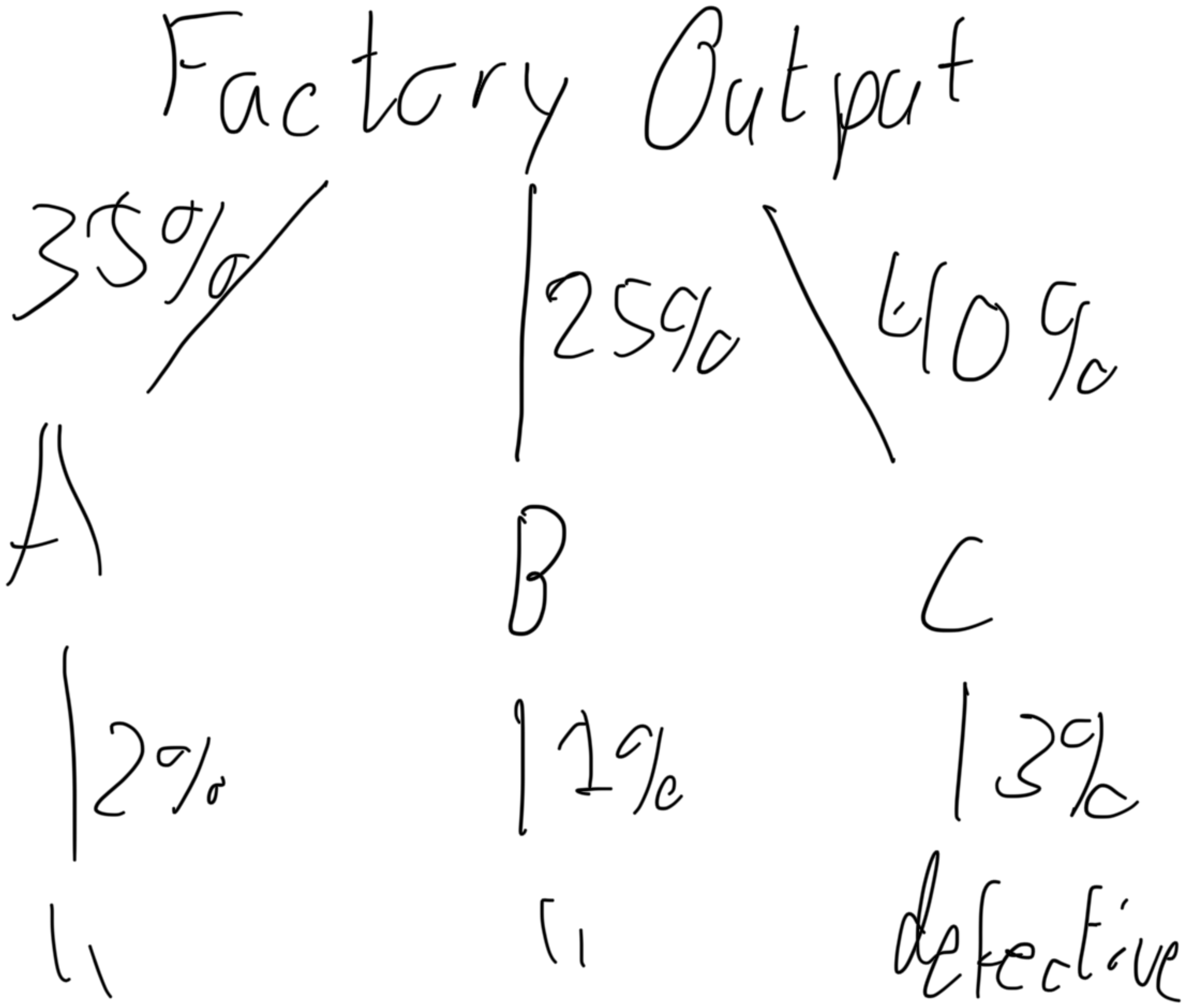
Execise 1



$$P(\text{Defect}) = P(A) \cdot P(\text{Defect} \mid A)$$

$$+ P(B) \cdot P(\text{Defect} \mid B)$$

$$+ P(C) \cdot P(\text{Defect} \mid C)$$

$$= 0{,}35 \cdot 0{,}02 + 0{,}25 \cdot 0{,}01$$

$$+ 0{,}4 \cdot 0{,}02 = 0{,}0215 = 2{,}15\%$$

$P(C \mid \text{Defect})$

$P(C \mid \text{Defect})$

$$= \frac{P(\text{defect} \mid C)\, P(C)}{P(\text{defect})}$$

$$= \frac{0.03 \cdot 0.4}{0.0215} \approx 0.558 = 55.8\%$$

Exercise 2

C
D
E
F

$P(F \mid C) = 1/4$, $P(F \mid D) = 1/3$

$P(F \mid E) = 2/3$

$P(C) + P(D) + P(E) = 1$

$P(C) = 2P(D)$, $P(D) = P(E)$

$2P(D) + P(D) + P(D) = 1$

$P(D) = 1/4$

$P(F) = P(F \mid C)P(C) + P(F \mid D)P(D) + P(F \mid E)P(E)$

$= 0.3750$

$$P(C|F') = \frac{P(C)(1-P(F|C))}{1-P(F)} = 0.600$$

$$P(F|E_s) = P(F|E_s, E)P(E) + 0$$

$$= 2/3 \cdot 1/4 = 1/6$$

$$P(F|D_s) = 1/3 \cdot 1/4 = 1/12$$

$$P(F|C_s) = 1/4 \cdot 1/2 = 1/8$$

$$P(F|E_s, E)P(E)$$

2) $$P(C|E_s, \bar{F}) = \frac{P(\bar{F}|C, E_s)P(C)}{P(\bar{F}|E_s)}$$

$$= \frac{1 \cdot 1/2}{5/6} = \frac{6}{10}$$

Bayesian Statistics

Let θ be quantity of interest

Learner starts with specifying a <u>prior distribution</u> $\pi(\theta)$, which

quantifies [illegible] uncertainty about θ befo[illegible] quantifies her uncertainty about θ before seeing the data.

Then she calculates the posterior distribution $\pi(\theta | z)$: the conditional distribution of θ given $Z = z$, by Bayes' theorem:

$$\pi(\theta | z) = \frac{\pi(\theta) f(z|\theta)}{\int_{\Theta} \pi(\theta') f(z|\theta') \, d\theta'}$$

$\pi(\theta)$ prior

$f(z|\theta)$ likelihood

$\int_{\Theta} \cdots \, d\theta'$ marginal density of Z

Bayes marginal (likelihood)

Model evidence

Ex 3. Formulate a general expression for the posterior mean and var.

($\pi(\theta|z)$ posterior)

$$\mathbb{E}(\theta | z) = \int_{\Theta} \theta \, \pi(\theta|z) \, d\theta$$

$$\mathrm{Var}(\theta | z) = \int_{\Theta} (\theta - \mathbb{E}(\theta|z))^2 \pi(\theta|z) \, d\theta$$

$$\pi(\theta|z) \propto \pi(\theta) f(z|\theta)$$

