13. 11. 2023

7.* Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ linear map s.t. $f(2,3) = (1,1)$
$f(4,2) = (6,8)$. Find $f(x,y)$, $\forall (x,y) \in \mathbb{R}^2$

$\left.\begin{array}{l} (2,3) \text{ and } (4,2) \text{ are linear independent} \\ \dim \mathbb{R}^2 = 2 \end{array}\right\} \Longrightarrow$

$\Rightarrow ((2,3),(4,2))$ basis of $\mathbb{R}^2$

generators and l.i.

$\forall$ vector can be uniquely written as a linear combination of the vectors in the basis

$\forall (x,y) \in \mathbb{R}^2 \ \exists \alpha, \beta \in \mathbb{R}$:

$(x,y) = \alpha \cdot (2,3) + \beta (4,2)$

$f(x,y) = f(\alpha \cdot (2,3) + \beta(4,2)) =$

$\overset{f \text{ linear map}}{=} f(\alpha \cdot (2,3)) + f(\beta(4,2)) =$

$$= \alpha \cdot f(2,3) + \beta \cdot f(4,2) =$$

$$= \alpha(1,1) + \beta \cdot (6,8)$$

$$= (\alpha + 6\beta, \alpha + 8\beta)$$

$$\begin{cases} x = 2\alpha + 4\beta \\ y = 3\alpha + 2\beta \end{cases} \iff \begin{cases} \alpha = \dfrac{x - 4\beta}{2} \\ y = 3\alpha + 2\beta \end{cases}$$

$$\iff \begin{cases} \alpha = \dfrac{x - 4\beta}{2} \\ y = \dfrac{3x - 12\beta}{2} + 2\beta \iff y = \dfrac{3x}{2} - 4\beta \iff \end{cases}$$

$$\iff \beta = \frac{\frac{3x}{2} - y}{4}$$

$$\alpha = \frac{x - \left(\frac{3x}{2} - y\right)}{2} = -\frac{x}{4} + \frac{y}{2}$$

$$f(x,y) = \left(-\frac{x}{4} + \frac{y}{2} + \frac{24x}{8} - \frac{6y}{8}, -\frac{x}{4} + \frac{y}{2} + 3x - 2y\right) =$$

$$= \left(\frac{11x}{4} - \frac{y}{5}, \frac{11x}{4} - \frac{3y}{2}\right)$$

7.5, 6 $\quad f \in \operatorname{End}_{\mathbb{R}}(\mathbb{R}^3)$

$$f(x,y,z) = (-y+5z,\ x,\ y-5z)$$

Determine the basis and dimension for Ker f and Im f. and complete these bases to bases of the ambient space.

$f : V \to V$ linear map

$\operatorname{Ker} f = \{ v \in V \mid f(v) = 0 \}$

$$\operatorname{Ker} f = \{ v \in \mathbb{R}^3 \mid f(v) = 0 \}$$

$$v = (x, y, z) \in \mathbb{R}^3$$

$$f(v) = 0 \iff (-y+5z,\ x,\ y-5z) = 0$$

$$\begin{cases} -y+5z = 0 \\ x = 0 \\ y-5z = 0 \end{cases} \iff \begin{cases} y = 5u \\ x = 0 \\ z = u \end{cases},\ u \in \mathbb{R}$$

$$\operatorname{Ker} f = \{ (0, 5u, u),\ u \in \mathbb{R} \}$$
$$= \{ u(0,5,1),\ u \in \mathbb{R} \}$$

$= \langle (0,5,1) \rangle$

$\dim \operatorname{Ker} f = 1$ bcs. the basis of $\operatorname{Ker} f$ has only one vector.

ambient space has dim $= 3 \Rightarrow$ we need 2 more vectors

We want to find $v_2, v_3 \in \mathbb{R}^3$ s.t. (v_1, v_2, v_3) is a basis of $\mathbb{R}^3$

We choose $v_2 \in \mathbb{R}^3 - \langle v_1 \rangle$

$\langle v_1 \rangle = \operatorname{Ker} f$

take $v_2 = (1,0,0) \notin \langle v_1 \rangle$ so v_1, v_2 are lin. indep.

