Universite Paris Dauphine

Master Statistic and Big Data

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Money Market modeling with a random-coefficient linear model

Introduction

In that problem, we are asked to find the time varying coefficients that better fit the following equation thanks to the Kalman filter algorithm :

$DM1_t = \beta_0, t + \beta_1, t * OBL_{t-1} + \beta_2, t * INF_{t-1} + \beta_3, t * SUR_{t-1} + \beta_4 * DM1_{t-1}, t * + \zeta_t$
where :

- $DM1_t$ is the log-ratio of US Money Supply

- OBL_t is the increment of short term rates

- INF_t is the log ratio of american consumer price index

- SUR_t is the surplus or deficit of the US federal government budget

1. Data Importation

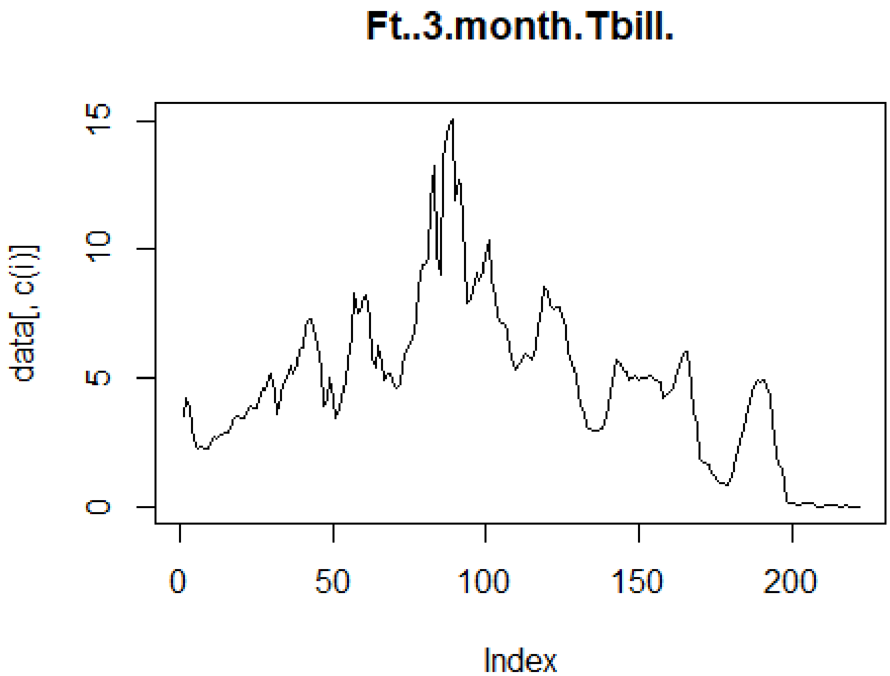
```
setwd("C:\\Users\\oussa\\Downloads\\Data Science\\Master Statistique Big Data
Dauphine\\Module 2\\Séries temporelles")

data=read.csv("data_DM2.csv")
data=data[-c(1),]#there is a missing value in the first row

head(data)

##   observation_date Ft..3.month.Tbill. M1t..Monetary.Supply.
## 2       1959-07-01               3.50                 140.2
## 3       1959-10-01               4.22                 142.0
## 4       1960-01-01               3.95                 140.5
## 5       1960-04-01               3.03                 138.4
## 6       1960-07-01               2.35                 139.6
## 7       1960-10-01               2.31                 142.7
##   CPIt..Consumer.Price. SURt..Federal.Government.
## 2             0.1338803                      -3.0
## 3             0.1346905                      -4.5
## 4             0.1348128                       3.8
```

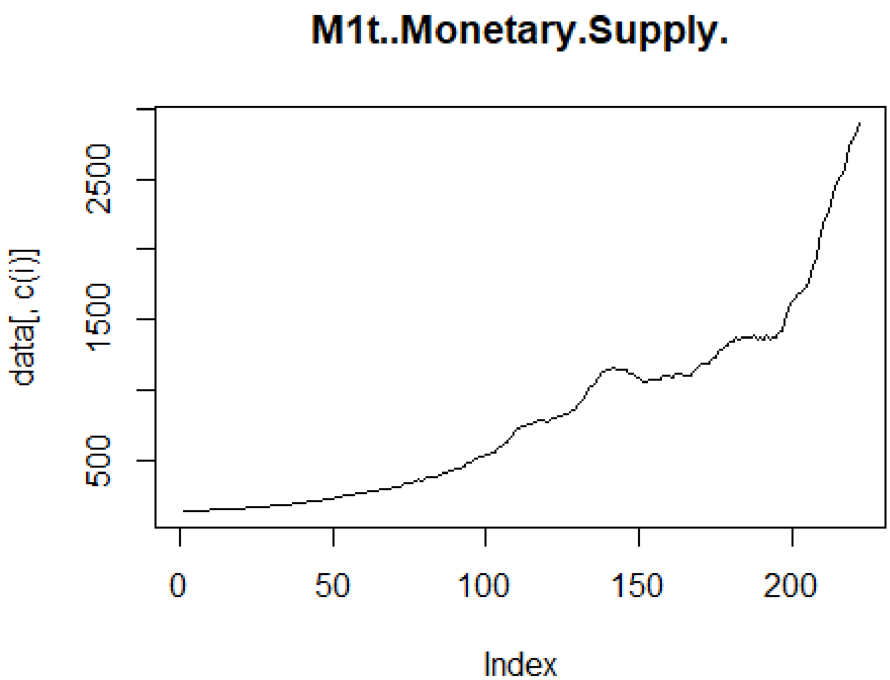
```
## 5             0.1356230                         4.4
## 6             0.1356994                        -0.8
## 7             0.1365707                        -3.9
```

