Solution

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June 2025

1. Solution to the Black-Scholes Equation

From the replication argument used in the derivation of the Black-Scholes model, we arrive at the following partial differential equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + rx\frac{\partial f}{\partial x} - rf = 0, \tag{1}$$

This expression is equivalent to the standard form commonly found in financial mathematics literature:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \tag{2}$$

where $C(t, S)$ represents the price of a European call option at time t with underlying asset price S, volatility σ, and risk-free interest rate r.

This PDE is solved with the terminal (boundary) condition:

$$f(T, x) = h(x) = \text{máx}\{x - K, 0\}, \tag{3}$$

where T is the maturity time and K is the strike price of the option.

To solve this partial differential equation, we will apply the Fourier transform to convert it into an ordinary differential equation in the frequency domain. After solving the resulting equation, we will use the inverse Fourier transform to recover the solution in the original domain.

To proceed, let us recall some well-known results regarding the Fourier transform. The Fourier transform $\mathfrak{F}$ of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as:

$$\hat{f}(w) = \mathfrak{F}[f](w) = \int_{-\infty}^{\infty} e^{-iwx} f(x)dx,$$

and the inverse Fourier transform is given by:

$$\mathfrak{F}^{-1}[\hat{f}(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw.$$

Additionally, we will make use of the following properties:

1. $\mathfrak{F}[\frac{\partial^n f}{\partial x^n}] = (iw)^n \mathfrak{F}[f]$

2. $\mathfrak{F}[f+g] = \mathfrak{F}[f] + \mathfrak{F}[g]$

3. $\mathfrak{F}[cf] = c\mathfrak{F}[f]$

4. $\mathfrak{F}^{-1}[\mathfrak{F}[f]] = f$

5. $\mathfrak{F}^{-1}[c\hat{f}] = c\mathfrak{F}^{-1}[\hat{f}]$

6. $\mathfrak{F}\left[\frac{1}{s\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2}\right] = e^{-iwm-\frac{s^2w^2}{2}}$

7. $\mathfrak{F}[f * g] = \mathfrak{F}[f] \cdot \mathfrak{F}[g]$

where c, s, and m are constants, and $*$ denotes convolution, defined by:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(z)g(x-z)dz.$$

Solution: We begin by making a change of variables to the Black-Scholes equation in order to convert the boundary condition into an initial condition. Let us define $\tau = T - t$, so that $\tau = 0$ implies $T = t$.

Now we can write the equation in terms of x and τ, but we can simplify further by defining $y = \log(x)$. Let us rewrite $f(t,x)$ as $g(\tau, y)$. Then, by applying the chain rule for functions of several variables, we obtain:

$$\begin{aligned}
f_t(t,x) &= \frac{\partial f(t,x)}{\partial t} = \frac{\partial g(\tau,y)}{\partial \tau}\frac{d\tau}{dt} = -\frac{\partial g(\tau,y)}{\partial \tau} = -g_\tau(\tau,y),\\
f_x(t,x) &= \frac{\partial f(t,x)}{\partial x} = \frac{\partial g(\tau,y)}{\partial y}\frac{dy}{dx} = g_y(\tau,y)\cdot\frac{1}{x},\\
f_{xx}(t,x) &= \frac{\partial^2 f(t,x)}{\partial x^2} = \frac{\partial}{\partial x}\left[g_y(\tau,y)\cdot\frac{1}{x}\right]\\
&= g_{yy}(\tau,y)\cdot\left(\frac{1}{x^2}\right) - g_y(\tau,y)\cdot\left(\frac{1}{x^2}\right)\\
&= \frac{1}{x^2}\left(g_{yy}(\tau,y) - g_y(\tau,y)\right).
\end{aligned}$$

Using this, we can rewrite the Black-Scholes equation in terms of $g(\tau, y)$ as follows:

$$\begin{aligned}
rg(\tau,y) &= rg_y(\tau,y)\cdot\frac{1}{x}\cdot x - g_\tau(\tau,y) + \frac{1}{2}\left[g_{yy}(\tau,y) - g_y(\tau,y)\right]\sigma^2\\
&= rg_y(\tau,y) - g_\tau(\tau,y) + \frac{1}{2}\sigma^2\left(g_{yy}(\tau,y) - g_y(\tau,y)\right).
\end{aligned}$$

