13.10.2023

1) $x_n \to \ell$: $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$:

$|x_n - \ell| < \varepsilon \ , \forall n \geq N_\varepsilon$

$$\frac{1}{\sqrt{n}} \to 0$$

Let $\varepsilon > 0$. We want to find $N_\varepsilon \in \mathbb{N}$ s.t.

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon, \ \forall n \geq N_\varepsilon$$

ℓ

$$\frac{1}{\sqrt{n}} < \varepsilon \iff \frac{1}{\varepsilon} < \sqrt{n} \iff \frac{1}{\varepsilon^2} < n$$

Take $N_\varepsilon = \left[\frac{1}{\varepsilon^2}\right] + 1 > \frac{1}{\varepsilon^2}$

Then $\forall n \geq N_\varepsilon > \frac{1}{\varepsilon^2} \iff \frac{1}{\sqrt{n}} < \varepsilon \ ; \forall n \geq N_\varepsilon$

2) a) $x_n = \sqrt{n+1} - \sqrt{n} = \dfrac{\cancel{n}+1 - \cancel{n}}{\sqrt{n+1} + \sqrt{n}} =$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n \to \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\infty} = 0$$

$$x_{n+1} \overset{?}{\leq} x_n$$

$$\frac{1}{\sqrt{n+2} + \sqrt{n+1}} \overset{?}{\leq} \frac{1}{\sqrt{n+1} + \sqrt{n}} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{n+2} + \cancel{\sqrt{n+1}} \overset{?}{\geq} \cancel{\sqrt{n+1}} + \sqrt{n}$$

$$\Leftrightarrow \sqrt{n+2} \overset{?}{\geq} \sqrt{n} \quad \text{„True" (Decreasing)}$$

$x_n \in (0, 1]$

Bounded

b) $x_n = \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dots + \dfrac{1}{n(n+1)} =$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$x_{n+1} - x_n \overset{?}{>} 0$$

$$1 - \frac{1}{n+2} - 1 + \frac{1}{n+1} \overset{?}{>} 0$$

$$\frac{1}{n+1} - \frac{1}{n+2} \overset{?}{>} 0 \quad \text{„True"}$$

$$x_{n+1} > x_n \ , \forall n \in \mathbb{N}$$

$$x_n < 1$$

$$x_n \in \left[\frac{1}{2}, 1\right)$$

d) $x_n = \dfrac{2^n}{n!}$

$$\frac{x_{n+1}}{x_n} = \frac{\cancel{2^{n+1}}^{\,2}}{\underbrace{\cancel{(n+1)!}}_{n+1}} \cdot \frac{\cancel{n!}^{\,1}}{\underbrace{\cancel{2^n}}_{1}} = \frac{2}{n+1} < 1 \text{ for } n+1 > 2 \Leftrightarrow$$

$\Leftrightarrow n > 1$

$1 \nearrow 2, 2 \searrow \frac{8}{6}$

$x_{n+1} < x_n$, for $n > 1$

$$\left.\begin{array}{l} x_n \in (0, 2] \Rightarrow \text{bounded} \\ x_n \text{ decreasing}, n > 1 \end{array}\right\} \Rightarrow \text{Convergent}$$

Ratio test

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$$

- if $l < 1$, then $x_n \to 0$
- if $l > 1$, then $x_n \to \infty$

(*) Behaves like a geometric prog.

In our case: $\frac{x_{n+1}}{x_n} = \frac{2}{n+1} \to 0 < 1 \Rightarrow$

$\Rightarrow x_n \to 0$

3) a) $x_n = \sqrt{n}\left(\underbrace{\sqrt{n+1} - \sqrt{n}}_{\sqrt{n+1}+\sqrt{n}}\right)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\,(\cancel{n}+1-\cancel{n})}{\sqrt{n+1}+\sqrt{n}} = \lim \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{\frac{n+1}{n}}+1\right)} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\underbrace{\tfrac{1}{n}}_{\to 0}}+1} =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

ex $\lim_{n\to\infty} (2^n + 3^n)^{\frac{1}{n}} = \lim_{n\to\infty} (3^n)^{\frac{1}{n}} \left[\underbrace{\left(\tfrac{2}{3}\right)^n}_{\to 0} + 1\right]^{\overset{\to 0}{\frac{1}{n}}} =$

$= 3 \cdot 1^0 = 3$

b) $\left(a_1^n + a_2^n + \dots + a_k^n\right)^{\frac{1}{n}}$, with $a_i \geq 0$

let $a_m = \max\{a_1, a_2, \dots, a_k\}$

$$(a_m^n)^{\frac{1}{n}} \cdot \underbrace{\left[\left(\frac{a_1}{a_m}\right)^n + \left(\frac{a_2}{a_m}\right)^n + \dots + \left(\frac{a_k}{a_m}\right)^n\right]^{\frac{1}{n}}}_{E} \stackrel{(*)}{=}$$

$$1 \leq E \leq k \quad |^{\frac{1}{n}}$$

$$1 \leq E^{\frac{1}{n}} \leq k^{\frac{1}{n}}$$

$$1 \leq \lim E \leq 1$$

$$\lim E = 1$$

$\stackrel{(**)}{=} \quad a_m$

c) $\sqrt[n]{n} = n^{\frac{1}{n}} = \infty^0$

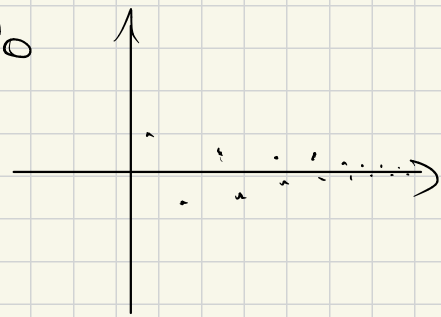
Hint $a = e^{\ln a}$, $f^g = e^{\ln f^g} = e^{g \cdot \ln f}$

