BU7142 Foundations of Business Analytics

Lecture 2

Set and Probability Theory

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Contents

Set Theory

- Definition. A set is a collection of objects, called elements or members.

- Examples:
  - R is the set of all real numbers between -∞ and ∞.
  - {1,2,4,7} is the set including the numbers 1,2,4,7.
  - Intervals are also sets, e.g. all numbers between 3 and 4
  - Set of daily prices of the stock of HP during the last year.

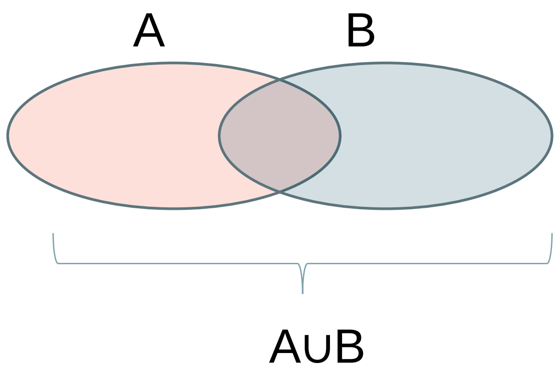
Set Theory

- Definition. A subset is any collection of elements in a set. An empty set is a set which contains no elements (Φ).

- Examples
  - (3,4] is a subset of R.
  - The price of HP's stock on the first day of each month is a subset of the set of its daily prices during a year.
  - The empty set is a subset of any set.

Set Theory: Union

- Union. Let A,B, be two sets. Their union is the set which includes all elements which are in A or in B. It is denoted by A∪B.



- Example:

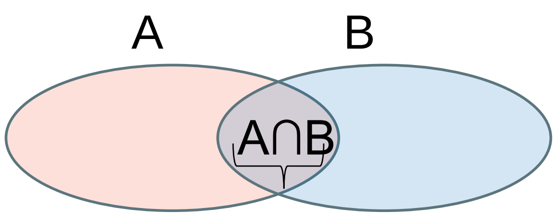
A: the set of daily prices of HP's stock between January and April 2014.

B: the set of HP's daily prices between March and May 2014.

A∪B: the set of HP's daily prices between January and May 2014.

Set Theory: Intersection

- Intersection. Let A,B, be two sets. Their intersection is the set which includes all elements which are in A and in B. It is denoted by A∩B.



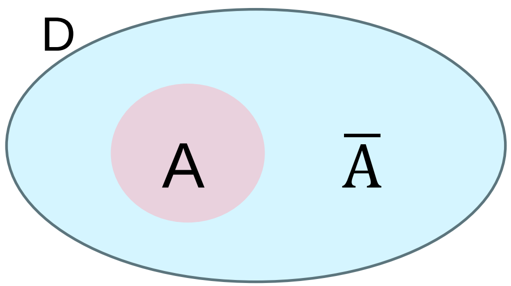
- Example:

A: the set of daily prices of HP's stock between January and April 2014.

B: the set of HP's daily prices between March and May 2014.

A∩B: the set of HP's prices between March and April.

Set Theory: Compliment



- Compliment. Let A be a subset of a set D.
  Its complement, $\overline{A}$, consists of all of the elements in D apart from those in A.

- Example.
  D is the set of a survey results.
  A is the set of men's answers.
  $\overline{A}$ is the set of the women's answers.

- An important set is the set of all possible subsets of a given set, A.

- Example.
  If A={1,2,3}, the set of all subsets consists of:
  { Φ ,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}.

Probability Theory: Definitions

- An experiment is any process or procedure for which more than one outcome is possible.
- Examples:
  - Tossing a coin: Possible outcome: head
  - Rolling a die: Possible outcome: 6
  - The price of IBM's stock tomorrow: Possible outcome: \$187.81

- Sample Space is the set of all possible outcomes of an experiment.
- Examples:
  - Tossing a coin: {head, tail}
  - Rolling a die: {1,2,3,4,5,6}
  - The price of IBM's stock tomorrow: $[0, \infty)$

- An event is a subset of a sample space.
- Examples:
  - The price of IBM's stock tomorrow is between \$180 and \$190.

Probability Theory: Definitions

- Probability is a numerical measure of the chance that an event will occur.

- Probability Measure is a function P from the set of all of the events of a sample space Ω to [0,1], which satisfies for all disjoint events
  $A_1, A_2, \ldots, A_n$:
  - $P(A) \geq 0$
  - $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.
  - $P(\Omega) = 1$

- A probability measure is a function which assigns a probability to each outcome in the sample space.

