Richard Mužík

richard@imuzik.cz

Introduction to Cooperative Game Theory
as part of series Cooperative game theory

Introduction

In this talk, I will present you the basics of cooperative game theory. It is an introduction to a series of talks solving different problems with the model.

Definition 1 (Cooperative game) *A cooperative game is an ordered pair (N, v), where N is a set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.*

Definition 2 (Payoff vector) *Payoff vector is $x \in \mathbb{R}^n$, where x_i represents payoff of player i. It is efficient, if $\sum_{i \in N} x_i = v(N)$. It is individually rational, if $x_i \geq v(i)$. We denote $x(S) = \sum_{i \in S} x_i$.*

Definition 3 (Core) *For a cooperative game (N, v), the core $\mathcal{C}(v)$ is*

$$\mathcal{C}(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \wedge (S) \geq v(S), \forall S \subseteq N\}.$$

Observation 4 (Emptyness of the core) *There are cooperative games (N, v) with empty core.*

Theorem 5 (Weak Bondareva-Shapley) *Cooperative game (N, v) has non-empty core if and only if*

$$v(N) \geq \sum_{S \subseteq N} y_S v(S) \text{ for all feasible } y \in \mathbb{R}^{(2^n - 1)}.$$

Definition 6 *For a cooperative game (N, v), the Shapley value $\varphi(v)$ of player i is*

$$\varphi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} (v(S \cup i) - v(S)).$$

Definition 7 (Marginal vector) *For cooperative game (N, v) and permutation $\sigma \in \Sigma_n$ is m_v^σ marginal vector, where $(m_v^\sigma)_i = v\left(S_{\sigma(i)} \cup i\right) - v\left(S_{\sigma(i)}\right)$ and $S_{\sigma(i)} = \{j \in N \mid \sigma(j) < \sigma(i)\}$.*

Definition 8 (Weber set) *For cooperative game (N, v) the Weber set is*

$$\mathcal{W}(v) = conv\,\{m_v^\sigma \mid \sigma \in \Sigma_n\}.$$

Lemma 9 (Shapley value and Weber set) *For a cooperative game (N, v) it holds:*

$$\varphi(v) \in \mathcal{W}(v).$$

Moreover, $\varphi(v)$ is the center of gravity of $\mathcal{W}(v)$.

Theorem 10 (Weber set and core) *For every cooperative game (N, v), it holds $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.*

Definition 11 (Classes) *The cooperative game (N, v) is said to be*

- *monotonic game* $\equiv (S \subseteq T \subseteq N)\,(v(S) \leq v(T))$.
- *superadditive game* $\equiv (S, T \subseteq N, S \cap T = \emptyset)\,(v(S) + v(T) \leq v(S \cup T))$.
- *convex game* $\equiv (S, T \subseteq N)\,(v(S) + v(T) \leq v(S \cap T) + v(S \cup T))$.
- *essential game* $\equiv v(N) \geq \sum_{i \in N} v(i)$.
- *balanced game* $\equiv \mathcal{C}(v) \neq \emptyset$.

Theorem 12 (Balanced and essential) *Balanced cooperative games are essential.*

Theorem 13 (Convex and superadditive) *Convex cooperative games are superadditive.*

Theorem 14 (Core of convex games) *For a convex cooperative game (N, v), it holds $\mathcal{C}(v) = \mathcal{W}(v)$.*

Corollary 15 (Shapley value and convex games) *For a convex cooperative game (N, v), it holds:*

1. $\varphi(v) \in \mathcal{C}(v)$.
2. $\varphi(v)$ *is the centre of gravity of* $\mathcal{C}(v)$.

Bibliography

[1] Hans Peters *Game theory: A multi-leveled approach.* Springer Texts in Business and Economics, 2nd edition, 2015.