Regra da Cadeia:
Se $y = f(u)$ e $u = g(x)$, f e u são funções diferenciáveis, então

$$\frac{dy}{dx} = f'(u)\, u'(x) \ \text{ ou } \ \frac{dy}{dx} = \frac{df}{du} . \frac{du}{dx}$$

Ou:

Se $y = (f \circ g)(x)$, f e g são funções diferenciáveis, então

$$\frac{dy}{dx} = f'(g(x)) \,. g'(x)$$

Dessa forma, se $f(x) = x^n$, $y = f(g(x)) = (g(x))^n$, temos que:

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

$f'(g(x)) = n(g(x))^{n-1}$

Pela regra da cadeia que acabamos de provar, resulta que:

$$\frac{dy}{dx} = f'(g(x)) \,. g'(x) = n(g(x))^{n-1} g'(x) \,.$$

Regras de Derivação

1) $y = x^n \Longrightarrow y' = nx^{n-1}$

2) $y = a^{bx} \Longrightarrow y' = b\ a^{bx}\ln(a)$

3) $y = e^{bx} \Longrightarrow y' = b\ e^{bx}$

4) $y = senh(ax) \Longrightarrow y' = a\ cosh(ax)$

5) $y = cosh(ax) \Longrightarrow y' = a\ senh(ax)$

6) $y = sen(ax) \Longrightarrow y' = a\ cos(ax)$

7) $y = cos(ax) \Longrightarrow y' = -a\ sen(ax)$

8) $y = log_a(bx) \Longrightarrow y' = \frac{1}{x} log_a(e)$

9) $y = ln(bx) \Longrightarrow y' = \frac{1}{x}$

11) $y = \mathrm{tg}(ax) \Longrightarrow y' = a\ sec^2(ax)$

12) $y = \mathrm{cotg}(ax) \Longrightarrow y' = -a\ cossec^2(ax)$

13) $y = sec(ax) \Longrightarrow y' = a\ sec(ax)\mathrm{tg}(ax)$

14) $y = cossec(ax) \Longrightarrow y' = -a\ cossec(ax)\mathrm{cotg}(ax)$

Regras de Derivação

1) $y = u^n \Longrightarrow y' = nu^{n-1}u'$

2) $y = a^u \Longrightarrow y' = u'\, a^u \ln(a)$

3) $y = e^u \Longrightarrow y' = u'e^u$

4) $y = senh(u) \Longrightarrow y' = u'cosh(u)$

5) $y = cosh(u) \Longrightarrow y' = u'senh(u)$

6) $y = sen(u) \Longrightarrow y' = u'cos(u)$

7) $y = cos(u) \Longrightarrow y' = -u'sen(u)$

8) $y = log_a(u) \Longrightarrow y' = \frac{u'}{u} log_a(e)$

9) $y = ln(u) \Longrightarrow y' = \frac{u'}{u}$

11) $y = \mathrm{tg}(u) \Longrightarrow y' = u'sec^2(u)$

12) $y = \mathrm{cotg}(u) \Longrightarrow y' = -u'cossec^2(u)$

13) $y = sec(u) \Longrightarrow y' = u'sec(u)\mathrm{tg}(u)$

14) $y = cossec(u) \Longrightarrow y' = -u'cossec(u)\mathrm{cotg}(u)$

Exemplos:

1) Obtenha a primeira derivada das funções:

a) $y = \frac{(x^3+4)^3}{(7-3x^6)^5}$

Pela regra do quociente: $y' = \left(\frac{u}{v}\right)' = \frac{v.u' - u.v'}{v^2}$

$$y' = \frac{(7-3x^6)^5.\left((x^3+4)^3\right)' - (x^3+4)^3.\left((7-3x^6)^5\right)'}{\left((7-3x^6)^5\right)^2}$$

Pela regra da cadeia, temos que:

$$y' = \frac{(7-3x^6)^5.3(x^3+4)^2(x^3+4)' - (x^3+4)^3.5(7-3x^6)^4(7-3x^6)'}{(7-3x^6)^{10}}$$

$$y' = \frac{(7-3x^6)^5.3(x^3+4)^2 3x^2 - (x^3+4)^3.5(7-3x^6)^4(-18x^5)}{(7-3x^6)^{10}}$$

$$y' = \frac{9x^2(7-3x^6)^4(x^3+4)^2\left(7-3x^6+10x^3(x^3+4)\right)}{(7-3x^6)^{10}}$$

$$y' = \frac{9x^2(7-3x^6)^4(x^3+4)^2(7-3x^6+10x^6+40x^3)}{(7-3x^6)^{10}}$$

$$\boxed{y' = \frac{9x^2(x^3+4)^2(7+7x^6+40x^3)}{(7-3x^6)^6}}$$

Outra opção: Reescrevendo a função

Aplicando logaritmo natural em ambos os lados, temos que:

$$\ln(y) = \ln\left(\frac{(x^3+4)^3}{(7-3x^6)^5}\right)$$

$$\ln(y) = \ln\left((x^3+4)^3\right) - \ln\left((7-3x^6)^5\right)$$

$$\ln(y) = 3\ln(x^3+4) - 5\ln(7-3x^6)$$

Derivando ambos os lados com relação a x, temos que:

