Trinity College Dublin
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BU7142 Foundations of Business Analytics

Lecture 5

Hypothesis Testing

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Hypothesis Testing: Example

Alice is a coffee lover, from a year's data, on average she drinks 10 coffee per week, with standard deviation 2. If there is a week during which she drinks 15 coffee, is it a normal week for her? How about the week during which she drinks 9 coffee?

Big Picture and Motivation

- Hypothesis testing is a **framework** that you can use to test a statement, a concern, or a belief; It is a probabilistic/statistical **language** of “evidence”; It is also the **basis** of scientific rigor.

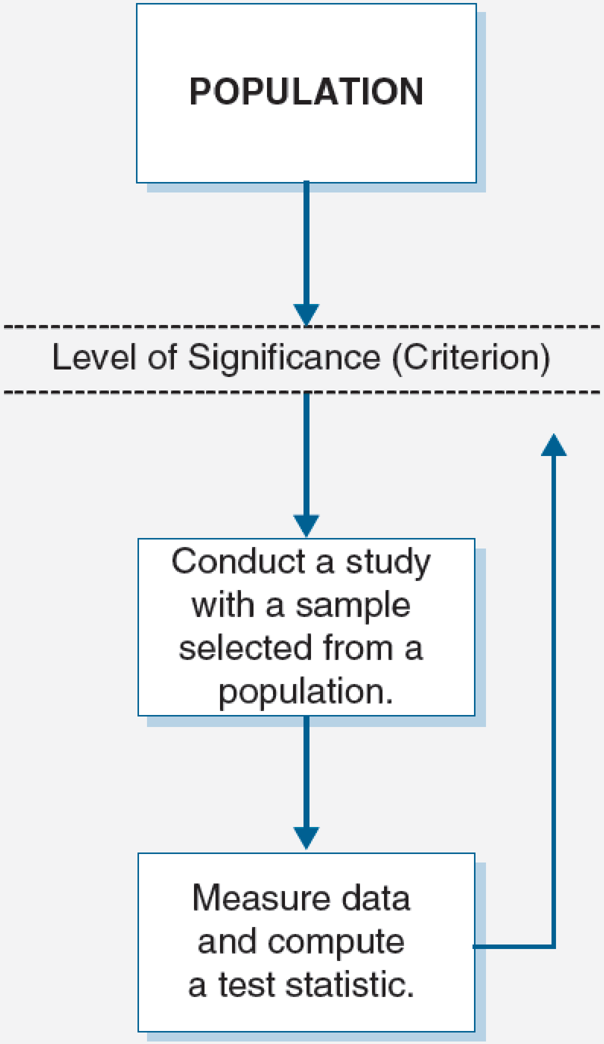
- Applications
  - Pharmaceutical drug testing, biomedical tests
  - Engineering
  - Strategy
  - Human resources management
  - Operations (production, manufacturing, etc.)
  - Marketing
  - Finance
  - …

Framework for Hypothesis Testing

STEP 1: State the hypotheses. A researcher states a null hypothesis about a value in the population (H_0) and an alternative hypothesis that contradicts the null hypothesis.

STEP 2: Set the criteria for a decision. A criterion is set upon which a researcher will decide whether to retain or reject the value stated in the null hypothesis.

A sample is selected from the population, and a sample mean is measured.



STEP 3: Compute the test statistic. This will produce a value that can be compared to the criterion that was set before the sample was selected.

STEP 4: Make a decision.
If the probability of obtaining a sample mean is less than 5% when the null is true, then reject the null hypothesis.
If the probability of obtaining a sample mean is greater than 5% when the null is true, then retain the null hypothesis.

Four Steps

- Step 1. Formulate the hypothesis
  - Null Hypothesis (H_0)
  - Alternate Hypothesis (H_A)

- Step 2. Set the criteria for a decision

- Step 3. Acquire an objective test statistic (e.g. from evidence)

- Step 4. Does the objective test statistic represent an overwhelming evidence against the Null Hypothesis? (i.e., p-value < α? or equivalently, compare the test statistic with the critical z-value or t-value)
  - If yes, reject the null (H_0) and accept the alternative (H_A) as the truth.
  - If no, accept the null (H_0) as the truth.

