# Ricci & scalar curvature

$\hookrightarrow$ Contract Riemann curvature tensor

**Definition:** Let $R$ be the Riemann tensor of $M$. The Ricci curvature tensor $Ric$ of $M$ in the contraction whose components relative to a coordinate system are $R_{ij} = R^m_{imj}$

$$\rightarrow Ric(X,Y) = \langle R(X,\partial_i)Y, \partial_i\rangle$$

**Notes:**

I. The only non-zero contractions of $R$ are $\pm Ric$ due to symmetries of $R$.

II. $Ric(X,Y) = Ric(Y,X)$ symmetry

**Proof:**
$$Ric(X,Y) = \langle R(X,\partial_i)Y, \partial_i\rangle$$
$$= \langle R(Y,\partial_i)X, \partial_i\rangle \quad \text{Symmetry by pairs.}$$
$$= Ric(Y,X).$$

In components:

$$R_{ij} = R^m_{imj} = g^{km} R_{imjk} = g^{km} R_{jkim} = R^k_{jki} = R_{ji}$$

**Definition:** If on a manifold $M$, its Ricci tensor is identically zero then $M$ is said to be Ricci flat.

**Definition:** The scalar curvature $S$ of $M$ is the contraction of its Ricci tensor.

$$S = \langle R(\partial_i, \partial_j)\partial_i, \partial_j\rangle$$

such that

$$S := R^i_i = g^{ij} R_{ij} = g^{ij} R^k_{ikj}$$

**THEOREM:** $dS = 2\,\mathrm{div}\,Ric.$

**Proof:** 2nd - Bianchi

$$\nabla_Z R(X,Y) + \nabla_X R(Y,Z) + \nabla_Y R(Z,X) = 0$$

$$\nabla_Z \langle R(X,Y)V,W\rangle + \nabla_X \langle R(Y,Z)V,W\rangle + \nabla_Y \langle R(Z,X)V,W\rangle = 0$$

Change $Z \longleftrightarrow X$ in the 3rd-term

$$\nabla_Z \langle R(X,Y)V,W\rangle + \nabla_X \langle R(Y,Z)V,W\rangle - \nabla_Y \langle R(X,Z)V,W\rangle = 0$$

Contracting $z, w$

$$\nabla_{\partial_i}\langle R(X,Y)V,\partial_i\rangle + \nabla_X\langle R(Y,\partial_i)V,\partial_i\rangle - \nabla_Y\langle R(X,\partial_i)V,\partial_i\rangle = 0$$

Contracting $X, V$

$$\nabla_{\partial_i}\langle R(\partial_j,Y)\partial_j,\partial_i\rangle + \nabla_{\partial_j}\langle R(Y,\partial_i)\partial_j,\partial_i\rangle - \nabla_Y\langle R(\partial_j,\partial_i)\partial_j,\partial_i\rangle = 0$$

$$\nabla_{\partial_i} Ric(Y,\partial_i) + \nabla_{\partial_j} Ric(Y,\partial_j) - \nabla_Y S = 0$$

$$2\nabla_{\partial_i} Ric(Y,\partial_i) = \nabla_Y S$$

$\blacksquare$

**Definition:** $G = Ric - \frac{1}{2} g S$ is called the Einstein tensor such that

$$G := R_{ab} - \frac{1}{2} g S$$

**Corollary:** $\nabla G = 0$ Einstein equations in vacuum!

In components
$$\nabla^a G_{ab} = \nabla^a\left(R_{ab} - \frac{1}{2} g_{ab} S\right) = 0.$$

where $\nabla^a := g^{ab}\nabla_b$.