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Vector spaces

V is a K-vector space

- $(V, +)$ abelian group
- $(K, +, \cdot)$ field (usually $K = \mathbb{R}$ | sometimes $K = \mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$)
- "$\cdot$" : $K \times V \to V$

  ↳ external operation | $(k, v) \mapsto k \cdot v$

Axioms (need to know)

- $\forall\, \alpha, \beta \in K,\ \forall\, v \in V$

  $$(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$$

- $\forall\, \alpha \in K,\ \forall\, v, w \in V$

  $$\alpha (v + w) = \alpha v + \alpha w$$

- $\forall\, \alpha, \beta \in K,\ \forall\, v \in V$

  $$(\alpha \cdot \beta) \cdot v = \alpha \cdot (\beta \cdot v)$$

  } describes the external operation

- $\forall\, v \in V,\ 1 \cdot v = v$

  neutral elem in K.

4) Let $V = \{x \in \mathbb{R} \mid x > 0\}$

$\forall x, y \in V \quad , \forall k \in \mathbb{R}$

$x \perp y = x \cdot y$

$k \top x = x^k$

Show that V is a $\mathbb{R}$-vector space

$\boxed{\bullet \ (V, \perp) \text{ abelian group}}$

$$\left.\begin{matrix} x \perp y = z \\ x, y > 0 \end{matrix}\right\} \Rightarrow z > 0 \Leftrightarrow z \in V, \forall x, y \in V$$

$\Rightarrow$ „$\perp$" stable operation (1)

- $x, y, z \in V$

$$\left.\begin{matrix} (x \perp y) \perp z = xy \perp z = xyz \\ x \perp (y \perp z) = x \perp yz = xyz \end{matrix}\right\} \Rightarrow \text{„}\perp\text{" asoc.} \quad (2)$$

- $$\left.\begin{matrix} 1 \in V \\ x \perp 1 = x \cdot 1 = x, \forall x \in V \\ 1 \perp x = 1 \cdot x = x, \forall x \in V \end{matrix}\right\} \Rightarrow e = 1 \quad (3)$$

- $\forall x, y \in V$

  $x \perp y = xy = yx = y \perp x \Rightarrow$ „$\perp$" comutative (4)

- $\forall x \in V, \exists x^{-1}$ s.t. $x \perp x^{-1} \overset{(4)}{=} x^{-1} \perp x = e$

$$x \perp x^{-1} = 1$$

$$x \cdot x^{-1} = 1 \quad | \cdot \frac{1}{x}, \quad x > 0$$

$$x^{-1} = \frac{1}{x} \Rightarrow x^{-1} \in V \text{ bec. } \frac{1}{x} > 0, \forall x \in V \Rightarrow$$

$\Rightarrow \mathcal{U}(V) = V$ (5)

(1), (2), (3), (4), (5) $\Rightarrow (V, \perp)$ ab. group

Axioms proof:

1. $T : \mathbb{R} \times V \xrightarrow{\lambda} V$

$$k T x = x^k > 0, \forall x \in V \text{ and } \forall k \in \mathbb{R} \Rightarrow$$

$$\Rightarrow x^k \in V$$

$\alpha, \beta \in \mathbb{R}$

$x \in V$

$$(\alpha + \beta) T x = (\alpha T x) \perp (\beta T x) = x^\alpha \perp x^\beta =$$

$$= x^\alpha \cdot x^\beta = x^{\alpha+\beta} \in V$$

2. $x, y \in V$

$$\alpha T(x \perp y) = \alpha T(x \cdot y) = (\alpha T x) \perp (\alpha T y) =$$

$$= x^{\alpha} \perp y^{\alpha} = (xy)^{\alpha}$$

3. $\alpha, \beta \in \mathbb{R}$

$x \in V$

$$(\alpha \cdot \beta) T x \stackrel{?}{=} \alpha T (\beta T x)$$

$$\left.\begin{aligned} (\alpha \cdot \beta) T x &= x^{\alpha\beta} \\ \alpha T (\beta T x) &= \alpha T x^{\beta} = (x^{\beta})^{\alpha} = x^{\alpha\beta} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ true

4. $1 T v \stackrel{?}{=} v$

$v^1 = v \Rightarrow$ true

V , K- vector space

$S \subseteq V$

Characterization theorem for subspaces:

$S \leq_K V$ (K - subspace) $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} (i)\ S \neq \emptyset \\ (ii)\ \forall\, x, y \in S : x+y \in S \text{ (stable under "+")} \\ (iii)\ \forall\, k \in K, \forall\, x \in S : k \cdot x \in S \text{ (stable under the external operation)} \end{cases}$$

equivalent

$$\Leftrightarrow \begin{cases} (i)\ S \neq \emptyset \\ (ii)\ \forall\, x, y \in S, \forall\, k_1, k_2 \in K\ k_1 x + k_2 y \in S \end{cases}$$

7) Which ones of the following sets are subspaces of $\mathbb{R}^3$?

