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Symmetry reduction for quantized diffeomorphism-invariant theories of connections

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Abstract. Given a symmetry group acting on a principal fibre bundle, symmetric states of the quantum theory of a diffeomorphism-invariant theory of connections on this fibre bundle are defined. These symmetric states, equipped with a scalar product derived from the Ashtekar–Lewandowski measure for loop quantum gravity, form a Hilbert space of their own. Restriction to this Hilbert space yields a quantum symmetry reduction procedure within the framework of spin-network states, the structure of which is analysed in detail. Three illustrating examples are discussed: reduction of (3 + 1)- to (2 + 1)-dimensional quantum gravity, spherically symmetric quantum electromagnetism and spherically symmetric quantum gravity. In the latter system the eigenvalues of the area operator applied to the spherically symmetric spin-network states have the form $A_n \propto \sqrt{n(n+2)}$, $n = 0, 1, 2, \ldots$, giving $A_n \propto n$ for large n. This result clarifies (and reconciles) the relationship between the more complicated spectrum of the general (non-symmetric) area operator in loop quantum gravity and the old Bekenstein proposal that $A_n \propto n$.

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1. Introduction

Over the last few years there have been many very active and partially successful attempts to quantize gravitational systems. One of them is loop quantum gravity (see the reviews [1]) which uses a (non-perturbative) canonical framework (Cauchy surfaces, canonical coordinates and momenta, constraints, etc) for diffeomorphism-invariant gravity formulated in terms of connection variables, without or with couplings to matter.

Among its achievements loop quantum gravity counts the following: the uncovering of a discrete structure of space [2–5]; a derivation of the Bekenstein–Hawking formula for the black hole entropy in terms of microscopic degrees of freedom [6–10]; and a regularization and partial solution of the constraints of general relativity [11–14]. All of these results rely heavily on the fundamental assumption that holonomies (Wilson loops) of the $SU(2)$ gauge connection of general relativity in the Ashtekar formulation with *real* connections [15, 16] become densely defined operators in the quantum theory.

In order to test the basic assumptions of that approach and in order to check whether the theory can have the correct classical limit, mini- and midi-superspace models [17] are very appropriate because their (strongly) reduced number of degrees of freedom make them more transparent and sometimes even solvable.

These models are usually obtained by a classical symmetry reduction of the full theory and they are selected in order to facilitate quantization, while keeping some of the basic problems faced in the quantization of general relativity. An example is the reduction of the $(3+1)$-dimensional theory to a $(2+1)$-dimensional one. Here a simplification occurs because $(2+1)$-dimensional gravity has only finitely many physical degrees of freedom and is exactly soluble [18]. This model has already been investigated in the loop quantization approach [19] starting from a classical $(2+1)$-dimensional theory of gravity in terms of Ashtekar variables.

Our strategy will be different. We want to isolate symmetric states of the already loop-quantized $(3+1)$-dimensional full theory. These states have to be exactly symmetric, not only symmetric at large distances compared with the Planck scale. Constraining the space of physical states with these symmetric ones amounts to a symmetry reduction on the quantum level. This procedure depends, of course, on the definition of symmetric states; all our discussions are based on definition 3.1 of section 3 below. It represents a quantum version of the symmetry concepts as developed in the context of classical Einstein–Yang–Mills theories and their associated fibre bundles and connections [20–22]. By making explicit use of invariant connections it is well suited to our concept of symmetry reduction and selects the correct number of physical degrees of freedom. This is illustrated by our example of spherically symmetric electromagnetism: classically a multipole expansion shows that there are only two parameters, electric and magnetic charge, corresponding exactly to the result for symmetric states of the quantum theory (see section 4.2). By definition, the space of symmetric states to be considered here is equivalent to the state space obtained by a loop quantization of a *classically* symmetry reduced theory *if* one is interested only in the reduced model; however, our construction provides more than just a quantization of a reduced model because we are able to interpret any quantum state of the reduced theory as a symmetric state of the full quantum theory. This justifies the name 'symmetry reduction at the quantum level' since we can equally well view the reduced quantum theory as a subsector of the non-symmetric quantum theory without recourse to the classical level.

One could also imagine defining symmetric states as states which are invariant with respect to an action of the symmetry group on the Hilbert space, which would be the usual procedure applied to conventional quantum mechanical systems. In a diffeomorphism-invariant quantum theory, however, this is inappropriate because the symmetry group is a subgroup of the diffeomorphism group. For a *non-compact* space manifold one can interpret the symmetries as global asymptotic transformations at spatial infinity which are not regarded as elements of the gauge group [23]. However, analogously to the diffeomorphism group which does not act strongly continuously, there are no non-trivial states which are invariant with respect to space transformations. One could use group-averaging techniques [11] to find invariant generalized states but, as will be discussed below for the rotation group, this does not lead to the desired results for the purposes of this paper. If, on the other hand, the space manifold is *compact*, the symmetry group is always a subgroup of the diffeomorphism gauge group and diffeomorphism-invariant states would automatically be invariant with respect to the symmetry group. It would then be impossible to select, for example, translation-invariant states for cosmological models in a non-trivial manner. Therefore, we use our more restrictive definition of symmetric states in a diffeomorphism-invariant quantum theory, which allows for such a selection.

Because the solutions to all the constraints of loop quantum gravity are not known we will carry out this symmetry reduction procedure on an auxiliary Hilbert space. Therefore, we will have to regularize and to solve the reduced constraints on our spaces of symmetric states. As always with reduced models, the hope is that this regularization and the search for solutions can be done with more ease. At the same time one hopes that the model under consideration can lead to new insights in the quantization of the full theory. However, one should also note that the properties of symmetric states in the *physical* Hilbert space on which all constraints are solved may differ from those symmetric states in the auxiliary Hilbert space which solve the *reduced* constraints. We have nothing to say about this problem in the present paper.

Our main motive for the analysis of the present paper has its origin in our interest in the reduction to spherical symmetry. The classically reduced Schwarzschild system has been quantized using Ashtekar variables (but not using loop quantization techniques) in [24, 25]. A corresponding analysis has been performed for the Reissner–Nordstrøm model [26]. These models are of physical interest because they are related to vacuum black holes. Similar to $(2 + 1)$-dimensional gravity they have only finitely many physical degrees of freedom, and it is an interesting question as to how this reduction of infinitely many degrees of freedom of the full theory takes place in a loop quantization using spin-network states.

Another motivation is to possibly find a way to calculate the degeneracy of energy levels. The levels are not degenerate in the quantization of the classically reduced theory; however, this approach makes use only of smooth fields, whereas loop quantum gravity relates the black hole entropy to distributional configurations (generalized functions). Also, non-spherical fluctuations should not be ignored. The degeneracy plays a crucial role in the calculation [27, 28] of black hole entropy using a canonical partition function approach. The entropy obtained in this way is proportional to the horizon area, but the constant of proportionality is only known to be $O(1)$, so that a quantitative comparison with the semiclassical $A/4$ -law is not immediate. A similar problem arises in the loop quantum gravity calculation [10] because of the so-called Immirzi parameter [29]. A comparison of the entropy of the full theory with that of the spherically symmetric one may shed some light on the origin of the degeneracies because degenerate states of the symmetric model may become non-degenerate for the non-symmetric one.

Another possible application is to study cosmological models within loop quantum gravity [30].

In order to attack these physical questions we have to define symmetric states in the Hilbert space of loop quantum gravity and determine their properties. This will be done in the present paper in a more general setting. We will investigate the case of a Lie symmetry group S acting on a principal fibre bundle with a compact Lie structure group G.

Before going to this general case let us dwell briefly on spherically symmetric quantum gravity. The states of the (non-spherically symmetric) full theory are given in terms of a polymer-like structure called a spin network, lying in the spacelike section of spacetime used to carry out a canonical quantization. We now have to face the problem of how to establish a symmetry of a discrete structure under a continuous symmetry group. A possible approach is suggested by the well known solution of the diffeomorphism constraint using group-averaging methods [11]. However, this cannot lead to the desired result here. We would have to average not only over rotated subgraphs of the graph underlying a given spin network which is to be averaged, but it would be necessary to average over all rotations of an edge while keeping the other edges fixed because these give the same holonomy when evaluated in an invariant connection. Otherwise the holonomy of an edge as a multiplication operator would not commute with rotations. Therefore, some averaging of parts of a given graph has to be done, which, however, runs into problems when gauge invariance has to be imposed. In some sense the rotation group is too rigid compared with the diffeomorphism group. It acts only simply transitively on its orbits, whereas the diffeomorphism group acts k-transitively for any $k \in \mathbb{N}$, i.e. any two given sets of k different points can be mapped one onto another by a single diffeomorphism.

A lesson, however, can nevertheless be drawn from group averaging. Symmetric states have to be distributions (generalized functions) on the function space over the quantum configuration space. This can be understood from the following intuitive picture. Constraining the support of a spin-network function to only symmetric connections yields a singular distribution. This observation will guide us in our definition of symmetric states.

In order to achieve this we will use the theory of invariant connections on symmetric principal fibre bundles [20–22]. The essential properties and results will be recalled in the next section, together with some mathematical techniques of loop quantization. That section will also serve to fix our notation. Section 3 deals with the definition of symmetric states and the analysis of their properties; it contains entirely new material. Our main result is theorem 3.2, by means of which we can identify spaces of symmetric states with certain spin-network spaces.

In section 4 we will discuss some examples: quantum symmetry reduction to (2 + 1)-dimensional gravity and spherically symmetric electromagnetism as well as gravitation. These examples are intended to illustrate the 'kinematical' framework of the symmetry reduction proposed here. Solving the corresponding constraints is another task.

As to (2+1)-dimensional gravity we shall show how our approach leads to the kinematical Hilbert space derived and used by Thiemann [19]. Spherically symmetric diffeomorphism-invariant electromagnetism will be treated for a vanishing gravitational field only. It is nevertheless an interesting example for the symmetry reduction of a diffeomorphism-invariant system, and it illustrates the reduction of degrees of freedom to finitely many ones and also the classifying role of the magnetic charge.

The discussion of spherically symmetric gravity is mainly restricted to kinematical aspects: the symmetry reduction is implemented, the Gauß constraint solved in the context of the appropriate spin networks and the solution of the diffeomorphism constraint by group averaging indicated. The problem of dealing with the more difficult Hamiltonian constraint (a well defined operator on spin-network states and the solution of the constraint) will be discussed elsewhere [31].

As an important application of our approach we determine in section 5 the spectrum of the area operator acting in the spherically symmetric sector of loop quantum gravity. The resulting eigenvalues are

$$A(j) \propto \sqrt{j(j+1)}, \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots.$$

For large j this leads to the old Bekenstein area spectrum obtained from a Bohr–Sommerfeld quantization.

2. Preparations

In this section we will recall some facts concerning loop quantization techniques and the theory of invariant connections on symmetric principal fibre bundles which will be used in the following sections.

2.1. Spin networks with Higgs field vertices

Let G be a compact Lie group, Σ an analytic manifold and $P(\Sigma, G, \pi)$ a principal fibre bundle over the base manifold Σ with structure group G. The affine space of connections on this fibre bundle will be denoted as $\mathcal{A}$, and the (local) gauge group as $\mathcal{G}$. Investigating invariant connections will lead us to the use of Higgs fields, which are sections of the adjoint bundle of P (the associated vector bundle employing the adjoint representation). The space of all smooth Higgs fields will be called $\mathcal{U}$. These three function spaces will have to be extended in the course of quantization. Their treatment is given in [11] for $\mathcal{A}$ and $\mathcal{G}$, and in [32] for $\mathcal{U}$. In the following we will combine these procedures.

Let $\mathcal{T}$ be the parallel transport algebra generated by elements of the matrix-valued parallel transporters associated with a fundamental representation of G along all piecewise-analytic paths in Σ, subject to appropriate relations ensuring the correct matrix multiplication under composition of paths and taking care of the group properties of G (incorporating Mandelstam identities).

Similarly, let $\mathcal{P}$ be the point holonomy algebra generated by elements of matrices, including the identity, in a fundamental representation of G of point holonomies. According to [32] a point holonomy is a function on the space $\mathcal{U}$ of classical Higgs fields obtained by exponentiating the value of a Higgs field at a given point.

The multiplication within the algebras is a multiplication of $\mathbb{C}$-valued functions on $\mathcal{A}$ and $\mathcal{U}$, respectively. Let $\overline{\mathcal{T}}$ and $\overline{\mathcal{P}}$ be their completions in the sup norm. These two algebras are Abelian $C^\star$-algebras with identity. We can build the product algebra $\overline{\mathcal{T}} \otimes \overline{\mathcal{P}}$ which is the completed tensor product space of the underlying vector spaces with pointwise multiplication. This, too, is an Abelian $C^\star$-algebra with identity and we can use Gel'fand–Neumark theory (see, e.g., [33]) to obtain the following isometry of $C^\star$-algebras:

$$\overline{\mathcal{T}} \otimes \overline{\mathcal{P}} \cong C(\overline{\mathcal{A}}) \otimes C(\overline{\mathcal{U}}) \cong C(\overline{\mathcal{A}} \times \overline{\mathcal{U}}). \tag{1}$$

$\overline{\mathcal{A}}$ and $\overline{\mathcal{U}}$ are the Gel'fand spectra of the respective algebras consisting of all continuous $\star$-homomorphisms of the algebras to $\mathbb{C}$. The isometry is given by the Gel'fand transform $\hat{}\colon \overline{\mathcal{T}} \to C(\overline{\mathcal{A}})$ defined below, and similarly for $\overline{\mathcal{U}}$. $\overline{\mathcal{A}}$ and $\overline{\mathcal{U}}$ are extensions of the spaces $\mathcal{A}$ and $\mathcal{U}$ which are densely embedded in the Gel'fand topology. This topology is uniquely defined by the following two conditions: $\overline{\mathcal{A}}$ should be compact and $\hat{T}\colon \overline{\mathcal{A}} \to \mathbb{C}$, $A \mapsto \hat{T}(A) = A(T)$ should be continuous for all $T \in \overline{\mathcal{T}}$. The compact Hausdorff space $\overline{\mathcal{A}} \times \overline{\mathcal{U}}$ will serve as the quantum configuration space.

There is, however, an alternative construction of these spaces which is better suited for calculations. Here the extensions are constructed as certain projective limit spaces. The partially ordered directed set used to define these limits is the set Γ of all piecewise-analytic graphs γ in Σ. The projective families $(\mathcal{A}_\gamma, p_{\gamma\gamma'})$ and $(\mathcal{G}_\gamma, p_{\gamma\gamma'}) = (\mathcal{U}_\gamma, p_{\gamma\gamma'})$ can be found in [11, 34]. $\mathcal{A}_\gamma$ is the space of all functions which assign elements of the group G to the edges of γ, and which obey certain relations ensuring the correct behaviour under inversion and composition of edges. The elements of $\mathcal{G}_\gamma$ and $\mathcal{U}_\gamma$ assign group elements to the vertices of γ. The projections $p_{\gamma\gamma'}\colon \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma$, $\gamma \subset \gamma'$ restrict the domain of definition of the connections in $\mathcal{A}_{\gamma'}$ to the edges of γ, and similarly for $\mathcal{G}_\gamma$ and $\mathcal{U}_\gamma$. $\mathcal{G}_\gamma$ acts on both $\mathcal{A}_\gamma$ and $\mathcal{U}_\gamma$ by usual gauge transformations. The projective families define the projective limits $\overline{\mathcal{A}}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{U}}$, where now $\overline{\mathcal{G}}$ acts on $\overline{\mathcal{A}}$ and $\overline{\mathcal{U}}$. These projective limit spaces are identical to the Gel'fand spectra constructed above, and their topology induced from the Tychonov topology is equivalent to the Gel'fand topology. This can be seen from the fact that $\overline{\mathcal{A}}$ is—as the projective limit of compact spaces—compact, and that the maps $\hat{T}$ are continuous. In the proof of theorem 3.1 we will make use of the equivalence of Gel'fand and Tychonov topologies.

Our quantum configuration space now is the space $\overline{\mathcal{A}} \times \overline{\mathcal{U}}$ and the auxiliary Hilbert space will consist of functions on this space. An important class of such functions is that of cylindrical functions which depend on the connection and Higgs field only via a finite number of edges and vertices in Σ. A function f, cylindrical with respect to the underlying graph γ, can be written as

$$f(A, U) = f_\gamma(A(e_1), \ldots, A(e_n), U(v_1), \ldots, U(v_m)) \tag{2}$$

where $e_1, \ldots, e_n$ are the edges of γ, and $v_1, \ldots, v_m$ its vertices. The functions f_γ representing a cylindrical function f have to obey certain consistency conditions. The auxiliary Hilbert space is $L_2(\overline{\mathcal{A}} \times \overline{\mathcal{U}}, \mathrm{d}\mu_{AL})$ obtained from the space of cylindrical functions by completion with respect to the Ashtekar–Lewandowski measure $\mathrm{d}\mu_{AL}$, which is, on a cylindrical subspace, the finite product of Haar measure on G for each edge and vertex of the respective graph γ.

