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New Hamiltonian formulation of general relativity

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The phase space of general relativity is first extended in a standard manner to incorporate spinors. New coordinates are then introduced on this enlarged phase space to simplify the structure of constraint equations. Now, the basic variables, satisfying the canonical Poisson-brackets relations, are the (density-valued) soldering forms $\tilde{\sigma}^{a}{}_{A}{}^{B}$ and certain spin-connection one-forms $A_{aA}{}^{B}$. Constraints of Einstein's theory simply state that $\tilde{\sigma}^{a}$ satisfies the Gauss law constraint with respect to A_a and that the curvature tensor $F_{abA}{}^{B}$ and A_a satisfies certain purely algebraic conditions (involving $\tilde{\sigma}^{a}$). In particular, the constraints are at worst quadratic in the new variables $\tilde{\sigma}^{a}$ and A_a. This is in striking contrast with the situation with traditional variables, where constraints contain nonpolynomial functions of the three-metric. Simplification occurs because A_a has information about both the three-metric and its conjugate momentum. In the four-dimensional space-time picture, A_a turns out to be a potential for the self-dual part of Weyl curvature. An important feature of the new form of constraints is that it provides a natural embedding of the constraint surface of the Einstein phase space into that of Yang-Mills phase space. This embedding provides new tools to analyze a number of issues in both classical and quantum gravity. Some illustrative applications are discussed. Finally, the (Poisson-bracket) algebra of new constraints is computed. The framework sets the stage for another approach to canonical quantum gravity, discussed in forthcoming papers also by Jacobson, Lee, Renteln, and Smolin.

I. INTRODUCTION

The presence of gauge freedom in a physical theory reflects itself in its Hamiltonian formulation through the fact that not all points of the phase space are accessible to the system; there are constraints. The canonical transformations generated by the constraint functionals correspond precisely to gauge motions in the phase-space variables. In Yang-Mills theory, for example, these transformations cause rotations of the *internal* indices under which physical observables remain unchanged. In general relativity, on the other hand, the canonical transformations generated by constraints correspond to motions in *physical* space-time and are therefore intertwined also with dynamics. Consequently, constraints play a qualitatively different role; they are vastly more powerful than in other gauge theories. This difference is expected to be crucial in the quantum theory. In the Yang-Mills case, it is relatively straightforward to impose the quantum constraints on physically admissible wave functions; it is the action of the Hamiltonian that is nontrivial. One can argue that the situation will be opposite once we bring in gravity. Now, the crucial problem is the imposition of quantum constraints; at least formally, the action of the Hamiltonian is rather simple once the quantum constraints are satisfied. This feature is independent of the type of matter fields present. It arises simply because the theory has no fixed, kinematical background geometry.

It is therefore important to arrange matters such that the constraints of general relativity acquire a simple form. The purpose of this paper is to introduce new variables on the gravitational phase space which bring about such a simplification. Furthermore, the use of these variables enables one to embed the constraint surface in the phase space of Einstein's theory into that of Yang-Mills phase space. In addition to being a useful tool to relate the mathematical structures of the two theories, this embedding opens up new avenues in canonical quantum gravity.[1-3]

The traditional[4] choice of the basic, canonically conjugate variables for general relativity consists of a positive-definite three-metric q_{ab} and its conjugate momentum, a tensor density p^{ab}, of weight 1, both defined on a three-manifold Σ. The strength of this choice lies in the direct geometrical significance of these fields: In a four-dimensional solution of Einstein's equation on $\Sigma\times R$, q_{ab} can be identified with the pullback to Σ of the four-metric and p^{ab} is related to the extrinsic curvature k^{ab} of Σ through

$$p^{ab}=(\det q)^{1/2}G^{-1}(K^{ab}-K^{mn}q_{mn}q^{ab})\ , \tag{1}$$

where G is Einstein's constant. [$G=(8\pi c^{-3})$ (Newton's constant). In what follows we use $c=1$ units.] However, the expression of the constraint functionals is quite complicated in terms of these variables (see Sec. II A). Indeed, the constraints are *nonpolynomial* in their dependence on q_{ab}. This feature makes it difficult to perform technical manipulations, and, in particular, to unravel the structure of the reduced phase space. In the spatially compact case, for example, the reduced phase space

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acquires certain conical singularities,[5] whose structure, however, turns out to be significantly simpler than what one is first led to expect from the form of the constraints. This simplicity came as a surprise and suggested that there should exist other variables in terms of which constraints become more manageable and the simplicity of the conical singularities transparent.[6] Another issue obscured by the complicated dependence of constraints on q_{ab} and p^{ab} is the simplicity of (anti-)self-dual solutions of Einstein's equation. Over the past decade, a complete analysis of (anti-)self-dual solutions has become available, thanks to the techniques introduced by Newman, Penrose, and Plebanski.[7] Considering the complexity of the full Einstein equation, it is remarkable that an exhaustive treatment of such a large class of solutions should be possible. Moreover, it turns out that the classical S matrix is trivial in the (anti-)self-dual case: in spite of nonlinearities, the classical S matrix is the same as in the linearized limit.[8] This suggests that the (anti-)self-dual system may be exactly integrable. Normally, the Hamiltonian formulation is well suited to address these issues. However, because of the complexity of the Einstein constraints as functionals of q_{ab} and p^{ab}, it has not been possible to investigate these ideas. Indeed, the mathematical simplicity of (anti-)self-dual equations has been a complete mystery in the Hamiltonian framework. Finally, the nonpolynomial dependence on q_{ab} has made it impossible in quantum theory to construct the momentum representation; all work in canonical and path-integral quantization has, consequently, been restricted to the configuration representation.

It is therefore tempting to look for other canonically conjugate pairs on the gravitational phase space. The rich structure of (anti-)self-dual solutions suggests a strategy: look for variables which capture the (anti-)self-dual part of the four-dimensional curvature. We will follow this strategy.

It turns out that the variables best suited for our purpose are certain spin connections which turn out to be potentials for the (anti-)self-dual part of the Weyl curvature when constraints are satisfied. Therefore, we begin in Sec. III by enlarging the gravitational phase space to incorporate spinors. Such an enlargement is needed, in any case, if one wishes to introduce spinorial matter, and has appeared in the literature in the guise of triad (or tetrad) formulations. In Sec. IV we introduce the new variables: the spin connections. These have information about both the three-geometry and the extrinsic curvature. Consequently, the standard constraints of Einstein's theory can be reexpressed merely as algebraic conditions on the curvature of the spin connections. (When all the constraints are satisfied, these connections reduce to those introduced by Sen[9] in a somewhat different context.) The algebraic conditions involve only the curvature and the Infeld–Van der Waerden symbols which solder spinors to the tangent space at each point of Σ. In addition, the Infeld–Van der Waerden forms and the spin connections may be thought of as canonically conjugate variables on the extended phase space; each set forms a (complete) set of commuting (with respect to Poisson-brackets) variables and the two have c-number Poisson brackets between each other. Section V discusses the various simplifications that arise from the use of these new variables in place of q_{ab} and p^{ab}. In particular, the result on the embedding of the constraint surface of Einstein's theory into that of Yang-Mills is obtained. In Sec. VI we show that the constraints form a first-class system. In Sec. VII we present the expression of the Hamiltonians—i.e., generators of (asymptotic) space-time translations—in terms of the new variables. Section VIII summarizes the situation and points out some of the possible applications of the framework. Some mathematical preliminaries are collected in Sec. II and in the Appendix.

For simplicity, throughout the technical discussion, we have restricted ourselves to the source-free case. It is clear from the presentation, however, that the incorporation of matter sources and/or cosmological constant is relatively straightforward. Our space-time signature is $-+++$ and conventions on curvature tensors are $D_{[a}D_{b]}k_c =: \frac{1}{2}R_{abc}{}^{d}k_d$, $R_{ac} := R_{abc}{}^{b}$, and $R = R_a{}^a$.

II. PRELIMINARIES

This section is divided into two parts. In the first part we briefly recall[4] the standard Hamiltonian formulation of general relativity. This summary will serve to fix notation and conventions and provide the point of departure for the discussion of the extended phase space in Sec. III. In the second part, we recall certain differential geometric results obtained by Sen[9] using SU(2) spinors. These motivate the definitions of the new variables in Sec. IV.

A. Phase space of general relativity

Fix a three-manifold Σ in which topological complications, if any, are restricted to a compact set. More precisely, we assume that either Σ is compact or the complement of a compact set of Σ is diffeomorphic to the complement of a ball in R^3. For brevity, throughout this article we shall focus only on the technically more difficult noncompact case. Results in the compact case are easily obtained by ignoring our boundary conditions and setting our boundary integrals to zero. Also, although our assumption implies that Σ has at most one asymptotic region, it is straightforward to extend the framework to allow more such regions (as, for example, in the Kruskal extension of the Schwarzschild space-time).

The configuration space $\tilde{\mathcal{C}}$ is the space of all positive-definite metrics q_{ab} on Σ with an appropriate asymptotic behavior. A possible choice of boundary conditions is the following. Fix a positive-definite metric e_{ab} on Σ which is Euclidean, i.e., flat outside some compact set. Let r denote a radial coordinate with respect to e_{ab}. Then, we let $q_{ab} \in \tilde{\mathcal{C}}$ iff it has the form[10]

$$q_{ab} = \left[1 + \frac{M(\theta,\phi)}{r}\right]^4 e_{ab} + O\left[\frac{1}{r^2}\right] . \quad (2)$$

Fix a point q_{ab} in $\tilde{\mathcal{C}}$. A tangent vector at q_{ab} is represented by a second-rank, symmetric tensor field

$(\delta q)_{ab}$, on Σ, which has the same falloff as $(q_{ab}-e_{ab})$. A cotangent vector is therefore represented by a second-rank, symmetric tensor density p^{ab} of weight 1, with the falloff

$$p^{ab}q_{ab}=O\left[\frac{1}{r^3}\right] \text{ and } p^{ab}-\tfrac{1}{3}pq^{ab}=O\left[\frac{1}{r^2}\right] \quad (3)$$

so that the action of the covector p^{ab} on any tangent vector $(\delta q)_{ab}$,

$$p^{ab}{}_0\,(\delta q)_{ab}:=\int_\Sigma p^{ab}(\delta q)_{ab}\ ,$$

is well defined. Thus, the falloff conditions on q_{ab} determine those on p^{ab}. Had we required all pieces of $(q_{ab}-e_{ab})$ to fall off only as $1/r$, we would have obtained only those p^{ab} which fall off faster than $1/r^2$ so that the three-momentum would have vanished identically on the entire phase space. However, our choice (2) of the boundary conditions is not unique. It is only that it provides a simple way to construct a phase space which is neither too large nor too small to be physically interesting. A rigorous treatment, which is feasible but would require considerably more space, would involve appropriately weighted Sobolev spaces and a careful handling of functional analysis.

The phase space $\tilde{\Gamma}$ is the cotangent bundle over $\tilde{\mathcal{C}}$. Thus, a point of $\tilde{\Gamma}$ is a pair (q_{ab},p^{ab}) satisfying (2) and (3). $\tilde{\Gamma}$ has a natural symplectic structure $\tilde{\Omega}$ whose action, at a point (q,p) of $\tilde{\Gamma}$, on tangent vectors $(\delta q,\delta p)$ and $(\delta q',\delta p')$ at that point, is given by

$$\tilde{\Omega}\mid_{(q,p)}((\delta q,\delta p),(\delta q',\delta p'))$$
$$:=\int_\Sigma (\delta p)^{ab}(\delta q)'_{ab}-(\delta p)'^{\,ab}(\delta q)_{ab}\ . \quad (4)$$

Not all points of $\tilde{\Gamma}$ are accessible to the vacuum (Einstein) gravitational field: There are constraints. These are given by

$$C_a(q,p):=-2q_{am}D_np^{mn}=0 \quad (5)$$

and

$$C(q,p):=-(\det q)^{1/2}G^{-1}R$$
$$+G(\det q)^{-1/2}(p^{ab}p_{ab}-\tfrac{1}{2}p^2)=0\ , \quad (6)$$

where D and R are, respectively, the derivative operator and the scalar curvature of q_{ab}. Given constraints, it is natural to ask for the canonical transformations they generate. Note, however, that canonical transformations are generated by (real-valued) functions on the phase space. C_a and C, on the other hand, are mappings from $\tilde{\Gamma}$ to vector and scalar fields on Σ. Therefore, to obtain functions from them, we have to smear them with vector and scalar fields. Set

$$C_{\mathbf{N}}(q,p):=\int_\Sigma N^aC_a(q,p) \quad (7)$$

and

$$C_N(q,p):=\int_\Sigma NC(q,p)\ . \quad (8)$$

From boundary conditions imposed on q_{ab} and p^{ab}, it follows that, at a general point of $\tilde{\Gamma}$, these integrals will converge only if the smearing fields, N^a and N, tend to zero at infinity.[11] We shall assume that they fall off as $1/r$ and their nth (D) derivatives fall off as $1/r^{n+1}$. Then, $C_{\mathbf{N}}a$ and C_N are smooth on Γ. The canonical transformation generated by $C_{\mathbf{N}}a$ corresponds to the one-parameter family of diffeomorphisms generated by N^a on Σ and that generated by C_N corresponds, *on the constraint surface*, to "time evolution" via Einstein's equation with lapse N. These constraints constitute a first-class system. The Poisson brackets are given by

$$\{C_{\mathbf{N}},C_{\mathbf{M}}\}=-C_{\mathbf{K}} \quad \text{with } K^a=(\mathcal{L}_{\mathbf{N}}M)^a\ ,$$

$$\{C_{\mathbf{N}},C_M\}=-C_K \quad \text{with } K=(\mathcal{L}_{\mathbf{N}}M)\ , \quad (9)$$

$$\{C_N,C_M\}=-C_{\mathbf{L}} \quad \text{with } L^a=q^{ab}(N\partial_bM-M\partial_bN)\ .$$

Finally, let us consider the generators of asymptotic space-time translations, i.e., shift-lapse pairs (T^a,T), which correspond, respectively, to translational Killing fields T^a of e_{ab} and asymptotically constant functions T on Σ. The Hamiltonian $H_{\mathbf{T}}$ generating the asymptotic space translation T^a is given by

$$H_{\mathbf{T}}(q,p)=\int_\Sigma(\mathcal{L}_Tq_{ab})p^{ab}\ . \quad (10)$$

One often integrates by parts and rewrites H_{Ta} as

$$H_{\mathbf{T}}(q,p)=-2\int_\Sigma T_aD_bp^{ab}+2\oint T_ap^{ab}dS_b$$
$$=\int_\Sigma T^aC_a+2\oint T_ap^{ab}dS_b\ , \quad (11)$$

to bring out the fact that, on the constraint surface, the numerical value of this Hamiltonian is just a surface term, the Arnowitt-Deser-Misner[4] (ADM) three-momentum. Note, however, that, because of the convergence problems mentioned above, the passage from (10) to (11) is subtle: one first performs the integration in (10) over a finite volume, integrates by parts and *then* takes the limit. The integral, particularly the volume integral, in (11) is to be understood in this limiting sense; one *first* integrates and *then* takes the limit. In the same sense, the Hamiltonian H_T generating an asymptotic time translation, is given by

$$H_T(q,p) = \int_\Sigma TC + \frac{1}{G}\oint_{\partial\Sigma} T(\partial_a q_{bc} - \partial_b q_{ac})e^{ac} dS^b \ , \tag{12}$$

where ∂ is the derivative operator of the background metric e_{ab}. Again, when the constraints are satisfied, the numerical value of $H_T(q,p)$ is given just by the surface term, the ADM energy.[4] In presence of matter sources, the expressions of constraint functionals acquire extra terms involving matter variables. The form (11) and (12) of the Hamiltonians, however, remains the same in absence of derivative couplings.[12] In particular, on the new constraint surface, the numerical values of the Hamiltonians are again given by the surface integrals in (11) and (12).