(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$

(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$

(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$

(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

(v) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$

(vi) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$

Solution

(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$

$(0, 1, 1) \in A \Rightarrow A \neq \emptyset$

$\forall k_1, k_2 \in \mathbb{R}, \forall v_i = (0, y_i, z_i) \in A$

$k_1 \cdot (0, y_1, z_1) + k_2 (0, y_2, z_2) =$

$= (0, k_1 y_1, k_1 z_1) + (0, k_2 y_2, k_2 z_2)$

$= (0, k_1 y_1 + k_2 y_2, k_1 z_1 + k_2 z_2) \in A$

$\Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$

(ii) $\overset{\in B}{(0,1,2)} + \overset{\in B}{(1,2,0)} = (1,3,2) \notin B$

$\Rightarrow \quad B \leq_{\mathbb{R}} \mathbb{R}^3$

(iii) $C = \{ (x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z} \}$

$C \neq 0$

let $k = \frac{1}{2} \in \mathbb{R}$

$v\,(1, 10, 11.5) \in C$

$k \cdot v = \frac{1}{2} \cdot (1, 10, 11.5) = (\underset{\notin \mathbb{Z}}{\underset{\downarrow}{\frac{1}{2}}}, 5, 5.75) \notin C$

$\Rightarrow \quad C \not\leq_{\mathbb{R}} \mathbb{R}^3$

(iv) $D = \{ (x,y,z) \in \mathbb{R}^3 \mid x+y+z=0 \}$

$(-1, 0, 1) \in D \Rightarrow D \neq \emptyset$

$\forall\, k_1, k_2 \in \mathbb{R}$ and $\forall\, v_1, v_2 \in D$

$k_1 \cdot v_1 + k_2 \cdot v_2 = (k_1 x_1, k_1 y_1, k_1 z_1) + (k_2 x_2 + k_2 y_2 + k_2 z_2)$

$$= (k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2, k_1 z_1 + k_2 z_2)$$

$$k_1 x_1 + k_2 x_2 + k_1 y_1 + k_2 y_2 + k_1 z_1 + k_2 z_2 =$$

$$k_1 (\underbrace{x_1 + y_1 + z_1}_{=0}) + k_2 (\underbrace{x_2 + y_2 + z_2}_{=0}) =$$

$$= 0 + 0 = 0 \Rightarrow$$

$$\Rightarrow k_1 v_1 + k_2 v_2 \in D$$

$$\Rightarrow D \leq_{\mathbb{R}} \mathbb{R}^3$$

(v) $E = \{ (x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1 \}$

$$(-1, 1, 1) \in E \Rightarrow E \neq \emptyset$$

$$(0, 0, 1) \in E$$

$$(-1, 1, 1) + (0, 0, 1) = (-1, 1, 2) \notin E$$

because $-1 + 1 + 2 = 2 \neq 1 \Rightarrow E \not\leq_{\mathbb{R}} \mathbb{R}^3$

$$(vi) \quad F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$$

$$(0, 0, 0) \in F \Rightarrow F \neq \emptyset$$

$$\forall\, v_1, v_2 \in F$$

$$v_1 + v_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2) =$$

$$= (x_1 + x_2,\; y_1 + y_2,\; z_1 + z_2) \xlongequal[x_2 = y_2 \text{ and } x_2 = z_2]{x_1 = y_1 \text{ and } x_1 = z_1} (2x_1, 2x_1, 2x_1) \in F$$

$$\forall\, k \in \mathbb{R},\; \forall\, v \in F$$

$$k \cdot v = k(x, y, z) = (kx, ky, kz) \xlongequal[z = x]{y = x}$$

$$= (kx, kx, kx) \in F$$

$$\Rightarrow F \leq_{\mathbb{R}} \mathbb{R}^3$$