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Loop Quantum Geometry: A primer

Alejandro Corichi*

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
A. Postal 70-543, México D.F. 04510, México

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Abstract

This is the written version of a lecture given at the "VI Mexican School of Gravitation and Mathematical Physics" (Nov 21-27, 2004, Playa del Carmen, México), introducing the basics of Loop Quantum Geometry. The purpose of the written contribution is to provide a Primer version, that is, a first entry into Loop Quantum Gravity and to present at the same time a friendly guide to the existing pedagogical literature on the subject. This account is geared towards graduate students and non-experts interested in learning the basics of the subject.

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*Electronic address: corichi@nucleares.unam.mx

I. INTRODUCTION

Loop Quantum Gravity (LQG) has become in the past years a mayor player as a candidate for a quantum theory of gravity. On the one hand it has matured into a serious contender together with other approaches such as *String/M Theory*, but on the other is it not as well understood, neither properly credited as a real physical theory of quantum gravity. The purpose of this contribution is to provide a starting point for those interested in learning the basics of the theory and to provide at the same time an introduction to the rich literature on the subject which includes very well written reviews and monographs. Given the space constraint we shall not attempt to write a comprehensive review of LQG, but to provide, we hope, and useful guide to the subject.

Let us start by providing a list of references that will be useful in the various stages. Firstly, there are several primer introductions to the subject, written for different purposes. For instance, there was for many years the canonical primer by Pullin [1]. Unfortunately, it is now somewhat dated. Good introductions to spin networks and recoupling theory needed in LQG are given by the primers by Rovelli [2] and Major [3]. There are recent up-to-date accounts written for non-experts that give nice motivation, historical perspective and an account of recent and in progress work from two different perspectives [4] and [5]. There are also technical reviews that give many details and are certainly a good read [6], [7], [8], and (from an outside perspective) [9].

The subject has matured enough so that several monographs have been written, including some recent and updated. These monographs approach and present the subject from different perspectives depending, of course, on the authors own taste. From these, it is worth mentioning two. The first one by Rovelli is physically motivated but not so heavy in its mathematical treatment, and can be found in [10]. A mathematically precise treatment, but not for the faint of heart is given by the monograph by Thiemann [11]. There have been also several nice reviews that motivate and give a birdseye view of the subject such as [12], [13] and [14]. Finally, there are several accounts on comparisons between loop quantum gravity and other approaches, such as string theory. On chronological order, we have a review by Rovelli [15], an entertaining dialog [17] and a critical assessment by Smolin [18].

The second purpose of this paper is to present an introduction to the formalism known as *(loop) quantum geometry*. The difference between quantum geometry and loop quantum gravity is that the former is to be thought of as the (new) formalism dealing with background-free quantum theories based on connections, whereas the later is a particular implementation where gravity, as defined by general relativity, is the theory under consideration. For instance, one could think of applying the same formalism to more general theories such as supergravity and/or higher derivative theories. In the remainder of this section we shall give a motivation for why one should study loop quantum gravity, when one is interested in the basic problem of uniting quantum mechanics and the theory of gravitation.

Why should one study loop quantum gravity? For one thing, it is based on two basic principles, namely the general principles of quantum theory and one of the main lessons from general relativity: that physics is diffeomorphism invariant. This means that the field describing the gravitational interaction, and the geometry of spacetime is fully dynamical and interacting with the rest of the fields present. When one is to consider its quantum description, this better be background independent. The fact that LQG is based in general principles of quantum mechanics means only that one is looking for a description based on the standard language of quantum mechanics: states are elements on a Hilbert space (well

defined, of course), observables will be Hermitian operators thereon, etc. This does not mean that one should use *all* that is already known about quantizing fields. Quite on the contrary, the tools needed to construct a background independent quantization (certainly not like the quantization we know), are rather new.

Another reason for studying LQG is that this is the most serious attempt to perform a full *non-perturbative* quantization of the gravitational field. It is an attempt to answer the following question: can we quantize the gravitational degrees of freedom without considering matter on the first place? Since LQG aims at being a physical theory, which means it better be falsifiable, one expects to answer that question unambiguously, whenever one has the theory fully developed. This is one of the main present challenges of the theory, namely to produce predictions that can be tested experimentally. Since the theory does not suffer from extra dimensions nor extended symmetries, one expects that the task will be feasible to complete, without the burden of getting rid on those extra features. Will this be the final theory describing the quantum degrees of freedom of the gravitational field? Only experiments will tell, but for the time being, LQG remains an intriguing possibility very well worth the trial.

This contribution is organized as follows: In Sec. II we provide some of material needed in order to be ready to fully grasp the details of LQG. In particular, we recall the canonical description go general relativity in the geometrodynamics language and perform the change of variables to go to the description in terms of connections. In Sec. III we consider within the classical description of the gravitational field, the observables that will be regarded as the basic objects for the quantization. We shall see that both background independence and diffeomorphism invariance lead us to select holonomies and electric fluxes as the basic objects. Section IV will be devoted to the discussion of the Hilbert space of the theory, its multiply characterizations and a particular basis that is very convenient, namely, the spin network basis. In Section V we provide a discussion of the new (quantum) geometry that the theory presents for us, focusing on basic geometrical operators. Section VI will be devoted to a list of accomplishments and open issues.

Finally, we should note that our list of references is minimalist, trying to concentrate on review articles or monographs rather than in the original articles. A more complete reference list can be found in the books [11] and [10] and in the bibliography compilation [20]. An online guide for learning LQG with references can be found in [19].

II. PRELIMINARIES

What are the pre-requisites for learning LQG? First of all, a reasonable knowledge of General Relativity (GR), specially its Hamiltonian formulation as in [21] and [22], acquaintance with QFT on flat space-time and preferably some notions of "quantization", namely the passage from a classical theory to a quantum one as in [23]. Finally, the language of gauge field theories is essential, including connections and holonomies. A good introduction to the subject, including the necessary geometry, is given in [24].

The first step is to introduce the basic classical variables of the theory. Since the theory is described by a Hamiltonian formalism, this means that the 4-dim spacetime M is of the form $M = \Sigma \times \mathbb{R}$, where Σ is a 3-dimensional manifold. The first thing to do is to start with the geometrodynamical phase space $\Gamma_{\rm g}$ of Riemannian metrics q_{ab} an Σ and their canonical momenta $\tilde{\pi}^{ab}$ (related to the extrinsic curvature K_{ab} of Σ into M by $\tilde{\pi}^{ab} = \sqrt{q}\,(K^{ab} - \frac{1}{2} q^{ab}\, K)$,

with $q = \det(q_{ab})$ and $K = q^{ab}K_{ab}$). Recall that they satisfy,

$$\{\tilde{\pi}^{ab}(x), q_{cd}(y)\} = 2\kappa\, \delta^a_{(c}\, \delta^b_{d)}\, \delta^3(x,y) \quad ; \quad \{q_{ab}(x), q_{cd}(y)\} = \{\tilde{\pi}^{ab}(x), \tilde{\pi}^{cd}(y)\} = 0 \qquad (1)$$

General Relativity in these geometrodynamical variables is a theory with constraints, which means that the canonical variables $(q_{ab}, \tilde{\pi}^{ab})$ do not take arbitrary values but must satisfy four constraints:

$$\mathcal{H}^b = D_a\,(\tilde{\pi}^{ab}) =\approx 0 \qquad \text{and,} \qquad \mathcal{H} = \sqrt{q}\,\left[R^{(3)} + q^{-1}(\tfrac{1}{2}\tilde{\pi}^2 - \tilde{\pi}^{ab}\tilde{\pi}_{ab})\right] \approx 0 \qquad (2)$$

The first set of constraints are known as the vector constraint and what they generate (its gauge orbit) are spatial diffeomorphisms on Σ. The other constraint, the scalar constraint (or super-Hamiltonian) generates "time reparametrizations". We start with 12 degrees of freedom, minus 4 constraints means that the constraint surface has 8 dimensions (per point) minus the four gauge orbits generated by the constraints giving the four phase space degrees of freedom, which corresponds to the two polarizations of the gravitational field.

In order to arrive at the connection formulation, we need first to enlarge the phase space $\Gamma_{\rm g}$ by considering not metrics q_{ab} but the co-triads e^i_a that define the metric by,

$$q_{ab} = e^i_a\, e^j_b\, \delta_{ij} \qquad (3)$$

where $i, j = 1, 2, 3$ are internal labels for the frames. These represent 9 variables instead of the 6 defining the metric q_{ab}, so we have introduced more variables, but at the same time a new symmetry in the theory, namely the $SO(3)$ rotations in the triads. Recall that a triad e^i_a and a rotated triad $e'^i_a(x) = U^i{}_j(x)\, e^j_a(x)$ define the same metric $q_{ab}(x)$, with $U^i{}_j(x) \in SO(3)$ a local rotation. In order to account for the extra symmetry, there will be extra constraints (first class) that will get rid of the extra degrees of freedom introduced. Let us now introduce the densitized triad as follows:

$$\tilde{E}^a_i = \tfrac{1}{2}\,\epsilon_{ijk}\tilde{\eta}^{abc}\, e^j_b\, e^k_c \qquad (4)$$

where $\tilde{\eta}^{abc}$ is the naturally defined levi-civita density one antisymmetric object. Note that $\tilde{E}^a_i \tilde{E}^b_j \delta^{ij} = q\, q^{ab}$.

