Homework:

I. $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$

II. $\mathcal{L}_X [Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$

Example: Killing vectors in $(\mathbb{R}^4, \delta)$

$$\Gamma\text{'s} = 0, \quad \partial_\mu X_\nu + \partial_\nu X_\mu = 0 \qquad \text{Killing equations.}$$

$X = X^\mu \partial_\mu$

$\mathcal{X} = \mathcal{X}^\mu \partial_\mu$

$\Longrightarrow$ X must depend linearly on x's

- $X^\mu_{(i)} = \delta^\mu_i \quad , \quad 0 \leq i \leq 3$

  $\hookrightarrow$ Spacetime translations

  $$x^\mu \longrightarrow x^\mu + \epsilon X^\mu$$

- $X_\mu = a_{\mu\nu} x^\nu \qquad (a_{\mu\nu}$ constant$)$

$$\partial_\mu X + \partial_\nu X_\mu = 0$$

$$\Longleftrightarrow \quad \partial_\mu (a_{\nu\rho} x^\rho) + \partial_\nu (a_{\mu\rho} x^\rho) = 0 \qquad \text{Affine Group}$$

$$a_{\nu\rho} \partial_\mu x^\rho + a_{\mu\rho} \partial_\nu x^\rho = 0$$

$$a_{\nu\rho} \delta^\rho_\mu + a_{\mu\rho} \delta^\rho_\nu = 0$$

$$a_{\nu\mu} + a_{\mu\nu} = 0$$

$$a_{\nu\mu} = -a_{\mu\nu} \qquad a\text{'s skew symmetric.}$$

$a_{\mu\nu}$ six quantities $\left\{ \begin{array}{c} 3 \text{ Boosts} \\ \oplus \\ 3 \text{ Rotations} \end{array} \right\}$ This characterises $SO(4)$

In m-dim Minkowski:

$$\frac{m(m+1)}{2} \qquad \text{Killing vector fields.}$$

$$\frac{m(m+1)}{2} = m + (m-1) + \frac{(m-1)(m-2)}{2}$$

- Translations.
- Boosts.
- Rotations.

Definition: Manifolds with $m(m+1)/2$ Killing vectors are called maximally symmetric spaces.

Properties of Killing:

I. $aX+bY$ $(a,b\in\mathbb{R})$ is Killing if X and Y are Killing.

II. $[X,Y]$ is Killing if X and Y are Killing.

$\Big\}$ Lie algebra

Proof:

$$\mathcal{L}_{[X,Y]} = \cancel{\mathcal{L}_X}\mathcal{L}_Y g - \cancel{\mathcal{L}_Y}\mathcal{L}_X g = 0$$

$$\mathcal{L}_{[X,Y]}f = \mathcal{L}_X\mathcal{L}_Y f = \mathcal{L}_X(Y(f)) = X(Y(f))$$

$$-\quad \mathcal{L}_Y\mathcal{L}_X f = Y(X(f))$$

$$[X,Y](f) = \mathcal{L}_{[X,Y]}f.$$

Example:

$$ds^2 = d\theta\otimes d\theta + \sin^2\theta\, d\varphi\otimes d\varphi$$

metric of S^2

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = \partial_\mu X_\nu + \partial_\nu X_\mu - 2\Gamma^\lambda_{\mu\nu}X_\lambda = 0.$$

Killing equations:

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\theta\phi} = \cot\theta$$

I. $\partial_\theta X_\theta + \partial_\theta X_\theta = 0$

II. $\partial_\phi X_\phi + \partial_\phi X_\phi + 2\sin\theta\cos\theta\, X_\theta = 0$

III. $\partial_\theta X_\phi + \partial_\phi X_\theta - 2\cot\theta\, X_\phi = 0.$

$\to$ X_θ does not depend on θ

$$\therefore X_\theta = f(\phi).$$

$\to$ $\partial_\phi X_\phi = -\sin\theta\cos\theta\, f(\theta)$

$$\Rightarrow X_\phi = -F(\phi)\sin\theta\cos\theta + g(\theta)$$

where $$F(\phi) = \int^\phi f(\phi')\,d\phi'$$

$$\partial_\theta X_\theta = -F(\phi)\left[\cos^2\theta - \sin^2\theta\right] + \frac{dg(\theta)}{d\theta}$$

$$\partial_\phi X_\theta = \frac{df(\phi)}{d\phi}$$

$$-F(\phi)(\cos^2\theta - \sin^2\theta) + \frac{dg}{d\theta} + \frac{df}{d\phi}$$

$$+2\cot\theta\left[F(\phi)\sin\theta\cos\theta - g(\theta)\right] = 0$$

$$-F(\phi)\left[\cos^2\theta - \sin^2\theta - 2\cos^2\theta\right] + \frac{dg}{d\theta} + \frac{df}{d\phi} - 2\cot\theta\, g(\theta) = 0$$

