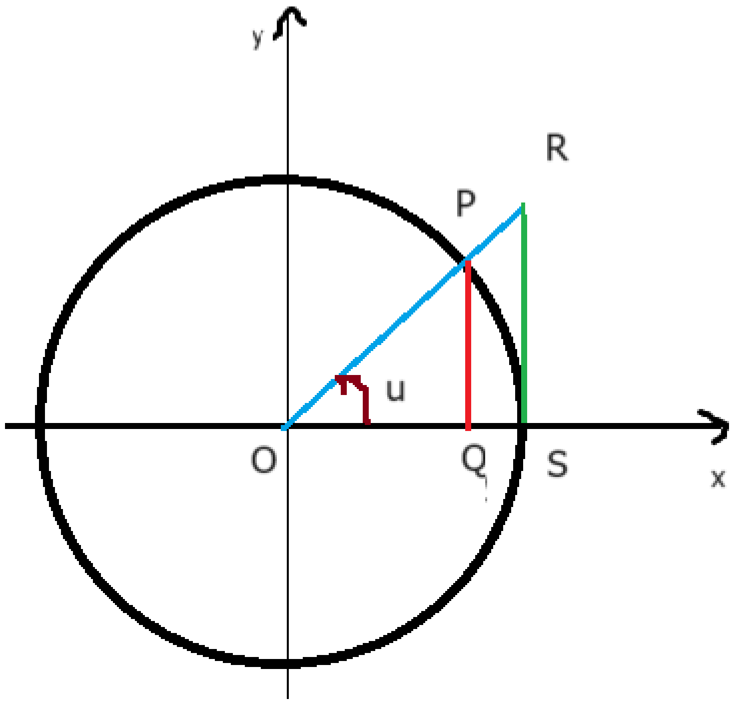
Limites Notáveis ou Fundamentais

Teorema 1: $\lim\limits_{u\to 0}\dfrac{\operatorname{sen}(u)}{u}=1.$

Demonstração:



Comparando os comprimentos, temos que:

$$\overline{PQ} < arco\ (PQ) < \overline{RS}) \quad \Rightarrow\ sen(u) < u < \operatorname{tg}(u) \ \Rightarrow\ sen(u) < u < \frac{sen(u)}{\cos(u)}$$

Considerando que $u \in \left(0, \frac{\pi}{2}\right)$, temos que: $sen(u) > 0$ e $\cos(u) > 0$

Multiplicando a desigualdade por $\frac{1}{sen(u)} > 0$, temos que: $1 < \dfrac{u}{\operatorname{sen}(u)} < \dfrac{1}{\cos(u)}$

Como todas as expressões são positivas, por propriedade: $1 > \dfrac{sen(u)}{\mathrm{u}} > \cos(\mathrm{u})$

Fazendo $u \to 0^+$, temos que: $\lim\limits_{u\to 0^+} 1 > \lim\limits_{u\to 0^+}\dfrac{sen(u)}{\mathrm{u}} > \lim\limits_{u\to 0^+}\cos(u)$

$$\Rightarrow \quad 1 > \lim_{u\to 0^+}\frac{sen(u)}{\mathrm{u}} > 1 \quad \xRightarrow[\textit{Pelo Teorema do Confronto}]{} \quad \lim_{u\to 0^+}\frac{sen(u)}{\mathrm{u}} = 1$$

De forma análoga, prova-se que $\lim\limits_{u\to 0^-}\dfrac{sen(u)}{\mathrm{u}} = 1$, para $u \in \left(-\frac{\pi}{2}, 0\right)$.