B. The Sen connections

Fix, as in Sec. II A, a three-manifold Σ and consider SU(2) spinor fields $\lambda^A, \mu_A, \ldots$ on it. The spinorial indices are raised and lowered by the alternating spinors ϵ^{AB} and ϵ_{AB}:

$$\lambda_A = \lambda^B \epsilon_{BA} \quad \text{and } \mu^A = \epsilon^{AB}\mu_B \ .$$

The Infeld–Van der Waerden fields $\sigma_{aA}{}^B$ solder the spinor indices to the tangent space at each point,

$$\lambda_a \equiv -\sigma_{aA}{}^B \lambda_B{}^A \ ,$$

and define a positive-definite metric q_{ab}, a compatible alternating tensor ϵ_{abc}, and a torsion-free derivative operator D_a (on tensor and spinor fields) via

$$q_{ab} = -\sigma_{aA}{}^B \sigma_{bB}{}^A \equiv -\mathrm{Tr}\sigma_a\sigma_b \ ,$$

$$\epsilon_{abc} = -\sqrt{2}\sigma_{aA}{}^B\sigma_{bB}{}^D\sigma_{cD}{}^A$$
$$= -\sqrt{2}\,\mathrm{Tr}\sigma_a\sigma_b\sigma_c \ ,$$

and

$$D_a\sigma_{bA}{}^B = 0 \ .$$

[For details on SU(2) spinors and their relation to SL(2,C) ones, see Ref. 9 or the Appendix.]

We can now recall Sen's results.[9] Fix on Σ a second-rank, symmetric tensor field k_{ab} and introduce two connections $^{\pm}D$ via

$$^{\pm}D_a\lambda_{bM} := D_a\lambda_{bM} \pm \frac{i}{\sqrt{2}} k_{aM}{}^N\lambda_N \ , \tag{13}$$

where $k_{aM}{}^N = k_{ab}\sigma^b{}_M{}^N$. These will be referred to as *Sen connections*. Let us compute the curvature of these connections. Since the action of $^{\pm}D$ on tensors is the *same* as that of (the metric compatible connection) D, let us focus only on the spinorial curvature. We have

$$^{\pm}f_{abM}{}^N\lambda_N := 2\,^{\pm}D_{[a}{}^{\pm}D_{b]}\lambda_M$$
$$= R_{abM}{}^N - k_{[a\,|M|}{}^P k_{b]P}{}^N\lambda_N$$
$$\pm\sqrt{2}iD_{[a}k_{b]M}{}^N\lambda_N \ , \tag{14}$$

so that

$$^{\pm}f_{abc} := -\mathrm{Tr}\,^{\pm}f_{ab}\sigma_c$$
$$= R_{abc} - \frac{1}{\sqrt{2}}\epsilon_{cde}k_{[a}{}^d k_{b]}{}^e \pm \sqrt{2}iD_{[a}k_{b]c} \ . \tag{15}$$

Here, $R_{abc} \equiv -R_{abM}{}^N\sigma_{cN}{}^M$ is the spinorial curvature of the σ-compatible D. The spinorial and tensorial curvatures of D are related by an identity

$$R_{ab}{}^{cd} = -\sqrt{2}R_{abp}\epsilon^{pcd} \ . \tag{16}$$

Thus, the curvature of $^{\pm}D$ depends not only on the Riemann tensor of q_{ab} but also on k_{ab} and its derivatives. Furthermore, this dependence is of a particularly convenient form. We have

$$q^{bc}\,^{\pm}f_{abc} = \mp\frac{i}{\sqrt{2}}D^b(k_{ab} - kq_{ab}) \tag{17}$$

and

$$\epsilon^{abc}\,^{\pm}f_{abc} = \frac{1}{\sqrt{2}}(-R + k^{ab}k_{ab} - k^2) \ . \tag{18}$$

Now, if we were to think of k_{ab} as the extrinsic curvature of Σ, (17) and (18) are precisely the left-hand sides of the constraint equations that the pair (q_{ab}, k_{ab}) has to satisfy in order to qualify as Cauchy data for Einstein's vacuum equation. [Using the definition (1) of p^{ab}, vanishing of (17) and (18) can be seen to be identical with (5) and (6), respectively.] Thus, the constraints of general relativity are coded in the algebraic structure of the curvature of $^{\pm}D$: they simply require that $^{\pm}f_{abc}\epsilon^{ab}{}_d$ be symmetric and trace-free.

Let us choose a pair (q_{ab}, k_{ab}) such that the initial-value constraints (17) and (18) are satisfied. Then, upon evolution by the remaining Einstein equations, one obtains a four-dimensional metric $^4g_{ab}$ of signature $(-+++)$ which is Ricci flat. It turns out[13] that there is a simple relation between the Weyl curvature of g_{ab}, evaluated on Σ, and the curvature forms $^{\pm}f_{abc}$:

$$^{\pm}f_{abc}\epsilon^{ab}{}_d = \sqrt{2}(-E_{cd} \pm iB_{cd}) \ , \tag{19}$$

where E_{cd} and B_{cd} are, respectively, the electric and the magnetic parts, relative to Σ, of the Weyl curvature. ($E_{ab} \equiv {}^4C_{ambn}\xi^m\xi^n$ and $B_{ab} \equiv {}^{*4}C_{ambn}\xi^m\xi^n$, where $^4C_{ambn}$ is the Weyl tensor, $^{*4}C_{ambn}$, its dual and ξ^a, the unit normal to Σ with respect to $^4g_{ab}$.) Thus, when constraints are satisfied, $^{\pm}D$ can be interpreted as potentials for the (anti-)self-dual part of the curvature of the four-dimensional space-time. It is remarkable that one can feed information about extrinsic curvature into the connections $^{\pm}D$ just in the way needed to code the Einstein constraints in certain (algebraically isolated) parts of $^{\pm}F_{abc}$ *and* the (anti-)self-dual part of the four-dimensional Weyl curvature in the remaining parts.

III. THE EXTENDED PHASE SPACE

Let Σ be a three-manifold as in Sec. II A. We wish to introduce an extension of the phase space $\tilde{\Gamma}$ to incorporate spinor fields. However, since Σ is not equipped with an *a priori* metric, we must first spell our the sense in which our fields are to be spinorial.

Consider, in addition to the tensor fields $T^{a\cdots b}{}_{c\cdots d}$ on Σ, objects such as $\lambda^{A\cdots B}{}_{M\cdots N}{}^{a\cdots b}{}_{c\cdots d}$ with internal SU(2) indices $A\cdots B, M\cdots N$. [Objects such as $\lambda^{A\cdots B}{}_{M\cdots N}$ with only internal indices are to be thought of as SU(2) "Higgs scalars."] Formally, one can regard $\lambda^{A\cdots B}{}_{M\cdots N}{}^{a\cdots b}{}_{c\cdots d}$ either as *generalized tensors* in the sense of Ref. 14 or as cross sections of suitable vector bundles over Σ. The SU(2) character of internal indices refers to the following structure. First, the space of fields (restricted to any one point of Σ) with one internal index λ^A is a two-dimensional, complex vector space. Second, there exists a preferred nowhere vanishing skew field ϵ^{AB}. We denote its inverse by ϵ_{AB}—$\epsilon^{AB}\epsilon_{AM}=\delta_M{}^B$—and raise and lower internal indices with these ϵ:

$$\lambda^A \equiv \epsilon^{AB}\lambda_B \quad\text{and}\quad \mu_A=\mu^B\epsilon_{BA} . \tag{20}$$

Finally, there exists an isomorphism, called an Hermitian conjugation and denoted by a dagger, from the space of objects with internal indices onto itself such that

$$\begin{aligned}&(\lambda_A+c\mu_A)^\dagger=\lambda^\dagger{}_A+\bar{c}\mu^\dagger{}_A,\quad (\lambda^\dagger{}_A)^\dagger=-\lambda_A ,\\ &\epsilon^\dagger{}_{AB}=\epsilon_{AB},\quad (\lambda^A)^\dagger\lambda_A\geq 0\\ &\qquad\qquad\text{equality holding iff } \lambda_A=0 ,\\ &(\lambda_A\mu_B)^\dagger=\lambda^\dagger{}_A\mu^\dagger{}_B ,\end{aligned} \tag{21}$$

where c is any complex number and $\bar{c}$ its complex conjugate. The group of isomorphisms from the system of Higgs scalars to itself which preserves its structure as a tensor algebra, the alternating tensor, and the dagger operation is precisely the group of local SU(2) transformations.

Note that the internal indices are *not* to be thought of as spinor indices since we do not yet have a soldering form to tie them down to the tangent space of Σ. Nonetheless, as in gauge theories, we can introduce connections on these generalized tensors.[15,14] A connection D_m maps a generalized tensor field with a given index structure (e.g., $\lambda^{A\cdots B}{}_{M\cdots N}{}^{a\cdots b}{}_{c\cdots d}$) to another one which has an additional covariant index m (written $D_m{}^{A\cdots B}{}_{M\cdots N}{}^{a\cdots b}{}_{c\cdots d}$) such that the following properties hold: (i) Additivity $D_m(\lambda^{\cdots}\ldots+\mu^{\cdots}\ldots)=D_m\lambda^{\cdots}\ldots+D_m\mu^{\cdots}\ldots$; (ii) Leibniz rule $D_m(\lambda^{\cdots}\ldots\mu^{\cdots}\ldots)=(D_m\lambda^{\cdots}\ldots)\mu^{\cdots}\ldots+\lambda^{\cdots}\ldots D_m\mu^{\cdots}\ldots$; (iii) $v^mD_mf=\mathcal{L}_vf$ for all functions f on Σ; (iv) torsion-free property $D_{[a}D_{b]}f=0$; and (v) annihilation of ϵ, $D_m\epsilon_{AB}=0$. It is straightforward to analyze the structure on the space of these connections. First, we ask "how many" connections are there? One can show that any two connections D_m and D'_m are related by a pair of fields, $C_{ab}{}^c$ and $C_{aA}{}^B$, satisfying $C_{[ab]}{}^c=0$ and $C_{a[AB]}=0$:

$$(D'_a-D_a)\lambda_{bA}=C_{ab}{}^c\lambda_{cA}+C_{aA}{}^B\lambda_{bB} . \tag{22}$$

Thus, there are as many connections as there are fields $C_{ab}{}^c$ and $C_{aA}{}^B$ with above algebraic symmetries. Next, we compute the curvature. The "tensorial part," $R_{abm}{}^N$, and the "internal part," $F_{abM}{}^N$, of the curvature are given by

$$2D_{[a}D_{b]}\lambda_{mM}=R_{abm}{}^n\lambda_{nM}+F_{abM}{}^N\lambda_{mN} . \tag{23}$$

They satisfy the identities

$$\begin{aligned}&R_{[abm]}{}^n=0,\quad D_{[a}R_{bd]m}{}^n=0 ,\\ &F_{ab[MN]}=0,\quad D_{[a}F_{bc]M}{}^N=0 .\end{aligned} \tag{24}$$

The fact that all of this structure can be introduced prior to a soldering form or a metric will be significant later in this section as well as in the next section.

We now introduce soldering forms which tie the abstractly defined internal indices to the tangent space of Σ, thereby making them spinor indices. Consider isomorphisms $\sigma_{aA}{}^B$ from the tangent vectors λ^a to Σ to the trace-free, second-rank, Hermitian spinors $\lambda_A{}^B$: $\lambda_A{}^B=\sigma_{aA}{}^B\lambda^a$. Denote the inverse mapping by $\sigma^a{}_A{}^B$. Properties of ϵ, Hermitian conjugation, and σ imply that

$$q_{ab}:=-\sigma_{aA}{}^B\sigma_{bB}{}^A\equiv-\mathrm{Tr}\sigma_a\sigma_b \tag{25}$$

is a positive-definite three-metric on Σ. Thus, given a specific σ, we are back to the standard spinorial scenario, discussed, e.g., in the Appendix. We now wish to regard σ as a basic dynamical variable and the structure outlined prior to the introduction of σ as the kinematical arena. In particular, the forms ϵ_{AB} and ϵ^{AB} (and the Hermitian conjugation operation) are to be thought of as c-number entities, fixed once and for all, independently of the choice of the dynamical variable. The metric q_{ab} is to be thought of as a secondary object, derived from the primary dynamical variable $\sigma_{AB}{}^B$.

