Ex 1 Trace : $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$

is it possible to have $AB - BA = I_n$?

sol the problem is harder if you don't know that you should use the trace. Let's take the trace of the equation above:

$$tr(AB - BA) \underset{\text{linearity of trace}}{=} tr(AB) - tr(BA) \underset{\text{invariability by rotation}}{=} tr(AB) - tr(AB) = 0$$

while $tr(I_n) = tr\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = n > 0$

so this equality cannot be true $\square$
↳ $AB - BA = I_n$

Ex 2 We consider $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ such that:

$$(*) \quad AB - BA = A$$

Calculate $tr(A^p)$ for $p \in \mathbb{N}^*$

sol Let's try to calculate it directly

$$tr(A^p) = tr(A^{p-1} . A) \underset{(*)}{=} tr(A^{p-1}(AB - BA)) = tr(A^{p-1} . A . B - A^{p-1} BA)$$

Trick
Because we only know an equation of A, we have to extract it

$$\underset{\text{linearity of trace}}{=} tr(A^p B) - tr(A^{p-1} BA)$$

$$\underset{\text{invariant by rotation}}{=} tr(A^p B) - \underbrace{tr(A A^{p-1} B)}_{tr(A^p . B)}$$

$$= \boxed{0}$$

$\square$

Ex3 (determinant)

We consider $A \in \mathbb{R}^{(2n+1)\times(2n+1)}$ and A being anti-symmetric

Show that $\det A = 0$

<u>sol</u> First let's extract the information we have above:

(i) A is antisymmetric means that : $A = -A^T$

(ii) A's size is odd $\Rightarrow$ it might be useful: it reminds us of $(-1)^{2n+1} = -1$

Let's try to calculate directly then :

$$\det A = \det(-A^t) = (-1)^{2n+1} \det A^t \overset{(ii)}{=} (-1) \det A$$

A antisymmetric (i) property of det property of det : $\det A^t = \det A$.

so we have $\det A = -\det A$

ie $2 \det A = 0$ ie $\boxed{\det A = 0}$ □

Ex4 A matrix A is said to be idempotent if $A^2 = A$

(i) Show that the only possible eigenvalues for A are 0 and 1.

(ii) Give an example of an idempotent matrix that has eigenvalues 0 and 1 (both).

<u>sol</u> : (i) This question can be resolved with this classic method

1) Assuming the existence of λ, an eigenvalue of A

2) Showing that λ is necessarily 0 or 1.

Let's consider (λ, x) an eigenvalue/eigenvector pair of A.

we have $Ax = \lambda x$ (definition)

$$\text{so } \lambda x = A^2 x = A.Ax = A\underset{\in \mathbb{R}}{\lambda} x = \lambda Ax = \lambda^2 x$$

$A = A^2$ separate A^2 definition $Ax = \lambda x$

and because x is a non zero vector (eigenvector)

we have $\lambda = \lambda^2 \Longrightarrow \boxed{\lambda = 0 \text{ or } \lambda = 1}$ □

(ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ works because $A^2 = A$ and we can read the eigenvalues on the diagonal $\Rightarrow \lambda_1 = 1 ; \lambda_2 = 0$

$\boxed{\text{Ex5}}$ We consider $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x \longmapsto \sum_{i=1}^{n} |x_i|$ " $\forall$ " = for all

Show that f is a norm in $\mathbb{R}^n$.

Sol The method to show that a function is a norm is :

① Show Positivity : ie $\forall x \in \mathbb{R}^n \; f(x) \geq 0$

② Show Definitness : ie $x = 0 \iff f(x) = 0$

③ Show Homogeneity : ie $\forall x \in \mathbb{R}^n \; \forall \lambda \in \mathbb{R} \; f(\lambda x) = |\lambda| f(x)$

most often, this is the hard part ④ Show triangular inequality : ie $\forall (x,y) \in (\mathbb{R}^n)^2 \; f(x+y) \leq f(x) + f(y)$

Let's do it :

① $\forall i \in [\![1, n]\!]$, we know that $|x_i| \geq 0$

so by summing them together we have : $\sum_{i=1}^{n} |x_i| \geq 0 \Rightarrow f(x) \geq 0$

② ($\Rightarrow$) if $x = 0$ then $\forall i \in [\![1, n]\!] \; x_i = 0$ because $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

then by summing all the x_i's we have

$$f(x) = \sum_{i=1}^{n} |x_i| = 0$$

⚠ don't forget the other side

($\Leftarrow$) if $f(x) = 0$ we have $\sum_{i=1}^{n} |x_i| = 0$

which means that $\forall i \in [\![1, n]\!] \; |x_i| = 0$

so $\forall i \in [\![1, n]\!] \; x_i = 0$ and thus $x = 0$

③ $\forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$

$$f(\lambda x) = \sum_{i=1}^{n} |\lambda x_i| = \sum_{i=1}^{n} |\lambda| |x_i| \overset{\text{factorize}}{=} |\lambda| \sum_{i=1}^{n} |x_i| = |\lambda| f(x)$$

④ $\forall (x,y) \in (\mathbb{R}^n)^2$,

because $\forall i \in [\![1, n]\!] \; |x_i + y_i| \leq |x_i| + |y_i|$

$$f(x+y) = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} (|x_i| + |y_i|) = f(x) + f(y)$$

We proved ①②③④ $\Rightarrow$ f is a norm!

Practice Prove that

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x \longmapsto \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$ is a norm

$g : \mathbb{R}^n \longrightarrow \mathbb{R}$, $x \longmapsto \max |x_i|$ is a norm

$\boxed{Ex6}$ Find the local extrema of $f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2$

~~sol~~ METHOD : ① Calculate $\nabla f(x)$ and solve $\nabla f(x) = \vec{0}$

② Calculate $\nabla^2 f(x)$ and apply it at specific points (the solutions α of $\nabla f(x) = 0$)

③ Check if $\nabla^2 f(\alpha)$ is Positive/Negative Semidefinite

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 3x_1^2 - 3x_2 \\ 3x_2^2 - 3x_1 \end{pmatrix}$$

$$\nabla f(x) = 0 \Rightarrow \begin{cases} x_1^2 = x_2 \\ x_2^2 = x_1 \end{cases} \Rightarrow \begin{cases} (x_1, x_2) = (0,0) \\ \text{or} \\ (x_1, x_2) = (1,1) \end{cases}$$

These are inflexion points or local extremas? (the Hessian will tell us)

$$H(x) = \nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 6x_1 & -3 \\ -3 & 6x_2 \end{pmatrix}$$

so $H(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ and $H(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$

$H(0,0)$ PSD? $\rightarrow$ Symmetric OK

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\rightarrow$ $x^T H(0,0) x = -6x_1x_2$ which can be > 0 (if x_1, x_2 different signs) or < 0 (if x_1, x_2 same signs)

so $(0,0)$ is not a local extrema

$H(1,1)$ PSD? $\rightarrow$ symmetric OK

$\rightarrow$ $x^T H(1,1) x = 6(x_1^2 - x_1x_2 + x_2^2) \geq 0$

to prove this look at $(x_1 + x_2)^2 \geq 0$, $(x_1 - x_2)^2 \geq 0$

So $(1,1)$ is a local ~~minima~~ minima