1.5.1 Decide whether the following statements are true or false.

~~a)~~ "All the solutions of $x'' + 3x' + x = 1$ satisfy $\lim_{t\to\infty} x(t) = 1$."

b) "The solution of the IVP $x'' + 4x = 1, \quad x(0) = 5/4, \quad x'(0) = 0$ satisfies $x(\pi) = 5/4$."

~~c)~~ "The equation $x' = 3x + t^3$ admits a polynomial solution. (*Hint.* Look for a polynomial solution of degree 3.)"

a) $\lim_{t\to\infty} x(t) = 1$

$$x = x_h + x_p$$

$$x_h \Rightarrow x'' + 3x' + x = 0$$

$$r^2 + 3r + 1 = 0$$

$$\Delta = 9 - 4 = 5$$

$$r_{1,2} = \frac{-3 \pm \sqrt{5}}{2} \longmapsto e^{\frac{\sqrt{5}-3}{2} \cdot t}, \; e^{\frac{-\sqrt{5}-3}{2} \cdot t}$$

$$x_h = c_1 \cdot e^{\frac{\sqrt{5}-3}{2} \cdot t} + c_2 e^{\frac{-\sqrt{5}-3}{2} \cdot t}, \quad c_1, c_2 \in \mathbb{R}$$

$$x_p = 1$$

$$x = c_1 \cdot e^{\frac{\sqrt{5}-3}{2} \cdot t} + c_2 \cdot e^{\frac{-\sqrt{5}-3}{2} \cdot t} + 1, \quad c_1, c_2 \in \mathbb{R}.$$

$$\lim_{t \to \infty} x(t) = 1 \quad \text{„True"}$$

c) $x_p = at^3 + bt^2 + ct + d$

$$x_p' = 3x_p + t^3$$

$$x_p' = 3at^2 + 2bt + c = 3at^3 + 3bt^2 + 3ct + 3d + t^3$$

$$\Leftrightarrow \begin{cases} 0 = 3a + 1 & \Rightarrow a = -\frac{1}{3} \\ 3a = 3b & \Rightarrow b = -\frac{1}{3} \\ 2b = 3c & \Rightarrow c = -\frac{2}{9} \\ c = 3d & \Rightarrow d = \frac{-2}{27} \end{cases}$$

$$x_p = -\frac{1}{3}t^3 - \frac{1}{3}t^2 - \frac{2}{9} \cdot t - \frac{2}{27}$$

x_p is a solution and a pol. function $\Rightarrow$ True

1.5.2 Let $\lambda \in \mathbb{R}$ be a parameter. Find the general solution of $x'' - x = e^{\lambda t}$ knowing that, depending on λ, it has a particular solution either of the form $ae^{\lambda t}$ or of the form $ate^{\lambda t}$.

$x = x_h + x_p$

$x_h : x'' - x = 0$

$\lambda^2 - 1 = 0 \; ; \; \lambda_{1,2} = \pm 1 \longmapsto e^{-t}, e^{t}$

$x_h = c_1 \cdot e^{-t} + c_2 \cdot e^{t} \quad , c_1, c_2 \in \mathbb{R}$

$x_p = a \cdot e^{\lambda t}$

$\Rightarrow \; x_p'' - x_p = e^{\lambda t}$

$x_p' = a \cdot \lambda \cdot e^{\lambda t}$

$x_p'' = a \cdot \lambda^2 \cdot e^{\lambda t}$

$a \cdot \lambda^2 \cdot e^{\lambda t} - a \cdot e^{\lambda t} = e^{\lambda t} \quad \Big| \cdot \frac{1}{e^{\lambda t}} > 0$

$a \cdot \lambda^2 - a = 1$

$a(\lambda^2 - 1) = 1$

$a = \frac{1}{\lambda^2 - 1} \quad , \quad \lambda \in \mathbb{R} \setminus \{\pm 1\}$

$$x_p = \frac{1}{\lambda^2-1} \cdot e^{\lambda t} \quad , \lambda \in \mathbb{R} \setminus \{\pm 1\}$$

$\lambda \in \{-1, 1\}$:

$$x_p = a \cdot t \cdot e^{\lambda t}$$

$$x_p' = a \cdot e^{\lambda t} + a \cdot \lambda \cdot t \cdot e^{\lambda t}$$

$$x_p'' = a \cdot \lambda \cdot e^{\lambda t} + a \cdot \lambda \cdot e^{\lambda t} + a \cdot \lambda^2 \cdot t \cdot e^{\lambda t}$$

$$\Leftrightarrow a \cdot \lambda \cdot e^{\lambda t} + a \cdot \lambda \cdot e^{\lambda t} + a \cdot \lambda^2 \cdot t \cdot e^{\lambda t} -$$

$$- a \cdot t \cdot e^{\lambda t} = e^{\lambda t} \quad \Big| \cdot \frac{1}{e^{\lambda t}}$$

$$\Leftrightarrow a\lambda + a\lambda + \cancel{a\underset{1}{\lambda^2} t} - \cancel{at} = 1$$

$$\Leftrightarrow 2a\lambda = 1$$

$$\Leftrightarrow a = \frac{1}{2\lambda}$$

$$x_p = \frac{1}{2\lambda} \cdot t \cdot e^{\lambda t}$$

$\lambda \in \mathbb{R} \setminus \{-1, 1\}$: $x = c_1 \cdot e^{-t} + c_2 \cdot e^{t} + \frac{1}{\lambda^2-1} \cdot e^{\lambda t}$

