Theorem: Let M be an n-dim manifold without boundary. Then, the adjoint exterior derivative is the adjoint of the exterior derivative, that is,

$$(\alpha,\delta\beta)=(d\alpha,\beta)$$

$$\alpha\in\Lambda^k(T^*M),\qquad \beta\in\Lambda^{k+1}(T^*M)$$

Proof: First consider $*^{-1}:\Lambda^k\longrightarrow\Lambda^{n-k}$, such that for $\gamma\in\Lambda^k(T^*M)$

$$*^{-1}\gamma=(-1)^{k(n-k)}(-1)^s*\alpha$$

Also

$$\delta=*d*=(-1)^k*^{-1}d*$$

Finally, we know

$$(\mu,\nu)\sigma=\mu\wedge*\nu,\qquad \sigma\in\Lambda^n$$

From the definition of $*$.

Let

$$\alpha\in\Lambda^k,\ \beta\in\Lambda^{k+1}$$

$$[(d\alpha,\beta)-(\alpha,\delta\beta)]\sigma=d\alpha\wedge*\beta-\alpha\wedge*\delta\beta$$

$$=d\alpha\wedge*\beta-\alpha\wedge*(-1)^k*^{-1}d*\beta$$

$$=d\alpha\wedge*\beta-(-1)^k\alpha\wedge(**^{-1})d*\beta$$

$$=d\alpha\wedge*\beta-(-1)^k\alpha\wedge d*\beta$$

$$=d(\alpha\wedge*\beta).$$

$$\int_M[(d\alpha,\beta)-(\alpha,\delta\beta)]\sigma=\int_M d(\alpha\wedge*\beta)=\int_{\partial M}\cancelto{0}{\alpha\wedge*\beta}$$

Stokes.

Definition: A p-form is co-closed if $\delta x=0$ and it is co-exact if $\alpha=\delta\beta$.

Proposition: Exact and co-exact forms are orthogonal with respect to the inner-product on $\Lambda^k(T^*M)$

Proof: Let $\alpha=\delta\beta$, $\eta=d\mu$

$$(\alpha,\eta)=(\delta\beta,d\mu)=\cancelto{0}{(\delta^2\beta,0)}=\cancelto{0}{(\beta,d^2\mu)}=0.$$

■

Contraction operator i_X

Let T^*M be the n-dim cotangent vector space, and let $\Lambda^p(T^*M)$ be the vector space of p-forms defined on it.

The interior product

$$i_X : \Lambda^p(T^*M) \longrightarrow \Lambda^{p-1}(T^*M)$$

$$i_X\omega(X_1, \ldots, X_p) := \omega(X, X_1, \ldots, X_{p-1})$$

where $\omega \in \Lambda^p(T^*M)$ and $X \in TM$.

take $X = X^\mu \partial_\mu$

$$\omega = \frac{1}{p!}\, \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

Then,

$$i_X\omega = \frac{1}{p!} \sum_{s=1}^{p} X^{\mu_s} \omega_{\mu_1 \ldots \mu_p} (-1)^{s-1} dx^{\mu_1} \wedge \cdots \wedge \underbrace{dx^{\mu_s}}_{\text{absent}} \wedge \cdots \wedge dx^{\mu_p}$$

Thecnically this is a contraction with each entry appearing on the p-form ω.

In particular $f \in \Lambda^0(T^*M)$

$$i_X f = 0.$$

$\alpha \in \Lambda^1(T^*M)$

$$i_X\alpha = \alpha(X) = (\alpha, X)$$

Example: (x, y, z) coordinates in $\mathbb{R}^3$

$$i_{e_x}(dx \wedge dy) = (i_{e_x} dx) \wedge dy - \cancelto{0}{(i_{e_x} dy) \wedge dx}$$

$$= dy$$

$$i_{e_x}(dy \wedge dz) = 0$$

$$i_{e_x}(dz \wedge dx) = -dz$$

Properties of i_{e_x}:

I. $i_X i_Y = -i_Y i_X$

II. $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^p \alpha \wedge i_X \beta.$

III. $i_X(\alpha_1 + \alpha_2) = i_X\alpha_1 + i_X\alpha_2$

Proof:

I. Let $\omega \in \Lambda^p(T^*M)$

$$\omega(X, Y, X_1, X_2, \dots, X_{p-2}) = i_X\omega(Y, X_1, \dots, X_{p-2})$$
$$= i_X i_Y \omega(X_1, \dots, X_{p-2})$$

due to the antisymmetry of ω we have

$$i_X i_Y = - i_Y i_X$$

in particular $i_X^2 = 0$.

