

$$W[\psi](p, q) = \text{Sym}[\hat{P}_\psi](p, q) = \int \psi\left(q + \frac{z}{2}\right) \overline{\psi\left(q - \frac{z}{2}\right)} e^{-ipz/\hbar} dz$$

$$\langle \psi, \hat{A}\psi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2} W[\psi](p, q) A(p, q) dp dq ; \quad |\psi\rangle \in \mathcal{H}, \langle \psi| \in \mathcal{H}^*$$

$\stackrel{:=}{=} I$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \psi\left(q + \frac{z}{2}\right) \overline{\psi\left(q - \frac{z}{2}\right)} e^{-ipz} dz \right] \left[\int_{\mathbb{R}} K_A\left(q + \frac{z}{2}, q - \frac{z}{2}\right) e^{-izp} dz \right] dp dq$$

Integrating in p brings $\delta(z + z')$, and thus integrating in z' .

$$I = \int_{\mathbb{R}^2} dq dz \psi\left(q + \frac{z}{2}\right) \overline{\psi\left(q - \frac{z}{2}\right)} K_A\left(q - \frac{z}{2}, q + \frac{z}{2}\right)$$

$$\text{Change } q' := q - \frac{z}{2}, \quad q := q + \frac{z}{2} \iff z = q - q', \quad q' = \frac{q+z}{2}.$$

$$dz dq = dz \wedge dq = (dq - dq') \wedge \frac{1}{2}(dq' + dq)$$

$$= \frac{1}{2} [dq \wedge dq' - dq' \wedge dq] = dq \wedge dq' = dq dq'$$

$$I = \int_{\mathbb{R}} dq dq' \overline{\psi(q')} K_A(q', q) \psi(q) = \int_{\mathbb{R}} dq' \overline{\psi(q')} \left[\int_{\mathbb{R}} K_A(q', q) \psi(q) dq \right]$$

$$= \int_{\mathbb{R}} dq' \overline{\psi(q')} (\hat{A}\psi)(q') = (\psi, \hat{A}\psi)$$

Note: Symbols may depend on \hbar ! ← This encloses the quantum behaviour at the phase space level.

Example:

$$\text{Sym}[\hat{p}\hat{q} + \frac{\hbar}{2i}] = \text{Sym}[\hat{p}\hat{q}] + \text{Sym}\left[\frac{\hbar}{2i}\right]$$

$$= \text{Sym}[\hat{p}, \hat{q}] + \frac{\hbar}{2i} = pq$$

$$\Rightarrow \text{Sym}[\hat{p}, \hat{q}] = pq - \frac{\hbar}{2i}$$

This may define a semiclassical observable if for any $A \in A_c$ (such that $\hat{A} \in \hat{A}_c = Q_\hbar^\omega[A]$). It allows an asymptotic expansion.

$$A(p, q) \sim \hbar \sum_{n=0}^{\infty} A_n(p, q) \hbar^n$$

$a \in \mathbb{R}$, $a_n(p, q)$ do not depend on \hbar .

$A_0(p, q)$ is the classical observable corresponding to the quantum observable \hat{A} .

Back to Weyl & Wigner: Some properties.

$$1) A(x, p) = \int dq e^{-ipq/\hbar} \left\langle x + \frac{q}{2}, \hat{A} \left(x - \frac{q}{2} \right) \right\rangle$$

Proof:

$$K_{\hat{A}}(q, q') = \langle q, Q_\hbar^\omega[A] q' \rangle$$

$$= \int dp A\left(p, \frac{q+q'}{2}\right) e^{ip(q-q')/\hbar}$$

Define

$$T_\hbar[\hat{A}] = T_\hbar[Q_\hbar^\omega[A]] := \int dq \langle q, \hat{A} q \rangle = \int dq \int dp A(p, q)$$

Using $K_{\hat{A}}(q, q')$

Now, taking $q = x + \frac{u\hbar}{2}$, $q' = x - \frac{u\hbar}{2}$ in the kernel.

$$\left\langle x + \frac{u\hbar}{2}, \hat{A} \left(x - \frac{u\hbar}{2} \right) \right\rangle = \int dp A(p, x) e^{iup/\hbar}$$

by Fourier

$$A(p, x) = \frac{1}{2\pi} \int du \left(\int dp A(p, x) e^{iup/\hbar} \right) e^{-iup/\hbar}$$

$$= \frac{1}{2\pi} \int du \left\langle x + \frac{u\hbar}{2}, \hat{A} \left(x - \frac{u\hbar}{2} \right) \right\rangle e^{-iup/\hbar}$$



$$2) A(p, q) = \frac{1}{2\pi} \int du e^{ixu/\hbar} \left\langle p + \frac{uh}{2}, \hat{A}\left(p, \frac{uh}{2}\right) \right\rangle$$

Proof: By completeness relation

$$1 = \int dp |p\rangle \langle p|, \text{ and } \langle p|p'\rangle = \delta(p-p')$$

$$A(p, q) = \int du e^{-iup/\hbar} \left\langle x + \frac{up}{\hbar} \mid p' \right\rangle \langle p'|A|p''\rangle \left\langle p'' \mid x - \frac{u}{\hbar} \right\rangle dp' dp''$$

$$\langle x|p\rangle = \frac{1}{\hbar^{1/2}} \exp\left(\frac{iXP}{\hbar}\right)$$

$$A(p, q) = \int du e^{-iup/\hbar} \left[\frac{1}{\hbar^{1/2}} \exp\left(\frac{iP(x+uh/2)}{\hbar}\right) \right] \langle p'|Ap''\rangle$$

$$\times \left[\frac{1}{\hbar^{1/2}} \exp\left(\frac{iP(x+uh/2)}{\hbar}\right) \right] dp' dp''$$

$$= \frac{1}{\hbar} \int du dp' dp'' e^{-iup/\hbar} e^{-ix(p''-p')/\hbar} e^{\frac{iu}{2}(p'+p'')} \langle p'|Ap''\rangle$$