An orthogonal basis is given by the set of spin-network functions with Higgs field vertices. These are cylindrical functions given by a graph γ together with a labelling j of its edges, and labellings j', j'' of its vertices with equivalence classes of irreducible representations of G, and a third labelling C of the vertices with certain intertwining operators. Given a vertex v, C_v is given by an intertwining operator from the tensor product of the representations j_e labelling incoming edges e and the representation j'_v to the tensor product of the representations j_e labelling outgoing edges and the representations j'_v and j''_v. The value of a spin-network function on an element $(A, U) \in \overline{\mathcal{A}} \times \overline{\mathcal{U}}$ is found by taking for each edge $e \subset \gamma$ the element $A(e)$ in the representation j_e and for each vertex $v \in \gamma$ the element $U(v)$ in the representation j'_v, and then contracting these matrices according to the intertwining operators in the vertices. The resulting function will transform according to the representation j''_v in each vertex v. In particular, the spin-network function will be gauge invariant if all the representations j''_v are trivial.

2.2. *Invariant connections on symmetric principal fibre bundles*

It is well known [20–22] that an invariant connection on a manifold Σ can be decomposed into a reduced connection of a reduced gauge group on a submanifold $B \subset \Sigma$ plus some scalar fields on B acted on by a group action determined by the symmetry reduction (a representation of the reduced structure group). The multiplet of scalar fields will be called a 'Higgs field' in the following. It arises because in general an invariant connection is not manifestly invariant, but only invariant up to gauge transformations. For example, the authors of [35, 36] make an ansatz for a spherically symmetric connection using the fact that a symmetry transformation can be compensated by a gauge transformation if the Lie algebra of the structure group contains an $su(2)$ subalgebra.

In this paper we will use a more general and more systematic approach which yields a complete classification of invariant connections on symmetric principal fibre bundles. It can be found in [20–22], and its main elements will be recalled in the present subsection. The method has the following advantages.

- The structure of the reduction of the gauge group and the appearance of Higgs fields becomes clearer.
- All partial gauge fixings (selections of a certain homomorphism $\lambda \in [\lambda]$ defined below) can be treated on the same footing (and eventually be relaxed), whereas the ansatz of [35, 36] amounts to selecting one special λ, i.e. a partial gauge fixing.
- A possible topological charge given by gauge inequivalent actions of the symmetry group in the fibres can be taken into account. This is excluded by the ansatz of [35, 36] from the outset by using trivial bundles and a fixed action of the symmetry group on the bundle only.

Whereas the first two points will be essential for constructing symmetry reductions in the spin-network context (section 3), the last point is needed for generality and allows us to describe, for example, a magnetic charge.

Now let $S < \mathrm{Aut}(P)$ be a Lie symmetry subgroup of bundle automorphisms acting on the principal fibre bundle $P(\Sigma, G, \pi)$ defined above. Using the projection $\pi\colon P \to \Sigma$ we obtain a symmetry operation of S on Σ. For simplicity we will assume that all orbits of S are of the same type; if necessary we will have to decompose the base manifold in several orbit bundles $\Sigma_{(F)} \subset \Sigma$, where $F = S_x$ is the isotropy subgroup of S consisting of elements fixing a point

x of the orbit bundle $\Sigma_{(F)}$. This amounts to a special treatment of possible symmetry axes or centres, respectively.

By restricting ourselves to one fixed orbit bundle we fix an isotropy subgroup $F \leqslant S$ and we require that the action of S on Σ is such that the orbits are given by $S(x) \cong S/F$ for all $x \in \Sigma$. This will be the case if S is compact. Moreover, we have to assume that the coset space S/F is reductive [20], i.e. $\mathcal{L}S$ can be decomposed as a direct sum $\mathcal{L}S = \mathcal{L}F \oplus \mathcal{L}F_\perp$ with $\mathrm{Ad}_F(\mathcal{L}F_\perp) \subset \mathcal{L}F_\perp$. If S is semisimple, $\mathcal{L}F_\perp$ is the orthogonal complement of $\mathcal{L}F$ with respect to the Cartan–Killing metric on $\mathcal{L}S$. The base manifold can then be decomposed as $\Sigma \cong \Sigma/S \times S/F$ where $\Sigma/S \cong B \subset \Sigma$ is the base manifold of the orbit bundle and it can be realized as a submanifold B of Σ via a section in this bundle.

Given a point $x \in \Sigma$, the action of the isotropy subgroup F yields a map $F: \pi^{-1}(x) \to \pi^{-1}(x)$ of the fibre over x commuting with the right action of the bundle. To each point $p \in \pi^{-1}(x)$ we can assign a group homomorphism $\lambda_p: F \to G$ defined by $f(p) =: p \cdot \lambda_p(f)$ for all $f \in F$. For a different point $p' = p \cdot g$ in the same fibre we obtain, using commutativity of the action of $S < \mathrm{Aut}(P)$ with right multiplication of G on P, the conjugated homomorphism $\lambda_{p'} = \mathrm{Ad}_{g^{-1}} \circ \lambda_p$. This construction yields a map $\lambda: P \times F \to G$, $(p, f) \mapsto \lambda_p(f)$ obeying the relation $\lambda_{p \cdot g} = \mathrm{Ad}_{g^{-1}} \circ \lambda_p$.

Given a fixed homomorphism $\lambda: F \to G$, we can build the principal fibre sub-bundle

$$Q_\lambda(B, Z_\lambda, \pi_Q) := \{p \in P_{|B} : \lambda_p = \lambda\} \tag{3}$$

over the base manifold B the structure group of which is the centralizer $Z_\lambda := Z_G(\lambda(F))$ of $\lambda(F)$ in G. $P_{|B}$ is the restricted fibre bundle over B. A conjugated homomorphism $\lambda' = \mathrm{Ad}_{g^{-1}} \circ \lambda$ will lead to an isomorphic fibre bundle.

The structure elements $[\lambda]$ and Q classify symmetric principal fibre bundles according to the following theorem [22].

Theorem 2.1. *A S-symmetric principal fibre bundle $P(\Sigma, G, \pi)$ with the isotropy subgroup $F \leqslant S$ of the action of S on Σ is uniquely characterized by a conjugacy class $[\lambda]$ of homomorphisms $\lambda: F \to G$ together with a* reduced bundle $Q(\Sigma/S, Z_G(\lambda(F)), \pi_Q)$.

Given two groups F and G we can make use of the relation [39]

$$\mathrm{Hom}(F, G)/\mathrm{Ad} \cong \mathrm{Hom}(F, T(G))/W(G) \tag{4}$$

in order to determine all conjugacy classes of homomorphisms $\lambda: F \to G$. Here $T(G)$ is a standard maximal torus and $W(G)$ is the Weyl group of G.

Now let ω be an S-invariant connection on the bundle P classified by $([\lambda], Q)$. The connection ω induces a connection $\tilde{\omega}$ on the reduced bundle Q. Because of the S-invariance of ω the reduced connection $\tilde{\omega}$ is a 1-form on Q with values in the Lie algebra of the reduced structure group. Furthermore, by using ω we can construct the linear map $\Lambda_p: \mathcal{L}S \to \mathcal{L}G$, $X \mapsto \omega_p(\tilde{X})$ for any $p \in P$. Here $\tilde{X}$ is the vector field on P given by $\tilde{X}(f) := \mathrm{d}(\exp(tX)^\star f)/\mathrm{d}t|_{t=0}$ for any $X \in \mathcal{L}S$ and $f \in C^1(P, \mathbb{R})$. For $X \in \mathcal{L}F$ the vector field $\tilde{X}$ is a vertical vector field, and we have $\Lambda_p(X) = \mathrm{d}\lambda_p(X)$ where $\mathrm{d}\lambda: \mathcal{L}F \to \mathcal{L}G$ is the derivative of the homomorphism defined above. This component of Λ is therefore already given by the classifying structure of the principal fibre bundle. Using a suitable gauge, λ can be held constant along B. The remaining components $\Lambda_p|_{\mathcal{L}F_\perp}$ yield information about the invariant connection ω. They are subject to the condition

$$\Lambda_p(\mathrm{Ad}_f(X)) = \mathrm{Ad}_{\lambda_p(f)}(\Lambda_p(X)) \qquad \text{for} \quad f \in F, \quad X \in \mathcal{L}S \tag{5}$$

which follows from the transformation of ω under the adjoint representation and which provides a set of equations which determine the Higgs field.

Keeping only the information characterizing ω we have, besides $\tilde{\omega}$, the Higgs field $\phi\colon Q \to \mathcal{L}G \otimes \mathcal{L}F_\perp^\star$ determined by $\Lambda_p|_{\mathcal{L}F_\perp}$. The reduced connection and the Higgs field suffice to characterize an invariant connection. This is the assertion of the following theorem [22].

Theorem 2.2 (Generalized Wang theorem). *Let $P(\Sigma, G)$ be an S-symmetric principal fibre bundle classified by $([\lambda], Q)$ according to theorem 2.1 and let ω be an S-invariant connection on P.*

Then the connection ω is uniquely classified by the reduced connection *$\tilde{\omega}$ on Q and the Higgs field $\phi\colon Q \times \mathcal{L}F_\perp \to \mathcal{L}G$ obeying equation (5).*

In general, ϕ will transform under some representation of the reduced structure group Z_λ. The Higgs field lies in the subspace of $\mathcal{L}G$ determined by equation (5). It forms a representation space of all group elements of G (which act on Λ) whose action preserves the Higgs subspace. These are precisely elements of the reduced group by definition.

The connection ω can be reconstructed from its classifying structure as follows. According to the decomposition $\Sigma \cong B \times S/F$ we have $\omega = \tilde{\omega} + \omega_{S/F}$, where $\omega_{S/F}$ is given by $\Lambda \circ \iota^\star \theta_{MC}$ in a gauge depending on the (local) embedding $\iota\colon S/F \hookrightarrow S$. Here θ_{MC} is the Maurer–Cartan form on S. For example, in the generic case (not in a symmetry centre) of spherical symmetry we have $S = SU(2)$, $F = U(1) = \exp\langle\tau_3\rangle$ ($\langle\cdot\rangle$ denotes the linear span), and the connection form can be gauged to be

$$A_{S/F} = (\Lambda(\tau_2)\sin\vartheta + \Lambda(\tau_3)\cos\vartheta)\,\mathrm{d}\varphi + \Lambda(\tau_1)\,\mathrm{d}\vartheta. \tag{6}$$

Here (ϑ, φ) are coordinates on $S/F \cong S^2$. The τ_j build a basis of $\mathcal{L}S$ and are given by $\tau_j := -\frac{1}{2}\mathrm{i}\sigma_j$, with σ_j being the Pauli matrices. $\Lambda(\tau_3)$ is given by $\mathrm{d}\lambda$, whereas $\Lambda(\tau_{1,2})$ are the Higgs field components.

Equation (6) contains as special cases the invariant connections found in [36]. These are gauge equivalent by gauge transformations depending on the angular coordinates (ϑ, φ), i.e. they correspond to homomorphisms λ which are not constant on the orbits of the symmetry group.

3. Symmetric states

Before describing the rather abstract construction of symmetric spin-network states we will present the general idea in a first subsection. The following subsections deal with the construction of symmetric states as generalized states of the unreduced theory and proofs of some of their properties.

3.1. Principal idea

The principal idea of our construction [37] described in this section is to make use of the reconstruction of an invariant connection from its classifying structure, namely by means of the pull-back of a function on the space of connections on Σ to a function on the space of connections plus Higgs fields on the reduced manifold B which in the context of analytic spin networks will be assumed to be an analytic submanifold of Σ.

However, some complications arise because of the classical partial gauge fixing by selecting a special homomorphism $\lambda \in [\lambda]$. The reduced gauge group and the space of Higgs fields depend on this selection. Moreover, the Higgs field does in general not transform under the adjoint representation of the reduced structure group, which would be helpful in spin-network quantization. In contrast, before imposing the constraint (5) it transforms in general under the adjoint representation of the *unreduced* structure group. Such a Higgs field can easily be implemented in the spin-network context using the rules recalled in section 2.1.

The inter-relation of partial gauge fixings and reductions of the gauge group makes it possible to eliminate partial gauge fixings by using the full gauge group on the reduced manifold. This is the essence of definition 3.3 below.

3.2. Construction

Let us now define the notion of symmetric states in the Hilbert space $\mathcal{H}_\Sigma := L_2(\overline{\mathcal{A}}_\Sigma, \mathrm{d}\mu_{AL})$. Because of the singular character of symmetric states mentioned in the introduction we will have to use the rigged Hilbert space $\Phi_\Sigma \subset \mathcal{H}_\Sigma \subset \Phi'_\Sigma$, where Φ_Σ denotes the space of cylindrical functions on the space $\overline{\mathcal{A}}_\Sigma$ of connections over Σ and Φ'_Σ its topological dual.

Definition 3.1 (Symmetric states). *Let P be an S-symmetric principal fibre bundle, classified by* $([\lambda], Q)$.

A $[\lambda]$-symmetric state *is a distribution $\psi \in \Phi'_\Sigma$ on Φ_Σ whose support contains only connections that are invariant under the S-action on P classified by* $[\lambda]$.

Although definition 3.1 catches the intuitive notion of a symmetric state, it is not well suited for a calculus. We have to develop some tools in analogy to the spin-network calculus. This will be done in the remaining part of this section by combining techniques collected in the preceding section. Application of these techniques will lead to several spaces of connections which are defined in

Definition 3.2. *Let $P(\Sigma, G)$ be a principal fibre bundle acted on by a symmetry group S according to the classification* $([\lambda], Q)$*, where Q is the reduced bundle over the manifold $B \subset \Sigma$.*

$\overline{\mathcal{A}}_\Sigma$ and $\overline{\mathcal{G}}_\Sigma$ are the space of generalized connections on P and the extended local gauge group, respectively. $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B$ is the space of generalized G connections and Higgs fields in the adjoint representation of G over B. $\overline{\mathcal{G}}_B$ is the extended local gauge group of generalized G gauge transformations over B.

For any $\lambda' \in [\lambda]$, $(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^{\lambda'}$ is defined to be the subset of $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B$ subject to the following constraints. The generalized connections take values in the structure group $Z_{\lambda'}$ of the reduced bundle Q, and the generalized Higgs fields take values in the submanifold of G obtained by exponentiating the linear solution space of equation (5). Here we have to use a separate Higgs field component for every element of a basis of $\mathcal{L}F_\perp$.

Remark. The space $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B$ is independent of the reduction of the structure group and the constraints (5) on the Higgs field. These affect only the definition of $(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^{\lambda'}$ which depends explicitly on the homomorphism λ', not only on its conjugacy class. Therefore, for any $\lambda' \in [\lambda]$ we have a separate space of connections and Higgs fields because already the reduced structure group may depend on λ'. We can eliminate this redundancy by factoring out the gauge group, but this has to be done with care due to the classical reduction of the gauge group.

In order to achieve our goal, we will make use of the classifying structure $(\tilde{\omega}, \phi)$ of a $[\lambda]$-invariant connection ω. Below theorem 2.2, we described the reconstruction of ω from its classifying structure. This reconstruction defines a continuous map

$$r^{(i)}_{\lambda'}\colon (\mathcal{A} \times \mathcal{U})^{\lambda'} \to \overline{\mathcal{A}}_\Sigma$$

which can be continued uniquely to a continuous map

$$r^{(\iota)}_{\lambda'} \colon (\overline{\mathcal{A}} \times \overline{\mathcal{U}})^{\lambda'} \to \overline{\mathcal{A}}_\Sigma.$$

As the notation indicates, this map depends not only on the homomorphism $\lambda' \in [\lambda]$, but also on the embedding $\iota \colon S/F \hookrightarrow S$.

Because a different ι would reconstruct a gauge-equivalent connection form, the dependence on ι can be eliminated by factoring out the gauge group on P. This leads us to the family of maps

$$r_{\lambda'} \colon (\overline{\mathcal{A}} \times \overline{\mathcal{U}})^{\lambda'} \to \overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma. \tag{7}$$

The dependence on λ' (as opposed to $[\lambda]$) is pure gauge: λ' can be changed arbitrarily in its conjugacy class $[\lambda]$ by applying a global transformation with a $g \in G$, $g \notin Z_\lambda$. This shows that the domains of definition of all the maps $r_{\lambda'}$ are, in fact, different, but that they are related by gauge transformations. This observation motivates the following.

