MAE 107 - Summer 2023
Take-home Final Exam
Due at 10 PM on Saturday September 9, 2023 in CANVAS

Open book, open note, open homework and midterm solution.
No access to internet. All work needs to be your own. No collaboration is allowed.

I, Model problem:

Consider a drainage sytem in which water flows out of a cylindrical tank through a rough pipe as rain falls as depicted in figure 1(a). Table 1 lists the description of parameters. The time-vayring precipitation is shown in figure 1(b). In this exam, you are to simulate how the water depth (w) in the tank and the velocity (u) at the pipe exit vary in time.

The governing equations for w and u are given in the following initial value problem (IVP):

$$\text{ODE's:} \quad \frac{dw}{dt} = -\frac{d_2^2}{d_1^2}u + S,$$

$$\frac{du}{dt} = \frac{gw}{l} - \left(1 + \kappa + \frac{fl}{d_2}\right)\frac{u^2}{2l}, \quad t \in [0, 900]$$

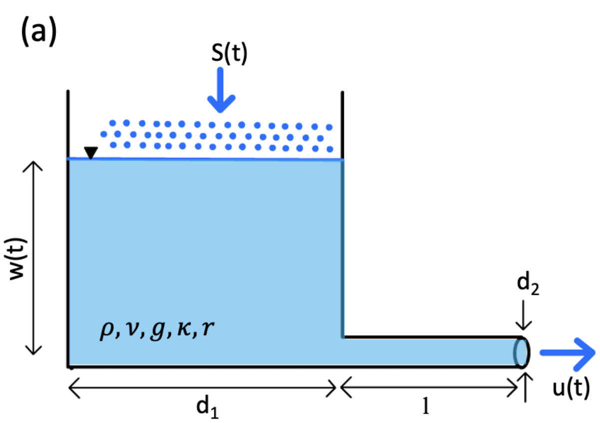
$$\text{IC:} \quad w(t=0) = 0.5 \quad \text{and} \quad u(t=0) = 1.9$$

II, Numerical methods:

In order to complete the exam, you need to implement the following numerical methods. Download **final_exam_template.zip** from CANVAS and use the template for guidance.

1, Root finding: The friction factor f can be obtained by solving the Colebrook equation:

$$\frac{1}{\sqrt{f}} = -2\,log_{10}\left(\frac{r}{3.72} + \frac{2.51}{Re\sqrt{f}}\right),$$



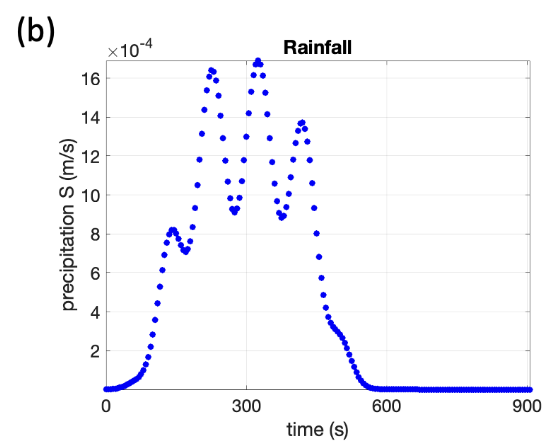


Figure 1: (a) Diagram of a drainage system where rain drops into a cylindrical tank and water flows out of a pipe. (b) Precipitation is given at every 5 seconds. Variable **time_rain** and **precipitaion** are stored in file **rain_data.mat**.

where r is roughness ratio and $Re = ud_2/\nu$ is Reynolds number. Use the secant method to find the root of the Colebrook equation. Complete function **colebrook** to obtain f.

2, Interpolation: Apply cubic splines method to interpolate the precipitation to smaller time step h as required by time integration. The function **clamped_cubic_spline** was written in homework 3. When use the function, set the derivatives at the end knots to zero.

3, Least squares: Complete function **best_fit_power_model** to fit a power law in the form $\bar{y} = \alpha\bar{x}^{\beta}$ through given $\bar{x}_i$ and $\bar{y}_i$ data points.

4, Numerical differentiation: Derivative must be approximated using second-order accurate finite-difference formulas.

5, Numerical integration: Integration must be approximated using one of the Simpson's rules or a combination of the rules if necessary. Use the method that has the smallest error based on the error formula.

