A Mean-Field Linear-Quadratic Leader-Follower Stochastic Differential Game

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Abstract: In this paper, a linear-quadratic leader-follower (LQLF) differential game is considered, where the game system is governed by a mean-field stochastic differential equation (MF-SDE). By stochastic maximum principle, the optimal solution to the LF stochastic differential game is expressed as a feedback form of the state and its mean with the aid of two systems of Riccati equations.

Key Words: MF-SDE, LQLF game, Feedback optimal control, Riccati equation.

1 INTRODUCTION

Let a standard Brownian motion $W(\cdot)$ valued in R is defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the natural filtration generated by $W(\cdot)$, and $T>0$ is a finite time duration. The argument of the functions will be simplified such that no confusion can induce from the context. For example, we denote $A(\cdot)$ by A. We let state $x^{v_1,v_2}(\cdot)$ satisfy the following controlled linear SDE

$$\begin{cases} dx^{v_1,v_2}(t) = \{A\,(t)\,x^{v_1,v_2}(t) + \bar{A}(t)E[x^{v_1,v_2}(t)] \\ \qquad\qquad + B_1\,(t)\,v_1\,(t) + B_2\,(t)\,v_2\,(t)\}dt \\ \qquad\qquad + \{C\,(t)\,x^{v_1,v_2}(t) \\ \qquad\qquad + \bar{C}\,(t)\,E[x^{v_1,v_2}(t)]\}dW\,(t)\,, \\ x^{v_1,v_2}(0) = x_0, \end{cases} \tag{1}$$

where $A, \bar{A}, B_1, B_2, C, \bar{C}$ are given deterministic and uniformly bounded functions. In the above equation, $x^{v_1,v_2}(\cdot)$ is the state process, with value in R. the follower's control is $v_1(\cdot)$, the leader's control is $v_2(\cdot)$, $v_1(\cdot)$ and $v_2(\cdot)$ take values in R. Let $\mathcal{V}_1[0,T] \triangleq L^2_{\mathcal{F}}(0,T;R)$ be the set of all $\{\mathcal{F}_t\}_{0\leq t\leq T}$-adapted and R-valued processes $v_1 : [0,T]\times\Omega \to R$ so that $E\int_0^T |v_1(t)|^2 dt < \infty$. The set $\mathcal{V}_2[0,T] \triangleq L^2_{\mathcal{F}}(0,T;R)$ is defined similarly. For given $v_2(\cdot) \in \mathcal{V}_2[0,T]$, the follower wishes to choose a $v_1^*(\cdot) \in \mathcal{V}_1[0,T]$ to minimize his cost functional

$$\begin{aligned} J_1(v_1(\cdot), v_2(\cdot)) = \frac{1}{2}E\Big[\int_0^T (Q_1(t)|x^{v_1,v_2}(t)|^2 \\ + \bar{Q}_1(t)|Ex^{v_1,v_2}(t)|^2 + R_1(t)|v_1(t)|^2)dt \\ + G_1|x^{v_1,v_2}(T)|^2 + \bar{G}_1|Ex^{v_1,v_2}(T)|^2\Big]. \end{aligned} \tag{2}$$

Then, considering that the follower will take $v_1^*(\cdot)$, the leader expects to find a $v_2(\cdot) \in \mathcal{V}_2[0,T]$ to minimize his cost functional

$$\begin{aligned} J_2(v_1^*(\cdot), v_2(\cdot)) = \frac{1}{2}E\Big[\int_0^T (Q_2(t)|x^{v_1^*,v_2}(t)|^2 \\ + \bar{Q}_2(t)|Ex^{v_1^*,v_2}(t)|^2 + R_2(t)|v_2(t)|^2)dt \\ + G_2|x^{v_1^*,v_2}(T)|^2 + \bar{G}_2|Ex^{v_1^*,v_2}(T)|^2\Big]. \end{aligned} \tag{3}$$

Here $Q_1, \bar{Q}_1, Q_2, \bar{Q}_2 \geq 0$, and $R_1, R_2 > 0$ are given deterministic and continuous functions; $G_1, \bar{G}_1, G_2, \bar{G}_2 \geq 0$ are constants. We observe that $E[x(\cdot)]$ emerges in the state equation and the cost functionals, such a problem is regarded as a mean-field linear-quadratic leader-follower (MFLQLF) stochastic differential game. In case an optimal control pair $(v_1^*(\cdot), v_2^*(\cdot))$ exists, we call it as an open-loop solution of the game.