Let us now consider the canonical variables. It turns out that the canonical momenta to the densitized triad $\tilde{E}^a_i$ is closely related to the extrinsic curvature of the metric,

$$K^i_a = \frac{1}{\sqrt{\det(\tilde{E})}}\, \delta^{ij}\tilde{E}^b_j\, K_{ab} \qquad (5)$$

For details see [7]. Once one has enlarged the phase space from the pairs $(q_{ab}, \tilde{\pi}^{ab})$ to $(\tilde{E}^a_i, K^j_b)$, the next step is to perform the canonical transformation to go to the $Ashtekar - Barbero$ variables. First we need to introduce the so called *spin connection* Γ^i_a, the one defined by the derivative operator that annihilates the triad e^i_a (in complete analogy to the Christoffel symbol that defined the covariant derivative D_a killing the metric). It can be inverted from the form,

$$\partial_{[a} e^i_{b]} + \epsilon^i{}_{jk}\, \Gamma^j_{\ a}\, e^k_b = 0 \qquad (6)$$

This can be seen as an extension of the covariant derivative to objects with mixed indices. The key to the definition of the new variables is to combine these two objects, namely the spin connection Γ with the object K^i_a (a tensorial object), to produce a new connection

$${}^{\gamma}A^i_a := \Gamma^i_a + \gamma\, K^i_a \qquad (7)$$

This is the *Ashtekar-Barbero Connection*. Similarly, the other conjugate variable will be the rescaled triad,

$$ {}^{\gamma}\tilde{E}^a_i = \tilde{E}^a_i/\gamma \tag{8}$$

Now, the pair $({}^{\gamma}A^i_a, {}^{\gamma}\tilde{E}^a_i)$ will coordinatize the new phase space Γ_γ. We have emphasized the parameter γ since this labels a one parameter family of different classically equivalent theories, one for each value of γ. The real and positive parameter γ is known as the Barbero-Immirzi parameter [25, 26]. In terms of these new variables, the canonical Poisson brackets are given by,

$$ \{{}^{\gamma}A^i_a(x), {}^{\gamma}\tilde{E}^b_j(y)\} = \kappa\, \delta^b_a\, \delta^i_j\, \delta^3(x,y)\,. \tag{9}$$

and,

$$ \{{}^{\gamma}A^i_a(x), {}^{\gamma}A^j_b(y)\} = \{{}^{\gamma}\tilde{E}^a_i(x), {}^{\gamma}\tilde{E}^b_j(y)\} = 0 \tag{10}$$

Let us summarize. i) We started with the geometrodynamical phase space where the configuration space was taken to be the space of 3-Riemannian metrics, ii) enlarged it by considering triads instead of metrics, and iii) performed a canonical transformation, and changed the role of the configuration variable; the connection ${}^{\gamma}A^i_a$ (that has the information about the extrinsic curvature) is now the configuration variable and the (densitized) triad is regarded as the canonical momenta. The connection ${}^{\gamma}A^i_a$ is a true connection since it was constructed by adding to the spin connection Γ^i_a a tensor field, that yields a new connection. We started by considering connection taking values in $so(3)$, the Lie algebra of $SO(3)$, but since as Lie algebras it is equivalent to $su(2)$, we will take as the gauge group the simply connected group $SU(2)$ (which will allow to couple fermions when needed). The phase space Γ_γ is nothing but the phase space of a $SU(2)$ Yang-Mills theory.

Let us now write down the constraints in terms of the new phase space variables. The new constraint that arises because of the introduction of new degrees of freedom takes a very simple form,

$$ G_i = \mathcal{D}_a\, \tilde{E}^a_i \approx 0 \tag{11}$$

that is, it has the structure of Gauss' law in Yang-Mills theory and that is the name that has been adopted for it. We have denoted by $\mathcal{D}$ the covariant defined by the connection ${}^{\gamma}A^i_a$, such that $\mathcal{D}_a\tilde{E}^a_i = \partial_a\tilde{E}^a_i + \epsilon_{ij}{}^k\, {}^{\gamma}A^j_a\tilde{E}^a_k$. The vector and scalar constraints now take the form,

$$ V_a = F^i_{ab}\, \tilde{E}^b_i - (1+\gamma^2) K^i_a\, G_i \approx 0 \tag{12}$$

where $F^i_{ab} = \partial_a {}^{\gamma}A^i_b - \partial_b {}^{\gamma}A^i_a + \epsilon^i{}_{jk}\, {}^{\gamma}A^j_a\, {}^{\gamma}A^k_b$ is the curvature of the connection ${}^{\gamma}A^j_b$. The other constraint is,

$$ \mathcal{S} = \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{\det(\tilde{E})}} \left[\epsilon^{ij}{}_k\, F^k_{ab} - 2(1+\gamma^2)\, K^i_{[a} K^j_{b]}\right] \approx 0 \tag{13}$$

Note incidently that if $(1+\gamma^2) = 0$, the constraints would simplify considerably. That was the original choice of Ashtekar, that rendered the connection complex (and thus the corresponding gauge group non-compact), and it had a nice geometrical interpretation in terms of self-dual fields. Historically the emphasis from complex to real connections (and a compact group as a consequence) was due to the fact that in this case the mathematical well defined construction of the Hilbert space has been completed, whereas the non-compact case remains open. For more details see [6] and specially Sec 3.2 of [7].

The next step is to consider the right choice of variables, now seen as functions of the phase space Γ_γ that are preferred for the non-perturbative quantization we are seeking. As

we shall see, the guiding principle will be that the functions (defined by an appropriate choice of smearing functions) will be those that can be defined without the need of a background structure, i.e. a metric on Σ.

III. HOLONOMIES AND FLUXES

We have seen that the classical setting for the formulation of the theory is the phase space of a Yang-Mill theory, with extra constraints. Since the theory possesses these constraints, the strategy to be followed is to quantize first and then to impose the set of constraints as operators on a Hilbert space. This Hilbert space is very important since it will be the home where the imposition of the constraints will be implemented and its structure will have some physical relevance. This is known as the Kinematical Hilbert Space $\mathcal{H}_{\rm kin}$. One of the main achievements of LQG is that this space has been rigourously defined, something that was never done in the old geometrodynamics program.

The main objective of this section is to motivate and construct the classical algebra of observables that will be the building blocks for the construction of the space $\mathcal{H}_{\rm kin}$. In QFT one could say that there are two important choices when quantizing a classical system. The first one is the choice of the algebra of observables, consisting of two parts, the choice of variables, and of functions thereof. The second choice is a representation of this chosen classical algebra into a Hilbert space. Both steps normally involve ambiguities. This is also true of the phase space we are starting with. Remarkably for our case, the physical requirements of background independence and diffeomorphism invariance will yield a choice of variables and of a representation that is in a sense, unique. The infinite freedom one is accustomed to in ordinary QFT is here severely reduced. Background independence and diffeomorphism invariance impose very strong conditions on the possible quantum theories.

Let us start by considering the connection A_a^i (from now on we shall omit the γ label). The most natural object one can construct from a connection is a holonomy $h_\alpha(A)$ along a loop α. This is an element of the gauge group $G = SU(2)$ and is denoted by,

$$h_\alpha(A) = \mathcal{P} \exp\left(\oint_\alpha A_a \, \mathrm{d}s^a\right) \tag{14}$$

The path-order exponential of the connection. Note that for notational simplicity we have omitted the 'lie-algebra indices'. The connection as an element of the lie algebra, in the fundamental representation, should be written as $A_a^i(\tau_i)_B^A$, with $A, B = 1, 2$ the 2×2-matrix indices of the Pauli matrices τ_i. From the holonomy, it is immediate to construct a gauge invariant function by taking the trace arriving then at the Wilson loop $T[\alpha] := \frac{1}{2} \operatorname{Tr} \mathcal{P} \exp\left(\oint_\alpha A_a \, \mathrm{d}s^a\right)$.

Several remarks are in order. i) When seen as a smearing of the connection, it is clear that the holonomy represents a one dimensional smearing (as compared with, say, the three dimensional smearings $\int A_a^i(x) g_i^a(x) \, \mathrm{d}^3x$ used in ordinary QFT); ii) The loop α can be seen as a label, but the holonomy *is* a function of the connection A_a^i; iii) Even when the holonomies are functions of the connection A, they only "prove" the connection along the loop α. In order to have a useful set of functions that can *separate points* of the space of smooth connections $\mathcal{A}$, one needs to consider for instance Wilson loop functions along *all* possible loops on Σ. The algebra generate by such functions is called the *holonomy algebra* $\mathcal{HA}$.

In order to implement the idea that one should look for generalized notions that will replace the loops, let us consider the most obvious extension. In recent years the emphasis

has shifted from loops to consider instead closed graphs Υ, that consist of N edges e_I ($I = 1, 2, \ldots, N$), and M vertices v_μ, with the restriction that there are no edges with 'loose ends'. As a aside remark one should mention that every graph Υ can be decomposed in independent loops α_i based at a chosen vertex v. Given a graph Υ, one can consider the parallel transport along the edges e_I, the end result is an element of the gauge group $g_I = h(e_I) \in G$ for each such edge. One can then think of the connection A_a^i as a map from graphs to N-copies of the gauge group: $A_a^i : \Upsilon \to G^N$. Furthermore, one can think of $\mathcal{A}_\Upsilon$ as the configuration space for the graph Υ, that is homeomorphic to G^N (one can regard $\mathcal{A}_\Upsilon$ as the configuration space of the 'floating lattice' gauge theory over Υ).

