E-values and Multiple Testing

ISI WSC Bernoulli Society New Researcher Award

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Bringing closure to FDR control: a general principle for multiple testing

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- Many hypotheses: $H_1, H_2, \ldots, H_m$
- Multiple testing goal:

  1) Many discoveries

  2) Control some error rate

  3) Post hoc flexibility

How to design a multiple testing procedure

- **e-Closure**
  A general recipe for making multiple testing methods

- **Building blocks**
  Intersection hypotheses and e-values

- **Contributions**

  - Recovers the Closure Principle for FWER
  - Extends to FDR
  - Uniformly improves eBH and BY
  - Introduces unprecedented flexibility in multiple testing

The e-variable

- **Definition**: e-variable

  An e-variable E for $\mathscr{P}$ is a non-negative random variable satisfying $\mathbb{E}_P[E] \leq 1$ for all $P \in \mathscr{P}$.

- The value taken by the e-variable after observing the data is called the e-value. However, often, as also happens with the infamous p-value (p-variable), the random variable E itself is also often called e-value.

Tests and the type I error guarantee

- **Definition:** binary test

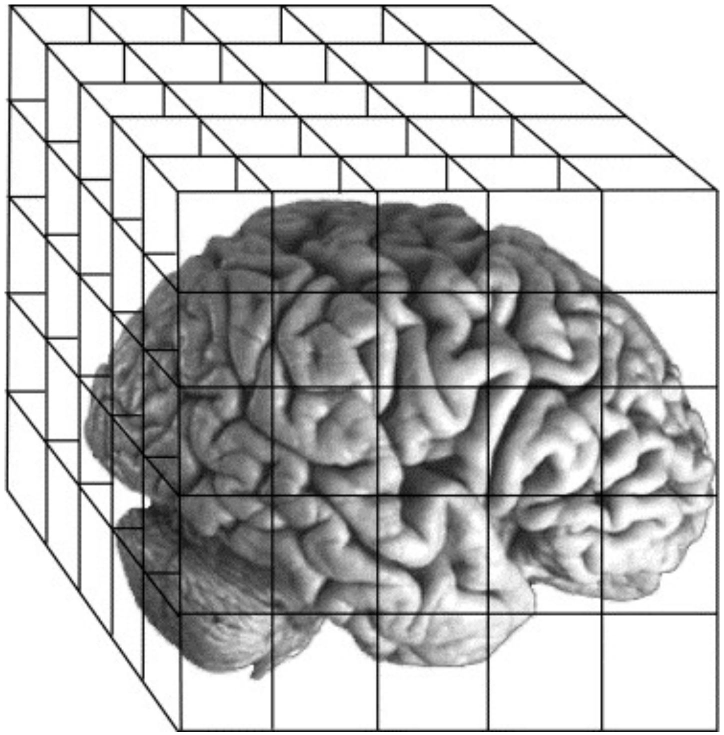
  A binary test ϕ is a $\{0,1\}$-valued random variable. The type-I error of a test ϕ for P is $\mathbb{E}_P[\phi]$. A test has level $\alpha \in [0,1]$ for $\mathscr{P}$ if its type-I error is at most α for every $P \in \mathscr{P}$.

