1) a) $\displaystyle\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}$

Ratio test: $\displaystyle\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{1\cdot 3\cdot \ldots (2n-1)}(2n+1)}{\cancel{2\cdot 4\cdot \ldots \cdot 2n}(2n+2)} \cdot \frac{\cancel{2\cdot 4\cdot \ldots \cdot 2n}}{\cancel{1\cdot 3\cdot \ldots \cdot (2n-1)}}$

$\displaystyle = \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1$ inconclusive $\Longrightarrow$

$\Rightarrow$ R-D test.

$\displaystyle\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right) =$

$\displaystyle = \lim_{n\to\infty} n \cdot \left(\cancel{1} + \frac{1}{2n+1} \cancel{-1}\right) = \frac{1}{2} < 1 \Rightarrow$

$\displaystyle\Rightarrow \sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}$ is DIVERGENT.

c) $\displaystyle\sum_{n\geq 1} a^{\ln n}, \ a>0$

Ratio test: $\displaystyle\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{\ln(n+1)}}{a^{\ln n}} = \lim_{n\to\infty} a^{\ln(n+1) - \ln n} =$

$\displaystyle = \lim_{n\to\infty} a^{\ln\frac{n+1}{n}} = a^{\ln 1} = a^0 = 1$ „inconclusive" $\Rightarrow$

$$\Rightarrow \text{R-D test: } \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\left(a^{\ln\frac{n}{n+1}}-1\right)$$

$$= \lim_{n\to\infty} n\cdot \underbrace{\frac{a^{\ln\frac{n}{n+1}}-1}{\ln\frac{n}{n+1}}}_{\to \ln a}\cdot \ln\frac{n}{n+1} = \boxed{-\ln a}$$

$$\boxed{\text{Recall: } \lim_{x\to 0}\frac{a^x-1}{x}=\ln a}$$

$$\lim_{n\to\infty} n\cdot\ln\frac{n}{n+1} = \lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^n = \ln\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n =$$

$$= \ln\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \ln\lim_{n\to\infty}\underbrace{\left(1-\frac{1}{n+1}\right)^{-(n+1)}}_{\to e}{}^{\cdot n\cdot\frac{-1}{n+1}} =$$

$$= \ln e^{-1} = -1$$

Since the limit depends on a, we discuss over a:

$$-\ln a<1 \iff \ln a>-1 \iff a>\frac{1}{e} \Rightarrow \text{CONV.}$$

$$0<a<\frac{1}{e} \Rightarrow \text{DIV}$$

$$a=\frac{1}{e} \Rightarrow \sum_{n\geq 1} e^{\ln n} = \sum_{n\geq 1} n \text{ which is DIV.}$$

27.10.2023

Raabe - Duhamel Test

$$\lim n\left(\frac{x_n}{x_{n-1}} - 1\right) = L \begin{cases} >1 : \text{CONV} \\ <1 : \text{DIV} \end{cases}$$

1) a), c) at the table (will redo above)

1) d) $\sum \frac{a^n \cdot n!}{n^n}$, $a > 0$

ratio test: $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \frac{\cancel{a^{n+1}}^{\,a} \cdot \cancel{(n+1)!}^{\,n+1}}{(n+1)^{\cancel{n+1}\, n}} \cdot \frac{n^n}{\cancel{a^n}_{1} \cdot \cancel{n!}_{1}} =$

$= \lim\limits_{n\to\infty} \frac{a \cdot n^n}{(n+1)^n} = \lim\limits_{n\to\infty} a \cdot \left(\frac{n}{n+1}\right)^n =$

$\to \frac{1}{e}$ (previous ex.)

$= a \cdot \frac{1}{e} = \frac{a}{e}$

- if $a < e \Rightarrow \frac{a}{e} < 1 \Rightarrow$ CONV.
- if $a > e \Rightarrow \frac{a}{e} > 1 \Rightarrow$ DIV.
- if $a = e \Rightarrow \frac{a}{e} = 1$ inconclusive $\Longrightarrow$

R-D test $\Rightarrow \sum \underbrace{\frac{e^n \cdot n!}{n^n}}_{x_n}$

$\lim_{n\to\infty} n\left(\frac{1}{e}\cdot\left(\frac{n+1}{n}\right)^n - 1\right)$ complicated :(

$$\boxed{\text{Recall}: \sum x_n \text{ conv.} \Rightarrow x_n \to 0 \qquad x_n \not\to 0 \Rightarrow \text{div.}}$$

check monotony of x_n

$$\frac{x_{n+1}}{x_n} = \frac{\cancel{e^{n+1}}^{\,e} \cdot \cancel{(n+1)!}^{\,n+1}}{(n+1)^{\cancel{n+1}\, n}} \cdot \frac{n^n}{\cancel{e^n}_{\,1} \cdot \cancel{n!}_{\,1}} =$$

$$= \frac{e \cdot n^n}{(n+1)^n} = e\left(\frac{n}{n+1}\right)^n > 1$$

$$\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e \qquad \left(1+\frac{1}{n}\right)^n < e$$

$$\left(\frac{n+1}{n}\right)^n < e$$

$\Rightarrow x_{n+1} > x_n \Rightarrow \lim_{n\to\infty} x_n > 0 \Rightarrow$ Div.

2) a) $\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

Leibnitz test

$x_n \searrow 0 \Rightarrow \sum (-1)^n \cdot x_n$ conv.

