Tangent space

In euclidean space, the directional derivative is defined as the rate at which the function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ changes at a point $p \in \mathbb{R}^n$ in the direction of $\vec{u}$.

$$\vec{\nabla}_{\vec{u}} f := \lim_{h \to 0} \frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} u_{x_i}$$

$$= \vec{\nabla} f \cdot \frac{\vec{u}}{|\vec{u}|}$$

We want to generalize this concept to a manifold!

Axiomatise it!

Definition: Let p be a point of a manifold M. A tangent vector space to M at p is a real valued function $v: \mathcal{F}(M) \longrightarrow \mathbb{R}$ that is

I. $\mathbb{R}$-linear: $v(af+bg) = av(f) + bv(g)$.

II. Leibnizian: $v(fg) = v(f)g + fv(g)$.

$\forall f, g \in \mathcal{F}(M)$; $a, b \in \mathbb{R}$

$\mathcal{F}(M) \ni f: M \to \mathbb{R}$.

$T_pM :=$ The set of all tangent vectors to M at p.

$\longrightarrow$ Tangent space at $p \in M$.

$$\left.\begin{aligned}(v+w)(f) &= v(f) + w(f)\\ (av)(f) &= av(f)\end{aligned}\right\} \forall f \in \mathcal{F}(M),\ a \in \mathbb{R}.$$

$\rightarrow T_pM$ is a vector space over $\mathbb{R}$.

Definition: Let $\xi = (x^1, \dots, x^n)$ be a coordinate system in M at p. If $f \in \mathcal{F}(M)$, let

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \xi)}{\partial u_i} \xi(p) \qquad i = 1, \dots, n.$$

where u^i are the natural coordinates on $\mathbb{R}^n$

$\xi: \mathcal{U} \longrightarrow \mathbb{R}^n$; $\mathcal{U} \subseteq M$ coordinates system.

$$f \circ \xi^{-1} : \xi(U) \longrightarrow \mathbb{R}$$

$$\Rightarrow \qquad \left.\frac{\partial}{\partial x_i}\right|_p : \mathcal{F}(M) \longrightarrow \mathbb{R}.$$

sending $f \in \mathcal{F}(M)$ to $\left(\frac{\partial f}{\partial x^i}\right)(p)$

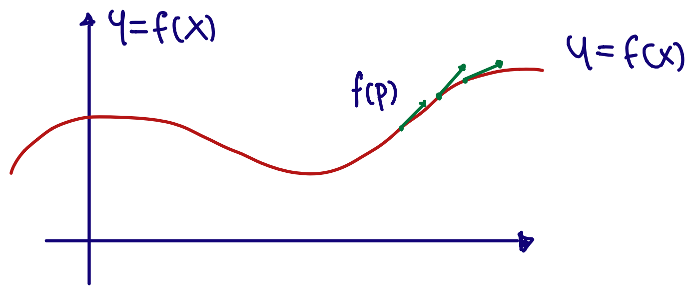
$$\frac{\partial}{\partial x^i} =: \partial_i\Big|_p$$

$\partial_i|_p$ is a tangent vector to M at p in the direction x^i

Theorem: (Basis) If $\xi = (x^1, \dots, x^n)$ is a coordinate system in M at p, then it's coordinate vectors $\partial_1|_p, \partial_2|_p, \dots, \partial_n|_p$ form a basis for the tangent space $T_p(M)$ and

$$V = \sum_{i=1}^{n} V(x^i)\,\partial_i|_p \ , \quad \forall\ V \in T_pM.$$

Proof: (sketch) first, consider a curve



then, consider a point $x = p + v$ very close to p taylor expand.

$$f(x - p + v) = f(p) + v\left.\frac{\partial f}{\partial x}\right|_{x=p} + \dots$$