The MFLQLF stochastic differential game model has extensive applications in physics, economics, finance and sociology, such as cooperative advertising problems, principal-agent and pricing problems.

The LF differential game is also named Stackelberg differential game, which was originally introduced by Stackelberg [1]. He defined the notion of a Stackelberg equlibrium in the context of static economic markets equlibrium. After then a lot of scholars focus attention on LF games. An LQLF differential game, where the diffusion does not include the state and control variables, was investigated by Bagchi and Basar [2]. Yong [3] considered a more general case with the state equation being a Itô's-type SDE and its diffusion containing the controls, and obtained an open-loop Stackelberg equilibrium. Bensousan, Chen and Sethi [4] derived global maximum principles with regard to open-loop and closed-loop LF differential games. Shi, Wang and Xiong [5] studied an LF differential game, where players have asymmetric characters, state feedback Stackelberg equilibrium was obtained, and an LQLF differential game with asymmetric information feature is originally introduced and addressed. Related research on LF differential games with asymmetric

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information feature, can be seen in [6–8]. Xu and Zhang [9] investigated a time-delay LF differential game, and obtained optimality condition of Stackelberg equilibrium. Whereas, the above literatures do not refer to the LF game with mean-field type. Motivated by this, we condider a MFLQLF stochastic differential game in this paper, which embodies the difference from the above literatures. The problem is new in LF differential game theory and has not been investigated in the previous literature.

Our paper is organized as follows. In Section 2, we give the main result. The state feedback form of optimal controls for the leader and the follower are explicitly obtained by two systems of Riccati equations. In Section 3, we give concluding remarks.

2 MAIN RESULT

2.1 MFLQ problem of the follower

Define the follower's Hamiltonian function

$$\begin{aligned}
&H_1 : [0,T]\times R\times R\times R\times R\times R\times R \to R \text{ by}\\
&H_1(t,x,\bar{x},v_1,v_2,h,k)\\
&= h[A(t)x+\bar{A}\bar{x}+B_1(t)v_1+B_2(t)v_2]\\
&+ k[C(t)x+\bar{C}(t)\bar{x}]\\
&- \frac{1}{2}Q_1(t)x^2-\frac{1}{2}\bar{Q}_1(t)\bar{x}^2-\frac{1}{2}R_1(t)v_1^2.
\end{aligned} \tag{4}$$

For any given $v_2(\cdot)$, assume that the follower has an optimal control $v_1^*(\cdot)$, and $x^{v_1^*,v_2}(\cdot)$ is the optimal state corresponding to $v_1^*(\cdot)$. By the maximum principle of MF-SDEs (see [10, 11]), (4) yields that

$$R_1v_1^*(t)-B_1h(t)=0, \tag{5}$$

where $\mathcal{F}_t$-adapted process $(h(\cdot),k(\cdot))$ is the unique solution to

$$\begin{cases}
-dh(t)=\{Ah(t)+\bar{A}E[h(t)]+Ck(t)+\bar{C}E[k(t)]\\
\qquad -Q_1x^{v_1^*,v_2}(t)-\bar{Q}_1E[x^{v_1^*,v_2}(t)]\}dt\\
\qquad -k(t)dW(t),\\
h(T)=-G_1x^{v_1^*,v_2}(T)-\bar{G}_1E[x^{v_1^*,v_2}(T)].
\end{cases} \tag{6}$$

See [12] for more details about the existence and uniqueness of solution to MF-BSDE. We except to derive the feedback form of $v_1^*(\cdot)$. Observing the terminal condition of (6) and the emergence of $v_2(\cdot)$, we put

$$h(t)=-P(t)x^{v_1^*,v_2}(t)-Q(t)E[x^{v_1^*,v_2}(t)]-\varphi(t), \tag{7}$$

where $P(\cdot),Q(\cdot)$ are both deterministic and differentiable functions, with values in R. $\varphi(\cdot)$ is an R-valued, $\mathcal{F}_t$-adapted process and satisfies the BSDE

$$\begin{cases}
d\varphi(t)=\gamma(t)dt+\eta(t)dW(t),\\
\varphi(T)=0.
\end{cases} \tag{8}$$

In the above equation, $\mathcal{F}_t$-adapted processes $\gamma(\cdot)\in R$ and $\eta(\cdot)\in R$ will be determined later.

