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Solving systems

Kronecker - Capelli theorem

(S) linear system

is compatible $\Leftrightarrow$ rank M_S = rank $\overline{M}_S$

Rouché

(S) compatible iff every characteristic minor is 0.

principal minor → non-zero and max size

col. of free terms

a row from M_S

↳ if all char. minors are 0, this will tell us that the col of free terms is a combination of the other. Doesn't give us any additional information.

6.2,3 Decide if the system is compatible and if so, solve it.

(i) $\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$; $A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 2 \end{pmatrix}$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} \underset{C_3 = C_3 - C_1}{\overset{C_2 = C_2 - C_1}{=}} \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -4 \\ 2 & -5 & -1 \end{vmatrix} =$$

$$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} -1 & -4 \\ -5 & -1 \end{vmatrix} = 1 - 20 = -19 \neq 0$$

rank $(A) = 3$

principal equations: (1), (2), (3)

principal unknowns: x_1, x_2, x_3

secondary unknowns: $x_4 := \alpha$

new system: $\begin{cases} x_1 + x_2 + x_3 = 5 + 2\alpha \\ 2x_1 + x_2 - 2x_3 = 1 - \alpha \\ 2x_1 - 3x_2 + x_3 = 3 - 2\alpha \end{cases}$

$$\Delta_{x_1} = \begin{vmatrix} 5+2\alpha & 1 & 1 \\ 1-\alpha & 1 & -2 \\ 3-2\alpha & -3 & 1 \end{vmatrix} = 5+2\alpha-6+4\alpha-3+$$

$$+3\alpha - 3+2\alpha - 30 - 12\alpha - 1+\alpha = -38$$

$$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{-38}{-19} = 2$$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 5+2\alpha & 1 \\ \alpha & 1-\alpha & -2 \\ 2 & 3-2\alpha & 1 \end{vmatrix} \quad \begin{array}{l} \underline{\underline{R_2 = R_2 - 2R_1}} \\ R_3 = R_3 - 2R_1 \end{array}$$

$$= \begin{vmatrix} 1 & 5+2\alpha & 1 \\ 0 & -9-5\alpha & -4 \\ 0 & -7-6\alpha & -1 \end{vmatrix} = 1\cdot(-1)^{1+1}\cdot \begin{vmatrix} -9-5\alpha & -4 \\ -7-6\alpha & -1 \end{vmatrix}$$

$$= 9+5\alpha - 28 - 24\alpha = -19 - 19\alpha$$

$$x_2 = \frac{\Delta_{x_2}}{\Delta} = \frac{-19-19\alpha}{-19} = 1+\alpha$$

$$\Delta_{x_3} = \begin{vmatrix} 1 & 1 & 5+2\alpha \\ 2 & 1 & 1-\alpha \\ 2 & -3 & 3-2\alpha \end{vmatrix} \quad \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 2R_1 \end{array}$$

$$= \begin{vmatrix} 1 & 1 & 5+2\alpha \\ 0 & -1 & \dots \\ 0 & -5 & \end{vmatrix} = \dots$$

Gauss method

8.5 Solve the following linear systems by using the Gauss or Gauss - Jordan.

(i) $$\begin{cases} 2x + 2y + 2z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \Rightarrow \text{row echelon form}$$

go from top to bottom, from left to right

pick a pivot, non-zero

used to make zeros

$$L_1 \leftrightarrow L_2 \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right) \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right)$$

$$L_2 \leftrightarrow L_3 \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 3 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - 4L_2}{\sim} \underbrace{\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right)}_{\text{row echelon form}}$$

If we are applying the Gauss method, then we now revert to the system

$$\begin{cases} x - y = 1 \\ y + z = 3 \\ -z = -11 \end{cases}$$

$\Rightarrow z = 11, \; y = -8, \; x = -7$

If we are applying Gauss-Jordan, then we continue with the transformations, from bottom to top, right to left)

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right) \overset{L_2 \leftarrow L_2 - L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right)$$

$$\overset{L_1 \leftarrow L_1 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right) \Rightarrow \begin{cases} x = -7 \\ y = -8 \\ z = 11 \end{cases}$$

ii) $\begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases}$

$$\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \sim$$

$$L_2 \leftarrow L_2 - 2L_1 \atop L_3 \leftarrow L_3 - L_1 \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x = 7\alpha + 1 \\ y = -3\alpha + 1 \\ z = \alpha \end{cases}$$

8.6 $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$, $\lambda \in \mathbb{R}$.

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \sim$$

$$\overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) {L_2 \leftarrow L_2 - 2L_1 \atop L_3 \leftarrow L_3 - L_1} \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda-2 \end{array}\right) \overset{L_3 \leftarrow L_3+L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda-5 \end{array}\right)$$

echelon form

syst. is compatible $\Rightarrow \lambda = 5$

for $\lambda \neq 5 \Rightarrow$ incompatible

$$\lambda = 5 \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \end{array}\right) \overset{L_1 \leftarrow L_1+\frac{2}{3}L_2}{\sim}$$

$$\sim \left(\begin{array}{cc|cc|c} 1 & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & -3 & 3 & -7 & -3 \end{array}\right)$$

$$\begin{cases} x_1 = -\alpha + \frac{2}{3}\beta = \dfrac{-3\alpha+2\beta}{3} \\ x_2 = \dfrac{-3\alpha+7\beta-3}{-3} \\ x_3 = \alpha \\ x_4 = \beta \end{cases}$$

8.7 $\begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases}$

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \underset{\sim}{L_1 \leftrightarrow L_2}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ a & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 - a \cdot L_1 \\ \sim \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a^2 & 1-a & 1-a^2 \\ 0 & 1-a & a-1 & a(a-1) \end{array}\right) \underset{\sim}{L_2 \leftrightarrow L_3}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a(a-1) \\ 0 & 1-a^2 & 1-a & 1-a^2 \end{array}\right) \underset{\sim}{L_3 \leftarrow L_3 - (1+a) L_2}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a(a-1) \\ 0 & 0 & 2-a-a^2 & -a^3-a^2+a+1 \end{array}\right)$$

$$1-a^2-\left(a(a-1)\cdot(1+a)\right) = 1-a^2-\left((a^2-a)(1+a)\right)$$

$$= 1-a^2-\left(\cancel{a^2}+a^3-a-\cancel{a^2}\right)$$

$$= -a^3-a^2+a+1$$

if $a=1 \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{cases} x = 1-\alpha-\beta \\ \alpha = y \\ \beta = z \end{cases}$$

if $a \neq 1 \Rightarrow$ if $2-a-a^2=0 \Rightarrow a=-2$ or $a=1 \Rightarrow$

$\Rightarrow a=-2 \ (a\neq 1) \Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ 0 & 3 & -3 & 6 \\ 0 & 0 & 0 & -5 \end{array}\right) \Rightarrow$

$\Rightarrow$ incompatible

if $a \neq -2 \Rightarrow z = \dfrac{1+a-a^2-a^3}{2-a-a^2} \Rightarrow$ we get x and y

by replacing it. upstairs