$$\int_{\circ} f(x)\,dx = \lim_{u \to \{e\}} \frac{(f : 1_u)}{(f : 1_u)}$$

$$\mathrm{Supp}(f) = \overline{\{x \in G / f(x) \neq 0\}}$$

Let $C_c(G)$ be the complex vectorial space of all the functions from G to $\mathbb{C}$, with compact support. For a complex vectorial space a linear map $L: V \longrightarrow \mathbb{C}$ it is called linear functional in V.

We say that $f \in C_c(G)$ is non-negative and we write $f \geqslant 0$, if $f \geq 0$, $\forall x \in G$. A linear functional I in $C_c(G)$ is the integral if

$$f \geqslant 0 \longrightarrow I(f) \geq 0.$$

Example: Let $x \in G$, and let $\delta_x(f) = f(x)$, $f \in C_c(G)$. Then δ_x is an integral, called the Dirac delta. If it is clear that integral to use, we write.

$$I(f) = \int_G f(x)\,dx.$$

If $f, g \in C_c(G)$ evalued in the reals, $f \geqslant g$. If $f - g \geqslant 0$, then $I(f) \geqslant I(g)$.

Lemma: for all integral in G

$$\left| \int_G f(x)\,dx \right| \leq \int_G |f(x)|\,dx.$$

Proof: Let $f_{\pm} = \max(\pm f, 0)$.

Then $f_{\pm} \in C_c(G)$, the function $f_{\pm}$ is non-negative.

$$f = f_+ - f_-,$$

$$|f| = f_+ + f_-.$$

Thus

$$\left| \int_G f(x)\,dx \right| = \left| \int_G f_+(x)\,dx - \int_G f_-(x)\,dx \right|$$

$$\leq \left| \int_G f_+(x)\,dx \right| + \left| \int_G f_-(x)\,dx \right|$$

$$= \int_G f_+(x)\,dx + \int_G f_-(x)\,dx$$

$$= \int_G |f(x)|\,dx. \qquad \blacksquare$$

Let $s \in G$, and $f \in C_c(G)$. for $x \in G$, we define

$$L_s f(x) = f(s^{-1}x)$$

The traslation by the left, by s. So, the function $L_s f$, again is in $C_c(G)$, with $L_s(L_t f) = L_{st} f$, $s,t \in G$.

$$\begin{aligned} L_s(L_t f)(x) &= (L_t f)(s^{-1}x) \\ &= f(t^{-1}s^{-1}x) \\ &= f((st)^{-1}x) \\ &= L_{st}(f)(x) \end{aligned}$$

and $L_1 f = f$.

A integral $I : C_c(G) \longrightarrow \mathbb{C}$, is called called invariant or invariant on the left if

$$I(L_s f) = I(f), \quad f \in C_c(G),\ s \in G$$

Using the integral notation, we say that $\int_G dx$, is invariant if and only if $\forall f \in C_c(G)$ and for all $y \in G$

$$\int_G f(yx)\,dx = \int_G f(x)\,dx$$

An example, $G \in \mathbb{R}$

$$\begin{aligned} I : C_c(\mathbb{R}) &\longrightarrow \mathbb{C} \\ f &\longmapsto I(f) = \int_{-\infty}^{\infty} f(x)\,dx. \end{aligned}$$

$$\int_{-\infty}^{\infty} f(x+a) = \int_{-\infty}^{\infty} f(x)\,dx.$$

Theorem: There is an invariant integral I of G. If I' is an invariant integral, so there is $c>0$, such that $I=cI$.

Any invariant integral, it is called Haar's integral.

Corollary: for all invariant integral I and for all $g\in C_c(G)$, with $g\geq 0$, $I(g)=0$, then $g=0$.

Proof: Let $g\in C_c(G)$, with $g\geq 0$ and $g\neq 0$. We have to prove that $I(g)\neq 0$. For this $f\in C_c(G)$, $f\geq 0$ and $I(f)\neq 0$.

As $g\neq 0$, there are $c_1, c_2, \dots, c_n>0$; $x_1, x_2, \dots, x_n\in G$ such that

$$f\leq \sum_{j=1}^{n} c_j L_{x_j} g.$$

Then $0<I(f)\leq \sum_{j=1}^{n} c_j I(L_{x_j} g)=(\sum_{j=1}^{n} c_j) I(g)$, therefore $I(g)\neq 0$.

∎

Lemma: The map $C_c(G)$ is a pre-Hilbert space with inner product

$$\langle f,g\rangle = \int_G f(x)\overline{g(x)}\,dx$$

Proof:

i) $\mathbb{C}$-linear

ii) $\langle v,w\rangle = \overline{\langle w,v\rangle}$

iii) We still need to prove that it is definite positive.

Let $f\in C_c(G)$, with $\langle f,f\rangle=0$. So, the function $|f|^2\in C_c(G)$ is positive, and because of the above corollary $|f|^2=0$; therefore, $f=0$.

∎

Note: The completation of $C_c(G)$ is a Hilbert space and it is called $L^2(G)$ (Does not depend on the invariant measure).

Example: 1) Haar's integral on $\mathbb{R}$.

$$I(f)=\int_{-\infty}^{\infty} f(x)\,dx.$$

2) On $\mathbb{R}/\mathbb{Z}$

$$I(f)=\int_0^1 f(x)\,dx.$$

3) On $\mathbb{R}_+^\times$

$$I(f)=\int_0^\infty f(x)\,\frac{dx}{x}.$$

4) On $GL_n(\mathbb{R})$ invertible matrices of $n\times n$.

$$I(f)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f\begin{pmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}\frac{da_{11}\cdots da_{nn}}{\det|a|^n}$$

Homework: Prove that the Haar's Integral of $GL_2(\mathbb{R})$

$$I(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\begin{pmatrix} x & y\\ z & w\end{pmatrix}\frac{dx\,dy\,dz\,dw}{|xw-zy|^2}$$

Hint: $I(L_s f)=I(f)$.