- Note. for any event A, $0 \leq P(A) \leq 1$:
  - $P(A)=0$, if A never happens.
  - $P(A)=1$, if A always happens.

Types of Probability

- Subjective Probability
- A-priori Classical Probability
- Empirical Classical Probability

Subjective Probability

- Subjective Probability: Probability values are assigned subjectively by people.

- Example:
  - I have a feeling that the probability that IBM's stock price tomorrow will be higher than $190 is 80%.
  - Someone less optimistic might estimate this probability by 50%.

- Advantage: Does not involve calculations

- Disadvantage: Subjective
  - Bilgin (2012) showed that people judge losses to have higher probabilities than gains
  - "It is very likely that this will happen" → Estimate the probability

A-priori Classical Probability

- A-priori classical probability: the probability is determined according to a certain theory.

- Example: Random walk model predicts that the probability that IBM's stock price will increase tomorrow is 50%, because according to the random walk model, the probability of an increase in the price equals to the probability of decrease.

- Advantages:
  - May be used to develop theories
  - Helps calculations

- Disadvantages:
  - One needs a trustworthy theory

Empirical Classical Probability

- Empirical classical probability:
  - Probability is based on observed data.

$$P(\text{event}) = \frac{\text{Number of cases in which the event occured}}{\text{The total number of cases}}$$

- Example: One monitors a very large number of trading days, e.g. 1000. and observes, 481 times that the price increased, then Prob.=481/1000=0.481.

- Advantages:
  - Does not require a theory
  - Based on real data

- Disadvantages:
  - Requires collection of data (time, money,…)
  - Based on the assumption that probability distributions do not change. (Is this assumption true?)

Exercise

In a marketing survey, 1000 consumers were asked whether they intend to buy a new product or not, and in a follow-up survey, the same group of consumers were asked whether they have bought this product. The data is shown below:

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	200	100	300
No	250	450	700
Total	450	550	1000

1. What is the probability that a person plans to purchase this product?
2. What approach did we use here?

Solution

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	200	100	300
No	250	450	700
Total	450	550	1000

1. The probability that a person plans to purchase this product within a year is:

$$P(\text{planned to purchase}) = \frac{\text{number of people who planned to purchase}}{\text{numberof people in the sample}}$$

$$= \frac{200+100}{1000} = 0.3$$

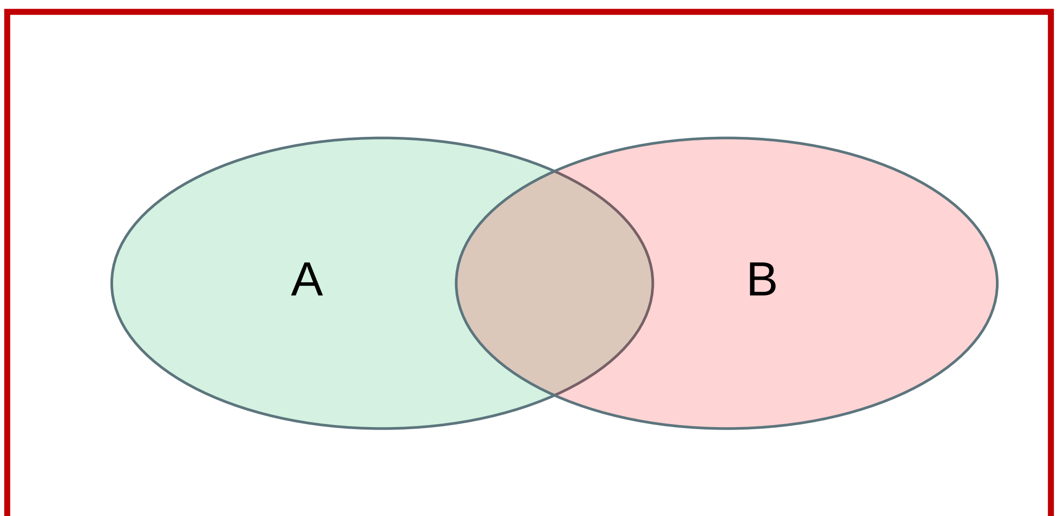
2. We used the empirical classical probability approach.

Probability Rules

1. $P(A)+P(\bar{A})=1$
2. $P(A \cap B)+P(A \cap \bar{B})=P(A)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

probar que estas tres son iguales

- Can you prove the above rules graphically?