Finally, we obtain:

$$rg(\tau,y) = \frac{1}{2}\sigma^2 g_{yy}(\tau,y) - g_\tau(\tau,y) + \left(r - \frac{1}{2}\sigma^2\right)g_y(\tau,y), \qquad (4)$$

with the initial condition:

$$g(0,y) = h(e^y) = \text{máx}\{e^y - k, 0\}. \tag{5}$$

Applying the Fourier transform to equation (4), we obtain:

$$\mathfrak{F}[rg(\tau,y)] = \mathfrak{F}\left[\frac{1}{2}\sigma^2 g_{yy}(\tau,y)\right] - \mathfrak{F}[g_\tau(\tau,y)] + \mathfrak{F}\left[\left(r - \frac{1}{2}\sigma^2\right) g_y(\tau,y)\right],$$

which leads to the following ordinary differential equation:

$$\begin{aligned}\frac{\partial \hat{g}}{\partial \tau} &= -\frac{w^2\sigma^2}{2}\hat{g} + iw\left(r - \frac{1}{2}\sigma^2\right)\hat{g} - r\hat{g}\\ &= \left[-\frac{w^2\sigma^2}{2} + iw\left(r - \frac{1}{2}\sigma^2\right) - r\right]\hat{g},\end{aligned}$$

whose solution is:

$$\hat{g} = e^{\left[-\frac{w^2\sigma^2}{2} + iw\left(r - \frac{1}{2}\sigma^2\right) - r\right]\tau}.$$

Taking into account the initial condition, we obtain:

$$\hat{g} = e^{-r\tau}\hat{h}\cdot e^{\left[iw\left(r-\frac{1}{2}\sigma^2\right) - \frac{\sigma^2 w^2}{2}\right]\tau}. \tag{6}$$

On the other hand, from property (6) of the Fourier transform (previously labeled **??**), using the substitutions $m = \left(\frac{\sigma^2}{2} - r\right)\tau$ and $s = \sigma\sqrt{\tau}$, we have:

$$e^{-iw\left(\frac{\sigma^2}{2}-r\right)\tau - \frac{(\sigma\sqrt{\tau})^2 w^2}{2}} = \mathfrak{F}\left[\frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{y-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}\right],$$

so we can write:

$$\hat{g} = e^{-r\tau}\hat{h}\cdot\mathfrak{F}\left[\frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{y-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}\right].$$

By property (7) of the Fourier transform (convolution theorem), and using the linearity of the transform:

$$\hat{g} = e^{-r\tau}\cdot\frac{1}{\sigma\sqrt{2\pi\tau}}\cdot\mathfrak{F}\left[h * e^{-\frac{1}{2}\left(\frac{y-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}\right].$$

Applying the inverse Fourier transform and the definition of convolution, we obtain:

$$
\begin{aligned}
g(y,\tau) &= e^{-r\tau}\cdot\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} h(z)e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}dz \\
&= e^{-r\tau}\cdot\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} \text{máx}(e^z-K,0)e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}dz.
\end{aligned}
$$

Now, we adjust the limits of integration to exclude the region where the integrand is zero. Specifically, $\text{máx}(e^z-K,0)>0$ when $e^z\geq K$, or equivalently when $z\geq\ln K$.