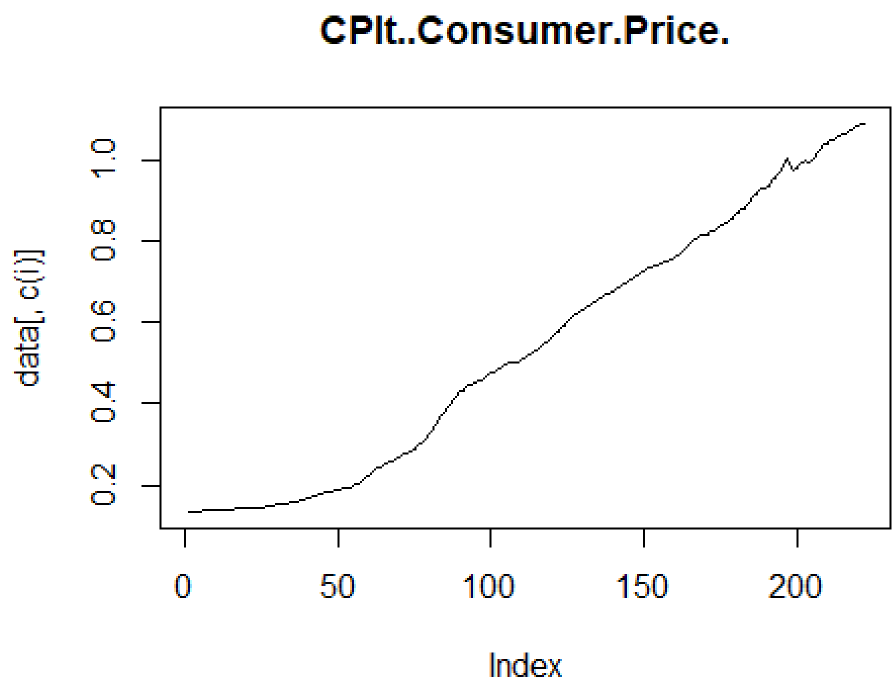
```r
colnames(data)
```

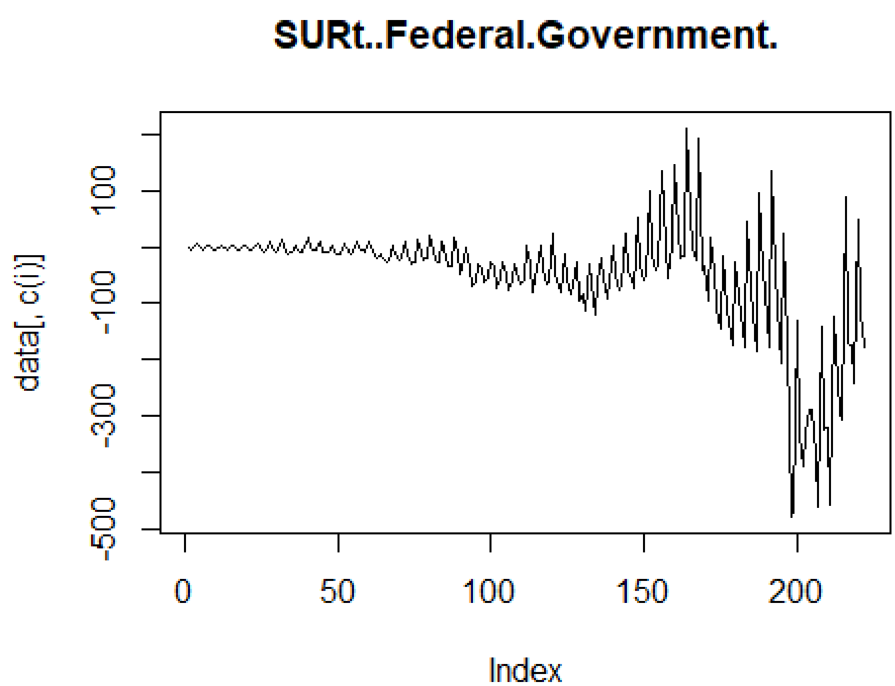
```
## [1] "observation_date"          "Ft..3.month.Tbill."
## [3] "M1t..Monetary.Supply."     "CPIt..Consumer.Price."
## [5] "SURt..Federal.Government."
```

```r
for (i in 2:ncol(data)){
plot(data[,c(i)],main=colnames(data)[i],type='l')
}
```


Ft..3.month.Tbill.
data[, c(i)]
0
5
10
15
0
50
100
150
200
Index


M1t..Monetary.Supply.
data[, c(i)]
500
1500
2500
0
50
100
150
200
Index





We can see an exponential tendancy on the Money Supply and the Consumer Price plots, which justifies the log-ratio operation

2. Density of the vectors ϵ_t

Let S the variance matrix of ϵ_t (as we have homoscedasticity S is constant)

$$S = \begin{bmatrix} {\sigma_0}^2 & 0 & 0 & 0 & 0 \\ 0 & {\sigma_1}^2 & 0 & 0 & 0 \\ 0 & 0 & {\sigma_2}^2 & 0 & 0 \\ 0 & 0 & 0 & {\sigma_3}^2 & 0 \\ 0 & 0 & 0 & 0 & {\sigma_4}^2 \end{bmatrix}, S^{-1} = \begin{bmatrix} \sigma_0^{-2} & 0 & 0 & 0 & 0 \\ 0 & \sigma_1^{-2} & 0 & 0 & 0 \\ 0 & 0 & \sigma_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & \sigma_3^{-2} & 0 \\ 0 & 0 & 0 & 0 & \sigma_4^{-2} \end{bmatrix}, N = 5 \text{ and the}$$

determinant $|S| = \prod \sigma_i^2$

Then, density of the ϵ_t is $f(x) = \frac{1}{(2\pi)^{N/2}|S|^{1/2}} \quad e^{-\frac{1}{2}x^\top S^{-1}x}$

3. Steps of the Kalman prediction algorithm

In this problem, we are in a state-space model with random coefficients, and under the normal condition. According to the Kalman theorem, if we chose Σ_0 $\hat{\beta}_0$ well, we can compute the following algorithm recursively : $D\hat{M}1_n = \Pi_{n-1}(DM1_n)$,

