Topics:

- Differential Geometry
- Einstein's Equations
- Black holes.
- Cosmology
- Quantum Gravity.

Bibliography:

Physics:

- D' Inverno
- Nakahara
- Schutz
- Misner, Thorne & Wheeler.
- Wald.

Maths:

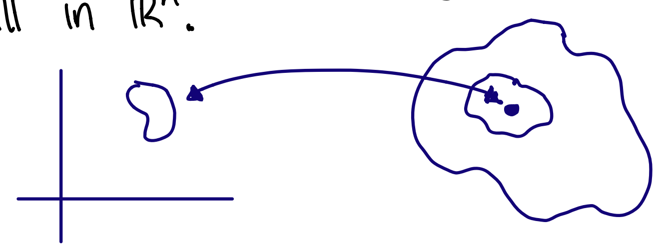
- Jost
- Dodson
- Petersen
- O' Neill
- Do Carmo
- Spivack
- Sachs & Wu.

Differential manifolds

$$\text{Calculus in } \mathbb{R}^n \longmapsto \text{Differential manifold.}$$

The most general object on which calculus can be conducted.

Definition: An n-dim manifold is a topological space that is locally Euclidean, that is, around every point there is a neighbourhood that is topologically the same as the open unit ball in $\mathbb{R}^n$.



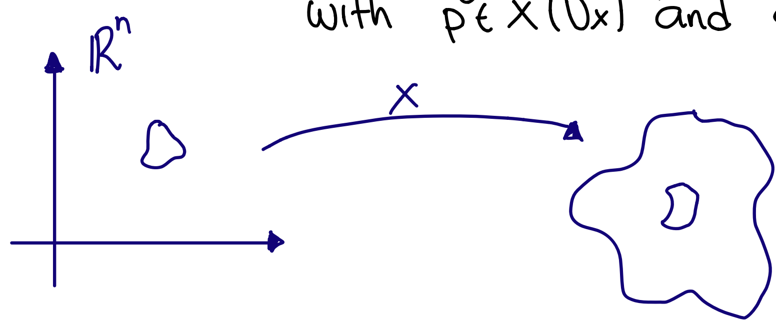
Note: We are understand topological space as: Convergence, connectedness and continuity defined.

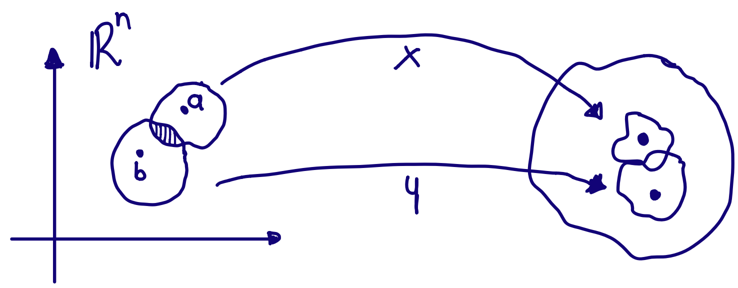
Definition: An n-dim manifold is a set furnished with a collection $\mathcal{P}$ of abstract patches (one-to-one functions $x: D \to M$, D open sets in $\mathbb{R}^n$) satisfying:

1) **Covering property:** The images of the patches in the collection $\mathcal{P}$ cover M.

2) **Smooth overlap property:** For any patches x, y in $\mathcal{P}$, the composite functions $y^{-1} \circ x$ and $x^{-1} \circ y$ are Euclidean.

3) **Hausdorff property:** For any points $p \neq q$ in M. There are disjoints patches x and y with $p \in x(D_x)$ and $q \in y(D_y)$.





$$(y^{-1} \circ x)(a) = y^{-1}(x(a)) = b$$

$$(x^{-1} \circ y)(b) = x^{-1}(y(b)).$$

$$x: D_1 \longrightarrow M_1$$

$$y: D_2 \longrightarrow M_2$$

$$y^{-1} \circ x = D_1 \longrightarrow M_1 \cap M_2 \longrightarrow D_2.$$

Euclidean n-space $\mathbb{R}^n$ is the set of all n-tuples $q = (q_1, \dots, q_n)$ of real numbers.

Natural inner product of $\mathbb{R}^n$ is the dot-product.

$$q \cdot \tilde{q} := \sum_i q_i \tilde{q}_i$$

With norm

$$|q| := \sqrt{q \cdot \tilde{q}}$$

and resulting metric

$$d(q, \tilde{q}) := |q - \tilde{q}|$$

A real valued function f defined on an open set U of $\mathbb{R}^n$ is smooth ($f \in \mathbb{R}^n$) for $f: U \longrightarrow \mathbb{R}$ provided all mixed partial derivatives of f (of all orders) exist and are continuous at every point of U.

Note: We want to extend these definitions to manifolds.

Definition: A manifold is a Hausdorff topological space such that every point has a neighbourhood homeomorphic to $\mathbb{R}^n$

Definition (Homeomorphism): A homeomorphism is a bijection (one-to-one and onto) f between topological spaces which is bicontinuous.

f and f^{-1} are continuous.