Full State Feedback

Linear State Space

$$\dot{\bar{x}} = A\bar{x}(t) + B\bar{u}(t)$$
$$\bar{y}(t) = C\bar{x}(t) + D\bar{u}(t)$$

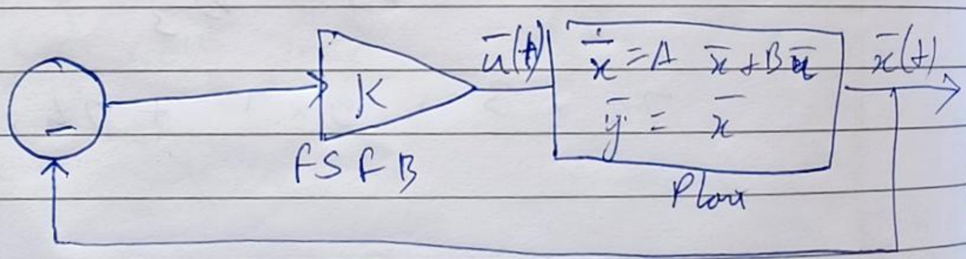
If $C = I$ & $D = 0$

$$\bar{y}(t) = \bar{x}(t)$$

short for FSFB

Consider Control Law

$$\bar{u}(t) = -K\bar{x}(t)$$



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* As you can see this is actually a regulator not a controller, as it regulates the non zero initial conditions back to origin

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t) \quad \text{where } \bar{u}(t) = -K\bar{x}$$

$$= A\bar{x} + B(-K\bar{x})$$

$$= A\bar{x}(t) - BK\bar{x}$$

$$\dot{\bar{x}}(t) = \underbrace{(A - BK)}_{\substack{\text{A Closed Loop} \\ A_{CL}}}\bar{x}$$

$$\therefore \dot{\bar{x}}(t) = A_{CL}\,\bar{x}(t) \qquad A_{CL} = A - BK$$

* So we will need a nonzero initial condition to get the state input.

* Behaviour of closed loop system is governed by A_{CL}, more specifically by the eigenvalues of A_{CL}, $\mathrm{eig}(A_{CL})$

Goal :- Can we move $eig(A_{CL})$?

Ans ~~Yes we can but only by changing the K matrix~~

Yes $eig(A_{CL})$ can be placed anywhere if A, B are controllable

~~Imp~~

How do we pick K to get the eigenvalues that we want

Example 1:- Controllable System

$$\dot{\bar{x}}(t) = \underbrace{\begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}}_{A} \bar{x}(t) + \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{B} \bar{u}(t)$$

Check controlability

$$P_C = \begin{bmatrix} B & AB \end{bmatrix}$$

$$P_C = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$$

~~1st~~ col is lineary independent of 2nd col

$\Rightarrow$ Rank = 2 = no. of states

$\Rightarrow$ Completely controllable

$\therefore$ Solve for K

$$\bar{u}(t) = -K \underset{n \times 1}{\bar{x}(t)}$$

Here $\bar{x}(t)$ is a $n \times 1$ vector
$\bar{u}(t)$ is a $m \times 1$ vector.

$n \longrightarrow$ no. of states
$m \longrightarrow$ no. of controls

$\therefore$ K should ~~by~~ be $(m \times n)$ order matrix

$\therefore$ In this case we have
1 control & 2 states
$\therefore$ 1×2 matrix K

$$K = [K_1 \quad K_2]$$

We Know that closed loop A maxic

$$A_{CL} = A - BK$$

$$= \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A_{CL} = \begin{bmatrix} 2K_1 & 3+2K_2 \\ 2-K_1 & 4-K_2 \end{bmatrix}$$

Now eigenvalues come from solving the characteristic eq^n

Closed loop eig

$$\lambda I - A_{cl} \Longrightarrow \text{singular}$$

$$\rightarrow \det(\lambda I - A_{cl}) = 0$$

$$\Rightarrow \left| \begin{bmatrix} \lambda - 2K_1 & 3+2K_2 \\ 2-K_1 & \lambda - 4 + K_2 \end{bmatrix} \right| = 0$$

