Pauli Paramagnetism

$$\mathcal{H}=\sum_{i=1}^{N}\left[\frac{1}{2m}\vec{p}_i^{\,2}-g\mu_B\vec{H}\cdot\vec{S}_i\right]$$

$\vec{S}_i :=$ Spin operator

$g:=2$ gyromacnetic ratio.

$\mu_B :=$ Bohr magneton

$$\mathcal{H}=\frac{\vec{p}^{\,2}}{2m}-g\mu_B H S_z \rightarrow \text{1 particle}$$

$$E_{\vec{k},\sigma}=\frac{\hbar^2 k^2}{2m}-\mu_B H\sigma\,; \qquad \sigma=\pm 1.$$

$$\ln(\Xi)=\sum_{\vec{k}}\sum_{\sigma}\ln\left\{1+z\exp\left(-\beta\frac{\hbar^2k^2}{2m}+\beta\mu_B H\sigma\right)\right\}$$

$$=VC\int_0^\infty \epsilon^{1/2}\left\{\sum_\sigma \ln\left[1+z\exp\left(-\beta\frac{\hbar^2k^2}{2m}+\beta\mu_B H\sigma\right)\right]\right\}dE$$

then,

$$\ln(\Xi)=\ln(\Xi)_+ + \ln(\Xi)_-$$

$$\ln(\Xi)_\pm := VC\int_0^\infty \epsilon^{1/2}\left\{\sum_\sigma \ln\left[1+z\exp\left(-\beta\frac{\hbar^2k^2}{2m}\pm\beta\mu_B H\sigma\right)\right]\right\}dE$$

$$N=z\frac{\partial}{\partial z}\ln(\Xi)=\langle N_+ + N_-\rangle$$

$$\langle N_\pm\rangle = VC\int_0^\infty \epsilon^{1/2}\left\{z^{-1}\exp\left(\beta E\mp\beta\mu_B H\right)+1\right\}^{-1}dE$$

$$M=\mu_B\langle N_+ + N_-\rangle \rightarrow \text{Magnetization}$$

Base state $\beta\rightarrow\infty$

$$\langle N_+\rangle = VC\int_0^{E_F-\mu_B H}\epsilon^{1/2}\,dE=\int_{-\mu_B H}^{E_F}(\epsilon-\mu_B H)^{1/2}\,dE$$

$$= \frac{2}{3} VC (E_F + \mu_B H)^{3/2}$$

$$\langle N_- \rangle = VC \int_0^{E_F + \mu_B H} \epsilon^{1/2} d\epsilon = \int_{-\mu_B H}^{E_F} (\epsilon + \mu_B H)^{1/2} d\epsilon$$

$$= \frac{2}{3} VC (E_F - \mu_B H)^{3/2}$$

$$N = \langle N_+ + N_- \rangle = \frac{2}{3} VC \left[(E_F + \mu_B H)^{3/2} + (E_F - \mu_B H)^{3/2} \right]$$

$$M = \mu_B \langle N_+ - N_- \rangle = \frac{2}{3} VC \mu_B \left[(E_F + \mu_B H)^{3/2} - (E_F - \mu_B H)^{3/2} \right]$$

If $\mu_B H \ll E_F$

$$N \approx \frac{4}{3} VC E_F^{3/2} \quad \text{and} \quad M \simeq 2VC \mu_B E_F^{3/2} \left(\frac{\mu_B H}{E_F} \right)$$

$$M = 2 \cdot \frac{3}{4} N \frac{\mu_B^2 H}{E_F} = \frac{3}{2} \frac{N \mu_B^2 H}{E_F}$$

$$\chi_0 = \left(\frac{\partial M}{\partial H} \right)_{T=0,V,N} = \frac{3}{2} \frac{N \mu_B^2}{E_F}$$

Degenerate limit

$$M = \mu_B \langle N_+ - N_- \rangle = -\frac{1}{\beta} \frac{\partial}{\partial H} \ln(\Xi)$$

$$= \mu_B VC \int_0^\infty \epsilon^{1/2} \left[f(\epsilon - \mu_B H) - f(\epsilon + \mu_B H) \right] d\epsilon$$

