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Operations Analytics

Tutorial #3: Queues

Solutions

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Exercise 4: A Petroleum Refinery

An integrated petroleum company is considering expansion of its one unloading facility at its Australian refinery. Due to random variations in weather, loading delays, and other factors, ships arriving at the refinery to unload crude oil arrive according to the Poisson distribution ($CV_a = 1$) with an average arrival rate of 8 ships per week. Service time is exponential with an average service rate of 12 ships per week ($CV_s = 1$).

4a) Current performance measures:

4a.i) What is the average number of ships waiting to gain access to the single unloading facility?

4a.ii) What is the average time a ship must wait before beginning to deliver its cargo to the refinery?

4a.iii) What is the average total time that a ship spends at the refinery?

4b) The company has under consideration a second unloading berth that could be rented for $\$6,000$ per week. The service time for this berth is also assumed to be exponential with the same service rate of 12 ships per week as the company's own berth. The opportunity cost of a ship not being at sea is $\$30,000$ per week.

4b.i) If the second berth is rented, what will be the average number of ships waiting?

4b.ii) What would be the average total time a ship would spend at the refinery?

4b.iii) Is the benefit of reduced waiting time (in dollars) worth the rental cost for the second berth?

4c) An alternative way to improve unloading facilities is for the company to modify the current operations to speed up the unloading process. Specifically, the new service time would still be exponential, but with a mean service rate of 15 ships per week. The incremental cost of the new operations would be $\$4,000$ per week.

4c.i) Compute the (new) expected waiting time and total time for ships.

4c.ii) Is the benefit worth the incremental cost?

4c.iii) How would you answer (4c.i) and (4c.ii) above if, instead of speeding up the service rate of the facility, the new operations made the service time constant, but still with the old service rate of 12 ships per week?

Solution

4a) We are given the following:

- Mean arrival rate of ships to facility: $\lambda = 8$ ships/week;
- Mean service rate per ship at facility (capacity of facility): $\mu = 12$ ships/week;
- Mean utilization rate of facility: $\rho = \frac{\lambda}{\mu} = \frac{8 \text{ ships/week}}{12 \text{ ships/week}} = 0.666$ (66.6% busy).

4a.i) The L_q can be obtained by the Little's Law:

$$L_q = \lambda \cdot W_q,$$

we are given $\lambda = 8$ ships/week, so we now need to find W_q.

The average waiting time, W_q, is given by the formula

$$W_q = \tau \frac{\rho}{1-\rho} \frac{1}{2} \left(CV_a^2 + CV_s^2 \right)$$

Note that the average service time, τ, is known as $\tau = \frac{1}{\mu} = \frac{1}{12}$ weeks $= 0.0833$ weeks.

Also, we are given that the arrivals follow the Poisson distribution, so $CV_a = 1$; the service times follow the Exponential distribution, so $CV_s = 1$.

Thus:

$$\begin{aligned} W_q &= \tau \frac{\rho}{1-\rho} \frac{1}{2} \left(CV_a^2 + CV_s^2 \right) \\ &= (0.0833 \text{ weeks}) \cdot \frac{0.666}{1-0.666} \cdot \frac{1}{2} \left(1^2 + 1^2 \right) \\ &= 0.167 \text{ weeks} \end{aligned}$$

So, Little's Law gives

$$\begin{aligned} L_q &= \lambda \cdot W_q \\ &= 8 \text{ (ships/week)} \cdot 0.167 \text{ (weeks)} \\ &= 1.336 \text{ ships} \end{aligned}$$

on average are waiting to be served.

4a.ii) Here, we are asked the average waiting time, W_q, which is already calculated as 0.167 weeks.

4a.iii) The average total time that a ship spends at the refinery equals the average time it is waiting in the queue, W_q, plus its average serving time, τ. Thus, on average a ship is spending a total of $W = W_q + \tau = 0.167 + 0.0833 \simeq 0.25$ weeks at the refinery.