Como os limites laterais existem e são iguais a 1, então $\lim\limits_{u\to 0}\dfrac{sen(u)}{\mathrm{u}} = 1.$

Teorema 2: $\lim\limits_{u \to 0} \dfrac{1 - \cos(u)}{u} = 0.$

Demonstração:

$$L = \lim_{u \to 0} \frac{1 - \cos(u)}{u} \cdot \frac{1 + \cos(u)}{1 + \cos(u)}$$

$$L = \lim_{u \to 0} \frac{1 - cos^2(u)}{u(1 + \cos(u))}$$

$$L = \lim_{u \to 0} \frac{sen^2(u)}{u(1 + \cos(u))}$$

$$L = \lim_{u \to 0} \left(\frac{sen(u)}{u} \cdot \frac{sen(u)}{1 + \cos(u)} \right)$$

$$L = \lim_{u \to 0} \frac{sen(u)}{u} \cdot \lim_{u \to 0} \frac{sen\ (u)}{1 + \cos(u)}$$

$$L = 1 . \frac{sen\ (0)}{1 + \cos(0)} \Rightarrow \boxed{L = 1.0 = 0}$$

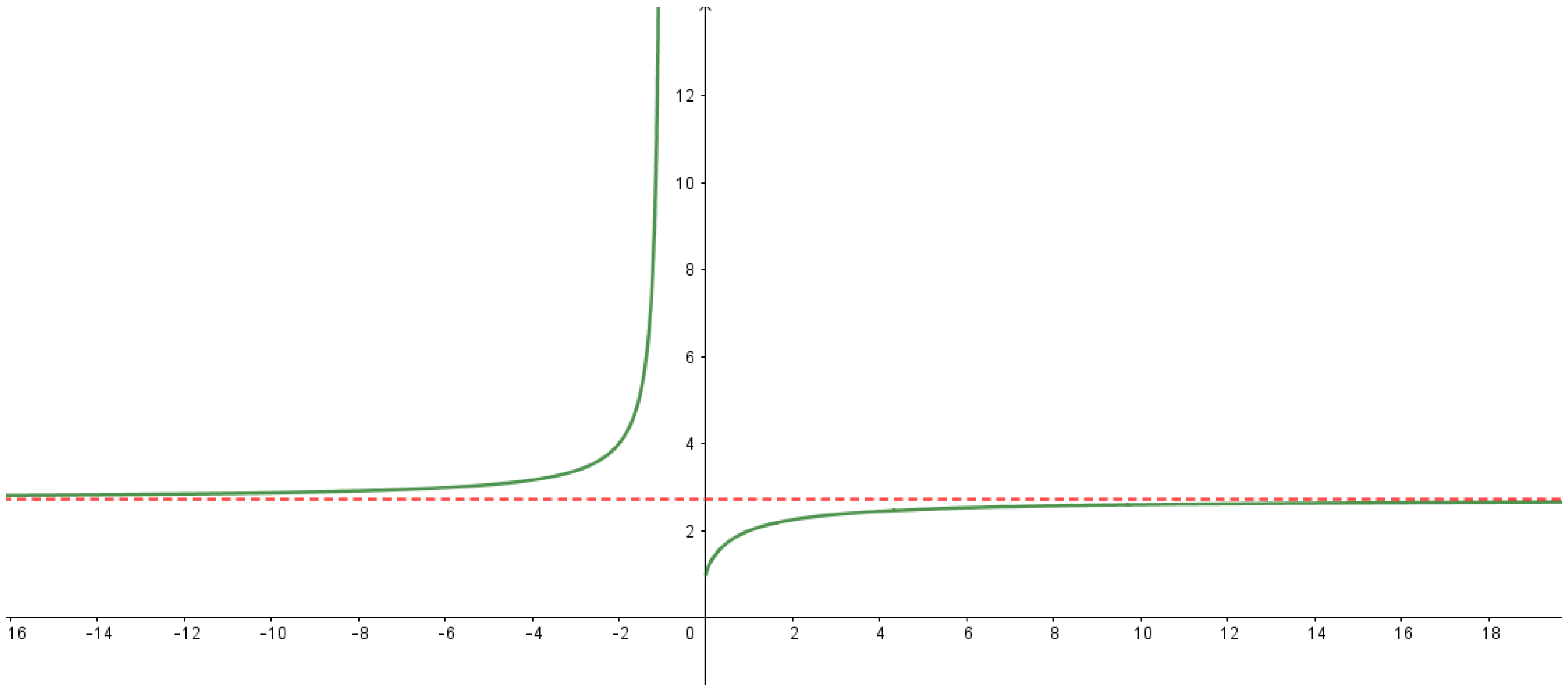
Teorema 3: $\lim_{u \to \pm\infty} \left(1 + \frac{1}{u}\right)^u = e.$

