

2.1 2.2 2.5 2.10 2.11 2.14 2.18

2.19 2.22 2.27, 2.30

3.1, 3.2, 3.3, 3.4, 3.7, 3.8, 3.10

3.12, 3.18, 3.20

Dot product (scalar product)

$\vec{v}, \vec{w} \in V$ vectors

$$\vec{v} \cdot \vec{w} = \begin{cases} 0, & \text{if } \vec{v} = \vec{0} \text{ or } \vec{w} = \vec{0} \\ |\vec{v}| \cdot |\vec{w}| \cdot \cos |\vec{v}, \vec{w}|, & \text{otherwise} \end{cases}$$

Notation: $\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle$

Properties

• Bilinearity: If $v_1, v_2, w \in V$ $\langle \alpha \vec{v}_1 + \beta \vec{v}_2, \vec{w} \rangle = \alpha \langle \vec{v}_1, \vec{w} \rangle + \beta \langle \vec{v}_2, \vec{w} \rangle$
 + $\alpha, \beta \in \mathbb{R}$:

$$(\alpha \vec{v}_1 + \beta \vec{v}_2) \cdot \vec{w} = \alpha \vec{v}_1 \cdot \vec{w} + \beta \vec{v}_2 \cdot \vec{w}$$

• Symmetry: If $\vec{v}, \vec{w} \in V$

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

• Positive definiteness: If $v \in V$

$$\vec{v} \cdot \vec{v} \in \mathbb{R} \geq 0, \text{ if } \vec{v} \neq \vec{0}, \vec{v} \cdot \vec{v} > 0$$

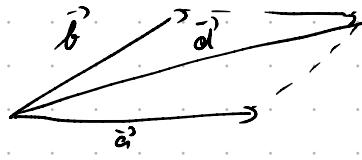
Consequences

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

$$\vec{v}^2 - \vec{w}^2 = (\vec{v} - \vec{w})(\vec{v} + \vec{w})$$

$$1. \vec{m}, \vec{n} \text{ unit vectors } \Rightarrow \langle \vec{m}, \vec{n} \rangle = 60^\circ \Rightarrow \cos(\vec{m}, \vec{n}) = \frac{1}{2}$$

Def lengths of the parallelogram spanned by the vectors $\vec{a} = 2\vec{m} + \vec{n}$, $\vec{b} = \vec{m} - 2\vec{n}$



$$\begin{aligned} d^2 &= \vec{a}^2 + \vec{b}^2 \\ &= 2\vec{m}^2 + \vec{n}^2 + \vec{m}^2 - 4\vec{m}\cdot\vec{n} \\ &= 3\vec{m}^2 - 2\vec{m}\cdot\vec{n} \end{aligned}$$

$$\begin{aligned} |\vec{d}|^2 &= \langle \vec{d}, \vec{d} \rangle = \langle \vec{a} + \vec{b}, \vec{a} + \vec{b} \rangle \\ &= |\vec{a} + \vec{b}|^2 \\ &= |3\vec{m} - \vec{n}|^2 \\ &= 9\langle \vec{m}, \vec{m} \rangle - 6\langle \vec{m}, \vec{n} \rangle + \langle \vec{n}, \vec{n} \rangle \\ &= 9|\vec{m}|^2 - 3|\vec{m}||\vec{n}| + |\vec{n}|^2 \\ &= 10 - 3 = 7 \end{aligned}$$

$$\Rightarrow |\vec{d}| = \sqrt{7}$$

$$\begin{aligned} \vec{d}_2 &= \vec{a} - \vec{b} = \vec{m} + 3\vec{n} \\ \langle \vec{d}_2, \vec{d}_2 \rangle &= |\vec{m} + 3\vec{n}|^2 \\ &= 2\langle \vec{m}, \vec{m} \rangle + 3\vec{m}^2 + 9\langle \vec{m}, \vec{n} \rangle \end{aligned}$$

$$= 13$$

$$\Rightarrow |\vec{d}_2| = \sqrt{13}$$

If we have an orthonormal basis of V , $B = (\vec{v}_1, \dots, \vec{v}_m)$ (i.e. $\forall i \neq j : \vec{v}_i \cdot \vec{v}_j = 0$), and $[\vec{w}]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$, $[\vec{w}]_B = \begin{pmatrix} a_1' \\ \vdots \\ a_m' \end{pmatrix}$

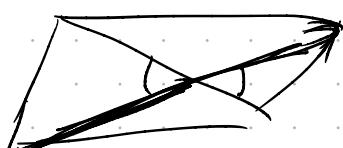
then:

$$\vec{w} \cdot \vec{w}' = a_1 a_1' + a_2 a_2' + \dots + a_m a_m'$$

So

$$|\vec{w}|^2 = \sqrt{a_1^2 + a_2^2 + \dots + a_m^2}$$

3. We are given $\vec{a} (2, 1, 0)$; $\vec{b} (0, -2, 1)$ w.r.t. to an orthonormal basis, find the angle between the diag. of the parallelog. spanned by \vec{a} and \vec{b} .



$$\vec{a} + \vec{b} = (2, -1, 1)$$

$$\vec{a} - \vec{b} = (2, 3, -1)$$

$$\langle \vec{d}_1, \vec{d}_1 \rangle = \langle \vec{a} + \vec{b}, \vec{a} + \vec{b} \rangle = 4 + 1 + 1 = 6 \Rightarrow |\vec{d}_1| = \sqrt{6}$$

$$\langle \vec{d}_1, \vec{d}_2 \rangle = 4 + 9 + 1 = 14 \Rightarrow |\vec{d}_2| = \sqrt{14} = 4$$

$$\begin{aligned}\langle \vec{d}_1, \vec{d}_2 \rangle &= \sqrt{6} \cdot \sqrt{14} \cdot \cos(\vec{d}_1, \vec{d}_2) \\ &= \sqrt{84} \cdot \cos(\vec{d}_1, \vec{d}_2) = 4 - 3 - 1 = 0 \\ &\Rightarrow \vec{d}_1, \vec{d}_2 \text{ are } \perp\end{aligned}$$

