Seminar I

1. a) $[-3,2) \cup \{3\} = A$

$lb(A) = (-\infty, -3]$

$up(A) = [3, \infty)$

$sup(A) = 3 = max(A)$

$inf(A) = -3 = min(A)$

b) $(-1,1] \cup (2, \infty)$

$lb(A) = (-\infty, -1]$

$\nexists up(A)$

$inf(A) = -1$

$sup(A) = \infty$

$\nexists max(A), \nexists min(A)$

c) $(-5,5) \cap \mathbb{Z} = \{ -4, \dots, 4\}$

$lb(A) = (-\infty, 4]$

$up(A) = [4, \infty)$

$inf(A) = -4 = min$

$sup(A) = 4 = max$

d) $A = \emptyset$

$x \in \mathbb{R},\ x \leq a,\ \forall a \in \emptyset$ | $\Rightarrow$ False $\Rightarrow$ $lb(A) = \mathbb{R}.\ ub(A) = \mathbb{R}$

Assume $\underbrace{\exists\ a \in \emptyset}_{\text{contradiction}}$ s.t. $x > \emptyset$

$inf(A) = \infty \quad sup = -\infty$

$\nexists min(A), \nexists max(A)$

2.

a) $\{x \in \mathbb{Q} \mid x^2 < 3\}$ $(-\sqrt{3}, \sqrt{3}) \cap \mathbb{Q}$

$lb(A) = (-\infty, -\sqrt{3}]$

$ub(A) = [\sqrt{3}, \infty)$

$\inf = -\sqrt{3}$
$\sup = \sqrt{3}$
$\nexists$ max, min

b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$

$f'(x) = 2x - 4 \Rightarrow x = 2$ root.

for $x \in (-\infty, 2]$, $f'(x) \geq 0 \Rightarrow f$ is increasing

— " — $[2, +\infty)$ — $f'(x) \leq 0$ f is decreasing.

$f(x) \geq f(2)$ $\forall x$ ..

$\Rightarrow f(x) = [-1, +\infty)$

c) $\left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \ldots \frac{n}{n+1} \right\}$

$\inf(A) = 0 = \min$
$\sup(A) = 1$
$\nexists \max(A)$

d) $D = \{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}\}$ $[\varepsilon, \infty)$

$\left\{ \left(\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)^m \mid k, m \in \mathbb{N} \right\}$

$(-\infty, -\varepsilon]$

$\sup(A) = 2 = \max$
$\inf(A) = 0$
$\nexists$ min

3. S - nonempty and bounded above

$-S := \{-x \mid x \in S\}$

$ub(S) \neq \emptyset$, $\exists x \in \mathbb{R} : x \geq s, \forall s \in S$

$\Rightarrow -x \leq -s \Rightarrow lb(-S) \neq \emptyset \Rightarrow$ we have a lb

- $\sup(S) \geq s$, $\forall s \in S$
- if $u \geq s$, $\forall s \in S$, then $\sup(S) \leq u$

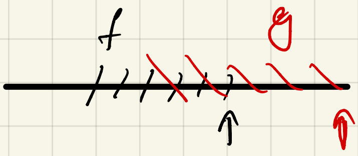
$\Rightarrow$ if $-u \leq -s, \ \forall s \in S$, then $-\sup(S) \geq -u$

$\Rightarrow$ $\boxed{-\sup(S) \leq -s \ , \ \forall s \in S}$ $-\sup(A) \in lb(-S)$

and $-\sup(S)$ is the greatest lower bound of $-S$

$\Rightarrow -\sup(S) = \inf(-S)$

4. $f : D \to \mathbb{R}$



$g : D \to \mathbb{R}$

$\sup(f(x) + g(x)) \leq \sup f(x) + \sup g(x)$

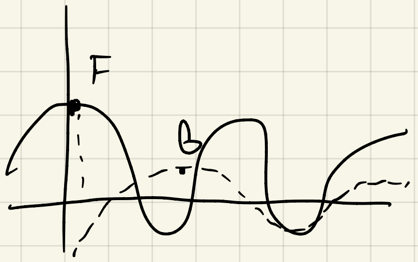
$f(x) \leq \sup_{x \in D} f(x) = F$

$g(x) \leq \sup(g(x)) = G$ (+)

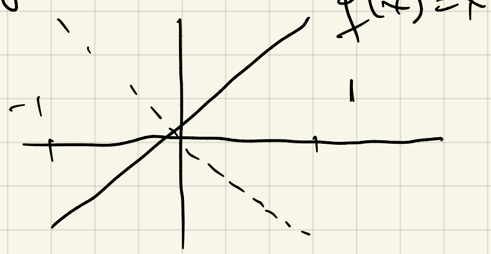
$f(x) + g(x) \leq F + G \Rightarrow F+G$ is an upper bound for $\{f(x) + g(x) \mid x \in D\}$

$\Rightarrow \sup(f(x) + g(x)) \leq F + G$ by definition

Ex 1.



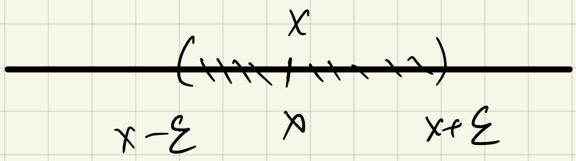
Ex 2 $g(x) = x$



$$\sup_{x \in [-1,1]} (f(x) + g(x)) = 0$$

$$\sup f(x) = 1 = \sup g(x)$$

6. $V \in \mathcal{V}(x)$ iff $\exists\, \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \subseteq V$



A. True

B. $\exists\, \varepsilon > 0, (-\varepsilon, \varepsilon) \subseteq V$

contains irrationals.

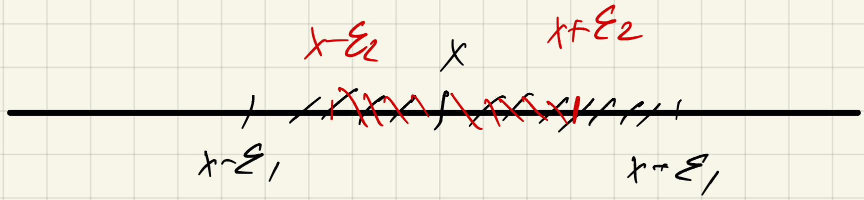
any neighborhood needs to also contain all IR (irrational numbers).

$$\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right] = 0 \Rightarrow \text{False} \Rightarrow \{0\} \in \mathcal{V}(0)$$

7. $x \in \mathbb{R}$, $U, V \in \mathcal{V}(x) \Rightarrow U \cap V \in \mathcal{V}(x)$

$U \in \mathcal{V}(x)$, $\exists\, \varepsilon_1 > 0$ s.t. $(x-\varepsilon_1, x+\varepsilon_1) \subseteq U$

$V \in \mathcal{V}(x)$, $\exists\, \varepsilon_2 > 0$ s.t. $(x-\varepsilon_2, x+\varepsilon_2) \subseteq V$



To prove $U \cap V \in \mathcal{O}(x)$, find $\varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq U \cap V$

$\Rightarrow \varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

9. (a) $(1,2]$

$\text{int}(A) = (1,2)$ - elements that are fully in A

$\text{cl}(A) = [1,2]$ -

(b) $[-3,2) \cup \{3\}$

$\text{int}(B) = (-3,2)$

$\text{cl}(B) = [-3,2] \cup \{3\}$

(d) $(-5,5) \cap \mathbb{Z} = \{-4, -3, \dots, 3, 4\} = D$

$\text{int}(D) = \emptyset$

$\text{cl}(D) = D$