$V_n^L = \mathbb{E}[(DM1_n - D\hat{M}1_n)^2] = \sigma^2 v_n^l$ the quadratic prediction error,

$\Sigma_n = \mathbb{E}[(\beta_n - \hat{\beta}_n)(\beta_n - \hat{\beta}_n)^\top]$, $\hat{\beta}_n = \Pi_{n-1}(\beta_n)$, η_n a strong white noise, $H\eta_n = \epsilon_n$

First we are asked to chose β_0 a random $\mathcal{N}(0{,}50 * I_5)$,then we can compute the following recursion (simplified since $A_t = I_5$): $\hat{\beta}_{n+1} = \hat{\beta}_n + \frac{\Sigma_n B_n^\top}{V_n^L} * (DM1_n - D\hat{M}1_n)$

(This is a stochastic gradient algorithm starting from $\hat{\beta}_0$) .Then $D\hat{M}1_{n+1} = \hat{\beta}_{n+1}^{\ \top} B_n^\top$, and $\Sigma_{n+1} = \Sigma_n + H_n H_n^\top - \frac{\Sigma_n * B_n^\top * B_n * \Sigma_n}{V_n^L}$ .At last : $V_{n+1}^L = B_n * \Sigma_{n+1} * B_n^\top + \sigma^2$

4. Likelyhood expression

For calibrating the hyperparameter $\theta = (\sigma_\zeta, \sigma_0, \ldots, \sigma_4)$, we can compute the likelihood contrast $L_t(\theta)$. First we initialize $\theta_0, \hat{\beta}_0$ which follows a random $\mathcal{N}(0{,}50 * I_5)$(the choice of the initialization is important for the algorithm convergence, it should correspond to the most likely fit on the observations),Σ_0 and L_0,then we compute the innovation

$I_n(\theta) = DM1_n - D\hat{M}1_n$. Then we update de QLIK loss

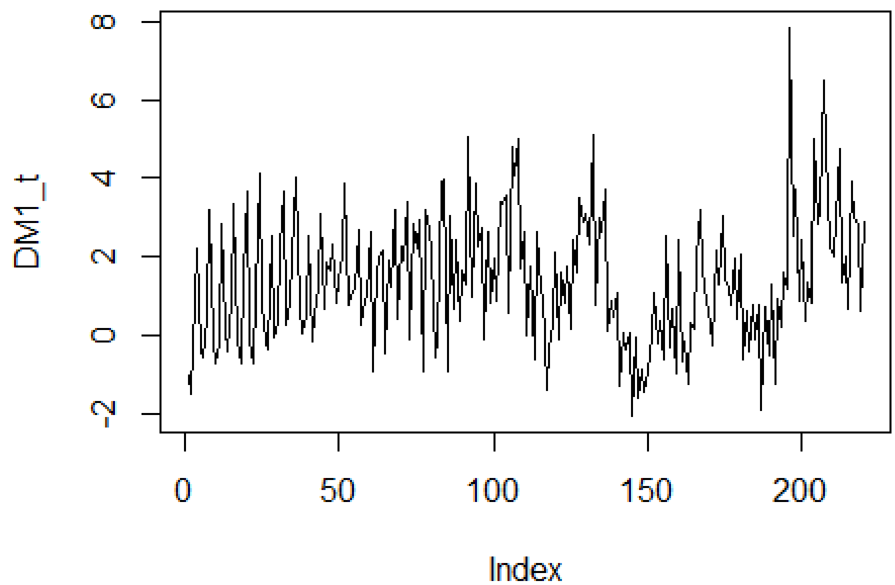
$L_n = L_{n-1}(\theta) + \frac{I_n^2(\theta)}{\sigma^2 * v_n^L(\theta)} + log(\sigma^2 * v_n^L(\theta))$. Finally, we compute the next linear prediction $D\hat{M}1_{n+1}(\theta)$ and the associated risk v_{n+1}^L . Moreover we can estimate σ^2 with

$\hat{\sigma}_n^2 = \frac{1}{n} * \sum_{t=1}^{n} \frac{(DM1_t - D\hat{M}1_t(\theta_n))}{v_t^L}$

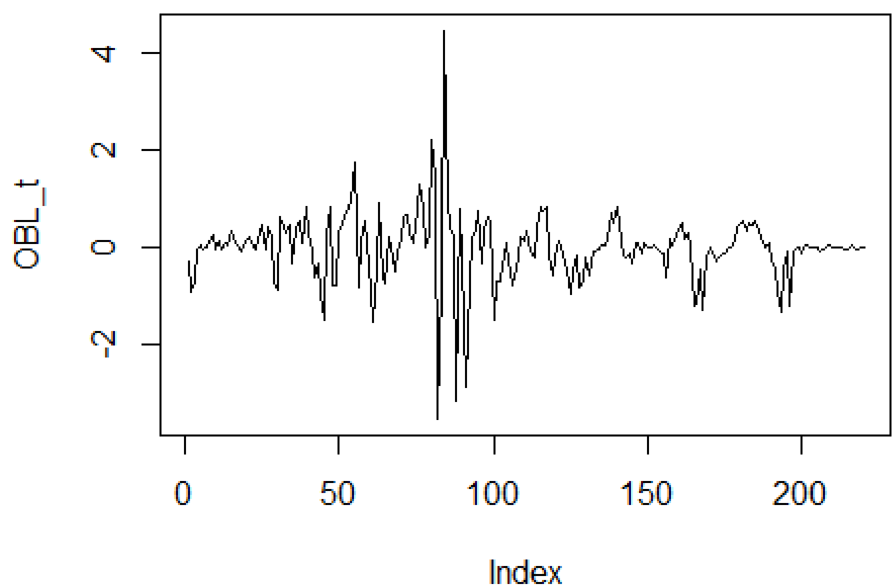
5. Implementation of the state-space model

