The Forward-Scattering Theorem Applied to the Scattering Dyadic

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I am delighted that this paper appears in the Special Issue honoring Prof. Leopold B. Felsen. I dedicate this paper to his memory. He was my friend and champion. When I introduced the singularity expansion method (SEM) for electromagnetic scattering in 1971 (Interaction Note 88), he immediately saw its significance. He encouraged me to publish this (and other contributions such as ultrawideband antennas), and invited me to publish an SEM chapter in a book which he edited [12]. I am sure that this had something to do with my winning the 2007 IEEE Electromagnetics Award (already announced), which he had also won before me. I am grateful for having known Leo.

***Abstract*—From the forward-scattering theorem we have relations between the absorption and scattering cross sections, and the forward scattering. The scattered fields are represented by a scattering dyadic times the incident plane wave. This allows one to reformulate the results in terms of the scattering dyadic, exhibiting some general characteristics of this dyadic. This is extended (for lossless scatterers) by analytic continuation away from the $j\omega$ axis out into the complex s plane and applied to poles in the singularity expansion method.**

***Index Terms*—Forward-scattering theorem, singularity expansion method.**

I. Introduction

The forward-scattering theorem is a classical result in electromagnetic theory [4], [9], [13], [14]. [Note that various authors also call this the cross-section theorem or optical theorem. There are also analogous forms for other types of waves (acoustic scattering, particle scattering).] While normally formulated in the frequency domain it can also be considered in the time domain as well [4]. Consulting the references, the reader can find yet more on this subject.

Physically this theorem tells us something about the scattering in all directions (4π steradians) by relating it to that in one particular direction, namely, that of the direction of propagation of the incident wave (i.e., forward scattering). This implies something special about the forward scattering direction. In this regard one can consult [10] for the general properties of the scattering dyadic, particularly in the forward direction, where various symmetries apply.

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In this paper we first define the scattering problem and quote the classical forward-scattering theorem (scalar). Then properties of the scattering dyadic are exhibited. The forward-scattering theorem is then recast in terms of the scattering dyadic. Using another result from the literature [8], the scalar form of the forward-scattering theorem is generalized to a dyadic form which is shown to be consistent with the scalar form. In these various forms we make the terms analytic functions of the complex frequency, s. This essential step allows us to continue the forward-scattering theorem out into the complex s plane away from the $j\omega$ axis. This comes from the fact that the terms are Laplace-transforms (two-sided) of real-valued time functions. Then the conjugate of a function of $j\omega$ becomes a function (no conjugate) of $-j\omega$ which can be replaced by-s.

Specializing the equations to the case of a lossless scatterer removes one term and gives a relation for the scattering for no energy loss in the scatterer. The scattering in all directions is related to the scattering in the forward direction. Going out into the s-plane this gives a relationship between parameters in the left half plane ((LHP) and those in the right-half plane (RHP). In the singularity expansion method (SEM) the scattering includes poles in the LHP. These can now be related to the scattering operator in the RHP, a new fundamental result.

II. The Scattering Problem

Fig. 1 shows a general scatterer in free space. The incident field is taken as a plane wave of the form

$$
\begin{aligned}
\tilde{\vec{E}}(\vec{r},\vec{1}_i,\vec{1}_p;s) &= E_0\tilde{f}(s)\tilde{\vec{e}}^{(\mathrm{inc})}(\vec{r},\vec{1}_i,\vec{1}_p;s)\\
\tilde{\vec{e}}^{(\mathrm{inc})}(\vec{r},\vec{1}_i,\vec{1}_p;s) &= \vec{1}_p e^{-\gamma\vec{1}_i\cdot\vec{r}}\\
\tilde{\vec{H}}^{(\mathrm{inc})}(\vec{r},\vec{1}_i,\vec{1}_p;s) &= \frac{E_0}{Z_0}\tilde{f}(s)\vec{1}_i\\
&\quad\times\tilde{\vec{e}}^{(\mathrm{inc})}(\vec{r},\vec{1}_i,\vec{1}_p;s)\\
&= \frac{E_0}{Z_0}\tilde{f}(s)\tilde{\vec{h}}^{(\mathrm{inc})}(\vec{r},\vec{1}_i,\vec{1}_p;s)\\
\vec{1}_i &\equiv \text{direction of incidence}\\
\vec{1}_p &\equiv \text{polarization}\\
&\qquad\text{(assumed linear (real valued))}\\
\vec{1}_i\cdot\vec{1}_p &= 0\ \text{(perpendicular)},\ \overleftrightarrow{1}_z=\overleftrightarrow{1}\\
&\quad-\vec{1}_i\vec{1}_i\equiv\text{transverse incidence dyadic}\\
\overleftrightarrow{1} &\equiv \vec{1}_x\vec{1}_x+\vec{1}_y\vec{1}_y+\vec{1}_z\vec{1}_z
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \text{identity dyadic} \\
\gamma &= \frac{s}{c} \equiv \text{propagation constant} \\
s &= \Omega + j\omega = \text{Laplace} - \text{transform} \\
&\quad \text{variable or complex frequency} \\
c &= [\mu_0 \varepsilon_0]^{1/2} \equiv \text{light speed} \\
Z_0 &= \left[\frac{\mu_0}{\varepsilon_0}\right]^{1/2} \equiv \text{wave impedance} \\
\tilde{\vec{e}}^{(\mathrm{inc})}&(\vec{r}, \vec{1}_i, \vec{1}_p; s) \equiv \text{normalized incident field} \\
&\quad (\text{dimensionless}).
\end{aligned} \tag{2.1}
$$