$\lambda \in \{-1, 1\}$: $x = c_1 \cdot e^{-t} + c_2 \cdot e^{t} + \frac{1}{2\lambda} \cdot t \cdot e^{\lambda t}$

1.5.3 Let $\omega > 0$ be a parameter and denote $\varphi(\cdot, \omega)$ the solution of the IVP

$$x'' + x = \cos(\omega t), \quad x(0) = x'(0) = 0.$$

~~(i)~~ When $\omega \neq 1$ find a solution of the form $x_p(t) = a\cos(\omega t) + b\sin(\omega t)$ for $x'' + x = \cos(\omega t)$. (Here you have to determine the real coefficients a and b.)

~~(ii)~~ Find a solution of the form $x_p(t) = t(a\cos t + b\sin t)$ for $x'' + x = \cos t$.

~~(iii)~~ Find $\varphi(\cdot, \omega)$ for any $\omega > 0$.

(iv) Prove that $\lim_{\omega \to 1} \varphi(t, \omega) = \varphi(t, 1)$ for each $t \in \mathbb{R}$.

i) $x_p' = -a \cdot w \sin(wt) + b \cdot w \cdot \cos(w \cdot t)$

$x_p'' = -a w^2 \cdot \cos(wt) - b w^2 \cdot \sin(wt)$

$x_p'' + x_p = \cos(wt) \Leftrightarrow$

$\Leftrightarrow -a w^2 \cdot \cos(w \cdot t) - b w^2 \cdot \sin(wt) + a \cdot \cos(wt) + b \cdot \sin(wt) = \cos(wt), \ \forall t \in \mathbb{R}$

$\Leftrightarrow \begin{cases} -a w^2 + a = 1 \Leftrightarrow a(-w^2+1) = 1 \\ -b w^2 + b = 0 \Leftrightarrow b(-w^2+1) = 0 \end{cases} \Rightarrow$

$w>0, w \neq 1$

$\Rightarrow \begin{cases} a = \dfrac{1}{-w^2+1} \\ b = 0 \end{cases}$

$x_p = \dfrac{1}{-w^2+1} \cdot \cos(w \cdot t)$

ii) $x_p = t(a\cdot\cos t + b\sin t)$

$x_p' = (a\cdot\cos t + b\sin t) + t(-a\cdot\sin t + b\cos t)$

$x_p'' = -a\sin t + b\cos t - a\sin t + b\cos t + t(-a\cos t - b\sin t)$

$x_p'' + x_p = \cos t \Leftrightarrow$

$\Leftrightarrow -2a\sin t + 2b\cos t - t(a\cos t + b\sin t) +$

$+ t(a\cos t + b\sin t) = \cos t, \forall t \in \mathbb{R}$

$$\begin{cases} -2a = 0 \Rightarrow a = 0 \\ 2b = 1 \Rightarrow b = \frac{1}{2} \end{cases}$$

$x_p = \frac{t}{2}\sin t$

iii) $x = x_h + x_p$

$x_h: x'' + x = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i \mapsto \cos t, \sin t$

$\alpha = 0, \beta = 1$

$e^{\alpha t}\cdot\cos\beta t, e^{\alpha t}\cdot\sin\beta t$

$$x_h = c_1 \cdot \cos t + c_2 \cdot \sin t$$

- $\omega > 0, \ \omega \neq 1$

$$x = c_1 \cdot \cos t + c_2 \sin t + \frac{1}{1-\omega^2} \cdot \cos(\omega t)$$

$$x' = -c_1 \sin t + c_2 \cos t - \frac{\omega}{1-\omega^2} \cdot \sin(\omega t)$$

$$\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases} \Rightarrow \begin{cases} c_1 + \frac{1}{1-\omega^2} = 0 \\ c_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c_1 = \frac{-1}{1-\omega^2} \\ c_2 = 0 \end{cases}$$

$$\varphi(t, \omega) = \frac{-1}{1-\omega^2} \cdot \cos t + \frac{1}{1-\omega^2} \cdot \cos(\omega t) =$$

