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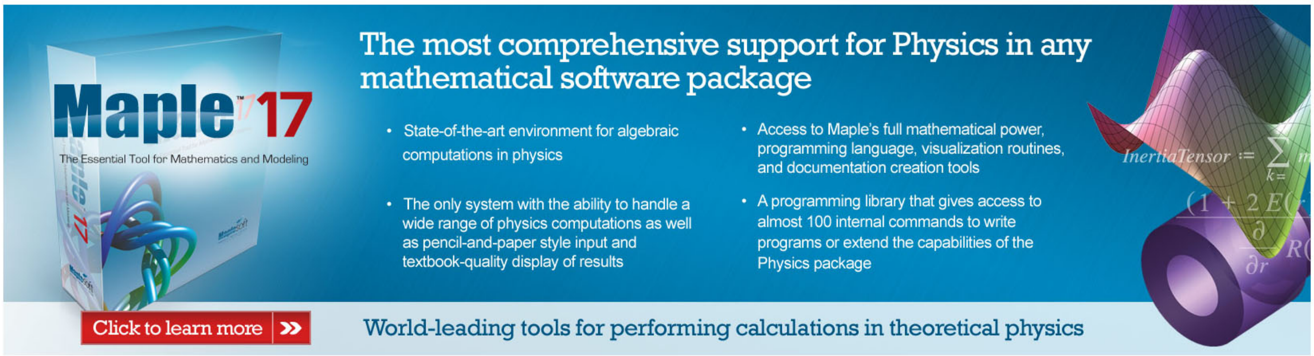

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Group actions on principal bundles and invariance conditions for gauge fields [a)]

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Invariance conditions for gauge fields under smooth group actions are interpreted in terms of invariant connections on principal bundles. A classification of group actions on bundles as automorphisms projecting to an action on a base manifold with a sufficiently regular orbit structure is given in terms of group homorphisms and a generalization of Wang's theorem classifying invariant connections is derived. Illustrative examples on compactified Minkowski space are given.

In the study of gauge field equations at the classical level a standard method of simplification involves the requirement that the fields be invariant under a group of space-time transformations.[1] Such a requirement leads to a reduction in the dimension of the free variables and a reduction of the gauge freedom to those changes of gauge which preserve the invariance condition. The specification of how the transformation group acts on the fields may involve an auxiliary gauge transformation. In local terms this gauge transformation will be determined by a function which we shall call a transformation function, depending on the group element and the space-time point and subject to an appropriate composition law. A change in gauge changes the local expression for the transformation function to an equivalent one. Since the form of the transformation function determines the form of the invariance equations and thus affects the difficulty in finding the invariant fields it is useful to have a reduction procedure for simplifying the invariance equations. An associated problem is determining all inequivalent transformation functions for a given transformation group. In this paper we study these problems and show how to find the most general gauge fields possessing a given symmetry using the language and methods of fiber bundle theory. Forgács and Manton[2] have studied the same problem from another point of view. For further applications to problems in symmetry breaking and dimensional reduction see Refs. 3–6.

Since a change of gauge can be interpreted as a change of fiber coordinates in a fiber bundle, our first step will be to formulate the problem in coordinate free language. So expressed, the problem of determining all inequivalent transformation functions is seen to be essentially the same as determining all inequivalent lifts of the transformation group action from the base to automorphisms on the bundle. For a homogeneous space, a known result[7] reduces the problem to a classification of group homomorphisms. For the general case, no result is known, however, provided the orbit structure is regular enough we can solve the problem under the additional hypothesis that the gauge group is compact. The gauge fields determine a connection on the bundle and the symmetry problem is equivalent to the classification of G-invariant connections. Again, for a homogeneous G space the solution is standard and may be extended to certain more general cases.

1. BASIC RESULTS FOR HOMOGENEOUS SPACES

Let H be the gauge group with Lie algebra $\mathscr{h}$, M a differentiable manifold, and G a Lie transformation group acting on M such that the map

$$G \times M \rightarrow M \ (g,x) \rightarrow f_g(x)$$

is differentiable and satisfies

$$f_e(x) = x, \quad f_{g_1}(f_{g_2}(x)) = f_{g_1 g_2}(x). \tag{1}$$

When no confusion can arise we shall write gx for $f_g(x)$.

The gauge fields which we consider are defined on an open covering $\{U_\alpha\}$ of M by a set of $\mathscr{h}$ valued 1-forms ω_α on U_α related by

$$\omega_\beta = \mathrm{Ad} k_{\alpha\beta}^{-1} \omega_\alpha + k_{\alpha\beta}^{-1} d k_{\alpha\beta},$$

where the functions $k_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow H$ satisfy $k_{\alpha\alpha} \equiv e$, $k_{\alpha\beta} k_{\beta\gamma} = k_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. The $k_{\alpha\beta}$ are transition functions for a principal H bundle E over M trivial over each U_α, that is, there are functions

$$\tau_\alpha : U_\alpha \times H \rightarrow E,$$

with $\tau_\beta^{-1}\tau_\alpha : U_\alpha \cap U_\beta \times H \rightarrow U_\alpha \cap U_\beta \times H$, such that

$$\begin{aligned}\tau_\beta^{-1}\tau_\alpha(x,h) &= (x, k_{\alpha\beta}(x)^{-1}h) \\ &= (x, k_{\beta\alpha}(x)h).\end{aligned} \tag{2}$$