We are now ready to define the new, extended phase space Γ. Fix, outside some compact region in Σ, a soldering form ${}^0\sigma^a{}_A{}^B$ (and its inverse ${}^0\sigma_{aA}{}^B$) whose connection D is flat. Thus, ${}^0\sigma$ is a soldering form of an Euclidean metric e_{ab}. Denote by $\mathcal{C}$ the space of all soldering forms $\sigma^a{}_A{}^B$ such that

$$\sigma^a{}_A{}^B=\left[1+\frac{M(\theta,\phi)}{r}\right]^2{}^0\sigma^a{}_A{}^B+O\left[\frac{1}{r^2}\right] . \tag{26}$$

Then $\mathcal{C}$ is the new configuration space. Given any σ in $\mathcal{C}$, we obtain a q in $\tilde{\mathcal{C}}$ via (25). Thus, there is a natural projection ψ from the new configuration space $\mathcal{C}$ to the traditional one $\tilde{\mathcal{C}}$: $\psi(\sigma^a{}_A{}^B)=q_{ab}$, where q_{ab} is the inverse of $q^{ab}\equiv-\mathrm{Tr}\sigma^a\sigma^b$. (The reason for choosing $\sigma^a{}_A{}^B$, rather than $\sigma_{aA}{}^B$, as the configuration variable will become clear in Sec. IV.) Let σ_1 and σ_2 project down to the same metric: q_{ab}. Then, it follows from (25) that σ_1 and σ_2 are related by a local SU(2) transformation. Thus, the enlargement of the configuration space from $\tilde{\mathcal{C}}$ to $\mathcal{C}$ has been brought about because of the freedom to perform internal SU(2) rotations. Indeed, while q_{ab} has six real components per space point, $\sigma^a{}_A{}^B$ has nine; the new three degrees of freedom correspond to precisely the three SU(2) rotations.

The momentum conjugate to $\sigma^a{}_A{}^B$ is a density of weight 1, $M_{aA}{}^B$, whose index structure is opposite of that of $\sigma^a{}_A{}^B$ and whose falloff is given by

$$\mathrm{Tr}M_a\sigma^a = O\left[\frac{1}{r^3}\right],$$
$$[M_{aA}{}^B + \tfrac{1}{3}(\mathrm{Tr}M_m\sigma^m)\sigma_{aA}{}^B] = O\left[\frac{1}{r^2}\right]. \tag{27}$$

The action of the (cotangent vector) $M_{aA}{}^B$ on any tangent vector $(\delta\sigma)^a{}_A{}^B$ at a point $\sigma^a{}_A{}^B$ of $\mathcal{C}$ is given by

$$M0\delta\sigma := \int_\Sigma -\mathrm{Tr}M_a\sigma^a . \tag{28}$$

Again, the falloff (27) of $M_{aA}{}^B$ is precisely such that the integral on the right-hand side of (28) converges.

The extended phase space Γ is the cotangent bundle over $\mathcal{C}$. Thus, a point of Γ is a pair $(\sigma^a{}_A{}^B, M_{aA}{}^B)$. The natural symplectic structure Ω on Γ is given by

$$\Omega\mid_{(\sigma,M)}((\delta\sigma,\delta M),(\delta\sigma',\delta M'))$$
$$:= \int_\Sigma \mathrm{Tr}[(\delta M_a')(\delta\sigma^a) - (\delta M_a)(\delta\sigma'^a)] , \tag{29a}$$

where $(\delta\sigma,\delta M)$ and $(\delta\sigma',\delta M')$ are any two tangent vectors at the point (σ,M) of Γ. Consequently, the Hamiltonian vector field X_f generated by an observable f is

$$X_f = \int_\Sigma \mathrm{Tr}\left[\frac{\partial f}{\delta\sigma^a}\frac{\delta}{\delta M_a} - \frac{\delta f}{\delta M_a}\frac{\delta}{\delta\sigma^a}\right], \tag{29b}$$

and the Poisson brackets between any two observables f and g are

$$(f,g) = \int \mathrm{Tr}\left[\frac{\delta f}{\delta\sigma^a}\frac{\delta g}{\delta M_a} - \frac{\delta f}{\delta M_a}\frac{\delta g}{\delta\sigma^a}\right]. \tag{29c}$$

Next, let us examine the constraints. In the transition from $\tilde{\mathcal{C}}$ to $\mathcal{C}$, we have added three degrees of freedom to the configuration variables. Since the physical degrees have not changed (we are still dealing with the vacuum Einstein equation) we have three new constraints. From a Lagrangian viewpoint, these arise because the Lagrangian is insensitive to SU(2) rotations on internal indices. More precisely, because the Lagrangian does not depend on the time derivatives of the three variables in $\sigma^a{}_A{}^B$ that undergo change under internal, SU(2) rotations, the corresponding momenta vanish. From a Hamiltonian viewpoint, the SU(2) rotations are gauge motions, whence their generating functionals should vanish. The three new constraints are

$$C_{ab} \equiv -\mathrm{Tr}M_{[a}\sigma_{b]} \equiv M_{[ab]} = 0 \tag{30a}$$

or

$$C^{AB} \equiv \sigma^a{}_M{}^{(A}M_{aN}{}^{B)}\epsilon^{MN} = 0 . \tag{30b}$$

Let us compute the corresponding canonical transformation. Given any trace-free, Hermitian spinor field $\Lambda_A{}^B$ we can define a constraint functional on Γ:

$$C_\Lambda(\sigma,M) \equiv \int_\Sigma -\Lambda_A{}^B C_B{}^A \tag{31}$$

which is differentiable on Γ only if $\Lambda_A{}^B$ tends to zero at infinity (faster than $1/r$). In this case, the Hamiltonian vector field is given by

$$X_\Lambda = \int_\Sigma -\mathrm{Tr}\tfrac{1}{2}\left[[\Lambda,\sigma^a]\frac{\delta}{\delta\sigma^a} + [\Lambda, M_a]\frac{\delta}{\delta M_a}\right]. \tag{32}$$

Thus, the infinitesimal changes in $\sigma^a{}_A{}^B$ and $M_{aA}{}^B$ caused by the canonical transformation are precisely the (infinitesimal) rotations of internal indices by $\Lambda_A{}^B$. Algebraic symmetries of $\Lambda_A{}^B$ imply that it is a generator of SU(2) transformations (see Appendix). Hence, the canonical transformations generated by the new constraints, Eqs. (30), generate small, i.e., tending to zero at infinity, SU(2) gauge transformations on the basic dynamical variables. Set

$$M^{(ab)} \equiv p^{ab} \tag{33}$$

so that, when (3) is satisfied, $p^{ab} = M^{ab}$. Then, the remaining constraints are the standard ones: Eqs. (5) and (6), where q_{ab} and p^{ab} are now regarded as functions of σ^a and M_a. Thus, we now have $3+3+1=7$ constraints. The configuration variable $\sigma^a{}_A{}^B$ has nine components per space point. Thus, we have two degrees of freedom per space point. The canonical transformations generated by (5) and (6) continue to retain their interpretation.

To summarize, the extended phase space Γ consists of pairs $(\sigma^a{}_A{}^B, M_{aA}{}^B)$ satisfying the boundary conditions (26) and (27). The Poisson brackets are given by (29). There are seven constraints. Six of them, (30) and (5), are linear in momentum and the seventh, (6), is quadratic. The Hamiltonians generating (asymptotic) space and time translations are given by (11) and (12).

We note the following.

(1) In terms of $\sigma^a{}_A{}^B$ and $M_{aA}{}^B$, and q_{ab} and p^{ab} given by (25) and (33) are secondary or derived quantities. One can therefore compute their Poisson brackets using (29). One obtains

$$\{q_{ab}(x), q_{cd}(y)\} = 0 ,$$
$$\{p^{ab}(x), q_{cd}(y)\} = 2\delta^a{}_{(c}\delta^b{}_{d)}\delta(x,y) ,$$
$$\{p^{ab}(x), p^{cd}(y)\} = \tfrac{1}{2}\delta(x,y)(M^{[ca]}q^{bd} + M^{[da]}q^{bc} + M^{[cb]}q^{ad} + M^{[db]}q^{ac}) .$$

Thus, modulo the new constraints (30), q_{ab} and p^{ab} have the same Poisson brackets as in Sec. II. That is, the enlargement of the phase space $\tilde{\Gamma}$ is compatible with the symplectic structure $\tilde{\Omega}$.

(2) In the above enlargement, we first extended the configuration space from $\mathcal{C}$ to $\mathcal{C}'$ by introducing internal indices (which, ultimately, play the role of spinorial indices) and then removed the gauge freedom corresponding to internal rotations by imposing new constraints (30). The passage from q_{ab} to σ^a is essential if one wishes to introduce spinorial matter and also for the spinorial variables for pure gravity, to be introduced in the next section. However, could we not have avoided the introduction and subsequent elimination of the freedom to perform internal SU(2) rotations by a "gauge-fixing procedure" which associates with each q_{ab} a canonical σ^a? Recall that there is a natural projection mapping ψ from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ which maps each σ^a to a q_{ab}. Thus, $\mathcal{C}$ may

be regarded as a fiber bundle over $\tilde{\mathcal{C}}$, each fiber representing the group of local SU(2) rotations. The natural question now is: does $\mathcal{C}$ admit natural horizontal cross sections? If it does, we could have used a horizontal cross section for the new configuration space. This space would be isomorphic with $\tilde{\mathcal{C}}$ so that there would be no additional gauge. At the same time, being a subspace of $\mathcal{C}$, it would provide soldering forms—and not just metrics—to enable the introduction of spinors. To analyze this issue, let us first consider tangent vectors $(\delta\sigma)^a$ at any one fixed point σ^a of $\mathcal{C}$. Using the "background" σ^a, we can convert $(\delta\sigma)^a$ into a tensor field $(\delta\sigma)_{mn}=\mathrm{Tr}(\delta\sigma)^a\sigma_n q_{am}$. The symmetric part $(\delta\sigma)_{(mn)}$ of this field gives the variation in the metric q_{ab} caused by $(\delta\sigma)$: Eq. (25) yields $\delta q_{ab}=2(\delta\sigma)_{(ab)}$. Thus, if $(\delta\sigma)_{mn}=(\delta\sigma)_{[mn]}$, the tangent vector $(\delta\sigma)^a$ generates a pure, internal gauge rotation which leaves the metric q_{ab} unaffected. On the other hand, if $(\delta\sigma)_{mn}=(\delta\sigma)_{(mn)}$, $(\delta\sigma)^a$ has "no internal gauge part." Hence, given any $(\delta\sigma)^a$, we can divide it into a "vertical part" $(\delta\sigma)_{[ab]}$ and a "horizontal part" $(\delta\sigma)_{(ab)}$. Thus, at each point of $\mathcal{C}$, there *are* naturally defined horizontal subspaces of the tangent space.

Unfortunately, however, as a simple calculation shows, these subspaces are *not* integrable.[16] Hence, there are no natural horizontal cross sections of $\mathcal{C}$ which could have served as internal-gauge-free but spinorial configuration spaces.

IV. THE NEW VARIABLES

The constraint equations (5) and (6) have remained intact in the transition from $\tilde{\Gamma}$ to Γ; the addition of new degrees of freedom does not, by itself, simplify the constraints. This is reminiscent of the situation in triad and tetrad frameworks. The key step in the simplification will be the introduction of certain variables on Γ. The extension to Γ is necessary, however, because these variables cannot be defined on $\tilde{\Gamma}$.

Fix a point $(\sigma^a{}_A{}^B, M_{aA}{}^B)$ of Γ. Then, we can introduce two connections, ${}^{\pm}\mathcal{D}$, which act on tensor and spinor fields on (Σ,σ):

$$ {}^{\pm}\mathcal{D}_a\lambda_{bM}=D_a\lambda_{bM}\pm\frac{i}{\sqrt{2}}\Pi_{aM}{}^N\lambda_N\ , \qquad (13') $$

where D_a is the connection which annihilates the given $\sigma^a{}_A{}^B$ and where $\Pi_{aM}{}^N$ is given by

$$ \Pi_{aM}{}^N\equiv G(\det q)^{-1/2}[M_{aM}{}^N+\tfrac{1}{2}(\mathrm{Tr}M_b\sigma^b)\sigma_{aM}{}^N] $$

or $\qquad (1')$

$$ M_{aM}{}^N=G^{-1}(\det q)^{1/2}[\Pi_{aM}{}^N+(\mathrm{Tr}\Pi_b\sigma^b)\sigma_{aM}{}^N] $$

Thus, $\Pi_{aA}{}^B$ is related to $M_{aA}{}^B$ in the same way that the extrinsic curvature K^{ab} is related to p^{ab} [Eq. (1)]. Note that $\Pi_{ab}\equiv-\mathrm{Tr}\Pi_a\sigma_b$ is *not* necessarily symmetric in a and b, whence the connections (13′) are *not* the same as the Sen connections (13), except when the constraint (30), $M_{[ab]}=0$, is satisfied. Why do we not simply use the symmetric part of Π_{ab} in (13′)? While this strategy seems at first attractive, it ruins certain crucial Poisson-brackets relations [see Eq. (35) below]. Consequently, for the passage to quantum theory to be manageable, it is important that (13′) be used as it is, without symmetrization.

