

Ch 6.

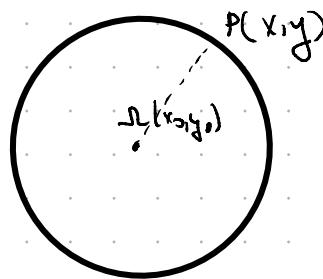
1, 3, 4, 5, 6, 7, 8, 9, 10

Quadratic curves (ellipses)

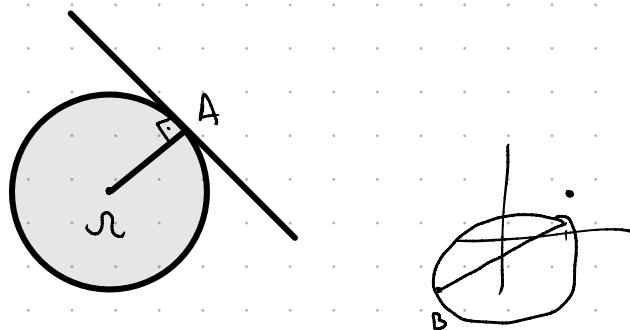
$$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$$

circle = locus of points whose distance to a fixed point $\mathcal{R}(x_0, y_0)$ (called the centre) is $R > 0$ (called the radius)

$$C(\mathcal{R}, R) : (x - x_0)^2 + (y - y_0)^2 = R^2 \quad (\text{implicit equation})$$



$$C(\mathcal{R}, R) : \begin{cases} x = x_0 + R \cos t \\ y = y_0 + R \sin t \end{cases} \quad (\text{parametric equation}) \quad t \in [0, 2\pi]$$



6.1 Find the eq of the circle :

a) of diameter $[AB]$, $A(1, 2)$, $B(-3, -1)$

c) circle passing through $A(3, 1)$, $B(-1, 3)$, having the centre on the line $C: 3x - y - 2 = 0$

$$|AB| = \sqrt{16 + 9} = \sqrt{25} = 5 \Rightarrow R = \frac{5}{2}$$

$$M = \left(\frac{1-3}{2}, \frac{2-1}{2} \right) = \left(-1, \frac{1}{2} \right)$$

$$\Rightarrow (x + 1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{25}{4}$$

$$2) \overrightarrow{AB}(-4, 2)$$

$$\text{mid } [AB] \Rightarrow M(1, 2)$$

$$\Rightarrow -4(x-1) + 2(y-2) = 0$$

$$-4x+4 + 2y - 4 = 0$$

$$-4x + 2y = 0 \Rightarrow y = 2x$$

$$\Rightarrow \begin{cases} 3x-y+2=0 \\ y=2x \end{cases} \Rightarrow 3x-2x+2=0$$

$$x=2$$

$$\Rightarrow y=4$$

$$\Rightarrow J(2, 4)$$

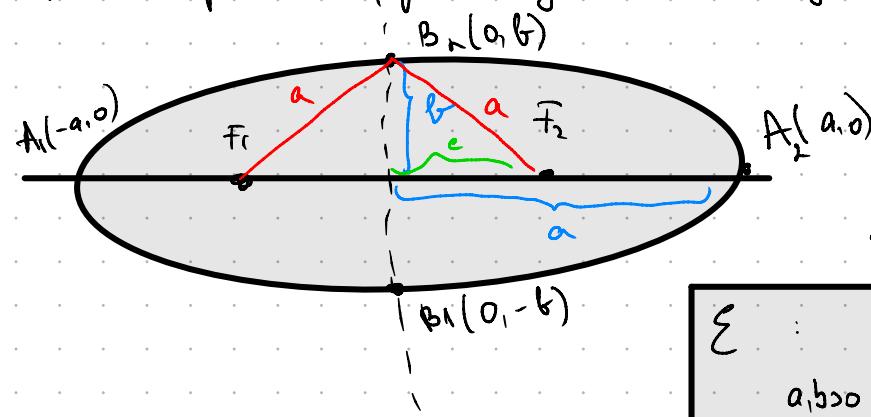
$$R = \sqrt{10}$$

$$\rightarrow C(c, \sqrt{10}) \Rightarrow (x-2)^2 + (y-4)^2 = 10$$

ellipse = locus of points in the plane whose sum of distances to two fixed points f_1, f_2 (which are called the focal points or foci) is a constant $2a$.

