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Task

Question 1: The exponential distribution has the probability density function (pdf) given by:

$$f(y;\ \lambda)\ =\ \lambda e\^{}(-\lambda y)\ for\ y\ >=\ 0, and\ 0\ otherwise.$$

$$Here, \lambda\ >\ 0\ is\ the\ rate\ parameter.$$

1. **Score Function**: The score function is the derivative of the log-likelihood with respect to the parameter. For a single observation y, the log-likelihood is given by:

$$l(\lambda)\ =\ log(f(y;\ \lambda))\ =\ log(\lambda)\ -\ \lambda y$$

Taking the derivative with respect to λ gives the score function:

$$S(\lambda)\ =\ \frac{d}{d\lambda}l(\lambda)\ =\ \frac{1}{\lambda}\ -\ y$$

2. **Estimating Equation**: The estimating equation is obtained by setting the score function equal to zero and solving for λ:

$$0\ =\ S(\lambda)\ =\frac{1}{\lambda}\ -\ y$$

Solving for λ gives the maximum likelihood estimate of λ:

$$\hat{\lambda}\ =\frac{1}{y}$$

However, this is for a single observation. If we have n observations, the maximum likelihood estimate would be:

$$\hat{\lambda}\ =\ \frac{n}{\Sigma(y_i)}\ for\ i\ =\ 1\ to\ n$$

3. **Information Matrix**: The information matrix is the negative of the expected value of the second derivative of the log-likelihood. The second derivative of the log-likelihood is:

$$\frac{d^2}{d\lambda^2}l(\lambda)\ =\ \frac{-1}{\lambda^2}$$

The expected value of this quantity is:

$$E[-\frac{d^2}{d\lambda^2}l(\lambda)]\ =\ -E[\frac{1}{\lambda^2}]\ =\ -\frac{1}{\lambda^2}$$

Therefore, the information matrix I(λ) is:

$$I(\lambda) = \frac{1}{\lambda^2}$$

These calculations assume that the observations are independent and identically distributed (i.i.d.) exponential random variables.

Question 3

C) **Fitting a generalized linear model (GLM) with a Poisson distribution and a log link function to these data using Fisher's scoring involves several steps. Here's a general outline of the process:**

1. **Score Function**: The score function is the derivative of the log-likelihood with respect to the parameters. For a Poisson regression model with a log link function, the score function is given by:

$$S(\beta) = \sum_{0}^{n} y_i - x_i e^{\beta_1 + \beta_2 \, x_i}$$

Estimating Equation: The estimating equation is obtained by setting the score function equal to zero and solving for the parameters. This results in a system of nonlinear equations that can be solved iteratively using Fisher's scoring.

2. **Information Matrix**: The information matrix is the negative of the expected value of the second derivative of the log-likelihood. For a Poisson regression model with a log link function, the information matrix is given by:

$$I(\beta) = \sum_{0}^{n} x_i{}^2 e^{\beta_1 + \beta_2 \, x_i}$$

Question 4

b. The link function that is appropriate in this case is the log link function. This is because the expected value μ_i is modeled as an exponential function of the predictors, which corresponds to a log link function in the context of a generalized linear model.

The log link function is defined as:

$$g(\mu) = ln(\mu)$$

where μ is the mean of the distribution function.

The inverse of the log link function, which is used to get the mean μ from the linear predictor η, is the exponential function:

$$\mu = exp(\eta)$$

So, if we have a linear predictor $\eta = \beta 0 + \beta 1 * x$, the mean μ would be calculated as:

$$\mu = exp(\beta 0 + \beta 1 * x)$$

This ensures that μ is always positive, as required for certain distributions like the Poisson or exponential distribution.

The derivative of the log link function, which is often needed for calculations like the score function or Fisher's information, is:

$$g'(\mu) = 1/\mu$$

c. The Exponential distribution is a special case of the Gamma distribution with shape parameter α = 1.

