# Dust (Non-interacting incoherent matter)

Check:

- D. Brown, CQG 10, 1579 1994, ArXiv: gr-qc/9304026v1
- S. Manoff, ArXiv: gr-qc/0303015v1 (2003)
- O. Minazzoli & T. Harko, ArXiv: 1209.2754v1

$$\mathcal{L} = -c\rho - \rho \int \frac{P(\rho)}{\rho^2}\, d\rho$$

Barotropic $\quad P = P(\rho)$

$\rho$ only depends on $P$

Let $\tau$ be a "proper-time" parameter

$$u^a := \frac{dx^a}{d\tau}$$ velocity vector field.

$$\rho_0 = \rho_0(x)$$ → Scalar field describing the proper density of the flow.

(Average mass density of matter).

$$T^{ab} := \rho_0 u^a u^b$$

Minkowski: $(1,-1,-1,-1)$

$$d\tau^2 = \frac{ds^2}{c^2} = \frac{1}{c^2}\left(c^2 dt^2 - d\vec{x}^2\right)$$

$$= dt^2\left(1 - \frac{\left(\frac{d\vec{x}}{dt}\right)^2}{c^2}\right) = dt^2\left(1 - \frac{v^2}{c^2}\right)$$

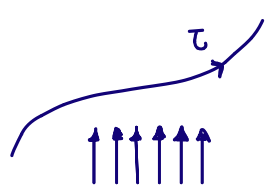
Define

$$\gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

then

$$= dt\, \gamma^{-1}.$$

$$u^a = \frac{dx^a}{d\tau} = \frac{dx^a}{dt}\frac{dt}{d\tau} = \gamma \frac{dx^a}{dt}$$



$$u^a = \gamma(1, \vec{v})$$

$$T^{00} = \rho_0 u^0 u^0 = \rho_0 \gamma^2 =: \rho$$ Relativistic energy density

**Theorem:**

$$\partial_b T^{ab} = 0 \quad \longleftrightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$$

Continuity equation.

**Proof:**

$$\partial_b T^{ab} = \partial_b (\rho_0 u^a u^b)$$

- $a = 0$

$$\partial_b (\rho_0 u^0 u^b) = \partial_0 (\rho_0 u^0 u^0) + \partial_i (\rho_0 u^0 u^i)$$
$$= \partial_0 (\rho_0 \gamma^2) + \partial_i (\rho_0 \gamma^2 v^i)$$
$$= \partial_0 (\rho) + \partial_i (\rho v^i)$$
$$\boxed{= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

- $a = i$ spatial components.

$$\partial_b (\rho_0 u^i u^b) = \partial_0 (\rho_0 u^i u^0) + \partial_j (\rho_0 u^i u^j)$$
$$= \partial_0 (\rho_0 \gamma^2 v^i) + \partial_j (\rho_0 \gamma^2 v^i v^j)$$
$$\boxed{= \partial_0 (\rho v^i) + \partial_j (\rho v^i v^j) = 0}$$

$$0 = (\partial_0 \rho) v^i + \rho (\partial_0 v^i) + \partial_j (\rho v^j) v^i + \rho v^j \partial_j (v^i)$$
$$= \cancel{v^i (\partial_0 \rho + \vec{\nabla} \cdot (\rho \vec{v}))}^{\to 0} + \rho \left[ \frac{\partial v^i}{\partial t} + \vec{v} \cdot \nabla (v^i) \right]$$

then,

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = 0$$

■

Navier-Stokes equation of motion for a perfect fluid

$$\rho \left( \frac{d\vec{v}}{dt} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = -\vec{\nabla} P + \rho X$$

Pressure. Extend force per unit mass.

Dust: Pressureless, forceless.

In a non-flat metric $\partial \longmapsto \nabla$, then

$$\nabla_a T^{ab} = 0$$

$$T^{ab} = \rho_0 u^a u^b$$

Energy conservation remains valid

## Perfect fluid

Again:

$$L_M = -C_\rho - \rho \int \frac{P(\rho)}{\rho^2} d\rho$$

Barotropic $P = P(\rho)$

$$\left.\begin{array}{l} u^a = \dfrac{dx^a}{d\tau} \\ \rho_0 := \rho_0(x) \\ P = P(x) \end{array}\right\} \text{such that} \qquad P(x) \longrightarrow 0 \quad \text{dust.}$$

Propose

$$T^{ab} = \rho_0 u^a u^b + P S^{ab}$$

$S^{ab}$ symmetric and constructed u's and g's

Assume:

$$S^{ab} := u^a u^b + \mu g^{ab}$$

$\mu$ constant.

