Chapter 5

1, 2, 5, 11, 12, 13, 14, 16

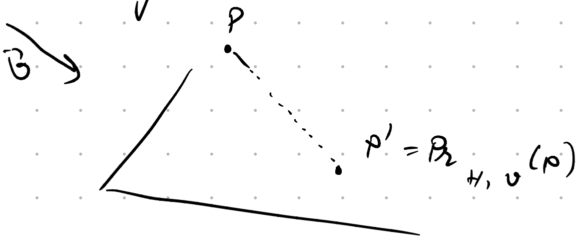
$f : \mathbb{A}^n \to \mathbb{A}^n$ - affine morphism

$\forall \; A, B \in \mathbb{A}^n$

$f(\vec{AB}) = \overrightarrow{f(A)\,f(B)}$

$f(P) = A \cdot P + b \quad , A \in M_{n \times n}(\mathbb{R}),\; b \in M_n(\mathbb{R})$

Projection a hyperplane along a dir $\vec{v}$



$H : a_1x_1 + \dots + a_nx_n + a_{n+1} = 0$

$a = (a_1, \dots, a_n) = \vec{n}_H$

$$Pr_{H,v}(P) = \left(I_n - \frac{v \otimes a}{\langle v, a \rangle}\right) \cdot P - \frac{a_{n+1}}{\langle v, a \rangle} \cdot a$$

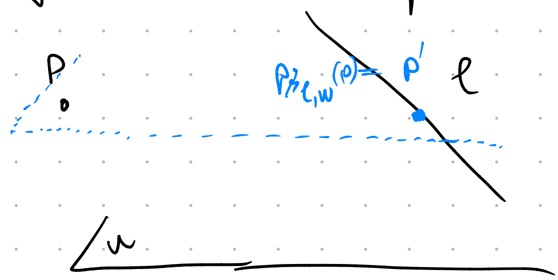
$$v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad w = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$v \otimes w = v \cdot w^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_n \\ \vdots & \vdots & & \vdots \\ x_ny_1 & \dots & \dots & x_ny_n \end{bmatrix}$$

$$\langle v, w \rangle = v^T \cdot w = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

$$Pr_H^{\perp} = Pr_{H,a} \Rightarrow Pr_H^{\perp}(P) = \left(I_n - \frac{a \otimes a}{\langle a, a \rangle}\right) \cdot P - \frac{a_{n+1}}{\langle a, a \rangle} \cdot a$$

Projection on a line ℓ parallel to a hyperplane w.



$Q \in P$, $v \in D(\ell)$

$$\Rightarrow Pr_{\ell,w}(P) = \frac{v \otimes a}{\langle v, a\rangle} \cdot P - \left(I_n - \frac{v \otimes a}{\langle v, a\rangle}\right) \cdot a$$

$$Pr_\ell^{\perp}(P) = \frac{a \otimes a}{\langle a, a\rangle} \cdot P - \left(I_n - \frac{a \otimes a}{\langle a, a\rangle}\right) \cdot Q$$

S.1. - coord planes
- coord axis.

$(xOy): z=0$

$(yOz): x=0$

$(zOx): y=0$

$Ox: \begin{cases} y=0 \\ z=0 \end{cases}$, $Oy: \begin{cases} z=0 \\ x=0 \end{cases}$, $Oz: \begin{cases} x=0 \\ y=0 \end{cases}$

$$Pr_{xOy,a}(P) = \left(I_n - \frac{a \otimes a}{\langle a, a\rangle}\right) \cdot P - \frac{a_{n+1}}{\langle a, a\rangle} \cdot a$$

$$a \otimes a = (0, 0, 1) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow Pr_{xOy,a}(P) = \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} - \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}\right) \cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot P$$

$$Pr_{xOz,a}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P$$

$$Pr_{y0z,a}(P) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P$$

We choose $Q = 0 = (0,0,0)$

$$P_{0x}^{\perp}(P) = \frac{(1,0,0) \otimes (1,0,0)}{\langle (1,0,0),(1,0,0) \rangle} P$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot P$$

$$P_{0y}^{\perp}(P) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P$$

$$P_{0z}^{\perp}(P) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P$$

$$Ref_{H,v}(P) = \left(I_n - \frac{2 (v \otimes a)}{\langle v, a \rangle} \right) \cdot P - \frac{2 a_{n+1}}{\langle v, a \rangle} \cdot a.$$

J.4. Orthogonal reflection of $P(6,-5,5)$ in $H: 2x - 3y + z - 4 = 0$ by det the matrix form