Once we have recognized that one can associate a configuration space for all graphs, one should not loose perspective that the relevant classical configuration space is still the space $\mathcal{A}$ of all (smooth) connections A_a^i. What we are doing at the moment is to construct relevant configuration functions, making use of the graphs and the space $\mathcal{A}_\Upsilon$. In particular, what we need is to consider generalizations of the Wilson loops $T[\alpha]$ defined previously. As we have mentioned before, every graph Υ can be decomposed into independent loops α_i and the corresponding Wilson loops $T[\alpha_i]$ are a particular example of functions defined over $\mathcal{A}_\Upsilon$. What we shall consider as a generalization of the Wilson loop are *all* possible functions defined over $\mathcal{A}_\Upsilon$ (in a sense the Wilson loops generate these functions, but are overcomplete). Thus, a function $c : G^N \to \mathbb{C}$ defines a *cylindrical* function C_Υ of the connection A as,

$$C_\Upsilon := c(h(e_1), h(e_2), \ldots, h(e_N)) \tag{15}$$

By considering all possible functions c and all possible embedded graphs Υ, we generate the algebra of functions known as Cyl (it is closely related to the holonomy algebra, and it can be converted into a C^*-algebra $\overline{\text{Cyl}}$, by suitable completion).

Even when we shall consider in what follows the full algebra Cyl, one should keep in mind that the basic objects that build it are precisely the holonomies, functions of the connection smeared along one dimensional objects. Let us now discuss why this choice of configuration functions is compatible with the basic guiding principles for the quantization we are building up, namely diffeomorphism invariance and background independence. Background independence is clear since there is no need for a background metric to define the holonomies. Diffeomorphism invariance is a bit more subtle. Clearly, when one applies a diffeomorphism $\phi : \Sigma \to \Sigma$, the holonomies transform in a covariant way

$$\phi_* \cdot h(e_I) = h(\phi^{-1} \cdot e_I)\,, \tag{16}$$

that is, the diffeomorphism acts by moving the edge (or loop). How can we then end up with a diffeo-invariant quantum theory? The strategy in LQG is to look for a *diffeomorphism invariant* representation of the diffeo-covariant configuration functions. As we shall see later, this has indeed been possible and in a sense represents the present 'success' of the approach.

Let us now consider the functions depending of the momenta that will be fundamental in the (loop) quantization. The basic idea is again to look for functions that are defined in a background independent way, that are natural from the view point of the geometric character of the object (1-form, 2-form, etc), and that transform covariantly with respect to the gauge invariances of the theory. Just as the the connection A_a^i can be identified with a one form that could be integrated along a one-dimensional object, one would like analyze the geometric character of the densitised triad $\tilde{E}$ in order to naturally define a smeared object. Recall that the momentum is a density-one vector field on Σ, $\tilde{E}_i^a$ with values in the dual of

the lie-algebra $su(2)$. In terms of its tensorial character, it is naturally dual to a (lie-algebra valued) two form,

$$E_{ab\,i} := \tfrac{1}{2}\,\eta_{abc}\,\tilde{E}^c_i \tag{17}$$

where η_{abc} is the naturally defined Levi-Civita symbol. It is now obvious that the momenta is crying to be integrated over a two-surface S. It is now easy to define the objects

$$E[S,f] := \int_S E_{ab\,i}\,f^i\,\mathrm{d}S^{ab}\,, \tag{18}$$

where f^i is a lie-algebra valued smearing function on S. This 'Electric flux' variable does not need a background metric to be defined, and it transforms again covariantly as was the case of the holonomies. The algebra generated by holonomies and flux variables is known as the *Holonomy-Flux* algebra $\mathcal{HF}$.

Perhaps the main reason why this Holonomy-Flux algebra $\mathcal{HF}$ is interesting, is the way in which the basic generators interact, when considering the classical (Poisson) lie-bracket. First, given that the configuration functions depend only on the connection and the connections Poisson-commute, one expects that $\{T[\alpha], T[\beta]\} = 0$ for any loops α and β. The most interesting poisson bracket one is interested in is the one between a configuration and a momenta variable,

$$\{T[\alpha], E[S,f]\} = \kappa \sum_\mu f^i(v)\,\iota(\alpha, S|_v)\,\mathrm{Tr}\left(\tau_i\,h(\alpha)\right) \tag{19}$$

where the sum is over the vertices v and $\iota(\alpha, S|_v) = \pm 1$ is something like the intersection number between the loops α and the surface S at point v. The sum is over all intersection of the loop α and the surface S. The most important property of the Poisson Bracket is that it is completely topological. This has to be so if we want to have a fully background independent classical algebra for the quantization.

A remark is in order. The value of the constant $\iota|_v$ depends not only on the relative orientation of the tangent vector of the loop α with respect to the orientation of Σ and S, but also on a further decomposition of the loop into edges, and whether they are 'incoming' or 'outgoing' to the vertex v. The end result is that is we have, for simple intersections, that the number $\iota|_v$ becomes insensitive to the 'orientation'. This is different to the $U(1)$ case where the final result *is* the intersection number. For details see [27] (and note the difference with the claims in [9]).

Let us now consider the slightly more involved case of a cylindrical function C_Υ that is defined over a graph Υ with edges e_I, intersecting the surface S at points p. We have then,

$$\{C_\gamma,\, E[S,f]\} = \frac{\kappa}{2}\sum_p \sum_{I_p} \iota(I_p)\,f^i(p)\,X^i_{I_p}\cdot c \tag{20}$$

where the sum is over the vertices p of the graph that lie on the surface S, I_p are the edges starting or finishing in p and where $X^i_{I_P}\cdot c$ is the result of the action of the i-th left (resp. right) invariant vector field on the I_p-th copy of the group if the I_p-th edge is pointing away from (resp. towards) the surface S. Note the structure of the right hand side. The result is non-zero only if the graph Υ used in the definition of the configuration variable C_Υ intersects the surface S used to smear the triad. If the two intersect, the contributions arise from the

action of right/left invariant vector fields on the arguments of c associated with the edges at the intersection.

Finally, the next bracket we should consider is between two momentum functions, namely $\{E[S,f], E[S',g]\}$. Just as in the case of holonomies, these functions depend only on one of the canonical variables, namely the triad $\tilde{E}$. One should then expect that their Poisson bracket vanishes. Surprisingly, this is *not* the case and one has to appropriately define the correct algebraic structure[1].

We have arrived then to the basic variables that will be used in the quantization in order to arrive at LQG. They are given by,

$$h(e_I) \qquad \text{Configuration function} \tag{21}$$

and

$$E[S,f] \qquad \text{Momentum function,} \tag{22}$$

subject to the basic Poisson bracket relations given by Eqs. (19) and (20). In the next section we shall take the Holonomy-Flux algebra $\mathcal{HF}$ as the starting point for the quantization.

IV. THE HILBERT SPACE

In this section we shall first outline the quantization strategy to arrive at the kinematical Hilbert space $\mathcal{H}_{\rm kin}$. Later on, we shall give certain details of the several parts that arise in the construction. As we have emphasized, the Kinematical Hilbert space is the starting point for the program of implementing the constraints as quantum operators. We shall consider briefly that issue later on.

A. General Considerations

The strategy is to build the Hilbert space as in the ordinary Schrödinger representation, where states are to be represented by wave-functions of the configuration space. Recall that in the present approach, one has decided that the space $\mathcal{A}$ is the space of (classical) configurations. The natural strategy is then to consider wave functions Ψ of the form,

$$\Psi = \Psi(A) \tag{23}$$

Following the analogy, one should expect that the Hilbert space will consist of wave-functions that are square integrable. This means that one has to introduce a measure $\mathrm{d}\mu$ on the space of connections, in order to define the inner product heuristically as,

$$\langle \Phi | \Psi \rangle = \int \mathrm{d}\mu \, \overline{\Phi} \, \Psi \tag{24}$$

[1] The end result is that one should not regard $E[S,f]$ as phase space functions subject to the ordinary Poisson bracket relations, but rather should be viewed as arising from vector fields X^α on $\mathcal{A}$. The non-trivial bracket is then due to the non-commutative nature of the corresponding vector fields. This was shown in [27] where details can be found

The Hilbert space would be denoted then as $\mathcal{H}_{\rm kin} = L^2(\mathcal{A}, {\rm d}\mu)$. In this configuration representation one expects that the connection and any function of it will act as a multiplication operator, and that the momenta will be represented as a derivation, with possible a correction term[2].

There are several different technical issues that need to be properly addressed in order to complete the construction as we have outlined it. Basically what needs to be done is to give mathematical meaning to the different aspects of the kinematical Hilbert space. Let us then make some general remarks for each of them.

i) *Configuration space.* From the experience that has been gained from the scalar field case, we know that the space where the wave functions have support is a much larger (functional) space as the space of classical configurations. In the scalar case, the classical configuration space involves smooth fields, whereas the *quantum configuration space* is made of tempered distributions (dual to the Schwarz space). In our case we expect that the quantum configuration space $\overline{\mathcal{A}}$ will also be an extension of the classical space. This is indeed the case as was shown in the mid-nineties by Ashtekar and collaborators. There are several characterizations of this space (that can be found in [6, 11, 29]), but here we shall mention only two of them (for a third one see below). The first one is to recall that one could think of a connection as an operator that acts on 1-dimensional objects (edges) and gives a group element. Certainly, a smooth connection has this property. It turns out that a *generalized connection*, an element of the quantum configuration space $\overline{\mathcal{A}}$, is precisely *any* such map (satisfying some tame conditions such as the composition: $h(e_1 \circ e_2) = h(e_1) \cdot h(e_2)\,;\,\forall\, e_I$). In particular, what is dropped is any notion of continuity of the connection. The other characterization is more algebraic. It is based on the observation that the (Abelian) algebra of cylindrical functions Cyl can be extended to a C^*-algebra $\overline{\rm Cyl}$ with unit. Standard theorems on representations of such algebras tell us that those algebras can be viewed as the space of continuous functions over a compact space Δ, called the spectrum of the algebra. It turns out that $\overline{\mathcal{A}}$ is precisely the spectrum of the algebra $\overline{\rm Cyl}$.

