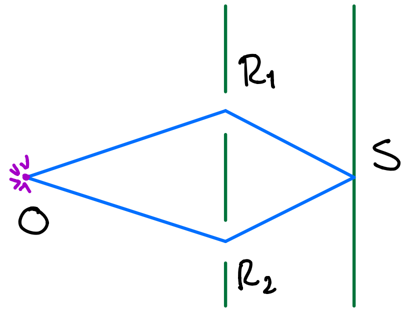
Path integral quantization

Double-slit problem:

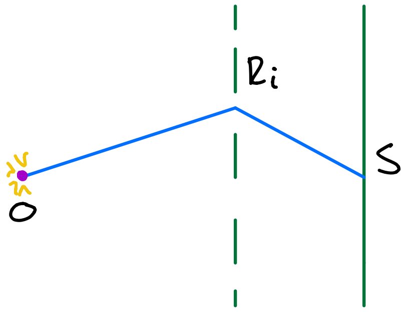


$$P(O \to S) = |A(O \to S)|^2$$

Quantum theory: Non-probabilistic

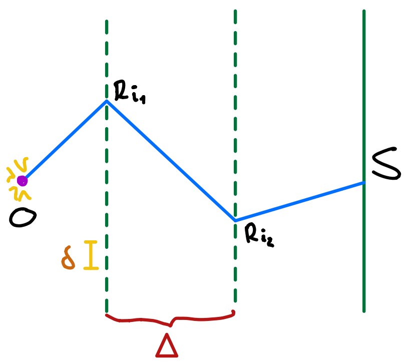
$$A(O \to S) = A(O \to R_1 \to S) + A(O \to R_2 \to S)$$

$$A \in \mathbb{C}.$$



n-slit:

$$A(O \to S) = \sum_i^n A(O \to R_i \to S)$$



$n_1 + n_2$:

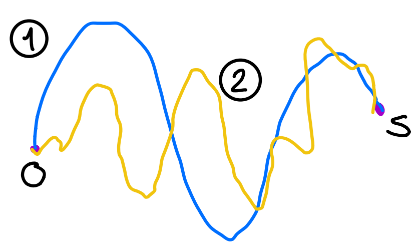
$$A(O \to S) = \sum_{i_2}^{n_2} \sum_{i_1}^{n_1} A(O \to R_{i_1} \to R_{i_2} \to S)$$

$$N = n_1 + n_2 + \dots$$

$n_i \to \infty : \delta \to 0$

$N \to \infty : \Delta \to 0$

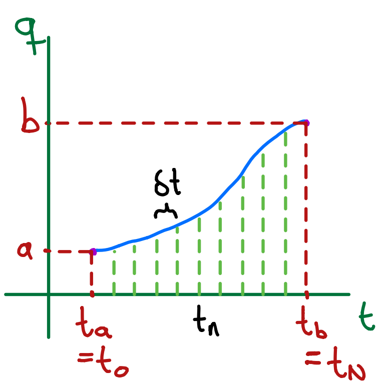
$$\boxed{A_T(O \to S) = \int_{\text{Path}} A_{\text{Path}}(O \to S)}$$



statistic formulation

$$|A_{\text{path}}(\text{Clas})| = \max$$

Formlize:



$$A_{ab}(t_b - t_a) = \langle b, t_b | a, t_a \rangle \equiv \langle b | a \rangle$$

along to the path: (q_n, t_n)

$$q_a = a, \quad q_b = b$$

$$H = \frac{p^2}{2m} + V(Q) \quad [H \neq H(t)]$$

$$\langle q_b, t_b | q_a, t_a\rangle = \langle q_b, e^{-iH(t_b-t_a)} | q_a\rangle$$

Using

$$\int dq_i \, |q_i, t_i\rangle\langle q_i, t_i| = \mathbb{1}$$

then,

$$\langle b|a\rangle = \prod_i^N \int dq_i \langle q_N, t_N | q_{N-1}, t_{N-1}\rangle \cdots \langle q_1, t_1 | q_0, t_0\rangle$$

each factor:

$$\langle q_{i+1}, t_{i+1} | q_i, t_i\rangle = \langle q_{i+1}, e^{-iH\delta t} | q_i\rangle$$

$1 - iH\delta t$.

$$\langle q_{i+1} | q_i\rangle = \delta(q_{i+1} - q_i) = \int \frac{dp_i}{2\pi} e^{ip_i(q_{i+1} - q_i)}$$

Similarly:

$$\langle q_{i+1} | V(Q) | q_i\rangle = V(q_i)\langle q_{i+1} | q_i\rangle$$

$$\langle q_{i+1}, t_{i+1} | P^2 | q_i, t_i\rangle = \int \frac{dp_i}{2\pi} p^2 e^{ip_i(q_{i+1} - q_i)}$$

$$\langle q_{i+1}, t_{i+1} | q_i, t_i\rangle \simeq \int \frac{dp_i}{2\pi} e^{ip_i\delta q_i} (1 - iH(p_i, q_i)\delta t + \cdots)$$

$$= \int \frac{dp_i}{2\pi} e^{ip_i \delta q_i} e^{-iH(p_i, q_i)\delta t}$$

therefore

$$\langle b|a\rangle = \prod_i^N \int dq_i \frac{dp_i}{2\pi} e^{i\left[p_i \frac{\delta q_i}{\delta t} - H(p_i, q_i)\right]\delta t}$$

In the limit:

$\delta t \to 0$

$N \to \infty$ with $t_b - t_a = N\delta t$.

$$\frac{\delta q}{\delta t} \to \dot{q} \quad ; \quad [\quad]\delta t \longrightarrow \int [\quad] dt$$

Definition: $\lim_{N\to\infty} \prod_i^{\infty} \int \frac{dq_i dp_i}{2\pi} \longrightarrow \int \mathcal{D}q(t)\,\mathcal{D}p(t)$ "Measure of the path"

therefore

$$\langle b|a\rangle = \int \mathcal{D}q\,\mathcal{D}p\, e^{i\int_{t_a}^{t_b} dt\,(p\dot{q} - H(p,q))} \longrightarrow \text{Path Integral}$$

where $q(t_a)=a$; $q(t_b)=b$

Notice,

$$H = \frac{p^2}{2m} + V(q)$$

$$\int dp\, e^{i[p\dot{q} - p^2/2m - V(q)]dt}$$

may be integrated $\sim \int dp\, e^{ap^2+bp+c}$

then,

$$\int dy\, e^{-1/2\,ay^2} = \sqrt{\frac{2\pi}{a}} \longrightarrow \sqrt{\frac{2\pi m}{i\delta t}}\, e^{iL(p,\dot{q})\delta t}$$

Divergent: $\delta t \to 0$

finally

$$\langle b|a\rangle \propto \int \mathcal{D}q(t)\, e^{i\int_{t_a}^{t_b} L(q,\dot{q})dt}$$

Finite

$$t \to it : \langle b|a\rangle \propto \int \mathcal{D}q\, e^{-S_E(q)}.$$