From the basic data we have to generate new features before implementing the state-space model :

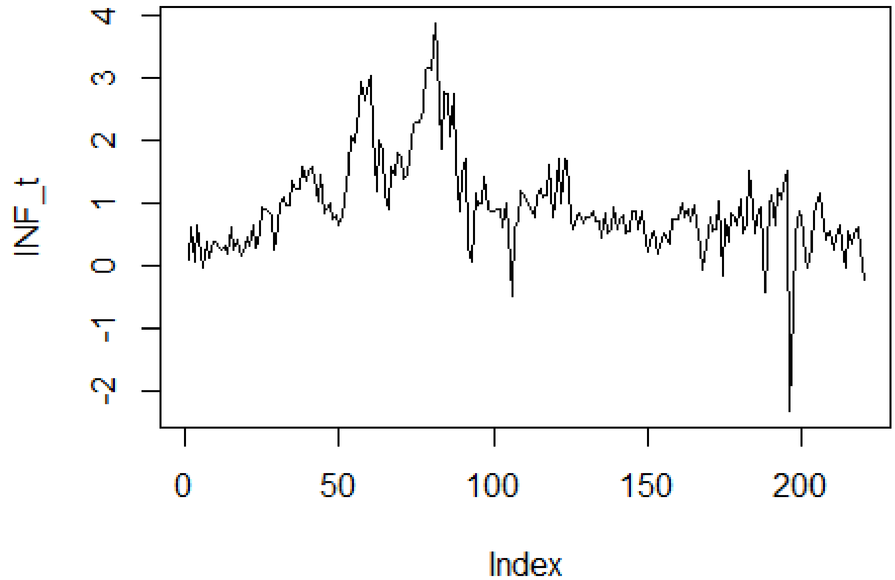
```
colnames(data)

## [1] "observation_date"          "Ft..3.month.Tbill."
## [3] "M1t..Monetary.Supply."     "CPIt..Consumer.Price."
## [5] "SURt..Federal.Government."

DM1_t=100*diff(log(data$M1t..Monetary.Supply.))[-1]
DM1_t_1=100*diff(log(data$M1t..Monetary.Supply.))[-(length(DM1_t)+1)]



OBL_t=diff(data$Ft..3.month.Tbill.)[-1]
OBL_t_1=diff(data$Ft..3.month.Tbill.)[-(length(OBL_t)+1)]


INF_t=100*diff(log(data$CPIt..Consumer.Price.))[-1]
INF_t_1=100*diff(log(data$CPIt..Consumer.Price.))[-(length(INF_t)+1)]

SUR_t=diff(data$SURt..Federal.Government.)[-1]
SUR_t_1=diff(data$SURt..Federal.Government.)[-(length(SUR_t)+1)]


Intercept=rep(1,length(DM1_t))#for Beta_o,t


plot(DM1_t,type='l')
```



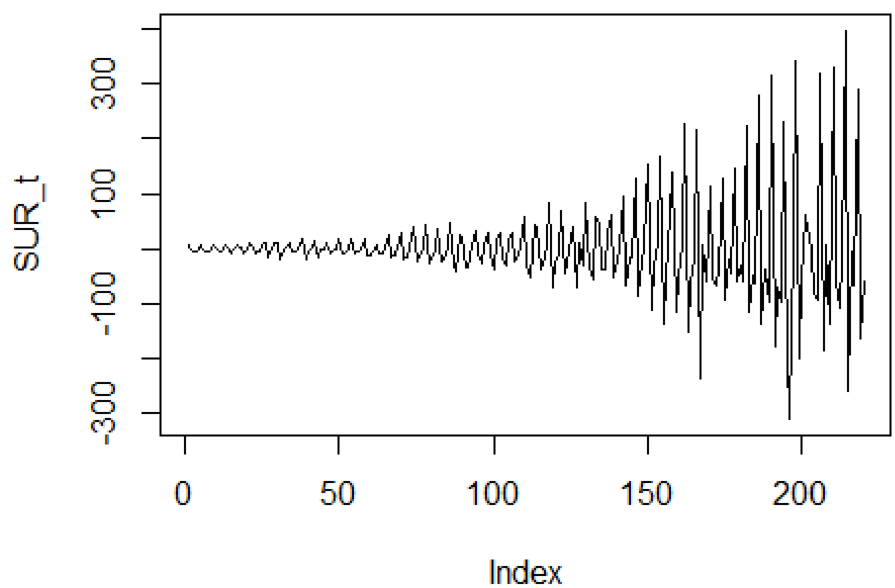

```r
plot(OBL_t,type='l')
```




```r
plot(INF_t,type='l')
```




```r
plot(SUR_t,type='l')
```




```r
library(KFAS)
```

```
## Warning: package 'KFAS' was built under R version 3.3.3
```

```r
?fitSSM
```

```
## starting httpd help server ...

##  done
```

```r
model=SSModel(DM1_t~SSMregression(~Intercept+OBL_t_1+INF_t_1+DM1_t_1+SUR_t_1,
Q=diag(NA,5),a1 = c(0,0,0,0,0),P1=diag(50^2,5))-1,H=NA)#a1 and P1 for
initialization of beta_o with N(0,50I)
```

6. Computation of the maximum Likelyhood estimator

```r
#calibration of the hyperparameter

fit=fitSSM(model,inits=c(0.5,0.1,0.1,0.1,0.1,0.1),method="BFGS")

model=fit$model
```