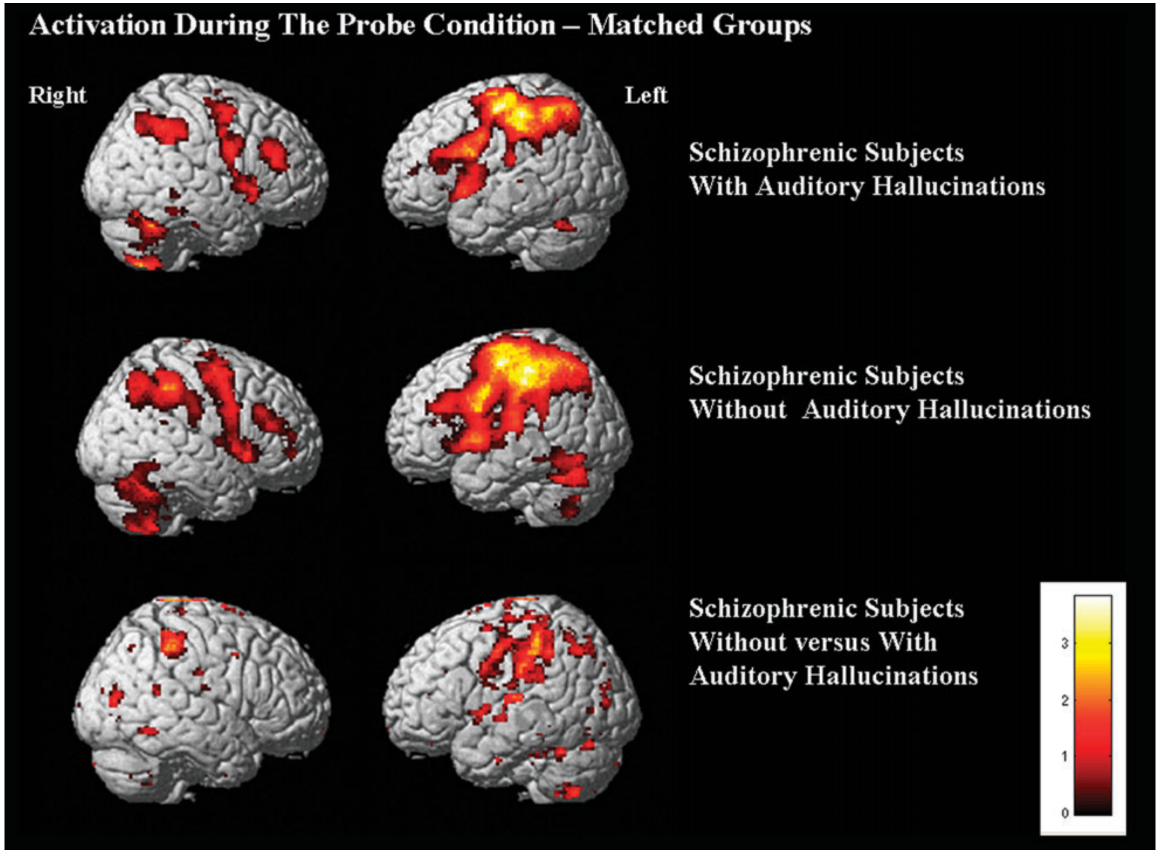
- **Markov's inequality for e-variables**

  Let E be an e-variable for $\mathscr{P}$. We have $P(E \geq 1/\alpha) \leq \alpha$ for all $P \in \mathscr{P}$ and $\alpha \in (0,1]$. Hence, $\mathbf{1}_{\{E \geq 1/\alpha\}}$ is a binary test of level α.

Example: Multiple testing in neuroimaging

130.000 voxels





Wible, Cynthia G. et al. “fMRI activity correlated with auditory hallucinations during performance of a working memory task: data from the FBIRN consortium study.” Schizophrenia bulletin 35 1 (2009): 47-57 .

Multiple testing: the problem

- If we test n true null hypotheses at level α, then on average we will (falsely) reject αn of them.
- Examples:
  - testing whether some of 20.000 genes are linked to a disease
  - fMRI: 100.000 voxels
  - DNA methylation: 500.000 sites
- We need other measures of acceptance/rejection errors.
- We need statistical procedures to control these measures of errors.