We choose $v_3 \in \mathbb{R}^3 - \langle v_1, v_2 \rangle$

$$\langle v_1, v_2 \rangle = \{ \alpha v_1 + \beta v_2 \mid \alpha, \beta \in \mathbb{R} \}$$

$$= \{ \alpha (0,5,1) + \beta (1,0,0) \mid \alpha, \beta \in \mathbb{R} \}$$

$$= \{ (\beta, 5\alpha, \alpha) \mid \alpha, \beta \in \mathbb{R} \}$$

$v_3 = (0,1,0) \notin \langle v_1, v_2 \rangle \Rightarrow v_1, v_2, v_3$ lin. indep.

so they form a basis.

$$\operatorname{Im} f = \{ f(x) \mid x \in \mathbb{R} \} =$$

$$= \{(-y+5z,\ x,\ y-5z),\ x, y, z \in \mathbb{R}\}$$

$$= \{(-y,\ 0,\ y) + (5z,\ 0,\ -5z) + (0, x, 0),\ x, y, z \in \mathbb{R}\}$$

$$= \{y(-1,0,1) + 5z(1,0,-1) + x(0,1,0),\ x, y, z \in \mathbb{R}\}$$

we now check for linear independence.

$$A = \begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \ \underline{\underline{L_1 = -L_3}} \quad 0 \Rightarrow \text{not lin. indep.}$$

$$d_p = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow (-1,0,1) \text{ and } (0,1,0)$$

are lin. indep $\Rightarrow$

$\Rightarrow$ they form a basis

$$\operatorname{Im} f = \langle \overbrace{(-1,0,1)}^{v_1}, \overbrace{(0,1,0)}^{v_2} \rangle \Rightarrow \dim \operatorname{Im} f = 2.$$

We want to find $v_3 \in \mathbb{R}^3 - \langle v_1, v_2 \rangle \Big\} \Longrightarrow$

$v_3 = (1, 2, 3) \notin \operatorname{Im} f$

$\Longrightarrow (v_1, v_2, v_3)$ is a basis of $\mathbb{R}^3$

7.7 Determine a complement for the following subspaces:

(i) $A = \{ (x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0 \}$

(ii) $B = \{ ax + bx^3 \mid a, b \in \mathbb{R} \}$ in $\mathbb{R}_3[x]$

V k-vector space

$W \to$ subspace of V

$\exists\, W' \leq V_k$ so that $V = W \oplus W'$

$\hookrightarrow$ the complement of W in V

i.e. $V = W + W'$ and $W \cap W' = 0$ (only one elem. in common)

(i) $A = \{ (x, y, z) \in \mathbb{R}^3 \mid x = -2y - 3z \}$

$A = \{ (-2y - 3z, y, z) \in \mathbb{R}^3 \}$

$A = \{(-2y, y, 0) + (-3z, 0, z\}, y, z \in \mathbb{R}\}$

$A = \{y(-2, 1, 0) + z(-3, 0, 1), y, z \in \mathbb{R}\}$

$A = \langle (-2, 1, 0), (-3, 0, 1) \rangle$

$$\begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} = M$$

$$d_p = \begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} = 3 \neq 0$$

$\operatorname{rank} M = 2 \Rightarrow \underbrace{(-2, 1, 0)}_{v_1}, \underbrace{(-3, 0, 1)}_{v_2}$ lin. indep. $\Rightarrow$

$\Rightarrow$ basis.