The far scattered fields (as $r \to \infty$ in direction $\vec{1}_r$) take the form

$$
\begin{aligned}
\tilde{\vec{E}}_f(\vec{r}, \vec{1}_i, \vec{1}_p; s) &= \frac{E_0 \tilde{f}(s)}{r} e^{-\gamma r} \tilde{\vec{\mathrm{v}}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) \\
\tilde{\vec{\mathrm{v}}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) &\equiv \text{normalized far field} \\
&\quad (\text{dimension meters}) \\
\vec{1}_r \cdot \tilde{\vec{\mathrm{v}}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) &= 0 \\
\overleftrightarrow{1}_r &\equiv \overleftrightarrow{1} - \vec{1}_r \vec{1}_r \equiv \text{transverse dyadic} \\
\tilde{\vec{H}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) &= \frac{E_0 \tilde{f}(s)}{Z_0 r} e^{-\gamma r} \vec{1}_r \\
&\quad \times \tilde{\vec{\mathrm{v}}}_f(\vec{1}_r, \vec{1}_i, \vec{1}p; s).
\end{aligned} \tag{2.2}
$$

The scattering is described by a scattering dyadic as [10]

$$
\begin{aligned}
&\tilde{\vec{E}}f(\vec{r}, \vec{1}_i, \vec{1}_p; s) \\
&\quad = \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \cdot \tilde{\vec{E}}^{(\mathrm{inc})}(\vec{0}, \vec{1}_i, \vec{1}_p; s) \\
&\tilde{\vec{\mathrm{v}}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s) \\
&\quad = \frac{1}{4\pi} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \cdot \tilde{\vec{e}}^{(\mathrm{inc})}(\vec{0}, \vec{1}_i, \vec{1}_p; s) \\
&\quad = \frac{1}{4\pi} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \cdot \vec{1}_p \\
&\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \\
&\quad \equiv \text{scattering dyadic (dimension meters)} \\
&\vec{1}_r \cdot \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \\
&\quad = \vec{0} = \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \cdot \vec{1}_p \text{ (transverse property)} \\
&\tilde{\overleftrightarrow{\Lambda}}^T(\vec{1}_r, \vec{1}_i; s) \\
&\quad = \tilde{\overleftrightarrow{\Lambda}}(-\vec{1}_i, -\vec{1}_r; s) (\text{reciprocity}).
\end{aligned} \tag{2.3}
$$

For convenience let us also introduce another form as

$$
\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s) = \frac{1}{4\pi} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \tag{2.4}
$$

which will simplify some of the later formulae. We can refer to the two forms interchangeably as the scattering dyadic (or scattering matrix). Note that $\vec{r} = \vec{0}$ has been taken as some convenient location in the vicinity of the scatterer (say the center of the minimum circumscribing sphere).

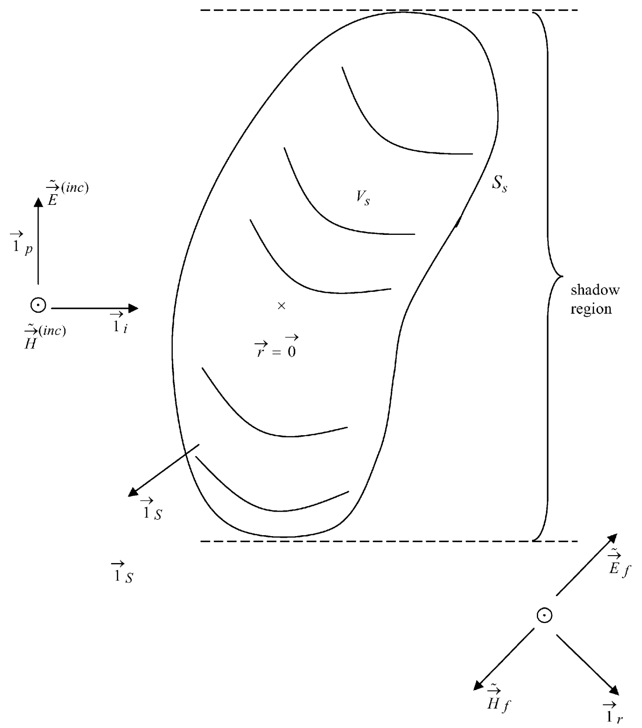

Fig. 1. Scattering of incident plane wave.

