Lie Group.

$$\rho : G \longrightarrow GL(V)$$

let G be a group and ρ be a representation of G to V, and ρ' in V'.

Let $\rho \oplus \rho'$ the direct sume of the representations ρ and ρ', is a representation that there is into the vector space $V \oplus V'$!

$$(\rho \oplus \rho')(g)(v, v') = (\rho(g)v, \rho'(g)v') \; ; \quad \forall \; v \in V, v' \in V'$$

other way is by using the tensor product, let V and V' vector spaces. Let $\{e_i\}$ be a basis of V and $\{e'_j\}$ a basis of V'.

The product $V \oplus V'$, is the vector space which basis are the elements of the form $e_i \otimes e'_j$; thus the dimension of $V \otimes V'$ is $(\dim V)(\dim V')$.

Given $v = v^i e_i \in V$, $v' = v'^j e'_j \in V'$, the tensor product of v and v' is

$$v \otimes v' = v^i v'^j \, e_i \otimes e'_j .$$

Thus the representation tensor product $\rho \otimes \rho'$ of G in $V \otimes V'$ is given by

$$(\rho \otimes \rho')(g)(v \otimes v') = \rho(g)v \otimes \rho'(g)v'$$

let ρ be a representation of G in V. Let's suppose V' being a subspace of V, invariant, i.e., if $v \in V'$, $\rho(g)v \in V'$, $\forall g \in G$.

we may define a representation ρ' of G in V', such that

$$\rho'(g)v = \rho(g)v \quad ; \forall \, v \in V'$$

If ρ has no invariant subspaces, we say that ρ is irreducible

Example: let $U(1)$, for all n, $U(1)$ has one representation ρ_n in $\mathbb{C}$.

$$\rho_n(e^{i\theta})v = e^{in\theta}v$$

$$\rho_n(e^{i\theta} e^{i\theta'})v = \rho_n(e^{i\theta})\rho_n(e^{i\theta})v$$

ρ_n is irreducible

$SU(2)$, is conformed by 2×2 unitary matrices with $\det = 1$.

Let the Pauli matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\{\sigma_1, \sigma_2, \sigma_3\} \cup \{\sigma_0\}$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Basis for the 2x2 hermitian matrices.

Homework: for $i = 1,2,3$. Show $\sigma^2 = 1$. If (i,j,k) is a cyclic permutation of $(1,2,3)$

$$\sigma_i \sigma_j = -\sigma_j \sigma_i = i\sigma_k$$

if we take

$$I = -i\sigma_1 \quad ; \quad J = -i\sigma_2 \quad ; \quad K = -i\sigma_3$$

$$I^2 = J^2 = K^2 = 1$$

$$IJ = -JI = K, \; JK = -KJ = I, \; KI = -IK = J$$

The algebra,

$$\mathbb{H} = \{a\mathbb{1} + bI + cJ + dK \,;\, a,b,c,d \in \mathbb{R}\}$$

is called the quaternions algebra

$$SU(2) = \{a\mathbb{1} + bI + cJ + dK : a^2 + b^2 + c^2 + d^2 = 1\}$$

i.e., $SU(2)$ is S^3 in quaternions.

The unitary representations of $SU(2)$ are called spin-0, spin-1/2, spin-1, ... where the spin-j is the representation of dimension $2j+1$.

Denote the spin-j representation as U_j.

Let H_j be the space of polynomials in $\mathbb{C}^2$, homogeneous of grade $2j$.