Definition 3.3. *Let* $[\lambda]$ *be a conjugacy class of homomorphisms.*

Then $(\overline{\mathcal{A}} \times \overline{\mathcal{U}}/\overline{\mathcal{G}})^{[\lambda]}$ *is the subset of* $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B$ *consisting of all* $\overline{\mathcal{G}}_B$ *gauge equivalence classes containing a representative which lies in some* $(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^{\lambda'}$*,* $\lambda' \in [\lambda]$*.*

Remark. Because we allow here for any local gauge transformation, we relax the condition that λ' should be constant on B, which is imposed in the classical symmetry reduction procedure.

$\overline{\mathcal{G}}$-equivariance of $r_{\lambda'}$, which means that for any $\overline{\mathcal{G}}_B$ gauge transformation g_B there is a $\overline{\mathcal{G}}_\Sigma$ gauge transformation g_Σ with $r_{\lambda'} \circ g_B = g_\Sigma \circ r_{\lambda'}$, now allows us to factor out gauge transformations in the domains of definition of the maps $r_{\lambda'}$. Thereby we obtain a further map

$$r_{[\lambda]} \colon (\overline{\mathcal{A}} \times \overline{\mathcal{U}}/\overline{\mathcal{G}})^{[\lambda]} \to \overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma \tag{8}$$

which depends only on the conjugacy class $[\lambda]$.

We then have

Lemma 3.1. *The subset of generalized gauge invariant,* $[\lambda]$*-invariant connections in* $\overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma$ *is given by* $\mathrm{Im}(r_{[\lambda]})$*, the image of the map* $r_{[\lambda]}$*.*

Proof. This is clear from the construction of $r_{[\lambda]}$. □

Recall that our goal is to develop a calculus on the manifold of $[\lambda]$-invariant connections. Given a continuous function on this manifold, we can pull it back via $r_{[\lambda]}$ and so obtain a continuous function on $(\overline{\mathcal{A}} \times \overline{\mathcal{U}}/\overline{\mathcal{G}})^{[\lambda]}$. If this function can be continued to a function on $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B$, we will have the desired calculus on that space at our disposal. An extension can indeed be achieved by expanding the pulled-back function in the spin-network basis of $C(\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B)$, the space of continuous functions on the space of connections and Higgs fields over B.

In order to achieve uniqueness of this expansion we have to truncate the spin-network basis $\mathcal{B}$ of $C(\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B)$ if necessary in such a way that

$$\hat{\mathcal{B}} := \{T|_{(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^{[\lambda]}}\}_{T \in \mathcal{B}} \tag{9}$$

is a set of independent functions, for example, we have to use only spin-network states with trivial Higgs vertices if equation (5) does not allow any non-vanishing Higgs field.

This extension procedure following the pull-back with $r_{[\lambda]}$ finally yields the map

$$\overline{r}^{\star}_{[\lambda]}\colon C(\overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma) \to C(\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B) \tag{10}$$

which will provide the key element in our investigation of symmetric states.

However, a function pulled back in such a way is quite singular on the space $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B$. Even if we constrain the domain of definition of $\overline{r}^{\star}_{[\lambda]}$ to Φ_Σ, the space of cylindrical functions, the pull-back may in general not lead to a cylindrical or AL-integrable function on $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B$. The holonomy to a generic edge depends on all components of an unreduced connection in all its points, which leads to a continuous distribution of Higgs vertices. We, therefore, have again to use a rigged Hilbert space, this time

$$\Phi_B \subset \mathcal{H}_B \subset \Phi'_B. \tag{11}$$

Here, Φ_B is the space of cylindrical functions on $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B$, Φ'_B its topological dual and $\mathcal{H}_B := L_2(\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B, \mathrm{d}\mu_{AL})$ (again modulo relations which solve the Higgs constraint (5), which will be dealt with in more detail elsewhere; examples can be found in section 4.3 and in [30]).

The restriction of $\overline{r}^{\star}_{[\lambda]}$ to Φ_Σ can now be interpreted as an antilinear map

$$\rho_{[\lambda]}\colon \Phi_\Sigma \to \Phi'_B,$$

reminiscent of a group-averaging map. The pull-back of a cylindrical function $f \in \Phi_\Sigma$ is interpreted as a distribution on Φ_B according to ($g \in \Phi_B$)

$$\rho_{[\lambda]}(f)(g) := \int_{\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B} \mathrm{d}\mu_{AL}\, \overline{\overline{r}^{\star}_{[\lambda]} f}\, g. \tag{12}$$

This integral is defined in the following way. The function $\overline{r}^{\star}_{[\lambda]} f$ on $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B$ can be expanded to a (in general very complicated) series of spin-network functions with Higgs field vertices in B (see also the proof of lemma 3.4). Only those terms contribute to the integral which are cylindrical with respect to the same graph as g, and so the integral amounts essentially to a projection on this finite part.

In a similar way, we can interpret a cylindrical function $g \in \Phi_B$ as a distribution on Φ_Σ according to

$$\sigma_{[\lambda]}(g)(f) := \int_{\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B} \mathrm{d}\mu_{AL}\, \overline{g}\, \overline{r}^{\star}_{[\lambda]} f \tag{13}$$

where $\sigma_{[\lambda]}\colon \Phi_B \to \Phi'_\Sigma$ is the antilinear map given by this interpretation.

The situation can be summarized in the diagram

$$\begin{tikzcd}
\Phi'_B & & \Phi'_\Sigma \\
\cup & & \cup \\
\mathcal{H}_B & & \mathcal{H}_\Sigma \\
\cup & & \cup \\
\Phi_B \arrow[uuuurr, "\sigma_{[\lambda]}"'] & & \Phi_\Sigma \arrow[uuuull, "\rho_{[\lambda]}"]
\end{tikzcd}$$

with the duality relation

$$\sigma_{[\lambda]}(\overline{g})(f) = \rho_{[\lambda]}(\overline{f})(g) \qquad \text{for} \quad f \in \Phi_\Sigma, g \in \Phi_B \tag{14}$$

between the maps $\sigma_{[\lambda]}$, $\rho_{[\lambda]}$ connecting the two Gel'fand triples.

In general (see the remarks preceding equation (11)), $\rho_{[\lambda]}(\Phi_\Sigma)$ is not contained in Φ_B, and we cannot compose $\rho_{[\lambda]}$ and $\sigma_{[\lambda]}$ to obtain a map from Φ_Σ to Φ'_Σ. This is the main difference to a group-averaging map, which is aimed to solve a gauge constraint. Only in very special situations can the symmetry reduction be formulated analogously to a group averaging (section 4.2 and [38]). However, in general we have the two maps $\rho_{[\lambda]}$, which restricts an asymmetric state to its symmetric part, and $\sigma_{[\lambda]}$, which identifies symmetric states with spin-network states over B thereby equipping the space of symmetric states with a calculus.

3.3. Properties of the symmetric states

The goal of this subsection is to prove that the construction of the previous subsection yields all symmetric states. In order to achieve this we need some preparations.

Lemma 3.2. *Let G be a compact topological group which is Hausdorff and H be a subgroup of G. The centralizer $Z_G(H) := \{g \in G : gh = hg$ for all $h \in H\}$ is a compact subgroup of G.*

Proof. It is well known that $Z_G(H)$ is a subgroup [39]. It can be written as

$$Z_G(H) = \bigcap_{h \in H} G_h$$

where G_h is the isotropy subgroup of $h \in H$ under the adjoint action $G \times G \to G$, $(g, h) \mapsto ghg^{-1}$ of G on itself. Because the intersection of an arbitrary set of closed sets is closed, it suffices to prove that all isotropy subgroups G_h are closed.

The action of G on itself leads, restricted to a fixed element $h \in G$, to the continuous map $c_h: G \to G, g \mapsto ghg^{-1}$. $G_h = c_h^{-1}(h)$ is the pre-image of a closed set (because G is Hausdorff) under a continuous map, and hence closed. Now, $Z_G(H)$ is a closed subset of a compact group and therefore compact. □

Lemma 3.3. *Let P be an S-symmetric principal fibre bundle and $[\lambda]$ be a conjugacy class of homomorphisms classifying P together with the reduced bundle Q.*

The set of generalized gauge-invariant $[\lambda]$-invariant connections on P is closed in $\overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma$.

Proof. According to lemma 3.1 we have to show that the image of r_λ, which is identical to the image of $r_{[\lambda]}$, is closed.

We will start by showing that the domain of definition of r_λ, i.e. $(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^\lambda$, is compact. The elements of this space take values in a compact set, because we know in the first place from lemma 3.2 that Z_λ is compact. The generalized Higgs fields take values in a compact manifold, too. Such a Higgs field has several components given by the linear map $\phi: \mathcal{L}F_\perp \to \mathcal{L}G$ subject to the linear equation (5). Therefore, the values of ϕ lie in a linear subspace of $\mathcal{L}G$, and exponentiating yields generalized Higgs fields taking values in a compact submanifold of G.

$(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^\lambda$ can be constructed as a projective limit along the lines described in section 2.1. The projective family consists of compact spaces, because the maps involved assign elements of compact spaces to a finite number of edges and vertices of graphs. Therefore, the projective limit is compact in its induced Tychonov topology which is equivalent to the induced topology as a subset of $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B$.

Now, as stated in section 2.1, the Tychonov topology is equivalent to the Gel'fand topology. However, r_λ is continuous in the Gel'fand topology (being by construction a continuation of a continuous function on the dense subspace $(\mathcal{A} \times \mathcal{U})^\lambda$), and so the image of r_λ is compact. As a compact subset of a Hausdorff space it is closed. □

Theorem 3.1. *Let* $[\lambda]$ *be a conjugacy class of homomorphisms.*
The image of $\sigma_{[\lambda]}$ *contains only* $[\lambda]$*-symmetric states.*

Proof. According to definition 3.1 we have to prove that the support of a distribution in the image of $\sigma_{[\lambda]}$ contains only $[\lambda]$-invariant connections. This amounts to showing that for any non-invariant generalized connection $\overline{A} \in (\overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma) \setminus \mathrm{Im}(r_{[\lambda]})$ there is a neighbourhood U of $\overline{A}$ so that the restriction of any distribution $\psi \in \mathrm{Im}(\sigma_{[\lambda]})$ to U is the zero distribution.

Because of lemma 3.3 $\overline{A}$ has a neighbourhood which is entirely contained in $(\overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma) \setminus \mathrm{Im}(r_{[\lambda]})$. If we restrict ψ to this neighbourhood, it will be the zero distribution due to its very definition in equation (13). The pull-back with $r_{[\lambda]}$ of any function supported on the neighbourhood of $\overline{A}$ will be zero. □

This theorem provides us with a rich class of $[\lambda]$-symmetric states. Even better, we have a calculus on this class of states, because they are identified by equation (13) with the space Φ_B of cylindrical functions. Elements of Φ'_B can be regarded as generalized symmetric states.

The following lemma states that we have found enough symmetric states.

Lemma 3.4. *Let* $[\lambda]$ *be a conjugacy class of homomorphisms.*
There is no cylindrical function $f \in \Phi_\Sigma$ *that is non-trivial on* $\mathrm{Im}(r_{[\lambda]})$ *and annihilated by all distributions in* $\sigma_{[\lambda]}(\Phi_B)$*. In particular, the space* $\sigma_{[\lambda]}(\Phi_B)$ *separates the elements of* Φ_Σ *that differ when restricted to* $[\lambda]$*-invariant connections.*

Proof. Let $\psi \in \Phi'_\Sigma$ be a symmetric state. If $f, g \in \Phi_\Sigma$ are two cylindrical functions that are identical when restricted to $[\lambda]$-invariant connections, then $\psi(f) = \psi(g)$. Let us, therefore, introduce the following equivalence relation which respects the algebraic and topological structure of Φ_Σ. Two cylindrical functions $f \sim g$ are equivalent if and only if $\overline{r}^\star_{[\lambda]} f = \overline{r}^\star_{[\lambda]} g$. Symmetric states can be seen as functions on the space of equivalence classes, and we need to show that, if $[f] \neq [g]$, there is a distribution $\psi \in \sigma_{[\lambda]}(\Phi_B)$ with $\psi(f) \neq \psi(g)$.

Let us look now at the function $\overline{r}^\star_{[\lambda]} f \in C(\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B)$ where $[f]$ is not trivial. As noted earlier, it may not be cylindrical, nor even integrable. However, if it is cylindrical, it will correspond to a symmetric state obeying $\sigma_{[\lambda]}(\overline{r}^\star_{[\lambda]} f)(f) \neq 0$. If it is not, we can approximate it by a sequence of cylindrical functions which is obtained by projecting it onto cylindrical subspaces, cylindrical with respect to graphs of an increasing net constructed as follows. If $\overline{r}^\star_{[\lambda]} f$ will lie in Φ'_B, but not in Φ_B, then it will be a countably infinite sum of terms, each being cylindrical with respect to a finite graph, the union of all these graphs being an infinite graph. The projections will be obtained by truncating to a finite number of these graphs, their number tending to infinity in the sequence mentioned. (This is reminiscent of the well known approximating sequences of the δ-distribution or, more generally, of an approximate identity in an algebra without identity.) All the functions f_i in the sequence will fulfil $\sigma_{[\lambda]}(f_i)(f) \neq 0$ proving our assertion.

Separation now follows from linearity, but can also be proved directly. We pick a representative for each equivalence class of cylindrical functions in Φ_Σ, and for each representative f the cylindrical function $\overline{r}^\star_{[\lambda]} f$, if it lies already in Φ_B, or else an appropriate element of the sequence approximating $\overline{r}^\star_{[\lambda]} f$. The term appropriate means that any linear relation between the chosen functions reflects a linear relation between equivalence classes.

An appropriate selection can always be done, because the sequences approximate the functions $\overline{r}^{\star}_{[\lambda]} f$ which are different for different classes. Thereby, we obtain a class of distributions separating the equivalence classes. □

The results of this lemma can be interpreted as (over-)completeness of the set $\sigma_{[\lambda]}(\Phi_B)$ of generalized functions on $\{f|_{\mathrm{Im}(r_{[\lambda]})} : f \in \Phi_\Sigma\}$ in the sense that $\sigma_{[\lambda]}(g)(f) = 0$ for all $g \in \Phi_B$ implies $f|_{\mathrm{Im}(r_{[\lambda]})} = 0$. Together with theorem 3.1 this can be summarized in

Theorem 3.2 (Quantum symmetry reduction). *Let P be an S-symmetric principal fibre bundle classified by* $([\lambda], Q)$*, Q being the reduced bundle over B.*

The space of $[\lambda]$*-symmetric states on $\overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma$ can, by means of the mapping $\sigma_{[\lambda]}$, be identified with the space Φ_B of cylindrical functions on $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B$.*

Proof. Let $\Phi_{\mathrm{symm}} := \Phi_\Sigma/\sim$ be the space of functions on the space $\mathrm{Im}(r_{[\lambda]})$ of generalized gauge-invariant and $[\lambda]$-invariant connections, and Φ'_{symm} its topological dual. Due to the definition in lemma 3.4 of $\sim$, Φ'_{symm} can be identified with the space of $[\lambda]$-symmetric states as defined in definition 3.1.

According to lemma 3.4, $\sigma_{[\lambda]}(\Phi_B)$ is in separating duality with Φ_{symm}. This implies that $\sigma_{[\lambda]}(\Phi_B)$ is dense in Φ'_{symm} in the weak topology [40, chapter II, section 6.2, corollary 4], and, therefore, in the space of $[\lambda]$-symmetric states. (Note, however, that the topology on Φ_B induced from the weak topology by the continuous map $\sigma_{[\lambda]}$ is coarser than the topology in which Φ_B is completed to $\mathcal{H}_B$.)

In conclusion, the map $\sigma_{[\lambda]}$ is injective (by construction of Φ_B, where the Higgs constraint (5) is assumed to be solved, and by the construction of $\sigma_{[\lambda]}$), and has a dense image in the space of $[\lambda]$-symmetric states. □

This theorem allows us to trade the space of symmetric states according to definition 3.1, which is not well suited for establishing a calculus, for the space Φ_B. This space and the given calculus thereon are only a bit more difficult to deal with than the space Φ_Σ of all states of the full theory, because we may have to use spin networks with Higgs field vertices.

