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Derive Score and Hessian for the Cauchy distribution:

1. Likelihood Function: Given a random sample $Y1, \ldots, Yn$ from the Cauchy distribution, the likelihood function $L(\theta;\, y)$ is the product of the probability density functions (pdf) for each observation:

$$L(\theta;\, y) = \prod_{i=1}^{n} f(y_i; \theta) = \prod_{i=1}^{n} \frac{1}{\pi(1 + (y_i - \theta)^2)}$$

2. Log-likelihood Function: Taking the natural logarithm of the likelihood function gives the log-likelihood function $l(\theta;\, y)$:

$$l(\theta;\, y) = ln(L(\theta;\, y)) = \sum_{i=1}^{n} \ln(f(y_i; \theta) = -nln(\pi) - \sum_{i=1}^{n} \ln(1 + (y_i - \theta)^2)$$

3. Score Function: The score function is the gradient (first derivative) of the log-likelihood function with respect to the parameter θ. Let's differentiate the log-likelihood:

$$S(\theta;\, y) = \frac{\partial l(\theta;\, y)}{\partial \theta} = \sum_{i=1}^{n} \frac{-2(y_i - \theta)}{(1 + (y_i - \theta)^2)}$$

4. Hessian Matrix: The Hessian matrix is the second derivative of the log-likelihood function, which in this case will be a scalar because we only have one parameter (θ). Let's differentiate the score function:

$$H(\theta;\, y) = \frac{\partial^2 \, l(\theta;\, y)}{\partial \theta^2} = \frac{\partial(S(\theta;\, y))}{\partial \theta} = \sum_{i=1}^{n} \frac{2(1 - (yi - \theta)^2)}{(1 + (yi - \theta)\text{^}2)^2}$$

So, the score function and the Hessian matrix provide important information about the shape of the log-likelihood function, which is used to estimate the parameter θ. The score function (gradient) gives the direction of the steepest ascent, while the Hessian provides information about the curvature of the function.