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On solving we get a 2nd order polynomial

$$\boxed{P(\lambda) = \lambda^2 + (-4 - 2k_1 + k_2)\lambda + 11k_1 - 6 - 4k_2} \quad \text{(1)}$$

Characteristic eq^n of closed Loop system under FSFB

Suppose we want

$$\lambda_{cl,1} = -5 + 2i$$
$$\lambda_{cl,2} = -5 - 2i$$

Then the characteristic eq^n should look like this.

$$P_{des}(\lambda) = (\lambda - (-5 + 2i))(\lambda - (-5 - 2i)) = 0$$

$$\Rightarrow P_d(\lambda) = \lambda^2 + 10\lambda + 29 = 0 \quad \text{(2)}$$

Comparing eq (1) & (2) coeff

$-4 - 2K_1 + K_2 = 10$

$11K_1 - 6 - 4K_2 = 29$

Solving both

$K_1 = 91/3$

$K_2 = \frac{224}{3}$

—x—

On Matlab.

```
A = [0 3;
     2 4];

B = [-2;
      1];
# check controllability
rank(ctrb(A, B))
```

perform pole placement.

p_desired = [-5+2*i   -5-2*i];

[K]=place (A, B, p_desired)

check that the CL eigenvalues are in the desired locatio.

A_CL = A - B*K;

eig (A_CL)

— x —

Note acker can be used too
~~see~~ run help acker in command window for note.

Eg. Instead of this you can also use
[K] = acker (A, B, p_desired)

However this is not reliable for higher order systems

Eg2) Uncontrollable System

$$\dot{\bar{x}}(t) = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0.5811 \\ 1 \end{bmatrix} \bar{u}(t)$$

A B

$$P_c = \begin{bmatrix} B & AB \end{bmatrix}$$

$$= \begin{bmatrix} 0.5811 & 3 \\ 1 & 2+\sqrt{10} \end{bmatrix}$$

rank = 1 $\neq$ 2

$\therefore$ Uncontrollable System

$$A_{CL} = A - BK.$$

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}(-2+\sqrt{10})K_1 & 3 - \frac{1}{2}(-2+\sqrt{10})K_2 \\ 2 - K_1 & 4 - K_2 \end{bmatrix}$$

Characteristic eqⁿ

$$\det(\lambda I - A_{cl}) = 0$$

$$\Rightarrow \boxed{\lambda^2 + \left(-4 - k_1 + \sqrt{\tfrac{5}{2}}\,k_1 + k_2\right)\lambda - 6 + 7k_1 - 2\sqrt{10}\,k_1 - 2k_2 + \sqrt{10}\,k_2}$$

Closed loop CE under FSFB

Say I want this system at

$$P_d(\lambda) = (\lambda + 5 - 2i)(\lambda + 5 + 2i)$$

$$= \lambda^2 + 10\lambda + 29$$

$$-4 - k_1 + \sqrt{\tfrac{5}{2}}\,k_1 + k_2 = 10$$

$$-6 + 7k_1 - 2\sqrt{10}\,k_1 - 2k_2 + \sqrt{10}\,k_2 = \underline{29}$$

In matrix for

$$\begin{bmatrix} \sqrt{\frac{5}{2}} - 1 & 1 \\ 7 - 2\sqrt{10} & \sqrt{10} - 2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} + \begin{bmatrix} -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 10 \\ 29 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sqrt{\frac{5}{2}} - 1 & 1 \\ 7 - 2\sqrt{10} & \sqrt{10} - 2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 14 \\ 35 \end{bmatrix}$$

But since rank $= 1$

So inverse doesn't exist.

Now Augmented matrix

$$\tilde{A} = [A \quad B]$$

$$= \begin{bmatrix} \sqrt{\frac{5}{2}} - 1 & 1 & 14 \\ 7 - 2\sqrt{10} & \sqrt{10} - 2 & 35 \end{bmatrix}$$

Row reduce this matrix

$$\begin{bmatrix} 1 & \frac{2}{-2\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Not good this basically says $0 \times K_1 + 0 \times K_2 = 1$

Eg. 3 Controllable with multiple controls.

$$\dot{\bar{x}}(t) = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \bar{u}(t)$$