$u^a u^b$ and $g^{ab}$ are the only one 2nd-rank tensors associated to the fluid!

$$T^{ab} = (\rho_0 + P) u^a u^b + \mu P g^{ab}$$

study $\nabla_b T^{ab} = 0$ (Minkowski)

- Velocity part: Define $\bar{\rho}_0 := \rho_0 + P$ and $\bar{\rho} := \gamma^2 \bar{\rho}_0$, then

$$\bar{\rho}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right]$$

• Metric part: $$\partial_b(\mu P g^{ab}) = \mu g^{ab}\partial_b P = \mu \partial^a P$$

Joining

$$\bar{\rho}\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\mu \vec{\nabla} P$$

$\mu = 1$

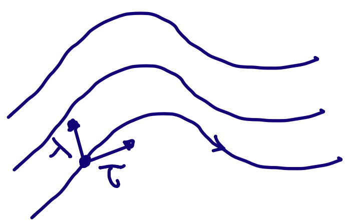
therefore $$\boxed{T^{ab} = \bar{\rho} u^a u^b + P g^{ab}}$$

situation in physics: $P = P(\rho, T)$ Equation of state

Temperature.

## Geodesic deviation

Let $x^a = x^a(\tau, \lambda)$ define a congruence of geodesics on M.




• Proper time
• Label of a geodesic

Convergence of Geodesics

Convergence of curves: Set of integral curves of a vector field

$X_{\gamma(t)} = \dot{\gamma}(t)$

Also $\nabla_{\vec{X}} \vec{X} = 0$

Define two vector fields

$$V^a := \frac{dx^a}{d\tau} \qquad \xi^a := \frac{dx^a}{d\lambda}$$

tangent to a geodesic — vector connecting two close geodesics.

$$[V, \xi]^a = V^b \partial_b \xi^a - \xi^b \partial_b V^a$$
$$= \frac{dx^b}{d\tau}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{d\lambda}\right) - \frac{dx^b}{d\lambda}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{d\tau}\right)$$
$$= \frac{\partial^2 x^a}{d\tau d\lambda} - \frac{\partial^2 x^a}{d\lambda d\tau} = 0$$

Also,

$$[v,\xi]^a = \mathcal{L}_v \xi^a$$
$$= v^b \partial_b \xi^a - \xi^b \partial_b v^a$$
$$= v^b \nabla_b \xi^a - \xi^b \nabla_b v^a$$
$$= \nabla_v \xi^a - \nabla_\xi v^a$$

Then,

$$\nabla_v \xi^a - \nabla_\xi v^a = 0$$
$$\nabla_v \xi^a = \nabla_\xi v^a$$
$$\nabla_v (\nabla_v \xi^a = \nabla_\xi v^a)$$
$$\nabla_v \nabla_v \xi^a = \nabla_v \nabla_\xi v^a$$

we want to find this!

$$R_{XY} Z := -\nabla_{[X,Y]} Z + [\nabla_X, \nabla_Y] Z.$$
$$R^a{}_{bcd} Z^b X^c Y^d = -\nabla_{[X,Y]} Z^a + [\nabla_X, \nabla_Y] Z^a$$

Set $\quad Z^a = v^a, \quad X = v, \quad Y = \xi.$

$$\nabla_v \nabla_\xi v^a - \cancelto{0}{\nabla_\xi \nabla_v v^a} - \cancelto{0}{\nabla_{[v,\xi]} v^a} = R^a{}_{bcd} v^b v^c \xi^d$$

therefore,

$$\nabla_v \nabla_v \xi^a = R^a{}_{bcd} v^b v^c \xi^d$$

finally

$$\boxed{\frac{D^2 \xi^a}{D\tau^2} - R^a{}_{bcd} v^b v^c \xi^d = 0}$$

Geodesic deviation

Tidal gravitational force.

Geodesic deviation describes how two geodesics tend to approach or recede from each other!

Conclusion: Geodesics in flat spacetime mantain their separation and those in curved spacetime don't!