Resolution in predicate logic

1) Transform the following formula into prenex,

$$U_1 = (\exists x)(\forall y)[(\exists z)P(y,z) \wedge (\forall u)(Q(x,u) \to (\exists z) R(u,z,x))]$$

$$\xrightarrow{\text{replace 1}} (\exists x)(\forall y)[\underline{(\exists z)}P(y,z) \wedge (\forall u)(\neg Q(x,u) \vee \underset{t}{\underline{(\exists z)}} R(u,\underset{t}{z},x))]$$

$$= (\exists x)(\forall y)[(\exists z)P(y,z) \wedge (\forall u)(\neg Q(x,u) \vee (\exists t) R(u,t,x))]$$

$$= (\exists x)(\forall y)[\underline{(\exists z)P(y,z)} \wedge \underline{(\forall u)(\exists t)(\neg Q(x,u) \vee R(u,z,x))}]$$

two independent groups => we can extract them in any order =>

=> we will have 2 prenex forms!

$$\mathcal{U} \equiv \mathcal{U}_1^P =$$

$$= (\exists x)(\forall y)\boxed{(\exists z)(\forall u)(\exists t)}\left[P(y,z) \wedge (\neg Q(x,u) \vee R(u,t,x))\right]$$

$$\equiv \mathcal{U}_2^P \; (\rightarrow \text{prenex form}) =$$

$$= \underbrace{(\exists x)(\forall y)\boxed{(\forall u)(\exists t)(\exists z)}}_{\text{prefix}}\underbrace{\left[P(y,z) \wedge (\neg Q(x,u) \vee R(u,t,x))\right]}_{\text{matrix}}$$

$\mathcal{U}_2^P$ provides $\mathcal{U}_2^S =$

$[x \leftarrow a, f(y,u), z \leftarrow g(y,u)]$, a - skolem const.

f, g - skolem functions

$$= (\forall y)(\forall u)\left[P(y,z) \wedge (\neg Q(a,u) \vee R(u, f(y,u), a))\right]$$

$\mathcal{U}_1^P$ provides $\mathcal{U}_1^S =$

$[x \leftarrow a, z \leftarrow f(y), t \leftarrow g(y,u)]$

$$= (\forall y)(\forall u)\left[P(y, f(y)) \wedge (\neg Q(a,u) \vee R(u, g(y,u), a))\right]$$

2.1) at the blackboard

3.1) Proove the inconsistency using <u>lock resolution</u>.

Clauses:

$$S_1 = \{ \overbrace{\underset{(1)}{\neg P(x)} \vee \underset{(2)}{Q(x)}}^{C_1},\ \overbrace{\underset{(3)}{P(a)}}^{C_2},\ \overbrace{\underset{(4)}{\neg Q(x)} \vee \underset{(5)}{\neg R(x)}}^{C_3},\ \overbrace{\underset{(6)}{\neg W(a)}}^{C_4},$$

$$\underbrace{\underset{(8)}{R(y)} \vee \underset{(7)}{W(y)}}_{C_5} \}$$

$$C_6 = Res^{lock}_{\theta_1}(C_1, C_2) = \underset{(2)}{Q(a)}, \quad \theta_1 = mgu(P(x), P(a)) =$$

$$= [x \leftarrow a]$$

$$C_7 = Res^{lock}_{\theta_1}(C_6, C_3) = \underset{(5)}{\neg R(a)}, \quad \theta_1 = mgu$$

$$C_8 = Res^{lock}_{\theta_2}(C_4, C_5) = \underset{(8)}{R(a)}, \quad \theta_2 = mgu(W(y), W(a)) = [y \leftarrow a]$$

$C_9 = \mathrm{Res}^{\mathrm{lock}}(C_7, C_8) = \square$

$S_1 \vdash^{\mathrm{lock}}_{\mathrm{Res}} \square$, so S_1 is inconsistent

4.1) $\vdash \underbrace{(\forall x) P(x) \vee (\forall x) Q(x) \to \forall(x)(P(x) \vee Q(x))}_{U_1}$

