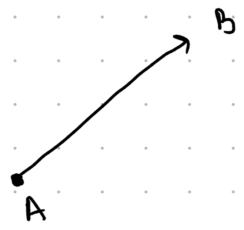
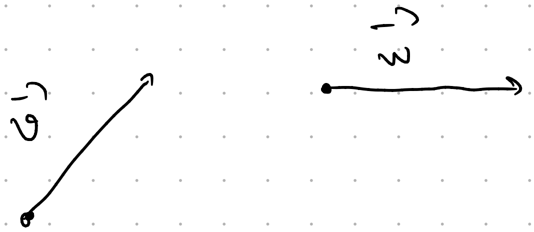
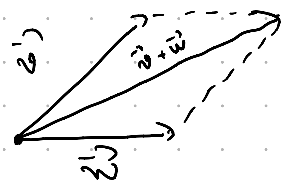
$\mathbb{E}^n$ - n - dimensional Euclidean space
(usually $n=2$ or $n=3$)

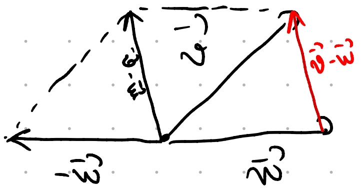


$\mathbb{V}^n$ - space of vectors in $\mathbb{E}^n$
($\mathbb{R}$ - vector space)





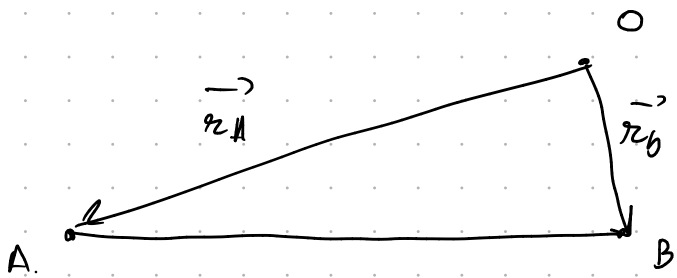
ADDITION
parallelogram rule



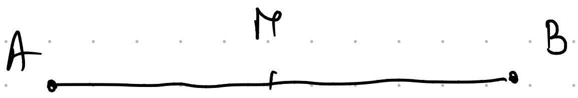
SUBTRACTION
triangle rule.

$A_1, A_2, \dots A_k \in \mathbb{E}^n \Rightarrow \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots + \overrightarrow{A_{n-1}A_n} = \vec{0}$

If we fix $O \in \mathbb{E}^n \Rightarrow \forall A \in \mathbb{E}^n$ we can define $\vec{r}_A = \overrightarrow{OA}$
= the position vector of A (w/ respect to O)



$\overrightarrow{AB} = \vec{r}_B - \vec{r}_A$

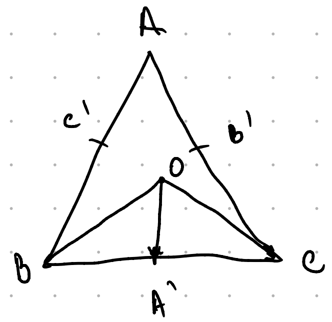


$A, M, B \in \mathcal{E}^n$

M - midpoint of $[AB]$ $\Longleftrightarrow$ $\vec{r}_M = \frac{1}{2}(\vec{r}_A + \vec{r}_B)$

6. A', B', C' mid points of sides of a triag. Show that $\forall\, O \in \mathcal{E}^n$ we have

$\vec{OA'} + \vec{OB'} + \vec{OC'} = \vec{OA} + \vec{OB} + \vec{OC}$



$\vec{OA'} = \frac{1}{2}(\vec{OB} + \vec{OC})$

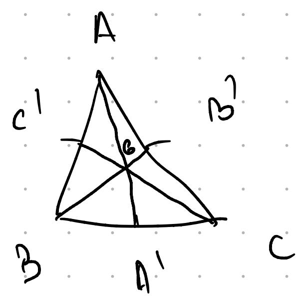
$\vec{OB'} = \frac{1}{2}(\vec{OA} + \vec{OC})$

$\vec{OC'} = \frac{1}{2}(\vec{OA} + \vec{OB})$ $+$

$\vec{OA'} + \vec{OB'} + \vec{OC'} = \frac{1}{2}(2\vec{OA} + 2\vec{OB} + 2\vec{OC})$

$= (\vec{OA} + \vec{OB} + \vec{OC})$

7.



