Lecture 8: Hypothesis Testing (contd.)

Defn (Best Critical Region of size α)

Consider the test of the simple null hypothesis

$$H_0: \theta = \theta_0 \quad vs \quad H_a: \theta = \theta_1$$

Let C be a critical region of size α, i.e. $\alpha = P[(X_1, X_2, \ldots, X_n) \in C; \theta_0]$. Then C is a <u>best critical region of size α</u> if, for every other critical region D of size α, i.e. $\alpha = P[(X_1, X_2, X_3, \ldots, X_n) \in D; \theta_0]$, we have

$$P[(X_1, X_2, \ldots, X_n) \in C; \theta_1] \geq P((X_1, X_2, \ldots, X_n) \in D; \theta_0)$$

That is, when $H_a: \theta = \theta_1$ is true, the probability of rejecting $H_0: \theta = \theta_0$ with the use of the critical region C is at least as great as the corresponding probability with the use of any other critical region D of size α.

Example 1:

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Poisson distribution with mean λ. A BCR for testing

$H_0: \lambda = 2$ vs $H_a: \lambda = 5$ is given by

$$\frac{L(\lambda=5)}{L(\lambda=2)} = \frac{5^{\sum x_i} e^{-5n} \; x_1! x_2! \ldots x_n!}{2^{\sum x_i} e^{-2n} \; x_1! x_2! \ldots x_n!} > k$$

$$\Rightarrow \left(\frac{5}{2}\right)^{\sum x_i} e^{-3n} > k$$

$$\Rightarrow \sum x_i \ln(5/2) - 3n > \ln k$$

$$\Rightarrow \sum x_i > \frac{\ln k + 3n}{\ln(5/2)} = c$$

Defn (Uniformly Most Powerful Test)

A test defined by a critical region C of size α is a uniformly most powerful test (UMPT) if it is a most powerful test against each simple alternative in H_a. The critical region C is called a uniformly most powerful critical region of size α.

Example 2

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, 36)$. Consider $H_0: \mu = 50$ vs $H_a: \mu = 55$

A BCR of size α is given by

$C = \{(x_1, x_2, \ldots, x_n) : \bar{x} > c\}$, where c

is selected so that the significance level

is α. Now consider testing

$H_0: \mu = 50$ vs $H_a: \mu > 50$

For each simple hypothesis $H_a: \mu = \mu_1 > 50$

$$\frac{L(\mu = \mu_1)}{L(\mu = 50)} = \exp\left\{ \frac{-1}{72}\left(2\sum x_i(50 - \mu_1) + n(\mu_1^2 - 50^2) \right) \right\} > k$$

$$\Rightarrow -\frac{1}{72}\left(2\sum x_i (50 - \mu_1) + n(\mu_1^2 - 50^2) \right) > \ln k$$

$$\Rightarrow \bar{x} > \frac{72 \ln k}{2n(\mu_1 - 50)} + \frac{\mu_1 - 50}{2} = c.$$

$\therefore \bar{x} > c$

Exercise

Let $X_1, X_2, \dots, X_n$ be a random sample of Bernoulli trials $b(1, p)$.

a) Show that a BCR for testing $H_0: p = 0.9$ against $H_a: p = 0.8$ can be based on the statistic $Y = \sum_{i=1}^{n} X_i$, which is distributed as $b(n, p)$

b) If $C = \left\{ (x_1, x_2, \dots, x_n) : \sum_{1}^{n} x_i \le n(0.85) \right\}$ and $Y = \sum_{1}^{n} X_i$, find the value of n such that $\alpha = P[Y \le n(0.85);\ p = 0.9] \approx 0.10$

(Hint: Use the normal approximation for the binomial distribution)

c) What is the approximate value of $\beta = P[Y > n(0.85);\ p = 0.8]$ for the test given in part (b)?

d) Is the test of part (b) a uniformly most powerful test when the alternative is $H_a: p < 0.9$?

Likelihood Ratio Tests

A more general test-construction method applicable when either or both of the null and alternative are <u>composite</u>.

Let Ω denote the total parameter space.

Let $H_0: \theta \in \omega$ vs $H_a: \theta \in \omega'$

<u>Defn (Likelihood Ratio):</u>

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

where $L(\hat{\omega}) = \max L$ under H_0

and $L(\hat{\Omega}) = \max L$ when $\theta \in \Omega$.

Note that $0 \le \lambda \le 1$. Small values of λ would lead to the rejection of H_0 and values close to 1 would support H_0.

Defn (Critical Region for LRT)

The set of points in the sample space for which $\lambda \le k$, where $0 < k < 1$. Here k is selected so that a desired significance level α is achieved.

