Teorema: REGRA DE L'HOPITAL

Se f e g são duas funções com primeiras derivadas contínuas em $x = x_0$, $\lim\limits_{x\to x_0} f\left(x\right) = 0$ e $\lim\limits_{x\to x_0} g\left(x\right) = 0$ e $\forall\ x \neq x_0$, $g'\left(x\right) \neq 0$ e $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$ existir então

$$\lim_{x\to x_0} \frac{f\left(x\right)}{g\left(x\right)} = \lim_{x\to x_0} \frac{f'\left(x\right)}{g'\left(x\right)}$$

Observação:

Esta regra pode ser aplicada somente para indeterminações do tipo $\frac{0}{0}$ ou $\dfrac{\infty}{\infty}$. As demais formas indeterminadas devem ser transformadas nas indeterminações do tipo $\frac{0}{0}$ ou $\frac{\infty}{\infty}$ mediante manipulações algébricas.

Exemplos:

1. Prove os limites notáveis utilizando a regra de L'Hôpital.

a) $\lim_{x \to 0} \frac{sen(x)}{x} = \frac{0}{0}$

Aplicando a regra de L'Hopital, temos que:

$$L = \lim_{u \to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

b) $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \frac{0}{0}$

$$\text{L} = \lim_{x \to 0} \frac{\text{sen}(x)}{1} = \text{sen}(0) = 0$$

$c)$ $\lim_{x \to 0} \frac{a^x - 1}{x} = \frac{0}{0}$

$$L = \lim_{x \to 0} \frac{a^x \ln(a)}{1} = a^0 \ln(a) = \ln(a)$$

$$d)\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = 1^{\infty}$$

$$\boxed{\ln(L) = 1 \Rightarrow L = e}$$

$$L = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

Aplicando a função logaritmo natural em ambos os lados, temos que:

$$\ln(L) = ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)$$

$$\ln(L) = \lim_{x\to\infty}\left(ln\left(1+\frac{1}{x}\right)^{x}\right)$$

$$\ln(L) = \lim_{x\to\infty}\left(x\ ln\left(1+\frac{1}{x}\right)\right) = \infty.0$$

$$\ln(L) = \lim_{x\to\infty}\frac{ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\ln(L) = \lim_{x\to\infty}\frac{ln(1+x^{-1})}{x^{-1}}$$

Aplicando a regra de L'Hopital, temos que:

$$\ln(L) = \lim_{x\to\infty}\frac{\frac{(1+x^{-1})'}{(1+x^{-1})}}{-x^{-2}}$$

$$\ln(L) = \lim_{x\to\infty}\frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}}$$

$$\ln(L) = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

Exemplo 2. Use a regra de L´Hopital para calcular os limites dados a seguir.

$$a)\ \lim_{x \to -1} \sqrt[5]{\frac{x^3+1}{5x^2+4x-1}} = \frac{0}{0}$$

$$L = \sqrt[5]{\lim_{x \to -1} \frac{x^3+1}{5x^2+4x-1}}$$

Aplicando a regra de L'Hopital, temos que:

$$L = \sqrt[5]{\lim_{x \to -1} \frac{3x^2}{10x+4}}$$

$$L = \sqrt[5]{\frac{3(-1)^2}{10(-1)+4}}$$

$$L = \sqrt[5]{\frac{3}{-6}} \Rightarrow \boxed{L = -\frac{1}{\sqrt[5]{2}}}$$

$$b.\ \lim_{x\to 5}\frac{ln(x)-ln(5)}{2^x-32}=\frac{0}{0}$$

Aplicando a regra de L'Hopital, temos que:

$$L=\lim_{x\to 5}\frac{\big(ln(x)\big)'-(\ln(5))'}{(2^x)'-32'}$$

$$L=\lim_{x\to 5}\frac{\frac{1}{x}}{2^x\ln(2)}$$

$$L=\lim_{x\to 5}\frac{1}{x\ 2^x\ln(2)}$$

$$L=\frac{1}{5.\,2^5\ln(2)}$$

$$\boxed{L=\frac{1}{160\ \ln(2)}}$$

c. $\lim_{x\to 0} \frac{tg^2 x}{x(\cos(x+3)-\cos(x-3))} = \frac{0}{0}$

Aplicando a regra de L'Hopital, temos que:

$$L = \lim_{x\to 0} \frac{2\ \mathrm{tg}(x)\ (\mathrm{tg}(x))'}{x(\cos(x+3)-\cos(x-3))' + x'(\cos(x+3)-\cos(x-3))}$$

$$L = \lim_{x\to 0} \frac{2\ \mathrm{tg}(x)\ sec^2(x)}{x(-(x+3)'\mathrm{sen}(x+3)+(x-3)'sen(x-3))+1(\cos(x+3)-\cos(x-3))}$$

$$L = \lim_{x\to 0} \frac{2\ \mathrm{tg}(x)\ sec^2(x)}{x(-\mathrm{sen}(x+3)+sen(x-3))+\cos(x+3)-\cos(x-3)}$$

