$$d: \Lambda^p(T^*M) \longrightarrow \Lambda^{p+1}(T^*M)$$

Example: In $\mathbb{R}^3$

$$\omega_0 = f(x,y,z)$$

$$\omega_1 = \omega_x dx + \omega_y dy + \omega_z dz$$

$$\omega_2 = \omega_{xy}\, dx \wedge dy + \omega_{yz}\, dy \wedge dz + \omega_{zx}\, dz \wedge dx.$$

$$\omega_3 = \omega_{xyz}\, dx \wedge dy \wedge dz.$$

$$d\omega_0 = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad = \nabla f \cdot d\vec{r} \quad (\text{grad})$$

$$d\omega_1 = \frac{\partial \omega_x}{\partial y} dy \wedge dx + \frac{\partial \omega_x}{\partial z} dz \wedge dx$$

$$+ \frac{\partial \omega_y}{\partial x} dx \wedge dy + \frac{\partial \omega_y}{\partial z} dz \wedge dy$$

$$+ \frac{\partial \omega_z}{\partial x} dx \wedge dz + \frac{\partial \omega_z}{\partial y} dy \wedge dz$$

$$d\omega_1 = \left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z}\right) dy \wedge dz$$

$$+ \left(\frac{\partial \omega_z}{\partial x} - \frac{\partial \omega_x}{\partial z}\right) dz \wedge dx \quad = (\vec{\nabla} \times \vec{\omega})_z \quad (\text{rot})$$

$$d\omega_2 = \frac{\partial \omega_{xy}}{\partial z} dz \wedge dx \wedge dy + \frac{\partial \omega_{yz}}{\partial x} dx \wedge dy \wedge dz + \frac{\partial \omega_{zx}}{\partial y} dy \wedge dz \wedge dx$$

$$= \left(\frac{\partial \omega_{yz}}{\partial x} + \frac{\partial \omega_{zx}}{\partial y} + \frac{\partial \omega_{xy}}{\partial z}\right) dx \wedge dy \wedge dz. \quad (\text{div})$$

$$d\omega_3 = 0$$

$$\text{Curl}(\text{grad } \omega_0) = 0$$

$$d(d\omega_0) = d^2\omega_0 = 0.$$

$$\text{div}(\text{curl } \omega_1) = 0$$

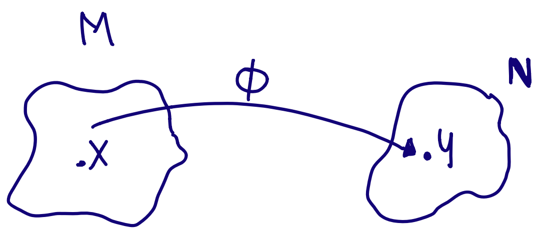
$$d(d\omega_1) = d^2\omega_1 = 0$$

Homework: Show $\text{div}(v \times u) = u \cdot \text{curl}\, v - v \cdot \text{curl}\, u$

$$\nabla \cdot (\vec{v} \times \vec{u}) = \vec{u} \cdot (\nabla \times \vec{v}) - \vec{v}(\nabla \times \vec{u})$$

Mappings

Let M and N be two manifolds, and let ϕ be a smooth mapping $\phi: M \longrightarrow N$. Let $x^1, \dots, x^m$ be local coordinates of M and $y^1, \dots, y^n$ be local coordinates of N.



If we consider a real-valued function $g \in \mathcal{F}(N)$

$$g: N \longrightarrow \mathbb{R}.$$

we may build the function $\phi^* g := g \circ \phi$

$$\phi^*: \Lambda^0(T^*N) \longrightarrow \Lambda^0(T^*M)$$

ϕ^* induced map by ϕ. "Pullback"