Applying Itô's formula to (7), we derive

$$\begin{aligned}
dh(t)=&-\{(\dot{P}+PA)x^{v_1^*,v_2}(t)\\
&+(\dot{Q}+\bar{A}P+AQ+\bar{A}Q)E[x^{v_1^*,v_2}(t)]\\
&+B_1Pv_1^*(t)+B_1QE[v_1^*(t)]\\
&+B_2Pv_2(t)+B_2QE[v_2(t)]+\gamma(t)\}dt\\
&-\{CPx^{v_1^*,v_2}(t)+\bar{C}PE[x^{v_1^*,v_2}(t)]+\eta(t)\}dW(t).
\end{aligned} \tag{9}$$

Comparing (6) with (9), we get

$$k(t)=-CPx^{v_1^*,v_2}(t)-\bar{C}PE[x^{v_1^*,v_2}(t)]-\eta(t), \tag{10}$$

and

$$\begin{aligned}
-\gamma(t)=&(\dot{P}+2AP+Q_1)x^{v_1^*,v_2}(t)\\
&+(\dot{Q}+2AQ+2\bar{A}Q+2\bar{A}P+\bar{Q}_1)E[x^{v_1^*,v_2}(t)]\\
&+B_1Pv_1^*(t)+B_1QE[v_1^*(t)]\\
&+B_2Pv_2(t)+B_2QE[v_2(t)]\\
&+A\varphi(t)+\bar{A}E[\varphi(t)]-Ck(t)-\bar{C}E[k(t)].
\end{aligned} \tag{11}$$

Plugging (7) into (5), we obtain

$$v_1^*(t)=-R_1^{-1}B_1(Px^{v_1^*,v_2}(t)+QE[x^{v_1^*,v_2}(t)]+\varphi(t)). \tag{12}$$

Substituting (10), (12) into (11), we can get that if

$$\begin{cases}
\quad 0=\dot{P}+(2A+C^2)P-B_1^2R_1^{-1}P^2+Q_1,\\
\quad 0=\dot{Q}+2(A+\bar{A}-B_1^2R_1^{-1}P)Q-B_1^2R_1^{-1}Q^2\\
\qquad +(2\bar{A}+2C\bar{C}+\bar{C}^2)P+\bar{Q}_1,\\
P(T)=G_1, Q(T)=\bar{G}_1,\\
\quad B_1\neq 0,\quad (2\bar{A}+2C\bar{C}+\bar{C}^2)P\geq 0
\end{cases} \tag{13}$$

admits a unique solution pair $(P(\cdot),Q(\cdot))$, then

$$\begin{aligned}
-\gamma(t)=&(A-B_1^2R_1^{-1}P)\varphi(t)+(\bar{A}-B_1^2R_1^{-1}Q)E[\varphi(t)]\\
&+C\eta(t)+\bar{C}E[\eta(t)]\\
&+B_2Pv_2(t)+B_2QE[v_2(t)].
\end{aligned} \tag{14}$$

Since (13) is decoupled, so it gets a unique solution pair $(P(\cdot),Q(\cdot))$, by the standard Riccati equation theory [13]. With (14), BSDE (8) takes the form

$$\begin{cases}
-d\varphi(t)=\{(A-B_1^2R_1^{-1}P)\varphi(t)\\
\qquad +(\bar{A}-B_1^2R_1^{-1}Q)E[\varphi(t)]\\
\qquad +C\eta(t)+\bar{C}E[\eta(t)]\\
\qquad +B_2Pv_2(t)+B_2QE[v_2(t)]\}dt\\
\qquad -\eta(t)dW(t),\\
\varphi(T)=0.
\end{cases} \tag{15}$$

Let us put (1) and (15) together. It is worthwhile to note that the equations of $(x^{v_1,v_1}(\cdot),\varphi(\cdot))$ are a decoupled MF-FBSDE. Therefore, they admit an $\mathcal{F}_t$-adapted solution $(x^{v_1,v_2}(\cdot),\varphi(\cdot),\eta(\cdot))$.

