}$

$r_2 = -1 \longmapsto e^{-t}$

$r_{3,4} = i \xmapsto[m=1]{\alpha=0,\beta=1} e^{0}\cdot \cos t \ , \ e^{0}\cdot \sin t$

$X = c_1 \cdot e^{t} + c_2 e^{-t} + c_3 \cdot \cos t + c_4 \cdot \sin t$

2) Find the sol. of the IVP (initial value problem)

$$\begin{cases} x'' + x = 0 \\ x\left(\frac{\pi}{2}\right) = 1 \\ x'\left(\frac{\pi}{2}\right) = -2 \end{cases}$$

$r^2 + 1 = 0$

$r_1 = \pm i \xmapsto[m=1]{\alpha=0,\beta=1} e^{0}\cdot \cos t \ , \ e^{0}\cdot \sin t$

$x\left(\frac{\pi}{2}\right) = c_1 \cdot \sin \frac{\pi}{2} + c_2 \cos \frac{\pi}{2} = 1$

$c_1 + 0 = 1 \Rightarrow c_1 = 1$

$x' = c_1 \cdot \cos t - c_2 \cdot \sin t$

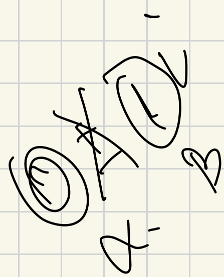
$$x'\left(\frac{\pi}{2}\right) = c_1 \cdot \cos\frac{\pi}{2} - c_2 \cdot \sin\frac{\pi}{2} = -2$$

$$-c_2 = -2 \Rightarrow c_2 = 2$$

$$x = \sin t + 2 \cdot \cos t$$

3) Find the sol. of the BVP

boundary value problem



a) $\begin{cases} x'' + x = 0 \\ x(0) = x(\pi) = 0 \end{cases}$

$$r^2 + 1 = 0$$

$$r_{1,2} = \pm i \xrightarrow[m=1]{\alpha=0,\ \beta=1} e^{0\cdot t} \cdot \cos t,\ e^{0\cdot t} \cdot \sin t$$

$$x = c_1 \cdot \cos t + c_2 \cdot \sin t$$

$$x(0) = c_1 = 0 \Rightarrow c_1 = 0;\quad x(\pi) = c_1 \cdot \cos\pi + c_2 \cdot \sin\pi = 0$$

$$\Rightarrow x = c \cdot \sin t,\ c \in \mathbb{R} \quad \Big\{ \Rightarrow c_2 \cdot 0 = 0 \Rightarrow c_2 \in \mathbb{R} \Rightarrow$$

b) $\begin{cases} x'' + x = 0 \\ x(0) = x(1) = 0 \end{cases}$

$r^2 + 1 = 0$

$r_{1,2} = \pm i \xrightarrow[m=1]{\alpha=0,\ \beta=1} e^{0\cdot t}\cdot\cos t,\ e^{0\cdot t}\cdot\sin t$

$x = c_1 \cdot \cos t + c_2 \cdot \sin t$

$x(0) = c_1 = 0 \Rightarrow c_1 = 0$

$x(1) = c_1 \cdot \cos 1 + c_2 \cdot \sin 1 \Rightarrow c_2 \cdot \sin 1 = 0 \Rightarrow c_2 = 0$

$x = 0$

4) Find the LHDE with CC of minimal order that has as solutions the following functions:

a) e^{-3t} and e^{5t}

b) $\sin(2t) \Rightarrow \alpha = 0,\ \beta = 2$

c) $2\cdot e^{-3t} + 5e^{5t}$ (reduces to a))

d) $t\cos t$ comes from a double root!

write the general sol. of each eq. that you found

a) $(r-5)(r+3)=0$

$r^2+3r-5r-15=0$

$r^2-2r-15=0 \iff x''-2x'-15x=0$

$x = c_1 \cdot e^{-3t} + c_2 \cdot e^{5t}$

b) $e^{\alpha t} \cdot \sin \beta t \Rightarrow \alpha=0,\ \beta=2$

$\left.\begin{array}{l} r_1 = +2i \\ r_2 = -2i \end{array}\right\} \Rightarrow$

$\Rightarrow (r+2i)(r-2i) \Rightarrow r^2+4=0 \Rightarrow x''+4x=0$

$x = c_1 \cdot \cos 2t + c_2 \cdot \sin 2t$

d) $\cos t = e^{\alpha t} \cdot \cos \beta t \Rightarrow \alpha=0,\ \beta=1 \Rightarrow$

