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On Symmetric Gauge Fields for arbitrary Gauge and Symmetry Groups

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Abstract. A classification of the possible symmetric principal bundles with a compact gauge group, a compact symmetry group and a base manifold which is regularly foliated by the orbits of the symmetry group is derived. A generalization of Wang's theorem (classifying the invariant connections) is proven and local expressions for the gauge potential of an invariant connection are given.

1 Introduction

A basic motivation to consider symmetric gauge fields is the reduction of the classical degrees of freedom of a gauge field theory. However, imposing a symmetry for gauge fields leads to a surprise similar to the "Kaluza-Klein miracle": The Yang-Mills action for symmetric fields is a Yang-Mills-Higgs action in a lower dimension. This "dimensional reduction" has been studied by several authors and is used, for instance, to construct generalized Kaluza-Klein theories with an additional Yang-Mills field and a homogeneous space as internal space. (For a review and key references see, e.g., [1].)

One approach for a systematic discussion of symmetric gauge fields is the following one. First, one constructs those principal bundles which admit an action of the symmetry group (by automorphisms), such that the induced action on the base agrees with a given one. Then, one classifies the invariant connections in these bundles and, finally, one derives local expressions for the gauge potentials of an invariant connection. The resulting fields are then, by construction, invariant up to a gauge transformation. (For an alternative approach see [2].)

On the basis of previous works [3], we have carried out these steps for arbitrary gauge und symmetry groups. Our results close a gap in the literature for the bundle classification and simplify the last two steps. Below we give a brief description of our investigations. (Details will be published elsewhere; for partial results see Appendix I of [4].)

2 Symmetric bundles with a regularly foliated base

For a symmetric principal bundle $P(M, G)$ with a compact gauge group G and a compact symmetry group K, we first consider the induced K-action on the base manifold M. Since the symmetry group is compact there exists an open and dense submanifold $M_{(H)} \subset M$ (the principal orbit bundle) which, at least locally, is regularly foliated by the orbits of K. That is, locally, $M_{(H)} \approx M_{(H)}/K \times K/H$ for a (compact) subgroup $H \subset K$ [5]. With a minor loss of generality, we may assume, therefore, that the base has the form

$$M = \tilde{M} \times K/H ,$$

where $\tilde{M}$ is a connected manifold, and that the K-action on M agrees with the canonical action of K on the homogeneous space K/H.

Theorem 1. *With the assumptions above, a K-symmetric principal bundle $P(M, G)$ is classified by a group homomorphism $\lambda\colon H \to G$ (out of a complete set of non-conjugate homomorphisms) and a principal bundle $\tilde{Q}(\tilde{M}, Z)$, where Z is the centralizer of the subgroup $\lambda(H) \subset G$.*

We wish to emphasize that this theorem represents a “global” result and that for a proof, no further assumptions are required.

The classifying bundle $\tilde{Q}$ is, as expected, a subbundle of $P|_{\tilde{M}}$, the portion of P over the submanifold $\tilde{M} \cong \tilde{M} \times \{eH\}$. To show this, we first observe that $\tilde{M} \times \{eH\}$ is a fixed point set of the subgroup $H \subset K$. Thus, each fiber of $P|_{\tilde{M}}$ is maped onto itself by the action of H. Next, let us use this property to introduce the map

$$\mu : P|_{\tilde{M}} \times H \to G , \qquad (p, h) \mapsto \mu_p(h) ,$$

where $\mu_p(h)$ is defined by

$$h \cdot p \ = \ p \cdot \mu_p(h) \ .$$

Then, for each $p \in P|_{\tilde{M}}$, the restriction $\mu_p\colon H \to G$ is a group homomorphism and for points in the same fibre of $P|_{\tilde{M}}$, the corresponding homomorphisms belong to the same conjugacy class. The following lemma now completes the construction.

Main Lemma. *Let p_0 be an arbitrary point of $P|_{\tilde{M}}$. Let $\lambda = \mu_{p_0}$ be the homomorphism corresponding to p_0 and let Z be the centralizer of the subgroup $\lambda(H) \subset G$. Then*

$$\tilde{Q}(\tilde{M}, Z) = \left\{ \ p \in P|_{\tilde{M}} \ \mid \ \mu_p = \lambda \ \right\}$$

is a reduced bundle of $P|_{\tilde{M}}$ *with structure group* Z.

