Classical field theory in flat background spacetime

Field $\rightsquigarrow$ Mechanical system in its own right.
$\hookrightarrow$ It fills the hole space
$\rightarrow$ Infinite dimensional.

Only realistic situation: Interaction of fields and particles.

Before	Now
Coordinates	Set of functions
$q_1(t), q_2(t), \dots, q_N(t) = \{q_i(t)\}$	$\varphi^\alpha(\vec{x}, t)$, $\alpha = 0, \dots, n-1$
$t \rightarrow$ Continuous independent parameter.	$x := (\vec{x}, t) \rightsquigarrow$ Continuous background independent parameters. } label the degrees of freedom of the system
$i \rightarrow$ discrete index.	$\alpha \longrightarrow$ Discrete index.

Let $\varphi^\alpha(x)$ be a set of real or complex field quantities:

$$\left|\begin{array}{l} \text{scalar} \\ \text{vector} \\ \text{tensor} \\ \text{spinor} \\ \text{twistor} \\ \vdots \end{array}\right| \text{ fields.}$$

Lagrangian

$$\mathcal{L} = \mathcal{L}(\varphi^\alpha(x), \varphi^\alpha{}_{,\mu}(x), x) \, ; \quad \varphi^\alpha{}_{,\mu} := \frac{\partial \varphi^\alpha}{\partial x^\mu}$$

Variational problem:

$$\delta \int_M \mathcal{L} \, dx = 0 \, ; \quad dx := d^n x = dx^0 \cdots dx^{n-1}.$$

$\hookrightarrow$ For a definition of functional (Gateaux) Derivative see Zeidler's, "QFT: A bridge between Maths & Physics" Vol. 1, or just "Geometry and Physics". or Molgado & Vallejo, comm Maths 2012 (?).

→ M is a n-dim manifold with (n-1)-dim boundary B.

a. Fixed a Boundary B.

$$\delta \int_M \mathcal{L}\, dx = \int_M \delta \mathcal{L}\, dx$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi^a}\delta\varphi^a + \frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\delta\varphi^a_{,\mu} + \cancel{\frac{\partial \mathcal{L}}{\partial \varphi^{\mu}}\delta x^{\mu}} \to 0$$

then,

$$\int_M \left(\frac{\partial \mathcal{L}}{\partial \varphi^a}\delta\varphi^a + \frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\delta\varphi^a_{,\mu} \right) dx$$

$$= \int_M \left(\frac{\partial \mathcal{L}}{\partial \varphi^a}\delta\varphi^a \right) dx + \int_M \frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\partial_\mu(\delta\varphi^a)\, dx$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\delta\varphi^a \right) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}} \right)\delta\varphi^a + \frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\partial_\mu \delta\varphi^a$$

$$0 = \delta \int_M \mathcal{L}\, dx = \int_M \left[\frac{\partial \mathcal{L}}{\partial \varphi^a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}} \right) \right]\delta\varphi^a dx + \int_M \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\delta\varphi^a \right) dx$$

$$0 = \int_M \left[\frac{\partial \mathcal{L}}{\partial \varphi^a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}} \right) \right]\delta\varphi^a dx + \int_B \left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\cancel{\delta\varphi^a} \right) d\sigma_\mu \to 0$$

If $\delta\varphi^a$ is arbitrary but vanishes on B, then

$$\delta \int_M \mathcal{L}\, dx = 0$$

is an extremum if and only if

$$\frac{\partial \mathcal{L}}{\partial \varphi^a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}} \right) = 0$$

Field equations

(Euler - Lagrange)

Assumption: These field equations are valid for any physical system.

b. Variations with a change of Boundary

$$\left.\begin{aligned} \varphi(x) &\longmapsto \varphi(x)+\delta\varphi(x) \\ x^\mu &\longmapsto x^\mu+\delta x^\mu \end{aligned}\right\}\bar{\delta}$$

Variation in spacetime points induce a variations in the fields.

Distinct variations:

$$\delta x = x' - x$$

$$\delta\varphi = \varphi'(x)$$

$$\begin{aligned} \bar{\delta\varphi} &= \varphi'(x') - \varphi(x) \\ &= \varphi'(x+\delta x) - \varphi(x) \end{aligned}$$

Taylor

$$\begin{aligned} &= \varphi'(x) + \frac{\partial x}{\partial x^\mu}\delta x^\mu - \varphi(x) \\ &= \delta\varphi(x) + \frac{\partial\varphi}{\partial x^\mu}\delta x^\mu \end{aligned}$$

$$\bar{\delta}\int_M \mathcal{L}\,dx = \int_M \delta\mathcal{L}\,dx + \int_{\bar{M}-M}\mathcal{L}\,dx$$

$M :=$ Original manifold.

$\bar{M} :=$ Varied manifold.

$$= \int_M \delta\mathcal{L}\,dx + \int_B \mathcal{L}\,d\sigma_\mu\,\delta x^\mu$$

But

$$\int_M \delta\mathcal{L}\,dx = \int_M \cancelto{0}{\left[\frac{\partial\mathcal{L}}{\partial\varphi^a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}\right)\right]}\delta\varphi^a\,dx$$

$$+ \int_B \partial\sigma_\mu\left(\frac{\partial\mathcal{L}}{\partial\varphi^a_{,\mu}}\delta\varphi^a\right)dx$$

then

$$\bar{\delta}\int_M \mathcal{L}\,dx = \int_B d\sigma_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\delta\varphi^a + \mathcal{L}\,\delta x^\mu\right)$$

assuming field equations are valid.

$$= \int_M dx\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\delta\varphi^a + \mathcal{L}\,\delta x^\mu\right) = 0$$

Define

$$J := \frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\delta\varphi^a + \mathcal{L}\,\delta x^\mu$$

then

$$\int_M x\, \partial_\mu J^\mu = 0$$

$$\partial_\mu J^\mu = 0 \longrightarrow \text{Generalised continuity equation.}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot J \doteq 0$$

$$J^0 \longrightarrow \text{Generalised charge.}$$

Define the "canonical" stress-energy tensor

$$T^{\mu\nu} := \frac{\partial \mathcal{L}}{\partial \varphi^a_{,\mu}}\varphi^{a,\nu} - \mathcal{L}\,g^{\mu\nu}$$

$$\partial_\mu T^{\mu\nu} = 0 \qquad \text{in curved spacetime}$$

$$\nabla_\mu J^\mu = 0$$

and

$$\nabla_\mu T^{\mu\nu} = 0$$

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{R}\,, \qquad \nabla_\mu G^{\mu\nu} = 0.$$

Theorem: The lagrangian determines the field equations up to an additive divergence term if variations with fixed B are considered.

Proof:

$$\mathcal{L} \longmapsto \mathcal{L} + \partial_\mu f^\mu$$

$$\delta \int_M (\mathcal{L} + \partial_\mu f^\mu)\, dx = \delta \int_M \mathcal{L}\, dx + \delta \int_M (\partial_\mu f^\mu)\, dx$$

$$= \delta \int_M \mathcal{L}\, dx + \cancel{\delta \int_B f^\mu d\sigma_\mu} \to 0$$

$\to \delta\varphi^a = 0$ on B.

Therefore, $\mathcal{L}$ and $\mathcal{L} + \partial_\mu f^\mu$ denote the same physical system.