Solution to Exercise

$H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$ $\quad X \sim \exp(\theta)$

$\omega = \{\theta : \theta = \theta_0\}$

$\Omega = \{\theta : \theta < \infty\}$

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\left(\frac{1}{\theta_0}\right)^n \exp\left\{-\sum_1^n x_i/\theta_0\right\}}{\left(\frac{1}{\bar{x}}\right)^n \exp\left\{-\sum^n x_i/\bar{x}\right\}}$$

$$= \left(\frac{\bar{x}}{\theta_0}\right)^n \exp\left\{-n\bar{x}/\theta_0 + n\right\}$$

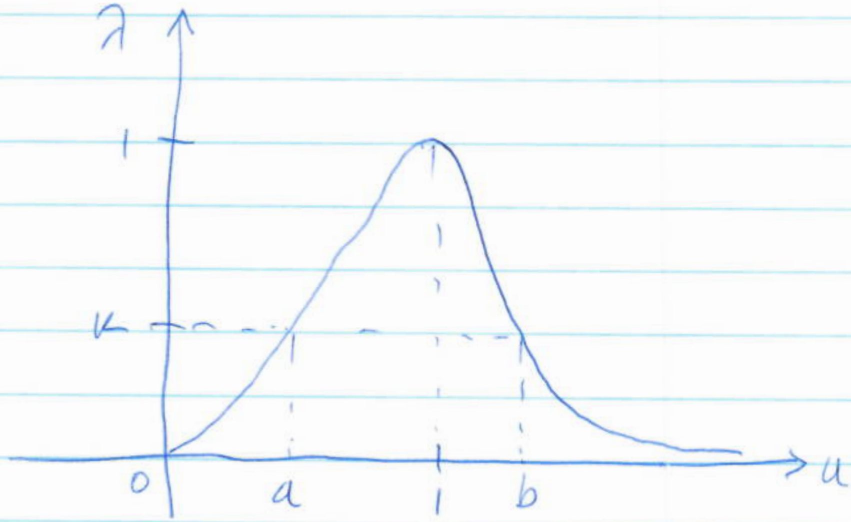
$$= \left(\frac{\bar{x}}{\theta_0}\right)^n \exp\left\{-n\left(\frac{\bar{x}}{\theta_0} - 1\right)\right\} = \lambda.$$

So $\lambda = u^n \exp\{-n(u-1)\}$ where $u = \bar{x}/\theta_0$.

Let us examine the function $\lambda = \lambda(u)$.

When $u = 0$, $\lambda = 0$ and $u = 1$, $\lambda = 1$ (maximum) since $0 \leq \lambda \leq 1$. As $u \to \infty$, $\lambda \to 0$

Plot of λ vs $u = \bar{x}/\theta_0$



$\lambda \le k \Rightarrow u \le a$ or $u \geq b$

i.e $\dfrac{\bar{x}}{\theta_0} \le a$ or $\dfrac{\bar{x}}{\theta_0} \geq b$

$\Rightarrow \bar{x} \le a(\theta_0) = c_1$ or $\bar{x} \geq b(\theta_0) = c_2$

Thus we reject H_0 when $\bar{x} \le c_1$ or $\bar{x} \geq c_2$

Now $X \sim exp(\theta) \Rightarrow Y = \dfrac{2X}{\theta} \sim \chi^2(2)$

$\Rightarrow \dfrac{2}{\theta_0} \sum_1^n X_i \sim \chi^2(2n)$ under H_0

So we reject H_0 if $\dfrac{2}{\theta_0} \sum X_i \le \chi^2_{1-\alpha/2}(2n)$

or $\dfrac{2}{\theta_0} \sum X_i \geq \chi^2_{\alpha/2}(2n)$.