Cooperative game theory

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Introduction

Cooperative game

A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

- Γ^n ... set of n-person cooperative games
- $S \subseteq N$ ... coalition
- $v(S)$ ... values of coalition
- usually $N = \{1, \ldots, n\}$
    - (S, v_S) is **subgame** (N, v):
        - $v_S\colon 2^S \to \mathbb{R}$
        - $v_S(T) := v(T)$ pro $T \subseteq S$

Cooperation - examples of models: *Minimal spanning-tree games*

Goal: *Find the best connection of players to a source*

- $N = N' \cup \{0\}$ ... set of players + source
- c_{ij} ... cost of connecting i, j
- solution: a network, where each $i \in N$ is connected to 0 with minimal sum of costs

Money first!

- **Payoff vector $\mathbf{x} \in \mathbb{R}^n$**
  - x_i represents payoff of player i
- Vector $\mathbf{x} \in \mathbb{R}^n$ is **efficient**, if $\sum_{i \in N} x_i = v(N)$
  - Usually, we distribute $v(N)$
    1. *value of cooperation* $v(N)$
    2. *shared costs* $c(N)$
- Vector $\mathbf{x} \in \mathbb{R}^n$ is **individually rational**, if $x_i \geq v(i)$
  - players prefer x_i over $v(i)$

- $\mathcal{I}^*(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$ ... **preimputation**
  - $x(S) := \sum_{i \in S} x_i$
- $\mathcal{I}(v) = \{x \in \mathcal{I}^*(v) \mid \forall i \in N : x_i \geq v(i)\}$ ... **imputation**

Solution concepts

Solution concepts

- Set of payoff vectors satisfying further properties are **solution concepts**
- Can reflect payoff distribution, which is
  - *...fair...*
  - *...non-discriminatory...*
  - *...stable (players will accept it)...*
  - ...

Formally:

1. sets of payoff vectors
   - $\Sigma(v) = \{x \in \mathbb{R}^n \mid \dots\}$
2. functions on games
   - $\Sigma\colon \Gamma^n \to 2^{\mathbb{R}^n}$

Formally:

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We distringush

1. **single-point** solution concepts
   - as a set: $\Sigma(v) = \{x\}$
     - we prefer: $\Sigma(v) = x$
   - as a function: $\Sigma\colon \Gamma^n \to \mathbb{R}$
2. **multi-point** solution concepts

Multi-point solution concept: The core

Idea: *Payoff distribution leads to cooperation...*

The core

For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\} .$$

- assumption: *homo economicus*
  - model of human as a player
  - strictly rational and selfish
  - follows his subjective goals
- $v(N)$ ... value, which is distributed among players
- $x(S) > v(S) \implies$ coalition S does not leave N
  - would lead to (S, v_S)
  - $v(S)$ ... distributed value

The core

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Reminder

Nash equilibrium

Strategy profile $(s_1, \ldots, s_n)$ is **Nash equilibrium**, if it holds for every player i,

$$v_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq v_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$$

for every $t_i \in S_i$.

The core

For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

$$\mathcal{C}(v) = \{x \in \mathcal{I}^*(v) \mid x(S) \geq v(S), \forall S \subseteq N\} .$$

Emptyness of the core

There are cooperative games (N, v) with empty core.

- Non-esential games
  - $v(N) < \sum_{i \in N} v(i)$
- Let $x \in \mathcal{C}(v)$
  - $x(N) = v(N) < \sum_{i \in N} \leq x(N)$
- x does not exist $\implies (v) = \emptyset$

Question: When is the core non-empty
We can encode the core as linear program (P) and determine the dual program (D)

- $(P) = \begin{cases} min_{x\in\mathbb{R}^n} & x(N) \\ \text{subject to} & x(S) \geq v(S) \text{ for } S \subseteq N \end{cases}$
- $(D) = \begin{cases} max_{y\in\mathbb{R}_+^{(2^n-1)}} & \sum_{S\subseteq N y_S v(S)} \\ \text{subject to} & \sum_{\emptyset\neq S\subseteq N} y_S \mathcal{X}_S = \mathcal{X}_N \end{cases}$

We derive the weak Bondereva-Shapley theorem

Weak Bondareva-Shapley theorem

Cooperative game (N, v) has non-empty core if and only if

$$v(N) \geq \sum_{S\subseteq N} y_S v(S) \text{ for all feasible } y \in \mathbb{R}^{(2^n-1)}$$

Single-point solution concept: The Shapley value

Idea: *Divide the profit in a fair way...*

The Shapley value

For a cooperative game (N, v), the **Shapley value** $\phi(v)$ of player i is

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S \cup i) - v(S) \right)$$

- $v(S \cup i) - v(S)$
  - *marginal contribution* (of player i in $S \cup i$)
- $\frac{s!(n-s-1)!}{n!}$
  - weights reflecting different sizes of coalitions
- $\sum_{S \subseteq N \setminus i}$
  - sum of all marginal contributions of i

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- The players agree om the following procedure:
  1. Form the grandcoalition N.
  2. Enter the the coalition individuallyand randomly.
  3. When player i enters coalition S, he receives $v(S \cup i) - v(S)$.
- $s!(n-s-1)!$ ... number of situations, in which i enters S
- $n!$ ... number of all possible ways to construct N
- $\phi_i(v)$ ... the average value of player i's payment

It is possible to define it using its properties...