More about Step 4: P-value & Significance

- P-value: probability of observing the statistic (or more extreme) given that the null hypothesis is true.
  - Must compute

- Significance level α: a threshold probability (e.g. 0.05, or 0.1) that determines whether or not the evidence is overwhelming.
  - Typically given

- If the p-value is less than the significance level α, then you can reject the null hypothesis

- How do we find the p-value? Use the sample distribution (normal)!

More about Step 4: T-Statistics

- Instead of comparing p-value with significance level α, we can also compare the test statistics with z-value (or t-value, if use t-distribution)

- We compare the test-statistic with the z-value, z_α, which corresponds to the required significance level α. Criteria depend on the type of test.

- Then, we decide to accept H_0 or not.

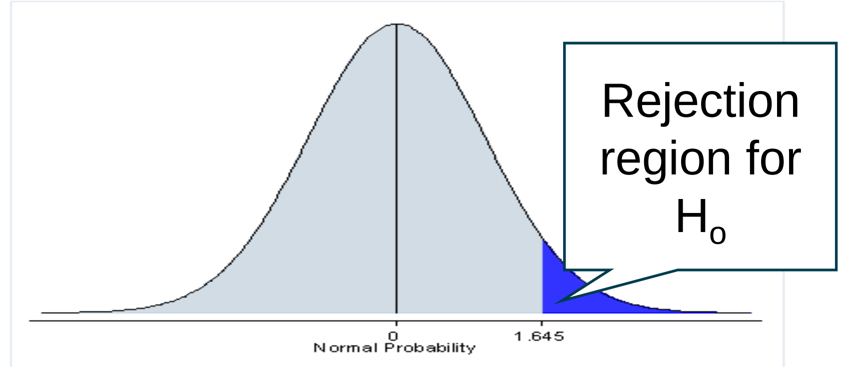
One-tailed and Two-tailed Tests

- One right-tailed test

  Hypotheses:

  H_0: $\mu \leq M$,

  H_1: $\mu > M$

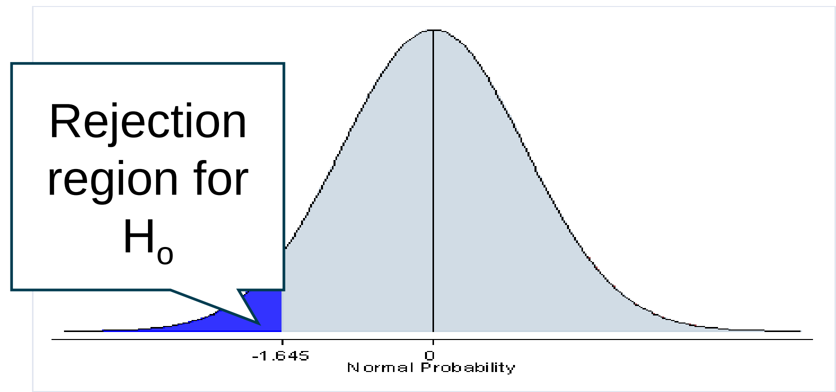


- One left-tailed test

  Hypotheses:

  H_0: $\mu \geq M$,

  H_1: $\mu < M$

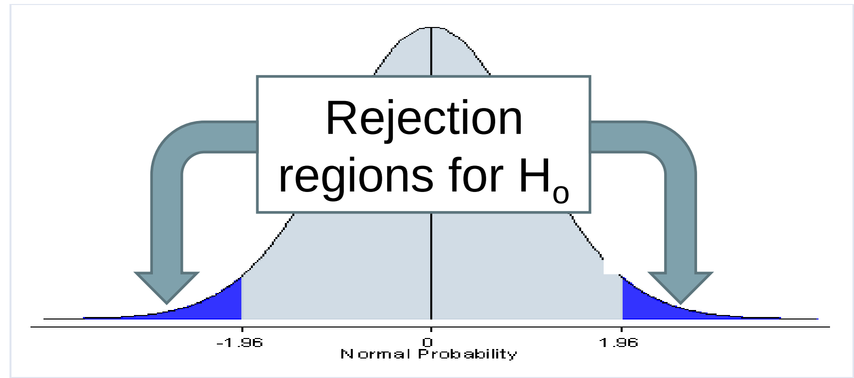


- Two-tailed test

  Hypotheses:

  H_0: $\mu = M$,

  H_1: $\mu \neq M$



One-tailed and two-tailed tests

Right tailed	Left tailed	Two-tailed
H_0: μ≤M, H_1: μ>M statistic=($\bar{X}$ -M)/(s/√n) P-value=P(Z>statistic) 1. If P<α, accept H_1 2. If statistic>Z_α, accept H_1	H_0: μ≥M, H_1: μ<M statistic=($\bar{X}$-M)/(s/√n) P-value= P(Z<statistic) 1.If P<α, accept H_1 2.If statistic<Z_α, accept H_1	H_0: μ=M, H_1: μ≠M statistic=($\bar{X}$-M)/(s/√n) If the statistic is positive: P-value=2*P(Z>statistic) If the statistic is negative: P-value=2*P(Z<statistic) 1.If P<α, accept H_1 2.If |statistic|>$Z_{\alpha/2}$, accept H_1 3.If M is outside the (1-α) level confidence interval for μ, accept H_1

Example 1: Connection with Confidence Intervals

A company that delivers packages within London claims that it takes an average of 28 minutes for a package to be delivered from your door to the destination. Suppose that you want to carry out a hypothesis test of this claim.

Set the null and alternative hypotheses:

H_0: $\mu = 28$

H_1: $\mu \neq 28$

Collect sample data:

$n = 100$

$\bar{x} = 31.5$

$s = 5$

Construct a 95% confidence interval for the average delivery times of *all* packages:

$$\bar{x} \pm z_{.025}\frac{s}{\sqrt{n}} = 31.5 \pm 1.96\frac{5}{\sqrt{100}}$$

$$= 31.5 \pm .98 = [30.52,\ 32.48]$$

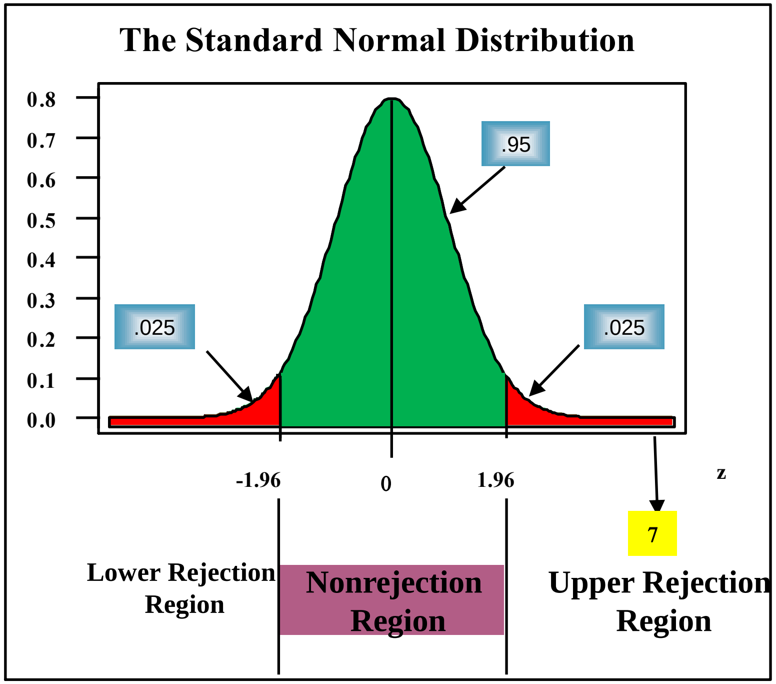
We can be 95% sure that the average time for all packages is between 30.52 and 32.48 minutes.

Since the asserted value, 28 minutes, is not in this 95% confidence interval, we may reasonably reject the null hypothesis.

Example 1: Two-tailed Testing

$$ts = \frac{\text{best guess - null value}}{\text{standard deviation of best guess at } H_0}$$

$$ts = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{31.5 - 28}{5/\sqrt{100}} = 7$$



$$\text{P-value} = 2P(Z > ts) = 2P(Z > 7) = 2*(0.5 - 0.499999999) \approx 0.000000002$$

Example 2: One-tailed testing

A certain kind of packaged food bears the following statement on the package: "Average net weight 12 oz." Suppose that a consumer group has been receiving complaints from users of the product who believe that they are getting smaller quantities than the manufacturer states on the package. The consumer group wants, therefore, to test the hypothesis that the average net weight of the product in question is 12 oz. versus the alternative that the packages are, on average, underfilled. A random sample of 144 packages of the food product is collected, and it is found that the average net weight in the sample is 11.8 oz. and the sample standard deviation is 6 oz. Given these findings, is there evidence the manufacturer is underfilling the packages?