Here we have used the radar convention in which the scattering dyadic is related to an incident plane wave. This is also sometimes considered as a scattering amplitude (as in amplitude or magnitude, and phase, not very appropriate to a dyadic- or matrix-valued function). Some authors (e.g., [8]) instead use this term (scattering matrix) to refer the scattering to some kind of incoming wave scattering into an outgoing wave. For present purposes, let us call this form a reversal matrix (or dyadic or operator) in which form we can also state the results of this paper.

III. Forward-Scattering Theorem

Following [4], [9], [13], [14] we have various cross sections. The "mixed" term gives a term involving forward scattering as

$$
\begin{aligned}
&\frac{1}{2}\left[\tilde{A}_\infty^{(\mathrm{mix})}(\vec{1}_i, \vec{1}_p; s) + \tilde{A}_\infty^{(\mathrm{mix})}(\vec{1}_i, \vec{1}_p; -s)\right] \\
&\quad = \frac{2\pi}{\gamma}[\vec{1}_p \cdot \tilde{\vec{\mathrm{v}}}_f(\vec{1}_i, \vec{1}_i, \vec{1}_p; s) - \vec{1}_p \\
&\qquad \cdot \tilde{\vec{\mathrm{v}}}_f(\vec{1}_i, \vec{1}_i, \vec{1}_p; -s)] \\
&\quad \equiv -\tilde{A}_f(\vec{1}_i, \vec{1}_p; s) \\
&\quad = \text{even function of } s.
\end{aligned} \tag{3.1}
$$

Physically, this mixed term exhibits the interaction of the scattered wave with the incident wave in the forward direction. Here the scattered field propagates with the same speed of light, c, as the incident wave, giving rise to a term accounting for the energy of those two waves interfering with each other. This is the essential term which identifies the result as a forward-scattering result, naming the theorem. Here, s is used as the analytic continuation of the conjugate of the function, or equivalently, the function of $-j\omega$ on the $j\omega$ axis out into the left and right halves of the complex s plane.

While this is derived by taking an asymptotic form for $r \to \infty$ of $\tilde{A}_{\infty}^{(\mathrm{mix})}$, the combination above avoids the divergence as $r \to \infty$. The scattered far field gives

$$\begin{aligned}\tilde{A}_{\infty}^{(\mathrm{sc})}(\vec{1}_i, \vec{1}_p; s) &= \int_{S_1} \tilde{\vec{\mathrm{v}}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; -s) \\ &\quad \cdot \tilde{\vec{\mathrm{v}}}_f(\vec{1}_r, \vec{1}_i, \vec{1}_p; s)\, dS \\ &= \text{scattering cross section} \\ &= \text{even function of } s \\ S_1 &= \text{surface of unit sphere} \\ &\quad (\text{integration over all } \vec{1}_r \text{ (real)}) \\ \tilde{A}_{\infty}^{(\mathrm{sc})}(\vec{1}_i, \vec{1}_p; j\omega) &\geq 0. \end{aligned} \tag{3.2}$$

The absorption in the scatterer is

$$\begin{aligned}&\tilde{A}_{\mathrm{ab}}(\vec{1}_i, \vec{1}_p; s) \\ &\quad = \frac{1}{2}[\tilde{A}_s(\vec{1}_i, \vec{1}_p; s) + \tilde{A}_s(\vec{1}_i, \vec{1}_p; -s)] \\ &\quad = \text{even function of } s \\ &\quad = \text{absorption cross section} \\ &\tilde{A}_{\mathrm{ab}}(\vec{1}_i, \vec{1}_p; j\omega) \\ &\quad \geq 0 \\ &\tilde{A}_s(\vec{1}_i, \vec{1}_p; s) \\ &\quad = \int_{S_s} [\tilde{\vec{e}}(\vec{r}, \vec{1}_i, \vec{1}_p; s) \times \tilde{\vec{h}}(\vec{r}, \vec{1}_i, \vec{1}_p; -s)] \cdot \vec{1}_S\, dS \end{aligned} \tag{3.3}$$

where the total fields, normalized as before to give dimensionless parameters, are used here. Again $-s$ is used instead of $-j\omega$ or the conjugate for analytic continuation. For a lossless scatterer this is zero for $s = j\omega$ and by analytic continuation throughout the s plane.