u	1	10	100	1000	10000	100000	1000000
(1+1/u)^u	2	2,59374246	2,704814	2,716924	2,718146	2,718268	2,71828

$$\lim_{u \to +\infty} \left(1 + \frac{1}{u}\right)^u = e$$

u	-1	-10	-100	-1000	-10000	-100000	-1000000
(1+1/u)^u	x	2,867971991	2,731999	2,719642	2,718418	2,718295	2,718283

$$\lim_{u \to -\infty} \left(1 + \frac{1}{u}\right)^u = e$$



Teorema 4: $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$, para $a > 0$ e $a \neq 1$.

Definindo: $u = a^x - 1 \quad \Rightarrow u + 1 = a^x \quad \Rightarrow \ln(u+1) = \ln(a^x) \quad \Rightarrow \ln(u+1) = x.\ln(a) \quad \Rightarrow x = \dfrac{\ln(u+1)}{\ln(a)}$

Se $x \to 0$, então $u \to 0$.

$$L = \lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{u\to 0} \frac{u}{\frac{\ln(u+1)}{\ln(a)}} = \lim_{u\to 0}\left(u.\frac{\ln(a)}{\ln(u+1)}\right) = \ln(a).\lim_{u\to 0}\frac{u}{\ln(u+1)} = \ln(a).\lim_{u\to 0}\frac{1}{\frac{\ln(u+1)}{u}}$$

$$L = \ln(a).\lim_{u\to 0}\frac{1}{\frac{1}{u}\ln(u+1)} = \ln(a).\frac{\lim\limits_{x\to 0} 1}{\lim\limits_{u\to 0}\left(\frac{1}{u}\ln(u+1)\right)} = \ln(a).\frac{1}{\lim\limits_{u\to 0}\left(\ln(u+1)^{\frac{1}{u}}\right)} = \ln(a).\frac{1}{\ln\left(\lim\limits_{u\to 0}(1+u)^{\frac{1}{u}}\right)}$$

Definindo: $\frac{1}{w} = u \quad \Rightarrow u = \dfrac{1}{w}$

Se u $\to$ 0, então w $\to \infty$.

$$L = \ln(a).\frac{1}{\ln\left(\lim\limits_{w\to 0}\left(1+\frac{1}{w}\right)^{w}\right)} = \ln(a).\frac{1}{\ln(e)} \quad \boxed{\Rightarrow L = \ln(a)}$$

Exemplos:

Use os limites notáveis ou fundamentais, sempre que possível para calcular os limites dados a seguir.

a. $\lim\limits_{x\to 0} \dfrac{sen(3x)}{sen(5x)} = \dfrac{0}{0}$

$$L = \lim_{x\to 0} \frac{sen(3x)}{sen(5x)} = \lim_{x\to 0} \frac{sen(3x)}{sen(5x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{sen(3x)}{x}}{\frac{sen(5x)}{x}} = \lim_{x\to 0} \frac{\frac{3\ sen(3x)}{3x}}{\frac{5\ sen(5x)}{5x}} = \frac{\lim\limits_{x\to 0} \frac{3\ sen(3x)}{3x}}{\lim\limits_{x\to 0} \frac{5\ sen(5x)}{5x}}$$

$$L = \frac{\lim\limits_{x\to 0} 3 \ \cdot \ \lim\limits_{x\to 0} \frac{sen(3x)}{3x}}{\lim\limits_{x\to 0} 5 \ \cdot \ \lim\limits_{x\to 0} \frac{sen(5x)}{5x}} = \frac{3 \lim\limits_{x\to 0} \frac{sen(3x)}{3x}}{5 \ \lim\limits_{x\to 0} \frac{sen(5x)}{5x}}$$