Let G,H be locally compact groups, σ-compacts, metrizables (LC). So the cartesian product $G\times H$ is a group of the same type, and then it has a Haar's measure.

Theorem (Fubini): Let $I_G(g)=\int_G g(x)dx$ a Haar's integral on G so for all $f\in C_c(G\times H)$, the function

$$y\longmapsto I(f(\cdot,y))=\int_G f(x,y)\,dx$$

$C_c(H)$. Let $I_H(h)=\int_H h(g)\,dy$ the Haar's integral on H. So, the Haar's integral on $G\times H$ it's given by

$$I(f)=\int_H\int_G f(x,y)\,dx\,dy=\int_G\int_H f(x,y)\,dy\,dx$$

Convolution

Let A be a LCA group, with Haar's integral $\int_A dx$.

Let $\hat{A}$ be your dual group, i.e., the characters group $\chi : A \to \mathbb{T}$
for $f \in L^1_{bc}(A)$, we have

$$L_{bc}(A) = \} f \text{ are continuous and bounded}, \|f\|_1 = \int |f(x)| dx < \infty \{$$

Let $\hat{f} : \hat{A} \longrightarrow \mathbb{C}$,

$$\hat{f}(x) = \int_A f(x) \overline{\chi(x)} dx$$

If $x \in \mathbb{R}$, φ_x the associated character to x i.e.,

$$\varphi_x(y) = e^{2\pi i x y}, \text{ thus } f \in L^1_{bc}(\mathbb{R}),$$

$$\hat{f}(\varphi_x) = \int_{\mathbb{R}} f(y) \overline{\varphi_x(y)} dy = \int_{-\infty}^{\infty} f(y) e^{-2\pi i x y} dy = \hat{f}(y)$$

Theorem: Let $f, g \in L^1_{bc}(A)$. So the integral

$$f \times g = \int_A f(xy^{-1}) g(y) dy$$

there is for each x in A, and it defines a function

$$f * g \in L^1_{bc}(A)$$

and

$$\widehat{(f * g)}(x) = \hat{f}(x) \hat{g}(x) \quad , \forall x \in \hat{A}$$

Proof: Let's assume that $|f(x)| \leq c \quad \forall x \in A$, then

$$\int_A |f(xy^{-1}) g(y)| dy \leq c \int_A |g(y)| dy = c \|g\|_1$$

because the integral exists and $f * g$ is bounded.
Now, we will prove that it is continuous.

Let $x_0 \in A$ and $|f(x)|, |g(x)| \leq c$, $\forall x \in A$ and let's assume $y \neq 0$.

for $\varepsilon > 0$, there is a function $\varphi \in C_c^+(A)$ such that $\varphi \leq |y|$ and

$$\int_A |g(y) - \varphi(y)|\,dy < \frac{\varepsilon}{4c}$$

In a f compact, is uniformly continuous, so there is a neighbourhood V, from the identity element such that

$$x \in Vx_0 \ , \ y \in \operatorname{supp} \varphi$$

Implies that

$$|f(xy^{-1}) - f(x_0 y^{-1})| < \frac{\varepsilon}{2\|g\|_1}$$

It follows that $x \in Vx_0$,

$$\int_A |f(xy^{-1}) - f(x_0 y^{-1})|\,\varphi(y)\,dy$$

$$\leq \frac{\varepsilon}{2\|g\|_1} \int_A \varphi(y)\,dy \leq \frac{\varepsilon}{2}$$

and therefore

$$\int_A |f(xy^{-1}) - f(x_0 y^{-1})|\,(|g(y)| - \varphi(y))\,dy$$

$$\leq 2c \int_A (|g(y)| - \varphi(y))\,dy$$

thus, $x \in Vx_0$

$$|f*g(x) - f*g(x_0)| = \left| \int_A (f(xy^{-1}) - f(x_0 y^{-1}))\,g(y)\,dy \right|$$

$$\leq \int_A |f(xy^{-1}) - f(x_0 y^{-1})|\,|g(y)|\,dy$$

$$= \int_A |f(xy^{-1}) - f(x_0 y^{-1})| \times (|g(y)| - \varphi(y) + \varphi(y))\, dy$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\|f * g\|_1 = \int_A |f * g(x)|\, dx$$

$$= \int_A \left| \int_A f(xy^{-1})\, g(y)\, dy \right| dx$$

$$\leq \int_A \int_A |f(xy^{-1})\, g(y)|\, dy\, dx$$

$$= \int_A \int_A |f(xy^{-1})||g(y)|\, dx\, dy$$

$$= \int_A |f(x)|\, dx \int_A |g(y)|\, dy$$

$$= \|f\|_1 \|g\|_1$$

The fourier transform

$$\widehat{(f * g)}(x) = \int_A (f * g)\, \overline{\chi(x)}\, dx.$$

$$= \int_A \int_A f(xy^{-1})\, g(y)\, \overline{\chi(x)}\, dx\, dy$$

$$= \int_A \int_A f(x)\, g(y)\, \overline{\chi(xy)}\, dx\, dy$$

$$= \int_A f(x)\, \overline{\chi(x)} \int_A g(y)\, \overline{\chi(y)}\, dy = \hat{f}(x)\, \hat{g}(x)$$

$\blacksquare$