Cooperative game theory

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Motivations and introduction

Goal: *To analyse and solve conflicting interaction between subjects*

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Fundamental question: *What strategy should player i choose?*

Nash equilibrium

Idea: Deviation from the actual strategy to a new strategy does not improve the outcome.

Nash equilibrium

Strategy profile $(s_1, \ldots, s_n)$ is **Nash equilibrium**, if it holds for every player i,

$$v_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq v_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$$

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- Is this model suitable for the analysis of *cooperation*?

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     - model of *cooperative games*

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  - w_i is continuous, concave function

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Cooperation - examples of models: *Market games*

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    - ζ may be negative
    - amount we paid to buy the commodities

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- What trade will occur within a given coalition?
    - *What is the profit of player **i** for a given market?*

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- $v_i = \begin{cases} 7 & i = 1, \ldots, 5 \\ 1 & i = 6, \ldots, 15 \end{cases}$

Cooperation - examples of models: *Cost-sharing problem*

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- solution: a network, where each $i \in N$ is connected to 0 with minimal sum of costs

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- usually $N = \{1, \dots, n\}$
  - (S, v_S) is **subgame** (N, v):
    - $v_S\colon 2^S \to \mathbb{R}$
    - $v_S(T) := v(T)$ pro $T \subseteq S$

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Cooperative game (N, v) is **market game**, if there is market $(N, \mathbb{R}^m_+, A, w)$ satisfying

$$v(S) = \max\{\sum_{i\in S} w^i(a^i_S) \mid (a^i_S)_{i\in S} \in \mathcal{A}_S\}.$$

Voting game

A **voting game** $(N, \mathcal{W})$ is given by:

- N ... set of players
- $\mathcal{W} \subseteq 2^N$ ... set of winning coalitions
  - $\emptyset \notin \mathcal{W}$
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Cooperation - examples of models: *Cost-sharing problem*

Goal: *Does cooperation lead to reduction of costs?*

- N ... set of players
- $c(S)$ ... value of costs for a service shared by $S \subseteq N$
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  - *...fair...*
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  - *...stable (players will accept it)...*
  - *...*

Idea: *Payoff distribution leads to cooperation...*

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For a cooperative game (N, v), the **core** $\mathcal{C}(v)$ is

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  - $v(S)$ ... distributed value

Examples of solution concepts: The Shapley value

Idea: *Divide the profit in a fair way...*

The Shapley value

For a cooperative game (N, v), the **Shapley value** $\phi(v)$ of player i is

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S \cup i) - v(S)\right)$$

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It is possible to define it using its properties...

The Shapley value

The **Shapley value** $\phi(v)$ is the only function $f\colon \Gamma^n \to \mathbb{R}$ satisfying for all games $(N, v), (N, w)$:

1. (AXIOM OF EFFICIENCE)
   - $\sum_{i \in N} f_i(v) = v(N)$
2. (AXIOM OF SYMMETRY)
   - $\forall i, j \in N\ (\forall S \subseteq N \setminus \{i, j\} : v(S \cup i) = v(S \cup j)) \implies f_i(v) = f_j(v)$
3. (AXIOM OF NULL PLAYER)
   - $\forall i \in N\ (\forall S \subseteq N : v(S) = v(S \cup i)) \implies f_i(v) = 0$
4. (AXIOM OF ADDITIVITY)
   - $v, w \in \Gamma^n : f(v + w) = f(v) + f(w)$

Cooperative games

A **cooperative game** is given by a set of players and real values representing the profit of each subset of players (*coalition*). These values are encoded by the *characteristic function* of a game. The goal of the cooperative game theory is to find payoffs (in form of **payoff vectors**) for individual players based on values of the game. Payoff vectors satisfying further properties form **solution concepts**. These can be expressed as:

1. sets of payoff vectors
2. functions on games
   - 2.1 defined by formula
   - 2.2 defined by its properties