Conditional Probability

- Definition: The probability of certain events depends on the probability of other events.

- Example:

  - Fruit and veges depend on the sun.

  (http://www.nzherald.co.nz/lifestyle/news/article.cfm?c_id=6&objectid=11723008)

- We denote the probability of A given B by P(A|B).

- If P(B)≠0, then $P(A\,|\,B) = \dfrac{P(A \cap B)}{P(B)}$

Exercise

In a marketing survey, 1000 consumers were asked whether they intend to buy a new product or not, and in a follow-up survey, the same group of people were asked whether they have bought this product. The data is shown below:

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	200	100	300
No	250	450	700
Total	450	550	1000

- What is the probability that a consumer purchased this product given that he or she planned to purchase?
- What is the probability that a consumer did not purchase this product given that he or she planned to purchase?

Solution

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	200	100	300
No	250	450	700
Total	450	550	1000

$$P(\text{Purchased} \mid \text{Planned}) = \frac{P(\text{Purchased and planned})}{P(\text{Planned})} = \frac{200/1000}{300/1000} = \frac{200}{300} = 0.6667$$

$$P(\text{Did not purchase} \mid \text{Planned}) = \frac{P(\text{Did not purchase and planned})}{P(\text{Planned})} = \frac{100/1000}{300/1000} = \frac{100}{300} = 0.3333$$

Bayes Theorem

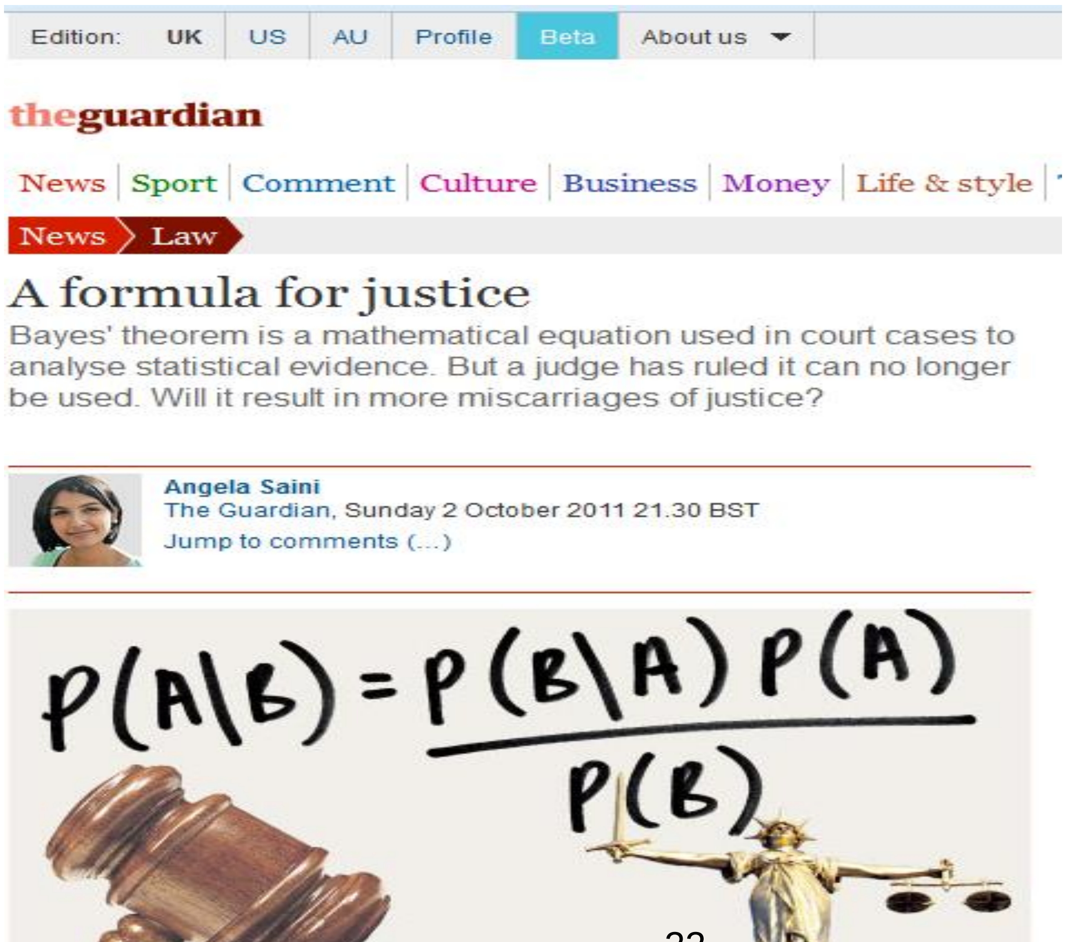
- Bayes' Theorem: $$\mathrm{P(B \mid A)} = \frac{\mathrm{P(A \mid B)P(B)}}{\mathrm{P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})}}$$
- Purpose: If we know P(A|B),P(B),P(A|$\overline{\mathrm{B}}$) and P($\overline{\mathrm{B}}$), then we can calculate P(B|A).
- Example: A factory manager uses quality test to identify defective products. The probability that a product is defective is 0.03. When the product is defective, the probability that this quality test will give a positive result is 0.90. When the product is not defective, the probability of a positive test result is 0.02. Suppose that the quality test has given a positive result. What is the probability that the product is actually defective?
- Solution: D: the product is defective, $\overline{D}$: the product is not defective
  T: test is positive, $\overline{\mathrm{T}}$: test is negative
  P(D)=0.03, P($\overline{\mathrm{D}}$)=1-0.03=0.97, P(T|D)=0.90, P(T|$\overline{D}$)=0.02