Integrating in u brings $\hbar \delta(p - \left(\frac{p'+p''}{2}\right))$

$$\text{change } u = p' - p'', z = p' + p'' \rightarrow p' = \frac{u+z}{2}, p'' = \frac{z-u}{2}$$

$$dp' dp'' \mapsto \frac{1}{2} dy dz.$$

$$A(p, q) = \frac{1}{2} \int dy dz \delta\left(p - \frac{z}{2}\right) e^{ixq/\hbar} \left\langle \frac{u+z}{2} \mid \hat{A}\left(\frac{z-u}{2}\right) \right\rangle$$

$$\delta[f(x)] = \frac{1}{|f'(x)|} \delta(x).$$

$$= \int dy e^{ixq/\hbar} \left\langle p + \frac{u}{2} \mid A\left(p - \frac{u}{2}\right) \right\rangle$$

$$3) Tr[\hat{A}\hat{B}] = \frac{1}{\hbar} \int dq dp A(q, p) B(q, p)$$

$$\text{Proof: } A(q,p) = \int dy e^{-ipy/\hbar} \left\langle q + \frac{y}{2} \mid \hat{A} \left(q - \frac{y}{2} \right) \right\rangle$$

$$B(q,p) = \int dy' e^{-ipy'/\hbar} \left\langle q + \frac{y'}{2} \mid \hat{B} \left(q - \frac{y'}{2} \right) \right\rangle$$

$$\int dp dq A(q,p) B(q,p) = \int dp dq dy dy' e^{-ipy/\hbar} \left\langle q + \frac{y}{2} \mid \hat{A} \left(q - \frac{y}{2} \right) \right\rangle \\ \times e^{-ipy'/\hbar} \left\langle q + \frac{y'}{2} \mid \hat{B} \left(q - \frac{y'}{2} \right) \right\rangle$$

Integrating in p brings $\hbar \delta(y - y')$, thus integrating in y'

$$I = \int dq dy \left\langle dq dy \left\langle q + \frac{y}{2} \mid \hat{A} \left(q - \frac{y}{2} \right) \right\rangle \left\langle q - \frac{y}{2} \mid \hat{B} \left(q + \frac{y}{2} \right) \right\rangle \right\rangle$$

change

$$u := q - \frac{y}{2}, \quad v := q + \frac{y}{2}$$

$$du dv = dq dy.$$

$$I = \hbar \int du dv \langle v | \hat{A}(u) \rangle \langle u | \hat{B}(v) \rangle \\ = \hbar \int dv \langle v | \hat{A} \hat{B}(v) \rangle = \hbar [\hat{A}, \hat{B}].$$

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$$4) W[11](p,q) = 1.$$

Proof:

$$W[11](p,q) = \int e^{-ipy/\hbar} \left\langle x + \frac{y}{2} \mid \mathbb{1} \left(x - \frac{y}{2} \right) \right\rangle dy.$$

$$= \int dy e^{-ipy/\hbar} \delta \left(\left(x + \frac{y}{2} \right) - \left(x - \frac{y}{2} \right) \right)$$

$$= \int dy e^{-ipy/\hbar} \delta(y)$$

$$= 1.$$

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5) Wigner function is normalized.

$$\int dq dp W(q, p) = 1.$$

Proof: $\int dq \int W[\psi](p, q) dp = \int dq |\psi(q)|^2 = 1.$

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6) Let W_a be the Wigner function associated to the operator

$$\hat{p}_a := |\psi_a\rangle\langle\psi_a|$$

Proof:

$$\text{Tr}[\hat{p}_a \hat{p}_b] = \frac{1}{\hbar} \int dp dq W_a(p, q) W_b(p, q)$$

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$$7) \text{ As } \text{Tr}[\hat{p}_a \hat{p}_b] = \hbar |\langle \psi_a | \psi_b \rangle|^2$$

Proof: Thus, for orthogonal states

$$\langle \psi_a | \psi_b \rangle = 0.$$

$$\Rightarrow \int dp dq W_a(p, q) W_b(p, q) = 0$$

∴ Wigner may be negative at some points in phase space

$$\langle \psi | A \psi \rangle = \int dp dq W(p, q) A(p, q).$$

8) Wigner function is real.

Proof:

$$W(q, p) = \int dz e^{-ipz/\hbar} \psi\left(q + \frac{z}{2}\right) \overline{\psi\left(q - \frac{z}{2}\right)}$$

$$W^*(q, p) = \int dz e^{ipz/\hbar} \overline{\psi\left(q + \frac{z}{2}\right)} \psi\left(q - \frac{z}{2}\right)$$

Change $z \mapsto q - z$

$$W^*(q, p) = \int dy e^{-ipy/\hbar} \overline{\psi\left(q - \frac{y}{2}\right)} \psi\left(q + \frac{y}{2}\right) = W(p, q).$$

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