If we fix a reference system $K=(0, i, j)$ where O is the midpoint of $[F_1, F_2]$

and $i \in D(F_1, F_2)$



then the eq of the ellipse is:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$F_1(-c, 0)$$

$$F_2(c, 0)$$

$$c = \sqrt{a^2 - b^2}$$

$$\text{If } M \in E: \lambda F_1 + \mu F_2 = 2a$$

$$e_i = \frac{c}{a} = \text{eccentricity (how far is the ellipse from a circle)}$$

$2 \times c = \text{the focal distance}$

6.3. Det. the foci of the ellipse $9x^2 + 25y^2 = 225$

$$\Rightarrow \frac{x^2}{5^2} + \frac{y^2}{3^2} = 1 \Rightarrow a=5, b=3$$

$$c = \sqrt{5^2 - 3^2} = 4 \Rightarrow F_1(-4, 0), F_2(4, 0)$$

6.4. Det. the intersection of the line $l: x + 2y - 7 = 0$

and the ellipse $E: x^2 + 3y^2 - 25 = 0$

$$\Rightarrow x = 7 - 2y$$

$$\Rightarrow 49 - 28y + 4y^2 + 3y^2 - 25 = 0$$

$$24 - 28y + 7y^2 = 0$$

$$\Delta = 28^2 - 28 \cdot 24 = 28(28 - 24) = 28 \cdot 4 = 112.$$

$$y_1 = \frac{28 - \sqrt{112}}{14}$$

$$y_2 = \frac{28 + \sqrt{112}}{14}$$

$$\sqrt{112} =$$

$$5\sqrt{112}$$

$$28\sqrt{112}$$

$$14\sqrt{112}$$

$$7 \cdot 2^4 = 2^2 \sqrt{7}$$

$$= \frac{28 - 4\sqrt{7}}{14} = \frac{14 - 2\sqrt{7}}{7}$$

$$= \frac{28 + 4\sqrt{7}}{14} = \frac{14 + 2\sqrt{7}}{7}$$

$$x_1 = \frac{49 - 28 + 2\sqrt{7}}{7} = \frac{21 + 2\sqrt{7}}{7}$$

$$x_2 = \frac{81 - 2\sqrt{7}}{7}$$

$$l: y = kx + m$$

$$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\begin{aligned} l \cap E: \quad & \left\{ \begin{array}{l} y = kx + m \\ \frac{x^2}{a^2} + \frac{(kx+m)^2}{b^2} = 1 \end{array} \right. \end{aligned}$$

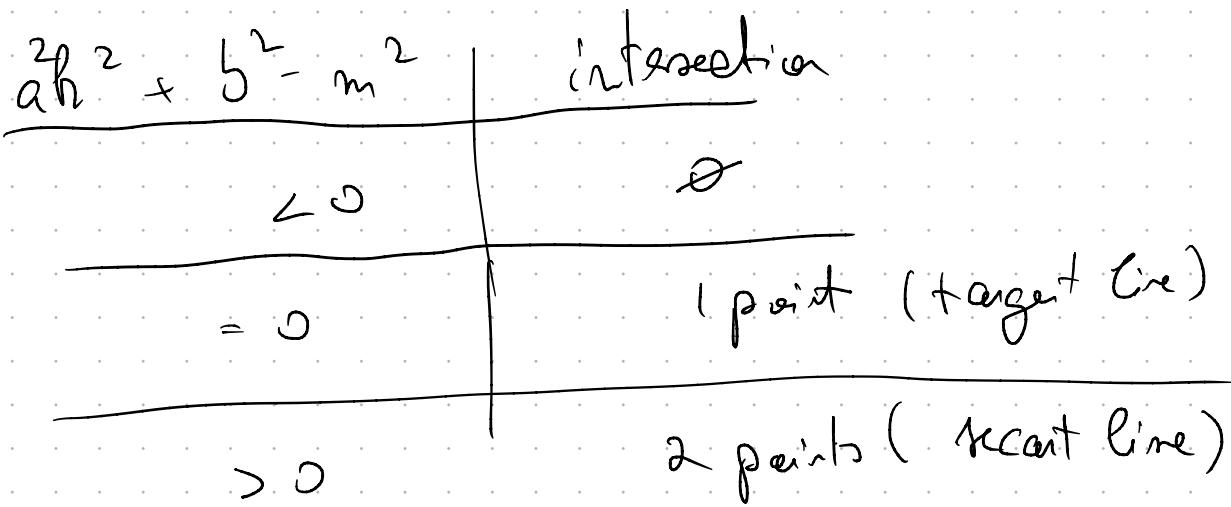
$$x^2 \left(\frac{1}{a^2} + \frac{k^2}{b^2} \right) + \frac{2km}{b^2} \cdot x + \frac{m^2 - 1}{b^2} = 0$$

