Bose-Einstein Condensate (BEC)

$$\ln(\Xi(T,V,\mu)) = -\sum_j \ln\{1-\exp(-\beta(E_j-\mu))\}$$

$$p(T,\mu) = -K_B T \lim_{V\to\infty} \frac{1}{V}\ln(\Xi(T,V,\mu))$$

$$\langle n_j\rangle = \frac{1}{\exp(\beta(E_j-\mu))-1}$$

$$N = \sum_j \langle n_j\rangle = \sum_j \frac{1}{\exp(\beta(E_j-\mu))-1}$$

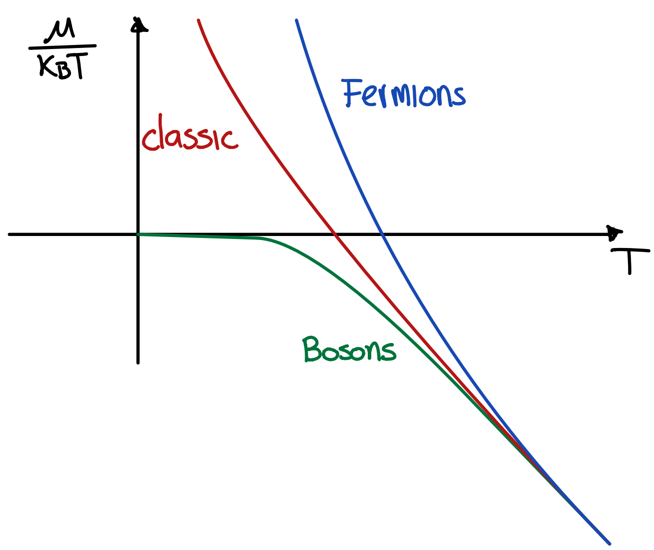
$E_j-\mu > 0 \to$ if $E_j = 0$ $\quad \mu < 0$

We already proved that in the classical limit

$$z = \exp(\beta\mu) = \frac{N}{V}\frac{h^3}{\gamma(2\pi m K_B T)^{3/2}}$$

then,

$$\frac{\mu}{K_B T} = \ln\left(\frac{1}{\gamma}\left(\frac{2\pi\hbar^2}{mK_B}\right)^{3/2}\right) + \ln\left(\frac{N}{V}\right) - \frac{3}{2}\ln(T)$$



In order to obtain T_0 we take $\mu = 0$ and $E_j = E_{\vec{k},\sigma} = \frac{\hbar^2 k^2}{2m}$

$$N = \gamma V C \int_0^\infty \frac{\varepsilon^{1/2}}{\exp(\beta_0 \varepsilon)-1}\, d\varepsilon \; ; \quad C := \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}, \quad \beta_0 = \frac{1}{K_B T_0}$$

Let $x = \beta_0 E$

Remember

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1}\, dx = \Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right) \quad ; \quad \zeta := \sum_{l=1}^{\infty} \frac{1}{l^n}$$

then,

$$N = \gamma \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)(K_B T_0)^{3/2}$$

$$T_0 = \frac{\hbar^2}{2mK_B}\left[\frac{4\pi^2}{\gamma\,\Gamma(3/2)\,\zeta(3/2)}\right]^{2/3}\left(\frac{N}{V}\right)^{2/3}$$

Under T_0

- Superfluidity
- Cooper pair

Suppose that $E_0 \simeq K_B T_0$

$$\Delta x \simeq \left(\frac{V}{N}\right)^{1/3}$$

$$\Delta x \Delta p \simeq h \rightarrow \Delta p \simeq h\left(\frac{N}{V}\right)^{1/3}$$

$$E_0 = K_B T_0 \simeq \frac{1}{2m}(\Delta p)^2 \longrightarrow T_0 \simeq \frac{h^2}{2mK_B}\left(\frac{N}{V}\right)^{2/3}$$

$$\frac{N}{V} = \left(\frac{1}{V}\frac{Z}{1-Z}\right) + \frac{1}{V}\sum_{j\neq 0}\frac{1}{Z^{-1}\exp(\beta E_j) - 1}$$

for $T \leq T_0$ and $\frac{N}{V}$ fixed.

$$\lim_{\substack{\mu \to 0 \\ V \to \infty}} \left[\frac{1}{V}\frac{Z}{1-Z}\right] \rightarrow \frac{N_0}{V}$$ Particles density of zero energy

$$\frac{1}{V}\sum_{j\neq 0}\frac{1}{Z^{-1}\exp(\beta E_j)-1}\xrightarrow{T<T_0}\gamma c\int_0^\infty \frac{E^{1/2}\,dE}{\exp(\beta E)-1}=N_e$$

$$N=N_0+N_e \rightarrow N_0=N-N_e$$

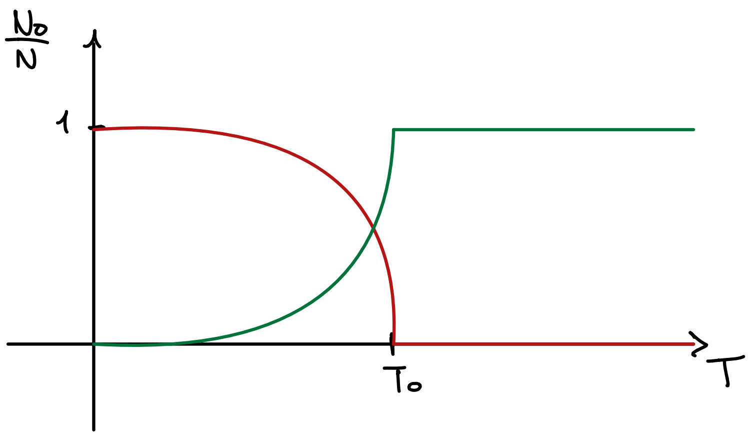
but,

$$\frac{N}{V}=\gamma c\int_0^\infty \frac{E^{1/2}\,dE}{\exp(\beta_0 E)-1}$$

then,

$$N_0=N-N_e=\text{constant}(T_0^{3/2}-T^{3/2})=\text{constant}\,T_0^{3/2}\left[1-\left(\frac{T}{T_0}\right)^{3/2}\right]$$

$$=N\left[1-\left(\frac{T}{T_0}\right)^{3/2}\right]$$



If $T\rightarrow T_0$ then $\frac{N_0}{N}\sim\frac{3}{2}\,\frac{T_0-T}{T}$.