The Shapley value

The **Shapley value** $\phi(v)$ is the only function $f\colon \Gamma^n \to \mathbb{R}$ satisfying for all games $(N, v), (N, w)$:

1. (AXIOM OF EFFICIENCE)
   - $\sum_{i \in N} f_i(v) = v(N)$
2. (AXIOM OF SYMMETRY)
   - $\forall i, j \in N \ (\forall S \subseteq N \setminus \{i, j\} : v(S \cup i) = v(S \cup j)) \implies f_i(v) = f_j(v)$
3. (AXIOM OF NULL PLAYER)
   - $\forall i \in N \ (\forall S \subseteq N : v(S) = v(S \cup i)) \implies f_i(v) = 0$
4. (AXIOM OF ADDITIVITY)
   - $v, w \in \Gamma^n : f(v + w) = f(v) + f(w)$

1. We fix one construction of N
   - $\sigma \in \Sigma_n$ ... represent it by permutation
   - $\sigma(i)$ ... the order, in which player i enters the coalition
2. We compute player's payments
   - m_v^σ ... **marginal vector**
   - $(m_v^\sigma)_i = v\left(S_{\sigma(i)} \cup i\right) - v\left(S_{\sigma(i)}\right)$
     - $S_{\sigma(i)} = \{j \in N \mid \sigma(j) < \sigma(i)\}$ ... predecessors of i under σ
3. We consider all combinations
   - $\mathcal{W}(v) = conv\left\{m_v^\sigma \mid \sigma \in \Sigma_n\right\}$ ... **Webber set**

- It holds: $\phi(v) = \sum_{\sigma \in \Sigma_n} \frac{m_v^\sigma}{n!}$

Relation between the Shapley value and the Webber set

For a cooperative game (N, v) it holds:

$$\phi(v) \in \mathcal{W}(v)$$

Moreover, $\phi(v)$ is the center of gravity of $\mathcal{W}(v)$.

Provable by induction:

The Weber set contains the core

For every cooperative game (N, v), it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$

Classes of games

- **monotonic game**

$$(S \subseteq T \subseteq N)\,(v(S) \leq v(T))$$

- **superadditive game**

$$(S, T \subseteq N, S \cap T = \emptyset)\,(v(S) + v(T) \leq v\,(S \cup T))$$

- **convex game**

$$(S, T \subseteq N)\,(v(S) + v(T) \leq v\,(S \cap T) + v\,(S \cup T))$$

- **essential game**

$$v(N) \geq \sum_{i \in N} v(i)$$

- **balanced game**

$$\mathcal{C}(v) \neq \emptyset$$

Question: *What are the relations between them*

Balanced and essential

Balanced cooperative games are essential.

- (N, v) is essential
  - $\mathcal{I}(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \wedge \forall i \in N : x_i \geq v(i)\} \neq \emptyset$
- (N, v) is balanced
  - $\emptyset \neq \mathcal{C}(v) \subseteq \mathcal{I}(v)$

Question: *What are the relations between them*

Convex and superadditive

Convex cooperative games are superadditive.

- (N, v) is convex
  - $(S, T \subseteq N)\,(v(S) + v(T) \leq v\,(S \cap T) + v\,(S \cup T))$
- (N, v) is superadditive
  - $(S, T \subseteq N, S \cap T = \emptyset)\,(v(S) + v(T) \leq v\,(S \cup T))$
- $S \cap T = \emptyset$
  - $(s \cap T) = 0$
  - $v(S) + v(T) \leq v\,(S \cup T)$

TODO: There will be beautiful picture

Convex games

$(S, T \subseteq N)\,(v(S) + v(T) \leq v\,(S \cap T) + v\,(S \cup T))$

Core of convex games

For a convex cooperative game (N, v), it holds $\mathcal{C}(v) = \mathcal{W}(c)$

Proof: We show only $m_v^{id} \in \mathcal{C}(v)$

1. $m_v^{id}(N) = \sum_{i \in N} v\,(\{1, \ldots, i\}) - v\,(\{1, \ldots, i-1\}) = V(N)$
2. $m_v^{id}(S) \geq v(S)$
   - $S = \{s_1, s_2, \ldots, s_k\}$
     - $s_1 < s_2 < \cdots < s_k$
   - $v(\{1, ..., s_i\}) - v(\{1, ..., s_{i-1}\}) \geq v(\{s_1, ..., s_i\}) - v(\{s_1, ..., s_{i-1}\})$
     - $\{s_1, s_2, \ldots, s_l\} \subseteq \{1, \ldots, s_l\}$ for $l \leq k$
   - $m_v^{id}(S) = \sum_{s_i \in S} v\,(\{1, \ldots, s_i\}) - v\,(\{1, \ldots, s_{i-1}\})$
   - $m_v^{id} \geq \sum_{s_i \in S} v\,(\{s_1, \ldots, s_i\}) - v\,(\{s_1, \ldots, s_{i-1}\}) = v(S)$

Convex games

$(S, T \subseteq N)\,(v(S) + v(T) \leq v\,(S \cap T) + v\,(S \cup T))$

Consequence:

The Shapley value and convex games

For a convex cooperative game (N, v), it holds:

1. $\phi(v) \in \mathcal{C}(v)$
2. $\phi(v)$ is the centre of gravity of $\mathcal{C}(v)$

Disadvantages of the model and motivation for next lectures

Disadvantages

Cooperative game

A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

1. **we might not be interested in forming** N
   - we want to analyze coalition formation
2. **not every coalition makes sense**
   - two players from completely different fields (why would they want to cooperate)
3. 2^n **real values representing** v
   - expensive to get all information
   - expensive to store all information
4. **not suitable for all scenerios**
   - unsuitable for more complex scenarios

Cooperative game

A **cooperative game** is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

1. **we might not be interested in forming** N
   - games with coalition structure
2. **not every coalition makes sense**
   - restricted games
3. 2^n **real values representing** v
   - uncertainty models
   - models with compact characteristic function
4. **not suitable for all scenerios**
   - bi-cooperative games and games with overlapping coalitions