Ex. 4 – find posterior $\pi(\theta|x)$

– show that the post. mean can be written as a weighted average of the prior mean and an estimate of θ from the data.

posterior dist.

$$p(\theta|x) = \frac{p(\theta)\, p(x|\theta)}{\int_\Theta \ldots d\theta}$$

$$\propto p(\theta)\, p(x|\theta)$$

$$= \frac{\beta^\alpha \theta^\alpha \exp(-\theta(x+\beta))}{\Gamma(\alpha)}$$

extend to n samples

$$= \theta^n \exp\left(-\theta \sum_{i=1}^n x_i\right) \theta^{\alpha-1} \exp(-\theta\beta)$$

$$= \theta^{n+\alpha-1} \exp\left(-\theta\left(\sum x_i + \beta\right)\right)$$

$$\rightarrow Ga\left(n+\alpha, \sum x_i + \beta\right)$$

posterior

Post mean: $$\frac{n+\alpha}{\sum x_i + \beta} = \left(\frac{\sum x_i}{\sum x_i + \beta}\right) \frac{n}{\sum x_i}$$

$$+ \frac{\beta}{\sum x_i + \beta} \frac{\alpha}{\beta}$$

$$= w \frac{n}{\sum x_i} + (1-w) \frac{\alpha}{\beta}$$

$$\left(w = \frac{\sum x_i}{\sum x_i + \beta} \right)$$

$\frac{n}{x_i}$ (estimate of ϑ from the data)

$\frac{\alpha}{\beta}$ PRIOR mean

$$\pi(\theta|x) = \frac{\pi(\theta) \cdot f(x|\theta)}{\#} \propto \frac{h^{\nu h} \theta^{\nu h-1} \exp(-h\theta)}{\Gamma(\nu h)} \cdot \frac{\theta^x}{x!} e^{-\theta}$$

$$\propto \theta^{\nu h - 1 + x} \cdot e^{-\theta(h+1)} \longrightarrow \theta^{\nu h - 1 + \sum x_i} \cdot e^{-\theta(h+n)}$$

$$\Downarrow$$

$$\pi(\theta | x_1, \dots, x_n) \sim \qquad \propto \theta^{21} \cdot e^{-\theta 8}$$

$$Ga(22, 8)$$

mean $\rightarrow$ $22/8 = 2{,}75$

variance $\rightarrow$ $\frac{22}{64} = 0.344$

Ex. 6 $\quad p(x|\vartheta) = \frac{1}{\vartheta}, \quad 0 < x < \vartheta$

$$p(x^n|\vartheta) = \frac{1}{\vartheta^n}, \quad \vartheta > \max_{i \in [n]} x_i$$

$$([n] := 1, 2, \ldots, n)$$

$$p(\theta) \propto \frac{1}{\theta^{\alpha+1}}, \quad \theta > \theta_0$$

(Pareto prior)

$$p(\theta | x) \propto \frac{1}{\theta^{n+\alpha+1}},$$

$$\theta > \max\{\theta_0, \max x_i\}$$

Let $t = \max\{\theta_0, \max x_i\}$

Then $p(\theta | x^n) = \dfrac{(\alpha+n)\, t^{\alpha+n}}{\theta^{n+k+1}}$

$\rightarrow$ Pareto posterior