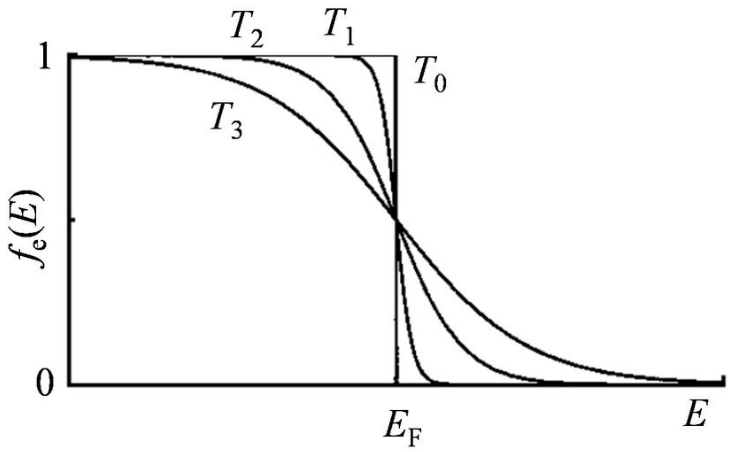
Degenerate Fermi gas



$$\Delta N \approx \gamma D(E_f) V K_B T$$

$$\Delta U \approx K_B T \Delta N \approx \gamma V D(E_f)(K_B T)^2$$

$$C_V = \frac{1}{N}\left(\frac{\partial U}{\partial T}\right) \approx 2\gamma \frac{V}{N} D(E_f) K_B^2 T$$

As $\quad N = \frac{2}{3}\gamma V E_f D(E_f)$, then

$$C_V \approx 3 K_B \frac{T}{T_F} \neq \frac{3}{2} K_B$$

At low temperatures $\quad C_V = \gamma T + \delta T^3$

Sommerfeld expansion

$$I = \int_0^\infty f(\epsilon)\phi(\epsilon)\,d\epsilon$$

$f(\epsilon) =:$ Fermi-Dirac distribution

$\phi(\epsilon) := A\epsilon^n \quad ; \quad n \geq \frac{1}{2}$

$f'(\epsilon)$ has a pronounced peak at $\epsilon = \mu$

$$I = f(\epsilon)\Psi(\epsilon)\Big|_0^\infty - \int_0^\infty \Psi(\epsilon) f'(\epsilon)\, d\epsilon$$

$$\Psi(\epsilon) = \int_0^\infty \phi(\epsilon')\, d\epsilon'$$

$$I = -\int_0^\infty \Psi(\epsilon) f'(\epsilon)\, d\epsilon$$

$$\Psi(\epsilon) = \Psi(\mu) + \left(\frac{d\Psi}{d\epsilon}\right)_{\epsilon=\mu}(\epsilon-\mu) + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{d^k\Psi}{d\epsilon^k}\right)_{\epsilon=\mu}(\epsilon-\mu)^k$$

$$I_k = -\int_0^\infty (\epsilon-\mu)^k f'(\epsilon)\, d\epsilon = \frac{1}{\beta^k}\int_{-\beta\mu}^{\infty} \frac{e^x x^k}{(e^x+1)^2}\, dx$$

$k = 0, 1, 2, 3, \ldots$

if $T \ll T_F$, $\mu \simeq \epsilon_F \gg k_B T$, then

$$I_k \approx \frac{1}{\beta^k}\int_{-\infty}^{\infty} \frac{e^x K^x}{(e^x+1)^2}\, dx$$

for odd k, the integral is zero, since $\frac{e^x}{(e^x+1)}$ is even.

$$I_0 = 1 \qquad I_2 = \frac{\pi^2}{3\beta^2}$$

$$I = \int_0^\mu \phi(\epsilon)\, d\epsilon + \frac{\pi^2}{6}(k_B T)^2 \left(\frac{d\phi}{d\epsilon}\right)_{\epsilon=\mu} + \ldots$$

$$U = \gamma V \int_0^\infty f(\epsilon)\, C \epsilon^{3/2}\, d\epsilon$$

$$= \gamma V C \left\{ \frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}(k_B T)^2 \mu^{1/2} + \ldots \right\}$$

$$N = \gamma V \int_0^\infty f(E)\, C E^{1/2}\, dE$$

$$= \gamma V C \left\{ \frac{2}{3}\mu^{3/2} + \frac{\pi}{12}(k_B T)^2 \mu^{-1/2} + \dots \right\}$$

therefore,

$$E_F^{3/2} = \mu^{3/2} \left\{ 1 + \frac{\pi^2}{8}\left(\frac{k_B T}{\mu}\right)^2 + \dots \right\}$$

$$E_f = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{\gamma}\right)^{2/3}\left(\frac{N}{V}\right)^{2/3}$$

finally,

$$\mu = E_f \left\{ 1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots \right\}$$

$$U = \frac{3}{5} N E_F \left\{ 1 + \frac{5}{12}\pi^2 \left(\frac{T}{T_F}\right)^2 + \dots \right\}$$

$$C_V = \frac{\pi^2}{2} k_B \frac{T}{T_F}$$