The forward scattering theorem then states

$$\tilde{A}_{\mathrm{ab}}(\vec{1}_i, \vec{1}_p; s) + \tilde{A}_{\infty}^{(\mathrm{sc})}(\vec{1}_i, \vec{1}_p; s) = \tilde{A}_f(\vec{1}_i, \vec{1}_p; s). \tag{3.4}$$

This is basically a statement of the conservation of real power when $s = j\omega$. The energy loss (absorption) in the scatterer, plus that lost to ∞ (scattering), is equal to a source term involving the incident field and the scattering in the forward direction. From this we conclude

$$\tilde{A}_f(\vec{1}_i, \vec{1}_p; j\omega) \geqslant 0. \tag{3.5}$$

The left side of (3.4) is often called the *extinction* cross section.

IV. High-Frequency Forward Scattering

Referring back to Fig. 1, one can define a shadow area or cross-section area as

$$\begin{aligned} A_c(\vec{1}_i) &= A_c(-\vec{1}_i) \\ &= \text{shadow region projected on a plane} \perp \vec{1}_i. \end{aligned} \tag{4.1}$$

Then following [9], we can evaluate the forward-scattering cross section at high frequencies for an "opaque" scatterer (a perfectly conducting scatterer being a special case).

For this purpose we can use some previous results derived for impulse-radiating antennas (IRAs) [1], [2]. In this case a uniform electric field on a planar aperture is taken as the negative of the incident electric field giving

$$\begin{aligned} \tilde{\vec{E}}_f(\vec{r}, \vec{1}_i, \vec{1}_p; s) &= -\frac{sE_0\tilde{f}(s)}{2\pi cr} e^{-\gamma r} A_c(\vec{1}_i)\vec{1}_p \\ \tilde{\vec{\mathrm{v}}}_f(\vec{1}_i, \vec{1}_i, \vec{1}_p; s) &= -\frac{sA_c(\vec{1}_i)}{2\pi c}\vec{1}_p = -\frac{\gamma}{2\pi}A_c(\vec{1}_i)\vec{1}_p \\ &= \frac{1}{4\pi}\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_i, \vec{1}_i; s)\cdot\vec{1}_p \\ \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_i, \vec{1}_i; s) &= -2\gamma A_c(\vec{1}_i)\overleftrightarrow{1}_i \\ &\text{as } s \to \infty \text{ in the RHP.} \end{aligned} \tag{4.2}$$

Traditionally, this has been stated on the $j\omega$ axis as $\omega \to \infty$. This is slightly more general in that causality allows this to be applied to dielectric scatterers which are not opaque, but merely delay the wave propagating through them. For $s = j\omega$ we still need an opaque scatterer. In the LHP we have singularities (poles) where the above result does not apply.

From (3.1) we then have the forward-scattering cross section as

$$A_f(\vec{1}_i, \vec{1}_p; j\omega) = 2A_c(\vec{1}_i) \quad \text{as } \omega \to \infty \tag{4.3}$$

this being nonnegative as required by (3.5). Note the limitation to the $j\omega$ axis here due to the functions of s and $-s$ in (3.1).

V. Low-Frequency Scattering

From [6] we have the low-frequency behavior of the scattering dyadic. This is dominated by the induced electric and magnetic dipoles giving

$$\begin{aligned}&\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_r, \vec{1}_i; s) \\ &\quad = \gamma^2[-\vec{1}_r \cdot \tilde{\overleftrightarrow{P}}(s)\cdot\vec{1}_i + \vec{1}_r \times \tilde{\overleftrightarrow{M}}(s)\times\vec{1}_i] \quad \text{as } s \to 0 \\ &\tilde{\overleftrightarrow{P}}(s) \\ &\quad = \tilde{\overleftrightarrow{P}}^T(s) \equiv \text{electric polarizability dyadic} \\ &\tilde{\overleftrightarrow{M}}(s) \\ &\quad = \tilde{\overleftrightarrow{M}}^T(s) \equiv \text{magnetic polarizability dyadic.} \end{aligned} \tag{5.1}$$

In general we have

$$\begin{aligned} &\tilde{\overleftrightarrow{P}}(0) \neq \overleftrightarrow{0} \text{ for conducting and dielectric targets} \\ &\tilde{\overleftrightarrow{M}}(0) \begin{cases} \neq \overleftrightarrow{0} \text{ for perfectly conducting targets} \\ \neq \overleftrightarrow{0} \text{ for permeable targets} \\ = \overleftrightarrow{0} \text{ for dielectric and imperfectly} \\ \quad \text{conducting targets.} \end{cases} \end{aligned} \tag{5.2}$$

We can see from (4.1) that, at least for low frequencies, the scattering dyadic is an even function of s going to zero at $s = 0$.