$$(\ln(u))' = \frac{u'}{u}$$

$$\left(\ln(y)\right)' = \left(3\ln(x^3+4)\right)' - \left(5\ln(7-3x^6)\right)$$

$$\frac{y'}{y} = 3\left(\ln(x^3+4)\right)' - 5\left(\ln(7-3x^6)\right)'$$

$$\frac{y'}{y} = 3\,\frac{(x^3+4)'}{(x^3+4)} - 5\,\frac{(7-3x^6)'}{(7-3x^6)}$$

$$\frac{y'}{y} = 3\,\frac{3x^2}{(x^3+4)} - 5\,\frac{-18x^5}{(7-3x^6)}$$

$$\frac{y'}{y} = \frac{9x^2}{(x^3+4)} + \frac{90x^5}{(7-3x^6)}$$

$$y' = y\left(\frac{9x^2}{(x^3+4)} + \frac{90x^5}{(7-3x^6)}\right)$$

$$y = \frac{(x^3+4)^3}{(7-3x^6)^5}$$

$$\boxed{y' = \frac{(x^3+4)^3}{(7-3x^6)^5}\left(\frac{9x^2}{(x^3+4)} + \frac{90x^5}{(7-3x^6)}\right)}$$

$$(\cos(u))' = -u'\,sen(u)$$

b) $y = cos^4\left(\frac{x}{2}\right) cos\left(\frac{x^4}{2}\right)$

Pela regra do produto, temos que: $y' = u.v' + u'.v$

$$y' = cos^4\left(\frac{x}{2}\right)\left(cos\left(\frac{x^4}{2}\right)\right)' + \left(cos^4\left(\frac{x}{2}\right)\right)' cos\left(\frac{x^4}{2}\right)$$

$$y' = cos^4\left(\frac{x}{2}\right)(-1)\left(\frac{x^4}{2}\right)' sen\left(\frac{x^4}{2}\right) + 4\left(cos\left(\frac{x}{2}\right)\right)^3\left(cos\left(\frac{x}{2}\right)\right)' cos\left(\frac{x^4}{2}\right)$$

$$y' = -cos^4\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)(x^4)' sen\left(\frac{x^4}{2}\right) - 4\left(cos\left(\frac{x}{2}\right)\right)^3\left(\frac{x}{2}\right)' sen\left(\frac{x}{2}\right) cos\left(\frac{x^4}{2}\right)$$

$$y' = -cos^4\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)4x^3 sen\left(\frac{x^4}{2}\right) - 4\left(cos\left(\frac{x}{2}\right)\right)^3\frac{1}{2} sen\left(\frac{x}{2}\right) cos\left(\frac{x^4}{2}\right)$$

$$y' = -2x^3 cos^4\left(\frac{x}{2}\right) sen\left(\frac{x^4}{2}\right) - 2cos^3\left(\frac{x}{2}\right) sen\left(\frac{x}{2}\right) cos\left(\frac{x^4}{2}\right)$$

$$\boxed{y' = -2cos^3\left(\frac{x}{2}\right)\left(x^3 cos\left(\frac{x}{2}\right) sen\left(\frac{x^4}{2}\right) + sen\left(\frac{x}{2}\right) cos\left(\frac{x^4}{2}\right)\right)}$$

$$c)\ y = \frac{2}{\sqrt[3]{e^{\sinh(4x)}}} - \ln\left(sen\left(\frac{\pi}{x}\right)\right)$$

Reescrevendo a função, temos que: $y = 2\left(e^{\sinh(4x)}\right)^{-\frac{1}{3}} - \ln\left(sen\left(\frac{\pi}{x}\right)\right)$

Aplicando as regras básicas de derivação, temos que:

$$y' = \left(2\left(e^{-\frac{1}{3}\sinh(4x)}\right)\right)' - \left(\ln\left(sen\left(\frac{\pi}{x}\right)\right)\right)'$$

$$y' = 2\left(\left(e^{-\frac{1}{3}\sinh(4x)}\right)\right)' - \frac{\left(sen\left(\frac{\pi}{x}\right)\right)'}{sen\left(\frac{\pi}{x}\right)}$$

$$y' = -\frac{2}{3}(\sinh(4x))'e^{\sinh(4x)} - \frac{\left(sen\left(\frac{\pi}{x}\right)\right)'}{sen\left(\frac{\pi}{x}\right)}$$

$$y' = -\frac{8}{3}\cosh(4x)\,e^{\sinh(4x)} - \frac{\left(\frac{\pi}{x}\right)'cos\left(\frac{\pi}{x}\right)}{sen\left(\frac{\pi}{x}\right)}$$

$$y' = -\frac{8}{3}\cosh(4x)\,e^{\sinh(4x)} + - \frac{(\pi x^{-1})'cos\left(\frac{\pi}{x}\right)}{sen\left(\frac{\pi}{x}\right)}$$