$\nvdash (\forall x)(P(x) \vee Q(x)) \to (\forall x) P(x) \vee (\forall x) Q(x)$

$\neg U_1 = \neg((\forall x) P(x) \vee (\forall x) Q(x) \to \forall(x)(P(x) \vee Q(x)))$

replace $\neg(\to)$

. . .

rename var

de Morgan's

extract quantifiers $\underbrace{(\exists z)(\forall x)(\forall y)}_{\text{prefix}} \underbrace{((P(x) \vee Q(y)) \wedge \neg P(z) \wedge \neg Q(z))}_{\text{matrix}} = (\neg U_1)^P$

$[z \leftarrow a]$, a = Skolem constant

$$(\neg U_1)^s = (\forall x)(\forall y)((P(x) \vee Q(y)) \wedge \neg P(a) \wedge \neg Q(a))$$

$$(\neg U_1)^c = \underbrace{(P(x) \vee Q(y))}_{C_1} \wedge \underbrace{\neg P(a)}_{C_2} \wedge \underbrace{\neg Q(a)}_{C_3}$$

$$S = \{ C_1, C_2, C_3 \}$$

$\vdots$ linear resolution $\Rightarrow []$

$(\neg U_1)^c \vdash_{Res} []$, so $\vdash U_1$

$$\neg U_2 = \neg((\forall x)(P(x) \vee Q(x)) \rightarrow (\forall x)P(x) \vee (\forall x)Q(x))$$

$$\xrightarrow[\neg(\rightarrow)]{\text{replace}} (\forall x)(P(x) \vee Q(x)) \wedge \neg((\forall x)P(x) \vee (\forall x)Q(x))$$

$$\xrightarrow[\text{de Morgan's}]{\text{rename}} (\forall x)(P(x) \vee Q(x)) \wedge (\exists y)\neg P(y) \wedge (\exists z)\neg Q(z)$$

extract quantifiers

$$\underline{(\exists y)(\exists z)(\forall x)\left(P(x) \vee Q(x)) \wedge P(y) \wedge Q(z)\right)}$$

this is a prenex form $(\neg U_2)^P$

$y \leftarrow a$; a skolem const
$z \leftarrow b$; b skolem const.

$$(\neg U_2)^S = (\forall x)((P(x) \vee Q(x)) \wedge P(a) \wedge Q(b))$$

$$(\neg U_2)^C = \underbrace{(P(x) \vee Q(x))}_{C_1} \wedge \underbrace{P(a)}_{C_2} \wedge \underbrace{Q(b)}_{C_3}$$

$$S_2 = \{C_1, C_2, C_3\}$$

$$C_4 = \underset{\theta_1 = [x \leftarrow a]}{Res}(C_1, C_2) = Q(a)$$

$$C_5 = \underset{\theta_2 = [x \leftarrow b]}{Res}(C_1, C_3) = P(b)$$

C_4 and C_3 don't resolve because the const. a and b are not unifiable

C_5 and C_2 don't resolve bcs a and b are not unifiable

So, no more resolvents can be generated and the $\square$ was not derived, so $(\neg U_2)^c \nvdash_{Res} \square \Rightarrow$

$\Rightarrow \nvdash U_2$

5.1) $U_1 = (\forall y) \leftrightarrow P(x,y) \leftrightarrow (\exists y)(\exists x) P(x,y)$

$\vdash U$?