VI. Application of the Forward Scattering Theorem to the Scattering Dyadic

Now we can write the various cross sections in terms of the scattering dyadic, substituting in the formulae of Section II. The forward scattering takes the form

$$\begin{aligned}\tilde{A}_f(\vec{1}_i, \vec{1}_p; s) &= \frac{2\pi}{\gamma}\vec{1}_p \cdot [\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_i; s)] \cdot \vec{1}_p \\ &= \frac{2\pi}{\gamma}\vec{1}_p \cdot [\tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i, \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i, \vec{1}_i; s)] \cdot \vec{1}_p. \end{aligned} \tag{6.1}$$

This has various alternate forms from the transpose, such as

$$\begin{aligned}\tilde{A}_f(\vec{1}_i, \vec{1}_p; s) &= \frac{2\pi}{\gamma}\vec{1}_p \cdot \left[\tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, -\vec{1}_i; s) - \tilde{\overleftrightarrow{\Sigma}}^T(-\vec{1}_i, -\vec{1}_i; -s)\right] \\ &\quad \cdot \vec{1}_p \\ &= \tilde{A}_f(-\vec{1}_i, \vec{1}_p; s) \\ &= \text{even function of } s \\ &\geq 0 \text{ for } s = j\omega \end{aligned} \tag{6.2}$$

showing symmetry on reversal of the direction of incidence. This can also be stated from general symmetry conditions in [10] as

$$\tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, -\vec{1}_i; s) = \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i, \vec{1}_i; s). \tag{6.3}$$

The far scattering gives

$$\begin{aligned}\tilde{A}_\infty^{(\mathrm{sc})}(\vec{1}_i, \vec{1}_p; s) &= \vec{1}_p \cdot \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, s) \cdot \vec{1}_p \\ \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, s) &= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_r, \vec{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s) dS \\ &= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, -\vec{1}_r; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s) dS \\ \tilde{\overleftrightarrow{\Xi}}^T(\vec{1}_i, s) &= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\vec{1}_i, -\vec{1}_r; s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; -s) dS \\ &= \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, -s) \\ \tilde{\overleftrightarrow{\Xi}}^\dagger(\vec{1}_i, j\omega) &= \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, j\omega) \text{ (Hermitian)} \\ \dagger &\equiv T* \equiv \text{adjoint.}\end{aligned} \tag{6.4}$$

The above has various alternate forms. Recall that

$$\begin{aligned}\tilde{A}_\infty^{(\mathrm{sc})}(\vec{1}_i, \vec{1}_p; s) &= \text{even function of s} \\ &\geq 0 \quad \text{for } s = j\omega.\end{aligned} \tag{6.5}$$

If the scatterer is lossless, then (3.4) gives

$$\begin{aligned}&\tilde{A}_\infty^{(\mathrm{sc})}(\vec{1}_i, \vec{1}_p; s) \\ &\quad = \tilde{A}_f(\vec{1}_i, \vec{1}_p; s) \\ &\vec{1}_p \cdot \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, s) \cdot \vec{1}_p \\ &\quad = \frac{2\pi}{\gamma}\vec{1}_p \cdot \left[\tilde{\overleftrightarrow{\Sigma}}^T(\vec{1}_i, \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\vec{1}_i, \vec{1}_i; s)\right] \cdot \vec{1}_p \\ &\quad = \frac{1}{2\gamma}\vec{1}_p \cdot \left[\tilde{\overleftrightarrow{\Lambda}}^T(\vec{1}_i, \vec{1}_i; -s) - \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_i, \vec{1}_i; s)\right] \cdot \vec{1}_p \\ &= \text{even function of } s.\end{aligned} \tag{6.6}$$

For $s = j\omega$ a lossless scatterer has

$$\begin{aligned}&\vec{1}_p \cdot \tilde{\overleftrightarrow{\Xi}}(\vec{1}_i, j\omega) \cdot \vec{1}_p \\ &\quad = -\frac{jc}{2\omega}\vec{1}_p \cdot \left[\tilde{\overleftrightarrow{\Lambda}}^T(\vec{1}_i, \vec{1}_i; -j\omega) - \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_i, \vec{1}_i; j\omega)\right] \cdot \vec{1}_p \\ &\quad = -\frac{c}{\omega}\vec{1}_p \cdot \mathrm{Im}(\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_i, \vec{1}_i; j\omega)) \cdot \vec{1}_p \\ &\geqslant 0.\end{aligned} \tag{6.7}$$

Letting $q = 1, 2$ denote the diagonal elements (transverse to $\vec{1}_i$), we then have

$$\tilde{\Xi}_{q,q}(\vec{1}_i, j\omega) = -\frac{c}{\omega}\mathrm{Im}(\tilde{\Lambda}_{q,q}(\vec{1}_i, \vec{1}_i; j\omega)) \geq 0 \tag{6.8}$$

For positive ω then the imaginary part of the diagonal elements of the forward scattering dyadic are negative. The off-diagonal elements are treated later.