$$\boxed{y' = \frac{-8}{3}\cosh(4x)\,e^{\sinh(4x)} + \frac{\pi}{x^2}cotg\left(\frac{\pi}{x}\right)}$$

d) $y = x^{x^2}$

Aplicando logaritmo natural em ambos os lados, temos que:

$$\ln(y) = \ln\left(x^{x^2}\right) \Longrightarrow \ln(y) = x^2 \ln(x)$$

Derivando ambos os lados com relação a x, temos que:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(x^2 \ln(x)\right)$$

$$\frac{y'}{y} = x^2 \left(\ln(x)\right)' + (x^2)' \ \ln(x)$$

$$\frac{y'}{y} = x^2 \frac{1}{x} + 2x \ \ln(x)$$

$$\frac{y'}{y} = x + 2x \ \ln(x)$$

$$\frac{y'}{y} = x\left(1 + 2 \ \ln(x)\right)$$

$$y' = yx\left(1 + 2 \ \ln(x)\right)$$

$$y' = x^{x^2} x\left(1 + 2 \ \ln(x)\right)$$

$$\boxed{y' = x^{x^2+1}\left(1 + 2 \ \ln(x)\right)}$$

Exemplo 2. Determine $\left.\frac{d^n y}{dx^n}\right|_{x=1} = y^{(n)}(1)$, sabendo que $y = f(x) = \frac{x}{ax+b}$, em que a e b são constantes reais tais que $ax + b \neq 0$.

Usando a regra do quociente, temos que: $y' = \left(\frac{u}{v}\right)' = \dfrac{v.u' - u.v'}{v^2}$

* Primeira derivada: $y' = \dfrac{(ax+b).x' - x.(ax+b)'}{(ax+b)^2} = \dfrac{(ax+b).1 - x.a}{(ax+b)^2} = \dfrac{b}{(ax+b)^2}$

* Segunda derivada: $y'' = \dfrac{d}{dx}\left(\dfrac{b}{(ax+b)^2}\right) = \dfrac{d}{dx}\left(b(ax+b)^{-2}\right) = b\dfrac{d}{dx}\left((ax+b)^{-2}\right)$

Pela regra da cadeia, temos que: $y'' = b(-2)(ax+b)^{-3}(ax+b)'$

$$y'' = -2ab(ax+b)^{-3} = \frac{-2ab}{(ax+b)^3}$$

* Terceira derivada: $y''' = \dfrac{d}{dx}\left(-2ab(ax+b)^{-3}\right) = -2ab\dfrac{d}{dx}\left((ax+b)^{-3}\right)$

$$y''' = -2ab(-3)(ax+b)^{-4}(ax+b)'$$

$$y''' = 6a^2b(ax+b)^{-4} = \frac{6a^2b}{(ax+b)^4}$$

Exemplo 2. Determine $\left.\frac{d^n y}{dx^n}\right|_{x=1} = y^{(n)}(1)$, sabendo que $y = f(x) = \frac{x}{ax+b}$, em que a e b são constantes reais tais que $ax + b \neq 0$.

* Quarta derivada:

$$y^{(4)} = \frac{d}{dx}(6a^2b(ax+b)^{-4}) = 6a^2b\frac{d}{dx}\left((ax+b)^{-4}\right)$$

$$y^{(4)} = 6a^2b(-4)(ax+b)^{-5}(ax+b)'$$

$$y^{(4)} = -24a^3b(ax+b)^{-5} = \frac{-24a^3b}{(ax+b)^5}$$

* Quinta derivada:

$$y^{(5)} = \frac{d}{dx}(-24a^3b(ax+b)^{-5}) = -24a^3b\frac{d}{dx}\left((ax+b)^{-5}\right)$$

$$y^{(5)} = -24a^3b(-5)(ax+b)^{-6}(ax+b)$$

$$y^{(5)} = 120a^4b(ax+b)^{-6} = \frac{120a^4b}{(ax+b)^6}$$

Para propormos a n-ésima derivada, observemos o formato das expressões obtidas até agora:

* Primeira derivada: $y' = \dfrac{b}{(ax+b)^2}$

* Segunda derivada: $y'' = \dfrac{-2ab}{(ax+b)^3}$

* Terceira derivada: $y''' = \dfrac{6a^2b}{(ax+b)^4} = \dfrac{2.3\ a^2b}{(ax+b)^4}$

* Quarta derivada: $y^{(4)} = \dfrac{-24a^3b}{(ax+b)^5} = \dfrac{-2.3.4\ a^3b}{(ax+b)^5}$

* Quinta derivada: $y^{(5)} = \dfrac{120a^4b}{(ax+b)^6} = \dfrac{2.3.4.5a^4b}{(ax+b)^6}$

* n-ésima derivada: $y^{(n)} = \dfrac{(-1)^{n+1}n!\,a^{n-1}\ b}{(ax+b)^{n+1}}$

Para $x = 1$, temos que: $\boxed{y^{(n)}(1) = \dfrac{(-1)^{n+1}n!\,a^{n-1}\ b}{(a+b)^{n+1}}}$