Exercise 1

In a certain factory machines A, B and C are all producing springs of the same length. Of their production, machines A, B and C, respectively produce 2%, 1% and 3% defective springs. Machine A produces 35% of the output of the factory, machine B 25% and machine C 40%.
If one spring is selected at random from the output of the factory find the probability it is defective. If it is defective find the probability it was manufactured on machine C.

Exercise 2

Long John Silver is searching for Captain Morgan's buried treasure. It could be on any one of three desert islands – Crystal Island, Diamond Island or Emerald Island. He thinks it is twice as likely to be on Crystal as on Diamond but equally likely to be on Diamond or Emerald. He believes that a one-day search of Crystal will find the treasure, if it is there, with probability 1/4. Similarly, one-day searches of Diamond and Emerald will find it with probabilities 1/3 and 2/3. Time is limited as Captain Morgan is known to be sailing to the area. Show that if he has only one day to search it is best to try Emerald. If a search of Emerald is unsuccessful find the probability that the treasure is on Crystal.

Exercise 3

Formulate a general expression for the posterior mean and variance.

Exercise 4

The random sample $x_1, \ldots, x_n$ is from an exponential distribution with density $p(x \mid \theta) = \theta \exp(-\theta x)$. The prior distribution is Ga(α, β), i.e.
$p(\theta) = \{\beta^\alpha \theta^{\alpha-1} \exp(-\beta\theta)\}/\Gamma(\alpha)$

Find the posterior distribution of θ.

Show that the posterior mean can be written as a weighted average of the prior mean and an estimate of θ from the data alone.

Exercise 5

It is believed that the number of accidents in a new factory will follow a Poisson distribution with mean θ per month. The prior distribution of θ is given by a gamma distribution with mean μ having density

$$p(\theta) = \frac{h^{\mu h}\theta^{\mu h-1}\exp(-h\theta)}{\Gamma(\mu h)}, \quad \theta \geq 0$$

If $\mu = 2$, $h = 2$ and there are 18 accidents in the first six months, derive the posterior distribution of θ and find its mean and variance.

Exercise 6

Observations $x_1, x_2, \ldots, x_n$ are a random sample from a uniform distribution on the interval $(0, \theta)$. the prior for θ is taken to be a Pareto distribution with density

$$p(\theta) = \begin{cases} \frac{\alpha\theta_0^\alpha}{\theta^{\alpha+1}} & \theta > \theta_0 \\ 0 & \theta \leq \theta_0 \end{cases}$$

Find the posterior distribution of θ.