The right action of the gauge group H on E is given by

$$R_k \tau_\alpha(x,h) = \tau_\alpha(x,hk), \quad \text{for } x \in M; \quad h,k \in H. \tag{3}$$

Define a local section σ_α by

$$\sigma_\alpha(x) = \tau_\alpha(x,e).$$

When there is no possibility of confusion we write $\sigma_\alpha(x)h$ for $R_h\sigma_\alpha(x)$. The form ω_α is the pull-back under σ_α of a connec-

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tion form ω on E. The pull-back of ω under τ_α is given by

$$(\tau_\alpha^*\omega)_{(x,h)} = \mathrm{Ad}h^{-1}(\omega_\alpha)_x + h^{-1}\,dh, \tag{4}$$

which in fact defines ω.

If the open sets U_α are G invariant the condition for G invariance of the ω_α up to gauge transformation is

$$(f_g^*\omega_\alpha)_x = \mathrm{Ad}\rho_\alpha(g,x)^{-1}(\omega_\alpha)_x + \rho_\alpha^{-1}(g,x)\,d\rho_\alpha(g,x), \tag{5}$$

where the differential in ρ_α is in the x variable. The function ρ_α is what we call a transformation function. The $\rho_\alpha: G\times U\to H$ satisfy

$$\rho_\alpha(g_1 g_2,x) = \rho_\alpha(g_2,x)\rho_\alpha(g_1,g_2x) \tag{6}$$

in order to satisfy the group composition law (1) and the compatibility condition, and

$$\rho_\alpha(g,x)k_{\alpha\beta}(gx) = k_{\alpha\beta}(x)\rho_\beta(g,x) \tag{7}$$

for the consistency of (5) under change of section. The functions ρ_α define a G action on E

$$\begin{aligned} G\times E\to E \quad (g,\tau_\alpha(x,h))&\to \tilde f_g\tau_\alpha(x,h)\\ &= \tau_\alpha(gx,\ \rho_\alpha(g,x)^{-1}h). \end{aligned} \tag{8}$$

[This is a valid G-action on E by virtue of (6) and independent of the local trivialization τ_α by virtue of (7).] Again writing $g\sigma_\alpha(x)$ for $\tilde f_g(\sigma_\alpha(x))$,

$$g\sigma_\alpha(x) = \sigma_\alpha(gx)\rho_\alpha(g,x)^{-1}. \tag{9}$$

The invariance condition (5) implies that the connection defined in (4) satisfies

$$\tilde f_g^*\omega = \omega. \tag{10}$$

This is the coordinate-free form of the invariance condition which we shall study.

Before proceeding, note that if the open sets U_α over which E is trivial cannot be chosen so that they are G invariant, then given $x\in U_\alpha$, we must restrict the $g\in G$ appearing in Eq. (5) to those for which $gx\in U_\alpha$. Alternatively we can find an infinitesimal invariance equation which can be expressed in local coordinates as follows.

Let $V(M)$ be the smooth vector fields on M. Denoting by $\mathscr{G}$ the left invariant vector fields on G (identified with the Lie algebra) define mappings:

$$\varphi:\mathscr{G}\to V(M) \quad \varphi(\xi)_x = \frac{d}{dt}\bigg|_0 \exp(-t\xi)x$$

and

$$r_\alpha:\mathscr{G}\times M\to \mathscr{h} \quad r_\alpha(\xi,x) = -\frac{d}{dt}\bigg|_0 \rho_\alpha(\exp t\xi,x).$$

The invariance equation in infinitesimal form becomes

$$\mathscr{L}_{\varphi(\xi)}\omega_\alpha = [r_\alpha(\xi,x),\omega_\alpha] - dr_\alpha(\xi,x), \tag{11}$$

where the left hand side denotes the Lie derivative and the differential on the right is in the x variable.

The function r_α satisfies the composition law

$$\begin{aligned} r_\alpha([\xi,\eta],x) = [r_\alpha(\xi,x),r_\alpha(\eta,x)] + \varphi(\xi)_x r_\alpha(\eta,x)\\ - \varphi(\eta)_x r_\alpha(\xi,x) \end{aligned} \tag{12}$$

and the compatibility condition

$$r_{\alpha\beta}(\xi,x) = \mathrm{Ad}k_{\alpha\beta}(x)^{-1}r_\beta(\xi,x) + k_{\alpha\beta}^{-1}\,dk_{\alpha\beta}. \tag{13}$$

The interpretation of the infinitesimal invariance condition on the bundle level is as follows.[8] Let

$$\Phi(\xi)_{\tau_\alpha(x,h)} = \tau_{\alpha*}(\varphi(\xi) + \mathrm{Ad}h^{-1}r_\alpha(\xi,x)). \tag{14}$$

Equation (13) guarantees that this defines unambiguously a vector field on E and Eq. (12) implies that $\Phi:\mathscr{G}\to V(M)$ is an algebra homomorphism

$$\Phi([\xi,\eta]) = [\Phi(\xi),\Phi(\eta)].$$

One checks that (11) is equivalent to

$$\mathscr{L}_{\Phi(\xi)}\omega = 0. \tag{15}$$

This infinitesimal form seems more general since it does not assume the existence of a group action in finite (integrated) form. However, if the infinitesimal action on M integrates and if the gauge group is compact the infinitesimal action on E given by Φ integrates.

