

## Riemann-Lebesgue Lemma.

Let  $f(t)$  be continuous in  $(a, b)$ , then

$$I(x) = \int_a^b f(t) e^{ixt} dt = O\left(\frac{1}{x}\right), \quad \text{as } x \rightarrow \infty$$

Provided that the integral.

$$\int_a^b |f(t)| dt$$

converges.

Proof:

$$I(x) = \int_a^{a+\pi/x} f(t) e^{ixt} dt + \int_{a+\pi/x}^b f(t) e^{ixt} dt.$$

$$I(x) = \int_a^{b-\pi/x} f(t) e^{ixt} dt + \int_{b-\pi/x}^b f(t) e^{ixt} dt$$

changing  $t = t - \frac{\pi}{x}$

$$\int_a^{a+\pi/x} f(t) e^{ixt} dt = - \int_a^{b-\pi/x} f\left(t - \frac{\pi}{x}\right) e^{ix\left(t - \frac{\pi}{x}\right)} dt$$

then

$$I(x) = \frac{1}{2} \int_a^{a+\frac{\pi}{x}} f(t) e^{ixt} dt + \frac{1}{2} \int_{a+\frac{\pi}{x}}^b f(t) e^{ixt} dt \\ + \frac{1}{2} \int_a^{b+\frac{\pi}{x}} [f(t) - f\left(t + \frac{\pi}{x}\right)] e^{ixt} dt, \quad \text{we want } x \rightarrow \infty.$$

Mean Value Theorem:  $f(t)$  continuous and bounded on  $[a, b]$ .

$$\int_a^b f(t) dt = f'(c)(b-a), \quad \text{for some real-number } c \in [a, b]$$

the first two integrals

$$\int_a^{a+\frac{\pi}{x}} f(t) e^{ixt} dt = f\left(\frac{\pi}{x}\right) \left(\frac{\pi}{x}\right) \sim O\left(\frac{1}{x}\right)$$

and

$$\int_{b-\frac{\pi}{x}}^b f(t) e^{ixt} dt = f\left(\frac{\pi}{x}\right) \left(\frac{\pi}{x}\right) \sim O\left(\frac{1}{x}\right).$$

Finally as  $f(t)$  is continuous  $\forall t \in [a, b]$

$$\lim_{x \rightarrow \infty} \int_0^{b-\pi/x} [f(t) - f(t + \frac{\pi}{x})] e^{ixt} dt = 0$$

$$\therefore I(x) \sim O\left(\frac{1}{x}\right)$$



We may extend the Riemann-Lebesgue Lemma to generalized Fourier integrals as long as  $|f(t)|$  is integrable,  $\psi(t)$  is continuous differentiable and  $\psi'(t) \neq 0$ . Take.

$$I(x) = \int_a^b f(t) e^{ix\psi(t)} dt$$

$$= \frac{1}{xi} \int_a^b \frac{f(t)}{\psi'(t)} \frac{d}{dt} (e^{ix\psi(t)}) dt$$

$$I(x) = \frac{1}{xi} \frac{f(t)}{\psi'(t)} e^{ix\psi(t)} \Big|_a^b - \frac{1}{xi} \int_a^b \frac{d}{dt} \left( \frac{f(t)}{\psi'(t)} \right) e^{ix\psi(t)} dt$$

This vanishes more rapidly than  $\frac{1}{x}$  as  $x \rightarrow \infty$ .

$$I(x) \sim \frac{1}{ix} \frac{f(t)}{\psi'(t)} e^{ix\psi(t)} \Big|_a^b \text{ as } x \rightarrow \infty$$

This method does not work for stationary points  $\psi'(t) = 0$ . The method of stationary phase will give the asymptotic behaviour of generalized Fourier integrals with stationary points.

Choose the integral such that  $\psi'(a)=0$ , and  $\psi'(t)=0$  for  $a \leq t \leq b$ .

$$I(x) = \int_a^b f(t) e^{ix\psi(t)} dt = \int_a^{a+\epsilon} f(t) e^{ix\psi(t)} dt + \int_{a+\epsilon}^b f(t) e^{ix\psi(t)} dt$$

$=: I_1(x).$

$\epsilon > 0$  is a small parameter

Note:

$$\int_{a+\epsilon}^b f(t) e^{ix\psi(t)} dt \sim \frac{1}{ix} \frac{f(t)}{\psi'(t)} e^{ix\psi(t)} \Big|_{a+\epsilon}^b \sim \frac{1}{x} \quad \text{as } x \rightarrow \infty.$$