According to the above arguments, we have the following theorem.

Theorem 2.1 Let $(P(\cdot), Q(\cdot))$ satisfy (13). For given $v_2(\cdot)$ for the leader, optimal control $v_1^*(\cdot)$ obtained by (12) is the state feedback form for the follower's problem, where $(x^{v_1^*,v_2}(\cdot), \varphi(\cdot), \eta(\cdot))$ is the unique $\mathcal{F}_t$-adapted solution to

$$\begin{cases} dx^{v_1^*,v_2}(t) = \{(A - B_1^2R_1^{-1}P)x^{v_1^*,v_2}(t) \\ \qquad + (\bar{A} - B_1^2R_1^{-1}Q)E[x^{v_1^*,v_2}(t)] \\ \qquad - B_1^2R_1^{-1}\varphi(t) + B_2v_2(t)\}dt \\ \qquad + \{Cx^{v_1^*,v_2}(t) + \bar{C}E[x^{v_1^*,v_2}(t)]\}dW(t), \\ -d\varphi(t) = \{(A - B_1^2R_1^{-1}P)\varphi(t) \\ \qquad + (\bar{A} - B_1^2R_1^{-1}Q)E[\varphi(t)] \\ \qquad + C\eta(t) + \bar{C}E[\eta(t)] \\ \qquad + B_2Pv_2(t) + B_2QE[v_2(t)]\}dt \\ \qquad - \eta(t)dW(t), \\ x^{v_1^*,v_2}(0) = x_0,\ \varphi(T) = 0. \end{cases} \tag{16}$$

2.2 MFLQ problem of the leader

Considering that the follower is willing to adopt his optimal control $v_1^*(\cdot)$ of form (12), the leader's state is as follows

$$\begin{cases} dx^{v_2}(t) = \{(A - B_1^2R_1^{-1}P)x^{v_2}(t) \\ \qquad + (\bar{A} - B_1^2R_1^{-1}Q)E[x^{v_2}(t)] \\ \qquad - B_1^2R_1^{-1}\varphi(t) + B_2v_2(t)\}dt \\ \qquad + \{Cx^{v_2}(t) + \bar{C}E[x^{v_2}(t)]\}dW(t), \\ -d\varphi(t) = \{(A - B_1^2R_1^{-1}P)\varphi(t) \\ \qquad + (\bar{A} - B_1^2R_1^{-1}Q)E[\varphi(t)] \\ \qquad + C\eta(t) + \bar{C}E[\eta(t)] \\ \qquad + B_2Pv_2(t) + B_2QE[v_2(t)]\}dt \\ \qquad - \eta(t)dW(t), \\ x^{v_2}(0) = x_0,\ \varphi(T) = 0, \end{cases} \tag{17}$$

where $x^{v_2}(\cdot) \equiv x^{v_1^*,v_2}(\cdot)$.
The leader is willing to find an $\mathcal{F}_t$-adapted optimal control $v_2^*(\cdot)$ to minimize his cost functional

$$\begin{aligned} J_2(v_2(\cdot)) = \frac{1}{2}E\Big[\int_0^T (Q_2(t)|x^{v_2}(t)|^2 \\ + \bar{Q}_2(t)|Ex^{v_2}(t)|^2 + R_2(t)|v_2(t)|^2)dt \\ + G_2|x^{v_2}(T)|^2 + \bar{G}_2|Ex^{v_2}(T)|^2\Big]. \end{aligned} \tag{18}$$