We can now formulate the problem in terms of fiber bundles as the determination of all principal H bundles with G action (as automorphisms) projecting to the given action on M and all invariant connections on such bundles. However the question posed in this form is too general since it involves the topological problem of classifying all H bundles over M. We restrict attention to the structure of the bundle over a neighborhood of an orbit in M and begin with the structure of E over a single orbit.

For $x\in M$ let G_x be the isotropy group at x and let $G(x)$ be the orbit through x. Assume the orbit is an imbedded submanifold of M then G/G_x is diffeomorphic to $G(x)$ and the structure of E over $G(x)$ is determined by (see e.g. Ref. 7).

Proposition 1: There is a one-to-one correspondence between

(a) Equivalence class of principal H bundles E over G/G_x admitting a G action which projects to left multiplication of G on G/G_x; and

(b) Conjugacy classes of homomorphisms $\lambda: G_x\to H$.

Equivalence in (a) means an isomorphism of bundles which commutes with the action of G and projects to the identity mapping.

We shall sketch a proof in order to clarify the result and establish notations.

Proof: Given a bundle E from one of the equivalence classes in (a) any $g\in G_x$ maps the fiber E_x over $x=eG_x$ into itself. If we pick a point $p\in E_x$ we have

$$gp = p\lambda(g),$$

where $\lambda: G_x\to H$. One sees immediately that λ is a homomorphism since the G and H actions commute and that if p is right translated by h then λ is conjugated by h. Also if $\varphi: E\to E'$ is a G equivariant bundle isomorphism so that E and E' are equivalent, the points p and $\varphi(p)$ determine the same homomorphism λ.

Conversely given $\lambda: G_x\to H$ we can construct a principal H bundle E_λ over G/G_x. On the set $G\times H$ define an equivalence relation

$$(g,h)\sim(gg_1,\lambda(g_1)^{-1}h), \quad \text{for } g_1\in G_x.$$

Let $[g,h]$ be the equivalence class of (g,h) and let E_λ be the set of equivalence classes. Another notation often used for E_λ is $G\times_{G_x}H$. Projection on the first factor $G\times H\to G$ defines a

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projection

$$\pi: G \times_{G_x} H \to G/G_x.$$

The left action of G and right action of H defined by

$$(g_1,(g,h)) \mapsto (g_1 g, h),$$

$$((g,h),h_1) \mapsto (g, hh_1)$$

preserve the equivalence relation and so define group actions of G and H on E_λ. The action of G on E_λ projects by π to left multiplication on the coset space G/G_x. The right action of H is transitive on the fibers of π. To verify the bundle structure, let $U \subset G/G_x$ be an open set on which there is a cross-section $\sigma: U \to G$ of $G \to G/G_x$. Then we can define a cross-section of E_λ over U by

$$y \to [\sigma(y), e]$$

and a corresponding local trivialization

$$(y,h) \mapsto [\sigma(y), h].$$

Since $G \to G/G_x$ itself has a bundle structure, there exists a covering of G/G_x by such open sets U. Having shown how to go from (a) to (b) and (b) to (a), we show that the composite in either order gives back the same equivalence class. If we pick the point $[e,e]$ in the fiber of E_λ over $x = eG_x$ we have for $g \in G_x$

$$g[e,e] = [g,e] = [e, \lambda(g)] = [e,e]\lambda(g).$$

Thus we recover the homomorphism λ from the bundle E_λ. Finally if E is a bundle and for $p \in E_x$ the associated homomorphism is λ, we define a G equivalent isomorphism:

$$E_\lambda = G \times_{G_x} H \to E,$$

$$[g,h] \to gph.$$

In local terms we can use this result to show how the transformation function depends on the homomorphism λ and the section σ of $G \to G/G_x$,

$$\begin{aligned} g[\sigma(y),e] &= [g\sigma(y),e] = [\sigma(gy)\sigma(gy)^{-1}g\sigma(y),e] \\ &= [\sigma(gy), \lambda(\sigma(gy)^{-1}g\sigma(y))] \\ &= [\sigma(gy),e]\lambda(\sigma(gy)^{-1}g\sigma(y)). \end{aligned}$$

Thus

$$\rho^{-1}(g,y) = \lambda(\sigma(gy)^{-1}g\sigma(y)) \tag{16}$$

if we use the section of E_λ

$$y \to [\sigma(y),e].$$

For a given homomorphism λ, the bundle E_λ need not be trivial and therefore the transformation function may not be defined throughout the orbit. The case when it *can* be is given by:

Corollary 1: The bundle E is trivial over G/G_x if and only if the homomorphism $\lambda: G_x \to H$ extends to a smooth function $\Lambda: G \to H$ such that

$$\Lambda(gg_1) = \Lambda(g)\lambda(g_1), \quad \text{for } g \in G, \quad g_1 \in G_x.$$

Proof: If σ is a section of E defined over all of G/G_x, define $\Lambda(g)$ by $g\sigma(x) = \sigma(gx)\Lambda(g)$. Conversely given Λ satisfying the hypotheses, $\sigma: gG_x \to [g, \Lambda^{-1}(g)]$ defines a section of the bundle $G \times_{G_x} H$ over G/G_x.