7. Computation of beta and sigma

The coefficients $\beta_{t/t-1}$ of the Kalman recursion are given by :

```r
#to get the Beta_hat:
out=KFS(model)
model$a1
```

```
##           [,1]
## Intercept    0
## OBL_t_1      0
## INF_t_1      0
## DM1_t_1      0
## SUR_t_1      0
```

```r
model$P1
```

```
##           Intercept OBL_t_1 INF_t_1 DM1_t_1 SUR_t_1
## Intercept      2500       0       0       0       0
## OBL_t_1           0    2500       0       0       0
## INF_t_1           0       0    2500       0       0
## DM1_t_1           0       0       0    2500       0
## SUR_t_1           0       0       0       0    2500
```

```r
summary(out$alphahat)#summary of the Beta t/t-1
```

```
##    Intercept          OBL_t_1            INF_t_1             DM1_t_1
##  Min.   :-0.3059   Min.   :-1.0582   Min.   :-0.40208   Min.   :-0.13261
##  1st Qu.: 0.9650   1st Qu.:-0.8805   1st Qu.:-0.38956   1st Qu.:-0.12843
##  Median : 1.9612   Median :-0.4685   Median :-0.31821   Median :-0.11977
##  Mean   : 1.7661   Mean   :-0.4801   Mean   :-0.22858   Mean   :-0.11456
##  3rd Qu.: 2.5510   3rd Qu.:-0.1737   3rd Qu.:-0.09273   3rd Qu.:-0.10028
##  Max.   : 3.4417   Max.   : 0.2150   Max.   : 0.13986   Max.   :-0.08781
```

```
##     SUR_t_1
##  Min.   :-0.00316
##  1st Qu.:-0.00316
##  Median :-0.00316
##  Mean   :-0.00316
##  3rd Qu.:-0.00316
##  Max.   :-0.00316
```

```
print(out$alphahat[1:5,])
```

```
##       Intercept    OBL_t_1    INF_t_1    DM1_t_1      SUR_t_1
## [1,] 0.1504809 -0.3394001 -0.4014659 -0.1251954 -0.003159826
## [2,] 0.1840718 -0.3355977 -0.4010254 -0.1251746 -0.003159826
## [3,] 0.3265539 -0.3364211 -0.4003699 -0.1252099 -0.003159826
## [4,] 0.4525352 -0.3348559 -0.3999295 -0.1252332 -0.003159826
## [5,] 0.4826046 -0.3230294 -0.3996067 -0.1252967 -0.003159826
```

```
out
```

```
## Smoothed values of states and standard errors at time n = 220:
##             Estimate    Std. Error
## Intercept   2.5828090   0.6573011
## OBL_t_1    -0.4682340   0.8368277
## INF_t_1     0.1298639   0.3616862
## DM1_t_1    -0.0889131   0.0857485
## SUR_t_1    -0.0031598   0.0009837
```

```
#diagonal terms on "out$P"" give us Sigma_t_t_1
```

The σ^2 of the Kalman recursion is given by :

```
print(model$H)
```

```
## , , 1
##
##           [,1]
## [1,] 1.695996
```

The variances of the coefficients, which are the diagonal terms of $\Sigma_{t/t-1}$ matrix is given by

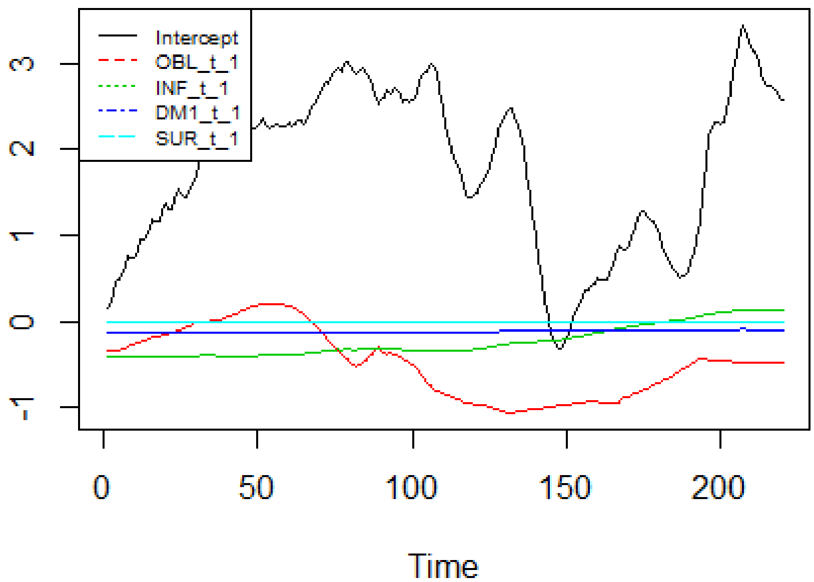
```
print(model$Q)
```

```
## , , 1
##
##            [,1]      [,2]        [,3]         [,4]         [,5]
## [1,] 0.09977607 0.0000000 0.000000000 0.000000e+00 0.000000e+00
## [2,] 0.00000000 0.0156983 0.000000000 0.000000e+00 0.000000e+00
## [3,] 0.00000000 0.0000000 0.002170704 0.000000e+00 0.000000e+00
## [4,] 0.00000000 0.0000000 0.000000000 4.842559e-05 0.000000e+00
## [5,] 0.00000000 0.0000000 0.000000000 0.000000e+00 2.615101e-40
```

We can see that the variance of the coefficients on the diagonal terms is very low,except perhaps for the "intercept".

