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2.1. Every student who makes good grades is brilliant or studies.

Let D = set of all people

unary predicate symbols

$$\begin{cases} P: D \to \{T, F\}; & P(x) = \text{"x is a student"} \\ Q: D \to \{T, F\}; & Q(x) = \text{"x makes good grades"} \\ R: D \to \{T, F\}; & R(x) = \text{"x is brilliant"} \\ S: D \to \{T, F\}; & S(x) = \text{"x studies"} \end{cases}$$

$$(\forall x)_{x \in D} \left(P(x) \wedge Q(x) \right) \longrightarrow \left(R(x) \vee S(x) \right)$$

1.8 The sum of two even numbers is an even number and then product is divisible by 4.

D = set of all integers $(= \mathbb{Z})$

Binary predicate symbol : $DIV: D \times D \to \{T, F\}$,

$DIV(x, y) = T$ if $x \vdots y$

$EQ: D \times D \to \{T, F\}$, $EQ(x, y) = T$ if $x = y$

Binary function symbols : $sum: D \times D \to D$, $sum(x, y) = x + y$

$prod: D \times D \to D$, $prod(x, y) = x \cdot y$

$$(\forall_{x,y\in D}\, x)(\forall y)\left[DiV(x,2)\wedge DiV(y,2) \longrightarrow DiV(sum(x,y),2)\wedge \wedge (DiV(prod(x,y)),4)\right]$$

Properties of DiV:

- reflexivity: $(\forall_{x\in D}\, x)\, DiV(x,x)$
- transitivity: $(\forall_{x,y,z\in D}\, x)(\forall y)(\forall z)\left[DiV(x,y)\wedge DiV(y,z)\rightarrow DiV(x,z)\right]$
- $(\forall_{x\in D}\, x)\; DiV(x,1)$

Properties of sum, prod:

- Commutativity: $(\forall x)(\forall y)\left[EQ(sum(x,y),\, sum(y,x)\right]$
- Associativity: $(\forall x)(\forall y)(\forall z)\, EQ(sum(sum(x,y),z),\, sum(x, sum(y,z)))$
- $(\forall x)\; EQ(sum(x,0),\, x)$
- Distributivity:

$$(\forall x)(\forall y)(\forall z)\left[EQ(prod(x, sum(y,z)),\, sum(prod(x,y),\, prod(x,z)))\right]$$

3.1 H_1: If x is the king and y is the oldest son, then y can become king.

H_2: If x is the king and y defeats x, then y will become king

H_3: Richard III is king

H_4: Henry VIII defeats Richard VII's oldest son

C: Can Henry VIII become king?

$os(x,y) = T$ if y is x's oldest son.

H_1: $(\forall x)(\forall y)\,(king(x) \wedge os(x,y) \to king(y))$

H_2: $(\forall x)(\forall y)\; king(x) \wedge df(x,y) \to king(y)$

H_3: $king(R_3)$

H_4: $df(R_3, H_7)$

H_5: $os(H_7, H_8)$

C: $king(H_8)$

$$\mathcal{U},\ \mathcal{U}\to\mathcal{V} \ \vdash_{mp}\ \mathcal{V}$$

$$(\forall x)\,\mathcal{U}(x) \longrightarrow \mathcal{U}(t)$$

universal instantiation $[x \leftarrow t]$

where t = term (constant or variable)

$$H_2 \ \vdash_{\substack{\text{univ.}\\ \text{instantiation}[x\leftarrow R_3]}} \ (\forall y)\,[\,king(R_3) \wedge df(R_3, y) \to king(y)\,] =$$

$= f_6$ (formula)

$$f_6 \ \vdash_{\substack{\text{univ.}\\ \text{instantiation}}} \ king(R_3) \wedge df(R_3, H_7) \to king(H_7) = f_7$$

$$f_8 = H_3 \wedge H_4 = king(R_3) \wedge df(R_3, H_7)$$

$$f_7,\ f_8 \ \vdash_{mp}\ king(H_7) \quad = f_9$$

$$H_1 \ \vdash_{\substack{\text{univ.}\\ \text{inst.}}} (\forall y)\ king(H_7) \wedge os(H_7, y) \to king(y) = f_{10}$$

$$f_{10} \ \vdash_{\substack{\text{univ.}\\ \text{inst.}}} \ king(H_7) \wedge os(H_7, H_8) \to king(H_8) = f_{11}$$

$f_{12} = f_8 \wedge H_5 = long(H_7) \wedge os(H_7, H_8)$

$f_{12}, f_{11} \vdash long(H_8) = C = f_{13}$

The sequence $H_1, \ldots, H_5, f_6, \ldots, f_{13}$ is the deduction (proof) of C from the hypothesis.

