28.11.2023

$A \in M_{m,n}(K)$

then $A \sim B$, then $\operatorname{rank} A = \operatorname{rank} B$

This is why if E is the echelon form of the matrix A, then $\operatorname{rank} A = \operatorname{rank} E$ = number of non-zero rows in E.

1) $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \sim$

$\underset{R_4 \leftarrow R_4 - 2R_1}{\overset{R_2 \leftarrow R_2 - 2R_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \overset{R_3 \leftarrow R_3 - R_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{R_4 \leftarrow R_4 - R_3}{\sim}$

$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \operatorname{rank} A = 3$

$$3)\ C=\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \overset{R_2 \leftrightarrow R_1}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \begin{matrix} R_2 \leftarrow R_2 - \beta R_1 \\ \sim \\ R_3 \leftarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \overset{R_2 \leftarrow R_2 + \beta R_3}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & 3-5\beta & 4-2\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \sim$$

$$\underset{\sim}{R_3 \leftarrow R_3 - \alpha R_2} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & 3-5\beta & 4-2\beta \\ 0 & 0 & x & y \end{pmatrix}$$

$$x = -2 - \alpha(3-5\beta) = -2 - 3\alpha + 5\alpha\beta$$

$$y = 1 - \alpha(4-2\beta) = 1 - 4\alpha + 2\alpha\beta$$

$$\operatorname{rank} A = \begin{cases} 2 & \text{iff } x = y = 0 \\ 3 & \text{otherwise} \end{cases}$$

$$\begin{cases} -2 - 3\alpha + 5\alpha\beta = 0 \\ 1 - 4\alpha + 2\alpha\beta = 0 \end{cases}, \quad \alpha\beta = \gamma$$

$$\begin{cases} -2 - 3\alpha + 5\gamma = 0 \\ 1 - 4\alpha + 2\gamma = 0 \end{cases} \Rightarrow \gamma = \frac{2+3\alpha}{5} \iff$$

$$\iff \begin{cases} \gamma = \frac{2+3\alpha}{5} \\ 1 - 4\alpha + 2 \cdot \frac{2+3\alpha}{5} = 0 \quad \Big| \cdot 5 \end{cases}$$

$$\iff \begin{cases} \gamma = \frac{2+3\alpha}{5} \\ 5 - 20\alpha + 4 + 6\alpha = 0 \end{cases}$$

$$\iff \begin{cases} \gamma = \frac{2+3\alpha}{5} \\ \alpha = \frac{9}{14} \end{cases} \Bigg| \Rightarrow \gamma = \frac{2 + 3\,\frac{9}{14}}{5} = \frac{\frac{\cancel{55}^{\,11}}{14}}{\cancel{5}_{\,1}} =$$

$$= \frac{11}{14} \; ; \quad \begin{rcases} \alpha\beta = \frac{11}{14} \\ \alpha = \frac{9}{14} \end{rcases} \Rightarrow \beta = \frac{\frac{11}{\cancel{14}}}{\frac{9}{\cancel{14}}} = \frac{11}{9}$$

$$\operatorname{rank} A = \begin{cases} 2, & \text{iff } \alpha = \frac{9}{14},\ \beta = \frac{11}{9} \\ 3, & \text{otherwise} \end{cases}$$

Inverting a matrix

$A \in M_n(K)$

$$\left(A \;\vdots\; I_n\right) \underset{\text{elim.}}{\overset{\text{Gauss-Jordan}}{\sim}} \left(I_n \;\vdots\; A^{-1}\right)$$

5) $$\left(\begin{array}{ccc:ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \quad R_2 \leftarrow -\frac{1}{5} \cdot R_2$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \quad R_3 \leftarrow R_3 + 12R_2$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \quad R_3 \leftarrow 5R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$$

We can use Gaussian elimination to extract a basis from a system of generators. We place them as rows in a matrix, which we bring to an echelon form. The rows of this matrix form a basis.

9) $S = \langle (1,0,4), (2,1,0), (1,1,-4) \rangle$

$T = \langle (-3,-2,4), (5,2,4), (-2,0,-8) \rangle$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{R_3 \leftarrow R_3 - R_1}{\overset{R_2 \leftarrow R_2 - 2R_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{R_3 \leftarrow R_3 - R_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

echelon form.

$\Rightarrow S = \langle (1,0,4), (0,1,-8) \rangle$
$(1,0,4), (0,1,-8)$ lin. indep
$\Big\}$ basis. $\Rightarrow \dim S = 2$

$$\begin{pmatrix} -2 & 0 & -8 \\ -3 & -2 & 4 \\ 5 & 2 & 4 \end{pmatrix} \overset{R_1 \leftarrow \left(-\frac{1}{2}\right) R_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ -3 & -2 & 4 \\ 5 & 2 & 4 \end{pmatrix} \underset{R_3 \leftarrow R_3 - 5R_1}{\overset{R_2 \leftarrow R_2 + 3R_1}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & -2 & 16 \\ 0 & 2 & -16 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 4 \\ 0 & -2 & 16 \\ 0 & 0 & 0 \end{pmatrix}$$

echelon form $\Rightarrow$ basis

$$\left.\begin{aligned} T &= \langle (1,0,4), (0,-2,16) \rangle \\ S &= \langle (1,0,4), (0,1,-8) \rangle \end{aligned}\right\} \Rightarrow S = T \Rightarrow$$

$$\begin{cases} \Rightarrow \dim(S+T) = \dim S = \dim T = 2 \\ \Rightarrow \dim(S \cap T) \;/\!/ \end{cases}$$

$$\dim S + \dim T = \dim(S+T) + \dim(S \cap T)$$