$$= \frac{1}{1-\omega^2} \left(-\cos t + \cos(\omega t) \right)$$

- $\omega = 1$

$$x = c_1 \cdot \cos t + c_2 \sin t + \frac{t}{2} \cdot \sin t$$

$$x' = -c_1 \cdot \sin t + c_2 \cos t + \frac{t}{2} \cdot \cos t$$

$$\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases} \Rightarrow \begin{cases} c_1 \cdot \cos t + \left(c_2 + \frac{t}{2}\right) \cdot \sin t = 0 \\ -c_1 \cdot \sin t + \left(c_2 + \frac{t}{2}\right) \cdot \cos t = 0 \end{cases}$$

$$\Rightarrow c_1 = c_2 = 0 \Rightarrow \varphi(t, 1) = \frac{t}{2} \cdot \sin t$$

iv) $$\lim_{\omega \to 1} \left(\frac{\cos(\omega t) - \cos(t)}{1 - \omega^2} \right) \underset{\frac{0}{0}}{\overset{l'H}{=}} \lim_{\omega \to 1} \frac{-t \sin(\omega t)}{-2\omega}$$

$$= \frac{-t \cdot \sin t}{-2} = \frac{t \sin t}{2} = \varphi(t, 1)$$

iv) Note that $\lim_{\omega \to 0} \varphi(t, \omega) = 1 - \cos t, \; \forall t \in \mathbb{R}$

Is $t \in \mathbb{R} \longmapsto 1 - \cos t$ the unique sol., denoted $\varphi(t, 0)$, of the ivp $\begin{cases} x'' + x = 1 \\ x(0) = x'(0) = 0 \end{cases}$?

YES

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$x = 1 - \cos t$

$x' = \sin t$

$x'' = \cos t$

$x'' + x = \cos t + 1 - \cos t = 1$

$x(0) = 1 - \cos(0) = 1 - 1 = 0$

$x'(0) = \sin(0) = 0$

True

uniqueness of the sol. by the ivp property

1.3.3 Find the general solution of $x' + \frac{1}{t}x = \frac{1}{t}e^{-2t+1}$ for $t \in (0, \infty)$. Justify the result in two ways.

first order LnHDE with variable coeff.

Method 1:

$X = X_h + X_p$

$X_h: \; x' + \frac{1}{t}x = 0$

Separation of variables

$$x' = -\frac{x}{t}$$

$$\frac{dx}{dt} = \frac{-x}{t}$$

$\cdot\; x \neq 0: \; \int \frac{dx}{x} = \int\left(-\frac{dt}{t}\right)$

$\ln|x| = -\ln|t| + c_0$, $c_0 \in \mathbb{R}$

$|x| = e^{-\ln|t| + c_0} = \frac{e^{c_0}}{|t|} = \frac{e^{c_0}}{t}$, $t > 0$

$\Rightarrow x = \frac{c_1}{t}$, $c_1 \in \mathbb{R}^*$

$x_0 = 0$ sol.

$\Rightarrow x = \frac{c}{t}$, $c \in \mathbb{R}$

x_p: Method of Lagrange

$x_p = \varphi(t) \cdot \frac{1}{t}$

$x_p' = \varphi'(t) \cdot \frac{1}{t} - \varphi(t) \cdot \frac{1}{t^2}$

$x_p' + \frac{1}{t} x_p = \frac{1}{t} \cdot e^{-2t+1}$, $\forall t > 0$

$\varphi'(t) \cdot \frac{1}{t} \cancel{- \varphi(t) \cdot \frac{1}{t^2} + \varphi(t) \cdot \frac{1}{t^2}} = \frac{1}{t} \cdot e^{-2t+1}$, $t > 0$

$\varphi'(t) = e^{-2t+1}$

Take $\varphi(t) = -\frac{1}{2} e^{-2t+1}$

$x_p = -\frac{1}{2t} \cdot e^{-2t+1}$

$$x = \frac{c}{t} - \frac{1}{2t} \cdot e^{-2t+1}, \quad c \in \mathbb{R}$$

<u>Methods</u>: Integrating factor method.

$$x' + \frac{1}{t}x = \frac{1}{t}e^{-2t+1}, \quad t>0$$

$$A \in \mathcal{C}^1((0,\infty), \mathbb{R})$$

$$(x \cdot e^{-A(t)})' = x' \cdot e^{-A(t)} - x \cdot A'(t) \cdot e^{-A(t)} =$$

$$= e^{-A(t)}(x' - A'(t) \cdot x)$$

$-A'(t) = \frac{1}{t}$, take $A(t) = -\ln(t) \Rightarrow$

$\Rightarrow e^{-A(t)} = e^{\ln t} = t \to$ integrating factor

We multiply the equation by the integrating factor

$$x' + \frac{1}{t}x = \frac{1}{t}e^{-2t+1} \quad | \cdot t$$

$$\underbrace{tx' + x}_{(tx)'} = e^{-2t+1} \quad | \int$$

$$tx = -\frac{1}{2} e^{-2t+1} + c \quad , c \in \mathbb{R}$$

$$x = -\frac{1}{2t} \cdot e^{-2t+1} + \frac{c}{t} \quad , c \in \mathbb{R}$$