Definindo $u = 3x$ e $v = 5x$, temos que:

$$L = \frac{3 \lim\limits_{u\to 0} \frac{sen(u)}{u}}{5 \ \lim\limits_{v\to 0} \frac{sen(v)}{v}} \Rightarrow \boxed{L = \frac{3}{5}}$$

$$b.\ \lim_{x\to 4}\frac{sen(x)-sen(4)}{3^{2(x-4)}-1}=\frac{0}{0}$$

Definindo $u = x - 4$, temos que:

$$L = \lim_{u\to 0}\frac{sen(u+4)-sen(4)}{3^{2u}-1}$$

$$L = \lim_{u\to 0}\frac{sen(u)\cos(4)+\cos(u)\,sen(4)-sen(4)}{3^{2u}-1}$$

$$L = \lim_{u\to 0}\frac{sen(u)\cos(4)-\text{sen}(4)\,[1-\cos(u)]}{3^{2u}-1}$$

$$L = \lim_{u\to 0}\frac{\dfrac{sen(u)\cos(4)-\text{sen}(4)\,[1-\cos(u)]}{u}}{\dfrac{3^{2u}-1}{u}}$$

$$L = \frac{\lim_{u\to 0}\dfrac{sen(u)\cos(4)-\text{sen}(4)\,[1-\cos(u)]}{u}}{\lim_{u\to 0}\dfrac{3^{2u}-1}{u}}$$

$$L = \frac{\lim_{u\to 0}\dfrac{sen(u)\cos(4)}{u}-\lim_{u\to 0}\dfrac{\text{sen}(4)\,[1-\cos(u)]}{u}}{\lim_{u\to 0}\dfrac{3^{2u}-1}{u}}$$

$$L = \frac{\cos(4)\lim_{u\to 0}\dfrac{sen(u)}{u}-\text{sen}(4)\lim_{u\to 0}\dfrac{1-\cos(u)}{u}}{\lim_{u\to 0}\dfrac{3^{2u}-1}{u}}$$

$$L = \frac{\cos(4)\,.\,1-\text{sen}(4)\,.\,0}{\lim_{u\to 0}\dfrac{(3^2)^u-1}{u}}$$

$$L = \frac{\cos(4)}{\ln(3^2)} \Rightarrow \boxed{L = \frac{\cos(4)}{2\ln(3)}}$$

c. $\lim_{x \to +\infty} \left[x \left(ln(x+7) - ln(x) \right) \right]$

$$L = \lim_{x \to +\infty} \left(x \, ln \left(\frac{x+7}{x} \right) \right) = \lim_{x \to +\infty} \left(x \, ln \left(1 + \frac{7}{x} \right) \right) = \lim_{x \to +\infty} ln \left(1 + \frac{7}{x} \right)^x = ln \underbrace{\left(\lim_{x \to +\infty} \left(1 + \frac{7}{x} \right)^x \right)}_{L_1}$$

$$L_1 = \lim_{x \to +\infty} \left(1 + \frac{7}{x} \right)^x$$

Definindo $\frac{1}{u} = \frac{7}{x}$, temos que:

$$L_1 = \lim_{u \to +\infty} \left(1 + \frac{1}{u} \right)^{7u}$$

$$L_1 = \lim_{u \to +\infty} \left(\left(1 + \frac{1}{u} \right)^u \right)^7$$

$$L_1 = \left(\lim_{u \to +\infty} \left(1 + \frac{1}{u} \right)^u \right)^7$$

$$\boxed{L_1 = e^7}$$

$$\boxed{L = \ln(e^7) = 7}$$