Cooperative game theory

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Introduction

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- $S \subseteq N$ ... coalition
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- usually $N = \{1, \ldots, n\}$
  - (S, v_S) is **subgame** (N, v):
    - $v_S\colon 2^S \to \mathbb{R}$
    - $v_S(T) := v(T)$ pro $T \subseteq S$

Cooperation - examples of models: *Minimal spanning-tree games*

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- $N = N' \cup \{0\}$ ... set of players + source
- c_{ij} ... cost of connecting i, j
- solution: a network, where each $i \in N$ is connected to 0 with minimal sum of costs

Goal of the model of cooperative games

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  - *...fair...*
  - *...non-discriminatory...*
  - *...stable (players will accept it)...*
  - ...

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Multi-point solution concept: The core

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$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\} .$$

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Nash equilibrium

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Reminder

Nash equilibrium

Strategy profile $(s_1, \ldots, s_n)$ is **Nash equilibrium**, if it holds for every player i,

$$v_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq v_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$$

for every $t_i \in S_i$.

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- x does not exist $\implies C(v) = \emptyset$

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We derive the weak Bondereva-Shapley theorem

Weak Bondareva-Shapley theorem

Cooperative game (N, v) has non-empty core if and only if

$$v(N) \geq \sum_{S \subseteq N} y_S v(S) \text{ for all feasible } y \in \mathbb{R}^{(2^n-1)}$$

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$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S \cup i) - v(S)\right)$$

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- $\phi_i(v)$ ... the average value of player i's payment

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   - $v, w \in \Gamma^n : f(v + w) = f(v) + f(w)$

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Relation between the Shapley value and the Webber set

For a cooperative game (N, v) it holds:

$$\phi(v) \in \mathcal{W}(v)$$

Moreover, $\phi(v)$ is the center of gravity of $\mathcal{W}(v)$.

Provable by induction:

The Weber set contains the core

For every cooperative game (N, v), it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$

Classes of games

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- **monotonic game**

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$$\mathcal{C}(v) \neq \emptyset$$

Question: *What are the relations between them?*

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  - $\emptyset \neq \mathcal{C}(v) \subseteq \mathcal{I}(v)$

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- $S \cap T = \emptyset$
  - $(S \cap T) = 0$
  - $v(S) + v(T) \leq v\,(S \cup T)$

TODO: There will be beautiful picture

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$(S, T \subseteq N)\,(v(S) + v(T) \leq v\,(S \cap T) + v\,(S \cup T))$

Consequence:

The Shapley value and convex games

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The Shapley value and convex games

For a convex cooperative game (N, v), it holds:

1. $\phi(v) \in \mathcal{C}(v)$
2. $\phi(v)$ is the centre of gravity of $\mathcal{C}(v)$

Disadvantages of the model and motivation for next lectures

Cooperative game

A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

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   - incomplete games (David Sychrovský + Filip Úradník)
   - interval games (Martin Kunst)
   - stochastic games (David Ryzák)
   - models with compact characteristic function (Martin Černý)