Aplicando a regra de L'Hopital novamente, temos que:

$$L = \lim_{x\to 0} \frac{2\ \mathrm{tg}(x)\left(sec^2(x)\right)' + (2\ \mathrm{tg}(x))'sec^2(x)}{x\left(-\mathrm{sen}(x+3)+sen(x-3)\right)' + x'\left(-\mathrm{sen}(x+3)+sen(x-3)\right) - (x+3)'\mathrm{sen}(x+3)+(x-3)'sen(x-3)}$$

$$L = \lim_{x\to 0} \frac{2\ \mathrm{tg}(x) 2\ sec^2(x)\mathrm{tg}(x) + 2sec^2(x)sec^2(x)}{x\left(-\cos(x+3)+cos(x-3)\right)+\left(-\mathrm{sen}(x+3)+sen(x-3)\right) - \mathrm{sen}(x+3)+sen(x-3)}$$

$$L = \frac{2}{-2\mathrm{sen}(3) + 2sen(-3)} \Longrightarrow \boxed{L = -\frac{1}{2\mathrm{sen}(3)}}$$

$sen(-x) = -sen(x), pois\ seno$ é $uma\ função\ ímpar$

d. $\lim_{x \to 0} \frac{1 - \cos(3x)}{x\, sen(3x)}$

Aplicando a regra de L´Hôpital, temos que:

$$L = \lim_{x \to 0} \frac{1' - (\cos(3x))'}{(x\, sen(3x))'} = \lim_{x \to 0} \frac{3sen(3x)}{x\big(sen(3x)\big)' + x' sen(3x)}$$

$$L = \lim_{x \to 0} \frac{3sen(3x)}{x\, 3\cos(3x) \;+ sen(3x)} = \frac{0}{0}$$

Aplicando a regra de L´Hôpital, temos que:

$$L = \lim_{x \to 0} \frac{3(sen(3x))'}{3x\, (\cos(3x))' + (3x)'\cos(3x) \;+ (sen(3x))'}$$

$$L = \lim_{x \to 0} \frac{9cos(3x)}{-9x\,\mathrm{sen}(3x) + 6\cos(3x)}$$

$$L = \frac{9cos(0)}{-9.0.\,\mathrm{sen}(0) + 6\cos(0)} \Rightarrow \boxed{L = \frac{9}{6} = \frac{3}{2}}$$

e. $\lim_{x \to 0} \frac{cossec^2(3x)}{cossec^2(5x)}$

Reescrevendo a função, temos que:

$$L = \lim_{x \to 0} \frac{\frac{1}{sen^2(3x)}}{\frac{1}{sen^2(5x)}} = \lim_{x \to 0} \frac{sen^2(5x)}{sen^2(3x)} = \frac{0}{0}$$

Aplicando a regra de L´Hôpital, temos que:

$$L = \lim_{x \to 0} \frac{2sen(5x)\ (sen(5x))'}{2\ sen(3x)(sen(3x))'} = \lim_{x \to 0} \frac{2sen(5x)\ 5cos(5x)}{2\ sen(3x)\ 3cos(3x)}$$

$$L = \frac{5}{3} \lim_{x \to 0} \frac{2sen(5x)\ cos(5x)}{2\ sen(3x)\ cos(3x)} = \frac{5}{3} \lim_{x \to 0} \frac{sen(10x)}{sen(6x)} = \frac{0}{0}$$