4.1 $\quad U = (\exists x)\, A(x) \wedge \exists(x)\, B(x) \to (\forall x)(A(x) \vee B(x))$

interpretation $i = \langle \overbrace{D, m}^{\text{pair}} \rangle$, where:

↳ domain ↳ function

D = the set of all straight lines of plane P

Let $d \in P$, a constant straight line belonging to D

$m(A) : D \to \{T, F\}$, $m(A)(x)$: "$x \perp d$"

$m(B) : D \to \{T, F\}$, $m(B)(x)$: "$x \parallel d$"

Evaluation:

$$V^i(U) = V^i((\exists x)\, A(x) \wedge (\exists x)\, B(x) \to (\forall x)(A(x) \vee B(x)))$$

$$V^i(U) = V^i((\exists x)\, A(x) \wedge (\exists x)\, B(x)) \to V^i((\forall x) A(x) \vee B(x))$$

can decompose further

↳ main quantifier ⇒ cannot decompose anymore

$$V^i(\mathcal{U}) = V^i((\exists x)A(x)) \wedge V^i((\exists x)B(x)) \to V^i((\forall x)A(x) \vee B(x))$$

$$V^i(\mathcal{U}) = \underbrace{\underset{x\in D}{(\exists x)} \text{"} x \perp d \text{"}} \wedge \underbrace{\underset{x\in D}{(\exists x)} \text{"} x \parallel d \text{"}} \to \underbrace{\underset{x\in D}{(\forall x)} (\text{"} x \perp d \text{"} \vee \text{"} x \parallel d \text{"})}$$

$$V^i(\mathcal{U}) = \underbrace{T \wedge T}_{T} \to F$$

$$V^i(\mathcal{U}) = T \to F$$

$$V^i(\mathcal{U}) = F \quad \Rightarrow \quad i \text{ is an anti-model of } \mathcal{U}$$

5.1) $\mathcal{U}_1 = (\forall x)(A(x) \Leftrightarrow B(x)) \to ((\forall x)A(x) \Leftrightarrow (\forall x)B(x))$

D = set of natural numbers $(= \mathbb{N})$, $i = \langle D, m \rangle$

$m(A)(x)$ = "x is a prime number"

$m(B)(x)$ = "x is a perfect square"

$$V^i(\mathcal{U}) = V^i(\underline{(\forall x)}(A(x) \Leftrightarrow B(x)) \to ((\forall x)A(x) \Leftrightarrow (\forall x)B(x))$$

cannot be decomposed anymore!

$$V^i(\mathcal{U}) = V^i(\forall x)(A(x) \Leftrightarrow B(x)) \to (V^i(\forall x)A(x) \Leftrightarrow V^i(\forall x)B(x))$$

$$V^i(\mathcal{U}) = \underset{x\in\mathbb{N}}{(\forall x)}(\text{"}x \text{ prime"} \Leftrightarrow \text{"}x \text{ p.s."}) \to (\underset{x\in\mathbb{N}}{(\forall x)} \text{"}x \text{ prime"} \Leftrightarrow \underset{x\in\mathbb{N}}{(\forall x)} \text{"}x \text{ p.s."})$$

$V^i(\mathcal{U}) = F \rightarrow (F \Leftrightarrow F)$

$V^i(\mathcal{U}) = F \rightarrow T$

$V^i(\mathcal{U}) = T \Rightarrow i$ is a model of $\mathcal{U}$

6.1) $\mathcal{U} = ((\exists x)P(x) \rightarrow (\exists x)Q(x)) \rightarrow (\forall x)(P(x) \rightarrow Q(x))$

antimodel?

$i = \langle D, m \rangle$

$D = \mathbb{N}$

$m(P)(x) =$ "x is even"

$m(Q)(x) =$ "x is odd"

$V^i(\mathcal{U}) = V^i((\exists x)P(x) \rightarrow (\exists x)Q(x)) \rightarrow V^i((\forall x)(P(x) \rightarrow Q(x)))$

$V^i(\mathcal{U}) = (V^i((\exists x)P(x)) \rightarrow V^i((\exists x)Q(x))) \rightarrow V^i((\forall x)(P(x) \rightarrow Q(x)))$

$V^i(\mathcal{U}) = (\underset{x\in\mathbb{N}}{(\exists x)}(\text{"x even"}) \rightarrow \underset{x\in\mathbb{N}}{(\exists x)}(\text{"x odd"})) \rightarrow \underset{x\in\mathbb{N}}{(\forall x)}(\text{"x even"} \rightarrow \text{"x odd"})$

$V^i(\mathcal{U}) = (T \rightarrow T) \longrightarrow F$

$V^i(\mathcal{U}) = T \rightarrow F$

$V^i(\mathcal{U}) = F \Rightarrow i$ is an anti-model of $\mathcal{U}$