Define the leader's Hamiltonian function

$$\begin{aligned} &H_2 : [0,T] \times R \times R \times R \times R \times R \times R \times R \times R \\ &\times R \times R \times R \to R \text{ by} \\ &H_2(t, x, \bar{x}, v_2, \bar{v}_2, \varphi, \bar{\varphi}, \eta, \bar{\eta}, q, y, z) \\ &= y[(A - B_1^2R_1^{-1}P)(t)x^{v_2} + (\bar{A} - B_1^2R_1^{-1}Q)(t)\bar{x}^{v_2} \\ &- B_1^2R_1^{-1}(t)\varphi + B_2(t)v_2] \\ &+ q[(A - B_1^2R_1^{-1}P)(t)\varphi + (\bar{A} - B_1^2R_1^{-1}Q)(t)\bar{\varphi} \\ &+ C(t)\eta + \bar{C}\bar{\eta} + B_2P(t)v_2 + B_2Q(t)\bar{v}_2] \\ &+ z[C(t)x^{v_2} + \bar{C}(t)\bar{x}^{v_2}] \\ &+ \frac{1}{2}Q_2(t)|x^{v_2}|^2 + \frac{1}{2}\bar{Q}_2(t)|\bar{x}^{v_2}|^2 + \frac{1}{2}R_2(t)v_2^2. \end{aligned} \tag{19}$$

Assume that the leader has an optimal control $v_2^*(\cdot)$, and $(x^*(\cdot), \varphi^*(\cdot), \eta^*(\cdot)) \equiv (x^{v_2^*}(\cdot), \varphi^*(\cdot), \eta^*(\cdot))$ is the optimal state corresponding to $v_2^*(\cdot)$. According to the stochastic maximum principle of MF-FBSDE [14], (19) yields that

$$R_2(t)v_2^*(t) + B_2y(t) + B_2Pq(t) + B_2QE[q(t)] = 0, \tag{20}$$

where $(q(\cdot), y(\cdot), z(\cdot))$ is the $\mathcal{F}_t$-adapted process and satisfies

$$\begin{cases} dq(t) = \{(A - B_1^2R_1^{-1}P)q(t) - B_1^2R_1^{-1}y(t) \\ \qquad + (\bar{A} - B_1^2R_1^{-1}Q)E[q(t)]\}dt \\ \qquad + \{Cq(t) + \bar{C}E[q(t)]\}dW(t), \\ -dy(t) = \{Q_2(t)x^*(t) + \bar{Q}_2(t)E[x^*(t)] \\ \qquad + (A - B_1^2R_1^{-1}P)y(t) \\ \qquad + (\bar{A} - B_1^2R_1^{-1}Q)E[y(t)] \\ \qquad + Cz(t) + \bar{C}E[z(t)]\}dt - z(t)dW(t), \\ q(0) = 0,\ y(T) = G_2x^*(T) + \bar{G}_2E[x^*(T)]. \end{cases} \tag{21}$$

We wish to obtain some feedback expression for the optimal control $v_2^*(\cdot)$ and $v_1^*(\cdot)$. For this target, we will consider the $(x^*(\cdot), q(\cdot))^T$ as the optimal state, set

$$\begin{aligned} &X = \begin{pmatrix} x^* \\ q \end{pmatrix},\ Y = \begin{pmatrix} y \\ \varphi^* \end{pmatrix}, \\ &Z = \begin{pmatrix} z \\ \eta^* \end{pmatrix},\ X_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ &\mathcal{F} = \begin{pmatrix} A - B_1^2R_1^{-1}P & 0 \\ 0 & A - B_1^2R_1^{-1}P \end{pmatrix}, \\ &\bar{\mathcal{F}} = \begin{pmatrix} \bar{A} - B_1^2R_1^{-1}Q & 0 \\ 0 & \bar{A} - B_1^2R_1^{-1}Q \end{pmatrix}, \\ &\mathcal{M} = \begin{pmatrix} 0 & -B_1^2R_1^{-1} \\ -B_1^2R_1^{-1} & 0 \end{pmatrix}, \mathcal{S}_1 = \begin{pmatrix} B_2 \\ 0 \end{pmatrix}, \\ &\mathcal{S}_2 = \begin{pmatrix} 0 \\ B_2P \end{pmatrix}, \bar{\mathcal{S}}_2 = \begin{pmatrix} 0 \\ B_2Q \end{pmatrix}, \\ &\mathcal{D} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \bar{\mathcal{D}} = \begin{pmatrix} \bar{C} & 0 \\ 0 & \bar{C} \end{pmatrix}, \\ &\mathcal{O} = \begin{pmatrix} Q_2 & 0 \\ 0 & 0 \end{pmatrix}, \bar{\mathcal{O}} = \begin{pmatrix} \bar{Q}_2 & 0 \\ 0 & 0 \end{pmatrix}, \\ &\mathcal{N} = \begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix}, \bar{\mathcal{N}} = \begin{pmatrix} \bar{G}_2 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{22}$$