In n-dim space the slope is changed to

$$\sum_{i=1}^{n} v_i \left.\frac{\partial f}{\partial x^i}\right|_{x=p}$$

where

$$v^i := v^i \left.\frac{\partial}{\partial x_i}\right|_{x=p}$$

then

$$\sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x_i}\right|_{x=p}$$

is the directional derivative $x = p$, such that

$$V(f) = \sum_{i=1}^{n} v^i \left.\frac{\partial f}{\partial x_i}\right|_{x=p}$$

as $f \in \mathcal{F}(M)$ is arbitrary, then

$$V = \sum_{i=1}^{n} V^i \frac{\partial}{\partial X_i}\Big|_{x=p}$$

to show linear independence, consider

$$\sum_{i=1}^{n} a^i \partial_i|_p = 0.$$

apply this to x^j

$$0 = \sum_{i=1}^{n} a^i \partial_i x^j|_{x=p} = \sum_{i=1}^{n} a^i \frac{\partial x^j}{\partial X_i}\Big|_{x=p} = \sum_{i=1}^{n} a^i \delta_i^j = a^j$$

which shows linear independence, therefore

$$\forall\, V \in T_pM, \quad v = \sum_{i=1}^{n} V^i \partial_i|_p$$

$$\dim(T_pM) = \dim M \qquad \blacksquare$$

No matter how curved the manifold may be, T_pM is always an n-dim vector space at each point p.

In classical mechanics

$$\mathcal{L} = \mathcal{L}(q^i(t), \dot{q}^i(t))$$

$\dot{q}$ Vector space

$$\dot{q} = \sum_{i=1}^{n} \dot{q}^i \frac{\partial}{\partial q^i}\Big|_{q \in M}$$

If M has coordinates $\{q^i\}$, Then T_qM has coordinates $\{\dot{q}^i\}$

$$TM := \bigcup_{q \in M} T_qM$$

Dual vector space

Given an n-dim vector space V with basis $\varepsilon_i, (i=1,\dots,n)$ the basis e^i of the dual vector space V^* is determined by the product

$$\langle \varepsilon_i, e^j \rangle = \delta_i^j$$

$$\alpha \in V^* \longrightarrow V \to \mathbb{R}, \text{ such that } \alpha(v) \in \mathbb{R}$$

Dual vector space to the tangent space T_pM i.e., linear maps

$$\lambda : T_pM \longrightarrow \mathbb{R}$$

$$\lambda(v) = \langle \lambda, v \rangle \in \mathbb{R} \longrightarrow \lambda \in T_p^*(M)$$

In our case: let f be any function on $\mathcal{F}(M)$ for each $X_p \in T_pM$

$$X_p(f) \text{ is a scalar}, \quad X_p : T_pM \longrightarrow \mathbb{R}.$$

$$\lambda(X_p) \ni \mathbb{R}$$

Define λ_f as df

$$df(P) = X_p(f)$$

$$df(X_p) := X_p(f) = \langle df, X_p \rangle$$

$$df : T_pM \longrightarrow \mathbb{R}$$

df will be called the differential or the gradient of the function f.

The coordinate differentials dx^i form a basis for T_p^*M

$$\langle dx^i, \partial_j|_p \rangle = dx^i(\partial_j|_p) = \partial_j x^i|_p = \delta^i_j|_p$$

$$\dim T_p^*M = \dim M = \dim T_pM$$

for any $\omega \in T_p^*M$

$$\omega = \omega_i dx^i$$

Cotagent space in classical mechanics

$\hookrightarrow$ momentum space with cotangent vector fields.

$$P = p_i dq^i \quad \text{where} \quad p_i := \mathcal{L}(q^i, \dot{q}^i)$$

such that

$$P(V_p) = \sum_{i,j} (p_i dq^i)(v^j \partial_j|_p) = \sum_{i,j} p_i v^j dq^i(\partial_j|_p)$$

$$= \sum_{i,j} p_i v^j \left(\frac{\partial q^i}{\partial q^j}\Big|_p \right) = \sum_{i,j} p_i v^j \delta^i_j|_p = \sum_i p_i v^i|_p = \langle P, V \rangle|_p$$

finally, $df(X_p) = X_p(f) = \langle df, X_p \rangle$