

Example:

$$I(x) = \int_0^x t^{-1/2} e^{-t} dt \quad \text{as } x \rightarrow \infty$$

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \operatorname{Re}(z) > 0.$$

$$I(x) = \underbrace{\int_0^\infty t^{-1/2} e^{-t} dt}_{=: \Gamma(1/2)} - \underbrace{\int_x^\infty t^{-1/2} e^{-t} dt}_{=: I_1}$$

$$\begin{aligned} I_1(x) &= - \int_x^\infty t^{-1/2} e^{-t} dt = \int_x^\infty t^{-1/2} \frac{d}{dt}(e^{-t}) dt \\ &= t^{-1/2} e^{-t} \Big|_x^\infty - \int_x^\infty \frac{d}{dt}(t^{-1/2}) e^{-t} dt \\ &= t^{-1/2} e^{-t} \Big|_x^\infty + \frac{1}{2} \int_x^\infty t^{-3/2} e^{-t} dt \end{aligned}$$

Leading order.

$$I(x) = \sqrt{\pi} - \frac{e^{-x}}{x^{1/2}} \quad \text{as } x \rightarrow \infty$$

Repeated integration by parts gives the asymptotic expansion

$$I(x) = \int_0^x t^{-1/2} e^{-t} dt \sim \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} \left[ 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (1)(3)(5)\cdots(2n-1)}{(2x)^n} \right]$$

as  $x \rightarrow \infty$

Example:

$$F(x) = \int_a^b f(t) e^{ixt} dt, \quad f(t) \in C^N$$

Munroe effect.

$$\begin{aligned}
 F(x) &= \int_a^b f(t) \left( -\frac{i}{x} \right) \frac{d}{dt} (e^{ixt}) dt \\
 &= \left( -\frac{i}{x} \right) f(t) e^{ixt} \Big|_a^b - \left( -\frac{i}{x} \right) \int_a^b \frac{d}{dt} f(t) e^{ixt} dt. \\
 &= \frac{i}{x} (f(a) e^{ixa} - f(b) e^{ibx}) + \frac{i}{x} \int_a^b \frac{d}{dt} f(t) e^{ixt} dt.
 \end{aligned}$$

Integrating by parts successively.

$$\begin{aligned}
 F(x) &= \sum_{n=0}^{\infty} \left( \frac{i}{x} \right)^{n+1} \left[ e^{ixa} f^{(n)}(a) - e^{ibx} f^{(n)}(b) \right] + \underbrace{\left( \frac{i}{x} \right)^n \int_a^b f^{(n)}(t) e^{ixt} dt}_{\mathcal{O}\left(\frac{1}{x^n}\right)}
 \end{aligned}$$

Example:

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt} dt, \quad x \rightarrow 0^+ \quad (\text{Stieltjes transform of } e^{-t})$$

$$I(x) = - \int_0^\infty \frac{1}{1+xt} \frac{d}{dt} (e^{-t}) dt.$$

$$= - \frac{1}{1+xt} e^{-t} \Big|_0^\infty + \int_0^\infty \frac{d}{dt} \left( \frac{1}{1+xt} \right) e^{-t} dt.$$

$$= 1 - \int_0^\infty \frac{x}{(1+xt)^2} e^{-t} dt$$

$$= 1 + \int_0^\infty \frac{x}{(1+xt)^2} \frac{d}{dt} (e^{-t}) dt$$

$$= 1 + x \left[ \frac{e^{-t}}{(1+xt)^2} \Big|_0^\infty - \int_0^\infty \frac{d}{dt} \left( \frac{1}{(1+xt)^2} \right) e^{-t} dt \right]$$

$$\begin{aligned}
 &= 1 - x - x(-2x) \int_0^\infty \frac{1}{(1+xt)^3} e^{-t} dt \\
 &= 1 - x + 2x^2 - \dots + (-1)^{n-1} (n-1)! x^{n-1} + (-1)^n n! x^n \int_0^\infty \frac{e^{-t}}{(1+xt)^{n+1}} dt
 \end{aligned}$$

then

$$I(x) \sim \sum_{n=0}^{\infty} (-1)^n n! x^n \quad \text{as } x \rightarrow 0^+$$