6, Time integration in IVP: Implement Runge-Kutta-Merson (RKM) method to obtain the evolution of w and u. The algorithm for the RKM method is given below:

$$\begin{aligned}
k_1 &= hf(t_n, y_n), \\
k_2 &= hf(t_n + \frac{1}{3}h, y_n + \frac{1}{3}k_1), \\
k_3 &= hf(t_n + \frac{1}{3}h, y_n + \frac{1}{6}k_1 + \frac{1}{6}k_2), \\
k_4 &= hf(t_n + \frac{1}{2}h, y_n + \frac{1}{8}k_1 + \frac{3}{8}k_3), \\
k_5 &= hf(t_n + h, y_n + \frac{1}{2}k_1 - \frac{3}{2}k_3 + 2k_4), \\
y_{n+1} &= y_n + \frac{1}{6}\left(k_1 + 4k_4 + k_5\right),
\end{aligned}$$

where $k's$ are the change in y at each substep.

Follow the template **ivpsys_fun.m** and **ivpsys_RKM.m**, to solve the ODE's. Function **ivpsys_fun.m** evaluates the right-hand-side of the ODE's while function **ivpsys_RKM.m** implements the Runge-Kutta-Merson method. <u>Use time step h = 0.3 s</u>.

III, <u>Analysis</u>:

(1) Numerically integrate the IVP to obtain the evolution of w(t) and u(t). Create **figure 1** to include three panels stacked on top of each other. In the top panel, plot w versus t with t on the horizontal axis. Plot u versus t in the middle panel, and the time rate of change of water depth (dw/dt) in the tank versus t in the bottom panel. Set **p1a = 'See figure 1'**. Put the first, second and last elements of dw/dt in **p1b**, **p1c** and **p1d**, respectively.

(2) Use function **colebrook** to obtain the evolution of friction factor f. Create **figure 2** to plot f versus Re with Re on the horizontal axis. Set **p2a = 'See figure 2'**.

Parameters	Definition
t	time in second
w	depth of water in the tank in meter
u	velocity at pipe exit in meter per second
S	precipitation in meter per second
f	friction factor f (unitless)
$Re = ud_2/\nu$	Reynolds number in the pipe (unitless)
$d_1 = 2$	tank diameter in meter
$d_2 = 0.05$	pipe diameter in meter
l = 0.5	pipe length in meter
r = 0.0001	pipe roughness ratio (unitless)
$\kappa = 1.5$	head loss due to pipe entrance and exit (unitless)
g = 9.81	gravity in meter per squared second
$\nu = 10^{-6}$	kinematic viscosity of water in squared meter per second
$\rho = 1{,}000$	density of water in kilogram per cubic meter

Table 1: Definition of parameters in drainage system.

(3) Use function **best_fit_power_model** to obtain the relationship between w and u in the form $u = \alpha w^{\beta}$. Create **figure 3** to plot u versus w with w on the horizontal axis. Use the power model fit to extrapolate the values of u for the depth range = 0.05:0.05:1. Also plot the extrapolated u in figure 3. Set **p3a = 'See figure 3'**. Put α and β into **p3b** and **p3c**, respectively.

(4) The law of mass conservation states that the change in mass in the drainage system is balanced by the mass input by rain and the mass outflow through the pipe:

$$\begin{aligned} m_{change} &= m_{rain} + m_{out} \quad \text{where} \\ m_{change} &= \frac{\pi\rho d_1^2}{4}\left[w(900) - w(0)\right], \\ m_{rain} &= \frac{\pi\rho d_1^2}{4}\int_0^{900} S(t)\,dt, \\ m_{out} &= -\frac{\pi\rho d_2^2}{4}\int_0^{900} u(t)\,dt. \end{aligned}$$

Compute m_{change}, m_{rain}, and m_{out} and put the answers in **p4a**, **p4b** and **p4c**, respectively. Note that the precipitation needs to be interpolated to the finer time step h before computing m_{rain}. Also, compute the expression $m_{change} - m_{rain} - m_{out}$ and put the answer in **p4d**.

Point distribution: Part (1) is worth 60 points; (2) 10 points; (3) 15 points; and (4) 15 points.

IV, Submission instructions: Follow **final.m** template. Put all answers in the **final.m** script. You will need to submit multiple files. Create a zip archive named **final.zip**. The zip archive should include the following files: **final.m**, **ivpsys_fun.m**, **ivpsys_RKM.m**, **clamped_cubic_spline.m**, **colebrook.m**, **best_fit_power_model.m**, **rain_data.mat**, academic integrity form (see next page) and any other files that you have created. Make sure that the three figures are plotted when your final exam script is executed. Your program must be free from syntax errors. Submit **final.zip** in CANVAS by 10 PM on Saturday 09/09/2023.

Academic Integrity: Read and sign the following form. You must include it in the final.zip file. Your exam will not be graded without the signed form.

I, Rishi Carlton (student name), confirm that I have displayed the utmost academic integrity while completing this exam. I did not give or receive assistance from anyone in any format. I acknowledge that the instructor has the right to give failure course grade and report to the UCSD Office of Academic Integrity if any violation occurs during the exam.

Signature: Rishi Carlton Date: 9/6/23