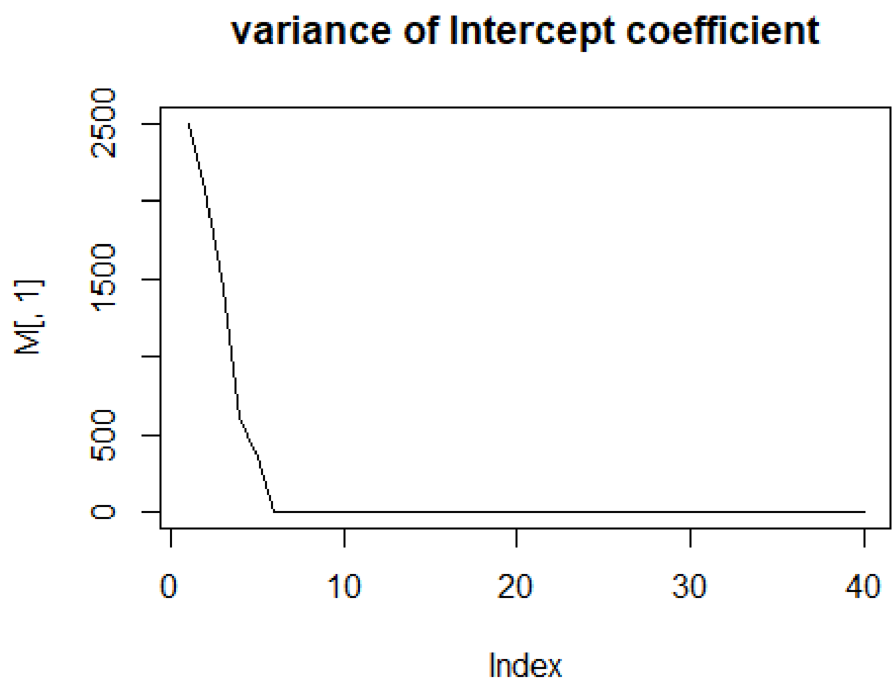
8. Curves and prediction

```
#plot of the Beta_t_t-1
ts.plot(out$alphahat,col=1:5)

legend("topleft", c("Intercept", "OBL_t_1", "INF_t_1","DM1_t_1","SUR_t_1"),
col = 1:5, lty = 1:5,cex=0.65)
```

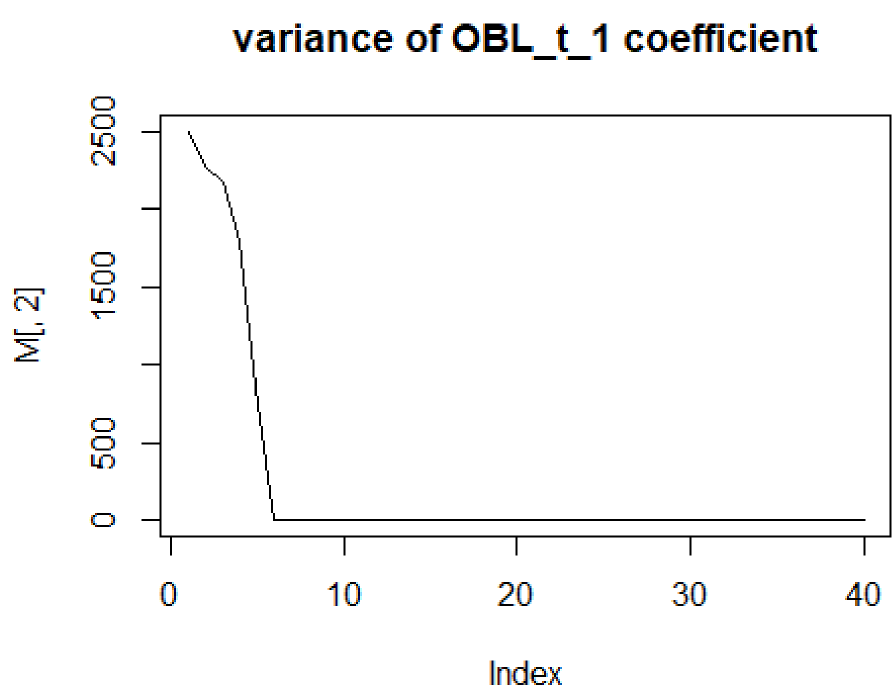



```
#plot of the Sigma_t_t-1 diagonal coefficients

M=matrix(0,40,5)
#M
#nrow(M)
for (i in 1:40)
{M[i,]=diag(out$P[,,i])}
M

##                [,1]         [,2]         [,3]         [,4]         [,5]
##  [1,] 2500.0000000 2500.0000000 2500.0000000 2.500000e+03 2.500000e+03
##  [2,] 2066.1116226 2275.0362395 2342.0236094 1.793714e+03 1.523527e+03
##  [3,] 1438.7891419 2166.9391742 2217.5165483 1.599913e+03 7.760363e+01
##  [4,]  606.8127183 1794.3746957 1883.8074101 7.047840e+02 1.149445e+01
##  [5,]  349.1852223  736.2512267 1065.0902681 3.423985e+02 1.143410e+01
##  [6,]    2.8543279    3.5545860    6.5626362 9.316699e-01 5.863289e-02
##  [7,]    2.3578577    3.3255163    6.1560137 9.175482e-01 5.803877e-02
```

