# Riemann-lebesgue lemma

Let $f(t)$ be continous in $(a,b)$. Then

$$I(x) = \int_a^b f(t) e^{ixt}\, dt = \mathcal{O}\left(\frac{1}{x}\right) \quad \text{as } x \longrightarrow \infty$$

Provided that the integral

$$\int_a^b |f(t)|\, dt \quad \text{converges.}$$

Proof:

$$I(x) = \int_a^{a+\pi/x} f(t) e^{ixt}\, dt + \int_{a+\pi/x}^b f(t) e^{ixt}\, dt$$

$$I(x) = \int_a^{b-\pi/x} f(t) e^{ixt}\, dt + \int_{b-\pi/x}^b f(t) e^{ixt}\, dt$$

Changing $t' = t - \frac{\pi}{x}$

$$\int_{a+\frac{\pi}{x}}^b f(t) e^{itx}\, dt = -\int_a^{b-\frac{\pi}{x}} f\left(t + \frac{\pi}{x}\right) e^{ixt}\, dt$$

$\Rightarrow$

$$I(x) = \frac{1}{2}\int_a^{a+\pi/x} f(t) e^{ixt}\, dt + \frac{1}{2}\int_{a+\pi/x}^b f(t) e^{ixt}\, dt$$
$$+ \frac{1}{2}\int_a^{b+\pi/x} \left[f(t) - f\left(t + \frac{\pi}{x}\right)\right] e^{ixt}\, dt$$

we want $x \rightarrow \infty$

Mean value theorem: ( $f(t)$ continuous and bounded on $[a,b]$ )

$$\int_a^b f(t)\,dt = f'(c)(b-a) \quad \text{for some real number } c \in [a,b]$$

the two first integrals

$$\int_a^{a+\pi/x} f(t)\,e^{ixt}\,dt = f'\left(\frac{\pi}{x}\right)\left(\frac{\pi}{x}\right) \sim O\left(\frac{1}{x}\right)$$

and

$$\int_{b-\frac{\pi}{x}}^{b} f(t)\,e^{ixt}\,dt = f'\left(\frac{\pi}{x}\right)\left(\frac{\pi}{x}\right) \sim O\left(\frac{1}{x}\right)$$

Finally, as $f(t)$ is continuous $\forall\ t \in [a,b]$

$$\lim_{x\to\infty} \int_0^{b-\pi/x} \left[f(t) - f\left(t + \frac{\pi}{x}\right)\right] e^{ixt} = 0$$

$$\therefore\ I(x) \sim O\left(\frac{1}{x}\right)$$

$\blacksquare$

We may extend the Riemann-Lebesgue lemma to generalised Fourier integrals as long as $|f(t)|$ is integrable, $\varphi(t)$ continuous differentiable and $\varphi'(t) \neq 0$.

Take

$$I(x) = \int_a^b f(t)\,e^{ix\varphi(t)}\,dt$$

$$= \frac{1}{ix}\int_a^b \frac{f(t)}{\varphi'(t)}\frac{d}{dt}\left(e^{ix\varphi(t)}\right)dt$$

$$I(x) = \frac{1}{ix}\frac{f(t)}{\varphi'(t)} e^{ix\varphi(t)}\Big|_a^b - \underbrace{\frac{1}{ix}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\varphi'(t)}\right) e^{ix\varphi(t)}\, dt}_{}$$

this vanishes more rapidly than $\frac{1}{x}$ as $x \to \infty$

$$I(x) \sim \frac{1}{ix}\frac{f(t)}{\varphi'(t)} e^{ix\varphi(t)}\Big|_a^b \quad \text{as } x \longrightarrow \infty$$

This method does not work for stationary points $\varphi'(t) = 0$.

The method of stationary phase will give the asymptotic behaviour of generalised Fourier integrals with stationary points.

Choose the integral such that $\varphi'(a) = 0$ and $\varphi'(t) \neq 0$ for $a < t \leq b$.

$$\Rightarrow \quad I(x) = \int_a^b f(t) e^{ix\varphi(t)}\, dt = \underbrace{\int_a^{a+\varepsilon} f(t) e^{ix\varphi(t)}\, dt}_{=: I_1(x)} + \int_{a+\varepsilon}^b f(t) e^{ix\varphi(t)}\, dt$$

$\varepsilon > 0$ is a small parameter.

Note:

$$\int_{a+\varepsilon}^b f(t) e^{ix\varphi(t)}\, dt \sim \frac{1}{ix}\frac{f(t)}{\varphi'(t)} e^{ix\varphi(t)}\Big|_{a+\varepsilon}^b \quad \text{as } x \to \infty$$

$$\sim \frac{1}{x}$$

In the first integral change

$$f(t) \longmapsto f(a)$$
$$\varphi(t) \longmapsto \varphi(a) + \frac{\varphi^{(p)}(a)(t-a)^p}{p!}$$

As before (for laplace) the leading contribution comes from a neighbourhood of the stationary point.