$H_0: \mu \geq 12$
$H_1: \mu < 12$
$n = 144$
For $\alpha = 0.05$, the critical value of z is -1.645

The test statistic is:

Do not reject H_0 if: $[z \geq -1.645]$
Reject H_0 if: $[z < -1.645]$

$n = 144$
$\bar{x} = 11.8$
$s = 6$

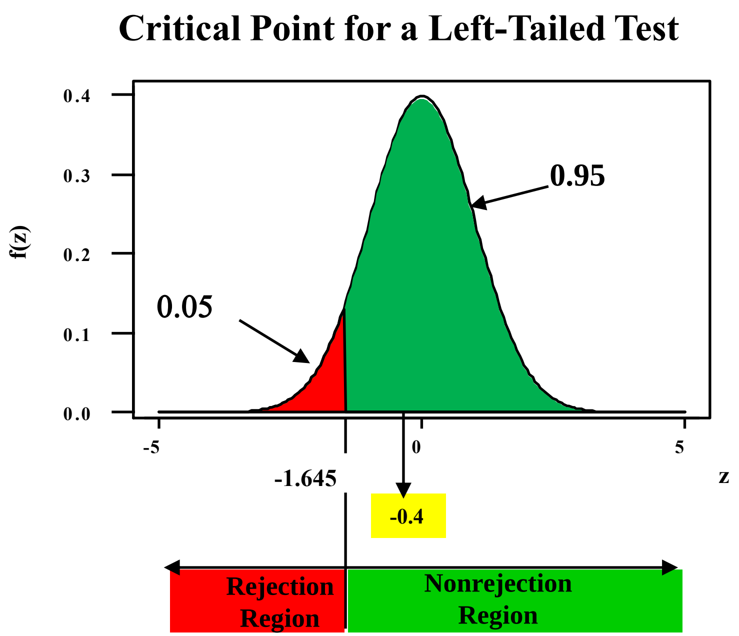
$$z = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{11.8 - 12}{\frac{6}{\sqrt{144}}}$$

$$= \frac{-.2}{.5} = -0.4 \quad \Rightarrow \text{Do not reject } H_0$$

Example 2: One-tailed testing

$$ts = \frac{\text{best guess - null value}}{\text{standard deviation of best guess at } H_0}$$

$$ts = z = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{11.8 - 12}{6/\sqrt{144}} = -0.4$$



$$\text{P-value} = P(Z < ts) = P(Z < -0.4) = 0.5 - 0.1554 \approx 0.3446$$

Exercise

A study was conducted to determine customer satisfaction from real estate deals. Suppose that before the financial crisis, the average customer satisfaction rating, on a scale of 0 to 100, was 77. A survey questionnaire was sent to a random sample of 50 residents who bought new houses recently. The average satisfaction rating for this sample was found to be 84. Assume that the standard deviation of satisfaction level in the population is 28. Did customer satisfaction increase? Answer to a significance level of $p=0.05$.

Exercise

Solution. Hypotheses: H_0: μ≤77, H_1: μ>77. Note that, this is a one-tailed test.

Calculation of test statistic

The sample size is larger than 30, so, by the Central Limit Theorem, the sample mean will be normally distributed with mean 77 and std 28/50^0.5=3.96.

