1) d) Interpret the slope and intercept of the regression line:

The slope represents the increase in weight (in pounds) for every 1-inch increase in height. The intercept represents the predicted weight (in pounds) when the height is 0 inches, which is not a meaningful value in this context.

2)a) To construct the method of moments estimator of θ, we will match the sample moments to the population moments. The first sample moment is the sample mean, which we denote as $\bar{X}$. We set the population mean μ equal to the sample mean $\bar{X}$ and solve for θ:

$$\bar{X} = \theta * sqrt(\pi/2)$$

$$\theta\_MM = \bar{X} / sqrt(\pi/2)$$

$$where$$

$$\theta\_MM \text{ } is \text{ } the \text{ } method \text{ } of \text{ } moments \text{ } estimator \text{ } for \text{ } \theta.$$

To determine if it is unbiased, we need to find the expected value of θ_MM:

$$E(\theta\_MM) = E(\bar{X} / sqrt(\pi/2))$$

Since the expected value of the sample mean $\bar{X}$ is equal to the population mean μ:

$$E(\theta\_MM) = \mu / sqrt(\pi/2)$$

Now, we substitute the expression for μ:

$$E(\theta\_MM) = (\theta * sqrt(\pi/2)) / sqrt(\pi/2)$$

$$E(\theta\_MM) = \theta$$

The expected value of the method of moments estimator is equal to the true parameter value, so the estimator is unbiased.

b) To construct a model-based estimator of the population variance, we first find the sample variance, denoted as s^2. Then, we set the population variance σ^2 equal to the sample variance s^2 and solve for θ:

$$s\hat{}2 = \theta\hat{}2 * (4 - \pi)/2$$

$$\theta\hat{}2 = 2 * s\hat{}2 / (4 - \pi)$$

$$\theta\_MB = sqrt(2 * s\hat{}2 / (4 - \pi))$$

$$where$$

$$\theta\_MB \text{ } is \text{ } the \text{ } model - based \text{ } estimator \text{ } for \text{ } \theta.$$

Now we will construct a model-based estimator for the population variance using the model-based estimator for θ:

$$\sigma\hat{}2\_MB = \theta\_MB\hat{}2 * (4 - \pi)/2$$

$$\sigma^2\_MB = (2 * s^2 / (4 - \pi)) * (4 - \pi)/2$$

$$\sigma^2\_MB = s^2$$

The model-based estimator for the population variance is equal to the sample variance. To check if it is unbiased, we need to find the expected value of σ^2_MB:

$$E(\sigma^2\_MB) = E(s^2)$$

The expected value of the sample variance s^2 is:

$$E(s^2) = [(n-1)/n] * \sigma^2$$

Therefore,

$$E(\sigma^2\_MB) = [(n-1)/n] * \sigma^2$$

The expected value of the model-based estimator for the population variance is not equal to the true population variance, so the estimator is biased.

3. A company manufacturing bike helmets wants to estimate the proportion p of helmets with a certain type of flaw. They decide to keep inspecting helmets until they find r = 5 flawed ones. Let X denote the number of helmets that were not flawed among those examined. a) Write the log-likelihood function and find the MLE of p. b) Find the method of moments estimator of p. c) If X = 47, give a numerical value to your estimators in (a) and (b)above.

This problem describes a negative binomial distribution. In this distribution, we are interested in finding the number of successes (non-flawed helmets) before observing r failures (flawed helmets). The probability mass function (PMF) of a negative binomial distribution is given by:

$$P(X = k) = C(k + r - 1, k) * p^k * (1 - p)^r$$

$$where$$

- $k\ is\ the\ number\ of\ successes,$
- $r\ is\ the\ number\ of\ failures,$
- $p\ is\ the\ probability\ of\ success, and$
- $C(n,k)\ is\ the\ binomial\ coefficient, which\ can\ be\ calculated\ as\ C(n,k) = n! / (k! * (n - k)!).$

a) To write the log-likelihood function, first write the likelihood function:

$$L(p) = C(X + r - 1, X) * p^X * (1 - p)^r$$

Now, take the natural logarithm of the likelihood function to get the log-likelihood function:

$$l(p) = log(C(X + r - 1, X)) + X * log(p) + r * log(1 - p)$$

To find the maximum likelihood estimator (MLE) of p, we need to maximize $l(p)$ with respect to p. Take the first derivative of l(p) with respect to p and set it equal to 0:

$$dl/dp = X/p - r/(1 - p) = 0$$

Solve for p:

$$X(1 - p) = rp$$

$$X - Xp = rp$$

$$X = p(X + r)$$

$$p\_MLE = X / (X + r)$$

b) To find the method of moments estimator of p, we first need to find the expected value and variance of the negative binomial distribution:

$$E(X) = r * (1 - p) / p\, Var(X) = r * (1 - p) / p\text{^}2$$

Now, match the sample mean (X) to the population mean $(E(X))$ and solve for p:

$$X = r * (1 - p) / p$$

$$p\_MM = r / (X + r)$$

c) If X = 47, we can find the numerical values for the estimators in (a) and (b) using the formulas derived above:

$$p\_MLE = X / (X + r) = 47 / (47 + 5) = 47 / 52 \approx 0.9038$$

$$p\_MM = r / (X + r) = 5 / (47 + 5) = 5 / 52 \approx 0.0962$$

So, the maximum likelihood estimator for p is approximately 0.9038, and the method of moments estimator for p is approximately 0.0962.

4) Let X1, ..., Xn be i.i.d Poisson(λ). a) Find the maximum likelihood estimator of λ. b) The number of surface imperfections for a random sample of 50 metal plates are summarized in the following table: Number of Scratches per Item 0 1 2 3 4 Frequency 4 12 11 14 9 Assuming that the imperfection counts have the Poisson(λ) distribution, compute the maximum likelihood estimate of λ. c) Give the model-based estimate of the population variance, and com- pare it with the sample variance. Assuming the Poisson model correctly describes the population distribution, which of the two estimates would you prefer and why?

a) Let $X1, \ldots, Xn$ be i.i.d Poisson(λ). The likelihood function for the Poisson distribution is given by:

$$L(\lambda) = \Pi\, P(Xi = xi) = \Pi\, (e\text{^}(-\lambda) * (\lambda\text{^}(xi)) / xi!)$$

where xi are the observed values of the random variables Xi. To find the maximum likelihood estimator (MLE) of λ, we take the natural logarithm of the likelihood function to get the log-likelihood function:

$$l(\lambda) = \Sigma[log(e\text{^}(-\lambda) * (\lambda\text{^}(xi)) / xi!)] = \Sigma[-\lambda + xi * log(\lambda) - log(xi!)]$$

Now, to find the MLE, we need to maximize l(λ) with respect to λ. Take the first derivative of l(λ) with respect to λ and set it equal to 0:

$$dl/d\lambda = \Sigma[-1 + xi/\lambda] = 0$$

Solve for λ:

$$\Sigma\, xi = \lambda * n$$

$$\lambda\_MLE = (\Sigma\, xi) / n$$

b) Given the number of surface imperfections and their frequencies, we can compute the maximum likelihood estimate of λ using the formula derived above:

Number of Scratches (xi): 0, 1, 2, 3, 4 Frequency (fi): 4, 12, 11, 14, 9

$$\Sigma\,(xi * fi) = (0 * 4) + (1 * 12) + (2 * 11) + (3 * 14) + (4 * 9) = 94$$

Total number of samples (n) = 4 + 12 + 11 + 14 + 9 = 50

$$\lambda\_MLE = (\Sigma\, xi * fi) / n = 94 / 50 = 1.88$$

c) The model-based estimate of the population variance for a Poisson distribution is equal to its mean, i.e., λ. So, the model-based estimate of the population variance is:

$$Var(\lambda\_MB) = \lambda\_MLE = 1.88$$

Now, let's calculate the sample variance:

Sample mean $(\bar{X}) = (\Sigma\, xi * fi) / n = 94 / 50 = 1.88$

Sample variance $(s\text{^}2) = (\Sigma\, fi * (xi - \bar{X})\text{^}2) / (n - 1)$

$$s\text{^}2 = (4 * (0 - 1.88)\text{^}2 + 12 * (1 - 1.88)\text{^}2 + 11 * (2 - 1.88)\text{^}2 + 14 * (3 - 1.88)\text{^}2 + 9 * (4 - 1.88)\text{^}2) / (50 - 1)$$

$$s\text{^}2 \approx 1.9459$$

The model-based estimate of the population variance is 1.88, and the sample variance is approximately 1.9459. If we assume the Poisson model correctly describes the population distribution, we would prefer the model-based estimate because it takes into account the underlying distribution and its properties. The sample variance might be affected by the specific sample and not fully reflect the properties of the Poisson distribution.