Reinforcement Learning and Optimal Control
Homework 1

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1 Solution 1:

1.1 Part a

$$f(x) = -e^{-(x-1)^2}, \quad x \in \mathbb{R}$$

First we find the derivative of this function and equate it to zero

$$f'(x) = e^{-(x-1)^2} \cdot 2(x-1) = 0$$

Solving for x gives us

$$x = 1$$

To check for minima and maxima we take second derivative

$$f''(x) = 2e^{-(x-1)^2}\left(-4x^2 + 8x - 2\right)$$

At $x = 1$, $f''(1) = 2$ which is more than 0 so there is a strict local minima at $x = 1$. Since there is only one value for which f' = 0. It is a global minima. Also since it is a convex function it is a global minima

1.2 Part b

$$f(x, y) = (1-x)^2 + 100(y - x^2)^2, \quad x, y \in \mathbb{R}$$

The gradient is given by:

$$\nabla f = \begin{bmatrix} -2(1-x) - 400x(y-x^2) \\ 200(y-x^2) \end{bmatrix}$$

Setting $\nabla f = 0$, we get:

$$\begin{bmatrix} -2(1-x) - 400x(y-x^2) \\ 200(y-x^2) \end{bmatrix} = 0$$

On solving we get y = x^2, and substituting it in the first equation we can solve for x and we get the value x = 1. The the point of extremity is (1,1). Taking the Hessian of f(x,y).

$$\nabla^2 f = H = \begin{bmatrix} 1200x^2 - 400y + 2 & -400x \\ -400x & 200 \end{bmatrix}$$

At (1,1)

$$H = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$$

By calculating the eigenvalues they all come out to be positive so this is a strict local minima.

1.3 Part c

$$f(x, y) = 20x + 2x^2 + 4y - 2y^2 \quad x, y \in \mathbb{R}$$

$$\nabla f = \begin{bmatrix} 4x + 20 \\ -4y + 4 \end{bmatrix} = 0$$

Solving the equations we get (x,y) = (-5,1)

$$\nabla^2 f = H = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

eigenvalues are 4 and -4. Since we have a negative eigenvalue this is a saddle point not a minima.

1.4 Part d

$$f(x, y) = x^T Qx + Lx$$

$$where \quad Q = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad L = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\nabla f = 2Qx + L \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6x + 2y - 1 \\ 2x + 6y + 1 \end{bmatrix}$$

Solving both for $\nabla f = 0$ we get (x,y) = (0.25,-0.25)

$$H = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$$

The eigenvalues of the hessian are 8 and 4 so positive definite so there is a strict local minima.

1.5 Part e

$$f = x^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 10 \end{bmatrix} x$$

where $x \in \mathbb{R}^2$.

The gradient is given by:

$$\nabla f = 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 10 \end{bmatrix} = 0$$

$$\nabla f = \begin{bmatrix} 2x + 4y + 1 \\ 4x + 2y + 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system of equations we get (x,y) = (-19/6, 4/3)

$$H = 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Since the Hessian has both positive and negative eigenvalues, the function does not have a strict local or global minimum.

1.6 Part f

$$f = \frac{1}{2} x^T \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} x - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

where $x \in \mathbb{R}^3$.

$$\nabla f = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} x - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\nabla f = \begin{bmatrix} x + y \\ x + y \\ 4z \end{bmatrix} = 0$$

Therefore the solution lies on x= -y on the plane z = 0

The Hessian matrix is:

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Since the Hessian is positive semi-definite (the matrix is not strictly positive definite but has non-negative eigenvalues), the function has a global minimum, but it is not strict.

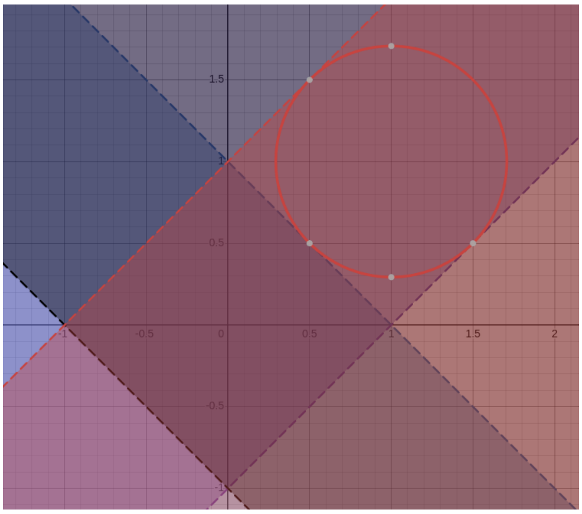

Figure 1: Question 2

2 Solution 2:

We want to find the point (x, y) that minimizes the distance to $(1, 1)$. Distance from the point is given by

$$f(x, y) = \sqrt{(x-1)^2 + (y-1)^2}$$

Here we can simply disregard the square root and minimize the function inside

$$\min_{x,y} \quad (x-1)^2 + (y-1)^2$$

under the constraints:

$$\begin{aligned} x + y &< 1 \\ y - x &< 1 \\ x - y &< 1 \\ -x - y &< 1 \end{aligned}$$

The function is a set of concentric circles centered at (1,1) and the constraints form a rhombus at the center. The closest distance would be just inside rhombus. The region of interest lies inside the rhombus in the center overlapping all the conditions. Since it is a strict inequality, the solution for the minimization would lie just barely inside the rhombus on the left side of the line $x + y < 1$.

The Lagrangian function is:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (x-1)^2 + (y-1)^2 + \lambda_1(x+y-1) + \lambda_2(y-x-1) + \lambda_3(x-y-1) + \lambda_4(-x-y-1)$$

The KKT conditions are as follows:

$$\begin{aligned} \nabla_x &= 2(x-1) + \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4 = 0 \\ \nabla_y &= 2(y-1) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \\ \nabla_{\lambda_1} &= x + y - 1 \\ \nabla_{\lambda_2} &= -x + y - 1 \\ \nabla_{\lambda_3} &= x - y - 1 \\ \nabla_{\lambda_4} &= -x - y - 1 \end{aligned}$$

For the limiting condition the minima would be at $(0.5 - \delta, 0.5 - \delta)$ where $\delta \to 0$. The value of f(x,y) at this point would be 0.70711. Only λ_1 would be an active inequality near the minima

3 Solution 3

We are given the function to minimize

$$\min_{x} \quad \frac{1}{2}x^T Qx$$

subject to:

$$Ax = b$$

where $Q \in \mathbb{R}^{n\times n}$ is positive definite and $A \in \mathbb{R}^{m\times n}$ is full rank with $m < n$.

The Lagrangian function is defined as:

$$\mathcal{L}(x, \lambda) = \frac{1}{2}x^T Qx + \lambda^T (Ax - b),$$

where $\lambda \in \mathbb{R}^m$ is the vector of Lagrange multipliers.

KKT Conditions

$$\nabla_x L = Qx + A^T \lambda = 0$$

$$\nabla_\lambda = Ax - b = 0$$

or we can rewrite this in matrix form as

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

We are given the condition $x_1 + x_2 + x_3 = 1$. This can be written in the form of Ax = b as

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1$$

For calculating the numerical solution for the values we are given

$$Q = \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

We can calculate the values of x using python and arrive at the values. Link to python code

$$x = \begin{bmatrix} -0.00404738 \\ -0.40069072 \\ 1.40444119 \end{bmatrix}$$

We can also verify that Ax = b holds true for these values.

3.1 Solution 4:

For Solution 4 please open the notebook named Exercise_4.ipynb