20.10.2023

1) a) $\sum_{n\geq 1} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \dots + \frac{2}{3^n} + \dots$

$q = \frac{1}{3}$; $\frac{q^n - 1}{q-1} = \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \dots + q^{n-1}$

$n \to \infty \Rightarrow \frac{-1}{q-1} = \frac{1}{1-q} = 1 + q + q^2 + \dots$

$\frac{2}{3} + \frac{2}{9} + \dots = \frac{2}{3} \cdot \underbrace{\left(1 + \frac{1}{3} + \dots\right)}_{\frac{1}{1-q}} = \frac{2}{3} \cdot \frac{1}{1-\frac{1}{3}} =$

$= \frac{2}{3} \cdot \frac{1}{\frac{2}{3}} = 1$

b) $\sum_{n\geq 1} \frac{2n+1}{n!} = \frac{2+1}{1!} + \frac{4+1}{2!} + \dots + \frac{2n+1}{n!} + \dots \Rightarrow$

$$\boxed{\sum_{n\geq 0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots = e}$$

!!

$\Rightarrow \sum_{n\geq 1} \frac{2n}{n!} + \underbrace{\sum_{n\geq 1} \frac{1}{n!}}_{e-1} = 2 \cdot \underbrace{\sum_{n\geq 1} \frac{1}{(n-1)!}}_{\sum_{n\geq 0} \frac{1}{n!}} + e - 1 =$

$$= 2e + e - 1 = 3e - 1$$

c) $$\sum_{n\geq 1} \frac{1}{4n^2-1} = \sum_{n\geq 1} \frac{1}{(2n-1)(2n+1)} =$$

$$= \frac{1}{2} \sum_{n\geq 1} \frac{(2n+1)-(2n-1)}{(2n-1)(2n+1)} =$$

$$= \frac{1}{2} \sum_{n\geq 1} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= \frac{1}{2} \left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots \right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-1}} - \underbrace{\frac{1}{2n+1}}_{\to 0} \right)$$

$$= \frac{1}{2}$$

d) $$\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$$

$$\frac{1}{n(n+1)(n+2)} = {}^{n+2)}\frac{1}{n(n+1)} - {}^{n+1)}\frac{1}{n(n+2)}$$

$$= \underbrace{\sum_{n\geq 1} \frac{1}{n(n+1)}}_{=1} - \underbrace{\sum_{n\geq 1} \frac{1}{n(n+2)}}_{\frac{3}{4}} \quad =$$

$$\sum_{n\geq 1} \frac{1}{n(n+1)} = \sum_{n\geq 1} \frac{n+1-n}{n(n+1)} =$$

$$= \sum_{n\geq 1} \frac{\cancel{n+1}}{n\cancel{(n+1)}} - \frac{\cancel{n}}{\cancel{n}(n+1)} = \sum_{n\geq 1} \frac{1}{n} - \frac{1}{n+1} =$$

$$= \lim_{n\to\infty} \left(1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n}} - \underbrace{\frac{1}{n+1}}_{\to 0} \right)$$

$$= 1 - 0 = 1$$

$$\sum_{n\geq 1} \frac{1}{n(n+2)} = \frac{1}{2} \sum \frac{n+2-n}{n(n+2)} =$$

$$= \frac{1}{2} \sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \frac{1}{2} \lim_{n\to\infty} \left(1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \right.$$

$$\left. - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6} + \ldots + \cancel{\frac{1}{n}} - \underbrace{\frac{1}{n+2}}_{\to 0} \right) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$$

$$= \quad 1 - \frac{3}{4} = \frac{1}{4}$$

e) $\sum_{n\geq 1} \frac{n}{2^n} = \left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \dots + \frac{n}{2^n} + \dots\right)$

$\sum x_n$, $\quad \frac{x_{n+1}}{x_n} \to l$ — if $l<1$, $\sum x_n$ conv.; if $l>1$, $\sum x_n$ div.

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$$

The series is convergent

$$\bullet\ S = \sum_{n\geq 1} x_n = \sum_{n\geq 1} \frac{n}{2^n} = \sum_{n\geq 1} \frac{n+1-1}{2^n} =$$

$$= \sum_{n\geq 1} \frac{n+1}{2^n} - \sum_{n\geq 1} \frac{1}{2^n}$$

$$= 2\cdot\sum_{n\geq 1} \frac{n+1}{2^{n+1}} - \underbrace{\sum_{n\geq 1} \frac{1}{2^n}}_{\frac{1}{2}+\frac{1}{4}+\dots = 1}$$

$$= 2\sum_{n\geq 1} \frac{n+1}{2^{n+1}} - 1 = 2\cdot\sum_{n\geq 2} \frac{n}{2^n} - 1 =$$

$$= 2\cdot\left(S-\frac{1}{2}\right)-1$$

excluded term

$$= 2S-2$$

$$S = 2S-2 \Rightarrow \boxed{S=2}$$

General method

$$S_n = \sum_{k=1}^{n} k\cdot x^k = x\sum_{k=1}^{n} k\, x^{k-1}$$

$$= x\sum_{k=1}^{n} (x^k)' = x\cdot\left(\sum_{k=1}^{n} x^k\right)'$$

$$\sum_{k=1}^{n} x^k = x+x^2+\dots+x^n = x\left(1+x+x^2+\dots+x^{n-1}\right)$$

$$= x\,\frac{x^n-1}{x-1} = \frac{x^{n+1}-x}{x-1}$$

$$\left(\sum_{k=1}^{n} x^k\right)' = \left(\frac{x^{n+1}-x}{x+1}\right)' =$$

$$= \frac{(x^{n+1}-x)'(x+1)-(x^{n+1}-x)(x+1)'}{(x-1)^2}$$

$$= \frac{\left[(n+1)\cdot x^n-1\right](x+1)-(x^{n+1}-x)\cdot 1}{(x-1)^2}$$

$$= \frac{\underbrace{n \cdot x^{n+1} - (n+1) \cdot x^{n}}_{\to 0} + 1}{(x-1)^2} \xrightarrow{n \to \infty} \frac{1}{(x-1)^2}$$