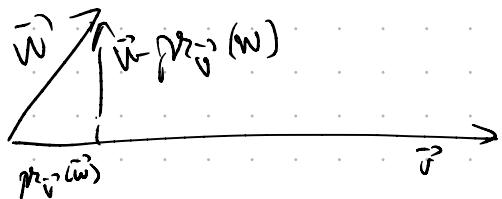
6. $\triangle ABC$
 $\vec{AB}^2 + \vec{AC}^2 - \vec{BC}^2 = 2 \cdot \vec{AB} \cdot \vec{AC}$ - law of cosines.

Proof:

$$\begin{aligned}\vec{AB}^2 + \vec{AC}^2 - \vec{BC}^2 &= (\vec{AB} - \vec{AC}) \cdot (\vec{AB} + \vec{AC}) + \vec{AC}^2 \\ &= (\vec{AB} - \vec{AC}) \cdot \vec{AC} + \vec{AC}^2 \\ &= (\vec{AB} - \vec{AC} + \vec{AC}) \cdot \vec{AC} \\ &= (\vec{AB} - \vec{AC}) \cdot \vec{AC} \\ &= (2 \vec{AB} - \vec{AC}) \cdot \vec{AC} \\ &= 2 \vec{AB} \cdot \vec{AC}.\end{aligned}$$

7. ABCD - tetrahedron,
 $\cos \angle (\vec{AB}, \vec{CD}) = \frac{\vec{AB}^2 + \vec{CD}^2 - \vec{AC}^2 - \vec{BD}^2}{2 \vec{AB} \cdot \vec{CD}}$

$$\begin{aligned}\vec{AD}^2 - \vec{AC}^2 + \vec{BC}^2 - \vec{BD}^2 &= (\vec{AD} - \vec{AC})(\vec{AD} + \vec{AC}) + (\vec{BC} - \vec{BD})(\vec{BC} + \vec{BD}) \\ &= \vec{CD} (\vec{AD} + \vec{AC} - \vec{BC} - \vec{BD}) \\ &= \vec{CD} (\vec{AD} + \vec{AC} + \vec{CB} + \vec{DB}) \\ &= \vec{CD} (\vec{AB} + \vec{AB}) \\ &= 2 \vec{AB} \cdot \vec{CD} \Rightarrow \text{true}.\end{aligned}$$



$$\begin{aligned}
 & |\vec{w}| \cdot \cos(\vec{v}, \vec{w}) = |\vec{v}_\perp(\vec{w})| = \\
 & = \frac{|\vec{w}| \cdot \cos(\vec{v}, \vec{w})}{|\vec{v}|} = \\
 & = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \cdot |\vec{v}| = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{v}|} \cdot |\vec{v}|
 \end{aligned}$$

Gramm Schmidt

$B = (v_1, \dots, v_n)$ basis of V

I Orthogonalization

$$1. \vec{v}_1' := \vec{v}_1$$

$$2. \vec{v}_2' := \vec{v}_2 - \text{pr}_{\vec{v}_1'}(\vec{v}_2)$$

$$3. \vec{v}_3' := \vec{v}_3 - \text{pr}_{\text{span}(\vec{v}_1', \vec{v}_2')}(\vec{v}_3)$$

$$= \vec{v}_3 - \text{pr}_{\vec{v}_1'}(\vec{v}_3) - \text{pr}_{\vec{v}_2'}(\vec{v}_3)$$

$$= \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{v}_1'}{|\vec{v}_1'| \cdot |\vec{v}_1'|} \vec{v}_1' - \frac{\vec{v}_3 \cdot \vec{v}_2'}{|\vec{v}_2'| \cdot |\vec{v}_2'|} \vec{v}_2'$$

- repeat process until we have a basis.

$B' = (\vec{v}_1', \dots, \vec{v}_n')$ - orthogonal.

II Normalization

$$\vec{v}_i'' := \frac{\vec{v}_i'}{|\vec{v}_i'|}$$

$$\rightarrow B'' = (\vec{v}_1'', \dots, \vec{v}_n'')$$

orthonormal basis.

3.10. In an orthonormal basis consider the vectors $\vec{v}_1(0,1,0)$, $\vec{v}_2(2,1,0)$, $\vec{v}_3(-1,0,1)$
 Use G-Schmidt to find an orthonormal basis containing \vec{v}_1

$$1. \vec{v}_1' = (0,1,0)$$

$$\vec{v}_2' = (2,1,0) - \text{proj}_{\vec{v}_1'}(\vec{v}_2) = (2,1,0) - \frac{(2,1,0) \cdot (0,1,0)}{(0,1,0) \cdot (0,1,0)} \cdot (0,1,0)$$

$$= (2,1,0) - (0,1,0)$$

$$= (2,0,0)$$

$$\vec{v}_3' = (-1,0,1) - \frac{(-1,0,1) \cdot (0,1,0)}{(0,1,0) \cdot (0,1,0)} (0,1,0) - \frac{(-1,0,1) \cdot (2,0,0)}{(2,0,0) \cdot (2,0,0)} (2,0,0)$$

$$= (-1,0,1) - 0 - \frac{-2}{4} \cdot (2,0,0)$$

$$= (-1,0,1) + (1,0,0)$$

$$= (0,0,1).$$

$$\Rightarrow v_1'' = (0,1,0)$$

$$v_2'' = (1,0,0)$$

$$v_3'' = (0,0,1)$$