If $\mu_B H \ll E_F$

$$f' = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2 \Delta x}$$

then,

$$M = \mu_B VC \int_0^\infty \epsilon^{1/2} \left[-2\mu_B H f'(\epsilon) \right] d\epsilon$$

$$= -2VC \mu_B^2 H \int_0^\infty \epsilon^{1/2} f'(\epsilon) d\epsilon$$

Remember that,

$$I = \int_0^{\mu} \phi(E)\,dE + \frac{\pi^2}{6}(K_BT)^2\left(\frac{d\phi}{dE}\right)_{E=\mu} + \dots$$

and

$$\psi(E) = \int_0^{\infty} \phi(E')\,dE'$$

then

$$I = -\int_0^{\infty} \psi(E) f'(E)\,dE$$

$$= \psi(\mu) + \frac{\pi^2}{6}(K_BT)^2\left(\frac{d^2\psi}{dE^2}\right)_{E=\mu} + \dots$$

$$= 2\mu_B^2 VCH\left[\mu^{1/2} + \frac{\pi^2}{6}(K_BT)^2 \frac{d}{dE}\left(\frac{1}{2}E^{-1/2}\right)\Big|_{E=\mu} + \dots\right]$$

$$= 2\mu_B^2 VCH\left[\mu^{1/2} + \frac{\pi^2}{6}(K_BT)^2\left(-\frac{1}{4}\mu^{-3/2}\right) + \dots\right]$$

$$N = VC\int_0^{\infty} E^{1/2}\left[f(E-\mu_BH) + f(E+\mu_BH)\right]dE$$

$$\approx 2VC\int_0^{\infty} E^{1/2} f(E)\,dE$$

$$= 2VC\left[\cancelto{0}{\frac{2}{3}E^{3/2} f(E)\Big|_0^{\infty}} - \int_0^{\infty} \frac{2}{3}E^{3/2} f'(E)\,dE\right.$$

$$= \frac{4}{3}\left[\mu^{3/2} + \frac{\pi^2}{6}(K_BT)^2\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\mu^{-1/2} + \dots\right]$$

$$= \frac{4}{3}VC\mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{K_BT}{\mu}\right)^2 + \dots\right]$$

As $N = \frac{2}{3}\gamma VCE_F^{3/2}$ and $\gamma = 2$

then,

$$\epsilon^{3/2} = \mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{K_BT}{\mu}\right)^2 + \dots\right]$$

$$\mu = \epsilon_F\left[1 + \frac{\pi^2}{8}\left(\frac{K_BT}{\mu}\right)^2 + \dots\right]^{-2/3}$$

$$\approx \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{K_BT}{\mu}\right)^2 + \dots\right]$$

$$\approx \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2 + \dots\right]$$

Changing μ in M

$$M = \mu_B^2 H N \mu^{-1}\left[1 + \frac{\pi^2}{8}\left(\frac{K_BT}{\mu}\right)^2 + \dots\right]^{-1}\left[1 - \frac{\pi^2}{24}\left(\frac{K_BT}{\mu}\right)^2 + \dots\right]$$

$$\approx \frac{3}{2}\mu_B^2 H N \mu^{-1}\left[1 - \frac{\pi^2}{6}\left(\frac{K_BT}{\mu}\right)^2 + \dots\right]$$

$$\approx \frac{3}{2}\mu_B^2 H N \epsilon_F^{-1}\left[1 - \frac{\pi^2}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2 + \dots\right]^{-1}\left[1 - \frac{\pi^2}{6}\left(\frac{K_BT}{\epsilon_F}\right)^2 + \dots\right]$$

$$\approx \frac{3}{2}\mu_B^2 \frac{HN}{\epsilon_F}\left[1 - \frac{\pi^2}{12}\left(\frac{K_BT}{\epsilon_F}\right)^2 + \dots\right]$$

then

$$\chi_0 = \left(\frac{\partial M}{\partial H}\right) = \frac{3}{2}\mu_B^2 \frac{N}{\epsilon_F}\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \dots\right]$$

Classical limit

$$z \ll 1 \rightarrow f(\epsilon) = \frac{1}{z^{-1}\exp(\beta\epsilon) + 1} \rightarrow z\exp(-\beta\epsilon)$$

then,

$$M = \mu_B V C \int_0^\infty \epsilon^{1/2} z\left(\exp(-\beta\epsilon + \beta\mu_B H) - \exp(-\beta\epsilon - \mu_B H)\right) d\epsilon$$

$$= \mu_B V C z \sinh(\beta\mu_B H) \int_0^\infty \epsilon^{1/2} \exp(-\beta\epsilon) d\epsilon$$

As,

$$N = VC\int_0^\infty \varepsilon^{1/2} z\left(\exp(-\beta\varepsilon+\beta\mu_B H)+\exp(-\beta\varepsilon-\mu_B H)\right)d\varepsilon$$

$$= VCz\cosh(\beta\mu_\mu H)\int_0^\infty \varepsilon^{1/2}\exp(-\beta\varepsilon)d\varepsilon$$

therefore,

$$M = N\mu_B\tanh(\beta\mu_B H)$$

If $\mu_B H \ll K_B T \rightarrow M \approx N\mu_B \frac{\mu_B H}{K_B T}$

then,

$$\chi_0 = \left(\frac{\partial M}{\partial H}\right)_T = \frac{N\mu_B^2}{K_B T}$$ Curie law