VE281

Data Structures and Algorithms

Open Addressing; Universal Hashing

Announcement

- Programming Assignment Two posted
  - Due time: midnight, Oct. 12th

Outline

- Collision Resolution: Open Addressing
  - Linear Probing
  - Quadratic Probing and Double Hashing
  - Performance of Open Addressing

- Pathological Data Sets and Universal Hashing

Open Addressing

- Reuse empty space in the hash table to hold colliding items.

- To do so, search the hash table in some systematic way for a bucket that is empty.
  - Idea: we use a sequence of hash functions h_0, h_1, h_2, . . . to probe the hash table until we find an empty slot.
    - I.e., we **probe** the hash table buckets mapped by h_0(key), h_1(key), …, in sequence, until we find an empty slot.
    - Generally, we could define $h_i(x) = h(x) + f(i)$

Open Addressing

- Three methods:
  - Linear probing:
    $h_i(x) = (h(x) + i) \% n$
  - Quadratic probing:
    $h_i(x) = (h(x) + i^2) \% n$
  - Double hashing:
    $h_i(x) = (h(x) + i*g(x)) \% n$

n is the hash table size

Linear Probing

$$h_i(key) = (h(key)+i) \% n$$

- Apply hash function h_0, h_1, …, in sequence until we find an empty slot.
  - This is equivalent to doing a linear search from **`h(key)`** until we find an empty slot.

- Example: Hash table size **`n = 9`**, **`h(key) = key%9`**
  - Thus $h_i(key) = (key\%9+i)\%9$
  - Suppose we insert 1, 5, 11, 2, 17, 21, 31 in sequence

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
	1	11			5			

How about 2?

Linear Probing

Example

- Hash table size **`n = 9`**, **`h(key) = key%9`**
  - Thus $\mathbf{h_i}$**`(key) = (key%9+i)%9`**
  - Suppose we insert 1, 5, 11, 2, 17, 21, 31 in sequence.

	1	11	2	21	5	31		17
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

  - $\mathbf{h_0}$**`(2) = 2.`** Not empty!
  - So we try $\mathbf{h_1}$**`(2) = 3.`** It is empty, so we insert there!
  - $\mathbf{h_0}$**`(21) = 3.`** Not empty!
  - $\mathbf{h_1}$**`(21) = 4.`** It is empty, so we insert there!
  - $\mathbf{h_0}$**`(31) = 4.`** Not empty!
  - $\mathbf{h_1}$**`(31) = 5.`** Not empty!
  - $\mathbf{h_2}$**`(31) = 6.`** It is empty, so we insert there!

Linear Probing

find()

	1	11	2	21	5	31		17
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

- With linear probing `h`$_i$`(key) = (key%9+i)%9`
  - How will you **search** an item with key = 31?
  - How will you **search** an item with key = 10?

- Procedure: probe in the buckets given by h_0(key), h_1(key), …, in sequence **until**
  - we find the key,
  - or we find an empty slot, which means the key is not found.

Linear Probing

remove()

	1	11	2	21	5	31		17
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

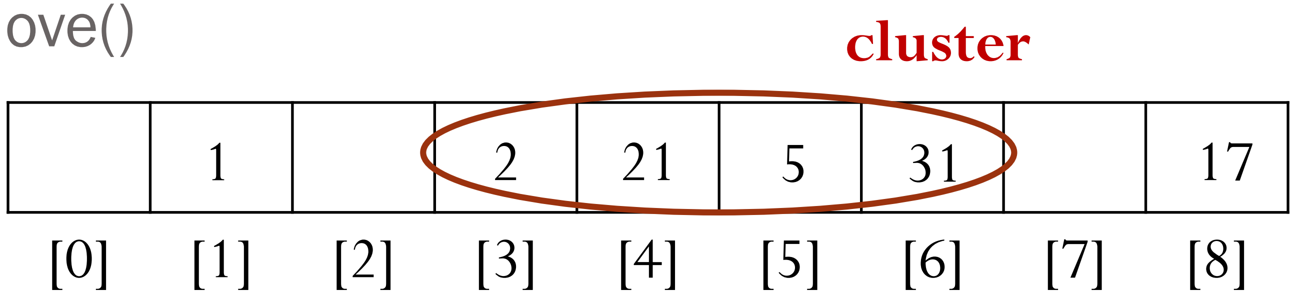
- With linear probing **`h`**$_i$ **`(key) = (key%9+i)%9`**
  - How will you **remove** an item with key = 11?
  - If we just find 11 and delete it, will this work?