Then (17) with (21) is rewritten as

$$\begin{cases} dX(t) = \{\mathcal{F}X(t) + \bar{\mathcal{F}}E[X(t)] \\ \qquad + \mathcal{M}Y(t) + \mathcal{S}_1v_2^*(t)\}dt \\ \qquad + \{\mathcal{D}X(t) + \bar{\mathcal{D}}E[X(t)]\}dW(t), \\ -dY(t) = \{\mathcal{O}X(t) + \bar{\mathcal{O}}E[X(t)] \\ \qquad + \mathcal{F}^TY(t) + \bar{\mathcal{F}}^TE[Y(t)] \\ \qquad + \mathcal{D}^TZ(t) + \bar{\mathcal{D}}^TE[Z(t)] \\ \qquad + \mathcal{S}_2v_2^*(t) + \bar{\mathcal{S}}_2E[v_2^*(t)]\}dt \\ \qquad - Z(t)dW(t), \\ X(0) = X_0,\ Y(T) = \mathcal{N}X(T) + \bar{\mathcal{N}}E[X(T)]. \end{cases} \tag{23}$$

From (20), we have

$$R_2(t)v_2^*(t) + \mathcal{S}_1^TY(t) + \mathcal{S}_2^TX(t) + \bar{\mathcal{S}}_2^TE[X(t)] = 0. \tag{24}$$

Substituting (24) into (23), we derive

$$\begin{cases} dX(t) = \{(\mathcal{F} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T)X(t) \\ \qquad + (\bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T)E[X(t)] \\ \qquad + (\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)Y(t)]\}dt \\ \qquad + \{\mathcal{D}X(t) + \bar{\mathcal{D}}E[X(t)]\}dW(t), \\ -dY(t) = \{(\mathcal{O} - \mathcal{S}_2 R_2^{-1}\mathcal{S}_2^T)X(t) \\ \qquad + (\bar{\mathcal{O}} - \mathcal{S}_2 R_2^{-1}\bar{\mathcal{S}}_2^T - \\ \qquad \bar{\mathcal{S}}_2 R_2^{-1}\mathcal{S}_2^T - \bar{\mathcal{S}}_2 R_2^{-1}\bar{\mathcal{S}}_2^T)E[X(t)] \\ \qquad + (\mathcal{F}^T - \mathcal{S}_2 R_2^{-1}\mathcal{S}_1^T)Y(t) \\ \qquad + (\bar{\mathcal{F}}^T - \bar{\mathcal{S}}_2 R_2^{-1}\mathcal{S}_1^T)E[Y(t)] \\ \qquad + \mathcal{D}^T Z(t) + \bar{\mathcal{D}}^T E[Z(t)]\}dt - Z(t)dW(t), \\ X(0) = X_0,\ Y(T) = \mathcal{N}X(T) + \bar{\mathcal{N}}E[X(T)]. \end{cases} \tag{25}$$