### Example: Laplace Integrals

$$I(x) := \int_a^b f(t) e^{x\phi(t)} dt \quad \xrightarrow{\text{In the literature they put } -x, \text{ but they are talking about minimums.}}$$

We want  $x \rightarrow \infty$ ,  $f$  and  $\phi$  continuous functions.

$$\begin{aligned}
 I(x) &= \frac{1}{x} \int_a^b \frac{f(t)}{\phi'(t)} \frac{d}{dt} (e^{x\phi(t)}) dt \\
 &= \frac{1}{x} \frac{f(t)}{\phi'(t)} e^{x\phi(t)} \Big|_a^b - \frac{1}{x} \int_a^b \frac{d}{dt} \left( \frac{f(t)}{\phi'(t)} \right) e^{x\phi(t)} dt.
 \end{aligned}$$

Integrating by parts again will introduce  $\frac{1}{x^2}$  terms... and so on!

leading terms

$$I(x) \sim \frac{1}{x} \frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{1}{x} \frac{f(a)}{\phi'(a)} \quad \text{as } x \rightarrow \infty$$

### Failure of integration by parts:

For Laplace type integrals, we expect to get an expansion of the form.

$$I(x) \sim e^{x\phi(b)} \sum_{n=1}^{\infty} A_n x^{-n} \quad \text{for } x \rightarrow \infty$$

If not, integration by parts is not a good technique!

Example:  $\int_0^\infty e^{-xt^2} dt = \frac{1}{2} \sqrt{\frac{\pi}{x}}$  (Gaussian)  $x \rightarrow \infty$

$$\left( \int_{-\infty}^\infty e^{-ay^2} dy \right)^2 = \left( \int_{-\infty}^\infty e^{-ay_1^2} dy_1 \right) \left( \int_{-\infty}^\infty e^{-ay_2^2} dy_2 \right)$$

$$= \iint_0^\infty e^{-a(y_1^2 + y_2^2)} dy_1 dy_2 = \int_0^{2\pi} \int_0^\infty e^{-ar^2} dr (r d\theta)$$

$$= 2\pi \int_0^\infty e^{-ar^2} r dr \quad s := ar^2$$

$$ds = 2ar dr \rightarrow dr = \frac{ds}{2a}$$

$$= \frac{\pi}{a} \int_0^\infty e^{-s} ds = \frac{\pi}{a} (-e^{-s}) \Big|_0^\infty = \frac{\pi}{a}$$

then

$$\int_0^\infty e^{-ay^2} dy = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

$$I(x) = \int_0^\infty e^{-xt^2} dt = \underbrace{\frac{e^{-xt^2}}{-2xt}}_0^\infty - \frac{1}{x} \int_0^\infty \frac{d}{dt} \left( \frac{1}{-2t} \right) e^{-xt^2} dt$$

$\hookrightarrow$  Divergent!

when  $f(t) = 1, \phi(t) = -t^2$ .

## Laplace method

We want to study Laplace Integrals.

$$I(x) = \int_a^b f(t) e^{x\phi(t)} dt$$

where  $f(t), \phi(t)$  are real continuous functions.

**Theorem: (Laplace)** Assume  $\phi(t)$  attains its maximum on  $t=c \in (a, b)$  and that  $f(c) \neq 0$ . Thus, the major contribution to the asymptotic expansion of  $I(x)$  ( $x \rightarrow \infty$ ) comes from a neighbourhood of the point  $t=c$ .