$\Rightarrow$ $\alpha + i\beta$ is a root of the charac. eq, but

$t \cdot \cos t \mapsto i$ which is a double root $\Rightarrow -i$ is also double root

$$(r-i)^2 \cdot (r+i)^2 = 0$$

$$(r^2+i)^2 = r^4+2r^2+1=0$$

$$x^{(IV)} + 2x'' + x = 0$$

$\pm i$ are double roots $\longmapsto$ $\cos t, \sin t, t\cos t, t\sin t.$

$$x = c_1 \cdot \cos t + c_2 \cdot \sin t + c_3 t \cdot \cos t + c_4 t \cdot \sin t$$

5) Find necessary and sufficient conditions for the params. $\mu \in \mathbb{R}$ and $\omega > 0$ s.t. any sol. of

$$x'' + \mu x' + \omega^2 x = 0$$

satisfies $\lim_{t\to\infty} x(t) = 0$

$e^{\alpha t}$: $\lim_{t\to\infty} e^{\alpha t} = 0 \Leftrightarrow \alpha < 0$

$$\lim_{t\to\infty} e^{\alpha t} \cdot \cos \beta t = 0 \Leftrightarrow \alpha < 0$$

$$\lim_{t\to\infty} e^{\alpha t} \cdot \sin \beta t = 0 \Leftrightarrow \alpha < 0$$

$$x'' + \mu x' + \omega^2 x = 0$$

$$r^2 + \mu r + \omega^2 = 0$$

$$\Delta = \mu^2 - 4 \cdot 1^2 \cdot \omega^2$$

$\underline{C_1}$ $\Delta > 0$ $\quad r_1 = \alpha_1 \in \mathbb{R}$

$\quad r_2 = \alpha_2 \in \mathbb{R}$

$$x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}, \quad c_1, c_2 \in \mathbb{R}$$

$$\lim_{t \to \infty} x(t) = 0 \iff \alpha_1 < 0 \text{ and } \alpha_2 < 0$$

$$\iff \mu > 0$$

$$\alpha_1 + \alpha_2 = -\mu$$

$$\alpha_1 \alpha_2 = \omega^2 > 0$$

$\underline{C_2}$ $\Delta = 0$ $\quad r_1 = r_2 = -\frac{\mu}{2}$

$$x = c_1 e^{-\frac{\mu}{2} t} + c_2 t e^{-\frac{\mu}{2} t}$$

$$\lim_{t \to \infty} x(t) = 0 \iff \mu > 0$$

$\underline{C_3}$ $\Delta < 0$ $\quad r_{1,2} = \alpha \pm i\beta$

Viette

$$r_1 + r_2 = 2\alpha \overset{\text{Viette}}{=} -\mu \quad , \quad r_1 \cdot r_2 = \alpha^2 + \beta^2$$

$$x = c_1 e^{\alpha t} \cdot \cos \beta t + c_2 \cdot e^{\alpha t} \cdot \sin \beta t$$

$$\lim_{t \to \infty} x(t) = 0 \iff \alpha < 0 \Rightarrow \mu > 0$$

<u>Conclusion</u> : $\mu > 0$

b) Find necessary and sufficient cond. for $\mu \in \mathbb{R}$ and $\omega > 0$ s.t. any sol. of $x'' + \mu x' + \omega^2 = 0$ is periodic.

$\underline{C_1}$ $\Delta > 0$

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$

<u>no periodic sol.</u>

$\underline{C_2}$ $\Delta = 0 \Rightarrow$ no periodic sol.

$\underline{C_3}$ $\Delta < 0$ $\lambda_{1,2} = \alpha \pm i\beta$

we have periodic sol. $\Leftrightarrow \alpha = 0 \Leftrightarrow \mu = 0$

$\Delta = \mu^2 - 4\omega^2$

<u>Conclusion</u>: $\mu = 0$.