Temario.

- Aproximaciones de integrales
- Series
- Espacios de Hilbert.
- Buscar temas

| SAT ↱ Con cita.

Asymptotic approximation of integrals.

[1] Wang (2001) Differential equations and Asymptotyc theory on mathematical physics.
[2] Bender and Orszag (1999) V1. Advanced mathematical methods for scientists and engineers.

Let f and g be two continuous complex functions defined on a subset of the complex plane, $H \subset \mathbb{C}$.

Let z_0 be a limit point of H

Def: $f(z) = \mathcal{O}(g(z))$ as $z \to z_0$ means that there is a constant $K > 0$ and a neighbourhood U of z_0 s.t.

$$|f(z)| \leq K |g(z)| \quad \text{for all} \quad z \in U \cap H$$

Def: $f(z) = o(g(z))$ as $z \to z_0$ means that for every $\epsilon > 0$ there exist a neighbourhood U_ϵ of z_0 s.t.

$$|f(z)| \leq \epsilon |g(z)| \quad \text{for all} \quad z \in U_\epsilon \cap H.$$

Physics: "f is of the same order of g"

$f(z) = \mathcal{O}(g(z)) \sim$ f is big-oh g $\sim$ How fast a function grows or declines

$f(z) = o(g(z)) \sim$ is smaller than.

We may consider

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} < \infty \qquad f(z) = O(g(z))$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0 \qquad f(z) = o(g(z))$$

Ex:

1) $f(x) = \underbrace{4x^3}_{\text{highest grow rate term}} - 3x^2 + 2x - 1 \qquad \text{as } x \to \infty$

$$\Rightarrow f(x) = O(x^3) \qquad \text{as } x \to \infty$$

Check:

$$|f(x)| = |4x^3 - 3x^2 + 2x - 1|$$
$$\leq |4x^3| + |3x^2| + |2x| + |1|$$
$$\leq 4x^3 + 3x^3 + 2x^3 + x^3 = 10x^3 = 10|x^3|$$

therefore

$$f(x) \leq 10|x^3|$$

Other way is:

$$\lim_{x \to \infty} \frac{f(x)}{x^3} = \lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 1}{x^3}$$

$$= \lim_{x \to \infty} \left(4 - \frac{3}{x} + \frac{2}{x^2} - \frac{1}{x^3}\right) = 4$$

2)

$$f(n) = 10 \log n + 5(\log n)^3 + 7n + 3n^2 + 6n^3 \qquad \text{as } n \to \infty$$

then $$f(n) = \Theta(n^3) \quad \text{as} \quad n \to \infty$$

$$|f n| = |10 \log n + 5(\log n)^3 + 7n + 3n^2 + 6n^3|$$
$$\leq |10 \log n| + |5(\log n)^3| + |7n| + |3n^2| + |6n^3|$$
$$\leq 10n^3 + 15n^3 + 7n^3 + 3n^3 + 6n^3 = 31n^3 = 31|n^3|$$

therefore.

$$|f(n)| = 31|n^3|$$

by the other way

$$\lim_{n \to \infty} \frac{f(n)}{n^3} = 6$$

3) $x^2 = o(x)$ as $x \to 0$

$$\lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} x = 0$$

4) $x - \sin x = o(x)$ as $x \to 0$

$$\lim_{x \to 0} \left(\frac{x - \sin x}{x}\right) = \lim_{x \to 0} \left(1 - \frac{\sin x}{x}\right) = 0$$