As in gauge theories, it is convenient to work with connection one-forms $A_{aA}{}^B$ in place of derivative operators. Let us therefore fix a fiducial connection ∂_a. For simplicity, we shall assume that ∂_a commutes with Hermitian conjugation, $\partial_a\lambda_B^\dagger=(\partial_a\lambda_B)^\dagger$, and has zero internal curvature, $\partial_{[a}\partial_{b]}\lambda_A=0$. Set

$$ {}^{\pm}\mathcal{D}_a\lambda_M=\partial_a\lambda_M+G\,{}^{\pm}A_{aM}{}^N\lambda_N\ , \qquad (34) $$

so that (13′) yields

$$ G\,{}^{\pm}A_{aM}{}^N=\Gamma_{aM}{}^N\pm\frac{i}{\sqrt{2}}\Pi_{aM}{}^N\ , \qquad (34') $$

where $\Gamma_{aM}{}^N$ are the spin connection one-forms of D; $(D_a-\partial_a)\lambda_M\equiv\Gamma_{aM}{}^N\lambda_N$. Thus ${}^{\pm}A$ contain information about both σ and M. We shall use either ${}^{+}A$ or ${}^{-}A$ as one of our new variables.

Before going on to investigate properties of $A^{\pm}$, it is useful to point out an analogy which provides an intuitive feeling for these variables. Consider the phase space of a harmonic oscillator labeled by pairs (q,p) of real numbers. It is often convenient to introduce a pair of complex-conjugate coordinates, $z=(m\omega)^{1/2}q+i(m\omega)^{-1/2}p$ and $\bar{z}=(m\omega)q-i(m\omega)^{-1/2}p$, where m is the mass and ω the frequency of the oscillator. The variables ${}^{+}A$ and ${}^{-}A$ on the gravitational phase space are analogous to z and $\bar{z}$. [In the case of the oscillator, parameters m and ω allow us to form a dimensionally meaningful combination of the basic variables q and p. In the gravitational case, the only available constant G does not enable one to form a dimensionally meaningful linear combination of σ and M. However, since Π is analogous to the extrinsic curvature, it has the same dimension as the derivative of σ. Hence, it *is* possible to add Π and the spinorial Christoffel symbols Γ as in the definition (34′) of ${}^{\pm}A$. In this sense, ${}^{\pm}A_{aM}{}^N$ on the gravitational phase space are as close to the variables z and $\bar{z}$ as it is dimensionally possible.] We shall return to this analogy at the end of this section.

Let us now compute the Poisson brackets between these connection one-forms.

First, we shall show that ${}^{+}A$ (or ${}^{-}A$) constitute a set of commuting variables. To see this, fix any C^∞ tensor densities of weight 1, $f^a{}_{AB}$ and $f'^a{}_{AB}$, with compact support on Σ and consider the functionals

$$ F(\sigma,M)\equiv\int_\Sigma f^a{}_{MN}({}^{+}A_a{}^{MN}) $$

and

$$ F'(\sigma,M)\equiv\int_\Sigma f'^a{}_{MN}({}^{+}A_a{}^{MN}) $$

on Γ. The Poisson brackets between these functionals is given by

$$ \{F,F'\}=\int_\Sigma\left[\frac{\delta F}{\delta M_a{}^{AB}}\frac{\partial F'}{\delta\sigma^a{}_{AB}}-\frac{\delta F}{\delta\sigma^a{}_{AB}}\frac{\delta F'}{\delta M_a{}^{AB}}\right]. \qquad (35) $$

The definition of ${}^{+}A_a{}^{AB}$ yields

$$\frac{\delta}{\delta M_m{}^{MN}{}_{(y)}}[{}^+A_a{}^{AB}(x)]=\frac{i}{\sqrt{2}}G(\det q)^{-1/2}\delta(x,y)(\delta_a{}^m\delta_M{}^{(A}\delta_N{}^{B)}-\tfrac{1}{2}\sigma_a{}^{AB}\sigma^m{}_{MN})\ , \tag{36a}$$

and for any $t^m{}_{MN}$,

$$\int_\Sigma\left[t^m{}_{MN}{}^0\frac{\delta}{\delta\sigma^m{}_{MN}}\right]{}^+A_a{}^{AB}=\frac{1}{2\sqrt{2}}\sigma^{bAB}[\epsilon_b{}^{mn}D_m\tau_{na}+\tfrac{1}{2}D_at_b+2i\tau\Pi_{ab}-i(\Pi_{cd}-\Pi q_{cd})t^{cd}q_{ab}+2i\Pi(t_{ab}-2\tau_{ab})] \tag{36b}$$

with $t^{ab}=t^a{}_{AB}\sigma^{bAB}$, $\tau_{ab}=t_{(ab)}$, and $t^c=\epsilon^{abc}t_{ab}$. Using (36a) to compute $\delta F/\delta M_m{}^{MN}$ and substituting the resulting expression for $t^m{}_{MN}$ in (36b), one obtains the first term in the Poisson brackets (35). The second term is readily obtained by interchanging $f^a{}_{MN}$ and $f'^a{}_{MN}$. A regrouping of terms now shows that the Poisson brackets vanish. Since $f^a{}_{MN}$ and $f'^a{}_{MN}$ are arbitrary, we have the result

$$\{{}^+A_{aA}{}^B(x),{}^+A_{mM}{}^N(y)\}=0\ . \tag{37a}$$

The calculation for ${}^-A$ is identical. Note that, had we used the symmetric part of Π_{ab} in the definition (13′) of ${}^\pm\mathcal{D}$ [as in Sen connections (13)], the relation (35) would *not* have held.

The Poisson brackets between ${}^+A_{aA}{}^B(x)$ and ${}^-A_{mM}{}^N(y)$ are straightforward to compute. However, the final expression is quite complicated. We shall not need this expression. We note only that the Poisson bracket is *not* a c number; it is a nonconstant function on Γ. It turns out, however, that the Poisson brackets between $\sigma^a{}_A{}^B$ and $A_{aA}{}^B$ are simple. Set

$$\tilde\sigma^a{}_A{}^B=(\det q)^{1/2}\sigma^a{}_A{}^B\ . \tag{38}$$

We note first that

$$\{\tilde\sigma^a{}_A{}^B(x),\tilde\sigma^m{}_M{}^N(x)\}=0\ . \tag{37b}$$

Next, using the fact that $\Pi_a{}^{AB}$ and $\tilde\sigma^m{}_{MN}$ are canonically conjugate, i.e., satisfy

$$\{\Pi_a{}^{AB}(x),\tilde\sigma^m{}_{MN}(y)\}=G\delta_a{}^m\delta_M{}^{(A}\delta_N{}^{B)}\delta(x,y)\ , \tag{39}$$

it immediately follows from (34) that

$$\{{}^\pm A_a{}^{AB}(x),\tilde\sigma^m{}_{MN}(y)\}=\pm\frac{i}{\sqrt{2}}\delta_a{}^m\delta_M{}^{(A}\delta_N{}^{B)}\delta(x,y)\ . \tag{37c}$$

Thus, $\tilde\sigma^a$ may be thought of as being "canonically conjugate" to ${}^\pm A_a$. (This is, however, a slight misuse of terminology since $\tilde\sigma^a$ is Hermitian while ${}^\pm A_a$ are not. See the following.) The fact that the Poisson brackets (39) turned out to be a c number is fortunate because, as we shall see in the next section, the constraints are most easily expressed in terms of ${}^+A_a$ (or ${}^-A_a$) and $\tilde\sigma^a$. Our basic variables will be therefore *either* $(\tilde\sigma^a{}_A{}^B,{}^+A_{aA}{}^B)$ *or* $(\tilde\sigma^a{}_A{}^B,{}^-A_{aA}{}^B)$. In what follows, we shall keep both signs; $\pm$ *will always stand for plus or minus.*

We make the following remarks.

(1) The variables ${}^+A_{aA}{}^B$ and ${}^-A_{aA}{}^B$ defined in (34) provide us a complex chart on Γ; analogous to the chart z^α, $\bar z^\alpha$ $(\alpha=1,\ldots,N)$ on the phase space of a N-dimensional oscillator. That is, $({}^+A_{aA}{}^B,{}^-A_{aA}{}^B)$ are completely determined by a pair $(\sigma^a{}_A{}^B,M_{aA}{}^B)$ and suffice to determine the pair from which they are constructed. The first part of this assertion follows by inspection of Eq. (34′). To prove the second, we first note that since ∂_a is fixed (i.e., is a c number), the sum and the difference of ${}^+A_{aA}{}^B$ and ${}^-A_{aA}{}^B$ determines the connection D and the field $\Pi_{aA}{}^B$, respectively. By construction of ${}^\pm A_{aA}{}^B$, we know that D is comparable with some soldering form. We need to show, however, that the soldering form is unique. For this, we use the boundary conditions. Let ${}^1\sigma={}^2\sigma$ be two elements of $\mathcal{C}$ which are both compatible with the connection. Then $D_a({}^1\sigma^b{}_M{}^N-{}^2\sigma^b{}_M{}^N)=0$. Since by the boundary condition (26) the difference must go to zero at infinity, we have ${}^1\sigma={}^2\sigma$. Now, using $\sigma^a{}_M{}^N$ and $\Pi_{aM}{}^N$, we can recover $M_{aM}{}^N$ using (1′). Thus, the pair (σ,M) which determines ${}^+A$ and ${}^-A$ can be recovered from ${}^+A$ and ${}^-A$.

(2) Using the above discussion and the Poisson-brackets relations (35) and (39) one can show that $\{{}^+A_{aA}{}^B\}$ and $\{{}^-A_{aA}{}^B\}$ each forms a complete set of commuting (with respect to the Poisson-brackets) variables. However, the analogy with z^α and $\bar z^\alpha$ does not go further. Whereas z^α and $\bar z^\alpha$ are canonically conjugate, ${}^+A_{aM}{}^N$ and ${}^-A_{aM}{}^N$ are not. Furthermore, even if they were canonically conjugate, we could not have worked just with $({}^+A_{aM}{}^N,{}^-A_{aM}{}^N)$ as our basic variables in an effective way since constraints cannot be simply expressed without recourse to $\tilde\sigma^a{}_A{}^B$. Our choice of $(\tilde\sigma^a{}_A{}^B,{}^\pm A_{aA}{}^B)$ is analogous to the choice (q^α,z^α) or $(q^\alpha,\bar z^\alpha)$ as basic variables. This choice is somewhat unconventional because one of the variables is real and the other complex. However, since any function on Γ can be unambiguously expressed as a function of $\tilde\sigma$ and ${}^\pm A$, the strategy is perfectly viable. In particular, the Poisson brackets between any two functions can be computed using only the basic Poisson brackets (35), (37), and (39), and, modulo standard factor ordering ambiguities, passage to quantum theory can be carried out by first promoting $\tilde\sigma$ and ${}^\pm A$ to operator valued distributions whose commutators are $\hbar/i$ times their Poisson brackets and then expressing other observables in terms of them.

(3) Note that the variables (σ^a,M_a) which determine ${}^\pm A_a$ can be recovered from $(\tilde\sigma^a,{}^\pm A_a)$ by purely algebraic manipulations; boundary conditions are *not* involved.

[See Eq. (34').] The use of $(\tilde\sigma^a, {}^\pm A_a)$ has another technical advantage over the use of $({}^+A_a, {}^-A_a)$. A pair $({}^+A_a, {}^-A_a)$ arises from some (σ^a, M_a) if and only if $\Gamma_a \equiv (G/2)({}^+A_a + {}^-A_a)$ is the spin connection of some σ^a, which is a *nonlocal* condition on ${}^\pm A_a$. On the other hand, $(\tilde\sigma^a, {}^\pm A_a)$ arises from some (σ^a, M_a) if and only if $(G{}^\pm A_a - \Gamma_a)$ is anti-Hermitian (where Γ_a is constructed locally from $\tilde\sigma^a$) which is a *local* condition on $(\tilde\sigma^a, {}^\pm A_a)$.

V. CONSTRAINTS IN THE YANG-MILLS FORM

In Sec. III, the constraints of Einstein's theory, Eqs. (30), (5), and (6) were expressed in terms of σ^a and M_a. Our purpose now is to reexpress them in terms of the new variables, $\tilde\sigma^a$ and ${}^\pm A_a$.

Let us begin with (30). First, we shall carry out a preliminary calculation to expand out ${}^\pm\mathcal{D}_a\tilde\sigma^a{}_{AB}$ in terms of $\sigma^a{}_{AB}$ and $M_a{}^{AB}$:

$$\begin{aligned}{}^\pm\mathcal{D}_a\tilde\sigma^a{}_{AB} &= D_a\tilde\sigma^a{}_{AB} \pm \frac{i}{\sqrt2}\Pi_{aA}{}^M\tilde\sigma^a{}_{MB} \pm \frac{i}{\sqrt2}\Pi_{aB}{}^M\tilde\sigma^a{}_{AM} \\ &= \pm\frac{i}{\sqrt2}(\det q)^{1/2}(\Pi_{am}\sigma^m{}_A{}^M\sigma^a{}_{MB} + \Pi_{am}\sigma^m{}_B{}^M\sigma^a{}_{AM}) \\ &= \pm\sqrt2 i(\det q)^{1/2}\Pi_{[am]}\sigma^m{}_A{}^M\sigma^a{}_{MB} \\ &= \pm\sqrt2 iGM_{[ab]}\sigma^a{}_A{}^M\sigma^b{}_{MB} \ . \end{aligned} \tag{40}$$

Hence (30) is completely equivalent to

$${}^\pm\mathcal{D}_a\tilde\sigma^a{}_A{}^B = 0 \ . \tag{30'}$$

It is useful to know that, since the divergence of a tensor density of weight 1 is independent of the choice of the derivative operator, one can expand out (30') knowing only the action

$${}^\pm\mathcal{D}_a\lambda_M = \partial_a\lambda_M + G{}^\pm A_{aM}{}^N\lambda_N \tag{34}$$

of ${}^\pm\mathcal{D}$ on internal indices. We have

$$\begin{aligned}{}^\pm\mathcal{D}_a\tilde\sigma^a{}_A{}^B &\equiv \partial_a\tilde\sigma^a{}_A{}^B + G{}^\pm A_{aA}{}^M\tilde\sigma^a{}_M{}^B - G{}^\pm A_{aM}{}^B\tilde\sigma^a{}_A{}^M \\ &\equiv \partial_a\tilde\sigma^a{}_A{}^B + G[{}^\pm A_a, \tilde\sigma^a]_A{}^B = 0 \ . \end{aligned} \tag{30''}$$

Thus, Eq. (30) has to be reexpressed in terms of the new variables.