We need to decouple (25), we use the idea of [15]. For this target, put

$$Y(t) = \hat{P}(t)X(t) + \hat{Q}(t)E[X(t)],\ t \in [0,T], \tag{26}$$

where $\hat{P}(\cdot), \hat{Q}(\cdot)$ are both deterministic and differentiable matrix-valued functions with

$$\hat{P}(T) = \mathcal{N}, \quad \hat{Q}(T) = \bar{\mathcal{N}}.$$

Applying Itô's formula to (26), we derive

$$\begin{aligned} dY(t) &= d(\hat{P}X(t) + \hat{Q}E[X(t)]) \\ &+ \{[\dot{\hat{P}} + \hat{P}(\mathcal{F} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T) \\ &+ \hat{P}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{P}]X(t) \\ &+ [\hat{P}(\bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T) \\ &+ \hat{P}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{Q} \\ &+ \dot{\hat{Q}} + \hat{Q}(\mathcal{F} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T) \\ &+ \hat{Q}(\bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T) \\ &+ \hat{Q}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{P} \\ &+ \hat{Q}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{Q}]E[X(t)]\}dt \\ &+ \{\hat{P}\mathcal{D}X(t) + \hat{P}\bar{\mathcal{D}}E[X(t)]\}dW(t) \\ &= -\{[\mathcal{O} - \mathcal{S}_2 R_2^{-1}\mathcal{S}_2^T \\ &+ (\mathcal{F}^T - \mathcal{S}_2 R_2^{-1}\mathcal{S}_1^T)\hat{P}]X(t) \\ &+ [\bar{\mathcal{O}} - \mathcal{S}_2 R_2^{-1}\bar{\mathcal{S}}_2^T - \bar{\mathcal{S}}_2 R_2^{-1}\mathcal{S}_2^T \\ &- \bar{\mathcal{S}}_2 R_2^{-1}\bar{\mathcal{S}}_2^T + (\mathcal{F}^T - \mathcal{S}_2 R_2^{-1}\mathcal{S}_1^T)\hat{Q}(t) \\ &+ (\bar{\mathcal{F}}^T - \bar{\mathcal{S}}_2 R_2^{-1}\mathcal{S}_1^T)\hat{P} \\ &+ (\bar{\mathcal{F}}^T - \bar{\mathcal{S}}_2 R_2^{-1}\mathcal{S}_1^T)\hat{Q}]E[X(t)] \\ &+ \mathcal{D}^T Z(t) + \bar{\mathcal{D}}^T E[Z(t)]\}dt \\ &+ Z(t)dW(t). \end{aligned} \tag{27}$$

Comparing the diffusion terms on both sides above, we arrive at

$$Z(t) = \hat{P}\mathcal{D}X(t) + \hat{P}\bar{\mathcal{D}}E[X(t)]. \tag{28}$$

Comparing the drift terms in (27), and plugging (28) into them, we have

$$\begin{cases} 0 = \dot{\hat{P}} + \hat{P}(\mathcal{F} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T) \\ \quad + (\mathcal{F} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T)^T\hat{P} \\ \quad + \hat{P}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{P} \\ \quad + \mathcal{D}^T\hat{P}\mathcal{D} + \mathcal{O} - \mathcal{S}_2 R_2^{-1}\mathcal{S}_2^T, \\ 0 = \dot{\hat{Q}} + \hat{Q}(\mathcal{F} + \bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T) \\ \quad + (\mathcal{F} + \bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T)^T\hat{Q} \\ \quad + \hat{Q}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{P} \\ \quad + \hat{P}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{Q} \\ \quad + \hat{Q}(\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{Q} \\ \quad + \hat{P}(\bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T) \\ \quad + (\bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T)^T\hat{P} \\ \quad + \bar{\mathcal{D}}^T\hat{P}\mathcal{D} + (\mathcal{D} + \bar{\mathcal{D}})^T\hat{P}\bar{\mathcal{D}} \\ \quad + \bar{\mathcal{O}} - \mathcal{S}_2 R_2^{-1}\bar{\mathcal{S}}_2^T - \bar{\mathcal{S}}_2 R_2^{-1}\mathcal{S}_2^T - \bar{\mathcal{S}}_2 R_2^{-1}\bar{\mathcal{S}}_2^T, \\ \hat{P}(T) = \mathcal{N},\ \hat{Q}(T) = \bar{\mathcal{N}}. \end{cases} \tag{29}$$