Error rates

$N \subseteq [m]$ hypotheses are true null; the rest are potential discoveries

Famous error rates:

- Familywise error rate (FWER): $P(|R \cap N| > 0)$
- Per-family error rate: $\mathbb{E}(|R \cap N|)$
- False Discovery rate: $\mathbb{E}\left(\frac{|R \cap N|}{R}\right)$

General form

Control some expected loss: $\mathbb{E}(f_N(R))$

Intersection hypotheses

Intersection hypothesis

For $S \subseteq [m], H_S = \bigcap_{i \in S} H_i$, which is true iff all $H_i, i \in S$ true

The e-collection

$E = (e_S)_{S \subseteq [m]}$: local e-values such that E$(e_N) \leq 1$

Sufficient

Each e_S is an e-value for $H_S\$, \$S \subseteq [m]$

The e-Closure procedure

- The e-Closed Procedure

$$\mathscr{R}_\alpha(E) = \left\{R \subseteq [m] : \alpha e_S \geq \mathrm{f}_S(R) \quad \forall S \subseteq [m]\right\}$$

- The e-Closure Principle

R controls $\mathrm{E}(\mathrm{f}_N(R)) \leq \alpha$ iff $R \in \mathscr{R}_\alpha(E)$ for e-collection E

- Simultaneous control

$\mathrm{E}(\mathrm{f}_N(R)) \leq \alpha$ simultaneously over $R \in \mathscr{R}_\alpha(E)$: $\quad \mathrm{E}\left(\max_{R \in \mathscr{R}_\alpha(E)} \mathrm{f}_N(R)\right) \leq \alpha$

Post hoc error rate

- All error rates

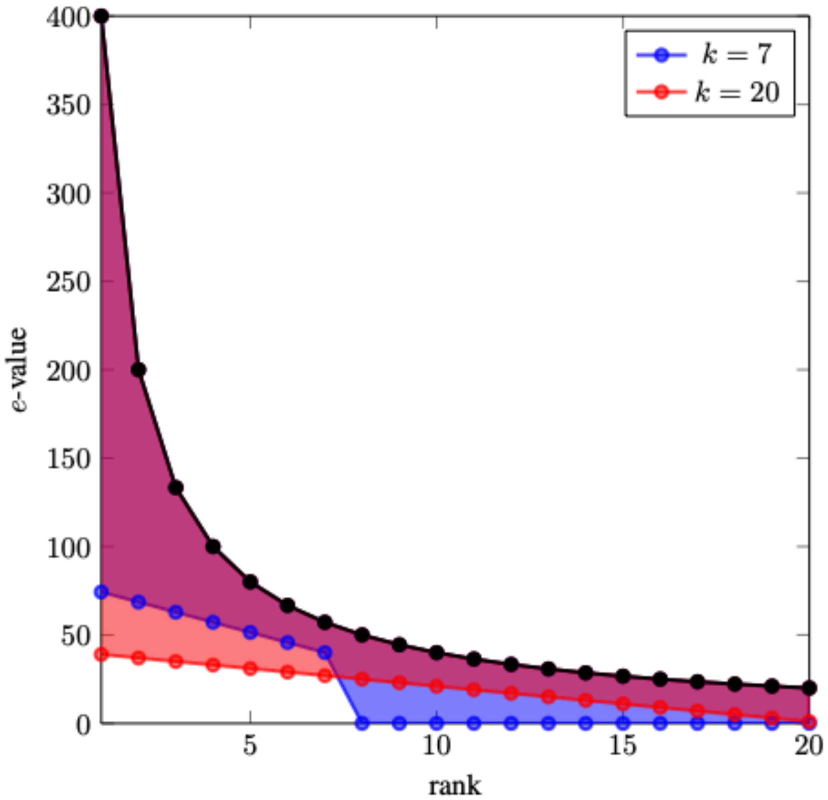
  $\mathscr{F} = \{\text{all functions } \mathrm{f}_N(R)\}$