Aplicando a regra de L´Hôpital, temos que:

$$L = \frac{5}{3} \lim_{x \to 0} \frac{10\ cos(10x)}{6\ cos(6x)} = \frac{5}{3} \frac{10\ cos(0)}{6\ cos(0)} = \frac{5}{3} . \frac{5}{3} \Rightarrow \boxed{L = \frac{25}{9}}$$

f. $\lim_{x\to+\infty}(x\ln(x+2) - x\ln(x)) = +\infty - \infty$

Reescrevendo a função, temos que:

$$L = \lim_{x\to+\infty}\big(x(\ln(x+2) - \ln(x))\big) = \lim_{x\to+\infty}\left(x\, ln\left(\frac{x+2}{x}\right)\right) = \lim_{x\to+\infty}\underbrace{\left(x\, ln\left(1+\frac{2}{x}\right)\right)}_{+\infty.\,0}$$

$$L = \lim_{x\to+\infty}\frac{ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

Aplicando a regra de L´Hôpital, temos que:

$$L = \lim_{x\to+\infty}\frac{\frac{(1+2x^{-1})'}{1+\frac{2}{x}}}{(x^{-1})'} = \lim_{x\to+\infty}\frac{\frac{-2x^{-2}}{1+\frac{2}{x}}}{-x^{-2}} = 2\lim_{x\to+\infty}\frac{1}{1+\frac{2}{x}}$$

$$L = 2\frac{1}{1+\frac{2}{+\infty}} \Rightarrow \boxed{L = 2}$$

g. $\lim_{x \to 0} \sqrt[x]{1+2x} = \lim_{x \to 0} (1+2x)^{\frac{1}{x}} = 1^{\infty}$

$$L = \lim_{x \to 0} (1+2x)^{\frac{1}{x}} \quad \Rightarrow \quad \ln(L) = ln\left(\lim_{x \to 0} (1+2x)^{\frac{1}{x}}\right)$$

Reescrevendo a função, temos que:

$$\ln(L) = \lim_{x \to 0} \left(ln(1+2x)^{\frac{1}{x}}\right) = \lim_{x \to 0} \left(\frac{1}{x} ln(1+2x)\right) = \lim_{x \to 0} \frac{ln(1+2x)}{x} = \frac{0}{0}$$

Aplicando a regra de L´Hôpital, temos que:

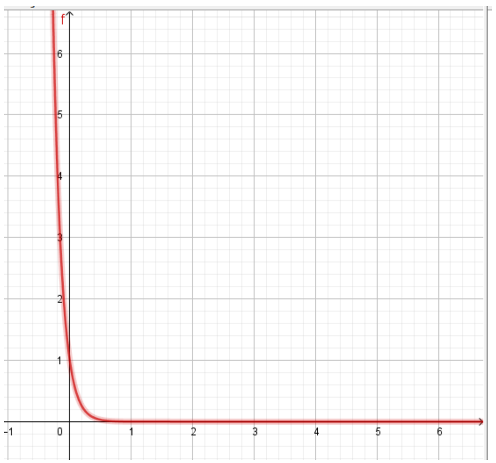
$$\ln(L) = \lim_{x \to 0} \frac{\frac{(1+2x)'}{1+2x}}{x'} = \lim_{x \to 0} \frac{\frac{2}{1+2x}}{1} = 2$$

$$\ln(L) = 2 \quad \Rightarrow \quad \boxed{L = e^2}$$

h. $\lim_{x \to \infty} (x^n e^{-7x}), n \in \mathbb{N}^*$

Se $x \to -\infty$, temos que:

$$L = \lim_{x \to -\infty} (x^n e^{-7x}) = (-\infty)^n . (+\infty) = (\pm\infty). (+\infty) = \pm\infty$$



Se $x \to +\infty$, temos que:

$$L = \lim_{x \to +\infty} (x^n e^{-7x}) = (+\infty)^n . 0 = +\infty . 0$$

$$L = \lim_{x \to +\infty} \frac{x^n}{e^{7x}} = \frac{+\infty}{+\infty}$$

Aplicando a regra de L´Hôpital, temos que:

$$L = \lim_{x \to +\infty} \frac{nx^{n-1}}{7e^{7x}} = \frac{+\infty}{+\infty}$$

Aplicando a regra de L´Hôpital, temos que:

$$L = \lim_{x \to +\infty} \frac{n(n-1)x^{n-2}}{7(e^{7x})'} = \lim_{x \to +\infty} \frac{n(n-1)x^{n-2}}{49e^{7x}} = \frac{+\infty}{+\infty}$$

Aplicando a regra de L´Hôpital, temos que:

$$L = \lim_{x \to +\infty} \frac{n(n-1)(n-2)x^{n-3}}{49(7e^{7x})} = \frac{+\infty}{+\infty}$$