If we write $(x,y) \in \mathbb{C}^2$; $x,y \in \mathbb{C}$. An element of H_j is a polynomial in x,y that is a linear combination of terms

$$f(x,y) = x^p y^q \quad , \quad \text{with } p+q = 2j$$

H_j has dimension $2j+1$ and its basis are

$$x^{2j}, x^{2j-1}y, x^{2j-2}y, \dots, y^{2j}$$

for $g \in SU(2)$, let $U_j(g)$, the linear transformation that acts on H_j

$$(U_j(g)f)(v) = f(g^{-1}v) \; ; \quad \forall\, f \in H_j,\; v \in \mathbb{C}^1$$

this is a representation,

$$(U_j(1)f)(v) = f(1^{-1}v) = f(v)$$

$$\begin{aligned}(U_j(g)U_j(h)f)(v) &= (U_j(h)f)(g^{-1}v) \\ &= f(h^{-1}g^{-1}v) \\ &= f((gh)^{-1}v) \\ &= (U_j(gh)f)(v)\end{aligned}$$

The representation spin-1 is of dim=3 (familiar to $\mathbb{R}^3$).

Actually there is an homomorphism.

$$\rho : SU(2) \longrightarrow SO(3).$$

let V be the vector space of 2x2 hermitian matrices without trace. We may identify with $\mathbb{R}^3$ because any matrix of this type

$$T = T^1\sigma^1 + T^2\sigma^2 + T^3\sigma^3 \quad ; \quad T^i \in \mathbb{R}$$

If $T \in V$, and $g \in SU(2)$

$$tr(gTg^{-1}) = tr(T) = 0$$

$$\begin{aligned}(gTg^{-1})^* &= (g^{-1})^* \, T^* \, g^* \\ &= g\,T\,g^{-1},\end{aligned}$$

then $gTg^{-1} \in V$

$$\rho(g)T = gTg^{-1}$$

ρ is a representation of $SU(2)$ in V

$$\rho(g)\rho(h)T = ghTh^{-1}g^{-1} = \rho(gh)T$$

$$\rho(1)T = 1T1^{-1} = T$$

i.e. an homomorphism

$$\rho : SU(2) \longrightarrow GL(V) = GL(3,\mathbb{R})$$

We can ensure that $\rho : SU(2)$ is in $O(3)$.

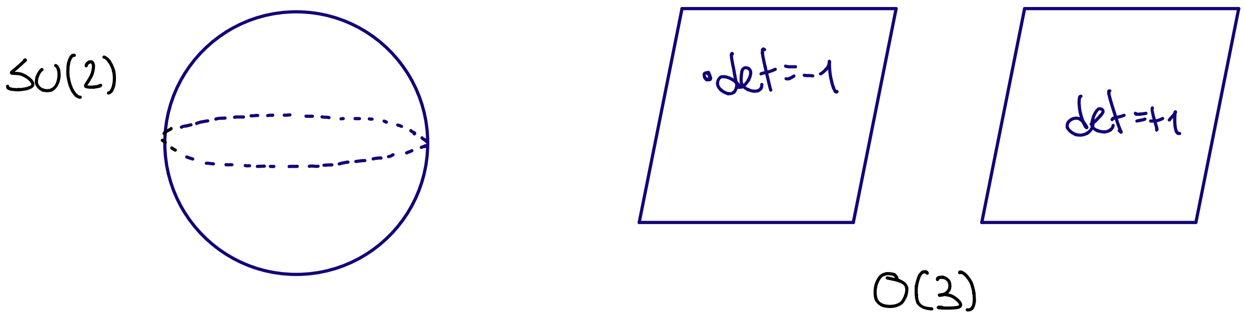
Homework:

$$\det T = \det(aI + bJ + cK)$$
$$= a^2 + b^2 + c^2$$
$$= 1.$$

$$\det(\rho(g)T) = \det(gTg^{-1})$$
$$= \det(g)\det(T)\det(g^{-1})$$
$$= \det(T)$$

$$\rho : SU(2) \longrightarrow O(3)$$

$O(3)$ has matrices of $\det = \pm 1$



$$\rho : SU(2) \longrightarrow SO(3)$$

is not an isomorphism

$$\rho(g)T = gTg^{-1}$$
$$\rho(-g)T = (-g)T(-g)^{-1}$$
$$= gTg^{-1}$$

$$SO(3,1) \longrightarrow SL(2,\mathbb{C}).$$