A' - mid BC

B' - mid AC

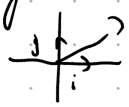
C' - mid AB

A, B, C collinear $\Leftrightarrow \exists\, \ell \in \mathcal{E}^n$, $A, B, C \in \ell \Leftrightarrow \vec{AB}, \vec{BC}$ linearly dependent $\Leftrightarrow \vec{AB} = \alpha \vec{BC}$

$K = (O, \mathcal{B})$ - reference system

$O \in \mathcal{E}^n$

$\mathcal{B}$ - basis of $\mathcal{V}^n$



$\vec{i}, \vec{j}$, basis of $\mathcal{V}^2$ $\Rightarrow [A]_K = [\vec{r}_A]_{\mathcal{B}}$

A, B, C collinear $\Leftrightarrow [\vec{AB}]_B = \alpha [\vec{AC}]_B$

8\. a) $A(3,-5)$, $B(-1,2)$, $C(-5,9)$

b) $\cancel{A(11,2)}$, $B(1,-3)$, $C(31,13)$

c) $A(1,0,-1)$, $B(0,-1,2)$, $C(-1,2,5)$

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$

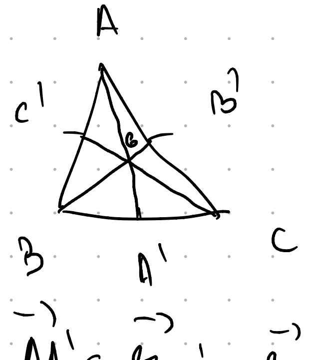
a) $\vec{AB} = \vec{r_B} - \vec{r_A} = (-1,2) - (3,-5) = (-4,7)$
$\vec{BC} = \vec{r_C} - \vec{r_B} = (-5,9) - (-1,2) = (-4,7)$
$\Rightarrow \vec{AB} = 1 \cdot \vec{BC}$
$\Rightarrow$ colinear.

b) $\vec{AB} = \vec{r_B} - \vec{r_A} = (1,-3) - (11,2) = (-10,-5)$
$\vec{BC} = \vec{r_C} - \vec{r_B} = (31,13) - (1,-3) = (30,16)$
$\Rightarrow$ they are not collinear.

c) $\vec{AB} = (0,-1,2) - (1,0,-1) = (-1,-1,3)$
$\vec{BC} = (-1,2,5) - (0,-1,2) = (-1,-1,3)$
$\Rightarrow$ lin. dependent
$\Rightarrow A, B, C$ collinear.

d) $\vec{AB} = (1,1,0) - (-1,-1,-4) = (2,2,4)$
$\vec{BC} = (2,2,2) - (1,1,0) = (1,1,2)$
$\Rightarrow$ lin. dep.
$\Rightarrow$ collinear.

7\.



$\vec{AA'} = \vec{r_{A'}} - \vec{r_A} = \frac{1}{2}(\vec{r_B} + \vec{r_C}) - \vec{r_A}$

$\vec{AG} = \vec{r_G} - \vec{r_A} = \frac{1}{3}(\vec{r_A} + \vec{r_B} + \vec{r_C}) - \vec{r_A}$

$\vec{AG} = \alpha \cdot \vec{AA'}$

$\frac{1}{3}\vec{r_A} + \frac{1}{3}\vec{r_B} + \frac{1}{3}\vec{r_C} - \vec{r_A} = \frac{\alpha}{2}\vec{r_B} + \frac{\alpha}{2}\vec{r_C} - \alpha\vec{r_A}$

$\Rightarrow -\frac{2}{3}\vec{r_A} + \frac{1}{3}\vec{r_B} + \frac{1}{3}\vec{r_C} = \frac{\alpha}{2}\vec{r_B} + \frac{\alpha}{2}\vec{r_C} - \alpha\vec{r_A}$