The section σ satisfies

$$\begin{aligned} g_1\sigma(gG_x) &= [g_1 g, \Lambda^{-1}(g)] \\ &= [g_1 g, \Lambda^{-1}(g_1 g)]\Lambda(g_1 g)\Lambda^{-1}(g) \end{aligned}$$

for $g_1, g \in G$ and so the associated transformation function is

$$\rho(g_1, gG_x) = \Lambda(g)\Lambda(g_1 g)^{-1}, \quad g_1, g \in G. \tag{17}$$

The condition for ρ to be independent of its second variable, the point in the orbit, is given by the following corollary.

Corollary 2: The following two conditions are equivalent.

(a) The bundle $E \to G/G_x$ is trivial with gauge function $\rho(g_1, gG_x)$ independent of the point gG_x.

(b) The homomorphism $\lambda: G_x \to H$ extends smoothly to a homomorphism $\Lambda: G \to H$.

Proof: Equation (17) shows that $\rho(g_1, gG_x) = \rho(g_1, G_x) = \Lambda(g_1)^{-1}$ if and only if Λ is a homomorphism.

The simplest transformation function is just the identity, the criterion for which is the following.

Corollary 3: The transformation function $\rho(g_1, gG_x)$ reduces to the trivial function $\equiv e$ if and only if it is trivial when restricted to the isotropy group G_x. That is, the image of λ in H is e.

One case in which this always occurs is when the G-action on M is free, i.e., $G_x = e$.

We continue with the discussion of G-invariant connections on $E \to G/G_x$. We shall give a proof of the theorem of Wang[9] classifying these connections, in which we make use of the bundle $E_\lambda = G \times_{G_x} H$.

Proposition 2: Let $\mathscr{G}$ be the Lie algebra of G, $\mathscr{G}_0$ the Lie algebra of $G_0 \subset G$ and $\mathscr{h}$ the Lie algebra of H. The G invariant connections on the bundle E_λ determined by $\lambda: G_0 \to H$ are in one to one correspondence with linear mappings $A: \mathscr{G} \to \mathscr{h}$ satisfying the following two equations:

$$A(\xi) = \lambda_*(\xi), \quad \text{for } \xi \in \mathscr{G}_0 \quad \text{and } \lambda_*, \tag{18a}$$

the homomorphism $\lambda_*: \mathscr{G}_0 \to \mathscr{h}$ determined by the differential of λ.

$$A(\mathrm{Ad}g^{-1}\xi) = \mathrm{Ad}\lambda(g)^{-1}(A(\xi)), \quad \text{for } \xi \in \mathscr{G} \text{ and } g \in G_0. \tag{18b}$$

Proof: Let ω be a G-invariant connection on $G \times_{G_0} H$, let $\psi: G \times H \to G \times_{G_0} H$ be defined by $\psi(g,h) = [g,h]$ and let $j: G \to G \times H$ be $j(g) = (g,e)$. Then $\psi^*\omega$ is a G-invariant connection on the trivial H bundle $G \times H$ and $j^*\psi^*\omega$, its pullback to the base space G, is a left G-invariant $\mathscr{h}$ valued form and thus is determined by its value at $T_e G$ which can be identified with $\mathscr{G}$. We conclude that if $\theta_{\mathscr{G}}$ is the left-invariant Maurer–Cartan form on G then there is a linear map $A: \mathscr{G} \to \mathscr{h}$ such that

$$j^*\psi^*\omega = A \circ \theta_{\mathscr{G}}.$$

Let $\psi^*\omega = \omega_1 + \omega_2$, where ω_1 acts on the tangents to the first factor and ω_2 on the tangents to the second factor in $G \times H$. If $\eta \in \mathscr{h}$ and $\tilde{\eta}$ is the vertical vector field on $G \times_{G_0} H$ generated by $R_{\exp t\eta}$, then $\omega(\tilde{\eta}) = \eta$ which implies ω_2 is the Maurer–Cartan form on $\mathscr{h}$, $\theta_{\mathscr{h}}$. From the equivariance condition

$$R_h^*\omega = \mathrm{Ad}h^{-1}\omega$$

we conclude

$$\psi^*\omega_{(g,h)} = \mathrm{Ad}h^{-1}(A \circ \theta_{\mathscr{G}}) + \theta_{\mathscr{h}}.$$