Going to high frequencies in the RHP then (4.2) also implies for a lossless opaque scatterer

$$\tilde{\Xi}_{q,q}(\vec{1}_i, j\omega) = 2A_c(\vec{1}_i) \text{ as } \omega \to \propto . \tag{6.9}$$

The off-diagonal elements are treated later.

VII. Extension of the Forward-Scattering Theorem

Following the procedure in [8] the scattering is cast into incoming and outgoing waves and takes the form

$$\begin{aligned}&\tilde{\vec{E}}^1_f(\vec{r}, \vec{1}_i, \vec{1}_p; s) \\ &\quad = \frac{2\pi}{\gamma}\delta(\vec{1}_i + \vec{1}_r)\vec{1}_p\frac{e^{\gamma r}}{r}\text{(incoming wave)} \\ &\qquad + [\tilde{\overleftrightarrow{\Sigma}}(\vec{1}_r, \vec{1}_i; s) \cdot \vec{1}_p - \frac{2\pi}{\gamma}\delta(\vec{1}_i - \vec{1}_r)\vec{1}_p] \\ &\qquad \times \frac{e^{-\gamma r}}{r}\text{(outgoing wave).}\end{aligned} \tag{7.1}$$

Note that this field $\tilde{\vec{E}}'_f$ is to be distinguished from the scattered far field as in (2.3) since it includes a far-field form of a special type of wave (not due to plane-wave scattering). It includes both incoming and outgoing waves, part of which is a special kind of incident wave. The reader should consult [8] for the complete derivation. This is found by extension of the procedure in [7] in which an incident plane wave is expanded in terms of spherical wave functions and the appropriate limit is taken as $r \to \infty$. This gives a special interpretation to $\tilde{\overrightarrow{E}}'$. (Such expansions are also given in ([3, Appendix B].) As we shall see this gives results consistent with the foregoing.

The outgoing wave can be related to the incoming wave by a dyadic as

$$\begin{aligned}
&\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s) - \frac{2\pi}{\gamma}\delta(\overrightarrow{1}_i - \overrightarrow{1}_r)\overleftrightarrow{1}_i \\
&\quad = -\tilde{\overleftrightarrow{R}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s)\frac{2\pi}{\gamma} \\
&\overrightarrow{1}_r \cdot \tilde{\overleftrightarrow{R}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s) \\
&\quad = \overrightarrow{0} = \tilde{\overleftrightarrow{R}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s) \cdot \overrightarrow{1}_i \\
&\tilde{\overleftrightarrow{R}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; s) \\
&\quad = \tilde{\overleftrightarrow{R}}(-\overrightarrow{1}_i, -\overrightarrow{1}_r; s)(\text{reciprocity}) \\
&\tilde{\overleftrightarrow{R}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s) \\
&\quad \equiv \text{wave reversal dyadic}
\end{aligned} \tag{7.2}$$

where $\delta(\overrightarrow{1}_r + \overrightarrow{1}_i)$ has been absorbed into the definition of $\tilde{\overleftrightarrow{R}}$ (by integration over S_1). We then have

$$\tilde{\overleftrightarrow{R}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s) = \overleftrightarrow{1}_i\delta(\overrightarrow{1}_i - \overrightarrow{1}_r) - \frac{\gamma}{2\pi}\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s). \tag{7.3}$$

As in [8] the sign is chosen so that $\tilde{\overleftrightarrow{R}}$ reduces to an identity when there is no scattering.

Next we form

$$\begin{aligned}
&\int_{S_1} \tilde{\overleftrightarrow{R}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; -s) \cdot \tilde{\overleftrightarrow{R}}(\overrightarrow{1}_r, \overrightarrow{1}_{i'}; s)\, dS_1 \\
&\quad = \int_{S_1} \left[\overleftrightarrow{1}_i\delta(\overrightarrow{1}_i - \overrightarrow{1}_r) + \frac{\gamma}{2\pi}\tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; -s)\right] \\
&\qquad \cdot \left[\overleftrightarrow{1}_{i'}\delta(\overrightarrow{1}_{i'} - \overrightarrow{1}_r) - \frac{\gamma}{2\pi}\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s)\right] dS \\
&\quad = \overleftrightarrow{1}_i\delta(\overrightarrow{1}_i - \overrightarrow{1}_{i'}) + \frac{\gamma}{2\pi}\tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_{i'}, \overrightarrow{1}_i; -s) \\
&\qquad - \frac{\gamma}{2\pi}\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, \overrightarrow{1}_{i'}; s) - \frac{\gamma^2}{4\pi^2} \\
&\qquad \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_{i'}; s)\, dS.
\end{aligned} \tag{7.4}$$