ii) *Measure.* A extremely important issue in the choice of quantum theory is the inner product on the Hilbert space. In the case of wave-functions this implies defining a measure on the configuration space $\overline{\mathcal{A}}$. At his point one should require that the measure –responsible in a sense for the resulting quantum representation– be diffeomorphism invariant. This would implement naturally the intuitive notion that the theory should be spatially diffeo-invariant. The construction of a measure with this property is one of the main results of the program since it was generally thought that no such measures existed for gauge theories. The diffeo-invariant measure is known as the Ashtekar-Lewandowski measure $\mu_{\rm AL}$.

iii) *Momentum operators.* The other important part of the quantum representation is of course, the way in which the functions $E[S, f]$ are promoted to operators on the Hilbert space $\mathcal{H}_{\rm kin}$. Intuitively, whenever the wave function Ψ is a function of the configuration variable (the connection), the conjugate variable acts as derivation $\hat{E}^a = \delta/\delta A_a$. However, one should also consider the possibility of adding a multiplicative term to the derivative. This term would commute with the (multiplicative) configuration operator $\hat{h}(e_I)$, so the commutation relations would be satisfied. This extra term would also account, for instance, for the existence of a non-trivial measure (See [28] for a discussion in the scalar case). Therefore, the choice of measure and the representation of the momentum operator are intertwined, and one should ensure that the measure not only be diffeomorphism invariant, but should

[2] For a discussion of the corresponding Schrödinger representation for a scalar field see [28].

also "support" the triad. A systematic study of these issues initiated by Sahlmann has yielded a uniqueness result: The only diffeo-covariant representation of the Holonomy-Flux algebra $\mathcal{HF}$ is given by the Ashtekar-Lewandowski representation [30].

iv) *Hybrid variables.* Recall that the basic functions we have chosen for the quantization are the pairs $(h(e_I), E[S,f])$. These generators of the classical algebra to be quantized (the Holonomy-Flux algebra), are not canonical. One of them, namely the holonomy, is a exponentiated version of the connection, whereas the electric flux is linear in the triad. This would be equivalent to a choice $U(\lambda) = \exp[i\lambda q]$ and $\pi(\nu) = \nu\, p$, for a quantum mechanical system with phase space $\Gamma = (q,p)$. This means that the basic functions of phase space are neither of the canonical type (q,p), nor of the Weyl form (both exponentiated); the quantization based on this functions is in a strict sense non-canonical. In finite dimensions the Stone von Neumann theorem assures us that the Weyl relations are equivalent to the Canonical Commutation Relations, provided the quantum representations are *regular*[3]. For a representation to be regular means that the exponentiated form of the operators –which are unitary– are continuous with respect to the parameters (λ in the $U(\lambda)$ above). One of the main features of the loop quantization is that the exponentiated variable, the holonomy, has a well defined associated operator $\hat{h}(e_I)$ on $\mathcal{H}_{\rm kin}$ that is however, discontinuous. Thus, the operator $\hat{A}_i^a$ does *not* exist! In the finite-dimensional system example, the equivalent statement would be to say that $\hat{U}(\lambda)$ is well defined but $\hat{q}$ is not. Such non-regular representations in quantum mechanics *do* exist [31] and form the basic starting point for Loop Quantum Cosmology [32, 33].

B. Ashtekar-Lewandowski Hilbert Space

Let us now consider the particular representation that defines LQG. As we have discussed before, the basic observables are represented as operators acting on wave functions $\Psi_\Upsilon(\overline{A}) \in \mathcal{H}_{\rm kin}$ as follows:

$$\hat{h}(e_I)\cdot\Psi(\overline{A}) = (h(e_I)\ \Psi)\,(\overline{A}) \tag{25}$$

and

$$\hat{E}[S,f]\cdot\Psi_\Upsilon(\overline{A}) = i\hbar\,\{\Psi_\Upsilon,\, {}^2E[S,f]\} = i\frac{\ell_{\rm P}^2}{2}\sum_p\sum_{I_p}\kappa(I_p)\,f^i(p)\,X^i_{I_p}\cdot\psi \tag{26}$$

where $\ell_{\rm P}^2 = \kappa\,\hbar = 8\pi\,G\,\hbar$, the Planck area is giving us the scale of the theory (recall that the Immirzi parameter γ does not appear in the basic Poisson bracket, and should therefore not play any role in the quantum representation). Here we have assumed that a cylindrical function Ψ_Υ on a graph Υ is an element of the Kinematical Hilbert space (which we haven't defined yet!). This implies one of the most important assumption in the loop quantization prescription, namely, that objects such as holonomies and Wilson loops that are smeared in one dimension are well defined operators on the quantum theory[4].

Let us now introduce the third characterization of the quantum configuration space which will be useful for constructing the Hilbert space $\mathcal{H}_{\rm kin}$. The basic idea for the construction

[3] For a nice discussion see for instance [23].

[4] this has to be contrasted with the ordinary Fock representation where such objects do not give raise to well defined operators on Fock space. This implies that the loop quantum theory is qualitatively different from the standard quantization of gauge fields.

of both the Hilbert space $\mathcal{H}_{\rm kin}$ (with its measure) and the quantum configuration space $\overline{\mathcal{A}}$, is to consider the projective family of *all* possible graphs on Σ. For any given graph Υ, we have a configuration space $\mathcal{A}_\Upsilon = (SU(2))^N$, which is n-copies of the (compact) gauge group $SU(2)$. Now, it turns out that there is a preferred (normalized) measure on any compact semi-simple Lie group that is left and right invariant. It is known as the Haar measure $\mu_{\rm H}$ on the group. We can thus endow $\mathcal{A}_\Upsilon$ with a measure μ_Υ that is defined by using the Haar measure on all copies of the group. Given this measure on $\mathcal{A}_\Upsilon$, we can consider square integrable functions thereon and with them the graph-Υ Hilbert space $\mathcal{H}_\Upsilon$, which is of the form:

$$\mathcal{H}_\Upsilon = L^2(\mathcal{A}_\Upsilon, {\rm d}\mu_\Upsilon) \tag{27}$$

If we were working with a unique, fixed graph Υ_0 (which we are not), we would be in the case of a lattice gauge theory on an irregular lattice. The main difference between that situation and LQG is that, in the latter case, one is considering all graphs on Σ, and one has a family of configurations spaces $\{\mathcal{A}_\Upsilon \,/\Upsilon \,{\rm a\,graph\,in}\,\Sigma\}$, and a family of Hilbert spaces $\{\mathcal{H}_\Upsilon \,/\Upsilon \,{\rm a\,graph\,in}\,\Sigma\}$. In order to have consistent families of configuration and Hilbert spaces one needs some conditions. In particular, there is a notion of when a graph Υ is 'larger' than Υ'. We say that if Υ contains Υ' then $\Upsilon > \Upsilon'$.[5]

Given this (partial) relation ">", we have a corresponding projection $P_{\Upsilon,\Upsilon'} : \Upsilon \mapsto \Upsilon'$, which in turn induces a projection operator for configuration spaces $P : \mathcal{A}_\Upsilon \mapsto \mathcal{A}'_\Upsilon$ and an inclusion operator for Hilbert spaces $\iota : \mathcal{H}'_\Upsilon \mapsto \mathcal{H}_\Upsilon$. The consistency conditions that need to be imposed are rather simple. The intuitive idea is that, if one has a graph Υ_o and a function thereon $\psi_\Upsilon[A]$ one should be able to consider larger graphs Υ_i where the function ψ be well defined; one should be able to talk of the 'same function', even when defined on a larger graph, and more importantly, its integral (and inner product with other functions) should be independent of the graph Υ_i we have decided to work on.

We are now in the position of giving a heuristic definition of the configuration space $\overline{\mathcal{A}}$ and $\mathcal{H}_{\rm kin}$: The quantum configuration space $\overline{\mathcal{A}}$ is the configuration space for the "largest graph"; and similarly, the kinematical Hilbert space $\mathcal{H}_{\rm kin}$ is the largest space containing all Hilbert spaces in $\{\mathcal{H}_\Upsilon \,/\Upsilon \,{\rm a\,graph\,in}\,\Sigma\}$. Of course, this can be made precise mathematically, where the relevant limits are called *projective*. For details see [6, 34] and [11]. The Ashtekar-Lewandowski measure $\mu_{\rm AL}$ on $\mathcal{H}_{\rm kin}$ is then the measure whose projection to any $\mathcal{A}_\Upsilon$ yields the corresponding Haar measure μ_Υ. The resulting Hilbert space can thus be written as

$$\mathcal{H}_{\rm kin} = L^2(\overline{\mathcal{A}}, {\rm d}\mu_{\rm AL})$$

Let us now see how it is that the cylindrical functions $\Psi_\Upsilon \in {\rm Cyl}$ belong to the Hilbert space of the theory. Let us consider a cylindrical function Ψ_Υ defined on the space $\mathcal{A}_\Upsilon$. If it is continuous, then it is bounded (since $\mathcal{A}_\Upsilon$ is compact), and thus it is square integrable with respect to the measure μ_Υ. Therefore, $\Psi_\Upsilon \in \mathcal{H}_\Upsilon$. Finally, we have the inclusion between Hilbert spaces given by $\mathcal{H}_\Upsilon \subset \mathcal{H}_{\rm kin}$ which implies that the cylinder function Ψ_Υ indeed belongs to the kinematical Hilbert space.

Let us now see how this inner product works for known functions such as Wilson loops. Consider $T[\alpha]$ and $T[\beta]$ two Wilson loops with $\alpha \neq \beta$ (and nonintersecting, for simplicity).

