RL Homework 2

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1 Exercise 3

1.1 Algorithm

- Reformulate the State cost function in the general form, obtain the Cost matrix
- Apply taylor series expansion to the contraints to linearize them, form the matrices G and g that symbolize the equality conditions. For now we are not focusing on making the inequality matrices H and h
- Form the Lagrangian, write down the KKT conditions and get the gradient of the cost function
- Find the hessian of the Lagrangian ignore the second order derivative of constraints and just put them down as zero(Given)
- Solve the KKT conditions with the gradients and Hessians you have obtained to get the step directions and lambda
- Take the sum of residual vector g to get the loss and do a backtracking line search to calculate the alpha
- Get the optimized variables from the minimization and use them for the control signal

1.1.1 Rewriting the State Cost Function

$$\min_{\theta_n,\omega_n,u_n} \sum_{n=0}^{300} 10(\theta_n - \pi)^2 + 0.1\omega_n^2 + 0.1u_n^2$$

This function can be rewritten in the form of

$$\min_{x} \sum_{n=0}^{300} \left(\frac{1}{2} x_n^T Q x_n - x_{des}^T R x_n \right)$$

$$Q = \begin{bmatrix} 20 & 0 \\ 0 & 0.2 \end{bmatrix} R = \begin{bmatrix} 0.2 \end{bmatrix} x_{des} = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}$$

We can then form the cost matrix G(Q,R) as a repeating diagonal matrix with alternating Q and R. Rewriting it the general notation we get

$$\min_{\bar{x}} \frac{1}{2} \bar{x}^T G \bar{x} + g^T \bar{x}$$

$$G(Q,R) = \begin{bmatrix} Q & 0 & 0 & \cdots \\ 0 & R & 0 & \cdots \\ 0 & 0 & Q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} g^T = \begin{bmatrix} -x_{des}^T G & -x_{des}^T G & -x_{des}^T G & \cdots \end{bmatrix}$$

1.2 Gradient and Hessian

Next we find the gradient and hessian of the Cost function G(Q,R), which can be easily found as the equations are of the form $\frac{1}{2}x^T Px + Qx$ which we know has the gradient $2Px + Q$ and the hessian $2P$

$$\nabla f(\bar{x}) = G\bar{x} + g$$

$$\nabla^2 f(\bar{x}) = G\bar{x}$$

$$G(Q,R) = \begin{bmatrix} Q & 0 & 0 & \cdots \\ 0 & R & 0 & \cdots \\ 0 & 0 & Q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} g^T = \begin{bmatrix} -(x_{des}^T G)^T \\ -(x_{des}^T G)^T \\ -(x_{des}^T G)^T \\ \vdots \end{bmatrix} \bar{x} = \begin{bmatrix} \bar{x_0} \\ \bar{x_1} \\ \bar{x_2} \\ \vdots \end{bmatrix}$$

where each x_n is a state vector.

1.3 Lagrangian

$$\mathcal{L}(x,\lambda) = f(x) + \lambda^T g(x)$$

KKT Condition would be

$$\nabla f(x) + \lambda \nabla g(x) = 0$$

The hessian would be given by

$$\nabla^2 \mathcal{L}(x,\lambda) = \begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial x^2} & \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{bmatrix}$$

$$\nabla^2_{xx} \mathcal{L} = \begin{bmatrix} Q & 0 & 0 & \cdots \\ 0 & R & 0 & \cdots \\ 0 & 0 & Q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

1.4 Linear Approximation equality constraints

The constraints given to us are non linear so we take Taylor series expansion to get an approximation at a guess $\bar{x}$ and reformulate the equation as $G(\bar{x})\Delta x = g(\bar{x})$ (Different G from Cost matrix) which can be written in matrix form as below. Later we can get the total constraint violation from the summation of the residual vector

$$\theta_0 = 0 \quad \omega_0 = 0$$

$$\theta_{n+1} = \theta_n + \Delta t \ \omega_n$$

$$\omega_{n+1} = \omega_n + \Delta t \ (u_n - g \sin \theta_n)$$

Taking Taylor series expansion

$$\Delta\theta_{n+1} = \Delta\theta_n + \Delta t \Delta\omega_n$$

$$\Delta\omega_{n+1} = \Delta\omega_n + \Delta t(\Delta u_n - g\Delta\theta_n \cos\theta_n)$$

$$\begin{bmatrix} 1 & \Delta t & 0 & -1 & 0 & 0 \\ -\Delta t g \cos \bar{\theta_n} & 1 & \Delta t & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\theta_n \\ \Delta\omega_n \\ \Delta u_n \\ \Delta\theta_{n+1} \\ \Delta\omega_{n+1} \\ \Delta u_{n+1} \end{bmatrix} = \begin{bmatrix} \theta_n + \Delta t \ \omega_n - \theta_{n+1} \\ \omega_n + \Delta t \ (u_n - g \sin\theta_n) - \omega_{n+1} \end{bmatrix}$$

Stacking them in order and the initial conditions as well

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & \Delta t & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\Delta t g \cos \bar{\theta}_0 & 1 & \Delta t & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \Delta t & 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\Delta t g \cos \bar{\theta}_1 & 1 & \Delta t & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Delta\theta_0 \\ \Delta\omega_0 \\ \Delta u_0 \\ \Delta\theta_1 \\ \Delta\omega_1 \\ \Delta u_1 \\ \Delta\theta_2 \\ \Delta\omega_2 \\ \Delta u_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \bar{\theta}_0 + \Delta t \ \bar{\omega}_0 - \bar{\theta}_1 \\ \bar{\omega}_0 + \Delta t \ (\bar{u}_0 - g \sin \bar{\theta}_0) - \bar{\omega}_1 \\ \bar{\theta}_1 + \Delta t \ \bar{\omega}_1 - \bar{\theta}_2 \\ \bar{\omega}_1 + \Delta t \ (\bar{u}_1 - g \sin \bar{\theta}_1) - \bar{\omega}_2 \\ \vdots \end{bmatrix}$$