Propositions of schwarzschild:

I. Spherically symmetric.

II. Static

III. Coordinates adapted for a timelike Killing vector field.

IV. Asymtotically flat $ds^2 = dt^2 - dr^2 - r^2 d\Omega^2$

V. $m := \frac{GM}{c^2}$ geometric mass

$$g_{00} \approx 1 + \frac{2\phi}{c^2} = 1 - \frac{2GM}{c^2 r}$$

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2 d\Omega^2$$

Description of the matter inside a star (interior solution)

Matter $\longrightarrow$ Perfect fluid.

$\rho = \rho(r)$ density of mass-energy

$P = P(r)$ Isotropic pressure

$n := n(r)$ Number density of baryons

$u^\mu := u^\mu(r)$ 4-velocity of fluid.

$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P g^{\mu\nu}$ stress-energy tensor

Static star $\longrightarrow$ Particles move along world lines of constant r, θ, φ.

$$u^r = \frac{dr}{d\tau} = 0$$

$$u^\theta = \frac{d\theta}{d\tau} = 0$$

$$u^\varphi = \frac{d\varphi}{d\tau} = 0$$

$$-1 = u_\alpha u^\alpha = g_{\mu\nu} u^\mu u^\nu = -e^{\nu} u^t u^t$$

$$(u^t)^2 = e^{-\nu} \qquad u^t = e^{-\nu/2}$$

$$u = e^{-\nu/2}\left(\frac{\partial}{\partial t}, 0, 0, 0\right)$$

$$T^{00} = \rho e^{-\nu}$$

$$T^{rr} = P e^{-\lambda}$$

$$T^{\theta\theta} = \frac{P}{r^2}$$

$$T^{\varphi\varphi} = \frac{P}{r^2 \sin^2\theta}$$

$$T^{\alpha\beta} = 0 \quad \text{if} \quad \alpha \neq \beta$$

Define on orthonormal basis

$$e_{\hat{t}} := \frac{1}{e^{\nu/2}} \frac{\partial}{\partial t}, \qquad e_{\hat{r}} := \frac{1}{e^{\lambda/2}} \frac{\partial}{\partial r}$$

$$e_{\hat{\theta}} := \frac{1}{r} \frac{\partial}{\partial \theta}, \qquad e_{\hat{\varphi}} := \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

$$u^\alpha = e^\alpha_{\hat{t}} \longrightarrow u^t = e^t_{\hat{t}} = 1, \quad u_{\hat{r}} = u_{\hat{\theta}} = u_{\hat{\varphi}} = 0$$

$$T_{\hat{t}\hat{t}} = \rho \qquad T_{\hat{r}\hat{r}} = T_{\hat{\theta}\hat{\theta}} = T_{\hat{\varphi}\hat{\varphi}} = P$$

$$T_{\hat{\alpha}\hat{\beta}} = 0 \quad \text{if} \quad \alpha \neq \beta$$

To determine ν, λ, ρ, P, n we need temperature or entropy.

Einstein equation's: $h = e^\lambda$, $f = e^\nu$

$$8\pi T_{00} = 8\pi \rho = G_{00} = \frac{1}{rh^2} h' + \frac{1}{r^2}\left(1 - \frac{1}{h}\right)$$

$$8\pi T_{\hat{r}\hat{r}} = 8\pi P = G_{11} = (rfh)^{-1} f' - r^{-2}\left(1 - \frac{1}{h}\right)$$

$$8\pi T_{\hat{\theta}\hat{\theta}} = 8\pi T_{\hat{\varphi}\hat{\varphi}} = 8\pi P = \frac{1}{2} \frac{1}{\sqrt{fh}} \frac{d}{dr}\left[(fh)^{-1/2} f'\right] + \frac{1}{2}(rfh)^{-1} f' - \frac{1}{2}(rh^2)^{-1} h'$$

