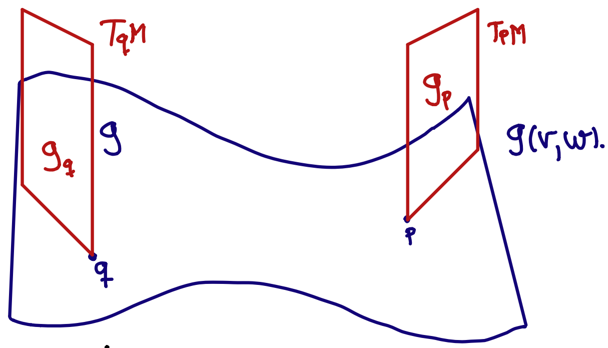
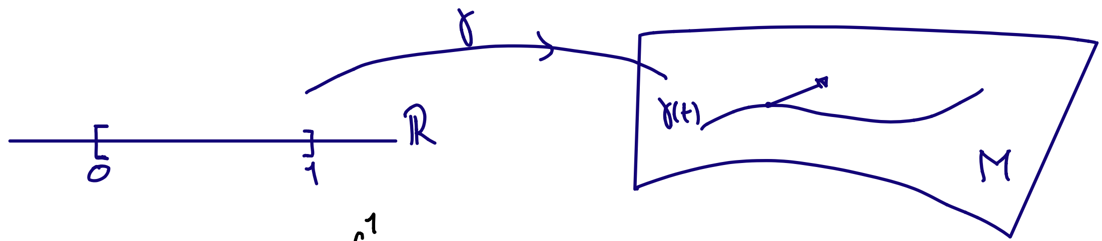
metric $g : V \times V \longrightarrow \mathbb{R}$

let $M$ be a manifold and consider the situation where the metric depends on where is. A metric $g$ on $M$, asign to each point $p \in M$, a metric $g_p$ in the tangent space $T_pM$, which varies smoothly.




The usual use of metrics is to measure time and distances for example, let $\gamma : [0,1] \longrightarrow M$ a spatial curve i.e., its tangent vector is spatial all around.




$$\int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}\, dt$$

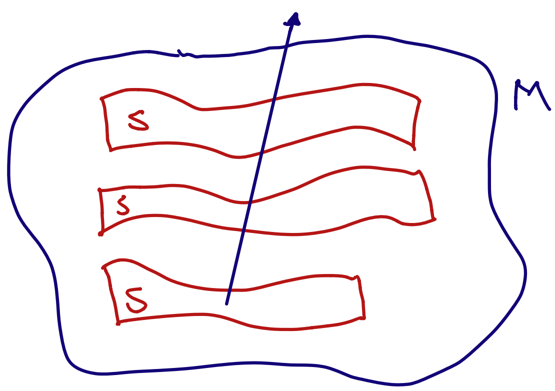
If $\gamma$ is a temporal curve

$$\int_0^1 \sqrt{-g(\gamma'(t), \gamma'(t))}\, dt$$

If the signature of $g$ is $(n,0)$, with $\dim M = n$, we say that $g$ is a Riemannian metric, if is $(n-1,1)$ $g$ is Lorentzian

Actually, the spacetime is a Lorentzian manifold.

$$M = S \times \mathbb{R}, \qquad (x^\mu) = (x^0, x^1, x^2, x^3)$$




$$g = g_t + g_s$$

If $V$ is a vector space with metric $g$, there exists a natural way to convert $v \in V$, in an element of $V^*$, using the linear functional.

$$g(v,\cdot): V \longrightarrow C^\infty(M)$$

Homework: Prove that the map from $V$ to $V^*$, given by

$$v \longmapsto g(v,\cdot)$$

is an isomorphism, i.e., is one-to-one and onto.

$$\}\partial_\mu\{.$$

we will work with a chart, let $e_\mu$ be a basis for the vector fields the components of the metric

$$g_{\mu\nu} \equiv g(e_\mu, e_\nu) \qquad \text{is } n \times n \text{ invertible matrix.}$$

(non-degenerate).

Let $g^{\mu\nu}$ be the inverse matrix.