The Gauß constraint is solved by using gauge-invariant functions, i.e. cylindrical functions on $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B/\overline{\mathcal{G}}_B$. The diffeomorphism constraint can be solved by group averaging. After imposing S-symmetry there are only those diffeomorphisms to average that respect this symmetry. These are precisely the diffeomorphisms of B, and the constraint can be solved by averaging over the diffeomorphism group of B acting on Φ_B.

We conclude this section with some remarks.

- We describe symmetric states by spin networks of the group G, not those of Z_λ, as might have been expected from the classical reduction. On the one hand, we are forced to do that because the Higgs field transforms, in general, according to the adjoint representation of G, not Z_λ. On the other hand, the reduced structure group Z_λ has its origin in a partial gauge fixing imposed by fixing λ. Our quantum symmetry reduced theory does not make use of such a partial gauge fixing. Instead we will have to enforce gauge invariance under the full group G.
- In quantizing classical expressions we have nevertheless to start from those which might be invariant only under the reduced gauge group. The first step of quantization will yield an operator acting on functions on $(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^\lambda$. It has then to be extended by gauge covariance to an operator on Φ_B. This is parallel to our construction of the map $r_{[\lambda]}$ in equations (7), (8) and (10). An example of the quantization of such an operator will be given in section 4.3.3.

- In order to actually be able to quantize classical expressions in the Hamiltonian formalism used so far, we will have to claim validity of the symmetric criticality principle in the theory under consideration. This means that critical points of the reduced action should correspond to critical points of the unreduced one. The validity of this principle cannot be taken for granted. It will be valid if the symmetry group S is compact, or if it is Abelian (more generally, unimodular) and acts freely [17]. These two cases cover our examples given below.

- Until now we have confined ourselves to a fixed conjugacy class $[\lambda]$. In general, there will be a family of such conjugacy classes. If it is not possible to select one by means of physical considerations, we will have to treat them on an equal footing. We will use the same space Φ_B for every class, but they will be differently represented as distributions on Φ_Σ by means of $\sigma_{[\lambda]}$. The different conjugacy classes give rise to different sectors, which are superselected as seen from the symmetric theory. They correspond to different subspaces of invariant connections embedded in $\overline{\mathcal{A}}_\Sigma/\overline{\mathcal{G}}_\Sigma$ via $r_{[\lambda]}$. We have to find a physical interpretation for the different conjugacy classes, the most natural being that of a topological charge. This is purely classical, because the different conjugacy classes arise already in the classical reduction. We will see in the next section that this case indeed occurs, namely for spherically symmetric electromagnetism where $[\lambda]$ gives the magnetic charge.

- Classically, a symmetric principal fibre bundle is classified by a conjugacy class of homomorphisms and a reduced bundle (see theorem 2.1). In the quantum theory, however, the conjugacy class of the homomorphisms will suffice, because the space $\overline{\mathcal{A}}_B$ of generalized connections contains connections on all the bundles over B, as proven in [41].

- The emergence of the Higgs field can be understood as a reflection of the information which is contained in those edges associated with a cylindrical state which are located entirely in the orbits of S. For example, in a spherically symmetric theory they represent edges in an S^2 orbit. This justifies even more the name 'point holonomies'. From this picture one can see that, in general, the function $\overline{r}^\star_{[\lambda]} f$ will involve an infinite number of Higgs vertices, where $f \in \Phi_\Sigma$ is a cylindrical function. If the underlying graph contains an edge that lies neither entirely in an orbit nor entirely in the manifold B, then f will depend on all components of a connection in points along the edge. In the symmetry-reduced theory some of the components are given by the Higgs field, so the full dependence of f on a connection requires a continuous set of Higgs vertices in the union of all graphs underlying $\overline{r}^\star_{[\lambda]} f$.

- The procedure of the present section can be reversed, using a Kaluza–Klein construction of a Higgs field as components of a connection in a higher-dimensional, symmetric manifold. Suppose we want to quantize a diffeomorphism-invariant theory of a connection and a Higgs field, for which there is a higher-dimensional Kaluza–Klein theory containing only connection fields. Then we can visualize the Higgs vertices arising in the quantization of the lower-dimensional theory as remnants of loops lying in the compactified extra dimensions. By 'blowing-up' a lower-dimensional spin network with Higgs vertices in this way in order to obtain a higher-dimensional ordinary spin network, we obtain a quantization of the Kaluza–Klein theory. This blowing-up preserves, like our symmetry reduction, gauge invariance, because the Higgs vertices transform under the adjoint representation of the structure group as do loops based on the vertices.

4. Examples

In this section we will give examples in order to illustrate some of the ideas of the preceding section and to test them.

4.1. (2 + 1)-dimensional gravity

(2 + 1)-dimensional gravity can be obtained in terms of a symmetry reduction of (3 + 1)-dimensional gravity by imposing the existence of one spacelike, hypersurface orthogonal Killing vector field of constant norm. The extra condition on the norm of the Killing field is necessary in order to eliminate a scalar field which is related to its norm. Otherwise one would obtain (2 + 1)-dimensional gravity coupled to a massless scalar field.

We therefore have the Abelian symmetry group $S = \mathbb{R}$ (or $S = SO(2)$), and we assume the space Σ to have the topology $\Sigma = B \times \mathbb{R}$ (or $\Sigma = B \times S^1$), where B is a two-dimensional manifold. The group S acts freely on the second component of Σ which means that $F = \{0\}$. Hence, there can only be one homomorphism $\lambda\colon F \to G$, $0 \mapsto 1$. Here $G = SU(2)$ is the gauge group of gravity, formulated as a gauge theory using real Ashtekar variables. In this case the reduced group $Z_\lambda = G$ is identical to the full structure group. Because F is trivial, equation (5) is trivially true and we have a one-component Higgs field ϕ taking values in $\mathcal{L}G$. If we use the Maurer–Cartan form $\theta = \mathrm{d}z$, where z coordinatizes $\mathbb{R}$, and the embedding $\iota = \mathrm{id}\colon \mathbb{R} \to \mathbb{R}$, the reconstruction is

$$r\colon (A^i_a\tau_i\,\mathrm{d}x^a, \phi^i\tau_i) \mapsto A^i_a\tau_i\,\mathrm{d}x^a + \phi^i\tau_i\,\mathrm{d}z. \tag{15}$$

Here the matrices $\tau_i = -\frac{1}{2}\mathrm{i}\sigma_i$ form a basis of $\mathcal{L}G$ and x^a, $a = 1, 2$ are coordinates on a chart of B. This shows that the Higgs field gives the z-component of the connection form.

Following the steps of the preceding section, we obtain the configuration space $\overline{\mathcal{A}}_B \times \overline{\mathcal{U}}_B$ of the reduced theory which consists of the fields A^i_a and ϕ^i on B appearing in (15).

Up to now we have imposed only the symmetry condition, not the condition on the norm of the Killing field. This second condition serves to remove the dynamical scalar field. An equivalent quantum condition can be formulated by allowing only trivial Higgs vertices. This will eliminate the dependence of the spin networks on generalized Higgs fields, thereby removing their local degrees of freedom. So we arrive at the quantum configuration space $\overline{\mathcal{A}}_B$ of vacuum (2 + 1)-dimensional gravity.

We can now build an auxiliary Hilbert space of integrable cylindrical functions on $\overline{\mathcal{A}}_B$, completely analogous to the unreduced theory. On this Hilbert space which is spanned by ordinary two-dimensional spin networks we can represent the constraints and search for solutions. This has already been done in [19] starting from the classical symmetry reduction and quantizing the classical configuration space. Our quantum symmetry reduction procedure gives the same results, but with a representation of the quantum states of the reduced theory as symmetric states of the full quantum theory.

The reduction of this subsection can be generalized straightforwardly to the case of an arbitrary gauge theory on $\Sigma = B \times \mathbb{R}^n$, where B is an arbitrary d-dimensional manifold and $S = \mathbb{R}^n$ acts on the second factor of Σ. A symmetry reduction will involve n components of a Higgs field, whereas a *quantum dimensional reduction* from $B \times \mathbb{R}^n$ to B can be obtained by postulating trivial Higgs vertices of spin networks in B.

4.2. Spherically symmetric electromagnetism

We will now reduce electromagnetism to a spherically symmetric one. Although spherically symmetric electromagnetism is almost trivial it is, nevertheless, quite instructive for our purposes because the quantum symmetry reduction can be carried out explicitly.

First, let us explain why we treat electromagnetism as a diffeomorphism-invariant theory. This can, of course, not be true for pure electromagnetism on, e.g., a Minkowski background. However, we are interested in the electromagnetic field as a field of a Reissner–Nordstrøm black hole. We therefore have to couple electromagnetism to gravity, which ensures diffeomorphism invariance. The dynamics of the electromagnetic field is then encoded in the Hamiltonian constraint. The diffeomorphism constraint will only contain gravitational fields, because in the spherically symmetric case diffeomorphism invariance of the electromagnetic field is already imposed by the $U(1)$ Gauß constraint. As a test model this coupled theory as a whole would be too complicated, so we discard the gravitational field by constraining it to be zero. This amounts to treating the electromagnetic field on a degenerate background which renders the Hamiltonian ill-defined. However, electric and magnetic fluxes will be well defined, so that we, nevertheless, may study the kinematics of the fields.

4.2.1. Classical symmetry reduction. We now have $G = U(1)$, $S = SU(2)$ and in general $F = U(1)$. The topology of space is $\Sigma \cong \mathbb{R} \times S^2$ or $\Sigma \cong \mathbb{R}^+ \times S^2$, implying $B = \mathbb{R}$ or $B = \mathbb{R}^+$, respectively.

We first determine all conjugacy classes of the homomorphisms $\lambda: U(1) \to U(1)$. Because such a homomorphism is a one-dimensional unitary representation of $U(1)$, it is given by its character. Therefore, we have the mappings $\mathrm{Hom}(U(1), U(1))/\mathrm{Ad} \cong \mathbb{Z}$ represented by the homomorphisms $\lambda_n: z \mapsto z^n$. For the Abelian group G the centralizers $Z_{\lambda_n} = U(1)$ are equal to the full group. Thus, a spherically symmetric principal fibre bundle with structure group $U(1)$ is classified by an integer n, and, of course, all reduced bundles are trivial, because B is contractible, and they are not needed for the classification.

Let us now determine the type of Higgs field allowed by equation (5). This field is given by the map $\Lambda: \mathcal{L}S \to \mathcal{L}G$, where $\Lambda|_{\mathcal{L}F} = \mathrm{d}\lambda_n$ is already fixed by the bundle classification. Here we have $\dim \mathcal{L}G = 1$, leaving only the possibilities $\dim\ker\Lambda = 2$ or $\dim\ker\Lambda = 3$. In the case $n \neq 0$, $\mathrm{d}\lambda_n$ is an isomorphism of vector spaces, forbidding $\dim\ker\Lambda = 3$. So we have $\dim\ker\Lambda = 2$ and the Higgs field $\phi = \Lambda|_{\mathcal{L}F_\perp} = 0$ vanishes. This kind of reasoning cannot be used in the case $n = 0$. However, looking at equation (5) we see that n enters this equation only in connection with $\mathrm{Ad}_{\lambda_n(f)}$ for $f \in F$. This occurrence is trivial in the Abelian case, so equation (5) is independent of n. Therefore, if it does not allow a Higgs field for $n \neq 0$ it also cannot allow a Higgs field for $n = 0$. This proves that the reduced theory is a theory of a $U(1)$ connection only.

Let us recall the reduced phase space structure from [26]. The canonical variables of the symmetry reduced theory are given by the radial components p and $\mathrm{i}\omega$ of the electric field and the $U(1)$ connection form, respectively. They are subject to the Gauß constraint $p' \approx 0$ which enforces p to be constant. The reduced phase space is two dimensional with the canonical variable p, which is proportional to the electric charge and the canonically conjugate $\Phi = -\int_B \mathrm{d}x\,\omega$. The electric charge is either a point charge sitting in $x = 0$ if $B = \mathbb{R}^+$, or the charge of the wormhole $\Sigma = \mathbb{R} \times S^2$.

4.2.2. Quantum symmetry reduction. According to theorem 3.2 the physical Hilbert space of spherically symmetric electromagnetism is given by $L_2(\overline{\mathcal{A}_B/\mathcal{G}_B}, \mathrm{d}\mu_{AL})$, the space of gauge invariant, Ashtekar–Lewandowski square-integrable functions on the space of generalized $U(1)$ connections over B. It is spanned by $U(1)$ spin networks, which are due to gauge invariance and the one-dimensional nature of B given by

$$(\eta_B)^K \qquad \text{where} \quad \eta_B := \exp\left(\mathrm{i}\int_B \mathrm{d}x\,\omega(x)\right) \tag{16}$$

is the holonomy along B of the $U(1)$ connection $\mathrm{i}\,\omega$ and $K \in \mathbb{Z}$ is the only parameter, corresponding to irreducible $U(1)$ representations labelling the basis states. We see that the reduced theory is particularly simple, having only two canonical degrees of freedom. The physical Hilbert space can be identified with $L_2(U(1), \mathrm{d}\mu_H)$, where $\mathrm{d}\mu_H$ is the Haar measure on $U(1)$.

Before coming to the observables of the theory we will investigate the quantum symmetry reduction $\overline{r}^{\star}_{[\lambda]}$. Let the symmetric principal fibre bundle be classified by the homomorphism λ_n, and let $F = U(1) \cong \exp\langle\tau_3\rangle \subset SU(2)$ be the isotropy subgroup. We then have to set $\Lambda(\tau_3) = \mathrm{d}\lambda_n(\tau_3) = \mathrm{i}\,n$ and $\Lambda(\tau_{1,2}) = 0$ in equation (6). The reconstruction of a $U(1)$ connection with x-component $\mathrm{i}\,\omega_x\colon B \to \mathrm{i}\,\mathbb{R} = \mathcal{L}U(1)$ on Q is given by the $[\lambda_n]$-invariant connection

$$\mathrm{i}\,\omega := r^{(\iota)}_{\lambda_n}(\mathrm{i}\,\omega_x\,\mathrm{d}x) = \mathrm{i}\,\omega_x\,\mathrm{d}x + \mathrm{i}\,n\cos\vartheta\,\mathrm{d}\varphi. \tag{17}$$

Here x is a (local) coordinate of B and (ϑ, φ) are the Killing parameters of the S-orbits. Together they yield a (local) coordinate system (x, ϑ, φ) of $B \times S^2$. We can calculate the curvature of this $[\lambda_n]$-invariant connection to obtain the (densitized) magnetic field of a Dirac monopole with magnetic charge n:

$$B_x = -n\sin\vartheta, \qquad B_\vartheta = B_\varphi = 0. \tag{18}$$

This confirms our claim that the conjugacy classes of homomorphisms correspond to the values of a topological charge.

Equation (17) will now be used to pull-back a spin-network state to a function on the space of invariant connections. To that end let $e\colon[0, 1] \to \Sigma$ be an edge in Σ, running from $e(0) = (x_1, \vartheta_1, \varphi_1)$ to $e(1) = (x_2, \vartheta_2, \varphi_2)$, chosen such that a parametrization $\vartheta(\varphi)$ is possible along e (otherwise we can cut e in pieces and set $\vartheta(\varphi) = 0$ if φ is constant along a piece without affecting the following). Then the $\mathrm{i}\,\omega$ holonomy along e is given by

$$\begin{aligned} h_e(\mathrm{i}\omega) &= \exp\left(\int_e \mathrm{d}t\,(\mathrm{i}\,\dot{x}\,\omega_x + \mathrm{i}\,n\,\dot{\varphi}\cos\vartheta)\right) \\ &= \exp\left(\mathrm{i}\int_{x_1}^{x_2}\mathrm{d}x\,\omega_x\right)\exp\left(\mathrm{i}n\int_{\varphi_1}^{\varphi_2}\mathrm{d}\varphi\,\cos\vartheta(\varphi)\right) \\ &=: h_{\pi(e)}(\mathrm{i}\omega_x)(\beta_e)^n. \end{aligned}$$

Here $\pi(e) := [x_1, x_2] = [x(e(0)), x(e(1))] \subset B$ is an edge in B, and $h_{\pi(e)}(\mathrm{i}\,\omega_x)$ is the holonomy of the reduced connection $\mathrm{i}\,\omega_x$ along it. β_e is a phase factor depending only on the geometry of e.