Therefore, $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}=\frac{\overline{X}-77}{3.96}$ will follow a standard normal distribution

Our test statistic is: $Z=\frac{\overline{X}-77}{3.96}=\frac{84-77}{3.96}=1.7676>1.64$

The p value is: P(Z>1.7676)=1-P(Z<1.7676)=0.0385<0.05

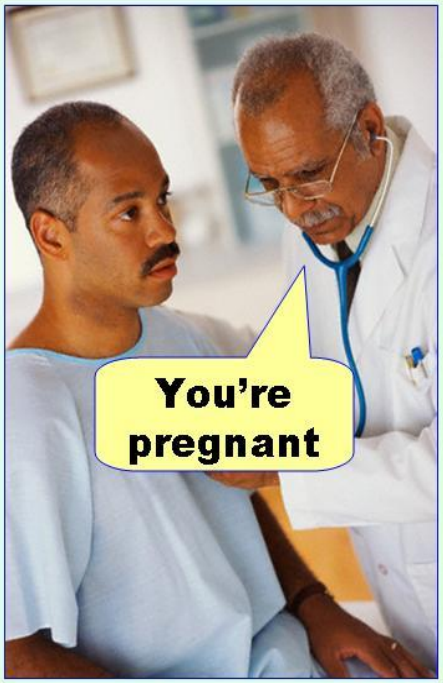
Conclusion: as the probability of obtaining a sample mean of 84 is smaller than the required significance level, we accept H_1 and reject H_0: customer satisfaction increased.

Types of errors

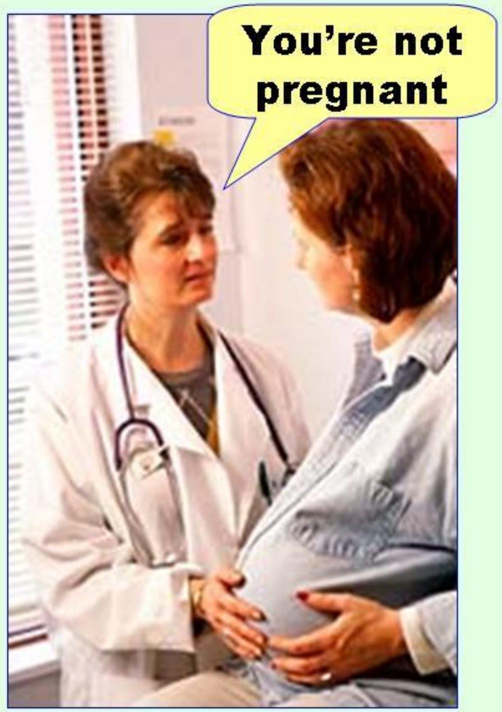
- We have to make an accept/reject decision
- We cannot work on a 100% confidence level, so there can always be errors.
- No error:
  - We accept H_0 and H_0 is true
  - We reject H_0 and H_0 is false
- Error:
  - We reject H_0 and H_0 is true: Type I error
  - We accept H_0 and H_0 is false: Type II error

Example

Type I error
(false positive)



Type II error
(false negative)



Not pregnant
=> H_0 is true

Pregnant
=> H_0 is false

H_0 : you are not pregnant
H_1 : you are pregnant

Type I error:
H_0 is true,
We reject H_0.

Type II error:
H_0 is false
We accept H_0

Types of Analysis and Tests

- There are many different ways to test a hypotheses

- It depends mostly on what you are testing
  - What is your research question?

- Useful analysis tool:
  - T-test
  - Chi-square test
  - ANOVA
  - Regression (Next lecture)

T-Test

Comparing the means of two groups

- To see whether there is any significant difference between the means for two groups in the variable of interest.
- Groups can be either two different groups or the same group after and before treatment

Example: Is the turnover rate higher in high-tech companies compared to other companies?

Note that you are ignoring all other variables that may be affecting turnover (e.g. company size).

T-Test: Using Statistics

- Inferences about differences between parameters of two populations
  - ✓ Observe the ***same*** group of persons or things (Paired-Observations)
    - – At two different times: “before” and “after”
    - – Under two different sets of circumstances or “treatments”
  - ✓ Independent Samples
    - Observe ***different*** groups of persons or things
      - – At different times or under different sets of circumstances

Comparisons of Two Population Means: Testing Situations

- I: Difference between two population means is 0
  - ✓ $\mu_1 = \mu_2$
    - H_0: $\mu_1 - \mu_2 = 0$
    - H_1: $\mu_1 - \mu_2 \neq 0$

- II: Difference between two population means is less than 0
  - ✓ $\mu_1 \leq \mu_2$
    - H_0: $\mu_1 - \mu_2 \leq 0$
    - H_1: $\mu_1 - \mu_2 > 0$