4b) Now, we have $s = 2$ servers, who share the same load, thus the overall utilization of the facility will be decreased (Why?[1]). The average utilization of the facility is

$$\begin{aligned} \rho &= \frac{\lambda}{s \cdot \mu} \\ &= \frac{8 \text{ (ships/week)}}{2 \text{ (servers)} \cdot 12 \text{ (ships/week per server)}} \\ &= 0.33 \text{ (33\%busy)} \end{aligned}$$

[1] Because 2 servers are now doing the same job of what 1 server was previously doing. Therefore, the average serving time, τ, will decrease; the total capacity will increase $\left(\mu = \frac{1}{\tau}\right)$, and the total utilization of the facility will *decrease* $\left(\rho = \frac{\lambda}{\mu}\right)$, for the same jobs' flow rate, λ.

Utilization rate (ρ)	Number of servers (s): 1	2	3
.10	.0111	.0020	.0004
.20	.0500	.0167	.0062
.30	.1286	.0593	.0300
.35	.1885	.0977	.0552

Table 1.1 L_q values for the multiserver queue.

4b.i) *If the second berth is rented, what will be the average number of ships waiting?*

We need to calculate L_q. Here we have multiple servers, so we can only compute L_q by referring to the table of values given. However, the 1.1 doesn't give us L_q-values for $\rho = 0.33$. The closest values we have are $\rho_1 = 0.30$ and $\rho_2 = 0.35$.

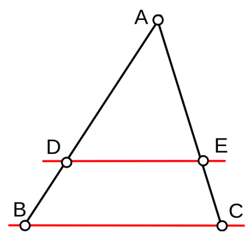

Fig. 1.1 Thales' Law: $\frac{AD}{AB} = \frac{AE}{AC}$

We note that:

for $\rho = 0.30 \longmapsto L_q = 0.0593$ (from the table)

for $\rho = 0.33 \longmapsto L_q =$???

for $\rho = 0.35 \longmapsto L_q = 0.0977$ (from the table)

Thus, using Thales' Law[2] (also known as "linear interpolation"):

$$\frac{0.33 - 0.30}{0.35 - 0.30} = \frac{L_q - 0.0593}{0.0977 - 0.0593}$$

or

$$\begin{aligned} L_q &= 0.0593 + \frac{0.0977 - 0.0593}{0.35 - 0.30}\,(0.33 - 0.30) \\ &= 0.0823 \text{ ships (approximately)} \end{aligned}$$

4b.ii) The average total time that a ship would spend at the refinery equals the average time it is waiting in the queue, W_q, plus its average serving time, τ. Using Little's Law, we have that $L_q = \lambda \cdot W_q$, or $W_q = \frac{L_q}{\lambda} = 0.01$ weeks. Thus, the total average time will be $W = W_q + \tau = 0.01 + 0.0833 = 0.0933$ weeks.

4b.iii) We have that:

Total Opp. Cost = (time spent at the refinery per ship)×(Opp. Cost per ship per week)×

[2] `http://en.wikipedia.org/wiki/Thales`: Thales was an ancient Greek philosopher and mathematician of the 7th century BC. Using the above "Thales' Law" he was able to measure the height of the Cheops' pyramid in the ancient Egypt.