$$n^{\frac{1}{n}} = e^{\frac{1}{n}\cdot \ln n} = e^{\frac{\ln n}{n}} = e^{0} = 1, \quad n\to\infty$$

e) $$\lim_{n\to\infty} \frac{2^n+(-1)^n}{3^n} = \lim_{n\to\infty}\left[\frac{2^n}{3^n}+\frac{(-1)^n}{3^n}\right] =$$

$$= \lim_{n\to\infty}\Big[\underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} + \underbrace{\left(-\frac{1}{3}\right)^n}_{\to 0}\Big] = 0$$

$$\left|\left(-\frac{1}{3}\right)^n\right| = \left(\frac{1}{3}\right)^n \to 0$$



f) $$\lim_{n\to\infty} \frac{(an+1)^2}{4n^2-2n+1}, \quad a\in\mathbb{R}$$

$$\lim_{n\to\infty} \frac{(an+1)^2}{4n^2-2n+1} = \lim_{n\to\infty} \frac{a^2n^2+2an+1}{4n^2+2n+1} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(a^2+\frac{2a}{n}+\frac{1}{n^2}\right)}{\cancel{n^2}\left(4+\frac{2}{n}+\frac{1}{n^2}\right)} = \frac{a^2}{4}, \text{ for } a\neq 0$$

$$a=0 \Rightarrow \lim_{n\to\infty} \frac{1}{4n^2-2n+1} = \frac{1}{\infty} = 0, \text{ for } a=0$$

4) $e_n = \left(1+\frac{1}{n}\right)^n$

$$\binom{n}{k} = C_n^k = \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

"n choose k"

$$(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+\dots+\binom{n}{k}x^k+\dots+x^n$$

$$e_n = 1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\dots+\binom{n}{k}\frac{1}{n^k}+\dots+\frac{1}{n^n}$$

$$= 1+1+\frac{n(n-1)}{2}+\dots+\frac{n(n-1)\dots(n-k+1)}{k!}\cdot\frac{1}{n^k}+\dots+\frac{1}{n^n}$$

$$= 1+1+\frac{1}{2}\cdot 1\cdot\underbrace{\left(1-\frac{1}{n}\right)}_{<1}+\dots+\frac{1}{k!}\cdot 1\cdot\underbrace{\left(1-\frac{1}{n}\right)}_{<1}\dots\underbrace{\left(1-\frac{k-1}{n}\right)}_{<1}+\dots+\frac{1}{n^n}$$

$$e_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{k!}+\dots+\frac{1}{n!}$$

$$\frac{1}{3!} = \frac{1}{2\cdot 3} < \frac{1}{2^2}$$

$$\frac{1}{k!} = \frac{1}{2\cdot 3\cdot\dots\cdot k} < \frac{1}{2^{k-1}}$$

$$e_n < 1+\underbrace{1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{k-1}}+\dots+\frac{1}{2^{n-1}}}$$

$$1+q+q^2+\dots+q^{n-1} = \frac{1-q^n}{1-q}$$

$$e_n < 1+\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1+2\left(1-\frac{1}{2^n}\right) < 1+2=3$$

$$e_{n+1} = 1+1+\frac{1}{2}\left(1-\frac{1}{n+1}\right)+\dots+\frac{1}{k!}\cdot 1\cdot\left(1-\frac{1}{k+1}\right)\left(1-\frac{2}{k+1}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{k+1}\right)+\dots$$

$$\Rightarrow \left.\begin{aligned} &e_{n+1} > e_n \\ &e_n < 3 \end{aligned}\right\} \Rightarrow \exists \lim_{n\to\infty} e_n$$

5) a) $$\lim_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n =$$

$$=\lim_{n\to\infty}\left(\frac{2n-1+2}{2n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{2n-1}\right)^n =$$

$$= \underbrace{\boxed{\lim_{n\to\infty}\left(1+\frac{2}{2n-1}\right)^{\frac{2n-1}{2}}}}_{\to e}{}^{\cdot\, n\cdot\frac{2}{2n-1}} = e^{\lim\limits_{n\to\infty}\frac{2n}{2n-1}} =$$

$$= e^1 = e$$

b) $\lim_{n\to\infty} n(\ln(n+2) - \ln(n+1)) =$

$$= \lim_{n\to\infty} n\left(\ln \frac{n+2}{n+1}\right) =$$

$$= \lim_{n\to\infty} \ln\left(\frac{n+2}{n+1}\right)^n = \ln \lim_{n\to\infty}\left(\frac{n+1+1}{n+1}\right)^n$$

$$= \ln \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \ln \underbrace{\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{(n+1)\cdot n\cdot\frac{1}{n+1}}}}_{\to e} =$$

$$= \ln e^{\lim_{n\to\infty}\frac{n}{n+1}} = \ln e^1 = \ln e = 1$$

7) a) $\lim_{n\to\infty} \frac{x_1+\dots+x_n}{n} \overset{S-C}{=} \lim_{n\to\infty} \frac{x_{n+1}}{1} =$

$$= \lim_{n\to\infty} x_n$$

ex $\quad x_n = (-1)^n, \quad x_1+x_2+\dots+x_n \in \{0, -1\}$

x_n average converges but the sequence does not!

b) $$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n} \overset{S-C}{=} \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n} \overset{S-C}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln \frac{n+1}{n}} =$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)\ln \frac{n+1}{n}} = 1$$