	1		2	21	5	31		17
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

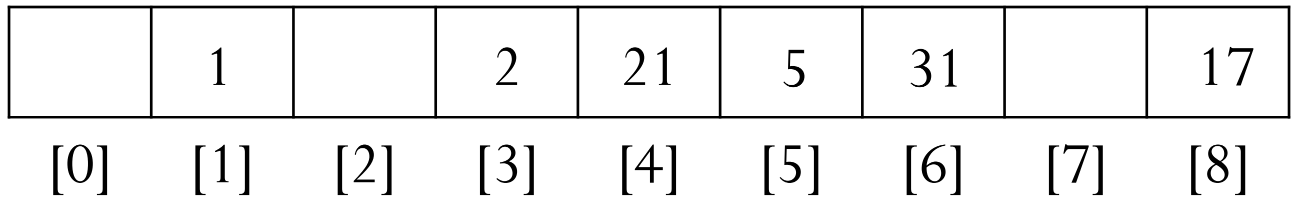
What is the result for searching key = 2 with the above hash table?

Linear Probing

remove()



- After deleting 11, we need to **rehash** the following “cluster” to fill the vacated bucket.
- However, we cannot move an item **beyond** its **actual** hash position. In this example, 5 cannot be moved ahead.



Linear Probing

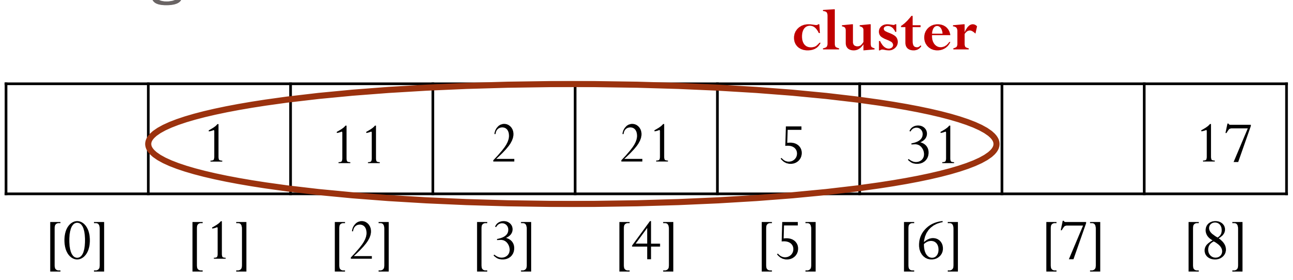
Alternative implementation of remove()

	1	~~11~~ **del**	2	21	5	31		17
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]

- **Lazy deletion**: we mark deleted entry as "**deleted**".
  - "deleted" is not the same as "empty".
  - Now each bucket has three states: "occupied", "empty", and "deleted".
- We can overwrite the "deleted" entry when inserting.
- When we **search**, we will keep looking if we encounter a "deleted" entry.

Linear Probing

Clustering Problem



- Clustering: when **contiguous** buckets are all occupied.
- **Claim**: Any hash value inside the cluster adds to **the end** of that cluster.
- Problems with a **large** cluster:
  - It becomes more likely that the next hash value will collide with the cluster.
  - Collisions in the cluster get more expensive to resolve.

Linear Probing

Clustering Problem

- Assuming input size N, table size 2N:
  - What is the best-case cluster distribution?

X		X		X		X	

  - What is the worst-case cluster distribution?

		X	X	X	X		

  - What's the average number of probes to find an empty slot in both cases?

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Quadratic Probing

$$\mathtt{h_i(key)} = \mathtt{(h(key)+i^2)\ \%\ n}$$

- It is less likely to form large clusters.
- Example: Hash table size $\mathtt{n = 7}$, $\mathtt{h(key) = key\%7}$
  - Thus $\mathtt{h_i(key) = (key\%7+i^2)\%7}$
  - Suppose we insert 9, 16, 11, 2 in sequence.

		9	16	11		2
[0]	[1]	[2]	[3]	[4]	[5]	[6]

  - $\mathtt{h_0(16) = 2}$. Not empty!
  - $\mathtt{h_1(16) = 3}$. It is empty, so we insert there.
  - $\mathtt{h_0(2) = 2}$. Not empty!
  - $\mathtt{h_1(2) = 3}$. Not empty!
  - $\mathtt{h_2(2) = 6}$. It is empty, so we insert there.