We want to find a vector $v_3 \in \underbrace{\mathbb{R}^3 \setminus \langle v_1, v_2 \rangle}_{\mathbb{R}^3 \setminus A}$

$v_3 = (0, 0, 1) \notin A \Rightarrow v_1, v_2, v_3$ is a basis of $\mathbb{R}^3 \Rightarrow$

$\Rightarrow A' = \langle v_3 \rangle \rightarrow$ complement of A in $\mathbb{R}^3$

(ii) $B = \{ ax + bx^3 \mid a, b \in \mathbb{R} \}$ in $\mathbb{R}_3[x]$

$B = \langle x, x^3 \rangle$

check for lin. indep. by using def.

Let $a, b \in \mathbb{R}$ s.t. $ax + bx^3 = 0 \Rightarrow a = b = 0 \Rightarrow$

$\Rightarrow x, x^3$ are lin. indep. $\Rightarrow (\underbrace{x}_{v_1}, \underbrace{x^3}_{v_2})$ is a basis for B

$\dim \mathbb{R}_3[x] = 4 \Rightarrow$ we need 2 more vectors v_3, v_4

We choose $v_3 \in \mathbb{R}_3[x] \setminus \langle v_1, v_2 \rangle$

Choose $v_3 = 1 + x + x^3 \Rightarrow (v_1, v_2, v_3)$ lin. indep. $\Rightarrow$

$\Rightarrow$ basis

Now we want $v_4 \in \mathbb{R}_3[x] \setminus \langle v_1, v_2, v_3 \rangle$

$\langle v_1, v_2, v_3 \rangle = \{ a(1+x+x^3) + bx + cx^3 \mid a, b, c \in \mathbb{R} \}$

$= \{ a + (a+b)x + (a+c)x^3 \mid a, b, c \in \mathbb{R} \}$

Choose $v_4 = x^2 \notin \langle v_1, v_2, v_3 \rangle$

$\Rightarrow v_1, v_2, v_3, v_4$ lim. indep $\}$

$\dim \mathbb{R}_3 = 4$ $\Big\}$ $\Rightarrow$

$\Rightarrow$ v_1, v_2, v_3, v_4 basis for $\mathbb{R}_3[x]$ $\Rightarrow$

$\Rightarrow$ the complement of B in $\mathbb{R}_3[x]$ is

$B' = \langle 1+x+x^3, x^2 \rangle$

1st dim. theorem (Rank - Nullity theorem)

$f : V \to V$ linear map

$$\dim V = \underbrace{\dim \operatorname{Ker} f}_{= \text{nullity}} + \underbrace{\dim \operatorname{Im} f}_{= \text{rank}}$$

2nd dim theorem

V - K vectorspace, $S, T \leq_K V$

$$\dim (S+T) = \dim S + \dim T - \dim (S \cap T)$$

7.10) Determine the dimensions of $S, T, S+T$ and $S\cap T$ for:

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

$$T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

$S+T = \langle S \cup T \rangle$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ lin. indep $\Rightarrow \dim S = 2$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ lin. indep. $\Rightarrow \dim T = 2$

$$S+T = \langle \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{\to M_1}, \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{\to M_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\to M_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}}_{\to M_4} \rangle$$

$a, b, c, d \in \mathbb{R}$ s.t.

$aM_1 + bM_2 + cM_3 + dM_4 = 0$

$$\left\{ \begin{array}{lll} a+b = 0 & \Rightarrow b = 0 \\ a+c = 0 & \Rightarrow a = \\ \left.\begin{array}{l} b+c+d = 0 \\ b+d = 0 \end{array}\right\} & \Rightarrow c = 0 \end{array} \right\} \Rightarrow d = 0 \Rightarrow M_1, M_2, M_3, M_4 \text{ are lin. indep.}$$

$\Rightarrow \dim(S+T) = 4$

$$\dim(S\cap T) \overset{2^{nd}\ \dim.\ th.}{=\!=} \dim S + \dim T - \dim(S+T)$$

$$= 2+2-4 = 0 \Rightarrow$$

$\Rightarrow S\cap T = 0 \Rightarrow$

$\Rightarrow \mathcal{M}_2(\mathbb{R}) = S \oplus T$

Concl. $S' = T$:)