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Proof: The argument is very close to that in the proof of Proposition 2 so we will omit most of the details. As in that proof we define $\psi: G\times H\times S \to G\times_{G_0} H\times S$ by $\psi(g,h,s)=([g,h],s)$ and find

$$\psi^*\omega(g,h,s)=\mathrm{Ad}h^{-1}(A_s\circ\theta_{\mathscr{G}}+\mu)+\theta_{\mathscr{h}},$$

where μ is a one-form on S. Left G invariance shows that there is no "g dependence" in the form μ. The conditions that the right-hand side define the pull-back of a form on $G\times_{G_0}H\times S$ impose in addition to Eqs. (18a) and (18b) on the linear mappings A_s the additional equation,

$$\mu=\mathrm{Ad}\lambda(g)^{-1}\mu,\quad \text{for } g\in G_0.$$

Thus μ must take values in the subalgebra of $\mathscr{h}$ of elements invariant under the adjoint action of $\lambda(G_0)$.

3. EXAMPLES

We now illustrate these results with some examples. let M_0 be compactified Minkowski space which we identify[1] with U(2), let $M=\mathrm{SU}(2)\times\mathrm{U}(1)$ be the twofold covering and let the gauge group H be SU(2). For the transformation group G also equal to SU(2) consider the following actions of G on M. Given $g\in G=\mathrm{SU}(2)$ and $(x,e^{i\psi})\in M=\mathrm{SU}(2)\times\mathrm{U}(1)$ define

$$\alpha_g(x,e^{i\psi})=(gx,e^{i\psi}),$$
$$\beta_g(x,e^{i\psi})=(xg^{-1},e^{i\psi}),$$
$$\gamma_g(x,e^{i\psi})=(gxg^{-1},e^{i\psi}).$$

Both α and β define simple actions with special cross-sections through $(x,e^{i\psi})$ given by $\varphi(s)=(x,e^{i(\psi+s)})$. The action defined by γ is not simple since there are two orbit types for the conjugation action of SU(2) on itself. Therefore we restrict to the open submanifold $M_1=(\mathrm{SU}(2)-\{\pm I\})\times\mathrm{U}(1)$ on which the action γ is simple, where

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}\in\mathrm{SU}(2).$$

On M_1

$$\varphi(s,t)=\left(x\begin{pmatrix}e^{is}&0\\0&e^{-is}\end{pmatrix},\ e^{i(\psi+t)}\right)$$

defines a special cross-section through $(x,e^{i\psi})$.

Since α and β commute there is a well defined action $\alpha\times\beta$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ on M which we shall also consider.

Example α: The isotropy group is the identity I and therefore the orbits are identifiable with G. By Corollary 3 the bundle structure over any orbit is trivial. By Theorem 1 the same is true over a neighborhood of an orbit and by Theorem 2 the connection form pulled back to the base space M under any G invariant section is given by

$$\tilde{\omega}=A_\psi\circ\theta_{\mathscr{G}}+Bd\psi,$$

where A_ψ is a smoothly parameterized family of linear maps $\mathscr{G}=\mathfrak{su}(2)\to\mathscr{h}=\mathfrak{su}(2)$, B is a smooth $\mathfrak{su}(2)$ valued function of ψ, and the Maurer–Cartan form $\theta_{\mathscr{G}}$ is regarded as defined, on a neighborhood of orbits, on the first term in $M\sim\mathrm{SU}(2)\times U(1)$. The triviality of the bundle in this case may be proved to be global (see Ref. 1).

Example β: This is completely equivalent to the previous example with the left invariant Maurer–Cartan form $\theta_{\mathscr{G}}$ replaced by the right invariant Maurer–Cartan form in the expression for $\tilde{\omega}$.

(For the above two examples, the gauge group SU(2) may be replaced by arbitrary H with algebra $\mathscr{h}$, with A_ψ interpreted as any smooth family of linear maps $A_\psi:\mathfrak{su}(2)\to\mathscr{h}$.

Example $\alpha\times\beta$: The transformation group G is $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and along the cross-section $\varphi(s)=(I,e^{is})$ the isotropy group is the diagonal subgroup $\Delta=\{(g,g)|g\in\mathrm{SU}(2)\}\subset G=\mathrm{SU}(2)\times\mathrm{SU}(2)$. Up to conjugacy in $H=\mathrm{SU}(2)$ there are two homomorphism $\lambda:\Delta\to H$

$$\lambda_0(g,g)\equiv I \text{ and } \lambda_1(g,g)=g.$$

These both extend to homomorphisms of $G\to H$ by choosing the extension independent of the second factor; therefore, by Corollary 2, there exists a section of E_{λ_0} and E_{λ_1} over the entire orbit. The bundle E_{λ_0} is defined by the equivalence relation

$$(g_1,g_2,h)\sim(g_1g_3^{-1},g_2g_3^{-1},h)$$

and E_{λ_1} is defined by the equivalence

$$(g_1,g_2,h)\sim(g_1g_3^{-1},g_2g_3^{-1},g_3h).$$

The group action on both is given by

$$\alpha\times\beta_{(g_1',g_2')}[g_1,g_2,h]=[g_1'g_1,g_2'g_2,h],$$

where $[\ \ ]$ denotes an equivalence class. We can identify the orbits with G/Δ and G/Δ can be identified with SU(2) by