For the case of a lossless scatterer [8] shows that

$$\int_{S_1} \tilde{\overleftrightarrow{R}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; -s) \cdot \tilde{\overleftrightarrow{R}}(\overrightarrow{1}_r, \overrightarrow{1}_{i'}; -s)\, dS = \overleftrightarrow{1}_i\delta(\overrightarrow{1}_i - \overrightarrow{1}_{i'}) \tag{7.5}$$

Hence, for the lossless case we have in our present notation

$$\begin{aligned}
&\int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_{i'}; s) dS \\
&\quad = \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\overrightarrow{1}_i, -\overrightarrow{1}_r; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_{i'}; s)\, dS \\
&\quad = \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_{i'}, \overrightarrow{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, \overrightarrow{1}_{i'}; s)\right] \\
&\quad = \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}(-\overrightarrow{1}_i, -\overrightarrow{1}_{i'}; -s) - \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, \overrightarrow{1}_{i'}; s)\right]
\end{aligned} \tag{7.6}$$

which is now a dyadic rather than a scalar equality. Note that the above have been derived for $s = j\omega$ so that frequency-domain energy conservation can be applied. The above form can be regarded as the analytic continuation away from the $j\omega$ axis into the s plane. Also observe that it combines two directions, $\overrightarrow{1}_i$ and $\overrightarrow{1}_{i'}$, for the incident field.

Specializing to the case of $\overrightarrow{1}_{i'} = \overrightarrow{1}_i$ we have for lossless scatterers

$$\begin{aligned}
\tilde{\overleftrightarrow{\Xi}}(\overrightarrow{1}_i, s) &= \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_i, \overrightarrow{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, \overrightarrow{1}_i; s)\right] \\
\tilde{\overleftrightarrow{\Xi}}(\overrightarrow{1}_i, s) &\equiv \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s)\, dS \\
&= \int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\overrightarrow{1}_i, -\overrightarrow{1}_r; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s)\, dS.
\end{aligned} \tag{7.7}$$

These are the same dyadic functions encountered in Section VI. Recall that in (6.4) we found a scalar form which only applied to the diagonal terms of $\tilde{\Xi}$. Now (7.7) gives a forward-scattering theorem (for lossless scatterers) by equating the two forms of $\tilde{\Xi}$ which is dyadic in nature, applying to both diagonal and off-diagonal parts of $\tilde{\overleftrightarrow{\Sigma}}$. Now we have relations for the off-diagonal elements as well. This is a generalized form of a Hermitian dyadic, analytically continuing the true Hermitian form for $s = j\omega$ out into the s-plane.

If, on the other hand, we look at backscattering, we have

$$\begin{aligned}
&\overrightarrow{1}_{i'} = -\overrightarrow{1}_i \\
&\int_{S_1} \tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_r, \overrightarrow{1}_i; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, -\overrightarrow{1}_i; s)\, dS \\
&\qquad = \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}^T(-\overrightarrow{1}_i, \overrightarrow{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, -\overrightarrow{1}_i; s)\right] \\
&\int_{S_1} \tilde{\overleftrightarrow{\Sigma}}(-\overrightarrow{1}_i, -\overrightarrow{1}_r; -s) \cdot \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, -\overrightarrow{1}_i; s)\, dS \\
&\qquad = \frac{2\pi}{\gamma}\left[\tilde{\overleftrightarrow{\Sigma}}(-\overrightarrow{1}_i, \overrightarrow{1}_i; -s) - \tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, -\overrightarrow{1}_i; s)\right]
\end{aligned} \tag{7.8}$$

for lossless scatterers. I suppose that this could be called a backscattering theorem. Again, this is a dyadic, rather than a scalar, result.