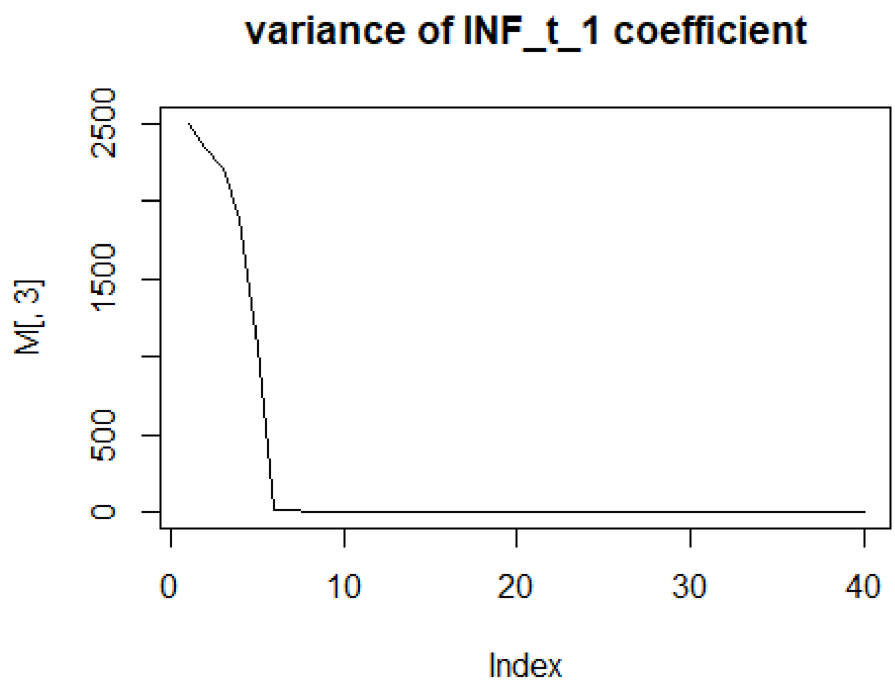
```
##  [8,]    0.8836018    2.0363010    4.0653158 5.421620e-01 3.229688e-02
##  [9,]    0.8445361    1.8662217    4.0277539 4.749432e-01 2.526632e-02
## [10,]    0.8591730    1.6951000    3.6358981 2.217951e-01 2.107880e-02
## [11,]    0.9094343    1.6768693    3.4385867 2.218399e-01 1.816462e-02
## [12,]    0.9058916    1.6764873    3.3954545 2.071158e-01 1.808676e-02
## [13,]    0.7650451    1.4485600    3.3249263 1.522121e-01 1.138262e-02
## [14,]    0.7862177    1.3803417    3.3140173 1.168379e-01 1.121187e-02
## [15,]    0.8337389    1.3881694    3.2855408 1.114980e-01 8.367179e-03
## [16,]    0.7484038    1.3905670    3.2247890 1.066572e-01 8.286768e-03
## [17,]    0.8096653    1.3238198    2.9933887 1.005054e-01 6.944619e-03
## [18,]    0.8431997    1.3307093    2.9861023 7.960714e-02 6.749895e-03
## [19,]    0.8976636    1.3347287    2.8874910 7.943841e-02 5.849004e-03
## [20,]    0.7642775    1.3499374    2.7713070 7.468911e-02 5.749612e-03
## [21,]    0.7126478    1.3655732    2.7271675 7.468090e-02 5.204889e-03
## [22,]    0.7966162    1.3652025    2.6353173 6.111896e-02 4.987383e-03
## [23,]    0.7972873    1.3635667    2.6373462 6.105155e-02 4.567846e-03
## [24,]    0.8673534    1.3690075    2.3297916 5.875796e-02 4.564836e-03
## [25,]    0.7881548    1.3844262    2.2958896 5.716384e-02 3.853544e-03
## [26,]    0.8646468    1.4001021    2.2567633 4.920533e-02 3.794052e-03
## [27,]    0.9640882    1.4017052    1.8457167 4.925351e-02 3.587647e-03
## [28,]    1.0597329    1.3264137    1.6386453 4.889091e-02 3.241847e-03
## [29,]    1.1055867    1.2432957    1.5717644 3.663486e-02 1.817113e-03
## [30,]    1.1762787    1.2478421    1.5347622 3.443481e-02 1.799641e-03
## [31,]    1.0412646    1.0211260    1.4977465 3.421266e-02 1.718162e-03
## [32,]    1.1314603    0.8099019    1.4383315 3.358101e-02 1.679967e-03
## [33,]    1.1737413    0.7981304    1.4059282 3.337617e-02 1.398287e-03
## [34,]    1.2687921    0.8062869    1.3344538 3.087295e-02 1.341647e-03
## [35,]    1.3076222    0.8112249    1.3199840 3.027944e-02 1.329778e-03
## [36,]    1.3438776    0.7791198    1.3210534 3.032785e-02 1.185497e-03
## [37,]    1.4348598    0.6371370    1.0605895 3.010662e-02 1.142673e-03
## [38,]    1.5231379    0.6524795    1.0416844 2.836954e-02 1.139376e-03
## [39,]    1.5695951    0.6382079    1.0348348 2.792490e-02 1.122170e-03
## [40,]    1.6626744    0.6484836    0.9030054 2.794475e-02 1.026399e-03

plot(M[,1],main='variance of Intercept coefficient',type='l')
```



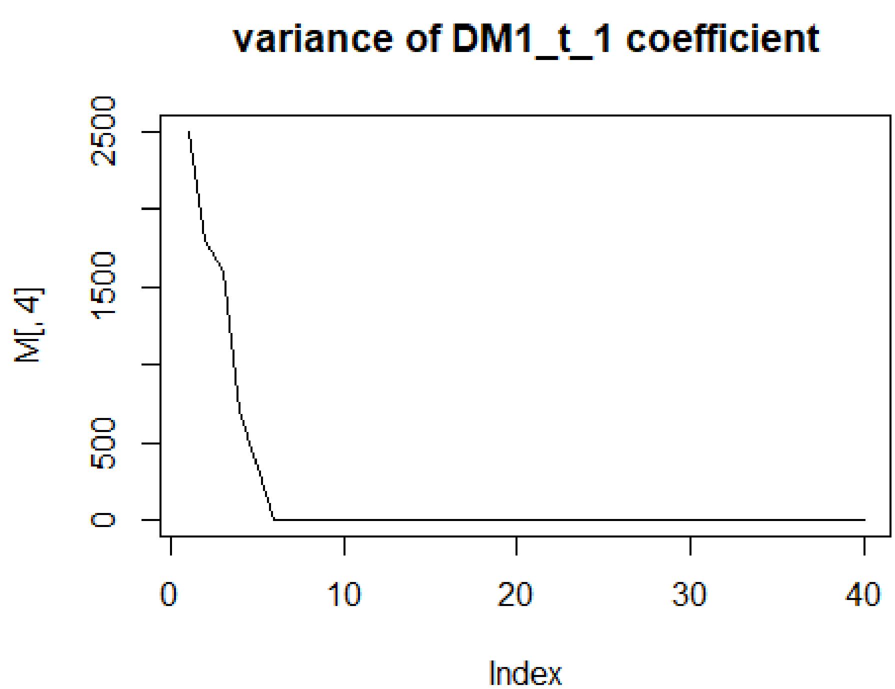

```
plot(M[,2],main='variance of OBL_t_1 coefficient',type='l')
```