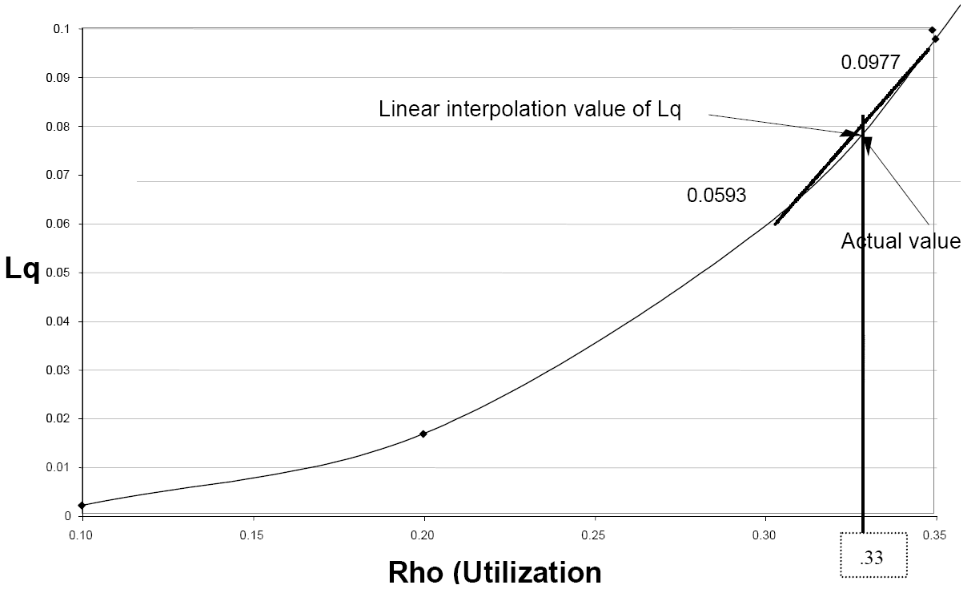

Fig. 1.2 Linear interpolation: approximating a curve by a line.

(average number of ships at the refinery per week)

BEFORE the expansion:

Total Opp. $\text{Cost}_1 = 0.25$ weeks $\times$ 30,000 \$/ship/week $\times$ 8 ships/week = \$60,000/week

AFTER the expansion:

Total Opp. $\text{Cost}_2 = 0.0933$ weeks$\times$30,000 \$/ship/week$\times$8 ships/week = \$22,392/week

Rental Cost = \$6,000/week

Using the second berth we have less costs (\$60,000/week vs. \$28,392/week), so it worths the consideration. We have a total estimated gain of \$60,000−\$22,392−\$6,000 = \$31,600 per week.

4c) We are given the following:

- Mean arrival rate of ships to facility: $\lambda = 8$ ships/week (same as before);
- Mean service rate per ship at facility (capacity of facility): $\mu = 15$ ships/week;
- Mean utilization rate of facility: $\rho = \frac{\lambda}{\mu} = \frac{8 \text{ ships/week}}{15 \text{ ships/week}} = 0.53$ (53% busy).

4c.i) The new expected waiting time, W_q, is given by the formula

$$W_q = \tau \frac{\rho}{1-\rho} \frac{1}{2} \left(CV_a^2 + CV_s^2 \right)$$

Note that the average service time, τ, is known as $\tau = \frac{1}{\mu} = \frac{1}{15}$ weeks = 0.0667 weeks per ship.

Also, we are given Poisson arrivals, so $CV_a = 1$, and exponential service times, so $CV_s = 1$.

Thus:

$$\begin{aligned} W_q &= \tau \frac{\rho}{1-\rho} \frac{1}{2} \left(CV_a^2 + CV_s^2 \right) \\ &= (0.0667 \text{ weeks}) \cdot \frac{0.53}{1-0.53} \cdot \frac{1}{2} \left(1^2 + 1^2 \right) \\ &= 0.075 \text{ weeks} \end{aligned}$$

and the total expected time for the ships at the facility equals

$$W = W_q + \tau = 0.075 + 0.067 = 0.14 \text{ weeks}$$

4c.ii) Similarly as before:

BEFORE:

Total Opp. $\text{Cost}_1 = 0.25$ weeks $\times$ 30,000 \$/ship/week $\times$ 8 ships/week = \$ 60,000/week

NOW:

Total Opp. $\text{Cost}_2 = 0.14$ weeks $\times$ 30,000 \$/ship/week $\times$ 8 ships/week = \$ 33,600/week

Rental Cost = \$4,000/week

Using this alternative we have less costs (\$ 60,000/week vs. \$33,600/week), so it worths the consideration. We have a total estimated gain of \$ 60,000 − \$ 33,600 − \$ 4,000 = \$ 22,400 per week.

Comparing the 4c) alternative with the 4b), we prefer modifying operations as in 4b).

4c.iii) Here, the utilization rate is the same as in 4a):

$$\rho = \frac{\lambda}{\mu} = \frac{8 \text{ ships/week}}{12 \text{ ships/week}} = 0.666 \ \ (66.6\% \text{ busy})$$

We note that since the new operations made now the service time constant, there no variability, and the coeffiecient of variation for service is zero, $CV_s = 0$.

