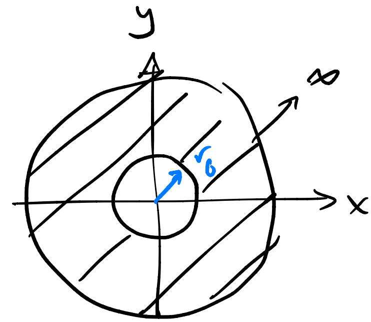
Circular Aperture



assume circular symmetry

so $\vartheta_x = \vartheta_y = \vartheta$

- we want the reciprocal space representation of

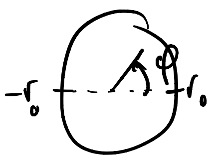
$\psi(x,y) = \psi_0$ (const.)

$k_x = k\vartheta_x$ etc.

then $$\psi(\vartheta_x, \vartheta_y) = \iint_{\text{aperture}} dx\,dy\; \psi_0\, e^{-i(k\vartheta_x x + k\vartheta_y y)}$$

$0 < x^2+y^2 < r_0^2$

Now:



$x = r_0\cos\varphi$; $\longrightarrow dx = -r_0 \sin\varphi\, d\varphi$

$y = r_0 \sin\varphi$

now we have assume circular symmetry s.t.

$\vartheta_x = \vartheta_y = \vartheta$

$$\mapsto \psi(\vartheta) = \psi_0 \int_{-r_0}^{r_0} e^{-ik\vartheta x}\, dx \int_{-r_0\sin\varphi}^{r_0\sin\varphi} dy$$

and $x = r_0 \cos\varphi \rightarrow dx = -r_0 \sin\varphi \, d\varphi$. Now

Stick in polars:

symmetry above/below $y = 0$

$$\Psi(\vartheta) = 2\psi_0 r_0^2 \int_0^{\pi} e^{-ik\vartheta r_0 \cos\varphi} \sin^2\varphi \, d\varphi$$

$$= 2\psi_0 r_0^2 \int_0^{\pi} \cos(k\vartheta r_0 \cos\varphi) \sin^2\varphi \, d\varphi$$

(other term $= 0$).

and a standard result is

$$J_n(z) = \frac{\left(\frac{1}{2} z\right)^n}{\pi^{1/2} \, \Gamma\left(n + \frac{1}{2}\right)} \int_0^{\pi} \cos(z \cos\varphi) \sin^{2n}\varphi \, d\varphi$$

So $z = k\vartheta r_0$ & $n = 1$ $\quad \left(\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\pi^{\frac{1}{2}}\right)$

Gamma fn.