$$x\mapsto(x,I)\Delta\in G/\Delta,\quad \text{for } x\in\mathrm{SU}(2).$$

Define sections σ_0 and E_{λ_0} and σ_1 of E_{λ_1} over an orbit by

$$\sigma_0(x)=[x,I,I] \text{ and } \sigma_1(x)=[x,I,I].$$

(The different notations used, distinguish between the two definitions of equivalence.) Then

$$\begin{aligned}\alpha\times\beta_{(g_1',g_2')}\sigma_0(x)&=[g_1'x,g_2',I]\\&=[g_1'xg_2'^{-1},I,I]\\&=\sigma_0(g_1'xg_2'^{-1})\end{aligned}$$

and

$$\begin{aligned}\alpha\times\beta_{(g_1',g_2')}\sigma_1(x)&=[g_1'x,g_2',I]\\&=[g_1'xg_2'^{-1},I,g_2']\\&=\sigma_1(g_1'xg_2'^{-1})g_2'.\end{aligned}$$

If we consider $E_{\lambda_0}\times S$ and $E_{\lambda_1}\times S$ and define σ_0 and σ_1 by

$$\sigma_0(x,e^{i\psi})=([x,I,I],e^{i\psi}).$$
$$\sigma_1(x,e^{i\psi})=([x,I,I],e^{i\psi}).$$

We get the same transformation equations. Write the Maurer–Cartan form $\theta_{\mathscr{G}}$ as $\theta_1+\theta_2$ corresponding to the direct product decomposition. An invariant connection on $E_{\lambda_0}\times S$ or $E_{\lambda_1}\times S$ pulled back to $G\times H\times S$ looks like

$$\omega_{(g_1,g_2,h,e^{i\psi})}=\mathrm{Ad}h^{-1}(A_\psi(\theta_1+\theta_2)+B_\psi\, d\psi)+\theta_{\mathscr{h}}$$

subject to compatibility with λ_0 or λ_1. For λ_0 the condition (18b) implies

$$A_\psi(\mathrm{Ad}(g,g)^{-1}(\theta_1+\theta_2))=A_\psi(\theta_1+\theta_2),$$

which implies $A_\psi\equiv 0$. The $\mathfrak{su}(2)$ valued function B_ψ is arbitrary and the connection pulled back to the base by σ_0 is just

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The mapping ψ defines a fibration of $G \times H$ over $G \times_{G_0} H$ with G_0 acting on the fibers. The conditions for a form on $G \times H$ to be the pull-back by ψ of a form on $G \times_{G_0} H$ are first that it vanish on tangents to the fibers and second that it be invariant under the action of G_0. The tangents to the fibers are given by differentiating in t the expression $[g \exp(t\xi)], \lambda(\exp(-t\xi))h]$ hence are of the form

$$[\xi, -\mathrm{Ad}h^{-1}\lambda_*(\xi)]_{(g,h)}, \quad \text{for } \xi \in \mathscr{G}.$$

The condition $\psi^*\omega\, \xi, -\mathrm{Ad}h^{-1}\lambda_*(\xi) = 0$ implies

$$A(\xi) = \lambda_*(\xi).$$

The G_0 action is given by

$$g_{1*}(\xi,\eta)_{(g,h)} = (\mathrm{Ad}g_1^{-1}\xi, \eta)_{(gg_1, \lambda(g_1)^{-1}h)}.$$

The invariance of $\psi^*\omega$ implies

$$\mathrm{Ad}\lambda(g) A(\mathrm{Ad}g^{-1}\xi) = A(\xi).$$

Since these conditions are necessary and sufficient the proposition is proved.

2.GENERALIZATION TO INTRASITIVE GROUP ACTIONS

Now we study the situation when the base space M is not a homogeneous space. The structure of the bundle over the orbit through x is determined by a homomorphism $\lambda_x : G_x \to H$. To put together the information over a set of orbits we need a smooth cross section, a submanifold intersecting each orbit in one point. Such a cross section may not exist even locally if the conjugacy class of isotropy group changes from orbit to orbit.

For $x \in M$, let $G(x)$ be the orbit of G through x, if $y \in G(x)$ then $y = gx$ and $G_y = gG_xg^{-1}$; the isotropy groups are conjugate. Associated to each orbit is a unique conjugacy class which we call the type of the orbit. If the action of G on M has just one orbit type one can often show that for all $x \in M$ there is a smooth imbedding of an open set $S \subset R^k$ ($k = \dim M - \dim G/G_0$) into M $\varphi : S \to M$ with $\varphi(0) = x$ and $\varphi(S)$ intersecting each orbit in a unique point, further the isotropy group of all the points $\varphi(S)$ is the same, $G_{\varphi(p)} = G_x$ for all $p \in S$. We call such a situation a simple G action and such an imbedding a special cross section. For a simple G action we can formulate a reasonable theorem without involved technical conditions on the orbit structure. One can best deal with the more complicated cases involving several orbit types individually.