Next, we consider (5) and (6). Here, the calculation is completely analogous to that with the Sen connections (Sec. II B). We begin by expressing the spinorial curvature of ${}^\pm\mathcal{D}$ in terms of $\sigma^a{}_A{}^B$ and $M_{aA}{}^B$. The spinorial curvature of ${}^\pm\mathcal{D}$ is defined via

$$G{}^\pm F_{abM}{}^N\lambda_N = 2{}^\pm\mathcal{D}_{[a}{}^\pm\mathcal{D}_{b]}\lambda_M \ , \tag{41}$$

so that

$${}^\pm F_{abM}{}^N = 2\partial_{[a}{}^\pm A_{b]M}{}^N + G[{}^\pm A_a, {}^\pm A_b]_M{}^N \ . \tag{41'}$$

(The factors of G ensure dimensional consistency.) Now, using the expression (13') of ${}^\pm\mathcal{D}$ in terms of σ and M in (41), one obtains

$$G{}^\pm F_{abc} = R_{abc} - \frac{1}{\sqrt2}\epsilon_{cde}\Pi_a{}^d\Pi_b{}^e \pm \sqrt2 iD_{[a}\Pi_{b]c} \ , \tag{15'}$$

where, as before, R_{abc} is the spinorial curvature of D. Hence, it follows that

$$\begin{aligned} G\,\mathrm{Tr}\sigma^a{}^\pm F_{ab} &= \frac{1}{2\sqrt2}(\Pi_{am}\Pi_{bn} - \Pi_{bm}\Pi_{an})\epsilon^{mna} \\ &\quad \mp\frac{i}{\sqrt2}D^a(\Pi_{ba} - \Pi q_{ba}) \\ &\simeq \mp\frac{i}{\sqrt2}D^a(k_{ab} - kq_{ab}) \ , \end{aligned} \tag{17'}$$

where $\simeq$ *stands for equality modulo constraint* (30). Thus, we can rewrite constraint (5) as

$$\begin{aligned} 0 = C_a(\sigma, M) &\equiv -2q_{am}D_nP^{mn} \\ &\simeq \mp2\sqrt2 i\,\mathrm{Tr}\tilde\sigma^{m\pm}F_{ma} \end{aligned} \tag{5'}$$

in terms of the new basic variables $\tilde\sigma^a{}_{AB}$ and ${}^\pm A_a{}^{AB}$.

Finally, to reexpress (6), we proceed as follows. Since

$$\begin{aligned} G\,\mathrm{Tr}\sigma^a\sigma^b{}^\pm F_{ab} &= -\frac{G}{\sqrt2}\epsilon^{abc}{}^\pm F_{abc} \\ &= \tfrac12(R + \Pi^2 - \Pi_{ab}\Pi^{ba}) \mp i\epsilon^{abc}D_a\Pi_{bc} \\ &\simeq \tfrac12(R + K^2 - K_{ab}K^{ab}) \ , \end{aligned} \tag{18'}$$

Eq. (6) becomes

$$0 = C(\sigma, M) \simeq -2(\det q)^{-1/2}\,\mathrm{Tr}\tilde\sigma^a\tilde\sigma^b{}^\pm F_{ab} \ . \tag{6'}$$

To summarize, the set of Einstein constraints can be recast in terms of the new variables simply as

$${}^\pm\mathcal{D}_a\tilde\sigma^a{}_A{}^B = 0 \ , \tag{30''}$$

$$\mathrm{Tr}\tilde\sigma^{a\pm}F_{ab} = 0 \ , \tag{5''}$$

$$\mathrm{Tr}\tilde\sigma^a\tilde\sigma^{b\pm}F_{ab} = 0 \ . \tag{6''}$$

A number of remarks are in order.

(1) Note that these constraints involve only our basic variables, $\tilde\sigma^a{}_A{}^B$ and ${}^\pm A_{aA}{}^B$, and their ∂ derivatives. In particular, we do not need to raise or lower the tensor

index on these fields; the inverse of $\tilde{\sigma}^{a}{}_{A}{}^{B}$ never enters the constraints. In fact, the constraints are at worst quadratic in each of the basic variables. This is a significant improvement because the dependence of constraints on old variables (q_{ab}, p^{ab}) was nonpolynomial. The simplification may be useful in a number of technical problems. For example, it may enable one to dispel the mystery surrounding the structure of conical singularities in the spatially compact case, referred to in the Introduction. More importantly, it makes the constraints more manageable in the quantum theory. In particular, it is feasible to construct both the $\tilde{\sigma}$ representation, in which quantum states are, to begin with, arbitrary real-valued functionals of $\tilde{\sigma}^{a}{}_{A}{}^{B}$, as well as the ${}^{\pm}A$ representation, in which they are holomorphic functionals of ${}^{\pm}A_{aA}{}^{B}$. By contrast, only the q representation is manageable in the traditional canonical quantization scheme because of the nonpolynomial dependence of constraints on q_{ab}. The ${}^{\pm}A$ representations are, furthermore, more interesting from a number of considerations. First, because ${}^{\pm}A$ is essentially given by $\partial\sigma \pm i\Pi$, wave packets with minimum uncertainty spread in the three-metric and the extrinsic curvature arise naturally in these representations. Second, they enable one to import qualitatively new ideas into quantum gravity from QCD.[3] Finally, they seem to lead one to a very interesting picture of quantum geometry in which the metric takes on its classical properties only on macroscopic scales and only on sufficiently complicated (excited) states.[2]

(2) The form of constraints (30″), (5″), and (6″) brings out the reason behind our initial choice of σ^{a} as the configuration variable rather than σ_{a}. A second reason for this choice is that, since the connection one-forms $A_{aA}{}^{B}$ naturally occur as covectors, the conjugate variable should have a contravariant vectorial index.

(3) In the final form of constraints, one needs to know the action of ${}^{\pm}\mathcal{D}$ only on internal indices. In (30″) ${}^{\pm}\mathcal{D}$ does act on the vector index of $\tilde{\sigma}^{a}{}_{A}{}^{B}$. However, as noted above, since $\tilde{\sigma}^{a}{}_{A}{}^{B}$ is a density of weight 1, this action is independent of which torsion-free connection is chosen to act on the vector index. In (5″) and (6″) only the spinorial curvature of ${}^{\pm}\mathcal{D}$ enters. Thus, although in the original definition (13′) of ${}^{\pm}\mathcal{D}$, we specified its action on both tensor and internal indices in the finished picture, we only need the action (34′) on internal indices; to operate on tensor indices we can choose *any* torsion-free extension of (34′) to tensors.

(4) Note that the left-hand side of (6″) contains a single term. The left-hand side of (6), on the other hand, contains a "kinetic term" quadratic in momenta and a "potential term" independent of momenta. It is only in the so-called strong-coupling limit, $G\to\infty$, that the potential term disappears and we are left with a single (kinetic) term. This limit has been studied in some detail in the literature. One knows, in particular, that many of the difficulties of the canonical quantization scheme with (q,p) variables can be overcome in this limit. Now, if one regards $\tilde{\sigma}^{a}$ as the new "momentum" variable [it is natural to do so since (5″) is linear in $\tilde{\sigma}^{a}$], the full constraint (6″) resembles the strong-coupling limit of (6). Hence, one may be able to take over to full theory, formulated with $(\tilde{\sigma}^{a}{}_{A}{}^{B}, {}^{\pm}A_{aA}{}^{B})$, some of the techniques developed to study the strong-coupling limit in terms of (q_{ab}, p^{ab}).

Next, we show that the constraints can be cast into Yang-Mills form. First, the connection one-forms ${}^{\pm}A_{aA}{}^{B}$ can be thought of as Yang-Mills connection one-forms on the three-manifold Σ, and its curvature, ${}^{\pm}F_{abA}{}^{B}$, as the dual of the magnetic field ${}^{\pm}B^{m}{}_{A}{}^{B}$:

$$ {}^{\pm}B^{m}{}_{A}{}^{B} = \epsilon^{mab}\, {}^{\pm}F_{abA}{}^{B} \; . $$

Since it is the electric field $E^{a}{}_{A}{}^{B}$ which satisfies the canonical Poisson-brackets relations with the connection one-form in the Yang-Mills theory, let us replace the symbol $\tilde{\sigma}^{a}{}_{A}{}^{B}$ by $E^{a}{}_{A}{}^{B}$. (Note that, in Yang-Mills theory, the electric field—i.e., the momentum conjugate to the vector potential—is also naturally a density. The density character can be ignored in Minkowski space but not in Einstein-Yang-Mills theory.) Then, constraints (30′), (5′), and (6′) become

$$ {}^{\pm}\mathcal{D}_{a} E^{a}{}_{A}{}^{B} = 0 \; , \tag{30‴} $$

$$ \mathrm{Tr}\mathbf{E}\times\mathbf{B} = 0 \; , \tag{5‴} $$

$$ \mathrm{Tr}\mathbf{E}\cdot\mathbf{E}\times\mathbf{B} = 0 \; . \tag{6‴} $$

The first of these equations is just the Yang-Mills Gauss law constraint. Thus, every initial datum $(\sigma^{a}{}_{A}{}^{B}, M_{aA}{}^{B})$ for Einstein's theory provides us with initial datum $({}^{\pm}A_{aA}{}^{B}, E^{a}{}_{A}{}^{B})$ for Yang-Mills theory which satisfies, in addition to the Gauss law constraint, four constraints (5‴) and (6‴) algebraic in Yang-Mills field strengths: We have an embedding of the Einstein constraint surface into the Yang-Mills constraint.

We note the following.

(1) Considered as a system of equations on Yang-Mills fields on a spacelike hypersurface Σ, Eqs. (30‴), (5‴), and (6‴) have the remarkable property that they do not require a background structure (such as a metric or volume element, or a derivative operator) on Σ. It is somewhat surprising that one can write such equations at all. The evolution equations for Yang-Mills theory, for example, do require a background three-geometry.

(2) The embedding, obtained above, is for initial data sets only, and not for the entire four-dimensional solutions of Einstein and Yang-Mills equations. Given a pair $({}^{\pm}A_{a}, E^{a})$ satisfying all constraints, we can choose to evolve it using either the Einstein Hamiltonian, or the Yang-Mills Hamiltonian. The Einstein evolution preserves all constraints. The Yang-Mills evolution, on the other hand, preserves only (30‴); in general, (5‴) and (6‴) will not continue to hold if $({}^{\pm}A_{a}, E^{a})$ are evolved using Yang-Mills equations.

(3) Note that the Yang-Mills data $({}^{\pm}A_{a}, E^{a})$ arising from some Einstein data (σ^{a}, M_{a}) must satisfy, in addition to (5‴) and (6‴), two conditions: (i) $E^{a}{}_{A}{}^{B}$ is an isomorphism between the tangent space of Σ and second-rank, trace-free, Hermitian Higgs scalars, and (ii) $\Pi_{aA}{}^{B}$ defined from ${}^{\pm}A_{aA}{}^{B}$ via (34′) is Hermitian. These conditions make it awkward to use the embedding directly to obtain explicit solutions to Einstein constraints from

Yang-Mills theory. If one is interested in complex general relativity, on the other hand, the situation is better. Now, one can drop Hermiticity conditions and *all* fields are SL(2,C) valued. Therefore, in addition to (5‴) and (6‴) one has only to ensure that $E^a{}_A{}^B$ is an isomorphism between the tangent space of Σ and second-rank, trace-free Higgs scalars.

(4) For Euclidean Einstein theory (i.e., with ${}^4g_{ab}$ of signature $+ + + +$), the situation is as follows. One can define (σ^a, M_a) as in Sec. III. The constraints are (30), (5), and a modified version of (6), where the modification consists only of changing the sign of terms quadratic in p^{ab}. One can define ${}^{\pm}\mathcal{D}$ and ${}^{\pm}A_a$ by equations analogous to (13′) and (34′), the only difference being that the factor of i in front of $\Pi_{aM}{}^N$ is dropped. Then, the Euclidean constraints are once again given by (5‴), (6‴), and (30‴). Thus, in the Euclidean case, $\sigma^a, M_a, \Pi_a, {}^{\pm}A_a$ are *all* Hermitian, whence the condition (ii) mentioned above in remark (3) is automatically satisfied.

(5) In general, i.e., independently of whether g_{ab} is Euclidean, or Lorentzian, or complex, the inverse σ_a of σ^a never features either in the constraints or (as we shall see) in the evolution equations, whence the equations continue to be meaningful even when σ^a is degenerate. Thus, this set of equations represents a generalization of Einstein's equations, reducing them when σ^a is nondegenerate. There are indications that this fact will play a significant role in quantum gravity. In particular, it may enable us to analyze the possibility of topology change while working in a canonical framework.

(6) For simplicity, let us consider complex general relativity and SL(2,C) Yang-Mills theory and count the number of degrees of freedom. Since SL(2,C) is three dimensional and since the zero rest-mass vector field has two degrees of freedom per internal degree, SL(2,C) Yang-Mills theory has six degrees of freedom. The imposition of the four additional conditions, (5‴) and (6‴), reduce the freedom to two. These are the degrees of freedom of (complex) general relativity. Thus, it is not surprising that four additional constraints are needed in the passage from Yang-Mills theory to Einstein's theory.

(7) Note that our boundary conditions on $\tilde{\sigma}^a$ and ${}^{\pm}A_a$ are different from those imposed on E^a and A_a in the Yang-Mills theory. While $\tilde{\sigma}^a$ approaches a constant value ${}^0\sigma^a$ as $1/r$ and ${}^{\pm}A_a$ fall off as $1/r^2$, the Yang-Mills electric field is normally assumed to fall as $1/r^2$, and the Yang-Mills potential, as $1/r$. This difference has physical consequence. For instance, while the (internal) color change is well defined in the Yang-Mills theory, it is *not* well-defined on the gravitational phase space.