[5] by containing we mean that the larger graph can be obtained from the smaller by adding some edges or by artificially dividing the existing edges (by proclaiming that an interior point $p \in e_I$ is now a vertex of the graph).

Each loop can be regarded as a graph on itself, with only an edge and no vertex (or many bi-valent vertices by artificially declaring them to be there). In order to take the inner product between these two function which looks like

$$\langle\, T[\alpha]\,|\,T[\beta]\,\rangle_{\rm kin} = \int_{\overline{\mathcal{A}}} {\rm d}\mu_{\rm AL}\; \overline{T[\alpha]}\; T[\beta] \tag{28}$$

we have to construct a graph Υ that contains both α and β. This is rather easy to do and in fact there is large freedom in doing that. The end result should be independent of the particular choice. The simplest possibility is to take an edge e that connects *any* point of α with any point of β (and for simplicity it does not intersect neither loop at another point). The resulting graph has now three edges (α, β, e), so we can construct its configuration space to be $\mathcal{A}_\Upsilon = (SU(2))^3 =$. The Wilson loops are now very simple cylinder functions on $\mathcal{A}_\Upsilon$ (or rather, the induced function by the inclusion of α into Υ). $T[\alpha]$ has associated a function $t_\alpha : (SU(2))^3 \to \mathbb{C}$ that only depends on the first argument (corresponding to α), whereas $T[\beta]$ has a function t_β that depends only on the second argument .Its functional dependence (the same for $T[\alpha]$ and $T[\beta]$), as a function of the holonomy in each edge, is very simple:

$$T[\alpha] = t_\alpha(h(\alpha), h(\beta), h(e)) = \tfrac{1}{2}{\rm Tr}(h(\alpha))\,.$$

Given that we are integrating a function $(\overline{T[\alpha]}\; T[\beta])$ that is defined on the graph $\Upsilon = \alpha \cup \beta \cup e$, and that the functional integral over the full space $\overline{\mathcal{A}}$ reduces to an integral over the minimal graph were the function is defined, the inner product (28) can then be rewritten as,

$$\langle\, T[\alpha]\,|\,T[\beta]\,\rangle_{\rm kin} = \int_{G_1} {\rm d}\mu_{\rm H}\; \overline{T[\alpha]} \int_{G_2} {\rm d}\mu_{\rm H}\; T[\beta] \int_{G_3} {\rm d}\mu_{\rm H} \tag{29}$$

Each integral is performed over each copy of the gauge group, where G_1 denotes the first entry in the functions t_i, namely the holonomy along α, and G_2 the holonomy along β. The third copy of the group, that associated to e has the property that the function we are integrating has no dependence on it, thus one is only integrating the unit function. We are assuming that μ_H is normalized so this integral is equal to one. The question now reduces to that of calculating the integral:

$$\int_{G_1} {\rm d}\mu_{\rm H}\; \overline{T[\alpha]} = \frac{1}{2} \int_{G_1} {\rm d}\mu_{\rm H}\; \overline{{\rm Tr}(h(\alpha))} \tag{30}$$

It turns out that the Haar measure is such that this integral vanishes. Thus, the inner product (28) is zero, for any two (different) loops α and β and for any edge e connecting them. Note that if we had assumed from the very beginning that the loops α and β are equal, then there would be no need to define a connecting edge and the inner product would be given by only one integral $\int_{G_1} {\rm d}\mu_{\rm H}\; \overline{T[\alpha]}\, T[\alpha] = \frac{1}{2}\int_{G_1} {\rm d}\mu_{\rm H}\; |{\rm Tr}(h(\alpha))|^2 \neq 0$

Several remarks are in order:

i) The fact that the inner product between any two Wilson loop functions vanishes is a clear signature that the measure $\mu_{\rm AL}$ is diffeomorphism invariant. More precisely, a diffeomorphism ϕ induces a transformation $U(\phi) : \mathcal{H} \to \mathcal{H}$ given by: $U(\phi) \cdot \Psi_\Upsilon = \Psi_{(\phi^{-1}\cdot\Upsilon)}$. The measure $\mu_{\rm AL}$ is such that the operator $U(\lambda)$ is unitary. Thus, diffeomorphism are unitarily implemented in this quantum theory.

ii) The integral (30) can be interpreted as the vacuum expectation value of the operator $\hat{T}[\alpha]$; The "vacuum" Ψ_0 in this representation is simply the unit function, so we have:

$$\langle\, \hat{T}[\alpha]\, \rangle_0 = \int_G T[\alpha]\, \mathrm{d}\mu_{\mathrm{H}} = 0\,.$$

We can then conclude that in the LQG representation, the vacuum expectation value of all Wilson loops vanishes exactly.

iii) As we mentioned before, the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$ can be regarded as the "largest" Hilbert space by combining the Hilbert spaces over all possible graphs. We shall use the following symbol to denote that idea. We write then:

$$\mathcal{H}_{\mathrm{kin}} = \otimes_\Upsilon\, \mathcal{H}_\Upsilon$$

Note that since the space of graphs Υ is uncountable, the Hilbert space $\mathcal{H}_{\mathrm{kin}}$ is non-separable. In the next part we shall se how a convenient choice of basis for each Hilbert space $\mathcal{H}_\Upsilon$ will allow us to have a full decomposition of the Hilbert space.

Let us then recall what is the structure of simple states in the theory. The vacuum or 'ground state' $|0\rangle$ is given by the unit function. One can then create excitations by acting via multiplication with holonomies or Wilson loops. The resulting state $|\alpha\rangle = \hat{T}[\alpha]\cdot|0\rangle$ is an excitation of the geometry but only along the one dimensional loop α. Since the excitations are one dimensional, the geometry is sometimes said to be *polymer like*. In order to obtain a geometry that resembles a three dimensional continuum one needs aa huge number of edges ($10^6 8$) and vertices.

C. A choice of basis: Spin Networks

The purpose of this part is to provide a useful decomposition of the Hilbert space $\mathcal{H}_\Upsilon$, for all graphs. In practice one just takes one particular graph Υ_o and works in that graph. This would be like restricting oneself to a fixed lattice and one would be working on the Hilbert space of a Lattice Gauge Theory. Thus, all the results of this subsection are also relevant for a LGT, but one should keep in mind that one is always restricting the attention to a little part of the total Hilbert space $\mathcal{H}_{\mathrm{kin}}$.

Let us begin by sketching the basic idea of what we want to do. For simplicity, let us consider just one edge, say e_i. What we need to do is to be able to decompose any function F on G (in this case we only have one copy of the group), in a suitable basis. In the simplest case, when the group is just $G = U(1)$ we just have a circle. In this case, the canonical decomposition for any function F is given by the Fourier series: $F = \sum_n A_n\, \mathrm{e}^{\mathrm{i}\, n\, \theta}$, with $A_n = \frac{1}{2\pi}\int F(\theta)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta$. As is well known, the functions $\mathrm{e}^{\mathrm{i}\, n\, \theta}$ represent a basis for any function on the group $G = U(1)$, that is also orthonormal with respect to the measure $\mathrm{d}\mu_{\mathrm{H}} = \frac{1}{2\pi}\mathrm{d}\theta$, which is nothing but the Haar measure on this group. For notational simplicity one can denote by $f_n(\theta) = \mathrm{e}^{\mathrm{i}\, n\, \theta}$, and they represent irreducible representations of the group $U(1)$, labelled by n. Then the series looks like $F(\theta) = \sum_n\, A_n\, f_n(\theta)$. Here we should think of assigning, to each of the basis functions f_n the trivial label n (for all integers), and the general function is a linear combination of the n-labelled basis.

In the case of the group $G = SU(2)$, there is en equivalent decomposition of a function

$f(g)$ of the group ($g \in G$). It reads,

$$f(g) = \sum_j \sqrt{j(j+1)}\, f_j^{mm'} \overset{j}{\Pi}_{mm'}\ (g) \tag{31}$$

where,

$$f_j^{mm'} = \sqrt{j(j+1)} \int_G \mathrm{d}\mu_H\ \overset{j}{\Pi}_{mm'}\ (g^{-1})\, f(g) \tag{32}$$

is the equivalent of the Fourier component. When doing harmonic analysis on groups the generalization of the Fourier decomposition is known as the Peter-Weyl decomposition. The functions $\overset{j}{\Pi}_{mm'}\ (g)$ play the role of the Fourier basis. In this case these are unitary representation of the group, and the label j labels the irreducible representations. In the $SU(2)$ case with the interpretation of spin, these represent the spin-j representations of the group. In our case, we will continue to use that terminology (spin) even when the interpretation is somewhat different.