The probability density function (pdf) of an Exponential distribution is given by:

$$f(y;\ \theta) = \theta e^{-\theta y}\ for\ y >= 0, and\ 0\ otherwise.$$

$$where\ \theta > 0\ is\ the\ rate\ parameter.$$

1. **Expected Value (E(Y))**: The expected value of a random variable Y following an Exponential distribution is given by the integral of y times the pdf over the range of Y. Mathematically, this is:

$$E(Y) = \int_0^\infty yf(y;\ \theta)\ dy$$

Substituting the pdf into the integral gives:

$$E(Y) = \int_0^\infty y\theta e^{-\theta y}\ dy$$

This integral can be solved by applying integration by parts

$$(let\ u = y\ and\ dv = \theta e^{-\theta y}\ dy)$$

which gives:

$$E(Y) = 1/\theta$$

2. **Variance (var(Y))**: The variance of a random variable Y following an Exponential distribution is given by the expected value of the square of Y minus the square of the expected value of Y. Mathematically, this is:

$$var(Y) = E(Y^2) - (E(Y))^2$$

The expected value of Y^2 is given by the integral of y^2 times the pdf over the range of Y:

$$E(Y^2) = \int_0^\infty y^2\theta e^{-\theta y}\ dy$$

This integral can also be solved by applying integration by parts

$$(let\ u\ =\ y^2\ and\ dv\ =\ \theta e^{(-\theta y)}\ dy)$$

, which gives:

$$E(Y^2)\ =\ \frac{2}{\theta^2}$$

Substituting $E(Y)\ =\ \frac{1}{\theta}$ and $E(Y\hat{}2)\ =\frac{2}{\theta^2}$ into the formula for the variance gives:

$$var(Y)\ =\ \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2\ =\ \frac{1}{\theta^2}$$

So, for an Exponential distribution with rate parameter θ, the expected value is $1/\theta$ and the variance is $1/\theta\hat{}2$.

Question 5

Let's denote $\beta\ =\ exp(b)$ for simplicity. Then, we have $Yi \sim N(ln(\beta), \sigma\hat{}2)$, which means that $ln(Yi) \sim N(b, \sigma\hat{}2)$.

The likelihood function for a sample $Y1, \ldots, YN$ from a normal distribution is given by:

$$L(b; Y_1, \ldots, Y_N) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N e^{\frac{-\sum_{i=0}^{N}(Y_i - b)^2}{2\sigma^2}}$$

Taking the natural logarithm of the likelihood function gives the log-likelihood function:

$$lnL(b; Y_1, \ldots, Y_N) =\ -\frac{N}{2}\ln(2\pi\sigma^2) - \sum_{i=0}^{N}\frac{(Y_i - b)^2}{2\sigma^2}$$

1. **Maximum Likelihood Estimator (MLE)**: The MLE of b is obtained by taking the derivative of the log-likelihood function with respect to b, setting it equal to zero, and solving for b. This gives:

$$\frac{d}{db}lnL(b; Y_1, \ldots, Y_N) = \sum_{i=0}^{N}\frac{Y_i - b}{\sigma^2} = 0$$

Solving for b gives the MLE of b as:

$$\hat{b} = \sum_{i=0}^{N}\frac{Y_i}{N}$$

This is simply the sample mean of the log-transformed Y values.

2. **Score Function**: The score function is the derivative of the log-likelihood function with respect to the parameter. In this case, it is:

$$S(b) = \frac{d}{db} lnL(b; Y_1, \ldots, Y_N) = \sum_{i=0}^{N} \frac{Y_i - b}{\sigma^2}$$

3. **Estimating Equation**: The estimating equation is obtained by setting the score function equal to zero and solving for the parameter. This gives the same result as the MLE:

$$\sum_{i=0}^{N} \frac{Y_i - b}{\sigma^2} = 0, \qquad \hat{b} = \sum_{i=0}^{N} \frac{Y_i}{N}$$

4. **Information Matrix**: The information matrix is the negative of the expected value of the second derivative of the log-likelihood. In this case, it is:

$$I(b) = -E\left[\frac{d^2}{db^2} \ln\big(L(b; Y_1, \ldots, Y_N)\big)\right] = \frac{N}{\sigma^2}$$

This is because the second derivative of the log-likelihood with respect to b is $-N \;/\; \sigma$^2. The negative expectation of this is $N \;/\; \sigma$^2.