Exercise: Let $v = v^\mu e_\mu$ a vector field. Prove that the following 1-form $g(v,\cdot)$ is equal to $v_\nu f^\nu$, where $f^\nu$ is the dual basis of the 1-forms and $v_\nu = g_{\mu\nu} v^\mu$

Answer:

$$g(v,\cdot) = g_\nu f^\nu$$

$$g(v^\mu e_\mu, w^\rho e_\rho) = v^\mu w^\rho g(e_\mu, e_\rho)$$

$$= v^\mu w^\rho g_{\mu\rho} = v^\mu w^\nu g_{\mu\nu}$$

$$v_\nu f^\nu(w^\rho e_\rho) = v_\nu w^\rho f^\nu(e_\rho)$$

$$= v_\nu w^\rho \delta^\nu_\rho$$

$$= v_\nu w^\rho$$

$$\Longrightarrow \quad v_\nu = g_{\mu\nu} v^\mu v^\nu$$

Homework: Let $w = w_\mu f^\mu$ be a 1-form, prove that the corresponding vector field is equal to $w^\nu e_\nu$, with

$$w^\nu = g^{\mu\nu} w_\mu.$$

Homework: Let $\eta$ be the Minkowski metric in $\mathbb{R}^4$, prove that its components in the canonical basis are.

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In general, if we have

$$A^{\alpha\beta\cdots\gamma}{}_{\delta\epsilon\cdots\rho}$$

$$g_{\alpha\mu} A^{\mu\beta\cdots\gamma}{}_{\delta\epsilon\cdots\rho} = A_{\alpha}{}^{\beta\cdots\gamma}{}_{\epsilon\cdots\rho}$$

or upper

$$g^{\delta\mu} A^{\alpha\beta\cdots\gamma}{}_{\mu\epsilon\cdots\rho} = A^{\alpha\beta\cdots\gamma\delta}{}_{\epsilon\cdots\rho}.$$

Homework: Prove that

$$g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Given two 1-forms $\omega, \mu$, we call to the function $\langle \omega, \mu \rangle$ the inner product of $\omega$ and $\mu$ (analogous to $g(v,w) = g_{\alpha\beta} v^{\alpha} w^{\beta}$).

$$\langle \omega, \mu \rangle \equiv g^{\alpha\beta} \omega_{\alpha} \mu_{\beta}$$

The inner product of p-forms. Take $e^1, \dots, e^p$ and $f^1, \dots, f^p$, 1-forms in $M$.

$$\langle e^1 \wedge e^2 \wedge \cdots \wedge e^p, f^1 \wedge \cdots \wedge f^p \rangle \equiv \det[g(e^i, f^j)]$$

pxp matrix form by $g(e^i, f^j)$

Homework: Let $E = E_x dx + E_y dy + E_z dz$ be a 1-form in $\mathbb{R}^3$ with the euclidean metric.

$$\langle E, E \rangle = E_x^2 + E_y^2 + E_z^2$$

If $B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy$ is a 2-form.

$$\langle B, B \rangle = B_x^2 + B_y^2 + B_z^2.$$

# Volume form

Let $V$ a vector space of dim $n$, with basis $\{e_\mu\}$, then

$$e_1 \wedge \cdots \wedge e_n \in \Lambda^n V \longrightarrow \text{Volume element.}$$

Suppose $\{f_\nu\}$ other basis of $V$,

$$f_\nu = T^\mu{}_\nu \, e_\mu$$

$$f_1 \wedge \cdots \wedge f_n = (T^1_1 e_1 + \cdots + T^n_1 e_n) \wedge \cdots \wedge (T^1_n e_1 + \cdots + T^n_n e_n)$$

$$= (\det T)\, e_1 \wedge \cdots \wedge e_n.$$

Since, is the sum of expressions of the form

$$\operatorname{sign}(\sigma)\, T^{\sigma(1)}_1 \cdots T^{\sigma(n)}_n \, e_1 \wedge \cdots \wedge e_n.$$

and $\sigma$ the permutation.

Let $M$ be a manifold of dim $n$, the volume form $\omega$ in $M$, is an $n$-form different than cero. In $\mathbb{R}^n$

$$\omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

$$\int_{\mathbb{R}^3} f\, dx\, dy\, dz = \int_{\mathbb{R}^3} f\, dx \wedge dy \wedge dz.$$

If $M$ is a manifold of dim $n$, with metric $g$, there exist a canonical volume form,

Cover $M$ with charts, $\varphi_\alpha : U^\alpha \to \mathbb{R}^n$

$$g(\partial_\mu, \partial_\nu) = g_{\mu\nu}$$

is defined

$$\mathrm{Vol} \equiv \sqrt{|\det g_{\mu\nu}|}\, dx^1 \wedge \cdots \wedge dx^n.$$

$$\int_M \sqrt{|\det g_{\mu\nu}|}\, f\, dx_1 \wedge \cdots \wedge dx_n$$

$$S = \int_M \sqrt{g}\, d^4x\, \mathcal{L}.$$