Given a cylindrical function $\Psi_\Upsilon[A] = \psi(h(e_1), h(e_2), \ldots, h(e_N))$, we can then write an expansion for it as,

$$\begin{aligned} \Psi_\Upsilon[A] &= \psi(h(e_1), h(e_2), \ldots, h(e_N)) \\ &= \sum_{j_1 \cdots j_N} f_{j_1 \cdots j_N}^{m_1 \cdots m_N, n_1 \cdots n_N}\ \phi^{j_1}_{m_1 n_1}(h(e_1)) \cdots \phi^{j_N}_{m_N n_N}(h(e_N)), \end{aligned} \tag{33}$$

where $\phi^j_{mn}(g) = \sqrt{j(j+1)}\ \overset{j}{\Pi}_{mn}\ (g)$ is the normalized function satisfying

$$\int_G \mathrm{d}\mu_{\mathrm{H}}\, \overline{\phi^j_{mn}(g)}\ \phi^{j'}_{m'n'}(g) = \delta_{j,j'}\delta_{m,m'}\delta_{n,n'}\,.$$

The expansion coefficients can be obtained by projecting the state $|\Psi_\Upsilon\rangle$,

$$f_{j_1 \cdots j_N}^{m_1 \cdots m_N, n_1 \cdots n_N} = \langle\, \phi^{j_1}_{m_1 n_1} \cdots \phi^{j_N}_{m_N n_N}\ |\ \Psi_\Upsilon \rangle \tag{34}$$

This implies that the products of components of irreducible representations $\prod_{i=1}^N \phi^{j_i}_{m_i n_i}[h(e_i)]$ associated with the N edges $e_I \in \Upsilon$, for all values of spins j and for $-j \leq m, n \leq j$ and for any graph Υ, is a complete orthonormal basis for $\mathcal{H}_{\mathrm{kin}}$. We can the write,

$$\mathcal{H}_\Upsilon = \otimes_j\, \mathcal{H}_{\Upsilon,j} \tag{35}$$

where the Hilbert space $\mathcal{H}_{\alpha,j}$ for a single loop α, and a label j is the familiar $(2j+1)$ dimensional Hilbert space of a particle of 'spin j'. For a complete treatment see [7].

We have been able to decompose the Hilbert space of a given graph, using the Peter-Weyl decomposition theorem, but how can we make contact with the so called spin networks? Let us for a moment focus our attention to the $U(1)$ case. Then, for each edge one could decompose any function in a Fourier series where the basis was labelled by an integer, that correspond to all possible irreducible unitary representations of the group. As we saw later, one could think of a basis of the graph Hilbert by considering the product over functions labelled by these representations. What one can do it to associate a label n_I for each edge, to denote the function made out of those basis functions corresponding to the chosen labels. A graph with the extra labelling, known as dressing, can be given different names. In

the $U(1)$ theory the labelling denotes the possible electric flux, so the graph $\Upsilon_{(n_1,n_2,\cdots,n_N)}$ with the labellings for each edge is called a *flux network*. In the case of geometry with group $SU(2)$, the graphs with labelling $j_I = \mathbf{j}$ are known as spin networks. As the reader might have noticed, in the geometry case there are more labels than the spins for the edges. Normally these are associated to vertices and are known as intertwiners. This means that the Hilbert spaces $\mathcal{H}_{\Upsilon,\mathbf{j}}$ is finite dimensional. Its dimension being a measure of the extra freedom contained in the intertwiners. One could then introduce further labelling $\mathbf{l}$ for the graph, so we can decompose the Hilbert space as

$$\mathcal{H}_\Upsilon = \otimes_j \mathcal{H}_{\Upsilon,\mathbf{j}} = \otimes_{\mathbf{j},\mathbf{l}} \mathcal{H}_{\Upsilon,\mathbf{j},\mathbf{l}} \tag{36}$$

where now the spaces $\mathcal{H}_{\Upsilon,\mathbf{j},\mathbf{l}}$ are one-dimensional. For more details see [35], Sec. 4.1.5 of [7] and Sec. 4.2 of [6].

Let us see the simplest possible example, the Wilson loop $T[\alpha]$. In this case there is only one edge, and the function only involves the lowest representation $j = 1/2$. Furthermore, the example is very easy since it is only the trace of the object $\phi^{1/2}_{mn}$. In terms of labelling it corresponds to a $(j = 1/2)$ label on the loop. Since there are no vertices, there is no choice for 'intertwiner'. The next simplest example of a spin network is for the simplest graph, namely a loops α but for higher spin j labellings. These functions correspond to taking the trace of the holonomy around α but in a higher representation j of the group. In general a spin-j Wilson loop $T^j[\alpha]$ can be written in terms of the fundamental one $T[\alpha]$ as a polynomial expansion of degree j: $T^j[\alpha] = \sum_{k=1,\ldots,j} A_k \, (T[\alpha])^k$, for some coefficients A_k.

Two final remarks are in order. While it is true that spin networks are naturally defined in terms of the harmonic analysis on G, and due to its properties as an eigen-basis of important operators (the subject of next chapter) have a special place in the theory, they are not, by themselves *the* theory. Dirac's transformation theory assures us that there are equivalent descriptions for the quantum theory that are not given by these functions. Another related issue is the existence of the so-called *loop representation* of the theory. In its early stages, non-perturbative quantum gravity (today known as LQG) was though to have two equivalent representations, the connection representation and the loop representation (precisely in the sense of Dirac's transformation theory). When the structure of the theory was properly understood and the Hilbert space was characterized, it was clear that the loop representation as originally envisioned was not well defined. The basic idea was to have loops α as the argument for wave-functions. Thus, if one had (in Dirac notation) a state $|\Psi\rangle$, the wave function of the connection would be $\Psi[A] = \langle A|\Psi\rangle$. If one had a 'basis of loops' $\langle\alpha|$, one could imagine having $\langle\alpha|\Psi\rangle$, the state in the loop representation. Furthermore, there was a proposal for defining this state via the 'loop transform':

$$\psi[\alpha] = \langle\alpha|\Psi\rangle = \int \mathcal{D}A \, \langle\alpha|A\rangle \, \langle A|\Psi\rangle := \int \mathcal{D}A \, \overline{T}[\alpha, A] \, \Psi[A] \tag{37}$$

It was then implied that the Wilson loops were the kernel of the transformation and were in a sense complete. From the discussion of this section, it is clear that those expectations were superseded. Namely, what we have seen is that loops by themselves are not enough; we would loose information by only considering loops[6]. We need the extra information contained in

[6] More precisely, when seen as graphs, loops are not enough. What was previously done was to consider

the spin networks, namely we need to consider all possible spins and intertwiners. Instead, what one has to consider is the 'spin network transform',

$$\psi[\Upsilon_{\mathbf{j},\mathbf{l}}] = \langle \Upsilon_{\mathbf{j},\mathbf{l}} | \Psi \rangle = \int \mathrm{d}\mu_{\mathrm{AL}} \, \langle \Upsilon_{\mathbf{j},\mathbf{l}} | A \rangle \, \langle A | \Psi \rangle := \int_{\overline{\mathcal{A}}} \mathrm{d}\mu_{\mathrm{AL}} \, \overline{\mathcal{N}}[\Upsilon_{\mathbf{j},\mathbf{l}}, A] \, \Psi[A] \tag{38}$$

Let us end this part with a two remarks regarding spin networks. So far we have only considered cylindrical functions that satisfy no further requirements. We know, on the other hand that in order to have physical states we need to impose the quantum constraints on the states. In particular Gauss' law implies that the states be gauge invariant, namely, invariant under the action of (finite) gauge transformations of the connection $A \mapsto g^{-1}Ag + g^{-1}\mathrm{d}g$. This condition, when translated to the language of holonomies, graphs and so on imposes some restrictions on the cylinder functions, that are conditions imposed only at the vertices of the graph:

$$\sum_{v} \sum_{e_v} X^i_{e_v} \cdot \mathcal{N}[\Upsilon_{\mathbf{j},\mathbf{l}}, A] = 0 \tag{39}$$

where the first sum is over vertices v of the graph and the second over edges e_v for each vertex v. Spin networks that satisfy these conditions are called gauge invariant spin networks (for a detailed treatment see Sec. 4.1.5 of [7]). Finally, there is a very convenient language for performing calculations involving (gauge invariant) spin networks, that employs graphical manipulations over the vertices of the graphs. This is known as *recoupling theory*, and (due to space restrictions here) it can be learned from the primers by Major [3] and Rovelli [2] and in Rovelli's Book [10].

In next section we shall explore the picture of the quantum geometry that the mathematical apparatus provides for us.

V. QUANTUM GEOMETRY

So far, we have constructed the kinematic Hilbert space of the quantum theory. This is the quantum analog of the classical phase space of the theory, the space of space-time metrics. Just as the spacetime, together with its geometrical constructs such as vectors, tensor fields, derivatives, etc. is the arena for doing classical geometry, the space $\mathcal{H}_{\mathrm{kin}}$ is the arena for quantum geometry. What we need is to define the quantum object, i.e. the operators, that will correspond to the geometric objects defined on this arena. Note that the background manifold Σ has many of the features of the classical geometry, namely all the structure that comes with Σ being a differentiable manifold: tangent spaces, vectors, forms, etc. What is *not* available are the notions arising from a metric on Σ: distance, angles, area, volume, etc. These are the properties of the geometry that are quantized. The purpose of this section is to explore this quantum geometry.

This section has three parts. In the first one we describe the nature of the operators associated to electric fluxes. In the second part, we present the simplest of the geometric operators, namely the area of surfaces and in the last part we discuss the non-commutative character of the quantum geometry.

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integer powers of loops $\alpha^n = \alpha \circ \alpha \cdots \circ \alpha$ in the 'basis' of loops. Graphs (that only care about the image of the embedding, and where $\alpha^2 = \alpha$) together with spin and intertwiners represent a more efficient (no redundancies) way of characterizing the *discrete* basis of the theory.