Problem of Quadratic Probing

- However, sometimes we will never find an empty slot even if the table isn't full!
- Luckily, if the **load factor** $L \leq 0.5$, we are guaranteed to find an empty slot.
  - Definition: given a hash table with n buckets that stores m objects, its **load factor** is

$$L = \frac{m}{n} = \frac{\text{\#objects in hash table}}{\text{\#buckets in hash table}}$$

More on Load Factor of Hash Table

- Question: which collision resolution strategy is feasible for load factor larger than 1?
  - Answer: separate chaining.
  - Note: for open addressing, we require $L \leq 1$.
- Claim: $L = O(1)$ is a necessary condition for operations to run in constant time.

Double Hashing

$$h_i(x) = (h(x) + i*g(x)) \% n$$

- Uses 2 distinct hash functions.

- Increment **differently** depending on the key.
  - If h(x) = 13, g(x) = 17, the probe sequence is 13, 30, 47, 64, …
  - If h(x) = 19, g(x) = 7, the probe sequence is 19, 26, 33, 40, …
  - For linear and quadratic probing, the incremental probing patterns are **the same** for all the keys.

Double Hashing
Example

- Hash table size $\mathtt{n = 7,\ h(key) = key\%7,}$ $\mathtt{g(key) = (5-key)\%5}$
  - Thus $\mathtt{h_i(key) = (key\%7+(5-key)\%5*i)\%7}$
  - Suppose we insert 9, 16, 11, 2 in sequence.

		9		11	2	16
[0]	[1]	[2]	[3]	[4]	[5]	[6]

  - $\mathtt{h_0(16) = 2.}$ Not empty!
  - $\mathtt{h_1(16) = 6.}$ It is empty, so we insert there.
  - $\mathtt{h_0(2) = 2.}$ Not empty!
  - $\mathtt{h_1(2) = 5.}$ It is empty, so we insert there.

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Performance of Open Addressing

- Hard to analyze rigorously.

- The runtime is dominated by the number of comparisons.

- The number of comparisons depends on the load factor L.

- Define the expected number of comparisons in an **unsuccessful search** as $U(L)$.
- Define the expected number of comparisons in a **successful search** as $S(L)$.

Expected Number of Comparisons

- Linear probing

$$U(L) = \frac{1}{2}\left[1 + \left(\frac{1}{1-L}\right)^2\right]$$
$$S(L) = \frac{1}{2}\left[1 + \frac{1}{1-L}\right]$$

L	$U(L)$	$S(L)$
0.5	2.5	1.5
0.75	8.5	2.5
0.9	50.5	5.5

$L \le 0.75$ is recommended.

Expected Number of Comparisons

- Quadratic probing and double hashing

$$U(L) = \frac{1}{1-L}$$
$$S(L) = \frac{1}{L}\ln\frac{1}{1-L}$$

L	$U(L)$	$S(L)$
0.5	2	1.4
0.75	4	1.8
0.9	10	2.6

Which Strategy to Use?

- Both separate chaining and open addressing are used in real applications.

- Some basic guidelines:
  - If space is important, better to use open addressing.
  - If need removing items, better to use separate chaining.
    - **`remove()`** is tricky in open addressing.
  - In mission critical application, prototype both and compare.

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Pathological Data Sets

- The **ideal** hash function spreads **every** data set out evenly.

- Does such **ideal** hash function exist?
  - No! For every hash function, there is a **pathological data set**.

- Reason: Fix a hash function $h: U \rightarrow \{0,1,\dots,n-1\}$
  - There exists a bucket i such that at least $|U|/n$ elements of U hash to i under h…
  - … if data set drawn only from these, everything collides!

Pathological Data Sets

- **Given**: A hash table with n buckets stores m items. Use separate chaining.
  - **Question**: If all m items are mapped to the same bucket, what's the time complexity of **`find()`**?

- **Answer**: The hash table degrades to a linked list.
  - The time complexity for **`find()`** is $O(m)$.
  - This is actually the **worst-case** time complexity.

Solution to Pathological Data Sets

- **Universal hashing**:
  - Design **a family** H of hash functions such that for **all** data set S, "**almost all**" functions $h \in H$ spread S out "**pretty evenly**".
  - Pick a hash function **randomly** from the family H.