The usual formula gives us:

$$\begin{aligned} W_q &= \tau \frac{\rho}{1-\rho} \frac{1}{2} \left(CV_a^2 + CV_s^2 \right) \\ &= (0.0833 \text{ weeks}) \cdot \frac{\frac{8}{12}}{1-\frac{8}{12}} \cdot \frac{1}{2} \left(1^2 + 0^2 \right) \\ &= 0.0833 \text{ weeks} \end{aligned}$$

Therefore:

$$W = W_q + \tau = 0.0833 + 0.0833 = 0.167 \text{ weeks}$$

Now, we compare the potential gains of this alternative.

BEFORE:

Total Opp. $\text{Cost}_1 = 0.25$ weeks $\times$ 30,000 \$/ship/week $\times$ 8 ships/week = \$ 60,000/week

NOW:

Total Opp. $\text{Cost}_2 = 0.167$ weeks $\times\, 30,000$ \$/ship/week $\times\, 8$ ships/week $= \$\,20,160$/week

Rental Cost $= \$4,000$/week

Using this alternative we have less costs (\$ 60,000/week vs. \$ 20,160/week), so it worths the consideration. We have a total estimated gain of $\$\,60,000 - \$\,20,160 - \$\,4,000 = \$\,16.160$ per week.

Note that the above net gain is less than the gains calculated in 4b) and 4c) above. However, all are better than the current situation in the refinery. □

Exercise 5: Voters' impatience to wait

During two former presidential elections in the United States, very long wait times have been witnessed at precincts (voting stations) in States that ultimately decided the election (Florida in 2000 and Ohio in 2004).

In Philadelphia as well, some voters complained about the long lines in some precincts, with most complaints coming from precinct A. In 2004, the average number of voters arriving at precinct A was of 35 per hour and the arrivals of voters was random with inter-arrival times that had a coefficient of variation of 1 ($CV_a = 1$).

Philadelphia had deployed 1 voting machine in precinct A. Suppose that each voter spent on average 100 seconds in the voting booth (this is the time needed to cast her/ his vote using a voting machine), with a standard deviation of 120 seconds.

5a) How long on average would a voter have to wait in line in precinct A in 2004 before entering in a booth to cast her/ his vote?

Given the long wait times for precinct A, the city of Philadelphia is thinking of alternative solutions to improve voting conditions. One of the proposed solutions is as follows.

Proposal 1: "Deploy three additional voting machines in precinct A. Assume that the voter turnout is expected to be a little more organized in 2008 as in the previous election and the time spent in the voting booth will now have a standard deviation of 75 seconds".

5b) Under Proposal 1, how long on average would a voter have to wait in line in precinct A in 2008 before casting her/ his vote?

Solution

5a) We need to calculate W_q, the average waiting time in the queue to be served (i.e. to cast a vote).

We are given the following:

- $s = 1$ server ("...1 voting machine...")
- Average service time: $\tau = \frac{100}{60}$ min $= 1.67$ min

- Average arrival rate: $\lambda = \frac{35}{60}$ voters/min $= 0.583$ voters/min
- Mean utilization rate: $\rho = \lambda\,\tau = \left(0.583\,\frac{\text{voters}}{\text{min}}\right) \cdot 1.67 \text{ min} = 0.97$ (97% busy)
- $CV_a = 1$
- $CV_s = \frac{\sigma}{\mu} = \frac{120 \text{ sec}}{100 \text{ sec}} = 1.2$

The expected waiting time, W_q, is given by

$$\begin{aligned} W_q &= \tau\,\frac{\rho}{1-\rho}\,\frac{1}{2}\,\left(CV_a^2 + CV_s^2\right) \\ &= (1.67 \text{ min}) \cdot \frac{0.97}{1-0.97} \cdot \frac{1}{2}\,\left(1^2 + 1.2^2\right) \\ &= 65.9 \text{ min} \end{aligned}$$

5b) Here, again we need to calculate W_q. However, this is a case of multiple servers ($s = 4$), so we should somehow use the values of the tables, since we don't have a formula for calculating W_q immediately. What we do have, indeed, is the Little's Law: $L_q = \lambda \cdot W_q$, or $W_q = \frac{L_q}{\lambda}$. So, we will calculate L_q by the tables and we have found W_q.