$x_n = \frac{1}{\sqrt{n(n+1)}} \to 0$

$\frac{x_{n+1}}{x_n} = \sqrt{\frac{n\cancel{(n+1)}}{\cancel{(n+1)}(n+2)}} = \sqrt{\frac{n}{n+2}} < 1$

$\left.\begin{matrix} x_n \text{ decreasing} \\ x_n \to 0 \end{matrix}\right\} \Rightarrow$

2b. $\Rightarrow \sum \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$ conv

$\sum \left| \frac{(-1)^{n+1}}{\sqrt{n(n+1)}} \right| = \sum \frac{1}{\sqrt{n(n+1)}}$

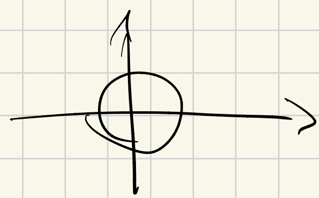
$\frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}} = \frac{n}{\sqrt{n(n+1)}} \to 1 \Rightarrow$

$\Rightarrow$ the two series which were compared above have the same nature $\Leftrightarrow$

$$\left.\begin{array}{l} \Leftrightarrow \dfrac{1}{\sqrt{n(n+1)}} \text{ "like" } \dfrac{1}{n} \\[2em] \sum \dfrac{1}{n} \text{ DiV} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \sum \frac{1}{\sqrt{n(n+1)}} \text{ DiV.}$$

b) $\sum (-1)^n \cdot \sin \frac{1}{n}$



$x_n = \sin \frac{1}{n}$

$\lim_{n \to \infty} \sin \frac{1}{n} = \sin 0 = 0$

$\frac{1}{n} \in [1, 0)$

sin increasing on $[0, \frac{\pi}{2}]$

$\frac{1}{n+1} < \frac{1}{n}$

$\sin \frac{1}{n+1} < \sin \frac{1}{n} \Rightarrow x_n \searrow$

$\Rightarrow$ conv. by the Leibnitz test.

$$\sum_{n\geq 1} \sin \frac{1}{n}$$

$\dfrac{\sin x}{x} \xrightarrow{x \to 0} 1$

$$\frac{\sin \frac{1}{n}}{\frac{1}{n}} \to 1 \Rightarrow \sum_{n\geq 1} \sin \frac{1}{n} \text{ "like" } \sum_{n\geq 1} \frac{1}{n}$$

can be any constant (one is equal to the other multiplied by a const.)

Comparison test

$$\sum \frac{1}{n} \text{ DIV.} \Rightarrow \sum \sin \frac{1}{n} \text{ DIV.}$$

fun fact of the day

e) $\displaystyle\sum_{n\geq 1} \frac{\sin n}{n^2}$ $\qquad 0 < |\sin n| < 1$

$$\sum_{n\geq 1} \left| \frac{\sin n}{n^2} \right| = \sum_{n\geq 1} \frac{|\sin n|}{n^2} < \underbrace{\sum_{n\geq 1} \frac{1}{n^2}}_{\text{CONV.}} \Rightarrow$$

$= \frac{\pi^2}{6}$

$\overset{C.T.}{\Longrightarrow} \sum_{n\geq 1} \frac{|\sin n|}{n^2}$ CONV. $\Longrightarrow$

$\Rightarrow \sum_{n\geq 1} \frac{\sin n}{n^2}$ absolutely convergent $\Rightarrow$ CONV.

3) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x}$

Radius of convergence $R = 1$

The series converges absolutely for $|x| < 1$

$\Longrightarrow$ we can differentiate it term by term

$$\sum_{n=1}^{\infty} (x^n)' = 0 + 1 + 2x + \dots = \sum_{n=1}^{\infty} n x^{n-1} =$$

$$= \left(\frac{1}{1-x}\right)'$$

$$\boxed{\left(\frac{1}{u}\right)' = \frac{-u'}{u^2}}$$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} n\cdot x^{n} = \frac{x}{(1-x)^2}$$

- Differentiate $\displaystyle\sum_{n=1}^{\infty} n\, x^{n-1} = \frac{1}{(1-x)^2}$

$$\sum_{n=2} n(n-1)\cdot x^{n-2} = \frac{2}{(1-x)^3} \quad \Big| \cdot x^2$$

$$\sum_{n=2} n(n-1)\, x^{n} = \frac{2x^2}{(1-x)^3}$$

4) $\displaystyle\sum_{n=0}^{\infty} t^n = 1+t+t^2+\dots = \frac{1}{1-t},\ |t|<1$

$$\int_0^x dt : \sum_{n=0}^{\infty} \int_0^x t^n\, dt = \int_0^x \frac{1}{1-t}\, dt$$

$$\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}\Big|_0^x = \sum_{n=0} \frac{x^{n+1}}{n+1} = -\ln(1-t)\Big|_0^x =$$

$$\| \quad \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \quad , \quad x < |1|$$

$$\sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t} \quad , |t| < 1$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot t^n = \frac{1}{1+t} \quad , \quad |t| < 1 \,\Big|\, \int_0^x dt$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \ln(1+x)$$

$$\sum_{n\geq 1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \ln(1+x)$$

OR: $x \to -x$: $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n} = \ln(1+x) \,\Big|\, \cdot (-1)$

5) $\sum_{n=1}^{\infty} (-1)^n t^n = \frac{1}{1+t}$, $|t|<1$

$t \to t^2$

$$\sum_{n=0}^{\infty} (-1)^n t^{2n} = \frac{1}{1+t^2} \quad \Big|\int_0^x dt.$$

$$\sum (-1)^n \frac{x^{2n+1}}{2n+1} = \operatorname{arctg} t \Big|_0^x = \operatorname{arctg} x,$$

for $|x|<1$

(bcs $|t|<1$).

$$x=1: \sum (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

by the Leibnitz test

$\operatorname{arctg} 1$

$x=1.$