VI. CONSTRAINT ALGEBRA

We shall carry out the calculation of Poisson brackets between the constraint functionals using only the basic Poisson brackets (35), (37), and (39) between $\tilde{\sigma}^a{}_A{}^B$ and ${}^{\pm}A_{aA}{}^B$ and the identities

$$\{A, B+\lambda C\} = \{A,B\} + \lambda\{A,C\}, \quad \{A,B\} = -\{B,A\} ,$$

and

$$\{AB, C\} = A\{B,C\} + \{A,C\}B$$

satisfied by any Poisson brackets.

Let us begin with (30″). Given a test field $N_A{}^B$—a (c-number) Hermitian, trace-free field tending to zero as $(1/r^2)$ at spatial infinity, we define a constraint functional $C^{\pm}{}_{\mathcal{N}}(\tilde{\sigma}, A)$ as

$$C^{\pm}{}_{\mathcal{N}}(\tilde{\sigma}, {}^{\pm}A) \equiv \pm\frac{\sqrt{2}}{Gi}\int_{\Sigma}(N_A{}^B)({}^{\pm}\mathcal{D}_a\tilde{\sigma}^a{}_B{}^A) , \tag{42}$$

where the overall numerical factor has been introduced for later convenience. Let us begin with the canonical transformation generated by this functional. Using (34) and integrating by parts, we have

$$C^{\pm}{}_{\mathcal{N}}(\tilde{\sigma}, {}^{\pm}A) = \pm\frac{\sqrt{2}i}{G}\int_{\Sigma}(\partial_a N_A{}^B + G[{}^{\pm}A_a, N]_A{}^B)\tilde{\sigma}^a{}_B{}^A . \tag{42′}$$

Using this expression and the Poisson brackets (35) and (37), we have

$$\{C^{\pm}{}_{\mathcal{N}}, \tilde{\sigma}^m{}_p{}^Q\} = [N, \tilde{\sigma}^m]_p{}^Q \tag{43a}$$

and

$$\{C^{\pm}{}_{\mathcal{N}}, {}^{\pm}A_{mp}{}^Q\} = -\frac{1}{G}{}^{\pm}\mathcal{D}_m N_p{}^Q . \tag{43b}$$

The second of these equations in turn implies

$$\{C^{\pm}{}_{\mathcal{N}}, {}^{\pm}F_{mnp}{}^Q\} = [N, {}^{\pm}F_{mn}]_p{}^Q . \tag{43c}$$

Thus, the infinitesimal canonical transformation generated by $\mathcal{C}_{\mathcal{N}}$ is precisely the infinitesimal SU(2) rotation generated by $N_A{}^B$. [This explains the choice of the overall numerical factor in Eq. (42).] Consequently, we have the following Poisson-brackets relations:

$$\{C^{\pm}{}_{\mathcal{N}}, C^{\pm}{}_{M}\} = \pm\frac{\sqrt{2}}{Gi}\int_{\Sigma}[M,N]_A{}^B({}^{\pm}\mathcal{D}_a\tilde{\sigma}^a{}_B{}^A) , \tag{44a}$$

$$\{C^{\pm}{}_{\mathcal{N}}, \mathrm{Tr}\tilde{\sigma}^a\,{}^{\pm}F_{ab}\} = 0 , \tag{44b}$$

and

$$\{C^{\pm}{}_{\mathcal{N}}, \mathrm{Tr}\tilde{\sigma}^a\tilde{\sigma}^b\,{}^{\pm}F_{ab}\} = 0 . \tag{44c}$$

These relations are not surprising; one could have anticipated them from the role of the Gauss law constraint in the Yang-Mills theory.

To compute the remaining Poisson brackets, it is convenient to obtain first the brackets between $\tilde{\sigma}^a(x)$ and ${}^{\pm}F_{mn}(y)$. Using the definition (41′) of ${}^{\pm}F_{mn}$ and the basic Poisson brackets between $\tilde{\sigma}^a$ and ${}^{\pm}A_a$, one has

$$\{\tilde{\sigma}^a{}_{AB}(x), {}^{\pm}F_{mn}{}^{MN}(y)\} = \mp(\sqrt{2}i)(\partial_{[m}\delta_{n]}{}^a\delta(x,y)\delta_A{}^{(M}\delta_B{}^{N)} - G\delta(x,y)\delta_{[n}{}^a\,{}^{\pm}A_{m]A}{}^{(M}\delta_B{}^{N)} - G\delta(x,y)\delta_{[n}{}^a\,{}^{\pm}A_{m]B}{}^{(M}\delta_A{}^{N)}) . \tag{45}$$

The computation of the remaining Poisson brackets is now straightforward since constraints (5′) and (6′) involve only algebraic combinations of $\tilde{\sigma}^a$ and ${}^{\pm}F_{mn}$. Set

$$C^{\pm}{}_{\mathbf{N}}(\tilde{\sigma},{}^{\pm}A)=\pm\frac{\sqrt{2}}{i}\mathrm{Tr}\int_{\Sigma}N^a\tilde{\sigma}^{b\pm}F_{ab} \tag{46}$$

and

$$C^{\pm}{}_{\underline{N}}(\tilde{\sigma},{}^{\pm}A)=\pm\frac{\sqrt{2}}{i}\mathrm{Tr}\int_{\Sigma}\underline{N}\tilde{\sigma}^{a}\tilde{\sigma}^{b\pm}F_{ab}\ , \tag{47}$$

where N^a is a vector field and $\underline{N}$ a scalar density of weight minus one, both (c numbers and) tending to zero as $(1/r)$ at spatial infinity. (Here, and in what follows, $\mathrm{Tr}\int\tilde{f}$ stands for the integral of trace of $\tilde{f}$.) Then, using the basic properties of Poisson brackets, relations (35), (37), and (45), suitable integrations by part, the falloff rate of fields involved, and the Bianchi identities satisfied by ${}^{\pm}F_{ab}$, one obtains

$$\begin{aligned}\{C^{\pm}{}_{\mathbf{N}},C^{\pm}{}_{\mathbf{M}}\}=\pm\frac{\sqrt{2}}{i}\mathrm{Tr}\int&(\mathcal{L}_{M}N)^a\tilde{\sigma}^{b\,\pm}F_{ab}\\&+(N^aM^{b\,\pm}F_{ab})({}^{\pm}\mathcal{D}_m\tilde{\sigma}^{m})\ ,\end{aligned} \tag{48}$$

$$\begin{aligned}\{C^{\pm}{}_{\mathbf{N}},C^{\pm}{}_{\underline{M}}\}=\pm\frac{\sqrt{2}}{i}\mathrm{Tr}\int_{\Sigma}&-(\mathcal{L}_{N}\underline{M})\tilde{\sigma}^{a}\tilde{\sigma}^{b\,\pm}F_{ab}\\&+\underline{M}N^a[\tilde{\sigma}^{b},{}^{\pm}F_{ab}](\mathcal{D}_m\tilde{\sigma}^{m})\ ,\end{aligned} \tag{49}$$

and

$$\begin{aligned}\{C^{\pm}{}_{\underline{N}},C^{\pm}{}_{\underline{M}}\}=\pm\frac{2\sqrt{2}}{i}\mathrm{Tr}\int_{\Sigma}&-(\underline{N}\partial_m\underline{M}-\underline{M}\partial_m\underline{N})\\&\times(\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{m})\tilde{\sigma}^{b\,\pm}F_{ab}\ .\end{aligned} \tag{50}$$

These results can be succinctly expressed as follows. Let $\bar{N}$ stand for the triplet ($N_A{}^B$,$\mathbf{N}$, and $\underline{N}$) and set

$$\begin{aligned}C^{\pm}{}_{\bar{N}}(\tilde{\sigma},{}^{\pm}A)=\pm\frac{\sqrt{2}}{i}\mathrm{Tr}\int_{\Sigma}&G^{-1}N\,{}^{\pm}\mathcal{D}_a\tilde{\sigma}^{a}\\&+\underline{N}\tilde{\sigma}^{a}\tilde{\sigma}^{b\,\pm}F_{ab}+N^a\tilde{\sigma}^{b\,\pm}F_{ab}\ .\end{aligned} \tag{51}$$

Then,

$$\{C^{\pm}{}_{\bar{N}},C_{\bar{M}}\}=C^{\pm}{}_{\bar{P}}\ ,$$

where

$$\begin{aligned}P_A{}^B&=[M,N]_A{}^B+GN^aM^{b\,\pm}F_{abA}{}^B\\&\quad-G(\underline{M}N^a-\underline{N}M^a)[\tilde{\sigma}^{b},{}^{\pm}F_{ab}]_A{}^B\ ,\\\underline{P}&=-\mathcal{L}_{\mathbf{N}}\underline{M}+L_{\mathbf{M}}\underline{N}\ ,\end{aligned} \tag{52}$$

and

$$P^a=-(\mathcal{L}_{\mathbf{N}}M^a)-2(\underline{N}\partial_m\underline{M}-\underline{M}\partial_m\underline{N})\mathrm{Tr}\tilde{\sigma}^{a}\tilde{\sigma}^{b}\ .$$

We make the following remarks.

(1) In the new Hamiltonian formulation, the lapse naturally arises as a density of weight -1 (Ref. 17) because the scalar constraint is a density of weight 2. The integrands in the expressions (42), (46), (47), and (52) of constraint functionals are all densities of weight 1, so that the integration can be carried out without reference to a specific volume element.

(2) If one gives the lapse and the shift dimensions of length and regards the SU(2) generators $N_A{}^B$ as dimensionless, the constraint functionals $C^{\pm}{}_{\bar{N}}$ have dimension of action. Hence, the Poisson brackets of these functionals with any observable on the phase space—which represents the infinitesimal change in the observable—has the same dimension as the observable.

(3) Equation (52) implies that the constraints are of first class: the Poisson brackets of constraint functionals vanish weakly. Note, however, that in the language of Becchi-Rouet-Stora-Tyutin (BRST) transformations, the system of constraints is *open*: the Poisson brackets involve structure functionals rather than structure constants. This situation occurs also in the usual Hamiltonian formulation of general relativity based on (q_{ab},p^{ab}) where the expression of Poisson brackets of two scalar constraint functionals contains a vector field constructed from the two lapses *and* the three-metric q_{ab}. In the present formulation, there are further complications: structure functionals involve *both* $\tilde{\sigma}^a$ and A_a and arise also in other Poisson brackets [(48) and (49)]. The BRST structure of (52) has not been explored. In particular, we do not know if the "second-order structure functions" vanish for this algebra.

(4) Note, however, that there *is* a technical improvement over the Poisson-brackets algebra based on (q_{ab},p^{ab}) variables. In the present case, structure functionals depend at most quadratically on the basic variables $\tilde{\sigma}^a$ and ${}^{\pm}A_a$: they involve only $\tilde{\sigma}^a$ and ${}^{\pm}F_{ab}$. In the usual Hamiltonian formulation, on the other hand, structure functionals have a nonpolynomial dependence on the basic variables through q^{ab}.

(5) It may come as a surprise that the Poisson brackets of two vector constraints, for example, involve the Gauss law constraint (30″). This comes about because the present vector and scalar constraints (5″) and (6″) are equivalent to the traditional vector and scalar constraints (5) and (6) modulo (30″). So, in fact, what is surprising is that just the right cancellations occur for the Poisson brackets (50) between new scalar constraints *not* containing (30″).

VII. DYNAMICS

We can now discuss dynamics. For reasons given in Sec. III, constraint functionals $C^{\pm}{}_{\mathbf{N}}$ and $C^{\pm}{}_{\underline{N}}$ [of Eqs. (46) and (47)] are differentiable on Γ only if N^a and $\underline{N}$ tend to zero at infinity as $1/r$. To obtain dynamical evolution, on the other hand, we need the shift and the lapse to go to constant values at infinity, corresponding to space and time translations. Consequently, to obtain the Hamiltonians generating dynamics, one must add suitable surface terms to the constraint functionals.

Let T^a be a translational Killing field of the flat metric ${}^0q^{ab} = -\mathrm{Tr}\,{}^0\sigma^a\,{}^0\sigma^b$ that we initially fixed outside a compact set of Σ and let $\underline{T}$ be a scalar density that equals $(\det q)^{-1/2}$ outside the compact set. Then, the pair $(T^a, \underline{T})$ defines a space-time translation. The Hamiltonian, $H_{\overline{T}}[\overline{T} \equiv (T_A{}^B = 0, T^a, \underline{T})]$, generating the corresponding canonical transformation on the phase space is given by

$$H^{\pm}{}_{\overline{T}}(\tilde{\sigma}, {}^{\pm}A_a) := \lim_{S\to\Sigma} \pm \mathrm{Tr} \int_S \underline{T}\tilde{\sigma}^a\tilde{\sigma}^b\,{}^{\pm}F_{ab} - iT^a\tilde{\sigma}^b\,{}^{\pm}F_{ab} \pm \lim_{S\to\Sigma} 2\,\mathrm{Tr} \oint_{\partial S} (-\underline{T}\tilde{\sigma}^{[a}\tilde{\sigma}^{b]}\,{}^{\pm}A_b + iT^{[a}\tilde{\sigma}^{b]}\,{}^{\pm}A_b) dS_a \quad (53)$$

$$= \pm \mathrm{Tr} 2 \int_\Sigma -(\partial_a \underline{T}\tilde{\sigma}^{[a}\tilde{\sigma}^{b]})\,{}^{\pm}A_b + G\underline{T}\tilde{\sigma}^{[a}\tilde{\sigma}^{b]}\,{}^{\pm}A_a\,{}^{\pm}A_b + i(\partial_a T^{[a}\tilde{\sigma}^{b]})\,{}^{\pm}A_b - iGT^{[a}\tilde{\sigma}^{b]}\,{}^{\pm}A_a\,{}^{\pm}A_b\,, \quad (53')$$

where the integrals in (53) are first evaluated on a finite portion S of Σ and the limit of the result is then taken as S expands out to fill all of Σ [to see how this subtlety arises, see discussion following Eq. (11)]. The form (53) of the Hamiltonian brings out its relation with constraints while the form (53′) brings out the fact that it is a well-defined quantity; since the integrand of (53′) falls off as $1/r^4$, the integral is manifestly convergent. Note that the Hamiltonian is polynomial in $\tilde{\sigma}^a$ and ${}^{\pm}A_a$. The numerical coefficients in (53) have been chosen to simplify the evolution equations for new variables. Consequently, there are relative numerical factors between (53) and (11), (12) and $\underline{T}$ and T^a in (53) have to be suitably rescaled to obtain geometrical lapse and shift fields: the geometrical lapse equals $(\det q)^{1/2}\underline{T}$ and the geometrical shift equals $(1/\sqrt{2})T^a$.