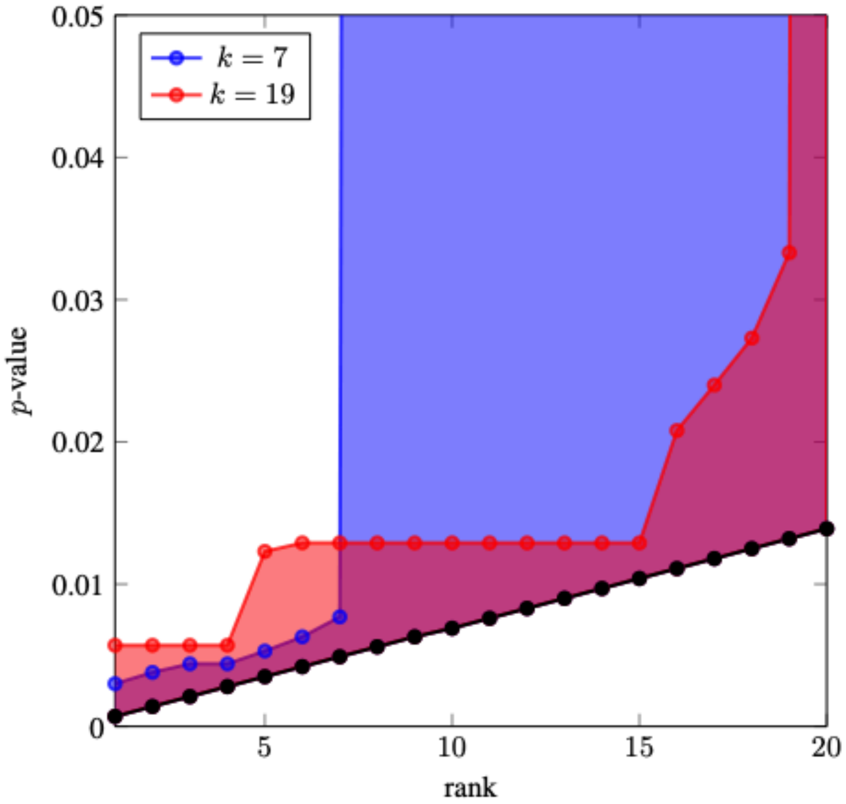
- Simultaneous (= post hoc) choice of error

  $$\mathrm{E}\left(\sup_{\mathrm{f}\in\mathscr{F}} \max_{R\in\mathscr{R}^{\mathrm{f}}_{\alpha}(E)} \mathrm{f}_N(R)\right) \leq \alpha$$

- So: possible to switch from FWER to FDR if not much signal present

Improving existing procedures: eBH and BY





BY vs closed BY in standard data sets

Dataset	m	BY / $\overline{BY}$ rejections		source
		$\alpha = 5\%$	$\alpha = 10\%$	
APSAC	15	3 / 3	3 / 5	BH '95
NAEP	34	6 / 8	8 / 11	BH '00
PADJUST	50	12 / 15	17 / 20	`p.adjust`
PVALUES	4289	129 / 145	225 / 275	`fdrtool`
VANDEVIJVER	4919	614 / 677	779 / 866	Goeman Solari '14
GOLUB	7128	617 / 648	743 / 799	Efron Hastie '16

More properties: post hoc α

- Choose rejected set post hoc
- Choose error loss post hoc
- One step further: choose α post hoc (Koning 2023)
- Requires: e-value does not depend on α. Then we have:

$$\mathrm{E}\left(\sup_{\alpha \in (0,1)} \sup_{\mathrm{f} \in \mathscr{F}} \max_{R \in \mathscr{R}_{\alpha}^{\mathrm{f}}(E)} \frac{\mathrm{f}_N(R)}{\alpha} \right) \leq 1$$

More properties: restricted combinations

- Logically related hypotheses, for example pairwise combinations

$$H_{1=2} : \mu_1 = \mu_2; \quad H_{1=3} : \mu_1 = \mu_3; \quad H_{2=3} : \mu_2 = \mu_3$$

- Logical relationships = gain in power

  Known only for FWER, unknown for FDR

Conclusion: e-Closure

- **General Principle**: unites all multiple testing methods
- **Simplifies multiple testing**: Choose how to summarise evicence against H_S; rest is computation
- **Flexibility**: Simultaneous over rejected sets, error rates, α
- **Power**: Uniformly improves known methods

A general recipe for making multiple testing methods