Para encerrar esse processo, precisamos encontrar a derivada n-ésima.

$$y = x^n$$

$$y' = nx^{n-1}$$

$$y'' = n(n-1)x^{n-2}$$

$$y''' = n(n-1)(n-2)x^{n-3}$$

$$y^{(4)} = n(n-1)(n-2)(n-3)x^{n-4}$$

$$\vdots$$

$$y^{(n)} = n(n-1)(n-2)(n-3)\ldots(n-(n-1))x^{n-n}$$

$$\boxed{y^{(n)} = n!}$$

$$y = e^{7x}$$

$$y' = 7e^{7x}$$

$$y'' = 7(e^{7x})' = 7(7e^{7x}) = 7^2 e^{7x}$$

$$y''' = 7^2(e^{7x})' = 7^2(7e^{7x}) = 7^3 e^{7x}$$

$$y^{(4)} = 7^3(e^{7x})' = 7^3(7e^{7x}) = 7^4 e^{7x}$$

$$y^{(5)} = 7^4(e^{7x})' = 7^4(7e^{7x}) = 7^5 e^{7x}$$

$$\vdots$$

$$\boxed{y^{(n)} = 7^n e^{7x}}$$

Dessa forma, após aplicar n vezes a regra de L'Hôpital, tem-se que:

$$L = \lim_{x \to +\infty} \frac{x^n}{e^{7x}} = \cdots = \lim_{x \to +\infty} \frac{n!}{7^n e^{7x}} = \frac{n!}{7^n(+\infty)} \Rightarrow \boxed{L = 0}$$

Exemplo 3. Supondo que f' é uma função contínua e $f(0) = 3$, se possível, determine o valor da constante m e o valor de $f'(0)$ para que

$$\lim_{x \to 0} \frac{mf(e^{4x} - 1) + [f(sen(3x))]^2}{x} = 12.$$

Note que:

$$L = \lim_{x \to 0} \frac{mf(e^{4x} - 1) + [f(sen(3x))]^2}{x} = \lim_{x \to 0} \frac{mf(e^{0} - 1) + [f(sen(0))]^2}{x}$$

$$L = \frac{mf(0) + [f(0)]^2}{0} = \frac{3m + 3^2}{0} = \frac{3m + 9}{0}$$

Se $3m + 9 \neq 0 \Longrightarrow L = \dfrac{número\ não\ nulo}{0} \Longrightarrow L = \infty$

Se $3m + 9 = 0 \Longrightarrow L = \frac{0}{0} \Longrightarrow$ $\text{Ainda, nada podemos afirmar sobre } L$

Assumindo $m = -3 \Longrightarrow L = \frac{0}{0} \Longrightarrow$ Podemos aplicar a regra de L'Hôpital

$$L = \lim_{x \to 0} \frac{-3f(e^{4x} - 1) + [f(sen(3x))]^2}{x} = \lim_{x \to 0} \frac{-3\left(f(e^{4x} - 1)\right)' + \left([f(sen(3x))]^2\right)'}{x'}$$

$$L=\lim_{x\to 0}\frac{-3\left(f\left(e^{4x}-1\right)\right)'+\left(\left[f\left(sen(3x)\right)\right]^2\right)'}{x'}$$

$$L=\lim_{x\to 0}\frac{-3f'\left(e^{4x}-1\right)\left(e^{4x}-1\right)'+2f\left(sen(3x)\right)\left(f\left(sen(3x)\right)\right)'}{1}$$

$$L=\lim_{x\to 0}\frac{-3f'\left(e^{4x}-1\right)4e^{4x}+2f\left(sen(3x)\right)f'\left(sen(3x)\right)\left(sen(3x)\right)'}{1}$$

$$L=\lim_{x\to 0}\frac{-3f'\left(e^{4x}-1\right)4e^{4x}+2f\left(sen(3x)\right)f'\left(sen(3x)\right)3\cos(3x)}{1}$$

$$L=\frac{-12f'(0)e^{0}+6f(0)f'(0)\cos(0)}{1}$$

$$L=\frac{-12f'(0)+6.3f'(0)}{1}$$

$$L=-12f'(0)+18f'(0)$$

$$\boxed{L=6f'(0)} \Rightarrow L=12 \iff 6f'(0)=12 \Rightarrow \boxed{f'(0)=2}$$

Conclusão: $L=12 \iff m=-3$ e $f'(0)=2$