$$\mathrm{P(D \mid T)} = \frac{\mathrm{P(T \mid D)P(D)}}{\mathrm{P(T \mid D)P(D) + P(T \mid \overline{D})P(\overline{D})}} = \frac{0.9 * 0.03}{0.9 * 0.03 + 0.02 * 0.97} = 0.582$$

- Exercise: Can you prove Bayes Theorem using conditional probability?

Bayes Theorem in Court

http://www.theguardian.com/law/2011/oct/02/formula-justice-bayes-theorem-miscarriage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A formula for justice

Bayes' theorem is a mathematical equation used in court cases to analyse statistical evidence. But a judge has ruled it can no longer be used. Will it result in more miscarriages of justice?

Angela Saini
The Guardian, Sunday 2 October 2011 21.30 BST
Jump to comments (...)

Independent Events

- Definition: Two events, A and B are called independent if P(A|B)=P(A). Namely, the outcome of B does not affect the probability of occurrence of A.

- Example: Two coin tosses are independent if the coins are fair.

- Note: The following necessary and sufficient conditions for independent events are equivalent:

$$P(B \mid A) = P(B) \qquad P(A \mid B) = P(A) \qquad P(A \cap B) = P(A) * P(B)$$

- Exercise: Can you prove the above note?

Exercise

In a marketing survey, 1000 consumers were asked whether they intend to buy a new product or not, and in a follow-up survey, the same group of consumers were asked whether they have bought this product. The data is shown below:

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	200	100	300
No	250	450	700
Total	450	550	1000

- Given that a consumer planned to purchase, what is the probability that he/she finally purchased the product?
- Given that a consumer planned not to purchase, what is the probability that he/she finally purchased the product?
- Are consumers' willingness to purchase and their actual purchase decisions independent or not? Why?

Solution

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	200	100	300
No	250	450	700
Total	450	550	1000

- Solution.

$$P(\text{Purchased} \mid \text{Planned}) = \frac{P(\text{Purchased and planned})}{P(\text{Planned})} = \frac{200/1000}{300/1000} = \frac{200}{300} = 0.6667$$

$$P(\text{Purchased} \mid \text{Planned Not}) = \frac{P(\text{Purchase and Planned Not})}{P(\text{Planned Not})} = \frac{250/1000}{700/1000} = \frac{250}{700} = 0.3571$$

$$P(\text{Purchased} \mid \text{Planned}) = 0.6667 \qquad P(\text{Purchased}) = 0.45$$

So the two events are not independent.

The Birthday Problem

To start, P(two people share birthday) = 1 - P(no people share birthday).