VIII. High-Frequency Extended Forward Scattering Theorem

Recall from (4.2)

$$\tilde{\overleftrightarrow{\Lambda}}(\overrightarrow{1}_i, \overrightarrow{1}_i; s) = -2\gamma A_c(\overrightarrow{1}_i)\overleftrightarrow{1}_i \quad \text{as } \mathrm{Re}(s) \to +\infty. \quad (8.1)$$

Then (7.7) gives

$$\frac{2\pi}{\gamma}\tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_i, \overrightarrow{1}_i; -s) + A_c(\overrightarrow{1}_i)\overleftrightarrow{1}_i = \tilde{\overleftrightarrow{\Xi}}(\overrightarrow{1}_i, s) \quad \text{as } \mathrm{Re}(s) \to +\infty. \quad (8.2)$$

IX. Application to Poles in the Singularity Expansion Method

The scattering dyadic has the form [5], [15], [11] (the well-known singularity-expansion-method (SEM) result, extended from [12])

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}(\overrightarrow{1}_r, \overrightarrow{1}_i; s) &= \sum_\alpha \frac{e^{-[s-s_\alpha]t_i}}{s-s_\alpha}\overrightarrow{c}_\alpha(-\overrightarrow{1}_r)\overrightarrow{c}_\alpha(\overrightarrow{1}_i) \\ &\quad + \text{entire function of s} \\ \overrightarrow{c}_\alpha(\overrightarrow{1}_i) &\equiv \text{scattering natural mode} \\ t_i &\equiv \text{turn on time}(=0 \text{ for simplicity}). \quad (9.1)\end{aligned}$$

Place this in (7.6) and take the limit as $s \to s_\alpha$ for some chosen s_α. This gives (for lossless scatterers)

$$\begin{aligned}\overrightarrow{c}_\alpha(\overrightarrow{1}_i)\int_{S_1}\overrightarrow{c}_\alpha(-\overrightarrow{1}_r)\cdot\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_{i'}; -s_\alpha)\,dS &= -\frac{2\pi}{\gamma_\alpha}\overrightarrow{c}_\alpha(\overrightarrow{1}_i)\overrightarrow{c}_\alpha(-\overrightarrow{1}_{i'}) \\ \gamma_\alpha = \frac{s_\alpha}{c}. \quad (9.2)\end{aligned}$$

Removing $\overrightarrow{c}_\alpha(\overrightarrow{1}_i)$ (common factor) gives

$$\int_{S_1}\overrightarrow{c}_\alpha(-\overrightarrow{1}_r)\cdot\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_r, \overrightarrow{1}_{i'}; -s_\alpha)\,dS = -\frac{2\pi}{\gamma_\alpha}\overrightarrow{c}_\alpha(-\overrightarrow{1}_{i'}) \quad (9.3)$$

relating natural modes in the LHP to the scattering dyadic in the RHP. A high-frequency asymptotic expansion of the scattering dyadic in the RHP can be used to evaluate this integral equation for specific scatterer shapes.

This can be cast in other forms as

$$\begin{aligned}-\frac{2\pi}{\gamma_\alpha}\overrightarrow{c}_\alpha(\overrightarrow{1}_i) &= \int_{S_1}\tilde{\overleftrightarrow{\Sigma}}^T(\overrightarrow{1}_r, -\overrightarrow{1}_i; -s_\alpha)\cdot\overrightarrow{c}_\alpha(-\overrightarrow{1}_r)dS \\ &= \int_{S_1}\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, -\overrightarrow{1}_r; -s_\alpha)\cdot\overrightarrow{c}_\alpha(-\overrightarrow{1}_r)dS \\ &= \int_{S_1}\tilde{\overleftrightarrow{\Sigma}}(\overrightarrow{1}_i, \overrightarrow{1}_r; -s_\alpha)\cdot\overrightarrow{c}_\alpha(\overrightarrow{1}_r)dS \quad (9.4)\end{aligned}$$

where changes of variables have been applied. Note that the $\overrightarrow{c}_\alpha$ have an arbitrary scaling constant in this integral equation.

For special cases the natural modes can be degenerate (say doubly degenerate for bodies of revolution). In such cases similar results hold, except now (9.4) has multiple solutions given by the order of the degeneracy. Also, while the poles in (9.1) have been taken as first order, higher order poles give the same result.

For the case of $\mathrm{Re}(s_\alpha) \to -\infty$ (far LHP) the scattering dyadic in (9.4) can be evaluated by high-frequency asymptotic methods (such as the geometrical theory of diffraction (GTD), etc.), valid in the RHP $(\mathrm{Re}(s) \to +\infty)$. This gives an integral equation for the $\overrightarrow{c}_\alpha$ and s_α and relates LHP and RHP behavior.

X. Concluding Remarks

The forward-scattering theorem and its extensions give various interesting results. This is particularly useful for the case of lossless scatterers, since we do not need to solve the scattering problem for the absorption by the scatterer.

One of the interesting extensions is that into the s plane away from the $j\omega$ axis. This gives some analytic properties of the scattering dyadic. Besides relating LHP and RHP behavior, there may be some additional properties to be discovered.

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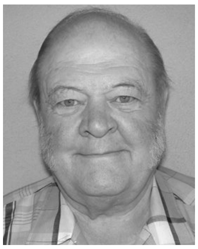

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