Universal Family of Hash Functions

- Definition: Let H be **a set of hash functions** from U to $\{0,1,2,\ldots,n-1\}$. H is **universal** if and only if:
  - **For all** $x, y \in U$ with $x \neq y$,

$$\Pr_{h\in H}\big(h(x) = h(y)\big) \leq \frac{1}{n}$$

- In other words, **any** two keys of U collide with probability at most $1/n$ when the hash function h is chosen **uniformly at random** from H
- Note: keys x and y fixed. Random on hash function picked
  - At most 1/n of the total functions map x and y to the same bucket
- Collision probability is as small as our “gold standard” of **completely random hashing**

Universal Family of Hash Functions
Example

- Example #1: The set of **all** functions that map from U to $\{0,1,2,\ldots,n-1\}$.
  - The family contain $n^{|U|}$ functions.
- Is this family universal?

- Yes! Because for any keys $x \neq y$, exactly $1/n$ of the total functions map x and y to the same bucket
  - Partition all the functions into n^2 subsets $S_{i,j}$ $(0 \leq i,j \leq n-1)$, where $S_{i,j}$ contains functions h such that $h(x) = i$ and $h(y) = j$
  - The numbers of functions in all subsets are equal.

Universal Family of Hash Functions
Example

- Example #2: $\{h_0, h_1, \dots h_{n-1}\}$ where $h_i : U \rightarrow i$, i.e., for any $u \in U, h_i(u) = i$.
- Is this family universal?

- No! Because for any keys $x \neq y$, all functions map x and y to the same bucket, i.e.,
$$\Pr_{h \in H}\big(h(x) = h(y)\big) = 1$$

Real Example: Hashing IP Addresses

- Let $U =$ IP address of the form (x_1, x_2, x_3, x_4) with each $x_i \in \{0,1,\dots,255\}$
- Let hash table size n be a **prime** number and $n > 255$.
  - Could be close to a multiple of #objects in the hash table.

- Define one hash function h_a per 4-tuple $a = (a_1, a_2, a_3, a_4)$ with each $a_i \in \{0,1,\dots,n-1\}$.
  - $h_a(x_1, x_2, x_3, x_4) = (a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)\ mod\ n$
  - There are n^4 such functions.

A Universal Family of Hash Functions

- **Define** the family H = all n^4 h_a's, i.e.,

$$H = \{h_a | a_1, a_2, a_3, a_4 \in \{0,1,\dots,n-1\}\}$$

$$h_a(x_1, x_2, x_3, x_4) = (a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)\ mod\ n$$

Theorem: Family H is universal.

Proof

- Consider distinct IP addresses (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4)
- Assume $x_4 \neq y_4$. We need to show the collision probability for a randomly chosen function $h_a \in H$ is at most $1/n$, i.e.,

$$\Pr_{h_a \in H}(h_a(x_1, \dots, x_4) = h_a(y_1, \dots, y_4)) \leq \frac{1}{n}$$

- Note: collision happens when

$$a_1 x_1 + \cdots a_4 x_4 = a_1 y_1 + \cdots a_4 y_4 \ (mod\ n)$$

$$\Leftrightarrow a_4(x_4 - y_4) = \sum_{i=1}^{3} a_i(y_i - x_i) \ (mod\ n)$$

Next: let's fix random choice a_1, a_2, a_3, but a_4 still random

Proof (cont.)

- Question: with a_1, a_2, a_3 fixed arbitrarily, how many choices of a_4 satisfy

$$a_4(x_4 - y_4) = \underbrace{\sum_{i=1}^{3} a_i(y_i - x_i)\ \ (mod\ n)}_{\text{fixed value}}\ ?$$

- Note:
  1. $0 \leq x_4 \neq y_4 \leq 255$
  2. $n > 255$ is prime
- Claim: For any $b \in \{0, \dots, n-1\}$, there is at most one $a_4 \in \{0, \dots, n-1\}$ that let $a_4(x_4 - y_4) = b\ (mod\ n)$
- Proof: for any $a_4{}' \in \{0, \dots, n-1\}$ and $a_4{}' \neq a_4$,

$$(a_4 - a_4{}')(x_4 - y_4) \neq 0 (mod\ n)$$

Imply $\Pr_{h_a \in H}(h_a(x_1, \dots, x_4) = h_a(y_1, \dots, y_4)) \leq \frac{1}{n}$

Q.E.D.