Thus, we split the integral as:

$$
\begin{aligned}
g(y,\tau) &= e^{-r\tau}\cdot\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty} (e^z-K)e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}dz \\
&= e^{-r\tau}\cdot\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty} e^z e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}dz \\
&\quad - e^{-r\tau}\cdot\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty} Ke^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}dz.
\end{aligned}
$$

Therefore, to find the solution, we only need to evaluate these two integrals. Let us now proceed step by step with each one, starting with the second integral, which turns out to be simpler. We consider:

$$
I = e^{-r\tau}\int_{\ln K}^{\infty} K\cdot e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2}dz.
$$

We perform a change of variables: let $u=\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}$, so that $du=\frac{-dz}{\sigma\sqrt{\tau}}$. When $z\to\ln K$, it follows that $u\to\frac{y-\left(\ln K+\left(\frac{\sigma^2}{2}-r\right)\tau\right)}{\sigma\sqrt{\tau}}$, and when $z\to\infty$, $u\to-\infty$.

Applying the change of variables, we obtain:

$$
Ke^{-r\tau}\cdot\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\frac{y-\left(\ln K+\left(\frac{\sigma^2}{2}-r\right)\tau\right)}{\sigma\sqrt{\tau}}}^{-\infty} e^{-\frac{1}{2}u^2}\cdot(-\sigma\sqrt{\tau})\,du.
$$

After canceling the term $\sigma\sqrt{\tau}$ and switching the integration limits, we get:

$$
Ke^{-r\tau}\int_{-\infty}^{\frac{y-\ln K+\left(-\frac{\sigma^2}{2}+r\right)\tau}{\sigma\sqrt{\tau}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du = Ke^{-r\tau}\Phi\left(\frac{y-\ln K+\left(-\frac{\sigma^2}{2}+r\right)\tau}{\sigma\sqrt{\tau}}\right),
$$

where Φ is the cumulative distribution function (CDF) defined by:

$$
\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du.
$$

The first integral follows a similar process, but we will need to complete the square to express it in terms of the cumulative distribution function.

Recall that the first integral is:

$$\begin{aligned}I &= e^{-r\tau} \cdot \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} e^{z} \cdot e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2} dz \\ &= \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} e^{z} \cdot e^{-r\tau} \cdot e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2} dz.\end{aligned}$$

Our task is to complete the square in the exponent with respect to the variable z. Let us expand and simplify the exponent expression:

$$\frac{2z\sigma^2\tau - 2\sigma^2 r\tau^2 - [y - (z + (\frac{\sigma^2}{2} - r)\tau)]^2}{2\sigma^2\tau}.$$

Squaring the term $[y - (z + (\frac{\sigma^2}{2} - r)\tau)]$ and expanding:

$$\frac{2z\sigma^2\tau - 2\sigma^2 r\tau^2 - y^2 + y\tau\sigma^2 - 2y\tau r - \frac{\sigma^4}{4}\tau^2 + \sigma^2 r\tau^2 - r^2\tau^2 + 2yz}{2\sigma^2\tau} + \frac{-z^2 - z\tau\sigma^2 + 2z\tau r}{2\sigma^2\tau}.$$

Grouping all the terms that involve z, we obtain:

$$\frac{-z^2 - z\tau\sigma^2 + 2z\tau r + 2z\sigma^2\tau + 2yz}{2\sigma^2\tau} = \frac{-z^2 + 2z\left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)}{2\sigma^2\tau}.$$

We now complete the square by adding and subtracting $\left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)^2$:

$$\begin{aligned}&\frac{-\left(z - \left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)\right)^2}{2\sigma^2\tau} + \frac{\left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)^2}{2\sigma^2\tau} \\ &= -\frac{1}{2}\left(\frac{z - \left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)}{\sigma\sqrt{\tau}}\right)^2 + \frac{\left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)^2}{2\sigma^2\tau}.\end{aligned}$$

Reintroducing the remaining terms not involving z, we add:

$$\frac{-2\sigma^2 r\tau^2 - y^2 + y\tau\sigma^2 - 2y\tau r - \frac{\sigma^4}{4}\tau^2 + \sigma^2 r\tau^2 - r^2\tau^2}{2\sigma^2\tau}.$$

Most of these constant terms cancel out when we expand $\left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)^2$, which yields:

$$\left(\left[\frac{\sigma^2}{2} + r\right]\tau + y\right)^2 = \frac{\sigma^4}{4}\tau^2 + \sigma^2 r\tau^2 + \sigma^2\tau y + 2r\tau y + y^2.$$