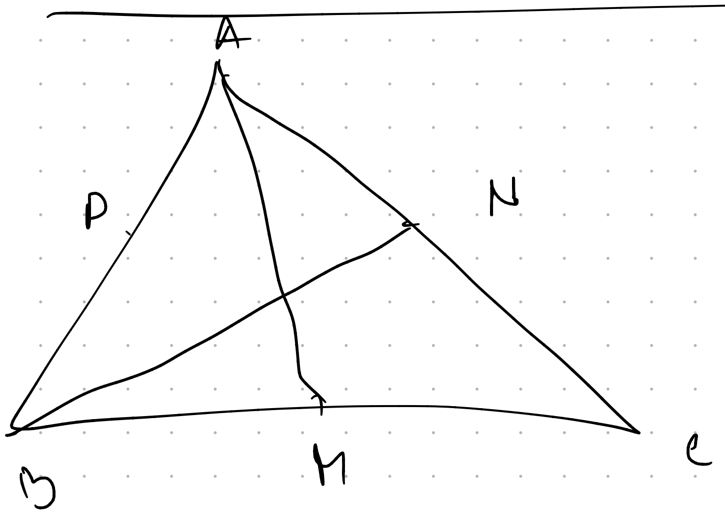
$\Rightarrow \alpha = \frac{2}{3} \Rightarrow$ collinear

$A, B \in \mathbb{E}^m$ distinct

$T \in AB \Rightarrow$

$\exists \lambda \in \mathbb{R}$ s.t.

$\vec{r_T} = \lambda \vec{r_A} + (1-\lambda)\vec{r_B}$ (the vector equation of the line)



$AM \cap BN = \{G\}$

show that $G \in CP$

$G \in AM \Rightarrow \exists \lambda \in \mathbb{R}$:

$\vec{r_G} = \lambda \vec{r_A} + (1-\lambda)\vec{r_M} =$

$= \lambda \vec{r_A} + \frac{1-\lambda}{2}\vec{r_B} + \frac{1-\lambda}{2}\vec{r_C}$

$G \in BN \Rightarrow \exists \mu \in \mathbb{R}$: s.t. $\vec{r_G} = \mu \vec{r_B} + \frac{1-\mu}{2}\vec{r_A} + \frac{1-\mu}{2}\vec{r_C}$

$$\Rightarrow \left(\lambda - \frac{1-\mu}{2}\right)\vec{r_A} + \left(\frac{1-\lambda}{2} - \mu\right)\vec{r_B} + \left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right)\vec{r_C}$$

$$= \vec{0}$$

ΔABC - non-degenerate $\Rightarrow$ $\vec{v} = AB$ and $\vec{w} = \vec{AC}$ are lin. indep.

$$\vec{r_B} = \vec{r_A} + \vec{v}$$
$$\vec{r_C} = \vec{r_A} + \vec{w}$$

$$\Rightarrow \left(\lambda - \frac{1-\mu}{2}\right)\vec{r_A} + \left(\frac{1-\lambda}{2} - \mu\right)(\vec{r_A} + \vec{v}) + \left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right)(\vec{r_A} + \vec{w}) = \vec{0}$$

$$\vec{r_A}\left(\cancel{\lambda} - \cancel{\frac{1-\mu}{2}} + \cancel{\frac{1-\lambda}{2}} - \cancel{\mu} + \cancel{\frac{1-\lambda}{2}} - \frac{1-\mu}{2}\right) + \left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right)\vec{w} + \left(\frac{1-\lambda}{2} - \mu\right)\vec{v}$$

$$= \vec{0} \Rightarrow \left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right)\vec{w} + \left(\frac{1-\lambda}{2} - \mu\right)\vec{v} = 0 \quad \Big\} \Rightarrow$$

w, v - lin. indep

$$\begin{cases} \frac{1-\lambda}{2} - \frac{1-\mu}{2} = 0 \Rightarrow 1-\lambda = 1-\mu \Rightarrow \lambda = \mu \\ \frac{1-\lambda}{2} - \mu = 0 \Rightarrow 1-\lambda = 2\mu \Rightarrow 3\lambda = 1 \Rightarrow \lambda = \mu = \frac{1}{3} \end{cases}$$

$$\Rightarrow \vec{r_G} = \frac{1}{3}\left(\vec{r_A} + \vec{r_B} + \vec{r_C}\right)$$

$$\vec{GP} = \frac{1}{2}\left(\vec{r_A} + \vec{r_B}\right) - \frac{1}{3}\left(\vec{r_A} + \vec{r_B} + \vec{r_C}\right)$$

$$= \frac{1}{6}\vec{r_A} + \frac{1}{6}r_B - \frac{1}{3}\vec{r_C}$$

$$\vec{CP} = \frac{1}{2}\left(\vec{r_A} + \vec{r_B}\right) - \vec{r_C}$$

$= 3\vec{GP} \Rightarrow G \in CP \Rightarrow AM, BN, CP$ - congruent in $P \Rightarrow$

$$\frac{PG}{CP} = \frac{1}{3}$$