Note that we are given the following:

- $s = 4$ servers ("...deploy three additional voting machines...")
- Average service time: $\tau = 1.67$ min
- Average arrival rate: $\lambda = 0.583$ voters/min
- Mean utilization rate: $\rho = \frac{\lambda\,\tau}{s} = \frac{\left(0.583\,\frac{\text{voters}}{\text{min}}\right)\cdot 1.67 \text{ min}}{4 \text{ servers}} = 0.24$ (24% busy) per server

Utilization rate (ρ)	Number of servers (s): 1	2	3	4
.10	.0111	.0020	.0004	.0001
.20	.0500	.0167	.0062	.0024
.30	.1286	.0593	.0300	.0159

Table 1.2 L_q values for the multiserver queue.

Using 1.2 for $s = 4$ servers, we need to use Thales' Law again, since we can't obtain the L_q value that corresponds to $\rho = 0.24$.

for $\rho = 0.20 \longmapsto L_q = 0.0024$ (from the table)

for $\rho = 0.24 \longmapsto L_q = \;???$
for $\rho = 0.30 \longmapsto L_q = 0.0159$ (from the table)
Thus, using Thales' Law:

$$\frac{0.24 - 0.20}{0.30 - 0.20} = \frac{L_q - 0.0024}{0.0159 - 0.0024}$$

or

$$\begin{aligned} L_q &= 0.0024 + \frac{0.0159 - 0.0024}{0.30 - 0.20}\,(0.24 - 0.20) \\ &= 0.0078 \text{ voters (approximately)} \end{aligned}$$

Now from Little's Law

$$\begin{aligned} W_q &= \frac{L_q}{\lambda} \\ &= \frac{0.0078 \text{ voters}}{0.583 \text{ voters/min}} \\ &= 0.013 \text{ min} \end{aligned}$$

So, the voters will now have an expected waiting time in the queue of approx. 1 sec (which is practically negligible). $\square$

Table: L_q values for the Multi-Server Queue

Values of L_q for s servers, with mean utilization rate ρ, assuming Poisson arrivals and Exponential service times.

Utilization rate (ρ)	Number of servers (s) 1	2	3	4	5
.10	.0111	.0020	.0004	.0001	.0000
.20	.0500	.0167	.0062	.0024	.0010
.30	.1286	.0593	.0300	.0159	.0086
.35	.1885	.0977	.0552	.0325	.0196
.40	.2667	.1524	.0941	.0605	.0398
.45	.3682	.2285	.1522	.1052	.0743
.50	.5000	.3333	.2368	.1739	.1304
.55	.6722	.4771	.3583	.2772	.2185
.60	.9000	.6750	.5321	.4306	.3542
.62	1.0116	.7743	.6213	.5109	.4269
.64	1.1378	.8880	.7246	.6051	.5130
.66	1.2812	1.0188	.8446	.7158	.6152
.68	1.4450	1.1698	.9847	.8461	.7367
.70	1.6333	1.3451	1.1488	1.0002	.8816
.72	1.8514	1.5500	1.3423	1.1834	1.0553
.74	2.1062	1.7914	1.5721	1.4025	1.2646
.76	2.4067	2.0785	1.8472	1.6668	1.5187
.78	2.7655	2.4237	2.1803	1.9887	1.8302
.80	3.2000	2.8444	2.5888	2.3857	2.2165
.82	3.7356	3.3661	3.0979	2.8832	2.7029
.84	4.4100	4.0265	3.7456	3.5190	3.3273
.86	5.2829	4.8852	4.5914	4.3526	4.1493
.88	6.4533	6.0414	5.7345	5.3834	5.2682
.90	8.1000	7.6737	7.3535	7.0898	6.8624
.92	10.5800	10.1392	9.8056	9.5290	9.2893
.94	14.7267	14.2712	13.9240	13.6344	13.3821
.96	23.0400	22.5698	22.2088	21.9060	21.6408
.98	48.0200	47.5350	47.1602	46.8439	46.5656
.99	98.0101	97.5176	97.1357	96.8127	96.4274