Noting that the Riccati equations system (29) is not s-tandard, and it has extensive open solvability. Owing to technical reasons, we can not have its solvability up to present. Therefore, we suppose that (29) admits a solution $(\hat{P}(\cdot), \hat{Q}(\cdot))$. We wish to study the general solvability of (29) in future.
Inserting (26) into (24), we obtain that

$$\begin{aligned} v_2^*(t) = -R_2^{-1}\{&[(B_2,\ 0)\hat{P} + (0,\ B_2P)]X(t) \\ &+ [(B_2,\ 0)\hat{Q} + (0,\ B_2Q)]E[X(t)]\}. \end{aligned} \tag{30}$$

The leader's optimal state $X(\cdot) = (x^*(\cdot), q(\cdot))^T$ satisfies the following SDE

$$\begin{cases} dX(t) = \{[\mathcal{F} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_2^T \\ \qquad + (\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{P}]X(t) \\ \qquad + [\bar{\mathcal{F}} - \mathcal{S}_1 R_2^{-1}\bar{\mathcal{S}}_2^T \\ \qquad + (\mathcal{M} - \mathcal{S}_1 R_2^{-1}\mathcal{S}_1^T)\hat{Q}]E[X(t)]\}dt \\ \qquad + \{\mathcal{D}X(t) + \bar{\mathcal{D}}E[X(t)]\}dW(t), \\ X(0) = X_0. \end{cases} \tag{31}$$

According to the above deduction, we have the following theorem.

Theorem 2.2 Let (29) admit a solution $(\hat{P}(\cdot), \hat{Q}(\cdot))$, and let $X(\cdot)$ be the $\mathcal{F}_t$-adapted solution to (31). Define $(Y(\cdot), Z(\cdot))$ via (26), (28), respectively. Furthermore (25) holds, and optimal control $v_2^*(\cdot)$ by (30) is a state feedback form for the leader.

Similarly, the optimal control $v_1^*(\cdot)$ for the follower can be expressed as state feedback form. In fact, by (12), (22), (26) we can obtain

$$\begin{aligned} v_1^*(t) = -R_1^{-1}\{&[(0,\ B_1)\hat{P} + (B_1P,\ 0)]X(t) \\ &+ [(0,\ B_1)\hat{Q} + (B_1Q,\ 0)]E[X(t)]\}. \end{aligned} \tag{32}$$

Therefore under the Theorems 2.1 and 2.2, the differential game gets a solution $(v_1^*(\cdot), v_2^*(\cdot))$, and they obtain a state feedback expression (30) and (32).

The rest of this section is focused on showing the above results by numerical calculation. We assume $B_1 = 0, A = \bar{A} = 0.02, C = \bar{C} = 0.2, G_1 = \bar{G}_1 = Q_1 = \bar{Q}_1 = B_2 = R_2 = 1$ and $T = 1$. Then (13) is reduced to a decoupled ODE

$$\begin{cases} \dot{P} + 0.08P = -1, \\ \dot{Q} + 0.08Q = -0.16P - 1, \\ P(1) = 1, Q(1) = 1. \end{cases} \tag{33}$$

Solving it, we obtain

$$P(t) = \frac{27}{2}e^{0.08(1-t)} - \frac{25}{2} \tag{34}$$

and

$$Q(t) = -e^{0.08(1-t)}(\frac{467}{50} + \frac{54}{25}t) + \frac{25}{2}. \tag{35}$$

With these data, it is not hard to see that

$$v_1^*(t) = 0$$

and

$$\begin{aligned} v_2^*(t) = &-\{[(1,\ 0)\hat{P} + (0,\ P)]X(t) \\ &+ [(1,\ 0)\hat{Q} + (0,\ Q)]E[X(t)]\}, \end{aligned}$$

where $\hat{P}$ and $\hat{Q}$ satisfy (29) with P and Q be determined by (34) and (35). The more general numerical example will be studied in the future work.

3 CONCLUSION

We have investigated an LQLF stochastic differential game with mean-field type in this paper. An interesting characteristic of the paper is that the optimal controls for the follower and the leader depend not only on the optimal state but also on its mean. In addition, to represent optimal solution, a new system of Riccati equations is introduced. It is worthy to study the general solvability of (29), and more mean-field cases, for example, the controls enter the diffusion, are highly necessary for further research. We hope to consider these challenging problems in future papers.

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