In the first integral change

$$\begin{array}{lcl} f(t) & \longrightarrow & f(a) \\ \psi(t) & \longrightarrow & \psi(a) + \frac{\psi^{(p)}(a)(t-a)^p}{p!} \end{array} \left\{ \begin{array}{l} \text{As before (for Laplace)} \\ \text{the leading contribution} \\ \text{comes from a} \\ \text{neighbourhood of the} \\ \text{stationary point.} \end{array} \right.$$

Where  $\psi'(a) = \psi''(a) = \dots = \psi^{(p-1)}(a) = 0$

$$I_1(x) = \int_a^{a+\epsilon} f(a) \exp \left\{ ix \left[ \psi(a) + \frac{1}{p!} \psi^{(p)}(a)(t-a)^p \right] \right\} dt$$

Next, replace  $\epsilon \rightarrow \infty$

(this will introduce error terms  $O(1/x)$ )

Let  $s = t - a$ ,  $ds = dt$

$$I_1(x) = f(a) e^{ix\psi(a)} \int_0^\infty \exp \left( ix \frac{\psi^{(p)}(a) s^p}{p!} \right) ds$$

Define  $\bar{u} = \frac{i s^p x \psi^{(p)}(a)}{p!} =: u$

upper sign if  $\psi^{(p)}(a) > 0$

lower sign if  $\psi^{(p)}(a) < 0$

$$s = \left\{ e^{\pm i\pi/2} \left[ \frac{p! u}{x |\psi^{(p)}(a)|} \right] \right\}^{1/p}$$

$$ds = e^{\pm i\pi/2p} \left( \frac{p!}{x |\psi^{(p)}(a)|} \right)^{1/p} \frac{1}{p} u^{\frac{1}{p}-1} du$$

$$\begin{aligned} \Rightarrow I_1(x) &= f(a) e^{i x \psi(a)} \int_0^\infty e^{-u} e^{\pm i\pi/2p} \left( \frac{p!}{x |\psi^{(p)}(a)|} \right)^{1/p} \frac{1}{p} u^{\frac{1}{p}-1} du \\ &= f(a) e^{i(x \psi(a) \pm \pi/2p)} \left( \frac{p!}{x |\psi^{(p)}(a)|} \right)^{1/p} \frac{1}{p} \int_0^\infty e^{-u} u^{\frac{1}{p}-1} du \end{aligned}$$

$$I_1(x) \sim f(a) e^{i(x \psi(a) \pm \pi/2p)} \left( \frac{p!}{x |\psi^{(p)}(a)|} \right)^{1/p} \frac{\Gamma(1/p)}{p} \quad \text{as } x \rightarrow \infty$$

Example

$$I(x) = \int_0^{\pi/2} e^{i x \cos t} dt$$

$$f(t) = 1 \quad \psi(t) = \cos t$$

$$0 = \psi'(t) = -\sin t \Rightarrow t=0 \text{ stationary point.}$$

$$\psi''(t) = -\cos t \Rightarrow \psi''(0) = -1 < 0$$

$$\Rightarrow p = 2$$

$$\Rightarrow I(x) \sim e^{i(x-\pi/4)} \left(\frac{2}{x}\right)^{1/2} \frac{\Gamma(1/2)}{2}$$

$$= e^{i(x-\pi/4)} \sqrt{\frac{\pi}{2x}} \quad \text{as } x \longrightarrow \infty$$

## Example (Bessel functions)

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(t) - nt) dt$$

We want the leading behaviour of  $J_n(n)$  as  $n \rightarrow \infty$

$$\Rightarrow J_n(n) = \operatorname{Re} \left( \frac{1}{\pi} \int_0^\pi e^{in(\sin t - t)} dt \right)$$

$$\psi(t) = \sin t - t \quad f(t) = 1$$

$$0 = \psi'(t) = \cos t - 1 \quad t = 0 \quad \text{stationary point.}$$

$$\psi''(t) = -\sin t \quad \psi''(0) = 0$$

$$\psi'''(t) = -\cos t \quad \psi'''(0) = -1 < 0$$

$$p = 3$$

$$J_n(n) \sim \frac{1}{\pi} \operatorname{Re} \left( e^{i(n(0) - \pi/6)} \left(\frac{3!}{n}\right)^{1/3} \frac{\Gamma(1/3)}{3} \right)$$

$$= \frac{1}{\pi} \frac{\sqrt{3}}{2} \left(\frac{6}{n}\right)^{1/3} \frac{\Gamma(1/3)}{3} = \frac{\Gamma(1/3)}{\pi} \frac{3^{1/2} \cdot 3^{1/3} \cdot 3^{-1}}{2^{1/3}} \frac{1}{n^{1/3}}$$

$$= \frac{\Gamma(1/3)}{\pi} \frac{1}{3^{1/6} 2^{2/3}} \frac{1}{n^{1/3}} \quad \text{steepest descend.}$$