$$\Delta = \frac{4k^2 m^2}{b^4} - 4 \cdot \frac{b^2 + k^2}{a^2 b^2} \cdot \frac{m^2 - 1}{b^2} =$$

$$= \frac{4}{a^2 b^4} \left(a^2 b^2 m^2 - \frac{(a^2 + a^2 k^2) \cdot (m^2 - 1)}{a^2} \right)$$

$$= \frac{4}{a^2 b^4} \left(a^2 k^2 m^2 - b^2 m^2 + b^4 - a^2 k^2 m^2 + a^2 k^2 b^2 \right)$$

$$= \frac{4}{a^2 b^2} \left(-m^2 + b^2 + a^2 k^2 \right)$$



If $a^2k^2 + b^2 - m^2 = 0$

$\Rightarrow m = \pm \sqrt{b^2 + a^2k^2}$ So the tangents are

$$y = kx \pm \sqrt{b^2 + a^2k^2}$$

The tangent to the ellipse \mathcal{E} : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

in the point (x_0, y_0) is:

$$T(x_0, y_0), \mathcal{E}: \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

$$\Leftrightarrow \underbrace{\frac{x_0}{a^2}(x-x_0)}_{\frac{\partial f}{\partial x}(x_0, y_0)} + \underbrace{\frac{y_0}{b^2}(y-y_0)}_{\frac{\partial f}{\partial y}(x_0, y_0)} = 0$$

where $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

EXAM.

b. b. Find the eq of a line which is orthogonal to the line $l: 2x - 2y - 13 = 0$ and tangent to the ellipse to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 : 4$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 : 5$$

$$\left. \begin{array}{l} \frac{x^2}{20} + \frac{y^2}{5} = 1 \Rightarrow a = 2\sqrt{5} \Rightarrow A_2(2\sqrt{5}, 0) \\ \qquad \qquad \qquad b = \sqrt{5} \Rightarrow B_2(0, \sqrt{5}) \end{array} \right\} \text{line that goes through the points}$$

$$ax + by + c = 0$$

$$a(2\sqrt{5}) + c = 0 \Rightarrow 2a\sqrt{5} + c = 0$$

$$b\sqrt{5} + c = 0$$

$$\Rightarrow c = -b\sqrt{5}$$

$$\Rightarrow 2a - b = 0$$

$$\Rightarrow b = 2a$$

$$\Rightarrow ax + 2a^2 + c = 0 \quad \text{D(l)} = (0, 1)$$

at this is orthogonal to

$$2x - 2y - 13$$

$$\Rightarrow B(l) = (-2, -2)$$

\Rightarrow

$$\left. \begin{array}{l} l_2 \cap E \Rightarrow \left\{ \begin{array}{l} x = x_A + 2t \\ y = y_A - 2t \\ x^2 + 4y^2 - 20 = 0 \end{array} \right. \end{array} \right\}$$

$$x_A^2 + h x_A x + h x^2 + 4y_A^2 - 16y_A x + 4t^2 - 20 = 0$$

$$x_A^2 + hy_A^2 + \lambda(hx_A - 16y_A) + 20x - 20 = 0$$

$$(hx_A - 16y_A)^2 - 4(x_A^2 + hy_A^2 - 20) \cdot 20 = 0$$

$$y^2 - 2yc + c^2 + hy - 20 = 0$$

$$5y^2 - 2cy - 20 + c^2 = 0$$

$$\Delta = 0 \Rightarrow c^2 = \frac{400}{16} \Rightarrow c = \pm 5$$

$$l_2: x+g+5=0$$

$$\text{or } x+g-5=0$$