II. If $\alpha \in \Lambda^p(T^*M)$ and $\beta \in \Lambda^q(T^*M)$; $p+q \leq n$

$$\alpha \wedge \beta \in \Lambda^{p+q}$$

Follows directly from the definition of exterior product

$$i_X \circ d : \Lambda^p \longrightarrow \Lambda^p$$
$$d \circ i_X : \Lambda^p \longrightarrow \Lambda^p$$

- $i_X d \neq d i_X$

$$\alpha = dx^i \longrightarrow \text{1-form}$$
$$i_X d(\alpha) = \cancelto{0}{i_X d^2 \alpha} = 0$$

$$d i_X \alpha = d\left(X^j \frac{\partial}{\partial x^j}(dx^i) \right)$$
$$= d\left(X^j\, dx^i(x^j) \right) = d(X^j \delta^i_j) = dX^i = \frac{\partial X^i}{\partial x^k}\, dx^k \neq 0.$$

- Neither $i_X d$ nor $d i_X$ are derivations

∴ They are not derivations

$$\alpha \in \Lambda^p \quad , \quad \beta \in \Lambda^q$$

$$i_X d(\alpha \wedge \beta) = i_X\left(d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \right)$$
$$= i_X d\alpha \wedge \beta + \cancelto{0}{(-1)^{p+1} d\alpha \wedge i_X \beta} + \cancelto{0}{(-1)^p i_X\alpha \wedge d\beta} + \alpha \wedge i_X d\beta$$

$$d i_X(\alpha \wedge \beta) = d\left(i_X\alpha \wedge \beta + (-1)^p \alpha \wedge i_X\beta \right)$$
$$= d i_X \alpha \wedge \beta + \cancelto{0}{(-1)^{p-1} i_X\alpha \wedge d\beta} + \cancelto{0}{(-1)^p d\alpha \wedge i_X\beta} + \alpha \wedge d i_X \beta$$

$$(d\circ i_X + i_X\circ d)(\alpha\wedge\beta) = (i_X d\alpha + d i_X\alpha)\wedge\beta + \alpha\wedge(i_X d\beta + d i_X\beta)$$

$$\mathcal{L} := d\circ i_X + i_X\circ d$$ Cartan's Identity.

↳ Lie's derivative

Homework: Show:

I. $[\mathcal{L}_X, d] = 0.$

II. $[\mathcal{L}_X, i_X] = 0.$

Example: Hamiltonian mechanics.

Let $H(p,q)$ be a Hamiltonian function and (p_μ, q^μ) be its phase space → T^*M

we may define a two-form

$$\omega = dp_\mu \wedge dq^\mu$$ called the symplectic two-form

→ Darboux theorem

Also, we may introduce a 1-form such that $\omega = d\theta$, that is

$$\theta = p_\mu dq^\mu$$

Given a function $f(p,q) \in \mathcal{F}(M)$, we may introduce the Hamiltonian vector field

$$X_f := \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu}$$

such that

$$i_{X_f}\omega = -\frac{\partial f}{\partial p_\mu}dp_\mu - \frac{\partial f}{\partial q^\mu}dq^\mu = -df$$

If now we take the vector field generated by the hamiltonian

$$X_H = \frac{\partial H}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu}\frac{\partial}{\partial p_\mu} = \frac{dq^\mu}{dt}\frac{\partial}{\partial q^\mu} + \frac{dp_\mu}{dt}\frac{\partial}{\partial p_\mu}$$

$$= \frac{d}{dt}$$

$$\dot{q}^\mu = \frac{\partial H}{\partial p_\mu} \quad , \quad \dot{p}_\mu = -\frac{\partial H}{\partial q^\mu}$$

- The symplectic form ω is left invariant along the flow generated by X_H

$$\begin{aligned}
\mathcal{L}_{X_H}\omega &= (i_{X_H} d + d\, i_{X_H})\omega \\
&= i_{X_H} d\omega + d(i_{X_H}\omega) \\
&= i_{X_H} \cancelto{0}{d(d\theta)} - \cancelto{0}{d(dH)} = 0
\end{aligned}$$

- Poisson brackets may be cost in terms of Hamiltonian vector fields.

$$\begin{aligned}
\{f,g\} := i_{X_f}(i_{X_g}\omega) &= -i_{X_f}(dg) \\
&= \frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial q^\mu}
\end{aligned}$$

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M).$$