$(2, 0, 0) \in H$

$\vec{v} = \vec{a} = (2,-3,1)$

$$Pr_H^{\perp}(P) = I_n \frac{(2,-3,1) \otimes (2,-3,1)}{\langle (2,-3,1) \cdot (2,-3,1) \rangle} \cdot P - \frac{-4}{\langle 2,-3,1) \cdot (2,-3,1) \rangle} \cdot a$$

$$= I_n - \begin{bmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{bmatrix} \frac{1}{14} P + \frac{4}{14} (2,-3,1)$$

$$= \begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix} \cdot P + \frac{1}{7} (4,-6,2)$$

$$= \begin{bmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 5 \end{bmatrix} \begin{bmatrix} \cancel{60 - 30 - 10} \; 20 \\ \cancel{36 - 25 + 15} \; 26 \\ \cancel{-12 - 15 + 65} \; 38 \end{bmatrix} = \begin{bmatrix} 20 \\ 26 \\ 38 \end{bmatrix} + \frac{1}{7}(4,-6,2) / :2$$

$$= \begin{bmatrix} 10 \\ 13 \\ 19 \end{bmatrix} + \frac{1}{7} \cancel{(2,-3,1)}$$

$$= \left[\cancel{\frac{72}{7}} \right.$$

Reflection $\Rightarrow$ $2 \cdot (2,1,3) - \begin{bmatrix} 6 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 1 \end{bmatrix}$

$\begin{bmatrix} \frac{88}{7} \\ \\ \end{bmatrix}$

$(2,1,3)$

$f(p) = Ap + b$

$\bar{f} = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$

$\bar{f}(\bar{p}) = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ 1 \end{pmatrix}$

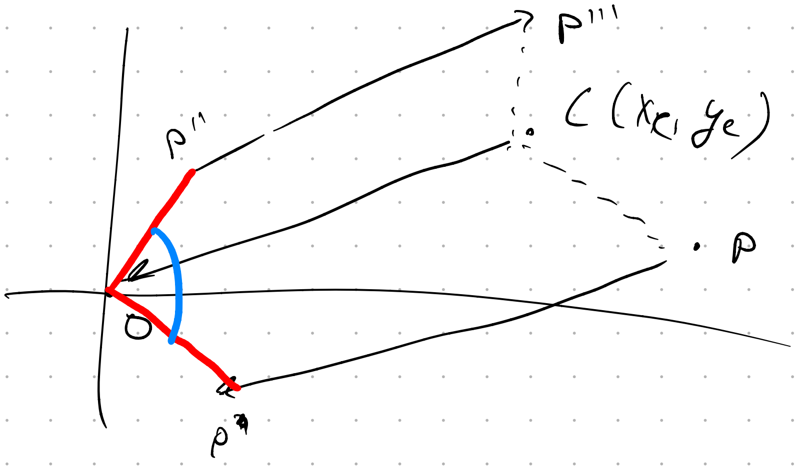
$= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} & b_1 \\ \vdots & & & & \vdots \\ a_{m1} & \dots & \dots & a_{mm} & b_m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$

$= \begin{pmatrix} a_{11}x_1 + \dots + a_{1m}x_m + b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mm}x_m + b_m \\ 1 \end{pmatrix}$

$\begin{pmatrix} Ap + b \\ 1 \end{pmatrix}$

$Rot_\theta(p) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot p$

$\overline{Rot_\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$Rot_{C,\theta}(P) = ?$

$P' = T_{(-x_c,-y_c)}(P)$

$P'' = Rot_\theta(P')$

$P''' = T_{(x_c,y_c)}(P'')$

$\Rightarrow rot_{C,\theta}(P) = T_{(x_c,y_c)} \circ Rot_\theta \circ T_{(-x_c,-y_c)}(P)$

$\overline{Rot}_{C,\theta}(P) = \overline{T_{(x_c,y_c)}} \circ \overline{Rot_\theta} \circ \overline{T_{(-x_c,-y_c)}}(P)$

$$= \begin{pmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$=$

14. $A(1,1)$, $B(4,1)$, $C(2,3)$

Find

$Ref_{AB} \circ Rot_{C,\frac{\pi}{2}}(ABC)$

$$\overline{Rot}_{c,\frac{\pi}{2}}(\bar{A}) = \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$Pr^{\perp}_{AB}(P) = \frac{\vec{AB} \otimes \vec{AB}}{\langle \vec{AB}, \vec{AB} \rangle} \cdot P - \left(I_2 - \frac{\vec{AB} \otimes \vec{AB}}{\langle \vec{AB}, \vec{AB} \rangle} \right) \cdot A$$

$$Ref^{\perp}_{AB}(P) = \left(2 \cdot \frac{\vec{AB} \otimes \vec{AB}}{\langle \vec{AB}, \vec{AB} \rangle} - I_2 \right) P - \left(I_2 - \frac{\vec{AB} \otimes \vec{AB}}{\langle \vec{AB}, \vec{AB} \rangle} \right) A$$

$$\vec{AB} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\vec{AB} \otimes \vec{AB} = \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\langle \vec{AB}, \vec{AB} \rangle = 9$$

$$\Rightarrow Ref^{\perp}_{AB}(P) = \left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot P - \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\overline{Rel_{AB}^{-1}} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right) \to Rel_{AB}^{-1} \circ Rec_{(\frac{\pi}{2})}(A)$$

$$= \begin{pmatrix} 1 & 0 & 7 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} \to A'(4, -3)$$