A. Flux operators

The operators $\hat{E}[S,f]$ corresponding to the electric flux observables, are in a sense the basic building blocks for constructing the quantum geometry. We have seen in Sec.III the action of this operators on cylindrical functions,

$$\hat{E}[S,f]\cdot\Psi_\Upsilon(\overline{A}) = -i\,\hbar\,\{\Psi_\Upsilon,\,{}^2E[S,f]\} = -i\,\frac{\ell_{\rm P}^2}{2}\sum_p\sum_{I_p}\kappa(I_p)\,f^i(p)\,X^i_{I_p}\cdot\psi \tag{40}$$

Here the first sum is over the intersections of the surface S with the graph Υ, and the second sum is over all possible edges I_p that have the vertex v_p (in the intersection of S and the graph) as initial of final point. In the simplest case of a loop α, there are only simple intersections (meaning that there are two edges for each vertex), and in the simplest case of only one intersection between S and α we have one term in the first sum and two terms in the second (due to the fact that the loop α is seen as having a vertex at the intersection point). In this simplified case we have

$$\hat{E}[S,f]\cdot\Psi_\alpha(\overline{A}) = \,-i\,\ell_{\rm P}^2\,f^i(p)\,X^i_{I_p}\cdot\psi \tag{41}$$

Note that the action of the operator is to 'project' the angular momentum in the direction given by f^i (in the internal space associated with the Lie algebra). As we shall see, this operator is in a sense fundamental the fundamental entity for constructing (gauge invariant) geometrical operators. For this, let us rewrite the action of the flux operator (40), dividing the edges that are above the surface S, as 'up' edges, and those that lie under the surface as 'down' edges.

$$\hat{E}[S,f]\cdot = \frac{\ell_P^2}{2}\sum_p f^i(p)\,(\hat{J}^p_{i(u)} - \hat{J}^p_{i(d)})\cdot \tag{42}$$

where the sum is over the vertices at the intersection of the graph and the surface, and where the 'up' operator $\hat{J}^p_{i(u)} = \hat{J}^{p,e_1}_i + \hat{J}^{p,e_2}_i + \cdots + \hat{J}^{p,e_u}_i$ is the sum over all the up edges and the down operator $\hat{J}^p_{i(d)}$ is the corresponding one for the down edges.

B. Area operator

The simplest operator that can be constructed representing geometrical quantities of interest is the *area operator*, associated to surfaces S. The reason behind this is again the fact that the densitized triad is dual to a two form that is naturally integrated along a surface. The difference between the classical expression for the area and the flux variable is the fact that the area is a gauge invariant quantity. Let us first recall what the classical expression for the area function is, and then we will outline the regularization procedure to arrive at a well defined operator on the Hilbert space. The area $A[S]$ of a surface S is given by $A[S] = \int_S {\rm d}^2x\,\sqrt{h}$, where h is the determinant of the induced metric h_{ab} on S. When the surface S can be parametrized by setting, say, $x^3 = 0$, then the expression for the area in terms of the densitized triad takes a simple form:

$$A[S] = \gamma\int_S {\rm d}^2x\,\sqrt{\tilde{E}^3_i\,\tilde{E}^3_j\,k^{ij}} \tag{43}$$

where $k^{ij} = \delta^{ij}$ is the Killing-Cartan metric on the Lie algebra, and γ is the Barbero-Immirzi parameter (recall that the canonical conjugate to A is ${}^{\gamma}\tilde{E}^a_i = \tilde{E}^a_i/\gamma$). Note that the functions is again smeared in two dimensions and that the quantity inside the square root is very much a square of the (local) flux. One expects from the experience with the flux operator, that the resulting operator will be a sum over the intersecting points p, so one should focus the attention to the vertex operator

$$\Delta_{S,\Upsilon,p} = -\left[(\hat{J}^p_{i(u)} - \hat{J}^p_{i(d)})(\hat{J}^p_{j(u)} - \hat{J}^p_{j(d)})\right] k^{ij}$$

with this, the area operator takes the form,

$$\hat{A}[S] = \gamma\, \ell_{\rm P}^2 \sum_p \widehat{\sqrt{\Delta_{S,\Upsilon,p}}} \tag{44}$$

We can now combine both the form of the vertex operator with Gauss' law $(\hat{J}^p_{i(u)} + \hat{J}^p_{i(d)})\cdot\Psi = 0$ to arrive at,

$$|(\hat{J}^p_{i(u)} - \hat{J}^p_{i(d)})|^2 = |2(\hat{J}^p_{i(u)})|^2 \tag{45}$$

where we are assuming that there are no tangential edges. The operator $\hat{J}^p_{i(u)}$ is an angular momentum operator, and therefore its square has eigenvalues equal to $j^u(j^u+1)$ where j^u is the label for the total 'up' angular momentum. We can then write the form of the operator

$$\hat{A}[S]\cdot\mathcal{N}(\Upsilon,\vec{j}) = \gamma\, \ell_{\rm P}^2 \sum_{v\in V} \sqrt{|\hat{J}^p_{i(u)}|^2}\cdot\mathcal{N}(\Upsilon,\vec{j}) \tag{46}$$

With these conventions, in the case of simple intersections between the graph Υ and the surface S, the area operator takes the well known form:

$$\hat{A}[S]\cdot\mathcal{N}(\Upsilon,\vec{j}) = \gamma\, \ell_{\rm P}^2 \sum_{v\in V} \sqrt{j_v(j_v+1)}\cdot\mathcal{N}(\Upsilon,\vec{j})$$

when acting on a *spin network* $\mathcal{N}(\gamma,\vec{j})$ defined over Υ and with labels $\vec{j}$ on the edges (we have not used a label for the intertwiners).

Let us now interpret these results in view of the new geometry that the loop quantization gives us. The one dimensional excitations of the geometry carry flux of area: whenever the graph pierces a surface it endows S with a quanta of area depending on the value of j. Furthermore, the eigenvalues of the operator are discrete, giving a precise meaning to the statement that the geometry is quantized: there a minimum (non-zero) value for the area given by taking $j = 1/2$ in the previous formula. Thus the area gap $a_{\rm o}$ is given by

$$a_{\rm o} = \gamma\, \ell_{\rm P}^2 \frac{\sqrt{3}}{2} \tag{47}$$

If the value of γ is of order unit, then we see that the minimum area is of the order of the Planck area. In order to get a macroscopic value for area we would need a very large number of intersections. The Immirzi parameter has to be fixed to select the physical sector of the theory. The current viewpoint is that the black hole entropy calculation can be used for that purpose.

There are operators corresponding to other geometrical objects such as volume, length, angles, etc. The main feature that they exhibit is that their spectrum is always discrete.

C. Non-commutative quantum geometry

One of the somewhat unexpected results coming from loop quantum geometry is the fact that the quantum geometry is inherently non-commutative. This means that the operators associated to geometrical quantities do not, in general, commute.

Let us for simplicity consider area operators. Given a surface S, any spin network state defined on graphs that do not have vertices on the surface are eigenstates of the area operator (bi-valent vertices are equivalent to no-vertices). There is no degeneracy. If the graph has trivalent vertices on the surface, spin networks are still eigenstates and there is again no degeneracy. It is for a four valent vertex that things start to get interesting. It is *not* true that any spin network state is an area eigenstate. There is, however, a choice of basis of eigenstates for any given area operator. This is true for any valence, namely that given one surface and any vertex thereon there is always a choice of basis that is an eigenbasis for that surface. There will be, however, degeneracy for higher valence vertices. For a four valent one, we know that for a fixed coloring of the edges, there will be some degeneracy. That is, the (finite dimensional) vector space spanned by the intertwiners can have dimension grater that one. The dimension will be given by the number of possible representations of $SU(2)$ compatible with the external colorings (via the triangle inequality). In order to break the degeneracy, it is enough to have one surface operator $A[S]$ acting on the vertex, provided that it has two vertices on each side of the surface. That choice of surface will then 'select' the decomposition of the vector space into eigenstates of $A[S]$, the eigenvalues given by the $\sqrt{j_{\rm int}(j_{\rm int}+1)}$ kind of formula, where the $j_{\rm int}$ means the 'internal' coloring of the representation used to write the intertwiner.

Let us now consider an explicit case of non-commutativity for a graph Υ having a four valent vertex v at the surface S. We assume that the edges e_1 and e_2 are *up* edges for S_1 (and (e_3, e_4) are down). We denote by j_5 the internal edge, and we choose as the state a linear combination of spin networks (with coefficients $(\alpha_0, \alpha_1, \alpha_2)$) of spin networks having $j_5 = 0, j_5 = 1$ and $j_5 = 2$ respectively as the internal index: $\Psi(\Upsilon, \mathbf{j_0}) = \alpha_0\ \mathcal{N}(\gamma, j_5 = 0) + \alpha_1\ \mathcal{N}(\gamma, j_5 = 1) + \alpha_2 \mathcal{N}(\gamma, j_5 = 2) = \sum_i \alpha_i \mathcal{N}_i$. The external edges (j_1, j_2, j_3, j_4) are all taken to be equal to one. For simplicity let us denote by $\mathcal{N}_i$ the eigenstates of the area operator: $\hat{A}[S_1] \cdot \mathcal{N}_i = a_i \mathcal{N}_i$, where $a_0 = 0, a_1 = \gamma \ell_{\rm P}^2\ \sqrt{2}$ and $a_3 = \gamma \ell_{\rm P}^2 \sqrt{6}$.

First we need to compute the action of the first surface $\hat{A}[S_1]$,

$$\hat{A}[S_1] \cdot \Psi(\Upsilon, \mathbf{j_0}) = \sum_i \alpha_i\, a_i\, \mathcal{N}_i \tag{48}$$

We know that in this case the vector space of intertwiners is three dimensional (spanned by $\mathcal{N}_i$), so we can think of the area operator acting on this space as a matrix $A^i_j = a_i\, \delta^i_j$ (no summation over i).