We will consider here  $\phi$  has only a MAXIMUM and thus  $\phi'(c)=0, \phi''(c) < 0$

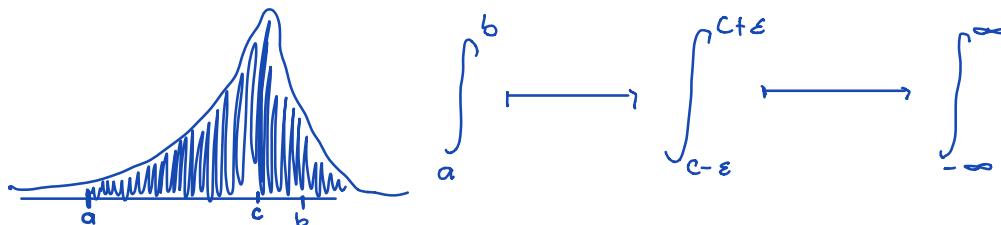
We may Taylor expand  $\phi(t)$  and  $f(t)$  around  $t=c$ .

$$\phi(t) = \phi(c) + \frac{1}{1!} \phi'(t) \Big|_{t=c} (t-c)^1 + \frac{1}{2!} \phi''(t) \Big|_{t=c} (t-c)^2 + \dots$$

by definition.

$$I(x) = \int_a^b f(c) e^{x(\phi(c) + \frac{1}{2}\phi''(c)(t-c)^2)} dt$$

$$= f(c) e^{x\phi(c)} \int_a^b e^{\frac{x}{2}\phi''(t-c)^2} dt$$



$$= f(c) e^{x\phi(c)} \int_{-\infty}^{\infty} e^{-as^2} ds$$

$$s := t - c$$

$$ds = dt$$

$$a := -\frac{1}{2} \phi''(c)$$

$$= f(c) e^{x\phi(c)} \frac{\sqrt{2\pi}}{\sqrt{|\phi''(c)|x}}$$

Example:

$$I(\lambda) = \int_{-1}^1 \frac{\sin(t)}{t} e^{-\lambda \cosh(t)} dt, \lambda \rightarrow 0$$

$$f(t) = \frac{\sin(t)}{t}, \phi(t) = -\cosh(t)$$

$$\phi'(t) = -\sinh(t), \phi''(t) = -\cosh(t)$$

$$\phi'(0) = 0, \phi''(0) = -1 < 0$$

$$I(\lambda) \sim (1) e^{-\lambda} \frac{\sqrt{2\pi}}{\sqrt{\lambda}} = \frac{\sqrt{2\pi}}{\sqrt{\lambda}} \left( 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots \right)$$

# Higher order asymptotics for Laplace Integrals

Take higher order in  $f(t)$  Taylor expanded.

$$I(x) = \int_a^b f(t) e^{x\phi(t)} dt \approx \int_a^b \left( f(c) + f'(c)(t-c) + f''(c) \frac{(t-c)^2}{2!} + \dots \right) e^{x(\phi(c) + \phi'(c)\frac{(t-c)^2}{2})} dt$$

$$t \rightarrow s = t - c$$

then

$$\begin{aligned} I(x) &\sim e^{x\phi(c)} f(c) \sqrt{\frac{2\pi}{x|\phi''(c)|}} + e^{x\phi(c)} \int_a^b f'(c) s e^{x\phi'(c)s^2} ds \\ &+ e^{x\phi(c)} \int_a^b \frac{1}{2} f''(c) s^2 e^{\frac{x}{2}\phi''(c)s^2} ds + \dots \end{aligned}$$

therefore

$$\begin{aligned} &\int s^2 e^{-as} ds \quad u := -\frac{x}{2} \phi''(c) s^2 \\ &= \int_0^\infty \frac{\sqrt{2u}}{\sqrt{x|\phi''(c)|}} e^{-u} du \quad du = -x\phi''(c)s ds \\ &= \frac{2^{1/2}}{(x|\phi''(c)|)^{3/2}} \int_0^\infty u^{1/2} e^{-u} du = \frac{2^{1/2}}{(x|\phi''(c)|)^{3/2}} \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \frac{1}{(x|\phi''(c)|)^{3/2}} \end{aligned}$$

$$\therefore I(x) = e^{x\phi(c)} \left[ \frac{f(c)\sqrt{2\pi}}{(x|\phi''(c)|)^{1/2}} + \frac{\sqrt{\pi}}{2} \frac{1}{(x|\phi''(c)|)^{3/2}} + \dots \right]$$