To obtain the evolution equations, we take the Poisson brackets of the basic variables with the Hamiltonian. One obtains

$$\dot{\tilde{\sigma}}^a \equiv \{H_{\overline{T}}, \tilde{\sigma}^a\} = \sqrt{2}\,{}^{\pm}\mathcal{D}_b(i\underline{T}\tilde{\sigma}^{[b}\tilde{\sigma}^{a]} + T^{[b}\tilde{\sigma}^{a]})\,, \quad (54)$$

$${}^{\pm}\dot{A}_a \equiv \{H_{\overline{T}}, {}^{\pm}A_a\} = \frac{1}{\sqrt{2}}([i\underline{T}\tilde{\sigma}^b, {}^{\pm}F_{ab}] - T^b\,{}^{\pm}F_{ab})\,. \quad (55)$$

Using the definition of the three-metric q_{ab} and the extrinsic curvature k_{ab} [Eqs. (25) and (34); recall also that $\Pi_{ab} = k_{ab}$ when the constraint (30) is satisfied] in terms of $\tilde{\sigma}^a$ and ${}^{\pm}A_a$, one can now obtain the evolution equations for q_{ab} and k_{ab}. One has, modulo constraints,

$$\dot{q}_{ab} = \mp 2Tk_{ab} + \frac{1}{\sqrt{2}}\mathcal{L}_{\mathrm{T}} q_{ab} \quad (56)$$

and

$$\dot{k}_{ab} = \mp D_a D_b T \pm TR_{ab} \mp 2Tk_a{}^m k_{mb} \pm Tkk_{ab} + \frac{1}{\sqrt{2}}\mathcal{L}_{\mathrm{T}} k_{ab}\,, \quad (57)$$

where $T \equiv (\det q)^{1/2}\underline{T}$. These are the standard evolution equations for Cauchy data.

On the constraint surface the numerical value of the Hamiltonian is given just by the surface terms. Even though the integrand of these terms contains both, Hermitian and anti-Hermitian pieces through ${}^{\pm}A_a$, it turns out that, due to the particular algebraic combinations involved and the boundary conditions satisfied by $\tilde{\sigma}^a$ and ${}^{\pm}A_a$, on the constraint surface the value of the surface integrals is, in fact, *real*. Substituting (34′) in (53) and using the boundary conditions, one obtains

$$H_{\overline{T}}{}^{\pm}(\tilde{\sigma}^a, {}^{\pm}A_a) \approx \mp \frac{1}{2G} \oint T(\partial_b q_{ac} - \partial_c q_{ab})\,{}^0q^{ab} dS^c + \frac{1}{\sqrt{2}} \oint T^a p_{ab} dS^b\,, \quad (58)$$

where $\approx$ stands for "equals on the constraint surface" and where ∂ is the derivative operator of the flat metric q_{ab}. Thus, except for overall factors, the numerical value of the Hamiltonian on physical states yields precisely the ADM energy and the ADM three-momentum. Note that, if one regards the phase space as a complex manifold, coordinatized by ${}^{\pm}A_a$, the surface (as well as volume) integrals in (53) are *holomorphic* functionals. However, they take on real values on the constraint surface and these coincide precisely with the ADM four-momentum.

We note the following.

(1) On the entire phase space, the Hamiltonian (53) is a holomorphic function. Therefore, on a generic point off the constraint surface, its numerical value is complex. Consequently, even on the constraint surface, the time evolution generated by $H^{\pm}{}_{\overline{T}}$ does not respect Hermiticity; the expression (54) for $\dot{\tilde{\sigma}}^a$, for example, has an anti-Hermitian piece (even when evaluated at a point of the phase space at which $\tilde{\sigma}^a$ and Π_a are both Hermitian). However, the anti-Hermitian piece is pure gauge, whence the evolution of q_{ab} and k_{ab} preserves reality [Eqs. (56) and (57)]. To obtain a Hermiticity-preserving evolution, therefore, we have only to add to the Hamiltonian a suitably weighted Gauss-law constraint functional (30′). Thus, for example, if the shift T^a is set to zero, the Hamiltonian

$$\overline{H}^{\pm}(\tilde{\sigma}, {}^{\pm}A) := \pm \mathrm{Tr} \int_\Sigma \underline{T}\,\mathrm{Tr}\tilde{\sigma}^a\tilde{\sigma}^b\,{}^{\pm}F_{ab} - \frac{1}{G}(D_a\underline{T})\tilde{\sigma}^a({}^{\pm}\mathcal{D}_b\tilde{\sigma}^b) \mp 2\,\mathrm{Tr} \oint_{\partial\Sigma} \underline{T}\tilde{\sigma}^{[a}\tilde{\sigma}^{b]\pm} A_b dS_a \quad (59)$$

$$\equiv \pm \frac{1}{2G} \int_\Sigma \underline{T}(\det q)(R - \Pi_{ab}\Pi^{ba} + \Pi^2) \mp \frac{1}{2G} \oint_{\partial\Sigma} T(\partial_a q_{bc} - \partial_b q_{ac}) q^{ac} dS^b \quad (59')$$

is real everywhere on the phase space and generates the Hermiticity-preserving evolution:

$$\dot{\tilde{\sigma}}^a = \sqrt{2}i\{{}^{\pm}\mathcal{D}_b(\underline{T}\tilde{\sigma}^{[b}\tilde{\sigma}^{a]}) - \tfrac{1}{2}[(D_b\underline{T})\tilde{\sigma}^b, \tilde{\sigma}^a]\} \qquad (54')$$

and

$$\dot{A}_a = \frac{i}{\sqrt{2}}\{[\underline{T}\tilde{\sigma}^b, {}^{\pm}F_{ab}] + (1/G){}^{\pm}\mathcal{D}_a[(D_b\underline{T})\tilde{\sigma}^b]\} \ . \qquad (55')$$

(2) Note that asymptotic translations (generated by $\underline{T}$ and T^a above) leave the background ${}^0\sigma^a{}_A{}^B$ invariant and preserve the boundary conditions on the phase-space variables $(\sigma^a, {}^{\pm}A_a)$. This property is *not* shared by an internal rotation $T_A{}^B$ which asymptotically approaches a constant nonzero value. That is, "global internal rotations" are incompatible with our boundary conditions. Hence, in contrast with the Yang-Mills theory, we do not have internal (color) charges on our phase space. This is to be expected on physical grounds since such charges, unlike energy momentum have no role in general relativity.

(3) Lee has pointed out (private communication) that the form of the Hamiltonian simplifies substantially if we set $\underline{T} = (\det q)^{-1/2}$ and $T^a = 0$. [Note incidently that (58) equals (59) for this choice since $D_a(\det q) = 0$.] Using Eqs. (13′), (34), and (34′) of Sec. IV, one obtains

$$\begin{aligned} H^{\pm}{}_{\underline{T}}(\tilde{\sigma}, {}^{\pm}A) &\equiv \pm \mathrm{Tr} \int_\Sigma \underline{T}\tilde{\sigma}^a \tilde{\sigma}^b\, {}^{\pm}F_{ab} \\ &\quad \mp 2\, \mathrm{Tr} \oint_{\partial\Sigma} \underline{T}\tilde{\sigma}^{[a}\tilde{\sigma}^{b]}\, {}^{\pm}A_b dS_a \\ &= \mp \int \underline{T}[({}^{\pm}A^{ab})^*({}^{\pm}A_{ba}) \\ &\qquad -({}^{\pm}A)^*({}^{\pm}A)](\det q) \ , \end{aligned}$$

where ${}^{\pm}A_{ab} = -\mathrm{Tr}\, {}^{\pm}A_a\sigma_b$ and ${}^{\pm}A = {}^{\pm}A_{ab}q^{ab}$. If one could show that it is always possible to go to a gauge in which ${}^{\pm}A_{ab}$ is symmetric and traceless, one would obtain a new proof of positivity of energy. This form of the Hamiltonian is useful especially in the weak-field and strong-coupling limits.

VIII. DISCUSSION

On the gravitational phase space Γ we performed a "canonical transformation" to pass from (σ^a, M_a) as the basic variables to $(\tilde{\sigma}^a, {}^{\pm}A_a)$. This transformation is analogous to the one which sends the variables (q,p) on the phase space of a harmonic oscillator to the variables $[q, z = (m\omega)^{1/2}q + i(m\omega)^{-1/2}p]$ [or, alternatively, to $(q,\bar{z})$]. Because the variables ${}^{\pm}A_a$ contain information about both (the connection compatible with) σ^a and M_a, the constraints and the Hamiltonian of general relativity simplify considerably in terms of $\tilde{\sigma}^a$ and ${}^{\pm}A_a$: while they depend nonpolynomially on (σ^a, M_a), their dependence on $\tilde{\sigma}^a$ and ${}^{\pm}A_a$ is at worst quadratic. This simplification is expected to play an important role in a number of problems in both classical and quantum gravity.

The key step in this simplification is the introduction of connections ${}^{\pm}\mathcal{D}$, or, connection one-forms ${}^{\pm}A_{aA}{}^B$. What is the interpretation of these variables? Let us consider an initial datum (σ^a, M_a) *satisfying all constraints* and compute the curvature two-forms ${}^{\pm}F_{ab}$ in terms of the corresponding Cauchy pair (q_{ab}, k_{ab}). As noted in Sec. V, we have

$$\begin{aligned} -G\, \mathrm{Tr}\, {}^{\pm}F_{ab}\sigma_c &\equiv G\, {}^{\pm}F_{abc} \\ &= -\frac{1}{2\sqrt{2}}R_{ab}{}^{mn}\epsilon_{cmn} - \frac{1}{\sqrt{2}}\epsilon_{cmn}k_a{}^m k_b{}^n \\ &\quad \pm \sqrt{2}iD_{[a}k_{b]c} \ . \end{aligned} \qquad (15'')$$

Hence, taking the dual on a and b, one obtains

$$\begin{aligned} W_{cd} := G\, {}^{\pm}F_{abc}\epsilon^{ab}{}_d &= -\sqrt{2}(R_{cd} - k_c{}^a k_{da} + kk_{cd}) \\ &\quad \pm \sqrt{2}i\epsilon^{ab}{}_d(D_a k_{bc}) \\ &= -\sqrt{2}(E_{cd} \mp iB_{cd}) \ , \end{aligned} \qquad (19')$$

where we have used the constraint (30) to set $\Pi_{ab} = k_{ab}$ and where E_{cd} and B_{cd} are the electric and the magnetic parts of the Weyl tensor of the vacuum solution obtained from the initial datum. Thus, ${}^{+}F_{ab}$ and ${}^{-}F_{ab}$ are essentially the anti-self-dual and the self-dual parts of the Weyl tensor and, consequently, ${}^{\pm}A_a$ is a potential for the (anti-)self-dual curvature. This property becomes transparent by noting that ${}^{\pm}\mathcal{D}$ are the restrictions to a three-surface Σ of the action of the space-time covariant derivative on (un)primed SL(2,C) spinors. (For details, see Ref. 9.)

This interpretation enables one to obtain a new characterization of half-flat (i.e., self-dual or anti-self-dual) solutions to Einstein's equation.[18] Since these solutions are either of Euclidean signature, $(+ + + +)$, or complex, one begins by either requiring that $(\tilde{\sigma}^a, {}^{\pm}A_a)$ be both SU(2), Hermitian fields [i.e., by dropping the factor i in front of Π_a in Eq. (34′)] or by letting them both be in the SL(2,C) Lie algebra. Then, self-duality is ensured by setting ${}^{+}F_{ab} = 0$. Now, if one uses $+$ variables, two of the three constraints, (5′) and (6′), as well as one of the evolution equations (55), are automatically satisfied and the entire content of Einstein's equations is captured in the first-order equations (30′) and (54). These equations can be further simplified by choosing the lapse appropriately and by introducing a suitable triad to expand $\mathrm{T}\tilde{\sigma}^a$ in terms of Pauli matrices. The final result is the following.[18] In the half-flat case, Einstein's equation is equivalent to the set of equations

$$\partial_a V^a{}_{\mathrm{a}} = 0 \ , \qquad (30'''')$$

$$\dot{V}^a{}_{\mathrm{a}} = \epsilon_{\mathrm{abc}}[V_{\mathrm{b}}, V_{\mathrm{c}}]^a \qquad (54'')$$

on a triad $V^a{}_a$, where [,] denotes Lie derivatives. These equations are remarkably simple. Their resemblance to Euler's equations for rigid bodies supports the conjecture of exact integrability of the half-flat equation. This issue is being investigated in collaboration with Mazur.

The new variables and their properties raise a number of other issues in classical gravity. We shall end by listing a few. Since the constraints are at worst quadratic in each of the new variables, one should be able to use them to "explain" why the conical singularities of the space of solutions to Einstein's equation (in the spatially compact case) are as simple as they turned out to be.[5,6] Is this indeed the case? Is the standard analysis of solutions with one or two Killing fields simplified by the use of new variables? Can one find new solutions to Einstein's and/or Yang-Mills equations by exploiting the embedding given in Sec. V? Are there freely specifiable "York-data" for new variables? Can the fact that ${}^{\pm}A_a$ are natural potentials for Weyl curvature be used to simplify perturbation analysis in general relativity? Are there convenient Lagrangian formulations of general relativity in terms of new variables?