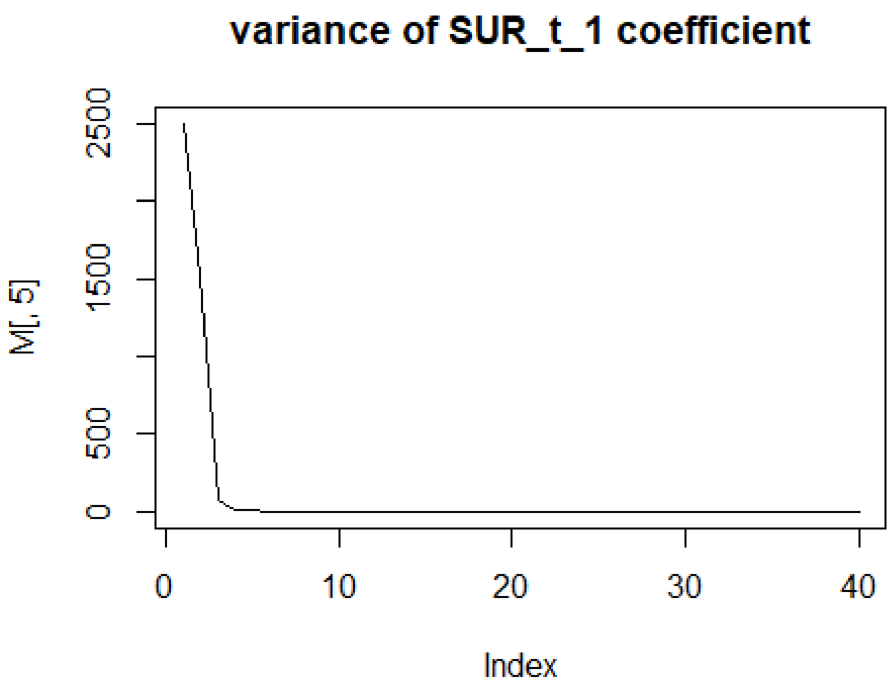

```
plot(M[,3],main='variance of INF_t_1 coefficient',type='l')
```



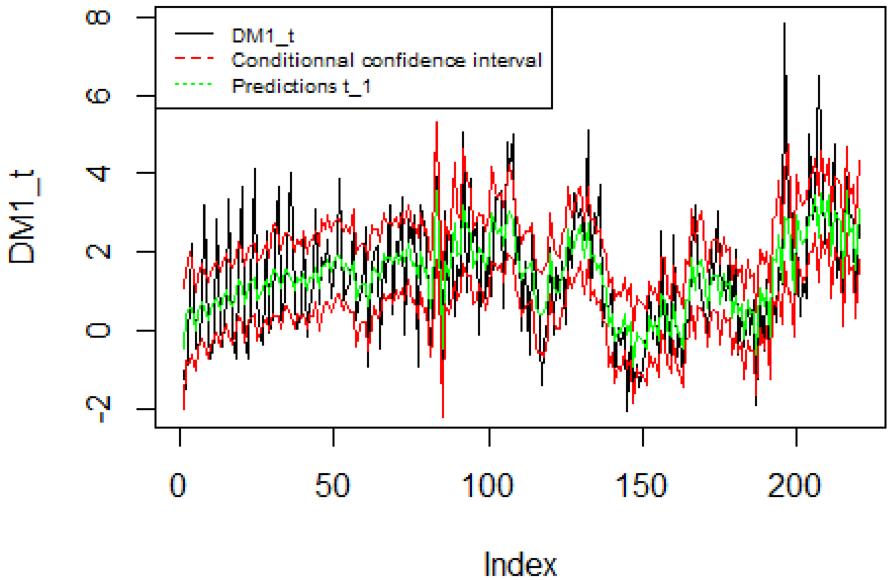

```
plot(M[,4],main='variance of DM1_t_1 coefficient',type='l')
```




```
plot(M[,5],main='variance of SUR_t_1 coefficient',type='l')
```




```
pred = predict(fit$model, interval = "conf", level = 0.95)
plot(DM1_t,type='l')
lines(pred[,1],col="green")
lines(pred[,2],col="red")
lines(pred[,3],col="red")
legend("topleft", c("DM1_t","Conditionnal confidence interval","Predictions
t_1"), col = c("black","red","green"), lty = 1:3,cex=0.65)
```




```
summary(pred)

##        fit                 lwr                 upr
##  Min.   :-0.9451   Min.   :-2.2448   Min.   :-0.01285
##  1st Qu.: 0.7442   1st Qu.:-0.2695   1st Qu.: 1.75265
##  Median : 1.3625   Median : 0.3653   Median : 2.35389
##  Mean   : 1.3713   Mean   : 0.3492   Mean   : 2.39332
##  3rd Qu.: 1.9066   3rd Qu.: 0.9259   3rd Qu.: 2.91951
##  Max.   : 3.5820   Max.   : 2.4890   Max.   : 5.30803
```

In the first plot, we can see that the coefficients of SUR_t_1 and DM1_t_1 are nearly equal to 0, the coefficient of INF_t_1 is slightly rising around 0.The coefficient of OBL_T_1 is varying between 0 and -1. At last, the Intercept coefficient β_0, t (which can be interpreted as a deterministic trend) varies a lot between 0 and 3. Moreover, we can see that the variances (the diagonal terms of $\Sigma_{t/t-1}$) converge very quickly, before 10 steps of the algorithm.

Remark : the values given by model$Q are not strictly the same as the last values of out$P, I cannot understand why.

To conclude, in view of the last graph, we can say that the prediction and the interval prediction fit very well to the DM1_t.