Quantum Statistics.

$$\{n_1, n_2, \dots, n_j, \dots\} =: \{n_j\}$$ number of occupation

Fermions $n_j = 0$ and 1

Bosons $n_j = 0, 1, 2, \dots, N$ Total number of particles.

$$E\{n_j\} = \sum_j \varepsilon_j n_j \qquad N = N\{n_j\} = \sum_j n_j$$

$$Z = Z(T,V,N) = \sum_{\{n_j\}} \exp\left[-\beta \sum_j \varepsilon_j n_j\right]$$

$$\Xi = \Xi(T,V,\mu) = \sum_{N=0}^{\infty} \exp(\beta \mu N) Z(T,V,N)$$

$$= \sum_{N=0}^{\infty} \exp(\beta\mu N) \sum_{\{n_j\}} \exp\left[-\beta\varepsilon_1 n_1 - \beta\varepsilon_2 n_2 - \dots\right]$$

$$= \sum_{N=0}^{\infty} \sum_{\{n_j\}} \exp(-\beta(\varepsilon_1-\mu)n_1 - \beta(\varepsilon_2-\mu)n_2 - \dots)$$

We can sum without restriction

$$\Xi = \sum_{n_1, n_2, \dots} \exp(-\beta(\varepsilon_1-\mu)n_1 - \beta(\varepsilon_2-\mu)n_2 - \dots)$$

$$= \left\{\sum_{n_1} \exp(-\beta(\varepsilon_1-\mu)n_1)\right\}\left\{\sum_{n_2} \exp(-\beta(\varepsilon_2-\mu)n_2)\right\} \dots$$

$$\Xi = \Xi(T,V,\mu) = \prod_j \left\{\sum_n \exp(-\beta(\varepsilon_j-\mu)n)\right\}$$

$$\ln(\Xi) = \sum_j \left(\ln\left\{\sum_n \exp(-\beta(\varepsilon_j-\mu)n)\right\}\right)$$

$$= \sum_j \left(\ln\left\{\frac{1}{1-\exp(-\beta(\varepsilon_j-\mu))}\right\}\right)$$

In the other hand

$$\frac{\partial \ln(\Xi)}{\partial \epsilon_j} = \frac{\sum_n \exp(-\beta(\epsilon_j-\mu)n)(-\beta n)}{\sum_n \exp(-\beta(\epsilon_j-\mu)n)} = -\beta\langle n_j\rangle$$

Therefore

$$\langle n_j\rangle = -\frac{1}{\beta}\frac{\partial \ln(\Xi)}{\partial \epsilon_j}$$

Bose-Einstein

$$\sum_n \exp(-\beta(\epsilon_j-\mu)n) = \{1-\exp(-\beta(\epsilon_j-\mu))\}^{-1}$$

only if $\exp(-\beta(\epsilon_j-\mu)) < 1 \quad \forall j$

if j such that $\epsilon_j = 0$, then $\exp(\beta\mu) < 1$.

therefore $\mu < 0$.

Now

$$\ln(\Xi(T,V,\mu) = \sum_j \ln\{1-\exp(-\beta(\epsilon_j-\mu))\}^{-1}$$

$$= -\sum_j \ln\{1-\exp(-\beta(\epsilon_j-\mu))\}$$

then,

$$\langle n_j\rangle = \left(-\frac{1}{\beta}\right)\left\{-\frac{-(-\beta)\exp(-\beta(\epsilon_j-\mu))}{1-\exp(-\beta(\epsilon_j-\mu))}\right\}$$

$$= \frac{\exp(-\beta(\epsilon_j-\mu))}{1-\exp(-\beta(\epsilon_j-\mu))} = \frac{1}{\exp(\beta(\epsilon_j-\mu))-1}$$

when $0 < \exp(-\beta(\epsilon_j-\mu)) < 1$, then $\langle n_j\rangle \geq 0 \; ; \forall j$

Fermi-Dirac

$$\sum_{n=0,1} \exp(-\beta(\epsilon_j-\mu)n) = 1+\exp(-\beta(\epsilon_j-\mu))$$

$$\ln(\Xi(T,V,\mu) = \sum_j \ln\{1+\exp(-\beta(\epsilon_j-\mu))\}$$

then,

$$\langle n_j \rangle = \frac{1}{\exp(\beta(\epsilon_j - \mu)) + 1}$$

$0 \leq \langle n_j \rangle \leq 1$ Pauli exclusion principle.

Finally,

$$\ln(\Xi)_{FD,BE} = \pm \sum_j \ln\left\{ 1 \pm \exp(-\beta(\epsilon_j - \mu)) \right\}$$

$$\langle n_j \rangle_{FD,BE} = \frac{1}{\exp(\beta(\epsilon_j - \mu)) \pm 1}$$

$+$ FD

$-$ BE