The application of this framework to nonperturbative canonical quantum gravity will be discussed in a series of papers by Jacobson, Lee, Mazur, Renteln, Smolin, Torre, and the author.

Note added in proof. (1) J. Samuel [Pramana - J. Phys. **28**, L429 (1987)] and T. Jacobson and L. Smolin [Phys. Lett. B (to be published)] have obtained Lagrangian formulations of general relativity in terms of the new variables introduced here. Many of our results are easier to obtain in these Lagrangian treatments. (2) A. Ashtekar, P. Mazur, and C. Torre [University of Utah report, 1987 (unpublished)] have completed the BRST analysis of general relativity in terms of the new variables. Of particular interest to the present paper is their result that the constraint algebra simplifies considerably if the vector constraint (5″) is replaced by $\mathrm{Tr}(\tilde{\sigma}^{\,b\pm}F_{ab} - {}^{\pm}A_a\, {}^{\pm}\mathcal{D}_b\tilde{\sigma}^{\,b}) = 0$. The new vector constraint generates "pure diffeomorphisms" thereby making it transparent that the second-order BRST structure functions can be made to vanish.

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APPENDIX: SU(2) SPINORS

Let Σ be a three-manifold and q_{ab} a positive-definite metric thereon. SU(2) spinors are two component objects, $\lambda^A, \mu_A, \ldots$ equipped with the usual operations of tensor algebra: addition, multiplication by functions, index substitution, and outer product.[14,15] Furthermore, we have the following additional structure. First, there exist second-rank, skew, nondegenerate spinors, ϵ^{AB} and ϵ_{AB}, with which we can raise and lower spinor indices:

$$\epsilon^{AB}\epsilon_{AC} = \delta_C^B,\quad \mu^A = \epsilon^{AB}\mu_B,\quad \mu_A = \mu^B\epsilon_{BA}\ . \tag{A1}$$

Second, associated with every spinor λ_A is its Hermitian conjugate, $\lambda^\dagger{}_A$, such that

$$(\lambda_A + c\mu_A)^\dagger = \lambda^\dagger{}_A + c^*\mu^\dagger{}_A,\quad (\lambda^\dagger{}_A)^\dagger = -\lambda_A$$

and (A2)

$$(\lambda^\dagger)^A\lambda_A \geq 0\ ,$$

where c is any complex number, c^* its complex conjugate, and where the equality in the last property holds if and only if $\lambda^A = 0$. Finally, there exists an isomorphism $\sigma^a{}_A{}^B$ between the space of complex tangent vectors v^a at any point of Σ and spinors $v^A{}_B$ at that point which are trace-free ($\Longleftrightarrow$ satisfy $\epsilon_{A[C}v^A{}_{B]} = 0$): $v^a = -\sigma^{aA}{}_B v^B{}_A = -\mathrm{Tr}\sigma^a v$. This isomorphism satisfies the following two properties: (i) it maps Hermitian spinors to real vectors and vice versa and (ii) the metric q_{ab} on Σ can be expressed as

$$q_{ab} = -\mathrm{Tr}\sigma_a\sigma_b = \sigma_a{}^{AB}\sigma_b{}^{CD}\epsilon_{AC}\epsilon_{BD} \tag{A3}$$

[all vector indices are raised and lowered by the metric (q^{ab} and q_{ab}) and all spinor indices by the alternating spinor (ϵ^{AB} and ϵ_{AB})]. σ solders the spinor indices to the tangent space at each point and is therefore called a *soldering form*. Because of (A3), the soldering form may be regarded as the "square root of the metric."

Finally, any spinor field $\Lambda^A{}_B$ which is nondegenerate (i.e., satisfies $\Lambda^A{}_B\alpha^B = 0$ iff $\alpha^B = 0$), Hermitian, and which satisfies $\epsilon_{AB}\Lambda^A{}_M\Lambda^B{}_N = \epsilon_{MN}$, defines an isomorphism of the spin system to itself. More precisely, we have the following. Λ defines a 1-1 linear mapping from the complex, two-dimensional spin space to itself, $\alpha^A \to \alpha'^A = \Lambda^A{}_B\alpha^B$, such that (i) $(\alpha^{\dagger A})' = (\alpha'^A)^\dagger$ and (ii) $\sigma'_a{}^{AB} = \Lambda^A{}_M\Lambda^B{}_N\sigma_a{}^{MN}$ is also a soldering form for the metric q_{ab}, $\mathrm{Tr}\sigma'_a\sigma'_b = \mathrm{Tr}\sigma_a\sigma_b = -q_{ab}$.

Now, if (Σ, q_{ab}) is embedded in a four-dimensional space-time $(M, {}^4g_{ab})$, at points of Σ, one has both the SU(2) as well as the SL(2,C) spinors. Recall[15] that the SL(2,C) spinors are of two types, unprimed and primed, e.g., λ_A and $\bar{\lambda}_{A'}$. The soldering form, $\sigma_a{}^{AA'}$, now has both types of indices and can be thought of as a square root of ${}^4g_{ab}$:

$${}^4g_{ab} = \sigma_a{}^{AA'}\sigma_b{}^{BB'}\epsilon_{AB}\bar{\epsilon}_{A'B'}\ . \tag{A4}$$

[As in the main text, g_{ab} has signature $(-+++)$, whence the $\sigma_a{}^{AA'}$ here is i times that used in Ref. 15.] Unprimed spinors, e.g., μ_A, can be regarded either as SU(2) or as SL(2,C) spinors. The primed spinors, e.g., $\bar{\mu}_{A'}$, on the other hand, belong only to the SL(2,C) category. However, together with $\zeta^a \equiv \zeta^{AA'}\sigma^a{}_{AA'}$, the unit timelike normal (with respect to ${}^4g_{ab}$) to Σ, they define an Hermitian-conjugation operation on the SU(2) (or unprimed) spinors:

$$\mu^{\dagger A} \equiv \sqrt{2}\zeta^{AA'}\bar{\mu}_{A'}\ . \tag{A5}$$

Finally, note that in the literature in general relativity,[15] one often suppresses the soldering forms σ and simply writes ${}^4g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'}$ or $v^a = v^{AA'}$. While this convention is convenient if one is dealing with only a fixed conformal class of metrics, it is impractical if one has to deal simultaneously with a wider class of metrics: For a given conformal class, one can fix a fiducial $\sigma_a{}^{AA'}$ and code the information of any one metric in the choice of

ϵ_{AB}, while in the case of a more general class, the freedom in the choice of ϵ_{AB} is simply not large enough to characterize a metric. [The situation is identical in the SU(2) case.] We will *not* follow this convention since we have to deal with all possible metrics on Σ. Thus, in our convention, ϵ_{AB} is fixed once and for all, without reference to a metric. The soldering form σ changes with the choice of the metric.

Finally, in the component notation, the structure developed above can be recast as follows. Let α^A denote any spinor such that $(\alpha^A)^\dagger \alpha_A = 1$. Then $e^A{}_{\mathrm{M}}, \mathbf{M}=1,2$ with $e^A{}_1 = \alpha^A$ and $e^A{}_2 = (\alpha^A)^\dagger$ is a normalized spin dyad. Given any spinor λ^A, we can write it as a linear combination of these basis vectors:

$$\lambda^A = \lambda^{\mathrm{A}} e^A{}_{\mathrm{A}} = \lambda^1 e^A{}_1 + \lambda^2 e^A{}_2 \,. \tag{A6}$$

Since $2\alpha^{[A}\alpha^{+B]} = \epsilon^{AB}$ it follows that the components $\epsilon^{\mathrm{MN}} = \epsilon^{AB} e_A{}^{\mathrm{M}} e_B{}^{\mathrm{N}}$ of ϵ^{AB} are $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The Hermitian conjugate of λ^A is given by

$$(\lambda^A)^\dagger = (\lambda^{\mathrm{M}})^* (e^A{}_{\mathrm{M}})^\dagger = (\lambda^1)^* e^A{}_2 - (\lambda^2)^* e^A{}_1 \,, \tag{A7}$$

where an asterisk denotes the operation of complex conjugation. Given a second-rank, trace-free, Hermitian spinor λ^{AB} (e.g., $\lambda^{AB} = i\lambda^{(A}\lambda^{+B)}$), the components $\lambda^{\mathrm{M}}{}_{\mathrm{N}} = \lambda^{AC}\epsilon_{CB} e^B{}_{\mathrm{N}} e_A{}^{\mathrm{M}}$ provide us a 2×2 *anti-Hermitian,* trace-free matrix. (Note that the dual basis $e_A{}^{\mathrm{M}}$ is given by $e_A{}^1 = -\alpha^\dagger{}_A$ and $e_A{}^2 = \alpha_A$.) Next, given a positive-definite metric q_{ab}, one can introduce a soldering form $\sigma_{aB}{}^A$ compatible with it as follows. Fix an orthonormal triad $e^a{}_m$ $(m=1,2,3)$ on Σ and set

$$\sigma_{aB}{}^A = i\sigma_{\mathrm{m}}{}^{\mathrm{M}}{}_{\mathrm{N}} e^{\mathrm{m}}{}_a e^A{}_{\mathrm{M}} e_B{}^{\mathrm{N}} \,, \tag{A8}$$

where $\sigma_m{}^{\mathrm{M}}{}_{\mathrm{N}}$ are the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to check that $\sigma_a{}^A{}_B$ is trace-free and Hermitian in spinor indices. Furthermore, the vector $\lambda^a = \sigma^a{}_{AB}\lambda^{AB}$ is real (i.e., its components $\lambda^{\mathrm{m}} = -i\sigma^{\mathrm{mM}}{}_{\mathrm{N}}\lambda^{\mathrm{N}}{}_{\mathrm{M}}$ are real) if λ^B is trace-free and Hermitian, in accordance with our requirements on the soldering forms. Finally consider isomorphisms $\Lambda^A{}_B$ from the spin system to itself. Using the fact that $\Lambda^A{}_B$ is nondegenerate, Hermitian and ϵ preserving, one can show that its components

$$\Lambda^{\mathrm{A}}{}_{\mathrm{B}} \equiv e^{\mathrm{A}}{}_A \Lambda^A{}_B e^B{}_{\mathrm{B}}$$

constitute a SU(2) matrix. Conversely, every SU(2) matrix $\Lambda^{\mathrm{A}}{}_{\mathrm{B}}$ defines an isomorphism of the spin system.

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[1]A. Ashtekar, in *Quantum Concepts in Space and Time,* edited by C. J. Isham and R. Penrose (Oxford University Press, Oxford, 1986); Phys. Rev. Lett. **57**, 2244 (1986); in *Constraint's Theory and Relativistic Dynamics,* edited by G. Longhi and L. Lusanna (World Scientific, Singapore, 1987).

[2]T. Jacobson and L. Smolin, Report No. YTP 87-29, 1987 (unpublished).

[3]P. Renteln and L. Smolin, Nucl. Phys. (to be published).

[4]See, e.g., R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation, An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962); K. Kuchǎr, in *Quantum Gravity 2,* edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford University Press, Oxford, 1980).

[5]A. Fischer, J. Marsden, and V. Moncrief, Ann. Inst. Henri Poincaré **33**, 147 (1980); J. Arms, J. Marsden, and V. Moncrief, Commun. Math. Phys. **78**, 455 (1982).

[6]A. Fischer (private communication to T. Jacobson).

[7]See, e.g., M. Ko, M. Ludvigsen, and K. P. Tod, Phys. Rep. **71**, 51 (1981).

[8]M. Ludvigsen, E. T. Newman, and K. P. Tod, J. Math. Phys. **22**, 818 (1981).

[9]A. Sen, J. Math. Phys. **22**, 1718 (1981); Phys. Lett. **119B**, 89 (1982).

[10]The falloff refers to components in a Cartesian chart (near infinity) of e_{ab}.

[11]If N and N^a approach constant values asymptotically as $1/r$, one can give meaning to the integrals in (7) and (8) by first integrating on a finite region of Σ and then, in the result, letting the boundary of the region approach spatial infinity. [See discussion after Eq. (11).] Even if this were done, the functional derivatives of $C_N\alpha(q,p)$ and $C_N(q,p)$, so defined, fail to exist, when $C_N\alpha$ and C_N cannot define an infinitesimal canonical transformation.

[12]The case with derivative couplings is discussed in K. Kuchǎr, J. Math. Phys. **18**, 1589 (1977).

[13]This result was also obtained by Ashtekar and Horowitz. See, e.g., A. Ashtekar, *Proceedings of the VIIth International Conference on Mathematical Physics,* Boulder, Colorado, 1983 (Elsevier, Amsterdam, 1984).

[14]A. Ashtekar, G. T. Horowitz, and A. Magnon, Gen. Relativ. Gravit. **14**, 411 (1982).

[15]See, e.g., R. Penrose and W. Rindler, *Spinors and Space-time* (Cambridge University Press, Cambridge, England, 1985), Vol. 1.

[16]K. S. Narain (private communication.)

[17]In the Hamiltonian framework, it is more natural to use a lapse density even if one works with the traditional variables (q_{ab}, p^{ab}): The equations of motion generated by the Hamiltonian (12) are equivalent to the projected Einstein equation, $q_a{}^m q_b^n G_{mn} = 0$, only on the constraint surface, whereas those generated by $H_T' \equiv \int T(\det q)^{1/2} C$ are equivalent to the projected Einstein equation everywhere on the phase space.

[18]A. Ashtekar, T. Jacobson, and L. Smolin, Report No. YTP 87-27, 1987 (unpublished); A. Ashtekar, in *Proceedings of the '86 Santa Cruz Conference on Mathematical General Relativity,* edited by J. Isenberg (American Mathematical Society, Providence, 1987).