K = as many rows as there are controls

$$K \Rightarrow \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$A_{cL} = A - BK$$

$$= \begin{bmatrix} 2k_{11} - k_{21} & 3 + 2k_{12} - k_{22} \\ 2 - k_{11} - k_{22} & 4 - k_{12} - k_{22} \end{bmatrix}$$

$$\det(\lambda I - A_{cL}) = 0$$

$$\lambda^2 + (-4 - 2k_{11} + k_{12} + k_{21} + k_{22})\lambda$$
$$+ (-6 + 11k_{11} - 4k_{12} + 3k_{12} \cdot k_{21} + 2k_{22} - 3k_{11}k_{22}) = 0$$

Let us use same characteristic eq

$$\lambda^2 + 10\lambda + 29 = 0$$

& Compare coeffn

Now 2eq^n 9un k.

~~$k_{21} = - \; -7-27k_{12}-6k_{11}^2-6k_{12}$~~

$$k_{21} = \frac{a}{}\left(\frac{-7-27k_{12}-6k_{11}^2-6k_{12}+3k_{12}k_{22}}{3(-1+k_{11}+k_{12})}\right)$$

$$k_{22} = -\left(\frac{49-9k_{11}-39k_{12}-6k_{11}+3k_{12}^2}{3(-1+k_{11}+k_{12})}\right)$$

Eg. What if u_1 was expensive.

$\bar{u} = -k\bar{x}$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then choose k_{11} & k_{12} very small

like $k_{11} = \frac{1}{10}$ here

$k_{12} = \frac{1}{2}$

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$$\Rightarrow K_{21} = \frac{-1033}{240} = -4.3$$

$$K_{22} = \frac{4417}{240} = 18.4$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4 \\ -4.3 & 18.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

Hence u_1 is very less her.

In MatLab

Eg. 3 Multiple input

```
B = = [-2 1;
       1 1];
rank (ctrb(A, B))
```

Ohuee

[K]=place(A, B, p_desired)

Matlab will give just a random answes that will work as there are ∞ sol^n.

Check that value is corre
eig (A - B * K)

Check whiteboard vale:

k_whiteboar = [-1/10 1/10 ;
-1033/240 4417/240]

eig (A - B * K_whiteboard)

Closing Thought.

FSFB ≃ Proportional Control.

FSFB has no root locus

—x—

LQR

Cost func^n

$$J = \int_0^\infty \underbrace{\bar{x}^T(t) \cdot Q\, \bar{x}(t)}_{\text{non-zero states}} + \underbrace{\bar{u}(t)^T R \bar{u}(t)\, dt}_{\text{non zero controls}}$$

~~x(t)dt~~

Q & R trade off b/w states & controls

⟨P⟩ minimise J

$\bar{u} \in R^m$

$Q \longrightarrow$ Semidefinate +ve

$R \longrightarrow$ Definate positive.

such that $\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t)$

1) If Q is bigger than $R \Rightarrow$ fast regulation of $\bar{x} \longrightarrow 0$

We will get large $\bar{u}$ (Aggressive)

2) If R is bigger than $Q \Rightarrow$ Slow regulation of $\vec{x} \longrightarrow 0$

We will get small $\bar{u}$ (Conservative)

Eg. 2 state 2 control system

$$\bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \bar{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}, \quad R = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}$$

$\bar{x}^T(t)\, Q\, \bar{x}(t) + \bar{u}^T(t)\, R\, \bar{u}(t)$

$= q_{11} x_1(t)^2 + q_{22} x_2(t)^2 + r_{11} u_1(t)^2 + r_{22} u_2(t)^2$

$q_{11} \rightarrow$ wt / penalty on a non zero $x_1(t)$

$q_{22} \rightarrow$ " " " $x_2(t)$ "

$r_{11} \rightarrow$ " " " $u_1(t)$ "

$r_{22} \rightarrow$ " " " $u_2(t)$ "