5) $x - \sin x \neq o(x^3)$

$$\lim_{x \to 0} \left(\frac{x - \sin x}{x^3}\right) = \lim_{x \to 0} \frac{1}{x^2} \left(\frac{x - \sin x}{x}\right)$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6}$$

therefore

$$x - \sin(x) = \Theta(x^3)$$

Def: Let $\{\varphi_n\}_{n\geq 0}$ be a sequence of continuous complex functions defined on $H \subset \mathbb{C}$. We say $\{\varphi_n\}_{n\geq 0}$ is an asymptotic sequence as $z \to z_0$ in H, if we have

$$\varphi_{n+1}(z) = o(\varphi_n(z)) \qquad \forall\, n \geq 0.$$

Def: If $\{\varphi_n\}_{n\geq 0}$ is an asymptotic sequence as $z \to z_0$ we say that

$$\sum_{n=1}^{\infty} a_n \varphi_n(z) \qquad (a_n \text{ constant})$$

is an asymptotic expansion of the function f if for each $N \geq 0$ we have

$$f(z) = \sum_{n=1}^{N} a_n \varphi_n(z) + o(\varphi_N(z)) \qquad \text{as } z \to z_0$$

In this case we write.

$$f(z) \sim \sum_{n=1}^{\infty} a_n \varphi_n(z) \qquad \text{as } z \to z_0.$$

HW: Show that $f(z)$ may be written as

$$f(z) = \sum_{n=1}^{N-1} a_n \varphi_n(z) + O(\varphi_N(z))$$

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0) Integration by parts: (Easy but not general method)

Ex:

$$I(x) = \int_x^{\infty} e^{-4t}\, dt \qquad \text{as } x \to \infty$$

First note that we do not that $x \to 0$. In the case $x \to 0$, Taylor e^{-4t} gives (an) (apparent) divergent. result.

$$e^{-4t} = 1 - t^4 + \frac{t^8}{2!} - \frac{t^{12}}{3!} + \dots \quad \leftarrow \text{Not.}$$

So we write.

$$I(x) = \underbrace{\int_0^\infty e^{-4t}\,dt}_{=:I_1} - \underbrace{\int_0^x e^{-4t}\,dt}_{=:I_2}$$

$$I_1 = \int_0^\infty e^{-4t}\,dt$$

$$s = t^4$$
$$ds = 4t^3\,dt$$
$$dt = \frac{1}{4} s^{-3/4}\,ds$$

$$I_1 = \frac{1}{4}\int_0^\infty e^{-s} s^{-3/4}\,ds = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

$$\Gamma(n) = \int e^{-z} z^{n-1}\,dz$$

$$z\,\Gamma(z) = \Gamma(z+1)$$

$$I_1 = \Gamma\left(\frac{5}{4}\right) \quad \text{as } x \to 0.$$

$$I_2 = \int_0^x e^{-4t}\,dt = \int_0^x \left(1 - t^4 + \frac{t^8}{2!} - \frac{t^{12}}{3!} + \dots\right) dt.$$

$$= \left[t - \frac{t^5}{5} + \frac{t^9}{18} - \frac{t^{13}}{78} + \dots\right]_0^x$$

$$= x - \frac{x^5}{5} + \frac{x^9}{18} - \frac{x^{13}}{78} + \dots$$

then

$$I(x) \sim \Gamma\left(\frac{5}{4}\right) \quad \text{as } x \to 0$$

However, in the case $x \to \infty$ we need to develop $I(x)$ in inverse powers of x.

$$I(x) = -\frac{1}{4}\int_x^{\infty} \frac{1}{t^3}\frac{d}{dt}\left(e^{-t^4}\right)dt$$

$$= -\frac{1}{4}\frac{e^{-t^4}}{t^3}\Bigg|_x^{\infty} - \left(-\frac{1}{4}\right)\int_x^{\infty}\frac{d}{dt}\left(\frac{1}{t^3}\right)dt$$

$$I(x) = \frac{1}{4x^3}e^{-x^4} - \frac{3}{4}\int_x^{\infty}\frac{e^{-t^4}}{t^4}dt$$

but,
$$\int_x^{\infty}\frac{1}{t^4}e^{-4t}dt < \frac{1}{x^4}\int_x^{\infty}e^{-t^4}dt = \frac{1}{x^4}I(x) \ll I(x)$$

leading behaviour of $I(x)$

$$I(x) \sim \frac{1}{4x^3}e^{-x^4}$$