$U_1 = (\forall y)(\exists x) P(x,y) \to (\exists y)(\exists x) P(x,y)$

$U_2 = (\exists y)(\exists x) P(x,y) \to (\forall y)(\exists x) P(x,y)$

$\vdash U \Leftrightarrow \vdash U_1$ and $\vdash U_2$

$\vdash U_1$ iff $(\neg U_1)^c \vdash \frac{Bn}{Res} \square$

$\vdash U_2$ iff $(\neg U_2) \vdash \frac{Bn}{Res} \square$

$\neg U_1 = \neg((\forall y)(\exists x) P(x,y) \to (\exists y)(\exists x) P(x,y))$

$\underset{\text{replace } \neg(\to)}{\equiv} (\forall y)(\exists x) P(x,y) \wedge \neg((\exists \underset{t}{y})(\exists \underset{z}{x}) P(\underset{z}{x},\underset{t}{y}))$

de Morgan's / rename var: $(\forall y)(\exists x)P(x,y) \wedge (\forall t)(\forall z)\neg P(z,t)$

extract quantifiers: $(\exists x)(\forall y)(\forall t)(\forall z)(P(x,y) \wedge \neg P(z,t)) =$

$= (\neg U_1)^p$

$(\neg U_1)^s$ replace $x \leftarrow f(y)$: $(\forall y)(\forall t)(\forall z)(P(f(y),y) \wedge \neg P(z,t))$

$(\neg U_1)^c$ $=$ $\underbrace{P(f(y),y)}_{C_1} \wedge \neg \underbrace{P(z,t)}_{C_2}$

$S_1 = \{C_1, C_2\}$

$mgu(C_1, C_2) = \{z \leftarrow f(y)\}\{y \leftarrow t\}$

$= \{z \leftarrow f(t), y \leftarrow t\} = \theta$

$P(f(t), t)$ is the common instance of C_1 and C_2

$Res_\theta(C_1, C_2) = \square \Rightarrow \vdash U_1$

$\neg U_2 = \neg((\exists y)(\exists x)P(x,y) \to (\forall y)(\exists x)P(x,y))$

$t \quad z \quad z\,t$

replace $\neg(\exists)$

de Morgan's

$$(\exists y)(\exists x)P(x,y) \wedge (\exists t)(\forall z)\neg P(z,t)$$

extract

quantifiers

$$\underbrace{(\exists y)(\exists x)(\exists t)(\forall z)\, P(x,y) \wedge \neg P(z,t)}_{(\neg U_2)^P}$$

$$(\neg U_2)^S_{\{x\leftarrow b,\ y\leftarrow a,\ t\leftarrow c\}} = (\forall z)\, P(b,a) \wedge \neg P(z,c)$$

$$(\neg U_2)^C = \underbrace{P(b,a)}_{C_1} \wedge \underbrace{\neg P(z,c)}_{C_2}$$

$$S_2 = \{C_1, C_2\}$$

C_1 and C_2 don't resolve bcs. $P(b,a)$ and $P(z,c)$ are not unifiable (a and c are diff. constants)

$(\neg U_2)^C \not\vdash \square \Rightarrow \not\vdash U_2$ (not a theorem)

Theory

Predicate resolution

$A \to B \equiv \neg A \vee B$

$C_1 = P(x) \vee \neg Q(x)$

$C_2 = \neg P(a)$

$\mathrm{Res}_{\theta}(C_1, C_2) = \neg Q(a)$, $\theta = [x \leftarrow a]$

Extraction of quantifiers

$(\forall x)\, A(x) \vee B(y) \equiv (\forall x)(A(x) \vee B(y))$

- U is inconsistent

iff U^P → prenex form — " — " —

iff U^S → skolem form — " — " —

iff U^C — " — " —

Substitution:

$\theta = [x \leftarrow a,\ y \leftarrow z,\ t \leftarrow f(a)]$

$x, y, z, t \in Var$, $a, b \in Const.$

$x \leftarrow a$ ✓	$x \leftarrow f(b)$ ✓
$\cancel{a \leftarrow x}$	$\cancel{f(x) \leftarrow a}$
$\cancel{a \leftarrow b}$	$\cancel{x \leftarrow f(x)}$
$x \leftarrow y$ ✓	
$x \leftarrow f(z)$ ✓	

$\vdash \mathfrak{A}_1$ iff $(\neg \mathfrak{A}_1) \vdash^{Rs} \square$