Solving the optimation ~~po~~ problem

Solution to (P) $\min J = \int_0^\infty \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}\, dt$

subject to $\dot{\bar{x}} = A\bar{x} + B\bar{u}$

$\bar{u}(t) = -K\bar{x}(t)$

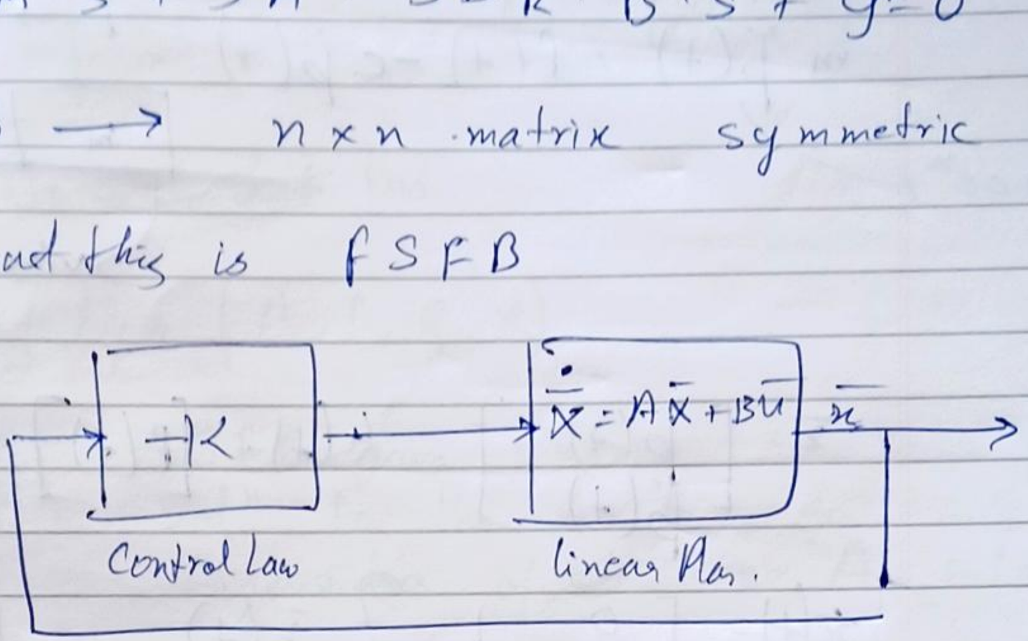
where $K = R^{-1} B^T S$

where S is sol^n of Algebraic Ricatti eq^n.

$$A^TS + SA - SBR^{-1}B^TS + Q = 0$$

$S \longrightarrow n \times n$ matrix symmetric

→ But this is FSFB



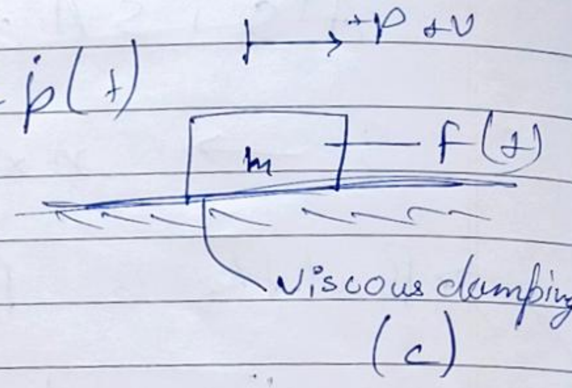
Procedure for LQR.

1) Given A & B (from plant)
2) Choose Q & R
3) Solve Algebraic Ricatti eq^n (ARE) for S
4) Compute optimal gain $K = R^{-1}B^TS$
5) Choose the K solution that yields a stable system

Eg. Mass / damper system

$$m\ddot{p}(t) = f(t) - c\dot{p}(t)$$

acc^n



$$\bar{x} = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} f(t) \end{bmatrix}$$

$$\dot{\bar{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix}}_{A} \bar{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} \bar{u}(t)$$

Let us choose values to calc
choose $m = 1$, $C = 0.2$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1/5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Chose Q & R

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [0.01]$$

In matlab has

$[K, S, E] = lqr(A, B, Q, R)$

K = FSFB gain matrix
S = Sol^n to the ARE
E = Eigenvalues of ~~closedloop~~ $A_{CL} = (A-BK)$

Summary.

$\langle P \rangle$ minimise $J = \int_0^\infty \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u} \, dt$

such that $\dot{\bar{x}} = A\bar{x} + B\bar{u}$

optimal sol^n is $\bar{u} = -K\bar{x}$ (FSFB)
where $K = R^{-1} B^T S$

$S \rightarrow$ Sol^n to ARE