Comparing with the reintroduced terms,

$$-\sigma^2 r\tau^2 - y^2 + y\tau\sigma^2 - 2y\tau r - \frac{\sigma^4}{4}\tau^2 - r^2\tau^2,$$

we obtain:

$$\left(\left[\frac{\sigma^2}{2}+r\right]\tau + y\right)^2 - \sigma^2 r\tau^2 - y^2 + y\tau\sigma^2 - 2y\tau r - \frac{\sigma^4}{4}\tau^2 - r^2\tau^2 = 2\tau y\sigma^2.$$

Returning to the integral:

$$\begin{aligned} I &= \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty} e^z e^{-r\tau} e^{-\frac{1}{2}\left(\frac{y-z-\left(\frac{\sigma^2}{2}-r\right)\tau}{\sigma\sqrt{\tau}}\right)^2} dz \\ &= \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty} e^{\frac{2\tau\sigma^2 y}{2\tau\sigma^2}} e^{-\frac{1}{2}\left(\frac{z-\left(\left[\frac{\sigma^2}{2}+r\right]\tau+y\right)}{\sigma\sqrt{\tau}}\right)^2} dz \\ &= \frac{1}{\sigma\sqrt{2\pi\tau}}e^y\int_{\ln K}^{\infty} e^{-\frac{1}{2}\left(\frac{z-\left(\left[\frac{\sigma^2}{2}+r\right]\tau+y\right)}{\sigma\sqrt{\tau}}\right)^2} dz. \end{aligned}$$

Now, making the change of variable $u = \frac{y-z+\left(\frac{\sigma^2}{2}+r\right)\tau}{\sigma\sqrt{\tau}}$, it follows that $-\sigma\sqrt{\tau}\,du = dz$. When $z \to \ln K$, then $u \to \frac{y-\ln K+\left(\frac{\sigma^2}{2}+r\right)\tau}{\sigma\sqrt{\tau}}$. Similarly, as $z \to \infty$, $u \to -\infty$.

So, we get:

$$\begin{aligned} I &= \frac{1}{\sigma\sqrt{2\pi\tau}}e^y\int_{\frac{y-\ln K+\left(\frac{\sigma^2}{2}+r\right)\tau}{\sigma\sqrt{\tau}}}^{-\infty} e^{-\frac{1}{2}u^2}(-\sigma\sqrt{\tau})du \\ &= e^y\int_{-\infty}^{\frac{y-\ln K+\left(\frac{\sigma^2}{2}+r\right)\tau}{\sigma\sqrt{\tau}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du \\ &= e^y\cdot\Phi\left(\frac{y-\ln K+\left(\frac{\sigma^2}{2}+r\right)\tau}{\sigma\sqrt{\tau}}\right), \end{aligned}$$

where Φ is the cumulative distribution function defined as:

$$\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du.$$

Therefore, the solution is:

$$g(y,\tau) = e^y\Phi\left(\frac{y-\ln K+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\Phi\left(\frac{y-\ln K+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right).$$

Returning to the variable x via $y = \ln x$, we get:

$$
\begin{aligned}
f(x,\tau) = x\Phi&\left(\frac{\ln x - \ln K + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
&- Ke^{-r\tau}\Phi\left(\frac{\ln x - \ln K + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right).
\end{aligned}
$$

Finally, using logarithmic properties, the solution to the equation becomes:

$$
f(x,\tau) = x\Phi\left(\frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}\Phi\left(\frac{\log(x/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \tag{7}
$$

When we return to the original variable $x = S$, equation (7) becomes the well-known solution of the Black-Scholes equation. It can be shown that this expression is equivalent to:

$$
f(x,t) = xN[d_1(x,t)] - e^{-r(T-t)}KN[d_2(x,t)], \tag{8}
$$

where N is the standard normal cumulative distribution function, and

$$
d_1(x,t) = \frac{\ln(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(x,t) = d_1(x,t) - \sigma\sqrt{T-t}.
$$