CAT 2023

Question 1

a) Distinguish between numerical and analytical methods in the solution of mathematical problems.

(3 Marks)

b) Describe the bisection method and explain how it differs from the Newton-Raphson method.

(3 Marks)

c) Given an initial guess *x=3*, find an approximate value of the root of the function $f(x) = x^2 - 2$ using 3 iterations. Provide a sample R code that you would use to solve this problem, stopping after 40 iterations.

(6 Marks)

d) Using the inverse-transform approach to explain (mathematically) how you would generate random numbers from the exponential distribution, $f(y) = \theta \exp - (\theta y)$, and further provide an R code that will be used to generate random numbers from this distribution.

(8 Marks)

Question 2

Starting with the Newton-Raphson formula

$$x^{(n+1)} = x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})}$$

shown that the order of convergence of the Newton-Raphson method is

$$|\varepsilon^{(n+1)}| = k|\varepsilon^{(n)}|^2$$

as $n \to \infty$, with $k = \frac{1}{2}\left|\frac{f''(r)}{f'(r)}\right|$ provided $|f'(r)| = 0$.

(20 Marks)

Question 3

a) Consider the following data:

x	1	2	4
y	1	3	1

Use quadratic spline interpolation to find the approximate value of y at x=3.

Hint:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 \\ 0 & 4 & 0 & 2 & 0 & 1 \\ 0 & 16 & 0 & 4 & 0 & 1 \\ 4 & -4 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0.5 & -0.5 & -0.25 & 0.25 & 0.5 & -0.5 \\ -1.0 & 1.0 & 0.00 & 0.00 & 0.0 & -3.0 \\ -3.0 & 3.0 & 1.00 & -1.00 & -3.0 & 3.0 \\ 2.0 & -1.0 & 0.00 & 0.00 & 0.0 & 2.0 \\ 4.0 & -4.0 & 0.00 & 1.00 & 4.0 & -4.0 \end{pmatrix}$$

(10 Marks)

b) Consider the following data

$$(x_1, y_1) = (1,7), (x_2, y_2) = (1,23), (x_3, y_3) = (3,100)$$

Use polynomial interpolation to determine the value of the function at x=2.7.

Hint:

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & -1 & 0.5 \\ -2.5 & 4 & -1.5 \\ 3.0 & -3 & 1.0 \end{pmatrix}$$

(10 Marks)

Question 4

Consider the general linear model

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \sim MVN(\boldsymbol{0}, \sigma^2 \boldsymbol{I}),$$

where $\mathbf{Y} = (Y_1, \dots, Y_n)'$ is an $(n \times 1)$ vector of response values, $\mathbf{X}$ is an $(n \times k)$ design matrix corresponding to the explanatory variables $X_1, \dots, X_k$, $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)'$ is the vector of parameters, and σ^2 is the variance.

Using a maximum likelihood approach, clearly showing the likelihood function, the log-likelihood function, and the score-vector, derive the maximum likelihood estimators of $\boldsymbol{\beta}$ and σ^2.

(20 Marks)

Question 5

a) Show that the probability density function of the Poisson distribution, $f(y) = \frac{e^{-\lambda}\lambda^y}{y!}$, y = 0,1, ... belongs to the exponential dispersion family

$$f(y;\theta) = \exp\left[\frac{y\theta - b(\theta)}{\phi}\right] + c(y;\phi).$$

(3 Marks)

b) Also show that the mean and the variance of the Poisson distribution are equal to $b'(\theta)$ and $b''(\theta)$, respectively.

(3 Marks)

c) Consider the generalized linear model

$$g(\mu_i) = \boldsymbol{X}_i\boldsymbol{\beta}$$

where $\mathbf{Y} = (Y_1, \dots, Y_n)'$ is an $(n \times 1)$ vector of response values belonging to the Poisson distribution, $\mathbf{X}$ is an $(n \times k)$ design matrix corresponding to the explanatory variables $X_1, \dots, X_k$, $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)'$ is the vector of parameters.
Derive an expression for the estimating equation and Hessian that would be used to estimate the vector of parameters $\boldsymbol{\beta}$.

(14 Marks)