The probability of two people sharing birthday is difficult to get, but we can get the probability of no people sharing birthday:

- One person
  This person can have any birthday. P(no)=(365/365)=1

- Two persons
  P(no)=(365/365) * (364/365)=99.73%

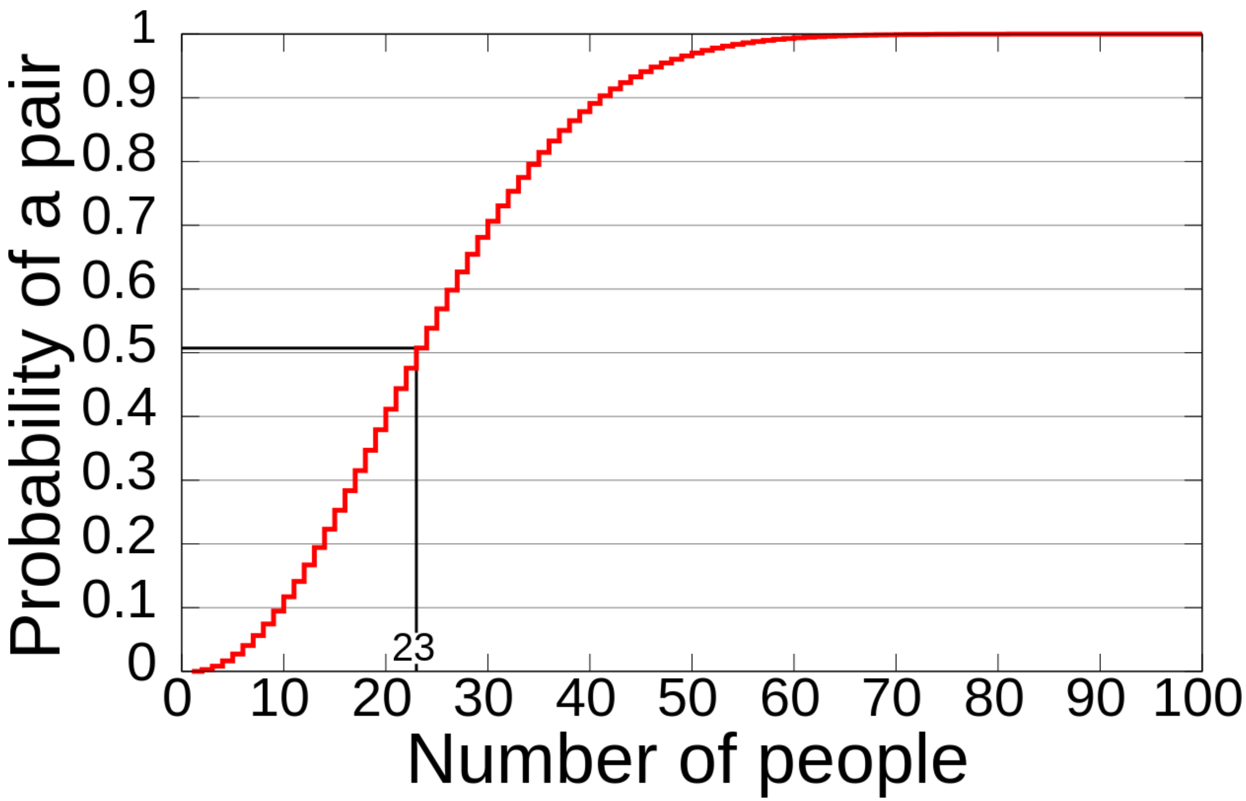
- Three persons
  P(no)=((365/365) * (364/365) * (363/365)=99.18%

……

- n persons
  P(no)=((365/365) * ((365-1)/365) * ((365-2)/365)*…*((365-n+1)/365)
  =365! / ((365-n)! * 365^n)=[365*364*…*(365-n+1)]/ 365^n

P(two people share birthday)=1- P(no people share birthday)
=1-[365*364*…*(365-n+1)]/ 365^n

The Birthday Problem



Seminar 2

Example

- If you flip a coin 3 times, what is the probability that you will get Heads on at least one of the 3 flips?

Exercise (Aczel, 2.11)

- According to an article in *Fortune*, institutional investors recently changed the proportions of their portfolios toward public sector funds. The article implies that 8% of investors studied invest in public sector funds and 6% in corporate funds. Assume that 2% invest in both kinds. If an investor is chosen at random, what is the probability that this investor has either public or corporate funds?

Exercise

200 people were interviewed regarding their intention to buy your product and whether or not they are financially able to do so. The following results were obtained:

	Intention to buy		
Can afford doing so	Yes	No	Total
Yes	40	20	60
No	80	60	140
Total	120	80	200

What is the probability that:

1. a customer is financially able and has a desire to buy?
2. a customer has a desire to buy?
3. a customer has an intention to buy given that she has the financial ability to pay?
4. Are customers intentions about buying the product independent of their ability to afford it?

Exercise

- An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The unconditional probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come?

Thank You!

Any Questions?