Where $\varphi'(a) = \varphi''(a) = \dots = \varphi^{(p-1)}(a) = 0$

$$I_1(x) = \int_a^{a+\varepsilon} f(a) \exp\left\{ ix \left[ \varphi(a) + \frac{1}{p!} \varphi^{(p)}(a)(t-a)^p \right] \right\} dt$$

Next, replace $\varepsilon \longrightarrow \infty$

(this will introduce error terms $\mathcal{O}(1/x)$)

Let $s = t-a$, $ds = dt$

$$I_1(x) = f(a) e^{ix\varphi(a)} \int_0^\infty \exp\left( \frac{ix}{p!} \varphi^{(p)}(a) s^p \right) ds$$

Define $\mp \frac{is^p x \varphi^{(p)}(a)}{p!} =: u$

upper sign if $\varphi^{(p)}(a) > 0$

lower sign if $\varphi^{(p)}(a) < 0$

$$s = \left\{ e^{\pm i\pi/2} \left[ \frac{p!\,u}{x|\varphi^{(p)}(a)|} \right] \right\}^{1/p}$$

$$ds = e^{\pm i\pi/2p} \left( \frac{p!}{x|\varphi^{(p)}(a)|} \right)^{1/p} \frac{1}{p} u^{\frac{1}{p}-1} du$$

$$\Rightarrow I_1(x) = f(a)\, e^{ix\varphi(a)} \int_0^\infty e^{-u} e^{\pm i\pi/2p} \left(\frac{p!}{x|\varphi^{(p)}(a)|}\right)^{1/p} \frac{1}{p} u^{\frac{1}{p}-1}\, du$$

$$= f(a)\, e^{i(x\varphi(a)\pm\pi/2p)} \left(\frac{p!}{x|\varphi^{(p)}(a)|}\right)^{1/p} \frac{1}{p} \int_0^\infty e^{-u} u^{\frac{1}{p}-1}\, du$$

$$I_1(x) \sim f(a)\, e^{i(x\varphi(a)\pm\pi/2p)} \left(\frac{p!}{x|\varphi^{(p)}(a)|}\right)^{1/p} \frac{\Gamma(1/p)}{p} \quad \text{as } x \to \infty$$

Example

$$I(x) = \int_0^{\pi/2} e^{ix\cos t}\, dt$$

$$f(t) = 1 \qquad \varphi(t) = \cos t$$

$$0 = \varphi'(t) = -\sin t \quad \Rightarrow \quad t = 0 \text{ stationary point.}$$

$$\varphi''(t) = -\cos t \quad \Rightarrow \quad \varphi''(0) = -1 < 0$$

$$\Rightarrow p = 2$$

$$\Rightarrow I(x) \sim e^{i(x-\pi/4)} \left(\frac{2}{x}\right)^{1/2} \frac{\Gamma(1/2)}{2}$$

$$= e^{i(x-\pi/4)} \sqrt{\frac{\pi}{2x}} \qquad \text{as } x \longrightarrow \infty$$

Example (Bessel functions)

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x\sin(t) - nt)\, dt$$

We want the leading behaviour of $J_n(n)$ as $n \to \infty$

$$\Rightarrow J_n(n) = \mathrm{Re}\left(\frac{1}{\pi}\int_0^\pi e^{in(\sin t - t)}\,dt\right)$$

$$\psi(t) = \sin t - t \qquad f(t) = 1$$

$$0 = \psi'(t) = \cos t - 1 \qquad t = 0$$ stationary point.

$$\psi''(t) = -\sin t \qquad \psi''(0) = 0$$

$$\psi'''(t) = -\cos t \qquad \psi'''(0) = -1 < 0$$

$$p = 3$$

$$J_n(n) \sim \frac{1}{\pi}\,\mathrm{Re}\left(e^{i(n(0) - \pi/6)}\left(\frac{3!}{n}\right)^{1/3}\frac{\Gamma(1/3)}{3}\right)$$

$$= \frac{1}{\pi}\,\frac{\sqrt{3}}{2}\left(\frac{6}{n}\right)^{1/3}\frac{\Gamma(1/3)}{3} = \frac{\Gamma(1/3)}{\pi}\;\frac{3^{1/2}\cdot 3^{1/3}\cdot 3^{-1}}{2^{1}\cdot 2^{-1/3}}\;\frac{1}{n^{1/3}}$$

$$= \frac{\Gamma(1/3)}{\pi}\;\frac{1}{3^{1/6}\,2^{2/3}}\;\frac{1}{n^{1/3}}$$

steepest descend.