If $T_{\gamma,k}$ is a spin network with a graph $\gamma \subset \Sigma$ and a labelling k of its edges with irreducible $U(1)$ representations (a labelling of the vertices is not necessary for gauge-invariant $U(1)$ spin networks because it is already given uniquely by the edge labelling), we can evaluate it on a $[\lambda_n]$-invariant connection. If $E(\gamma)$ denotes the edge set of γ we have

$$\begin{aligned} T_{\gamma,k}(\mathrm{i}\omega) &= \prod_{e\in E(\gamma)} h_e(\mathrm{i}\omega)^{k_e} = \prod_{e\in E(\gamma)} h_{\pi(e)}(\mathrm{i}\omega_x)^{k_e}\prod_{e\in E(\gamma)}(\beta_e)^{nk_e} \\ &= (\beta_{\gamma,k})^n\prod_{e\in E(\gamma)} h_{\pi(e)}(\mathrm{i}\omega_x)^{k_e}. \end{aligned} \tag{19}$$

Here $\beta_{\gamma,k} := \prod_{e\in E(\gamma)}\beta_e^{k_e}$ is a phase factor depending only on the geometry of T and its labelling.

The right-hand side of the last equation can be written as the evaluation of a spin-network state in B on the connection $\mathrm{i}\omega_x$. In order to do this we will define a projection π which assigns to a spin-network $T_{\gamma,k}$ in Σ a spin-network $\pi(T_{\gamma,k})$ in B. The set of vertices $V(\gamma)$ of γ can be projected on a finite set

$$\pi(V(\gamma)) := \{x(v) : v \in V(\gamma)\} =: \{x^{(i)}\}_{1\leqslant i\leqslant|\pi(V(\gamma))|},$$

where we have ordered the elements of $\pi(V(\gamma))$ such that $x^{(i)} < x^{(j)}$ for $i < j$. Their number is bounded by $|\pi(V(\gamma))| \leqslant |V(\gamma)|$. Let γ be chosen—if necessary by inverting edges and splitting edges by introducing new vertices—such that each edge $e \in E(\gamma)$ either lies entirely in an orbit of S, in which case we define $\pi(e) := \emptyset$, or it has a projection $\pi(e) = [x^{(i)}, x^{(i+1)}]$ for some $1 \leqslant i \leqslant |\pi(V(\gamma)|$ and $x(e(t))$ increases monotonically in t. To every projected edge $\pi(e)$ we assign the point $x_m(\pi(e)) := \frac{1}{2}(x(e(0)) + x(e(1)))$ in its interior. We can now define the projected spin network.

Definition 4.1. *Let $T_{\gamma,k}$ be a $U(1)$ spin network in $\Sigma = B \times S^2$, the graph γ be chosen as above.*

The projected graph $\pi(\gamma) \subset B$ *is given by its edge set*

$$E(\pi(\gamma)) := \{\pi(e) : e \in E(\gamma), \pi(e) \neq \emptyset\} = \{[x^{(i)}, x^{(i+1)}] : 1 \leqslant i < |\pi(V(\gamma))|\}$$

and its vertex set

$$V(\pi(\gamma)) := \pi(V(\gamma)).$$

The labelling of the projected graph descending from the spin network $T_{\gamma,k}$ is given by

$$\pi(k)_{\pi(e)} := \sum_{|e'\cap S_{x_m(\pi(e))}|=1} k_{e'}$$

for any edge $\pi(e) \in E(\pi(\gamma))$. Here $S_{x_m(\pi(e))}$ is the S-orbit through $x_m(\pi(e))$, and the condition $|e' \cap S_{x_m(\pi(e))}| = 1$ in the sum ensures that we count only the charge of edges running transversally through $S_{x_m(\pi(e))}$.

The projected spin network *is given by $\pi(T_{\gamma,k}) := T_{\pi(\gamma),\pi(k)}$.*

By means of the projected spin network, we can write equation (19) as

$$\begin{aligned} T_{\gamma,k}(\mathrm{i}\omega) &= (\beta_{\gamma,k})^n \prod_{e_B\in E(\pi(\gamma))} \prod_{e'\in E(\gamma):\pi(e')=e_B} h_{e_B}(\mathrm{i}\omega_x)^{k_{e'}} \\ &= (\beta_{\gamma,k})^n \prod_{e_B\in E(\pi(\gamma))} h_{e_B}(\mathrm{i}\omega_x)^{\pi(k)_{e_B}} = (\beta_{\gamma,k})^n \pi(T_{\gamma,k})(\mathrm{i}\omega_x). \end{aligned}$$

This equation enables us to write

$$\overline{r}^{\star}_{[\lambda_n]} T_{\gamma,k} = (\beta_{\gamma,k})^n \pi(T_{\gamma,k}). \tag{20}$$

We can see that $\overline{r}^{\star}_{[\lambda_n]} T \in \Phi_B$ is cylindrical on $\overline{\mathcal{A}}_B$ for any spin network $T \in \Phi_\Sigma$. This convenient circumstance is related to the vanishing of the Higgs field and cannot be taken for granted in cases of other structure groups. Here this fact allows us to compose the maps $\rho_{[\lambda]}$ and $\sigma_{[\lambda]}$ in the diagram at the end of section 3.2 to obtain a map

$$\sigma_{[\lambda]} \circ \rho_{[\lambda]} \colon \Phi_\Sigma \to \Phi'_\Sigma$$

reminiscent of a group-averaging map. In fact, the symmetry reduction considered here can be imposed by supplementing electrodynamics with a further constraint besides the Gauß constraint to arrive at an Abelian BF-theory [38]. Thus, the symmetry can be reduced by means of a 'rigging' map analogous to $\sigma_{[\lambda]} \circ \rho_{[\lambda]}$. The vanishing Higgs field also causes the appearance of the topological charge n as a power of the phase factor only. Furthermore, the above calculations exhibit the fact that a projected spin-network state is gauge invariant if and only if the original spin-network state in Σ is gauge invariant. This follows from the fact that gauge invariance in Σ forces the spin-network labelling to fulfil

$$\sum_{|e\cap S_1|=1} k_e = \sum_{|e\cap S_2|=1} k_e$$

for any two S-orbits S_1 and S_2 not containing a vertex of the graph γ (for simplicity we assume that all edges are oriented outwards). Therefore, a gauge-invariant spin network will project to a spin network with labelling $k_{e_B} = K$ for each edge e_B of the projected graph. This yields the gauge-invariant spin network $(\eta_B)^K$ defined above as a projected spin network.

Let us now write down the action of a $[\lambda_n]$-symmetric state as a distribution on Φ_Σ. For a spin network $T_{\gamma,k}$ it is given by

$$\begin{aligned}\sigma_{[\lambda_n]}((\eta_B)^K)(T_{\gamma,k}) &= \int_{\overline{\mathcal{A}}_B} \mathrm{d}\mu_{AL}\, (\overline{\eta}_B)^K\, \overline{r}^{\star}_{[\lambda_n]} T_{\gamma,k} = (\beta_{\gamma,k})^n \int_{\overline{\mathcal{A}}_B} \mathrm{d}\mu_{AL}\, (\overline{\eta}_B)^K\, T_{\pi(\gamma),\pi(k)} \\ &= (\beta_{\gamma,k})^n\, \delta_{\{B\},\pi(\gamma)}\, \delta_{\{K\},\pi(k)}. \end{aligned} \tag{21}$$

It is non-vanishing only if $T_{\gamma,k}$ is gauge invariant implied by $\pi(\gamma) = \{B\}$, and if the charge $\pi(k) = \sum_{|e\cap S_x|=1} k_e$ equals the labelling K of the symmetric state.

4.2.3. Observables. We will now quantize the observables p and Φ found in the classical reduction. p is canonically conjugate to the connection ω, and standard quantization rules lead in the connection representation to the quantization $\hat{p} = -\mathrm{i}\hbar\frac{\delta}{\delta\omega}$. Acting on a gauge-invariant state (16) this yields $\hat{p}\,(\eta_B)^K = \hbar K(\eta_B)^K$. We see that spin-network states are eigenstates of $\hat{p}$. All the eigenvalues of $\hat{p}$ are real, which shows that it is essentially self-adjoint. Furthermore, they are discrete exhibiting electric charge quantization as integer multiples of an elementary charge (this fact has already been observed in [42] in the asymmetric theory). The value of this elementary charge is, however, not fixed because p is only proportional to the electric charge. The factor of proportionality ensures the correct dimension and scales the elementary charge. This factor is arbitrary and is related to the normalization of the electromagnetic action.

In the following we will denote the basis states of the Hilbert space $L_2(U(1), \mathrm{d}\mu_H)$ as

$$|K\rangle := (\eta_B)^K.$$

These states span an orthonormal basis with respect to the Haar measure on $U(1) = \overline{\mathcal{A}}_B/\overline{\mathcal{G}}_B$ which derives from the Ashtekar–Lewandowski measure on $\overline{\mathcal{A}}_B$. They are labelled by the charge eigenvalues according to $\hat{p}\,|K\rangle = \hbar K\,|K\rangle$.

Now we have to quantize $\Phi = -\int_B \omega = \mathrm{i}\log\eta_B$. Recall that holonomies are well defined as operators in the loop quantization. This fact suggests using η_B instead of Φ. η_B can be promoted straightforwardly to the 'creation' operator

$$\hat{\eta}_B\,|K\rangle = (\eta_B)^{K+1} = |K+1\rangle,$$

which is unitary because it is invertible and preserves the norm of states it acts on. In view of $\eta_B = \exp(-\mathrm{i}\Phi)$ the operator $\hat{\eta}_B$ has indeed to be unitary. Note that it is the Haar measure on $U(1)$, which derives from the Ashtekar–Lewandowski measure on the unconstrained space (before solving the Gauß constraint) that incorporates the classical reality conditions correctly.

We will now close this subsection by showing that $\hat{p}$ and $\hat{\eta}_B$ have the correct commutator.

Theorem 4.1. $(p, \exp(-\mathrm{i}\Phi)) \mapsto (\hat{p}, \hat{\eta}_B)$ *is a representation of the classical Poisson $\star$-algebra on the Hilbert space* $L_2(U(1), \mathrm{d}\mu_H)$.

Proof. We have already seen that the $\star$-relations $p^\star = p$ and $\exp(-\mathrm{i}\Phi)^\star = \exp(\mathrm{i}\Phi)$ are represented properly.

Because p and Φ are canonically conjugate, we have the Poisson bracket

$$\{p, \exp(-\mathrm{i}\Phi)\} = \frac{\mathrm{d}\exp(-\mathrm{i}\Phi)}{\mathrm{d}\Phi} = -\mathrm{i}\exp(-\mathrm{i}\Phi).$$

The only non-vanishing matrix elements of the commutator $[\hat{p}, \hat{\eta}_B]$ are given by

$$\left\langle K \left| [\hat{p}, \hat{\eta}_B] \right| K-1 \right\rangle = \langle K | \hbar K | K\rangle - \langle K | \hbar(K-1) | K\rangle = \hbar.$$

We, therefore, have the relation

$$[\hat{p}, \hat{\eta}_B] = \hbar\hat{\eta}_B = \mathrm{i}\hbar\{p, \exp(-\mathrm{i}\Phi)\}\hat{} \tag{22}$$

implementing the correct representation of the Poisson structure. $\square$

4.3. Spherically symmetric quantum gravity

Our last example deals with spherically symmetric quantum gravity. Undoubtedly this one of our three examples is of the greatest physical interest because it describes quantum properties of spherically symmetric black holes. A first application in this direction will be presented in section 5. The classical symmetry and constraint reduction reveals that there is only one physical configuration degree of freedom, the mass (or a canonical pair: mass and time). This is similar to spherically symmetric electromagnetism discussed above which has only the electric charge as a physical degree of freedom. However, it can be anticipated that gravity will be more difficult since its constraints are more complicated.

In the present paper we treat in detail only the Gauß constraint and comment briefly on the solution of the diffeomorphism constraint. The more complicated Hamiltonian constraint will be dealt with elsewhere [31].

4.3.1. Symmetry reduction. We now specialize our general framework to the case $S = SU(2)$, $F = U(1)$ and $G = SU(2)$ (we will use real Ashtekar variables). The topology of the space Σ will be $\Sigma = B \times S^2$ with $B = \mathbb{R}$ or $B = \mathbb{R}^+$.

First, we have to find all conjugacy classes of homomorphisms λ: $F = U(1) \to SU(2) = G$. In order to do that we can make use of equation (4). We need the following information about $SU(2)$ (see, e.g., [39]).

Lemma 4.1. *The* standard maximal torus *of $SU(2)$ is given by*

$$T(SU(2)) = \{\mathrm{diag}(z, z^{-1}) | z \in U(1)\} \cong U(1).$$

The Weyl group *of $SU(2)$ is the permutation group of two elements,*

$$W(SU(2)) \cong S_2,$$

its generator acting on $T(SU(2))$ by $\mathrm{diag}(z, z^{-1}) \mapsto \mathrm{diag}(z^{-1}, z)$.

All homomorphisms in $\mathrm{Hom}(U(1), T(SU(2)))$ are then given by

$$\lambda_k: z \mapsto \mathrm{diag}(z^k, z^{-k})$$

for any $k \in \mathbb{Z}$, as in the electromagnetic example. However, here we have to divide out the action of the Weyl group, leaving only the maps λ_k, $k \in \mathbb{N}_0$, as representatives of all conjugacy classes of homomorphisms. We see that spherically symmetric gravity has a topological charge taking values in $\mathbb{N}_0$ (in the dreibein, not in the metric formulation, as we will see below).

We will represent F as the subgroup $\exp\langle\tau_3\rangle < SU(2)$ of the symmetry group S, and use the homomorphisms $\lambda_k\colon \exp t\tau_3 \mapsto \exp kt\tau_3$, out of each conjugacy class. This amounts to a partial gauge fixing called the τ_3 gauge in the following. As the reduced structure group we obtain $Z_G(\lambda_k(F)) = \exp\langle\tau_3\rangle \cong U(1)$ for $k \neq 0$ and $Z_G(\lambda_0(F)) = SU(2)$ ($k = 0$ leads to the manifestly symmetric connections of [35]). The map $\Lambda|_{\mathcal{L}F}$ is then given by $\mathrm{d}\lambda_k\colon \langle\tau_3\rangle \to \mathcal{L}G$, $\tau_3 \mapsto k\tau_3$. The remaining components of Λ which give us the Higgs field, are determined by $\Lambda(\tau_{1,2}) \in \mathcal{L}G$, and are subject to equation (5). This equation can be written here as

$$\Lambda \circ \mathrm{ad}_{\tau_3} = \mathrm{ad}_{\mathrm{d}\lambda(\tau_3)} \circ \Lambda.$$

Using $\mathrm{ad}_{\tau_3}\,\tau_1 = \tau_2$ and $\mathrm{ad}_{\tau_3}\,\tau_2 = -\tau_1$ we obtain the equation

$$\Lambda(a_0\tau_2 - b_0\tau_1) = k(a_0[\tau_3, \Lambda(\tau_1)] + b_0[\tau_3, \Lambda(\tau_2)]),$$

where $a_0\tau_1 + b_0\tau_2$, $a_0, b_0 \in \mathbb{R}$ is an arbitrary element of $\mathcal{L}F_\perp$. Since a_0 and b_0 are arbitrary this is equivalent to the two equations

$$k[\tau_3, \Lambda(\tau_1)] = \Lambda(\tau_2) \qquad \text{and} \qquad k[\tau_3, \Lambda(\tau_2)] = -\Lambda(\tau_1).$$

The general ansatz

$$\Lambda(\tau_1) = a_1\tau_1 + b_1\tau_2 + c_1\tau_3, \qquad \Lambda(\tau_2) = a_2\tau_1 + b_2\tau_2 + c_2\tau_3$$

with arbitrary parameters $a_i, b_i, c_i \in \mathbb{R}$ yields the equations

$$\begin{aligned} k(a_1\tau_2 - b_1\tau_1) &= a_2\tau_1 + b_2\tau_2 + c_2\tau_3, \\ k(-a_2\tau_2 + b_2\tau_1) &= a_1\tau_1 + b_1\tau_2 + c_1\tau_3. \end{aligned} \tag{23}$$

These equations have a non-trivial solution only if $k = 1$, for which we obtain

$$b_2 = a_1, \qquad a_2 = -b_1 \qquad \text{and} \qquad c_1 = c_2 = 0.$$

We shall discuss this main case first where we will also see how the present approach is related to that of [24, 25] and will comment on the physically uninteresting ones ($k \neq 1$) which have a vanishing Higgs field afterwards.

The configuration variables of the system are the above fields $a, b, c\colon B \to \mathbb{R}$ of the $U(1)$ connection form $A = c(x)\,\tau_3\,\mathrm{d}x$, on the one hand, and the two Higgs field components

$$\begin{aligned} &\Lambda|_{\langle\tau_1\rangle}\colon B \to \langle\tau_1, \tau_2\rangle,\ x \mapsto a(x)\tau_1 + b(x)\tau_2 \\ &\quad = \frac{1}{2}\begin{pmatrix} 0 & -b(x) - \mathrm{i}a(x) \\ b(x) - \mathrm{i}a(x) & 0 \end{pmatrix} =: \begin{pmatrix} 0 & -\overline{w}(x) \\ w(x) & 0 \end{pmatrix} \end{aligned} \tag{24}$$

on the other hand.