Given an H bundle $\pi : E \to M$ with G action projecting to a simple G action on M, let φ be a special cross section and $\sigma : S \to E$ be a "section of E over φ" that is $\pi\sigma(s) = \varphi(s)$. Define $\lambda_s : G_0 \to H$ by

$$g\sigma(s) = \sigma(s)\lambda_s(g), \quad g \in G_{\varphi(s)} = G_0.$$

Lemma: If G_0 and H are compact the section σ can be chosen so that λ_s is independent of s, equal to its value at $s = 0$.

Proof: Let T_0 be a maximal torus in G_0 and let t be an element such that $\{t^n\}$ is dense in T_0. We shall show that there is a smooth function $h : S' \to H$ $h(0) = e$, with $S' \ni 0$ open in S, such that $h(s)\lambda_s(t)h(s)^{-1}$ is constant. Then the homomorphism λ'_s corresponding to the section $\sigma(s)h(s)^{-1}$ is constant in S on T_0.

First observe that if χ is any character on H then $\chi \circ \lambda_s$ is a trigonometric polynomial on T_0 whose coefficients are integer valued functions continuous in s, hence constant. Thus $\chi \circ \lambda_s(t)$ is constant and letting χ vary over all characters we see that all $\lambda_s(t)$ are in the same conjugacy class. Let Z be the centralizer of $\lambda_0(t)$. What we have shown is that S is mapped smoothly into one orbit of the conjugation action of H on itself. That orbit is diffeomorphic to H/Z. Hence we have a smooth map $S \to H/Z$ and composing with a local section of $H \to H/Z$ we have, after possibly restricting to an open subset $S' \subset S$, a function $h : S \to H$ such that

$$h(s)\lambda_s(t)h(s)^{-1} = \lambda_0(t).$$

Since G_0 and H are compact each is a product of a torus and a compact semisimple group $-$ $G = A \times F$ and $H = B \times K$ with A, B tori and F, K compact semisimple groups. We can assume that $A \subset T_0$ and $T_0 \cap F = T_1$ is a maximal torus in F and that for some maximal torus T in H, $B \subset T$ and $T \cap K = T_2$ is a maximal torus in K. The restricted homomorphism (that is, $\lambda_s : T_0 \to T$ composed on the right with the inclusion $T_1 \to T_0$ and on the left with projection $T \to T_2$) $\lambda_s : T_1 \to T_2$ is constant in S and using the results of Dynkin[10] this shows that all the subgroups $\lambda_s(F)$ are conjugate in K. More precisely, the condition that $\lambda_s : T_1 \to T_2$ is constant in s implies that all $\lambda_s(F)$ are equivalent in every representation of K. For all semisimple groups there are at most finitely many conjugacy classes of semisimple subgroups of K which are equivalent in every representation. The continuity in s of λ_s implies that the conjugacy classes cannot vary with s and therefore all the $\lambda_s(F)$ are conjugate. That the conjugacy can be carried out with smooth dependence on s follows from the existence of smooth sections of $K/N(\lambda_0(F))$ the coset space of K by the normalizer of the subgroup $\lambda_0(F)$.

Combining this lemma with Propositions 1 and 2 gives us the following two theorems.

Theorem 1: Let M be a manifold with simple G-action and compact isotropy groups. Let E be a principal H bundle with G-action projecting on the G-action on M. Assume H is compact. Let $\varphi : S \to M$ be a special cross-section through $x \in M$ and $U = G \cdot \varphi(S) \subset M$. Then there is an isomorphism

$$E|_U \cong E_\lambda \times S, \quad \text{for some } \lambda : G_x \to H.$$

This theorem together with Proposition 1 and its corollaries completely analyzes the structure of a bundle with G action over the neighborhood of an orbit in space with a simple G action.

Proof: Let $\varphi : S \to M$ be a special cross-section and σ a section of E over φ such that the homomorphism λ_s is independent of s. Then define a mapping

$$f : E_{\lambda_0} \times S \to E|_U,$$

$$f([g,h],s) = g\sigma(s)h.$$

It is immediate to check that this is a G equivariant isomorphism.

Theorem 2: The G invariant connections on $E_\lambda \times S$ are determined by

(i) A family of linear maps $A_s : \mathscr{G} \to \mathscr{h}$ depending smoothly on s and satisfying (18a) and (18b).

(ii) A one-form μ on S with values in the subalgebra of $\mathscr{h}$ of elements invariant under the adjoint action of $\lambda_0(G_0)$.

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$B_\psi\, d\psi$. For λ_1 the condition that B_ψ takes values in the Ad-invariants of the image of λ_1 implies $B_\psi \equiv 0$. The remaining conditions are (18a)

$$A_\psi(\mathrm{Ad}(g,g)^{-1}(\theta_1+\theta_2)) = \mathrm{Ad}g^{-1}A_\psi(\theta_1+\theta_2),$$

which implies

$$A_\psi(\theta_1+\theta_2) = a_\psi\theta_1 + b_\psi\theta_2$$

for a_ψ, b_ψ scalar function of ψ and the right-hand side interpreted as taking values in su(2). By condition 18b, for $\xi \in$su(2),

$$A(\theta_1+\theta_2)(\xi,-\xi) = \lambda_*(\xi,-\xi) = \xi,$$

thus $a_\psi - b_\psi = 1$. The connection pulled back to the base is

$$\widetilde{\omega} = a_\psi\theta_1.$$

Example γ: This is an example which, in view of the orbit structure in M, goes beyond the scope of Theorems 1 and 2. We therefore begin by considering only the dense submanifold M_1.