$n x^n \to$ $\boxed{0 \quad \text{if } x<1}$

$$\lim_{n \to \infty} S_n = \frac{x}{(x-1)^2} = \sum_{n \geq 1} n x^n$$

ex $\quad x = \frac{1}{2} \Rightarrow \frac{\frac{1}{2}}{\frac{1}{4}} = 2$

f) $$\sum_{n \geq 1} \frac{1}{\sqrt[3]{n}} = \sum_{n \geq 1} \frac{1}{n^{\frac{1}{3}}}$$

$$\sqrt[2]{n} < n \Rightarrow \frac{1}{\sqrt[3]{n}} > \frac{1}{n}$$

$$\sum_{n \geq 1} \frac{1}{n} = \infty \Rightarrow \sum_{n \geq 1} \frac{1}{\sqrt[3]{n}} = \infty$$

3) a) $$\sum_{n \geq 2} \frac{1}{\ln n}$$

$$n > \ln n \Leftrightarrow \frac{1}{\ln n} > \frac{1}{n} \Bigg\} \Rightarrow \sum_{n \geq 2} \frac{1}{\ln n} = \infty$$

$$\sum_{n\geq 2} \frac{1}{n} = \infty$$

b) $$\sum_{n\geq 1} \frac{1}{n\sqrt{n+1}}$$

$$n\sqrt{n+1} < n^2$$

$$\frac{1}{n\sqrt{n+1}} > \frac{1}{n^2} \quad ; \quad \sum_{n\geq 1} \frac{1}{n^2} \text{ converges}$$

$$n\sqrt{n+1} > n\sqrt{n}$$

$$\frac{1}{n\sqrt{n+1}} < \frac{1}{n\sqrt{n}}$$

$\sum_{n\geq 1} \frac{1}{n^p}$ conv. when $p>1$

$$\sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{\frac{3}{2}}} \xrightarrow{\frac{3}{2}>1} \text{converges} \Rightarrow$$

$$\underset{\text{comparison}}{\Rightarrow} \sum \frac{1}{n\sqrt{n+1}} \text{ converges}$$

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n+1}}}{\frac{1}{n^{3/2}}} = 1$$

$\lim \frac{x_n}{y_n} = l \in (0, +\infty)$, then $(x_n), (y_n)$ have the same nature

c) $\sum_{n \geq 1} \frac{\ln(1+\frac{1}{n})}{n}$

$$\left(1+\frac{1}{n}\right)^n \to e$$

$$n \ln\left(1+\frac{1}{n}\right) \to 1$$

$$\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} \to 1, \quad \ln\left(1+\frac{1}{n}\right) \approx \frac{1}{n}$$

$$\sum_{n\geq 1} \frac{\ln(1+\frac{1}{n})}{n}, \quad \frac{\ln(1+\frac{1}{n})}{n} \approx \frac{1}{n^2}$$

$\sum_{n\geq 1} \frac{1}{n^2}$ converges $(2>1) \Rightarrow \sum_{n\geq 1} \frac{\ln(1+\frac{1}{n})}{n}$ conv.

d) $\sum \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim \frac{\cancel{(n+1)!}^{\,n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{n!}_{\,1}} =$$

$$= \lim \frac{\cancel{(n+1)} \cdot n^n}{(n+1)^{\cancel{n+1}\, n}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^{n}_{(-n+1)\left(-\frac{1}{(n+1)}\right)\cdot n} =$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) =$$

$$= e^{\lim_{n\to\infty} -\frac{n}{n+1}} = e^{-1} = \frac{1}{e} < 1 \Rightarrow$$

$\Rightarrow \sum \frac{n!}{n^n}$ conv.

e) $\sum \left(\frac{n}{n+1}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = l \begin{cases} l > 1 \Rightarrow \text{conv} \\ l < 1 \Rightarrow \text{div.} \end{cases}$$

$$\sqrt[n]{x_n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1 \Rightarrow \text{conv.}$$

f) $\sum_{n\geq 2} \frac{1}{n \ln n}$ "like" $\sum_{n\geq 1} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \cdot \ln 2^n}$

Ratio test: $\frac{n \ln n}{(n+1)\ln(n+1)} \to 1$? (not helpful)

<u>Cauchy Condensation Test:</u>

$\sum_{n\geq 1} x_n$ "like" $\sum_{n\geq 0} 2^n x_{2^n}$

$$\sum_{n\geq 0} \frac{1}{n \ln 2} = \frac{1}{\ln 2} \sum_{n\geq 0} \frac{1}{n} = \infty \Rightarrow$$

$\Rightarrow$ our series is also divergent.