Under a local $U(1)$ gauge transformation $z(x) = \exp t(x)\tau_3$ they transform as

$$c \mapsto c + \frac{\mathrm{d}t}{\mathrm{d}x} \qquad \text{and} \qquad w(x) \mapsto \exp(-\mathrm{i}t)w$$

which can be read off from

$$\begin{aligned} A &\mapsto z^{-1}Az + z^{-1}\,\mathrm{d}z = A + \tau_3\,\mathrm{d}t, \\ \Lambda(\tau_1) &\mapsto z^{-1}\Lambda(\tau_1)z = \begin{pmatrix} 0 & -\exp(\mathrm{i}t)\overline{w} \\ \exp(-\mathrm{i}t)w & 0 \end{pmatrix}. \end{aligned}$$

Comparing with [24] we see that the above variable c transforms as the connection coefficient A_1 there and the variables (a, b) as (A_2, A_3). However, the reconstructed connection form

$$A(x, \vartheta, \varphi) = c(x)\tau_3\,\mathrm{d}x + (-a(x)\,\mathrm{d}\vartheta + b(x)\sin\vartheta\,\mathrm{d}\varphi)\tau_1 \\ -(b(x)\,\mathrm{d}\vartheta + a(x)\sin\vartheta\,\mathrm{d}\varphi)\tau_2 + \cos\vartheta\,\mathrm{d}\varphi\,\tau_3 \tag{25}$$

is different from the connection form

$$\left[A_1 n_x^i\,\mathrm{d}x + 2^{-1/2}(A_2 n_\vartheta^i + (A_3 - \sqrt{2})n_\varphi^i)\,\mathrm{d}\vartheta + 2^{-1/2}(A_2 n_\varphi^i - (A_3 - \sqrt{2})n_\vartheta^i)\sin\vartheta\,\mathrm{d}\varphi\right]\tau_i \tag{26}$$

used in [24, 25] (now expressed in terms of real Ashtekar variables) where n_x^i, n_ϑ^i and n_φ^i are the standard unit vectors in polar coordinates. The two connection forms differ, however, only by a gauge rotation $g_1 := \exp(-\vartheta n_\varphi^i \tau_i)$ followed by a second gauge rotation $g_2 := \exp(-\varphi n_x^i \tau_i)$.

We note that the term $\cos\vartheta\,\mathrm{d}\varphi\,\tau_3$ in equation (25), which is the only term independent of the fields a, b and c, in the connection form $A(x, \vartheta, \varphi)$ derived above leads automatically to the appearance of A_3 as the combination $A_3 - \sqrt{2}$ in the connection form (26) of [24, 25]. This subtraction of $\sqrt{2}$ (which does not appear in [35]) has been added by hand in [24] in order to ensure the correct transformation of (A_2, A_3) in the defining representation of $SO(2)$.

The calculation above shows that c has to be identified with A_1 and (up to a rescaling with $\sqrt{2}$ used in [24] in order to obtain the standard symplectic structure) (a, b) with (A_2, A_3). We will rescale the variables here, too, and denote the rescaled ones by (A_1, A_2, A_3). Their conjugate variables are (E^1, E^2, E^3), and the symplectic structure is—adapted to our notation using the Immirzi parameter ι—given by

$$\{A_I(x), E^J(y)\} = \frac{\kappa\iota}{4\pi}\delta_I^J\delta(x, y) \tag{27}$$

where $\kappa = 8\pi G$ is the gravitational constant.

We recall [24, 25] that the metric on Σ is given by

$$(q_{ab}) = \mathrm{diag}\left(\frac{E}{2E^1}, E^1, E^1\sin^2\vartheta\right), \qquad E = (E^2)^2 + (E^3)^2. \tag{28}$$

The functions E^1 and E are closely related to two important geometrical quantities.

The surface $A(x)$ of the two-dimensional spherical orbit generated by the symmetry group S at $x \in B$ is given by

$$A(x) = 4\pi\,E^1(x) \tag{29}$$

and the spherically symmetric three-dimensional volume element $\mathrm{d}V$ 'transverse' to those orbits is

$$\mathrm{d}V = \frac{4\pi}{\sqrt{2}}\sqrt{E^1 E}\,\mathrm{d}x. \tag{30}$$

We finally turn briefly to the case $k \neq 1$ which is associated with vanishing Higgs fields $\Lambda(\tau_i) = 0$, $i = 1, 2$ or, equivalently, $A_2 = A_3 = 0$. The vanishing of the canonical variables A_2 and A_3 implies that their canonically conjugate momenta E^2 and E^3 become irrelevant, too, and we may put them to zero, $E^2 = E^3 = 0$. This means that the metric (28) and the volume (30) become degenerate and that we do not have a non-vanishing volume element.

Thus, the sectors with $k \neq 1$ describe degenerate sectors which are, however, different from that found in [24]. There a degenerate sector with vanishing volume was found in our $(k = 1)$-sector after solving the constraints. On our $(k \neq 1)$-sectors the diffeomorphism and

Hamiltonian constraint will be trivially fulfilled, but the $k = 1$-sector has non-trivial constraints whose solution manifold has several sectors.

That the degenerate sectors for $k \neq 1$ here are different from the degenerate one of [24] can be seen as follows. For $k \neq 1$ the last term $(\cos\vartheta \, \mathrm{d}\varphi \, \tau_3)$ in equation (25) gets multiplied by k and consequently the subtracted constant $-\sqrt{2}$ in equation (26), too, leading to different connection forms even for $A_2 = A_3 = 0$.

Thus, if we are interested in geometrically interesting systems with non-vanishing volumes we have to stick to the sector $k = 1$.

4.3.2. Undoing the partial gauge fixing. When deriving the form of the reduced fields, we made use of the τ_3 gauge. We now describe how this partial gauge fixing can be undone. This will be an essential ingredient when we quantize the spherically symmetric area in section 5.

The only $SU(2)$ gauge transformations fixing λ_1 are generated by τ_3, all other gauge transformations change λ in its conjugacy class. In this τ_3 gauge we have $\mathrm{d}\lambda_1(\tau_3) = \tau_3$, whereas in an arbitrary λ-gauge, $\lambda \in [\lambda_1]$, we have $\mathrm{d}\lambda(\tau_3) = g^{-1}\tau_3 g$, $g \in SU(2)$. In order to characterize this general gauge by coordinates we parametrize $SU(2)$ by Euler angles: $g = g_3(\psi)g_1(\vartheta)g_3(\varphi)$ where $g_i(t) := \exp t\tau_i$. This yields

$$\mathrm{d}\lambda(\tau_3) = \sin\vartheta \sin\varphi \, \tau_1 + \sin\vartheta \cos\varphi \, \tau_2 + \cos\vartheta \, \tau_3 =: n^i \tau_i$$

with $n^i n_i \equiv \vec{n}^2 = 1$, $n_i = \delta_{ij} n^j$.

Fixed $\vec{n}$ corresponding to a fixed $\lambda \in [\lambda_1]$ is analogous to a fixed direction in $SU(2)$, which has been introduced in a similar context in [43]. However, as shown above, a chosen $\vec{n}$ represents pure gauge and physical states and observables should, of course, be independent of the choice.

The classical phase space of the symmetry-reduced theory consists of fields (A_I, E^I), $1 \leqslant I \leqslant 3$. We will, for simplicity, consider here only the fields A_1, E^1, which in the reduced theory have the role of a connection and its canonical conjugate. In the λ-gauge we have the $U(1)$ connection form $A_1 n^i \tau_i \, \mathrm{d}x$, where x is a (local) coordinate on B, and the $\mathcal{L}U(1)$-valued field $E^1 n^i \tau_i$. Their symplectic structure is given by

$$\{A_1(x), E^1(y)\} = \frac{\kappa\iota}{4\pi}\delta(x, y). \tag{31}$$

Without partial gauge fixing the fields would be $SU(2)$ valued and given by $A^i \tau_i \, \mathrm{d}x$ and $E_i \tau^i$ with

$$A^1 = A \sin\vartheta_A \sin\varphi_A, \qquad A^2 = A \sin\vartheta_A \cos\varphi_A, \qquad A^3 = A\cos\vartheta_A,$$
$$E_1 = E \sin\vartheta_E \sin\varphi_E, \qquad E_2 = E\sin\vartheta_E \cos\varphi_E, \qquad E_3 = E\cos\vartheta_E$$

in spherical coordinates and with symplectic structure

$$\{A^i(x), E_j(y)\} = \frac{\kappa\iota}{4\pi}\delta^i_j \delta(x, y). \tag{32}$$

(The indices $I = 1$ in equation (31) denote space indices, whereas the indices i, j in equation (32) are $SU(2)$ indices which are lowered or raised in terms of the Killing metric (δ_{ij}).) By using the spherical coordinates we can symplectically embed the phase space (A_1, E^1) in the λ-gauge as a ray in the phase space of $SU(2)$-valued fields: $A_1 \mapsto A_1 n^i = A^i$, $E^1 \mapsto E^1 n_i = E_i$ with $\vartheta_A = \vartheta_E$ and $\varphi_A = \varphi_E$ fixed such that the direction of n^i is given by the angles ϑ_A, φ_A.

If we start the quantization from the phase space (A_1, E^1) we would arrive at $U(1)$ spin networks. However, this renders the partial gauge fixing manifest and even worse, a

Higgs field cannot be included easily in Higgs vertices using the framework of [32] because it transforms in the adjoint representation of $SU(2)$, not of $U(1)$ which is, of course, the trivial representation. As described in the general framework we can undo the partial gauge fixing in the quantum theory by extending the spin networks by $SU(2)$ gauge invariance to spin-network functions on the space of $SU(2)$ connections and appropriate Higgs fields over B. The spin-network functions then depend not only on A_1 (and Higgs field components) but on all $SU(2)$ components A^i introduced above. Accordingly, the $SU(2)$ components E_i get quantized to functional derivatives

$$\hat{E}_i(x) = \frac{\hbar\kappa\iota}{4\pi\mathrm{i}}\frac{\delta}{\delta A^i}(x).$$

This will be crucial for the quantization of the area operator.

Finally, we mention that the extension to $SU(2)$-invariant spin networks can be understood as integrating the partial gauge fixings $\vec{n}$ over S^2 for any edge in the graph underlying the spin-network state. In order to make this precise we use coherent states on $SU(2)$ introduced in [44]. These are defined for a fixed irreducible representation of $SU(2)$ with weight j and a state $|j, m\rangle$ therein by

$$|m, \vec{n}\rangle_j := \pi^{(j)}(g_3(\varphi)g_1(\vartheta))\,|j, m\rangle \qquad \text{for all} \quad \vec{n} \in S^2. \tag{33}$$

Here $\pi^{(j)}$ is the irreducible $SU(2)$ representation of weight j, and ϑ, φ are coordinates of $\vec{n}$ in S^2. These coherent states are (over-)complete for any fixed j, m, namely

$$\frac{2j+1}{4\pi}\int_{S^2} \mathrm{d}^2 n\ |m, \vec{n}\rangle_j\, \langle m, \vec{n}|_j = \pi^{(j)}(1) \tag{34}$$

is the identity operator in the j representation. We can now project each edge holonomy in an $SU(2)$ spin network to a $U(1)$ holonomy by inserting the projector $|m, \vec{n}\rangle_j\, \langle m, \vec{n}|_j$ in each edge with spin j, where m, $-j \leqslant m \leqslant j$ is arbitrary but fixed (for each j). At this point there arises an arbitrariness because any $SU(2)$ representation splits into several representations (labelled by m) of a $U(1)$ subgroup. This yields the holonomies of a $U(1)$ spin network in the λ-gauge where λ can be chosen arbitrarily for each edge (in the classification one uses only λ which are constant along B for simplicity. Such a choice is always possible by choosing an appropriate section in $P|_B$. However, λ is defined by the action of F in each point of P and can, of course, vary in different points). Note, however, that we have no such projection procedure for Higgs vertices; a projection of a full spin network and, therefore, a partial gauge fixing in the quantum theory can be done completely only in the degenerate sectors which have no Higgs vertices.

Arriving at a $U(1)$ spin network (and disregarding Higgs vertices), we can multiply the corresponding states for each edge by $(4\pi)^{-1}(2j+1)$ and integrate $\vec{n}$ over S^2. Using the completeness relation (34) we see that we get back the original $SU(2)$ spin network.

4.3.3. Quantization of the Gauß constraint. Although the Gauß constraint can be solved by restricting to gauge-invariant spin networks, here we give a regularization in order to exhibit more concretely the role of the reduced gauge group. Therefore, we will start with the constraint in the reduced $U(1)$ theory, and only after quantization extend the operator to an $SU(2)$ gauge-covariant one. The quantization of the area in section 5 gives an example how to quantize an operator after undoing the partial gauge fixing.

Classically, one uses a partial gauge fixing reducing the structure group $SU(2)$ to $U(1)$. This is explicit in the symmetry reduction of the Gauß constraint

$$\mathcal{G}^E[\Lambda] = \frac{4\pi}{\kappa\iota}\int_B \mathrm{d}x\ \Lambda(x)\left[(E^1)' + A_2E^3 - A_3E^2\right]$$

which is taken from [24] adapted to our notation. $\Lambda(x)$ is a Lagrange multiplier. In a pure, one-dimensional $U(1)$ gauge theory one would encounter the constraint $(E^1)' \approx 0$. Here, we have an additional term which provides coupling to the Higgs field. For the sake of clarity we will regularize these two terms one after another.

By standard methods we obtain

$$\begin{aligned}\frac{4\pi}{\kappa\iota}\int_B \mathrm{d}x\, \Lambda(\hat{E}^1)' f_\gamma &= \frac{\hbar}{\mathrm{i}}\int_B \mathrm{d}x\, \Lambda \frac{\mathrm{d}}{\mathrm{d}x}\sum_e \int_e \mathrm{d}t\, \dot{e}^x \delta(x, e(t))\, \mathrm{tr}\left[(h_e(0,t)\tau_3 h_e(t,1))^T \frac{\partial}{\partial h_e}\right] f_\gamma \\ &= \mathrm{i}\hbar \sum_e \int_e \mathrm{d}t\, \dot{e}^x \int_B \mathrm{d}x\, \frac{\mathrm{d}\Lambda}{\mathrm{d}x}\delta(x, e(t))\, \mathrm{tr}\left[(h_e(0,t)\tau_3 h_e(t,1))^T \frac{\partial}{\partial h_e}\right] f_\gamma \\ &= \mathrm{i}\hbar \sum_e \int_e \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\left\{\Lambda(e(t))\, \mathrm{tr}\left[(h_e(0,t)\tau_3 h_e(t,1))^T \frac{\partial}{\partial h_e}\right]\right\} f_\gamma \\ &= \mathrm{i}\hbar \sum_e \left[\Lambda(e(1)) X_L^3(h_e) - \Lambda(e(0)) X_R^3(h_e)\right] f_\gamma .\end{aligned}$$

In the first row, we used the standard quantization rule

$$\hat{E}^1 = \frac{\kappa\hbar\iota}{4\pi\mathrm{i}}\frac{\delta}{\delta A_1}$$

taking into account the symplectic structure (27). The functional derivative acts on a cylindrical function f_γ which depends on edge and point holonomies. As remarked in section 3.1 we start the quantization procedure on the space $(\overline{\mathcal{A}} \times \overline{\mathcal{U}})^\lambda$ due to classical gauge fixing. With λ chosen as in the present section a holonomy on this space takes the form

$$h_e = \exp\left(\int_e \mathrm{d}x\, A_1(x)\tau_3\right)$$

giving rise to the above derivative.

In the second step above we integrated by parts to be able to integrate over the δ-distribution in the following step. Because there the holonomies are Abelian, the trace does not actually depend on t so that we can include it into the argument of the t-derivative in order to integrate over t. Because of the Abelian nature, the left and right invariant vector field components X_L^3 and X_R^3 are identical. However, we keep both of them to make possible a comparison with $SU(2)$ theory.

