$$\dot{r}_i(t) = \frac{\dot{a}(t)}{a(t)} r_i(t) = H(t)\, r_i(t) \qquad r_i(t) = a(t)\, r_i(t_0)$$

$$V = Hd \qquad \frac{r_i}{a} < \infty$$

$$E = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2 - G\sum_{i<j}^{n} \frac{m_i m_j}{|r_i - r_j|} - \frac{\Lambda}{6}\sum_{i=1}^{n} m_i r_i^2 = \text{constant.}$$

$$A := \frac{1}{2}\sum_{i=1}^{n} m_i [r_i(t_0)]^2$$

$$B := G\sum_{\substack{i,j \\ i<j}}^{n} \frac{m_i m_j}{[r_i(t_0) - r_j(t_0)]} \qquad ;\ A, B, D \text{ constant.}$$

$$D := \frac{\Lambda}{6}\sum_{i=1}^{n} m_i [r_i(t_0)]^2 = \frac{1}{3}\Lambda A$$

$$E = A[\dot{a}(t)]^2 - \frac{B}{a(t)} - D[a(t)]^2$$

Cosmological DE

Special cases:

I. $\Lambda = 0 \longrightarrow D = 0.$

If $a(t)$ is an increasing function (Expanding Universe) then, $\frac{B}{a(t)}$ decreases.

So, as E is constant, $A[\dot{a}(t)]^2$ must be decreasing as well, and expansion must slow down!

II. $\Lambda > 0$ (Cosmic repulsion)

Galaxies are scaping away from the origin out to infinity.

Contributes to the expansion positively

III. $\Lambda < 0$

Cosmic attraction towards the origin

Contributes to the expansion negatively.

(Big crunch solution)

rescaling $a(t) \longmapsto \mu a(t)$; $\mu = \text{constant}$.

$$\boxed{\dot{a}^2 = \frac{C}{a} + \frac{1}{3}\Lambda a^2 - K}$$

$$\cancel{C := \frac{B\mu^3}{A}}$$

$$\cancel{K := -\mu^2 \frac{E}{A}}$$

Newtonian analogous to Friedmann equations

$$E = \mu^2 A \dot{a}^2 - \frac{B}{\mu a} - D\mu^2 a^2$$

$$\frac{E}{\mu^2 A} + \frac{B}{\mu^3 a A} + \frac{Da^2}{A} = \dot{a}^2$$

$$-K \qquad \frac{C}{a} \qquad \frac{1}{3}\Lambda a^2$$

$$\longrightarrow \quad K \begin{cases} 1, & E < 0 \\ 0, & E = 0 \\ -1, & E > 0 \end{cases}$$

Standard cosmological model

$$ds^2 = -dt^2 + a^2(t) \begin{cases} d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi) & K > 0 \\ dx^2 + dy^2 + dz^2 & K = 0 \\ d\psi^2 + \sinh^2(\psi)\,(d\theta^2 + \sin^2(\theta)\, d\phi) & K < 0 \end{cases}$$

Robertson-Walker metric

$$= -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\right] \quad ; K = 1, 0, -1$$

S, F, H.

Consider $t = \text{constant}$ surface, namely at $t = t_0$.

$${}^{(3)}ds^2 = a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\right]$$

$${}^{(3)}g_{rr} = \frac{a_0^2}{1-Kr^2} \qquad {}^{(3)}g_{\theta\theta} = a_0^2 r^2 \qquad {}^{(3)}g_{\phi\phi} = a_0^2 r^2 \sin^2(\theta).$$

where $a_0 := a(t_0)$. Now, for a 3D-space of constant curvature

$${}^{(3)}g^{\alpha\gamma}\, {}^{(3)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\beta\delta} = {}^{(3)}K\, {}^{(3)}g^{\alpha\gamma}\left({}^{(3)}g_{\alpha\beta}\, {}^{(3)}g_{\beta\delta} - {}^{(3)}g_{\alpha\delta}\, {}^{(3)}g_{\beta\gamma}\right)$$

$$= {}^{(3)}K\left(3\, {}^{(3)}g_{\beta\delta} - \delta^{\alpha}_{\beta}\, {}^{(3)}g_{\alpha\delta}\right) = {}^{(3)}K\left(2\, {}^{(3)}g_{\beta\delta}\right)$$

$$= 2\, {}^{(3)}K\, {}^{(3)}g_{\beta\delta}$$

Also, the non-vanishing components of Ricci tensor are

$$ {}^{(3)}R_{rr} = \frac{\lambda'}{r} \qquad {}^{(3)}R_{\theta\theta} = 1 + \frac{1}{2} r e^{-\lambda}\lambda' - e^{-\lambda} $$

$$ {}^{(3)}R_{\phi\phi} = \sin^2(\theta)\, {}^{(3)}R_{\theta\theta} \qquad \text{where } e^{\lambda} = \frac{a_0}{1-Kr^2} $$

then $\boxed{rr}$

$$ {}^{(3)}R_{rr} = 2\, {}^{(3)}K g_{rr} \longrightarrow \frac{\lambda'}{r} = \frac{2K}{1-Kr^2} = 2\, {}^{(3)}K \frac{a_0^2}{1-Kr^2} $$

$$ \lambda' e^{\lambda} = \frac{a_0(2Kr)}{(1-Kr^2)^2} \qquad \lambda' = \frac{2\cancel{a_0}Kr}{\cancel{(1-Kr^2)}} \frac{\cancel{(1-Kr^2)}}{\cancel{a_0}} $$

therefore, $\boxed{{}^{(3)}K = \frac{K}{a_0^2}}$ For $K=-1$ or $K=0$, space is infinite while for $K=1$ it is compact.

For the case $K=1$, the proper circunference goes like ${}^{(3)}L \propto a_0$ and proper volume goes like ${}^{(3)}V \propto a_0^3$.

In conclusion, a_0 measures the "radius of the Universe".

We will adopt this interpretation also for $K \neq 1$.

Geometrically K describes the curvature of 3-dim spatial sections.

Conformal time:

$$ \tau(t) := \int^{t} \frac{dt'}{a(t')} $$

$$ d\tau = d\int^{t} \frac{dt'}{a(t')} = \frac{dt}{a(t)} $$

$$ \rightarrow dt = a(\tau)\, d\tau $$

$$ \longrightarrow ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2(\theta) d^2\phi\right)\right] $$

Conformal time does not measure the proper time of any particular observer but only simplifies calculations.