We want to define $\phi^*_p: \Lambda^p(T^*N) \longrightarrow \Lambda^p(T^*M)$

Let $\omega \in \Lambda^1(T^*N)$, then $\omega = \sum a_i(y)\, dy^i$

$$\phi^*_1 \omega = \sum a_i(y(x)) \frac{\partial y^i}{\partial x^j} dx^j$$

$$\phi^*_1: \Lambda^1(T^*N) \longrightarrow \Lambda^1(T^*M)$$

Extend this map to p-forms by the relation

$$\phi^*_p: \Lambda^p(T^*N) \longrightarrow \Lambda^p(T^*M)$$

$$\phi^*_p(dy^1 \wedge \dots \wedge dy^p) = (\phi^*_1 dy_1) \wedge (\phi^*_1 dy_2) \wedge \dots \wedge (\phi^*_1 dy_p)$$

$$= \frac{\partial y_1}{\partial x^{j_1}} dx^{j_1} \wedge \frac{\partial y_2}{\partial x^{j_2}} dx^{j_2} \wedge \dots \wedge \frac{\partial y_p}{\partial x^{j_p}} dx^{j_p}$$

$$= \left(\frac{\partial y_1}{\partial x^{j_1}} \frac{\partial y_2}{\partial x^{j_2}} \cdots \frac{\partial y_p}{\partial x^{j_p}} \right) dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

$$= \frac{1}{p!} \left(\frac{\partial(y_1, \dots, y_p)}{\partial x^{j_1} \dots \partial x^{j_p}} \right) dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

Proposition: If ω is a p-form, then $d(\phi^*\omega)=\phi^*(d\omega)$

Proof: Take 0-forms

$$dg=\frac{\partial g}{\partial y^i}\,dy^i$$

$$\phi^* dg=\frac{\partial g(\psi(x))}{\partial y^i}\;\frac{\partial y^i}{\partial x^j}$$

$$=\frac{\partial}{\partial x^j}(\phi^* g)\,dx^j$$

$$=d(\phi^* g)$$

$\blacksquare$

Proposition: $(\psi\circ\phi)^*=\phi^*\circ\psi^*$

Proof:

$$[(\psi\circ\phi)^* g](x)=(g\circ(\psi\circ\phi)^*)(x)=g(\psi(\phi(x)))$$

$$=(g\circ\psi)(\phi(x))=(\psi^* g)(\phi(x))$$

$$=((\psi^* g)\circ\phi)(x)=\phi^*(\psi^* g)(x)$$

$$=((\phi^*\circ\psi^*)g)(x).$$

$\blacksquare$

Example:

I. $\phi:\mathbb{R}\longrightarrow\mathbb{R}^2$

$t\longmapsto(x,y)$

$x=t^2,\quad y=t^3$

If $\omega=x\,dy$ is a 1-form in $\mathbb{R}^2$.

$$\phi^*\omega=x(t)\frac{dy(t)}{dt}\,dt=t^2(3t^2)\,dt=3t^4\,dt$$

II. $\psi:\mathbb{R}^2\longrightarrow\mathbb{R}$.

$(x,y)\longmapsto t=x-y$

$$\psi^*(dt)=\frac{dt}{dx}\,dx+\frac{dt}{dy}\,dy=dx-dy.$$

Homework:

I. Consider

$$\phi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$$

$$\phi(x^1,\dots,x^n)=(y^1,\dots,y^n)$$

where $y^i=a^i_j x^j+b^i$, $a^i_j b_j$ constants.

Find $\phi^*(dy^1\wedge\cdots\wedge dy^n)=\det a\,(dx^1\wedge\cdots\wedge dx^n)$

II. Let $\phi:\mathbb{R}^2\longrightarrow\mathbb{R}^2$

$$\phi(x,y)\longmapsto(xy,1)$$

Compute $\phi^*(dx)$, $\phi^*(dy)$ and $\phi^*(y\,dx)$

Theorem: Let α be a p-form and $\{x^i\}$ and $\{y^i\}$ two different coordinates systems in M. The operator d is independent of the coordinate system.

Proof: Let $\alpha=\alpha_H dx^H=\beta_H dy^H$

$$dx^H=\frac{\partial x^H}{\partial y^K}dy^K=J^H_K\,dy^K$$

$$\alpha_H=(J^{-1})^H_K\,\beta_H$$

$$(d\alpha)_x=\frac{\partial\alpha_H}{\partial x^i}dx^i\wedge dx^H=\left(\frac{\partial\alpha_H}{\partial y^j}\frac{\partial y^j}{\partial x^i}\right)\left(\frac{\partial x^i}{\partial y^m}dy^m\right)\left(J^H_K\,dy^K\right)$$

$$=\frac{\partial\alpha_H}{\partial y^m}dy^m\wedge J^H_K\,dy^K$$

$$=\frac{\partial\beta_K}{\partial y^m}dy^m\wedge dy^K$$

$$=(d\alpha)_y$$

$\blacksquare$

Definition: A p-form ω such that $d\omega=0$ is called a closed form.