- III: Difference between two population means is less than D
  - ✓ $\mu_1 \leq \mu_2 + D$
    - H_0: $\mu_1 - \mu_2 \leq D$
    - H_1: $\mu_1 - \mu_2 > D$

Comparisons of Two Population Means: Testing Situations

- IV: Difference between two population means is greater than 0
  - ✓ $\mu_1 \geq \mu_2$
    - H_0: $\mu_1 - \mu_2 \geq 0$
    - H_1: $\mu_1 - \mu_2 < 0$

- V: Difference between two population means is greater than D
  - ✓ $\mu_1 \geq \mu_2 + D$
    - H_0: $\mu_1 - \mu_2 \geq D$
    - H_1: $\mu_1 - \mu_2 < D$

Testing Equality of Distributions: Chi-square Test

- It tests a null hypothesis stating that the frequency distribution of certain events observed in a sample is consistent with a particular theoretical distribution.

- Test for equality of distribution

Chi-square Test: Example

- 100 people were asked their stress level depending on their working hours per week. Is there a relationship between stress level and working hours?

Working Hours	Stress Level		Total
	High	Low	
Under 40 hours	28	26	54
Over 40 hours	14	32	46
Total	42	58	100

- We need to compare the distribution of Stress Level (High or Low) in the two age groups

- Use Chi-square Test

Chi-square Test: Statistics

- Chi-square (χ 2) test calculate the test value based on observed and expected values.

- $\chi\ 2 = \sum (O - E)\ \hat{}2\ /\ E$

- Where O is the count of observed data in each category and E is expected count of data in each category.

ANOVA: Analysis of Variance

ANOVA tests the following hypotheses:

- H_0 (null hypothesis): The means of all the groups are equal.

- H_a: Not all the means are equal

ANOVA: How does it work

ANOVA measures and compares two sources of variation in the data:

- Variation BETWEEN groups

  For each data value, check the difference between its group mean and the overall mean

- Variation WITHIN groups

  For each data value, check the difference between that value and the mean of its group

ANOVA: F-Statistic

- The F-statistic is a ratio of the Between-Group Variation divided by the Within-Group Variation:

$$\boxed{F = \frac{Between\ Group\ Variation}{Within\ Group\ Variation}}$$

- A large F-statistic indicates that there is a larger difference between groups than within groups, thus we may reject H_0,

Mini Case: A delayed flight

- Please download mini case and data file.

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Seminar: Hypothesis Testing

Exercise 1

- A milk company wants to determine whether the milk-fill process is working properly (that is, whether the mean fill throughout the entire packaging process remains at the specified 500 grams, and no corrective action is needed). To evaluate the 500-gram requirement, you take a random sample of 49 bottles, and weigh each box. The mean of the sample was 497 gram.

  What are the null and alternative hypotheses for a. one right-tailed test, b. one left-tailed test, c. a two-tailed test?

Exercise 2

- A baker claims that his bread height is more than 15 cm, on the average. Several of his customers do not believe him. To persuade his customers that he is right, the baker decides to do a hypothesis test. He bakes 30 loaves of bread. The average height of the sample loaves is 17 cm. The baker knows from baking hundreds of loaves of bread that the standard deviation for the height is 0.5 cm and that it is normally distributed. Could a test help the baker?

Exercise 3

- Late payment of medical claims can add to the cost of health care. It was reported that the mean time from the date of service to the date of payment for one insurance company was 41.4 days during a recent period. Suppose that a sample of 100 medical claims is selected during the latest time period. The sample mean time from the date of service to the date of payment was 39.6 days, and the sample standard deviation was 7.4 days.

a. Using the 0.05 level of significance, is there evidence that the population mean has changed from 41.4 days?

b. What is your answer in (a) if you use the 0.01 level of significance?

c. What is your answer in (a) if the sample mean is 38.2 days and the sample standard deviation is 10.7 days?

Exercise 4

Prices, in general, change from day to day. However, a certain commodity is known to have a price that is stable through time. The mean daily price is believed to be $14.25. To test the hypothesis that the average price is $14.25 versus the alternative hypothesis that it is not $14.25, a random sample of 16 daily prices is collected. The results have mean $16.50. Assume that the population daily price is normally distributed and the sample std of price is $5.8. Using significance level of 0.05, can you reject the null hypothesis?

Thank You!

Any Questions?