Before acting with $\hat{A}[S_2]$ we need to change basis from the basis diagonal on j_5 to the one diagonal to j_6. Thus we need to expand $\mathcal{N}_i$ in terms of the other basis. The 3×3 matrix $U^j{}_k$ that changes basis has to be an element of $SO(3, \mathbb{C})$. Its precise form is not relevant in what follows but it is known that its components are given by $6 - j$ symbols [10]. We can then write,

$$\mathcal{N}_j = U^k{}_j\, \tilde{\mathcal{N}}_k \tag{49}$$

We can now compute

$$\hat{A}[S_2]\cdot\hat{A}[S_1]\cdot\Psi(\Upsilon,\mathbf{j_0}) \;=\; \hat{A}[S_2]\cdot\sum_{i,k}\alpha_i\,a_i U^k{}_i\,\tilde{\mathcal{N}}_k \tag{50}$$

$$=\;\sum_{i,k}\alpha_i\,a_i U^k{}_i\,a_k\,\tilde{\mathcal{N}}_k \tag{51}$$

Let us now act with the operators in the reverse order, namely let us act with $\hat{A}(S_2)$ first,

$$\hat{A}[S_2]\cdot\Psi(\Upsilon,\mathbf{j_0})=\sum_{i,k}\alpha_i U^k{}_i\,a_k\,\tilde{\mathcal{N}}_k \tag{52}$$

Now, before acting with $\hat{A}[S_1]$ we need to change basis from the basis diagonal on j_6 to the basis diagonal to j_5. Thus we need to expand $\tilde{\mathcal{N}}_i$ in terms of the other basis. The matrix that defines this change of basis will be the inverse $U^{(-1)k}{}_j$ such that $\tilde{\mathcal{N}}_j = U^{(-1)k}{}_j\,\mathcal{N}_k$. We can now compute

$$\hat{A}[S_1]\cdot\hat{A}[S_2]\cdot\Psi(\Upsilon,\mathbf{j_0}) \;=\; \hat{A}[S_1]\cdot\sum_{i,k,m}\alpha_i\,U^k{}_i\,a_i U^{(-1)m}{}_k\,\mathcal{N}_m \tag{53}$$

$$=\;\sum_{i,k,m}\alpha_i\,U^k{}_i\,a_i U^{(-1)m}{}_k\,a_m\,\mathcal{N}_m \tag{54}$$

We can then write the commutator as,

$$[\hat{A}[S_1],\hat{A}[S_2]]\cdot\Psi(\Upsilon,\mathbf{j_0})=\sum_{i,k,m}\alpha_i\,\left(U^k{}_i\,a_i U^{(-1)m}{}_k\,a_m - a_i\,U^k{}_i\,a_k\,U^{(-1)m}{}_k\right)\,\mathcal{N}_m \tag{55}$$

Which is, in general, different from zero (for instance, the area matrix $A^i_j = a_i\delta^i_j$ does not commute with $U^i{}_k$). This implies clearly that there is an uncertainty relation for intersecting areas

$$(\Delta\hat{A}[S_1])^2\,(\Delta\hat{A}[S_2])^2\geq 0$$

For more details on non-commutativity see [36].

VI. RESULTS AND OPEN ISSUES

So far we have dealt with the kinematical aspects of the theory, in which the states are (gauge invariant) states labelled by embedded graphs on the three manifold Σ. However, the fact that general relativity is diffeomorphism invariant as a classical field theory provides a good motivation for trying to implement diffeomorphism invariance at the quantum level. In the $3+1$ setting this means implementing the constraints as quantum conditions on the quantum states.

Let us first consider the diffeomorphism constraint, that is, the generator of spatial diffeomorphisms. In the canonical setting, this means that we should ask for diffeomorphism invariant states. The answer is rather easy: the only diffeomorphism invariant state in $\mathcal{H}_{\rm kin}$ is the vacuum Ψ_0! This means that the intuitive idea that diff-invariant states as obtained by taking the quotient by the diffeomorphisms in the Hilbert space does not work. One needs to find a subtler prescription. Fortunately, it is now well understood that, in general,

solutions to the quantum constraints do not lie on the original kinematical Hilbert space where the constraints were originally posed. Instead, one has to look for solutions in a larger space. In the case of LQG, diffeo-invariant states live in $(\mathrm{Cyl})^*$, the dual space to the cylindrical functions, where a solution is found by summing over all possible diffeomorphism related states (recall that the action of the diffeomorphisms is a unitary map in the kinematical Hilbert space). Even when the sum is over an uncountable number of states, as a *distribution* its action on cylindrical functions is well defined. The procedure also includes specifying a diffeo-invariant inner product on solutions, and by completing, one arrives at $\mathcal{H}_{\mathrm{diff}}$, the Hilbert space of diffeomorphism invariant states. Roughly speaking, instead of considering states labelled by embedded graphs as in $\mathcal{H}_{\mathrm{kin}}$, diffeo-invariant states are labelled by diffeomorphism classes of graphs (apart from the internal labellings $(\mathbf{j}, \mathbf{i})$). Having solved the diffeomorphism invariance, in a rigorous manner and without anomalies, is one of the main achievements of the formalism. For details see [6].

Due to lack of space, we will refrain from discussing in detail two of the main applications of LQG, namely Black Holes and Cosmology, so we will only give a brief summary. For black holes, a detailed treatment of the boundary conditions of the theory in the presence of an isolated horizon has lead to a detailed description of the quantum geometry associated with the horizon, to an identification of the quantum degrees of freedom responsible for the entropy, and to a counting of states that yields the entropy proportional to the area. The proportionality factor is set, for large black holes, to 1/4 by adjusting the only free parameter of the theory, that is, the Barbero-Immirzi parameter. The same value can account for the entropy of very general BH, including Einstein-Maxwell, Einstein-Dilaton and non-minimally coupled scalar fields. For details and references see [6].

In the case of cosmology, a detailed program known as *Loop Quantum Cosmology* (LQC), has been developed in the past years (see for instance Bojowald's contribution to this volume and [33]). It represents a dimensional reduction, at the classical level, of the gravitational degrees of freedom to instances of great symmetry, as is expected to be the case in the cosmological setting. The quantization is performed via non-standard representations of the classical observable algebra, much in the spirit of LQG, with certain input coming from the full theory. As a result, the theory possesses very different behavior as the standard mini-superspace quantization: the space-time curvature has been shown to be bounded above and thus, the classical singularity is avoided. Furthermore, there is some evidence that the solutions may exhibit an inflationary period.

More recently, a new physical picture for the evaporation of black holes has emerged both from a detailed treatment of dynamical horizons and the end result of gravitational collapse (borrowing from results in LQC). These new developments suggests that one should reconsider the traditional viewpoint towards Hawking evaporation and the information loss problem [37].

Loop quantum gravity is a theory in the making. There are still unsolved issues and things that remain to be done. The most notorious issue that has eluded a complete solution is the implementation of the Hamiltonian constraint. There is no proposal that is fully satisfactory. There is a consistent, free of anomalies quantization by Thiemann, but is far from unique and there is no control (yet) on the physical properties of its solutions (see [11]). More recently Thiemann has proposed the 'master constraint program' to implement the constraints. There is a completely different approach to the subject, where a projector onto physical states is sought by summing over 4-dim discrete structures known as spin-foams (see for instance [7]). Finally, a new program that tries to solve the constraint in

a similar fashion as the diffeomorphism invariance is implemented, namely by considering finite actions of the constraint rather than defining and implementing its generator, is being pursued nowadays.

Another open issue is the semi-classical limit of the theory. That is, one needs to find states within the theory that approximate, in a low-energy/macroscopic limit, classical configurations such as Minkowski spacetime. To deal with this challenge there have been several proposals for semi-classical states, from the so-called weave states, to coherent states [11], WKB-like states [5], and more recently Fock-motivated states [6]. In most of these cases, the states belong to the kinematical space of states. A systematic approach to the definition of semiclassical states for constrained systems and to the question of whether one can approximate physical semi-classical states with kinematical ones has only begun [38].

In previous sections we have defined LQG in terms of what it is and does. To end this section we will part from this tradition and make some remarks about what LQG is *not*. This can be seen as a complement to the FAQ section of [5].

- *Renormalizability.* It is sometimes asserted that since LQG is a reformulation of General Relativity, then it is non-renormalizable. The first obvious observation is that LQG *is* a non-perturbative quantum theory, and as such, it has been shown to be finite (see for instance [6, 11]). There is no inconsistency with the fact that Einstein gravity is perturbatively non-renormalizable. The level of rigor with which LQG is defined, as a mathematically well defined theory, is superior to any interacting quantum field theory in 4 dimensions. As such, the theory has also shown to be quite independent of "infrared" cutoffs.

- *What is LQG not?* One sometimes sees statements like: "various alternative formulations of LQG such as random lattice or spin foam theory..." This claim is incorrect. Spin foams and random lattices are not alternative formulations of LQG. In particular, random lattices are not directly related to LQG, and spin foams are tools (under development) to compute projection operators onto physical states of the theory, i.e. as a tool for solving the quantum scalar constraint of LQG (see [7] for details).

- *Is LQG a discrete theory?* In this regard, it is important to stress, again, that LQG is a well defined theory on the continuum. It is *not* a discretized gravity model such as Regge calculus or dynamical triangulations, where the discrete structure is assumed from the beginning. It is true that a convenient basis of states in the Hilbert space of LQG is given by graphs and spins (the spin networks basis), but the theory is defined in the continuum, and the quantized character of the geometry (the discrete eigenvalues for the operators) are found as predictions of the theory. There is no continuum limit to be taken. All details and mathematical proofs of these assertions can be found in Refs. [6, 11].

To end this primer, let us point out to further references. Once the reader is somewhat familiar with the basic assumptions of the theory, she should move on to move detailed description of the formalism. In this regard, the book by Rovelli [10] is a good read for the conceptual background and spirit of LQG. For a good degree of detail, rigor and current status of the program see [6]. For a completely detailed treatment of the formalism see the monograph by Thiemann [11].

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