Using the special cross section defined for $0<s<\pi$ and $0\leqslant t<2\pi$

$$\varphi(s,t) = \left(\begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is}\end{pmatrix}, e^{it}\right), \quad \text{we find}$$

$$G_0 = \left\{\begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is}\end{pmatrix}\right\} \subset \mathrm{SU}(2).$$

For $p\in M_1$ and q in the fiber of E over p define $\lambda_q: G_0 \to H$ by

$$\tilde{\gamma}_g q = q\lambda_q(g),$$

where $\tilde{\gamma}$ is the action on E. The homomorphism λ_q is conjugate to some $\mu_n: G_0 \to H$, $n\in Z$, defined by

$$\mu_n\begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is}\end{pmatrix} = \begin{pmatrix} e^{ins} & 0 \\ 0 & e^{-ins}\end{pmatrix}$$

hence, by continuity, the integer n characterizing the homomorphism of the isotropy group into the gauge group is independent of p. There exists an extension of λ_q to $\bar{\lambda}_q: G\to H$ if and only if $n(p)=0$ or 1. Suppose the bundle E over M_1 extends to $\bar{E}$ over M and the G action on E extends to an action on $\bar{E}$ projecting to the γ action on M. We can find a section σ of $\bar{E}$ near $p(\psi) = (I, e^{i\psi})$ and since the isotropy group at $p(\psi)$ is SU(2) we have a homomorphism λ_ψ:SU(2)$\to$SU(2). Restricted to G_0

$$\lambda_\psi\begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is}\end{pmatrix} = \begin{pmatrix} e^{in(\psi)s} & 0 \\ 0 & e^{-in(\psi)s}\end{pmatrix}$$

where $n(\psi)\equiv 1$ or $n(\psi)\equiv 0$.

By continuity the first case implies $n(p)\equiv 1$ for $p\in M_1$ and the second case implies $n(p)\equiv 0$. In either case the homomorphism λ extends to $\bar{\lambda}_q: G\to H$ and we conclude that the transformation function can be chosen independent of the point in M_1. This implies that we can choose the transformation function to be either

$$\rho_0(g,y) \equiv I \in \mathrm{SU}(2),$$
$$\rho_1(g,y) = g^{-1} \in \mathrm{SU}(2).$$

The invariant connections corresponding to these transformation functions may be determined through Theorem 2 or by applying the theory of orthogonal invariants directly in the base space. The pull-backs ω_0 and ω_1 of the generic invariant connections corresponding to ρ_0 and ρ_1 respectively may be expressed as:

$$\omega_0 = M\,ds + N\,dt$$

and

$$\omega_1 = A dt + B\omega + C[U,\omega] + D(U,\omega)U,$$

where M, N are $\mathfrak{su}(2)$-algebra valued functions and A,B,C,D scalar functions depending on the invariants s and t only, U is an $\mathfrak{su}(2)$-valued function on M defined in the standard anti Hermitian representation[1] by

$$U(x, e^{i\psi}) = \tfrac{1}{2}(x - \tfrac{1}{2}\,\mathrm{Tr}x),$$

and ω is the Maurer–Cartan form in the first factor under the identification $M \sim \mathrm{SU}(2)\times\mathrm{U}(1)$.

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[1] J. Harnad, S. Shnider, and Luc Vinet, J. Math. Phys. **20**, 931 (1979); also in *Complex Manifold Techniques in Theoretical Physics*, edited by D. Lerner and P. Sommers (Pitman, New York 1979), pp. 219–30 and Refs. therein.

[2] P. Forgács and N. Manton, Comm. Math. Phys. **72**, 15 (1980).

[3] R. Jackiw and N. Manton, Ann. Phys.

[4] N. Manton, Nucl. Phys. B **158**, 141 (1979).

[5] M. Mayer, Univ. Cal. Irvine preprint.

[6] J. Harnad, S. Shnider, and J. Tafel, Lett. Math. Phys. **4**, 107 (1980).

[7] G. Bredon, *Introduction to Compact Transformaton Groups* (Academic, New York, 1972), Chap. 2.

[8] For 1-parameter groups, the infinitesimal transformation conditions on vector bundles are given by P. G. Bergmann and E. J. Flaherty, J. Math. Phys. **19**, 212 (1978).

[9] H. C. Wang, Nagoya Math. J. **13**, 1 (1958), cited in S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Wiley, New York, 1969), Vol. 1, p. 106.

[10] E. B. Dynkin, A.M.S. Translations, Series 2 vol. 6, p. 111–214.

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