The Higgs coupling term has to be rephrased before proceeding. We introduce polar coordinates (A, α) in the (A_2, A_3)-plane, i.e. $(A_2, A_3) = (A\cos\alpha, A\sin\alpha)$. These variables have the advantage that A is gauge invariant, whereas α can be gauged arbitrarily. After replacing the variables E^2 and E^3 with the respective derivative operators in the course of quantization, we encounter the derivation

$$A_2\frac{\partial}{\partial A_3} - A_3\frac{\partial}{\partial A_2} = \frac{\partial}{\partial\alpha}$$

the action of which on a point holonomy $h_v(A, \alpha) = \exp A(\cos\alpha\tau_1 + \sin\alpha\tau_2)$ (in our τ_3 gauge) can be calculated, using

$$h_v(A, \alpha) = \exp(\alpha\tau_3) h_v(A, 0) \exp(-\alpha\tau_3),$$

to be

$$\frac{\partial h_v}{\partial\alpha} = \tau_3 h_v - h_v\tau_3.$$

Now we are in a position to quantize the remaining part of the Gauß constraint:

$$\begin{aligned}\frac{4\pi}{\kappa\iota}\int_B \mathrm{d}x\,\Lambda(x)(A_2E^3-A_3E^2)\hat{}f_\gamma &= \frac{\hbar}{\mathrm{i}}\int_B \mathrm{d}x\,\Lambda(x)\frac{\delta}{\delta\alpha(x)}f_\gamma\\ &= \frac{\hbar}{\mathrm{i}}\int_B \mathrm{d}x\,\Lambda(x)\sum_{v\in B}\delta(x,v)\,\mathrm{tr}\left[(\tau_3h_v-h_v\tau_3)^T\frac{\partial}{\partial h_v}\right]f_\gamma\\ &= \mathrm{i}\hbar\sum_{v\in B}\Lambda(v)\left(X_L^3(h_v)-X_R^3(h_v)\right)f_\gamma.\end{aligned}$$

We can now combine the operators to obtain the quantization

$$\hat{\mathcal{G}}^E[\Lambda]=\mathrm{i}\hbar\sum_{v\in B}\Lambda(v)\left(\sum_{e(1)=v}X_L^3(h_e)-\sum_{e(0)=v}X_R^3(h_e)+X_L^3(h_v)-X_R^3(h_v)\right) \tag{35}$$

of the Gauß constraint which has a finite action on cylindrical functions and is therefore densely defined.

The structure of the operator is as expected. It is a sum over vertices weighted with the Lagrange multiplier. In each vertex it acts as the sum of a left invariant vector field for each incoming edge, a right invariant vector field for each outgoing edge, and a left as well as right invariant vector field for the point holonomy in the vertex. That the point holonomy contributes by left and right invariant vector fields represents the fact that the Higgs field transforms in the adjoint representation of $SU(2)$. The sum of vector fields is in complete agreement with the definition of gauge-invariant spin networks with Higgs. However, $\hat{\mathcal{G}}^E$ is a sum only of the third components of the vector fields in accordance with our τ_3 gauge. Recall that $\hat{\mathcal{G}}^E$ is just the quantization on the space of functions over $(\overline{\mathcal{A}}\times\overline{\mathcal{U}})^\lambda$, and that it has to be extended to our full reduced Hilbert space of functions on $(\overline{\mathcal{A}}\times\overline{\mathcal{U}})^{[\lambda]}$ by gauge covariance.

We can change the partial gauge fixing by performing a global $SU(2)$ gauge transformation with $g\in SU(2)\backslash\lambda(F)$. This transformation will conjugate the holonomies appearing in the vector fields. Using the formula

$$\frac{\partial}{\partial g^{-1}hg}=g^T\frac{\partial}{\partial h}(g^{-1})^T$$

which can be proved by using the chain rule, we obtain the transformation rule

$$X_L^i(g^{-1}hg)=\mathrm{tr}\left[(g^{-1}hg\tau_i)^T\frac{\partial}{\partial(g^{-1}hg)}\right]=\mathrm{tr}\left[(hg\tau_ig^{-1})^T\frac{\partial}{\partial h}\right]=\mathrm{Ad}_{ij}(g)X_L^j(h)$$

where $\mathrm{Ad}_{ij}(g)$ are matrix elements in the adjoint representation defined by $\mathrm{Ad}_g\,\tau_i=g\tau_ig^{-1}=:\mathrm{Ad}_{ij}(g)\tau_j$. The right invariant vector fields transform analogously. The transformed Gauß constraint in the $g\tau_3g^{-1}$ gauge is now

$$\begin{aligned}\hat{\mathcal{G}}_3'[\Lambda]&=\mathrm{i}\hbar\sum_{v\in B}\Lambda(v)\Bigg(\sum_{e(1)=v}X_L^3(g^{-1}h_eg)-\sum_{e(0)=v}X_R^3(g^{-1}h_eg)\\ &\quad+X_L^3(g^{-1}h_vg)-X_R^3(g^{-1}h_vg)\Bigg)\\ &=\mathrm{Ad}_{3i}(g)\hat{\mathcal{G}}_i[\Lambda]=:\hat{\mathcal{G}}[\Lambda_i].\end{aligned}$$

Here $\mathcal{G}_i$ are the components of the full $SU(2)$ Gauß constraint smeared with a function Λ, whereas $\mathcal{G}$ is the full Gauß constraint smeared with a function $\Lambda_i:=\mathrm{Ad}_{3i}(g)\Lambda$. Allowing arbitrary $SU(2)$ gauge transformations g, we can change Λ_i arbitrarily. This shows that the

Gauß constraint on Φ_B is the full $SU(2)$ constraint forcing $SU(2)$ spin networks with Higgs to be gauge invariant.

We want to stress here the necessity of the mathematical apparatus developed in section 3. It enabled us to take the role of the reduced gauge group into account, it provides us with an interpretation of symmetric states as generalized states of the unreduced theory and theorem 3.2 shows that all symmetric states can be obtained by using the one-dimensional spin networks used in the present subsection.

4.3.4. Quantization of the diffeomorphism constraint. We conclude by making a few remarks on the diffeomorphism constraint. It can be solved by group averaging where the diffeomorphism group acts by dragging the Higgs vertices. However, we can alternatively regularize the diffeomorphism constraint and solve it infinitesimally. The no-go theorem of [11, appendix C] is evaded by the one-dimensional nature of our graphs. Diffeomorphisms act only by dragging the ends of edges and the Higgs vertices attached to them. However, they cannot deform an edge transversally, and a longitudinal deformation can always be absorbed by a reparametrization of the edge. The quantized constraint is given by Lie derivatives on the Higgs vertex positions leading to the well known solutions in Φ'_B.

5. The area operator in the spherically symmetric sector of loop quantum gravity

In this section we apply our framework to the spectrum of the operator associated with two-dimensional areas. The problem is of considerable physical interest because the two-dimensional horizon of a Schwarzschild black hole constitutes such a system, the area of which is, up to a factor, a measure for the entropy of the black hole.

The quantum area spectrum of the horizon of (Schwarzschild) black holes in four-dimensional spacetime has a longer history. Already in 1974 Bekenstein, using Bohr–Sommerfeld-type arguments [45], suggested that the area $A = 4\pi R_S^2$, $R_S = 2GM^2/c^2$ of a (spherically symmetric) Schwarzschild black hole of mass M has an angular-momentum-like quantum area spectrum, $A(n) \propto n$, $n \in \mathbb{N} \equiv \{n = 1, 2, \ldots\}$, yielding an energy spectrum $E_n \propto \sqrt{n}$. In the meantime such a spectrum has been argued for by many authors (for details and the corresponding literature see [27, 28, 46]).

A very recent group-theoretical quantization based on the classical canonical structure of the Schwarzschild system in D ($\geqslant 4$) spacetime dimensions and the group $SO^{\uparrow}(1, 2)$ yields the spectrum [47]

$$A_{D-2}(k; n) \propto (k + n), \qquad n \in \mathbb{N}_0 \equiv \{n = 0, 1, 2, \ldots\}, \tag{36}$$

where k characterizes the irreducible unitary representation of $SO^{\uparrow}(1, 2)$ or its covering groups. For $SO^{\uparrow}(1, 2)$ itself we have $k \in \mathbb{N}$, for its twofold coverings $SU(1, 1) \cong SL(2, \mathbb{R})$ we have $k \in \frac{1}{2}\mathbb{N}$ and for the universal covering group k may be any positive real number.

On the other hand, the spectrum of the general (non-symmetric) area operator in loop quantum gravity is more complicated [2, 3]. Possible eigenvalues of the area operator in this theory are

$$A \propto \sum_p \sqrt{j_p(j_p + 1)}, \tag{37}$$

where p labels points at which the surface is intersected by a spin network and $j_p \in \frac{1}{2}\mathbb{N}_0$ is the spin of the edge intersecting the surface in p. Here we have ignored the singular case that the surface is intersected in a vertex of the spin network.

There is an important difference between the spectra (36) and (37). Whereas for the former the distance between successive eigenvalues remains the same for any n, that distance becomes smaller and smaller with increasing area for the spectrum (37) [3]. This result has led to expressions of doubts [1, 48, 49] as to the physical validity of the spectrum (36) and its possible implications for the structure of the semiclassical Hawking radiation [50].

Using the framework developed in the present paper we shall show how the spectra (36) and (37) are related and how the two approaches are to be reconciled.

5.1. Partial gauge fixing

Before quantizing the (spherically symmetric) area operator without gauge fixing let us first rephrase the results of [43] in terms of our framework by using the coherent states (33) and quantize the area in its λ-gauge-fixed form.

The angular component of the metric tensor is given by $|E^1|\,\mathrm{d}\Omega^2$, which leads to the classical expression $A(x) = 4\pi|E^1(x)|$ for the area of an S^2-orbit intersecting the radial manifold B at the point x (in a spherically symmetric theory these are the only surfaces whose area can be defined). Writing

$$A(x) = 4\pi|E^1(x)n_i\,n^i| = 4\pi|E_i(x)\,n^i|$$

we can readily quantize it on a gauge-fixed edge (projected from an $SU(2)$ edge of spin j) containing x by using

$$n^i\,\hat{E}_i(x) := \frac{\hbar\kappa\iota}{4\pi}\,|m,\vec{n}\rangle_j\,n^i\,J_i\,\langle m,\vec{n}|_j = \frac{\iota l_P^2}{4\pi}m\,|m,\vec{n}\rangle_j\,\langle m,\vec{n}|_j$$

where J_i is the angular momentum operator acting on the coherent state (see section 4.3.2).

However, this quantization depends on what quantum number m we choose for the projection by means of the coherent state. We can justify the choice $m = \pm j$ by demanding that we should be able to recover the spin of the edge uniquely from the projected data. The simplest way of doing so is given by such a selection of $m(j)$, namely choosing $m = j$ is analogous to the extremization used in [43]. In this way we obtain the spectrum $\frac{1}{2}\iota\,l_P^2\,\mathbb{N}_0$ for the operator, analogously to [43].

In doing so we have used a partial gauge fixing which, as discussed above, is inappropriate in the non-degenerate sector because of the Higgs vertices. If x is a Higgs vertex then we cannot use this operator because we cannot project at that point to a $U(1)$ spin network. This quantization is appropriate only in the degenerate sectors.

5.2. The area operator

We now quantize the area in the non-degenerate sector by using $SU(2)$ gauge-invariant spin networks and at the same time undoing any λ-gauge fixing. We begin by rewriting the area into the form

$$A(x) = 4\pi|E^1(x)| = 4\pi\sqrt{(E^1)^2\,n^i\,n_i} = 4\pi\sqrt{E^i\,E_i} \tag{38}$$

which is similar to the area

$$A(S_x) = \int_{S_x}\mathrm{d}^2\sigma\,\sqrt{E^i\,E_i}$$

of the orbit S_x intersecting B in x in the non-symmetric theory.

From now on we can proceed analogously to the quantization of the area operator in the non-symmetric theory [3]. As discussed in section 4.3.2, E_i gets quantized to

$$\frac{\iota l_P^2}{4\pi\mathrm{i}}\frac{\delta}{\delta A^i}$$

which acts on the $SU(2)$ holonomy $h_e = \mathcal{P}\exp\int_e \mathrm{d}x\, A^i\tau_i$ along the edge e: $[0,1]\to B$.

In order to consider a general point x which can be a Higgs vertex we assume that each edge containing x starts in x. We then have two outgoing radial edges, one oriented like B itself which we denote as e_+ and one oriented oppositely to B which we denote as e_-, and possibly a Higgs vertex in x which does not depend on A^i and which can be understood as representing edges tangential to the surface S_x. By applying the functional derivative $\delta/\delta A^i(x)$ to a cylindrical function f_γ with γ containing the edges e_+, e_- and the Higgs vertex x we obtain

$$\begin{aligned}\hat{E}_i(x) f_\gamma &= \frac{\iota l_P^2}{4\pi\mathrm{i}} \sum_{\epsilon\in\{+,-\}} \int_{e_\epsilon} \mathrm{d}y\, \delta(x,y)\, \mathrm{tr}\left((\tau_i\, h_{e_\epsilon})^T \frac{\partial}{\partial h_{e_\epsilon}}\right) f_\gamma \\ &= \frac{\iota l_P^2}{4\pi} \sum_{\epsilon\in\{+,-\}} \tfrac{1}{2}\epsilon J^i_{e_\epsilon} f_\gamma\,. \end{aligned} \tag{39}$$

Here $J^i_e = -\mathrm{i}X^i_e$ is given by the ith component of the right invariant vector field on $SU(2)$. This leads to the area operator

$$\hat{A}(x) = \tfrac{1}{2}\iota\, l_P^2\sqrt{(J_{e_+} - J_{e_-})^2} = \tfrac{1}{2}\iota l_P^2\sqrt{2J_{e_+}^2 + 2J_{e_-}^2 - (J_{e_+} + J_{e_-})^2}, \tag{40}$$

with eigenvalues

$$\tfrac{1}{2}\iota\, l_P^2\sqrt{2j_+(j_+ + 1) + 2j_-(j_- + 1) - j_v(j_v + 1)}. \tag{41}$$

Here the edges e_+ and e_- carry the spin j_+ and j_-, respectively, and j_v labels the vertex contractor. If x is a Higgs vertex the associated Higgs point holonomy labelled by a spin j can be visualized as a loop with spin j based in x. This is in accordance with the Gauß constraint which we regularized to a sum of invariant vector fields containing a left ($J_H^{(L)}$) and a right ($J_H^{(R)}$) invariant one for the Higgs field:

$$J_{e_+} + J_{e_-} =: J_{e_v} = J_H^{(L)} - J_H^{(R)}.$$

Thus x becomes a 4-vertex whose contractor can be determined by splitting the vertex into two 3-vertices with a new edge e_v connecting the edges e_+ and e_- with the Higgs loop. It is labelled by the spin j_v appearing in the eigenvalue of the area operator. Of course, the Higgs loop as well as the j_v-edge have no spatial extension in the manifold B. Because the Higgs field contributes by left and right invariant vector fields leading to the loop the spin j_v can only be integer valued. This fact has an immediate consequence on the topology dependence of the area operator discussed below. Here we note that in an appropriate topology of Σ the area spectrum is given by all values of the form (41) where $j_v \in \mathbb{N}_0$ and $j_+ \in \frac{1}{2}\mathbb{N}_0$ are arbitrary, whereas j_- is restricted by $|j_+ - j_v| \leqslant j_- \leqslant j_+ + j_v$. In general, however, the topology can impose restrictions on the possible values of the form (41) occurring in the area spectrum.

5.3. Comparison with the non-symmetric operator

The above area operator in the spherically symmetric sector which we obtained by restoring the full $SU(2)$ gauge invariance resembles that in the non-symmetric theory. The only crucial difference comes from the simpler topology of one-dimensional graphs. Therefore, we have no sum over vertices lying on the surface, but only one vertex which represents the whole surface. This difference influences the area spectrum considerably. Disregarding vertex contributions we obtain the spectrum

$$A(j) = \iota l_P^2 \sqrt{j(j+1)}, \qquad j \in \tfrac{1}{2}\mathbb{N}_0, \tag{42}$$

which is only a small subset of the corresponding spectrum (37) in the non-symmetric theory. In particular, for large j the spectrum becomes not dense but equidistant, and in the large-j limit it is consistent with the spectrum (36) of the horizon area.

Thus we have shown that loop quantum gravity in its spherically symmetric sector reproduces for large spins, i.e. in the assumed semiclassical regime, the older results, while it leads to corrections for small j, i.e. at the Planck scale.