Example 3

Let $X \sim N(\mu, 5)$ and $H_0: \mu = 162$ vs $H_a: \mu \neq 162$

$$\Omega = \{\mu : -\infty < \mu < \infty\} \text{ and } \omega = \{162\}$$

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{L(162)}{L(\bar{x})}, \text{ since } \hat{\mu} = \bar{X}$$

$$= \frac{\exp\left[-\left(\frac{1}{10}\right)\sum(x_i - \bar{x})^2 - \left(\frac{n}{10}\right)(\bar{x} - 162)^2\right]}{\exp\left[-\left(\frac{1}{10}\right)\sum(x_i - \bar{x})^2\right]}$$

$$= \exp\left[-\left(\frac{n}{10}\right)(\bar{x} - 162)^2\right] \le k$$

$$\Rightarrow -\left(\frac{n}{10}\right)(\bar{x} - 162)^2 \le \ln k$$

$$\Rightarrow \frac{|\bar{x} - 162|}{\sqrt{5/n}} \ge \frac{\sqrt{-(10/n)\ln k}}{\sqrt{5/n}} = c$$

Since $Z = (\bar{X}-162)/\sqrt{5/n} \sim N(0,1)$ under H_0,

let $c = z_{\alpha/2}$. Thus, $C = \left\{ \bar{x} : \frac{|\bar{x}-162|}{\sqrt{5/n}} \geq z_{\alpha/2} \right\}$

For $\alpha = 0.05$, $z_{\alpha/2} = 1.96$.

Now, suppose $X \sim N(\mu, \sigma^2)$, (μ, σ^2 unknown).

To test $H_0: \mu = \mu_0$ vs $H_a: \mu \neq \mu_0$,

define $\omega = \{(\mu, \sigma^2) : \mu = \mu_0, 0 < \sigma^2 < \infty\}$

$\Omega = \{(\mu, \sigma^2) : -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$

$$L(\hat{\Omega}) = \left(\frac{n e^{-1}}{2\pi \sum (x_i - \bar{x})^2} \right)^{n/2}$$

$$L(\hat{\omega}) = \left(\frac{n e^{-1}}{2\pi \sum (x_i - \mu_0)^2} \right)^{n/2}$$

$$\Rightarrow \lambda = L(\hat{\omega})/L(\hat{\Omega}) = \left(\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \mu_0)} \right)^{n/2}$$

$$= \left[\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2} \right]^{n/2}$$

$$= \left[\frac{1}{1 + \dfrac{n(\bar{x}-\mu_0)^2}{\sum (x_i-\bar{x})^2}} \right]^{n/2}$$

$$\Rightarrow \frac{1}{1 + \dfrac{n(\bar{x}-\mu_0)^2}{\sum (x_i-\bar{x})^2}} \leq k^{2/n}$$

$$\Rightarrow \frac{n(\bar{x}-\mu_0)^2}{\frac{1}{n-1}\sum (x_i-\bar{x})^2} \geq (n-1)(k^{-2/n}-1)$$

Under H_0, $\sqrt{n}(\bar{X}-\mu_0)/\sigma \sim N(0,1)$

and $\sum (X_i-\bar{X})^2/\sigma^2 \sim \chi^2(n-1)$

$$\therefore \; T = \frac{n(\bar{X}-\mu_0)/\sigma}{\sqrt{\sum (X_i-\bar{X})^2/(n-1)\sigma^2}} = \frac{\bar{X}-\mu_0}{S/\sqrt{n}}$$

has a $t(n-1)$ distribution

So we reject H_0 if $T^2 \geq (n-1)(k^{-2/n}-1)$

i.e $|T| \geq t_{\alpha/2}(n-1)$.

Exercise (fill in missing steps)

Let $X \sim N(\mu, \sigma^2)$, where both μ and σ^2 are unknown.

Let $H_0: \sigma^2 = \sigma_0^2$ vs $H_a: \sigma^2 \neq \sigma_0^2$

Here $\omega = \{(\mu, \sigma^2): -\infty < \mu < \infty,\ \sigma^2 = \sigma_0^2\}$

$\Omega = \{(\mu, \sigma^2): -\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}$

$$L(\hat{\Omega}) = \left(\frac{n e^{-1}}{2\pi \sum (x_i - \bar{x})^2}\right)^{n/2}$$

$$L(\hat{\omega}) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} \exp\left[-\frac{\sum (x_i - \bar{x})^2}{2\sigma_0^2}\right]$$

$$\lambda = L(\hat{\omega}) / L(\hat{\Omega})$$

$$= \left(\frac{v}{n}\right)^{n/2} \exp\left(-\frac{v}{2} + \frac{n}{2}\right) \leq k \quad \text{———} \quad (1)$$

where $v = \sum (x_i - \bar{x})^2 / \sigma_0^2$

Solving (1) for $v \Rightarrow v \leq c_1$ or $v \geq c_2$

But $V = \sum_1^n (X_i - \bar{X})^2 / \sigma_0^2 \sim \chi^2(n-1)$ under H_0.

Thus $c_1 = \chi^2_{1-\alpha/2}(n-1)$ and $c_2 = \chi^2_{\alpha/2}(n-1)$.

Exercise

Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with mean θ. Show that the likelihood ratio test of

$$H_0: \theta = \theta_0 \quad \text{vs} \quad H_a: \theta \neq \theta_0$$

has a critical region of the form

$$\sum_1^n x_i \le c_1 \quad \text{or} \quad \sum_1^n x_i \geq c_2 .$$

How would you modify this test so that chi-square tables can be used easily?