Conversely, a symmetric bundle P can be recovered from the classifying pair $(\lambda, \tilde{Q})$ by means of a standard construction. More precisely, P is "equivalent" to a bundle which is associated to the product bundle

$$P'(M, G') = \tilde{Q}(\tilde{M}, Z) \times K(K/H, H)$$

and has G as typical fiber. For later use, we give some details of the construction. The product bundle P' is, by definition, a principal bundle over $M = \tilde{M} \times K/H$ with structure group $G' = Z \times H$. Moreover, since the second factor is K-symmetric, the same is true for P'. Next, recall that Z is the centralizer of the subgroup $\lambda(H) \subset G$. Thus, the homomorphism $\lambda\colon H \to G$ naturally extends to a homomorphism from the structure group G' of P' to the structure group G of P:

$$\rho : G' = Z \times H \longrightarrow G\ , \qquad (z, h) \longmapsto z\, \lambda(h)\ .$$

Now, let G' act on G by

$$G' \times G \longrightarrow G\ , \qquad g' \longmapsto \rho(g')\, g\ .$$

Then the associated bundle $P' \times_{G'} G$ is a K-symmetric G-principal bundle over M which is equivalent to the given bundle P. (By an equivalence we mean a K- and G-equivariant bundle isomorphism which induces the identity of the base.)

3 Invariant connections and symmetric gauge potentials

Theorem 2. (Generalized Wang theorem) *Let P be a K-symmetric principal bundle classified by $(\lambda, \tilde{Q})$ and let ω be a connection in P which is invariant under the action of K. Then ω is classified by a connection $\tilde{\omega}$ in $\tilde{Q}$ and a scalar field $\tilde{\phi}$ over $\tilde{Q}$ with values in the linear subspace of $LG \otimes LH_\perp^*$ defined by*

$$\mathrm{Ad}(\lambda(h)) \circ \tilde{\phi} = \tilde{\phi} \circ \mathrm{Ad}(h)$$

for all $h \in H$. Here, LG denotes the Lie algebra of G and $LH_\perp$ is the complement of $LH \subset LK$ with respect to an invariant scalar product.

The target space of the scalar field $\tilde{\phi}$ is the fixed point set of a real representation of the isotropy group H. This offers the possibility for a group theoretical discussion of the "constraint equation" for $\tilde{\phi}$. Some technical subtleties arise, however, since the relevant representation is real.

A transparent proof of this theorem is obtained when the structure of the symmetric bundle P is used. To give the basic idea, let us first recall that P is equivalent to the bundle

$\pi\colon P' \times_{G'} G \mapsto M$ which is associated to the principal bundle $\psi\colon P' \to M$. Hence, the diagram

$$\begin{array}{ccc} P' \times G & \xrightarrow{\Psi} & P' \times_{G'} G \\ {\scriptstyle \mathrm{Pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ P' & \xrightarrow{\psi} & M \end{array}$$

where $\Psi\colon (p, g) \mapsto [p, g]$ is the canonical projection, is commutative. On the one hand, Ψ is a K-equivariant bundle map between principal G-bundles. Additionally, Ψ is the projection of a principal bundle with structure group G'. Now, if ω is an invariant connection in $P \cong P' \times_{G'} G$ then, due to the first property of Ψ, the pull-back $\Psi^*\omega$ is an invariant connection in the trivial bundle $\mathrm{Pr}_1\colon P' \times G \to P'$. From this we obtain that $\Psi^*\omega$ is determined by a K-invariant one-form over $P' = \tilde{Q} \times K$. Next, taking advantage of the second property of Ψ, we consider $\Psi^*\omega$ as the pull-back of a one-form by the canonical projection of a principal bundle. This time it follows that $\Psi^*\omega$ is a horizontal one-form (i.e., $\Psi^*\omega$ vanishes along the fibres of the bundle $\Psi\colon P' \times G \to P' \times_{G'} G$), which is invariant under the action of the structure group G'. With these properties of $\Psi^*\omega$ it is now quite easy to complete the proof.

The final step consists in the construction of local gauge potentials for an invariant connection. As it turns out, using results from the proof sketched above, this can easily be achieved. One finds

Corollary 1. *Let P be a K-symmetric principal bundle classified by $(\lambda, \tilde{Q})$, let ω be an invariant connection in P classified by $(\tilde{\omega}, \tilde{\phi})$ and let $\tilde{\sigma}$ and $\hat{\sigma}$ be local sections of the bundles $\tilde{Q}$ and $K(K/H, H)$, respectively. Then there is a local section σ of P such that the gauge potential $A = \sigma^*\omega$ is given by*

$$A = \tilde{\sigma}^*\tilde{\omega} + (\tilde{\sigma}^*\tilde{\phi} + L\lambda) \circ (\hat{\sigma}^{-1}\, d\hat{\sigma}) \; .$$

The presented theorems, together with this corollary, reduce the construction of symmetric gauge potentials to well studied, purely group theoretical problems. However, we have assumed that the symmetry group acts by bundle automorphisms. Physically, it would be more natural to require that the action is realised only projectively, whereby the projective factors are global gauge transformations. Correspondingly, one should then also weaken the invariance condition for the connection.

References

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