5.4. Topology dependence

As in the case of the area operator in the non-symmetric theory the spectrum of the spherically symmetric one depends on the topology of space. Here any surface whose area we can measure in a spherically symmetric theory has, of course, the topology of S^2. However, there are essentially two possible space topologies. A topology with two (or more) boundary components and a second homology $H_2(\Sigma) = \mathbb{Z}$, and one with a single boundary component and $H_2(\Sigma) = 0$ (we regard spatial infinity as a boundary). In the first case we have two physical realizations. The wormhole topology $B = \mathbb{R}$, $\Sigma = \mathbb{R} \times S^2$ represents a spacelike hypermanifold in the Kruskal extension of Schwarzschild (or Reissner–Nordstrøm) spacetime and has two boundary components at $\pm\infty$, whereas the topology $B = \mathbb{R}_+$, $\Sigma = \mathbb{R}^3 \backslash \{0\}$ can be seen as simulating an external, non-dynamical gravitational source sitting in the origin, i.e. in one of the two boundary components of Σ.

The topology with only one boundary component is given by $B = \mathbb{R}_+ \cup \{0\}$, $\Sigma = \mathbb{R}^3$ and has only the boundary at spacelike infinity. Here we have to treat the symmetry centre in 0 along the general framework of symmetry reduction. The isotropy subgroup is $F = S = SU(2)$ and the homomorphism $\lambda: F \to G$ is either $\overline{\lambda}_0: g \mapsto 1$, $\overline{\lambda}_1^{(\prime)}: g \mapsto g \cdot \{\pm 1\} \in G\backslash\{\pm 1\} \cong SO(3)$ or $\overline{\lambda}_1: g \mapsto g$ (up to conjugacy) for all $g \in S$. This can be seen from the fact that the kernel of λ is an invariant subgroup of S which can only be S, $\{\pm 1\}$ or $\{1\}$. By continuity, in the rest of B we must use the homomorphism λ_k if in 0 we use $\overline{\lambda}_k$ ($\overline{\lambda}_1^{(\prime)}$ and $\overline{\lambda}_1$ make no difference). This shows that we have only the possibilities $k = 0, 1$ if the symmetry centre is contained in Σ. The two possibilities lead to manifestly invariant connections ($k = 0$) and to connections invariant up to a gauge ($k = 1$). In all of these cases there can be no Higgs field in 0, which is in accordance with the fact that the Higgs field represents components of an invariant connection tangential to the S-orbits which is a single point in the case of 0. An immediate consequence of this fact is that $F = S$, which implies $\mathcal{L}F_\perp = \{0\}$ (in the Cartan–Killing metric) and therefore the Higgs field which is a map $\phi: \mathcal{L}F_\perp \to \mathcal{L}G$ vanishes.

We can now consider the implications of these considerations as to gauge-invariant spin networks. The crucial observation is that in a Higgs vertex the spins of the neighbouring edge holonomies can differ only by an integer value because the spin j_v mentioned above is integer-valued due to the Higgs loop in the vertex. If Σ has two boundary components we do not have to enforce $SU(2)$ gauge invariance in the two boundary points of B and the edges can be either all integer valued or all half-integer valued. This leads to the full spectrum (41) given above. However, if $0 \in B$ corresponds to the symmetry centre, i.e. an inner point of Σ implying that no Higgs vertex can lie there, we have to impose $SU(2)$ gauge invariance in 0. The edge incident in 0 can only have spin 0, which implies that the other edge spins can only be integer valued. This fact allows only a subset of (41) as the area spectrum.

5.5. Spectroscopy

As a last remark we note that the spectroscopy for spherically symmetric black holes (cf, e.g., [46]) is unaltered by our area spectrum (42) because it becomes uniformly spaced for large j, like the spectrum (36). This is not the case for the full area spectrum (37) of the non-symmetric theory which becomes almost continuous.

The large discrepancy between these two spectra may be understood as a line splitting due to a broken symmetry. Because of the discrete structure of space made explicit by a spin-network, spherical symmetry is strongly broken by a state in the non-symmetric theory. As is well known in quantum theory, breaking a symmetry can lead to a splitting of levels which were previously degenerate. In our case the degeneracy of the levels of a black hole is expected to be huge, growing exponentially with j (see section 3 of [28] and the literature mentioned therein). Splitting of these strongly degenerate levels by a broken symmetry may lead to an almost continuous spectrum as observed in the non-symmetric theory. This observation may also open up a new way to calculate the degeneracy of the energy levels of black holes in loop quantum gravity.

6. Conclusions

The main part of this paper is intended to define a space of symmetric states in a diffeomorphism-invariant theory of connections, to investigate its properties and, in particular, to equip it with a calculus. This is achieved in theorem 3.2. The results show that there are some subtleties, mainly due to the classical partial gauge fixing, which should be relaxed in the quantum theory.

In order to illustrate different points of the general framework we discussed several concrete examples.

(2 + 1)-dimensional gravity and spherically symmetric electromagnetism are quite easy to deal with. Quantum symmetry reduction of these models yields the expected results. Holonomy variables turned out to be well suited in order to quantize the electromagnetic model and its observables. Furthermore, the Ashtekar–Lewandowski measure incorporates the classical reality conditions correctly. The electromagnetic model exhibits a simple reduction of the degrees of freedom to finitely many ones corresponding to the classical theory.

Spherically symmetric gravity is the only example which has necessarily a Higgs field in physically meaningful applications. This property allows a non-trivial action of constraints. The Gauß and diffeomorphism constraint are as easy to deal with as in the full (3 + 1)-dimensional theory. Unfortunately, the Hamiltonian constraint does not seem to be more easily solvable within the framework of loop quantum gravity with its spin-network states than in the full theory. In [24, 25] it was solved, but it, together with the diffeomorphism constraint, had to be rephrased into a form not well suited for loop quantization.

However, regularized analogously to [13] the Hamiltonian constraint operator might be defined and analysed more easily, because of the simplicity of the one-dimensional graphs it acts on. There is no place for it to create new edges and it will create only Higgs vertices in the neighbourhood of a Higgs vertex it acts on, thereby changing the spin of the edge connecting the old and the new Higgs vertex. Note in this context that the appearance of 'Simon's subgraphs' of [51] is generic in the symmetry-reduced theory. The newly created vertices can always be shifted to their neighbouring vertices. Details and further developments are left for forthcoming publications [31].

As a first important physical application of the kinematical framework we calculated the spectrum of the area operator in the spherically symmetric sector of loop quantum gravity. We also discussed the implications for black hole spectroscopy, in particular, the reconciliation of the area spectrum of spherically symmetric loop quantum gravity with the Bekenstein spectrum.

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References

[1] Rovelli C 1997 Loop quantum gravity *Living Rev. Rel.* **1**
Rovelli C 1998 Strings, loops and others: a critical survey of the present approaches to quantum gravity *Plenary Lecture at GR15 Conf. (Pune), Preprint* gr-qc/9803024
Ashtekar A and Krasnov K 1999 *Black Holes, Gravitational Radiation and the Universe (Essays in honour of C V Vishveshwara)* ed B R Iyer and B Bhawal (Dordrecht: Kluwer)
(Ashtekar A and Krasnov K 1998 Quantum geometry and black holes *Preprint* gr-qc/9804039)

[2] Rovelli C and Smolin L 1995 Discreteness of area and volume in quantum gravity *Nucl. Phys.* B **442** 593–619
Rovelli C and Smolin L *Nucl. Phys.* B **456** 753
Loll R 1995 The volume operator in discretized quantum gravity *Phys. Rev. Lett.* **75** 3048–51
Loll R 1996 Spectrum of the volume operator in quantum gravity *Nucl. Phys.* B **460** 143–54
De Pietri R and Rovelli C 1996 Geometry eigenvalues and scalar product from recoupling theory in loop quantum gravity *Phys. Rev.* D **54** 2664–90
Frittelli S, Lehner L and Rovelli C 1996 The complete spectrum of the area from recoupling theory in loop quantum gravity *Class. Quantum Grav.* **13** 2921–32
De Pietri R 1997 On the relation between the connection and the loop representation of quantum gravity *Class. Quantum Grav.* **14** 53–70

[3] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry I: area operators *Class. Quantum Grav.* **14** A55–82

[4] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry II: volume operators *Adv. Theor. Math. Phys.* **1** 388–429

[5] Thiemann T 1998 A length operator for canonical quantum gravity *J. Math. Phys.* **39** 3372–92

[6] Rovelli C 1996 Black hole entropy from loop quantum gravity *Phys. Rev. Lett.* **14** 3288–91

[7] Rovelli C 1996 Loop quantum gravity and black hole physics *Helv. Phys. Acta* **69** 582–611

[8] Krasnov K 1997 Geometrical entropy from loop quantum gravity *Phys. Rev.* D **55** 3505–13

[9] Krasnov K 1998 On statistical mechanics of Schwarzschild black holes *Gen. Rel. Grav.* **30** 53–68

[10] Ashtekar A, Baez J C, Corichi A and Krasnov K 1998 Quantum geometry and black hole entropy *Phys. Rev. Lett.* **80** 904–7

[11] Ashtekar A, Lewandowski J, Marolf D, Mourão J and Thiemann T 1995 Quantization of diffeomorphism invariant theories of connections with local degrees of freedom *J. Math. Phys.* **36** 6456–93

[12] Thiemann T 1996 Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity *Phys. Rev. Lett.* B **380** 257–64

[13] Thiemann T 1998 Quantum spin dynamics (QSD) *Class. Quantum Grav.* **15** 839–73

[14] Thiemann T 1998 Quantum spin dynamics (QSD) II: the kernel of the Wheeler–DeWitt constraint operator *Class. Quantum Grav.* **15** 875–905

[15] Ashtekar A 1987 New Hamiltonian formulation of general relativity *Phys. Rev.* D **36** 1587–602

[16] Barbero G J F 1995 Real Ashtekar variables for Lorentzian signature space-times *Phys. Rev.* D **51** 5507–10
Thiemann T 1996 Reality conditions inducing transform for quantum gauge field theory and quantum gravity *Class. Quantum Grav.* **13** 1383–404

[17] Torre C G 1999 Midi-superspace models of canonical quantum gravity *Int. J. Theor. Phys.* **38** 1081–102

[18] Witten E 1988 2 + 1 dimensional gravity as an exactly soluble system *Nucl. Phys.* B **311** 46–78

[19] Thiemann T 1998 QSD IV: 2 + 1 Euclidean quantum gravity as a model to test 3 + 1 Lorentzian quantum gravity *Class. Quantum Grav.* **15** 1249–80

[20] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 (New York: Wiley) ch II.11
Kobayashi S and Nomizu K 1969 *Foundations of Differential Geometry* vol 2 (New York: Wiley) ch X

[21] Harnad J, Shnider S and Vinet L 1980 Group actions on principal bundles and invariance conditions for gauge fields *J. Math. Phys.* **21** 2719–24

[22] Brodbeck O 1996 On Symmetric gauge fields for arbitrary gauge and symmetry groups *Helv. Phys. Acta* **69** 321–4

[23] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of general relativity *Ann. Phys.* **88** 286

[24] Thiemann T and Kastrup H A 1993 Canonical quantization of spherically symmetric gravity in Ashtekar's self-dual representation *Nucl. Phys.* B **399** 211–58

[25] Kastrup H A and Thiemann T 1994 Spherically symmetric gravity as a completely integrable system *Nucl. Phys.* B **425** 665–86

[26] Thiemann T 1995 The reduced phase space of spherically symmetric Einstein–Maxwell theory including a cosmological constant *Nucl. Phys.* B **436** 681–720

[27] Kastrup H A 1997 Canonical quantum statistics of an isolated Schwarzschild black hole with a spectrum $E_n = \sigma\sqrt{n}\,E_P$ *Phys. Lett.* B **413** 267–73
To the list of papers discussing the spectrum (36) mentioned in this reference the following ones should be added:
Brotz T and Kiefer C 1996 Semiclassical black hole states and entropy *Phys. Rev.* D **55** 2186–91
Vaz C and Witten L 1999 Mass quantization of the Schwarzschild Black Hole *Phys. Rev.* D **60** 024009
Hod S 1998 Bohr's correspondence principle and the area spectrum of quantum black holes *Phys. Rev. Lett.* **81** 4293–6
Mäkelä J and Repo P 1998 How to interpret black hole entropy? *Preprint* gr-qc/9812075
Hod S 1999 Best approximation to a reversible process in black-hole physics and the area spectrum of spherical black holes *Phys. Rev.* D **59** 024014
Vaz C 2000 Canonical quantization, conformal fields and the statistical entropy of the Schwarzschild black hole *Phys. Rev.* D **61** 064017
Gour G 2000 Quantum mechanics of a black hole *Phys. Rev.* D **61** 124007

[28] Kastrup H A 1999 *Ann. Phys., Lpz.* to appear
(Kastrup H A 1999 Schwarzschild black hole quantum statistics from $Z(2)$ orientation degrees of freedom and its relations to ising droplet nucleation *Preprint* gr-qc/9906104)

[29] Immirzi G 1997 Real and complex connections for canonical gravity *Class. Quantum Grav.* **14** L177–81
Rovelli C and Thiemann T 1998 The Immirzi parameter in quantum general relativity *Phys. Rev.* D **57** 1009–14

[30] Bojowald M 2000 Loop quantum cosmology: I. Kinematics *Class. Quantum Grav.* **17** 1489–508
Bojowald M 2000 Loop quantum cosmology: II. Volume operators *Class. Quantum Grav.* **17** 1509–26

[31] Bojowald M 2000 Quantum geometry and symmetry *PhD thesis* RWTH Aachen, to be published

[32] Thiemann T 1998 Kinematical Hilbert spaces for fermionic and Higgs quantum field theories *Class. Quantum Grav.* **15** 1487–512

[33] Larsen R 1973 *Banach Algebras, an Introduction* (New York: Dekker)
Conway J B 1997 *A Course in Functional Analysis* (*Graduate Texts in Mathematics* vol 96) (Berlin: Springer)

[34] Ashtekar A and Lewandowski J 1995 Projective techniques and functional integration for gauge theories *J. Math. Phys.* **36** 2170–91
Ashtekar A and Lewandowski J 1995 Differential geometry on the space of connections via graphs and projective limits *J. Geom. Phys.* **17** 191–230

[35] Cordero P and Teitelboim C 1976 Hamiltonian treatment of the spherically symmetric Einstein–Yang–Mills system *Ann. Phys.* **100** 607–31

[36] Cordero P 1977 Canonical formulation of the spherically symmetric Einstein–Yang–Mills–Higgs system for a general gauge group *Ann. Phys.* **108** 79–98

[37] Bojowald M 1998 Rotationssymmetrische Quantengravitation im Rahmen der Spinnetzwerke *Diploma Thesis* RWTH Aachen

[38] Bojowald M 2000 Abelian BF-theory and spherically symmetric electromagnetism *J. Math. Phys.* **41** 4313–29
(Bojowald M 1999 Abelian BF-theory and spherically symmetric electromagnetism *Preprint* hep-th/9908170)

[39] Bröcker T and tom Dieck T 1985 *Representations of Compact Lie Groups* (Berlin: Springer)

[40] Bourbaki N 1987 *Topological Vector Spaces* (Berlin: Springer)

[41] Ashtekar A and Lewandowski J 1994 Representation theory of analytic holonomy $C^\star$-algebras *Knots and Quantum Gravity* ed J C Baez (Oxford: Oxford University Press)

[42] Corichi A and Krasnov K 1998 Loop quantization of Maxwell theory and electric charge quantization *Mod. Phys. Lett.* A **13** 1339–46

[43] Krasnov K 1998 The area spectrum in quantum gravity *Class. Quantum Grav.* **15** L47–53

[44] Perelomov A M 1972 Coherent states for arbitrary Lie groups *Commun. Math. Phys.* **26** 222–36

[45] Bekenstein J D 1974 The quantum mass spectrum of a Kerr black hole *Lett. Nuovo Cimento* **11** 467–70

[46] Bekenstein J D 1998 Black holes: classical properties, thermodynamics and heuristic quantization *Preprint* gr-qc/9808028

[47] Bojowald M, Kastrup H A, Schramm F and Strobl T 2000 *Phys. Rev.* D July
(Bojowald M, Kastrup H A, Schramm F and Strobl T 1999 Group theoretical quantization of a phase space $S^1 \times \mathbb{R}^+$ and the mass spectrum of Schwarzschild black holes in D space-time dimensions *Preprint* gr-qc/9906105)

[48] Smolin L 1996 Deviations from Hawking radiation? *Matters Grav.* **7** 10–1

[49] Barreira M, Carfora M and Rovelli C 1996 Physics with non-perturbative quantum gravity: radiation from a quantum black hole *Gen. Rel. Grav.* **28** 1293–9

[50] Bekenstein J D and Mukhanov V F 1995 Spectroscopy of the quantum black hole *Phys. Lett.* B **360** 7–12

[51] Lewandowski J and Marolf D 1998 Loop constraints: a habitat and their algebra *Int. J. Mod. Phys.* D **7** 299–330