Definition: A p-form ω such that $\omega=d\theta$ is called an exact form.

$$\text{Exact}\;\substack{\longrightarrow\\ \not\longleftarrow}\;\text{Closed}$$

$$\omega=d\theta\rightarrow d\omega=d^2\theta=0.$$

Example: Consider $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ in $\Lambda^1(\mathbb{R}^2)$

$$d\omega = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx\wedge dy + \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) dy\wedge dx$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right)\right)\right]$$

$$dx\wedge dy = 0\, dx\wedge dy.$$

In polar coordinates,

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$\omega = \frac{x\,dy - y\,dx}{r^2} = \frac{r\cos\theta(\sin\theta\, dr + r\cos\theta\, d\theta) - r\sin\theta(\cos\theta\, dr - r\sin\theta\, d\theta)}{r^2}$$

$$= \cos^2\theta\, d\theta + \sin^2\theta\, d\theta = d\theta$$

ω is not exact as it is not defined at the point $(0,0)$.

Example: Maxwell

$$A = A_\mu dx^\mu \ni \Lambda^1(T^*\mathbb{R}^{1,3})$$

$$F := dA = \frac{\partial A_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu = \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}\right) dx^\nu \wedge dx^\mu = F_{\mu\nu}\, dx^\mu \wedge dx^\nu.$$

$$\boxed{dF = d^2 A = 0}$$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$dF = \frac{\partial F_{\mu\nu}}{\partial x^\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu = 0.$$

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0$$

Bianchi's Identity.

If we identify $E_i = F_{i0}$, and $B_i = \frac{1}{2}\epsilon_i^{jk} F_{jk}$

$$\nabla\times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla\cdot B = 0$$

let's fix $\rho=0$

$$F_{\mu\nu,0}+F_{\nu 0,\mu}+F_{0\mu,\nu}=0$$

$\nu=0$

$\nu=i$

$$F_{\mu 0,0}+\cancel{F_{00,\mu}}^{\to 0}+F_{0\mu,0}=0$$

$$F_{\mu i,0}+F_{i0,\mu}+F_{0\mu,i}=0$$

$\mu=0$

$\mu=j$

$$F_{ji,0}+F_{i0,j}+F_{0j,i}=0$$

$$\frac{\partial F_{ji}}{\partial t}+\left(\frac{\partial E_i}{\partial x^j}-\frac{\partial E_j}{\partial x^i}\right)=0.$$

The other two Maxwell

$$S_m[A]:=-\frac{1}{4}\int_{\mathbb{R}^{1,3}} F_{\mu\nu}F^{\mu\nu}\,d^4x$$

$$=-\frac{1}{4}\int_{\mathbb{R}^{1,3}} F\wedge * F$$

Definition: Let $d:\Lambda^p(T^*M)\longrightarrow\Lambda^{p+1}(T^*M)$ be the exterior derivative operator.

The adjoint operator

$$\delta:\Lambda^p(T^*M)\longrightarrow\Lambda^{p-1}(T^*M)$$

is defined by

$$\delta=*d*$$

$$\delta: p\xrightarrow{*} n-p\xrightarrow{d}(n-p)+1\xrightarrow{*} n-(n-p+1)=p-1.$$

Proposition: $\delta^2 = 0$

Proof:

$$
\begin{aligned}
\delta^2 &= (*d*)(*d*) \\
&= *d(**)d* \\
&\propto *\cancelto{0}{d^2 *} \\
&= 0.
\end{aligned}
$$

Homework: Show

$$d = (-1)^{np} * \delta *$$

Definition: The Laplacian

$$\nabla^2 : \Lambda^p(T^*M) \longrightarrow \Lambda^p(T^*M)$$

is defined by

$$\Delta = -(\delta + \delta)^2 = -(d\delta + \delta d)$$

$\Box$ D'Alembertian in Minkowski.