$$F(\phi) + \frac{df}{d\phi} = -\left[\frac{dg}{d\theta} - 2\cot\theta\, g(\theta)\right] = -C$$

a) $$\left[\frac{dg}{d\theta} - 2\cot\theta\, g(\theta) = C\right]\frac{1}{\sin^2\theta}$$

$$F = \int f\, d\phi$$

$$\to F' = f$$

$$\frac{d}{d\theta}\left(\frac{g(\theta)}{\sin^2\theta}\right) = \frac{C}{\sin^2\theta}$$

$$\frac{g(\theta)}{\sin^2\theta} = C\int \csc^2\theta\, d\theta = -C\cot\theta + C_1.$$

$$g(\theta) = (C_1 - C\cot\theta)\sin^2\theta$$

b) $$F(\phi) + \frac{df}{d\phi} = -C$$

$$\frac{d}{d\theta}\left(F(\phi) + \frac{df}{d\phi}\right) = \frac{d}{d\phi}(-C) = 0$$

$$F' + f'' = 0$$

$$f + f'' = 0$$

$$f(\phi) = A\sin\phi - B\cos\phi$$

$$F(\phi) = -A\cos\phi - B\sin\phi - C$$

$\therefore$

$$X_\theta(\phi) = A\sin\phi - B\cos\phi$$

$$X_\phi(\theta,\phi) = -(-A\cos\phi - B\sin\phi - C)\sin\theta\cos\theta$$
$$+ (C_1 - C\cot\theta)\sin^2\theta$$
$$= (A\cos\phi + B\sin\phi)\sin\theta\cos\theta + C_1\sin^2\theta.$$

General Killing vector:

$$X = X_\theta \frac{\partial}{\partial\theta} + X_\phi \frac{\partial}{\partial\phi}$$

$$X = (A\sin\phi - B\cos\phi)\partial_\theta + \left[\frac{(A\cos\phi + B\sin\phi)\sin\theta\cos\theta + C_1\sin^2\theta}{\sin^2\theta}\right]\partial_\phi$$

$$= A\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) - B\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) + C_1\frac{\partial}{\partial\phi}$$

$$= AL_y + BL_x + CL_z$$

where L_x, L_y, L_z generates rotations

(Lie algebra $so(3) \simeq su(2)$)

∴ Metric in S^2 remain invariant under $SO(3)$

Definition: Let M be a pseudo-Riemannian manifold with metric g, and let $X \in \mathfrak{X}(M)$. If the infinitesimal displacement ϵX generates a conformal transformation, then X is called a conformal Killing vector.

$$\mathcal{L}_X g = \psi g \quad ; \quad \psi \in \mathcal{F}(M)$$
$$\psi = e^{2\sigma}$$

$$\frac{\partial}{\partial x^\mu}(x^\rho + \epsilon X^\rho)\frac{\partial}{\partial x^\nu}(x^\lambda + \epsilon X^\lambda) g_{\rho\lambda} = e^{2\sigma} g_{\mu\nu}$$

set $\sigma = \frac{\epsilon\psi}{2}$, $\qquad e^{2\sigma} \approx 1 + \epsilon\psi + \mathcal{O}(\epsilon^2)$

$$\longrightarrow \mathcal{L}_X g_{\mu\nu} = X^\lambda\partial_\lambda g_{\mu\nu} + \partial_\mu X^\lambda g_{\lambda\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = \psi g_{\mu\nu}$$

$$g^{\mu\nu}(\psi g_{\mu\nu}) = g^{\mu\nu}[X^\lambda\partial_\lambda g_{\mu\nu} + \partial_\mu X^\lambda g_{\lambda\nu} + \partial_\nu X^\lambda g_{\mu\lambda}]$$

$$m\psi = g^{\mu\nu}X^\lambda\partial_\lambda g_{\mu\nu} + 2\partial_\mu X^\mu$$

$$\varphi = \frac{g^{\mu\nu} X^\lambda \partial_\lambda g_{\mu\nu} + 2\partial_\mu X^\mu}{m}$$

Example: let X^μ be the coordinates of $\mathbb{R}^m$, The vector

$$D := X^\mu \frac{\partial}{\partial X^\mu}$$

(Dilaton) is a CKV.

$$\mathcal{L}_D g = \varphi g$$

Answer:

$$0 + \partial_\mu X^\lambda \delta_{\lambda\nu} + \partial_\nu X^\lambda \delta_{\mu\lambda}$$

$$= \delta_\mu^\lambda \delta_{\lambda\nu} + \delta_\nu^\lambda \delta_{\mu\lambda}$$

$$= \delta_{\mu\nu} + \delta_{\nu\mu}$$

$$= 2\delta_{\mu\nu} = \varphi \delta_{\mu\nu}$$

$$\varphi = 2.$$