$$8\pi \rho = \frac{1}{r^2} \frac{d}{dr}\left[r(1 - h^{-1})\right]$$

$$8\pi\rho r^2 = \frac{d}{dr}\left[r(1-h^{-1})\right]$$

$$\int 8\pi r^2 \rho(r)\, dr + a = r\left(1-\frac{1}{h}\right)$$

constant

Define

$$m(r) := 4\pi \int_0^r \rho(r') r'^2 dr' + a$$

$$\frac{2m(r)}{r} = 1 - \frac{1}{h} \qquad ds^2 = \dots - h dr^2 - \dots$$

$$\frac{1}{h} = 1 - \frac{2m(r)}{r}$$

$$\boxed{h = \left[1 - \frac{2m(r)}{r}\right]^{-1}} = e^{\lambda}$$

As $r \to 0, h \to 1$

$$\lim_{r\to 0} m(r) = a$$

$$\lim_{r\to 0} \frac{m(r)}{r} \longrightarrow \frac{a}{0}$$

L'Hôpital's

$$\lim_{r\to 0} \frac{4\pi\rho(r)r^2}{1} \longrightarrow 0$$

Take $a=0$; $\quad \rho(r) = r^{-n}, \ n>2$

Since Σ must be a spacelike manifold for static.

Then

$h>0$ if and only if $r \geq 2m(r)$

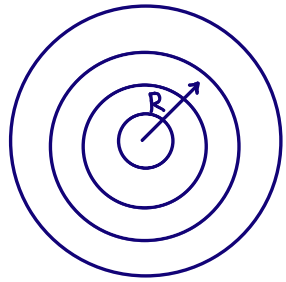
Assume $R :=$ Radius of the star.

Then,

If $\rho = 0$ for $r > R$.

Then,

h vacuum Schwarzschild with total mass.



$$M := m(R) = 4\pi \int_0^R \rho(r)\, r^2\, dr$$

$\longrightarrow$ Total Newtonian mass

Define proper mass

$$M_p := 4\pi \int_0^R \rho(r)\, r^2 \left(1 - \frac{2m(r)}{r^2}\right)^{-1/2} dr$$

$$\sqrt{^{(3)}g}\, d^3x = h^{1/2} r^2\, dr\, d\Omega$$

$E_B : M_p - M > 0$

Binding energy

$$\frac{1}{(1-x)^{1/2}} \approx 1 + \frac{1}{2}x - \dots$$

$8\pi T_{11} = G_{11}$ for $f = e^{\nu}$

$$8\pi P = (rfh)^{-1} f' - r^{-2}(1 - h^{-1})$$

$$\frac{f'}{f} = \frac{\frac{d\nu}{dr} e^{\nu}}{e^{\nu}} = \frac{d\nu}{dr}$$

$$8\pi P = \frac{1}{r}\left(1 - \frac{2m(r)}{r}\right)\frac{d\nu}{dr} - \frac{1}{r^2}\left(\frac{2m(r)}{r}\right)$$

$$\left(8\pi P + \frac{2m(r)}{r^3}\right)\frac{r}{1 - \frac{2m(r)}{r}} = \frac{d\nu}{dr}$$

$$\frac{d\nu}{dr} = \frac{2m(r) + 8\pi r^3 P(r)}{r[r - 2m(r)]}$$

$$\boxed{\nu(r) = \int \frac{2m(r) + 8\pi r^3 P(r)}{r[r - 2m(r)]}\, dr}$$

$PV = nRT$

$F = -\nabla\phi$

$$-\frac{d\phi}{dr} = \frac{m}{r^2}$$

In Newtonian limit $r^3 P << m(r)$

$2m(r) << r$

$$\frac{d\nu}{dr} \simeq \frac{2m(r)}{r^2} \rightsquigarrow$$ Symmetric Poisson equation for Newtonian Potential.

$$\phi := \frac{\nu}{2}$$

TOV equations

$$\nabla \cdot T = 0 \qquad \nabla_a T^{ab} = 0$$

$$T = (\rho + P) u \otimes u + Pg$$

then,

$$0 = \nabla \cdot T = [\nabla(\rho + P) \cdot u] u + (\rho + P)[\nabla u] u + (\rho + P) u \cdot \nabla u + \nabla P \cdot g$$

$u^a \nabla_a = \nabla_u$

$$= [\nabla_u \rho + \nabla_u P + (\rho + P) \nabla \cdot u] u + (\rho + P) \nabla_u u + \nabla P \qquad \text{(I)}$$

Now, consider this last equation along the 4-velocity

$$u = e^{-\nu/2}\left(\frac{\partial}{\partial t}, 0, 0, 0\right)$$

$$0 = u \cdot (\nabla \cdot T) \underset{u \cdot u = 1}{=} -[\nabla_u \rho + \cancel{\nabla_u P} + (\rho + P) \nabla \cdot u] + \cancel{\nabla_u P}$$

$$u \cdot \nabla_u u = \frac{1}{2} \nabla_u u^2 = \frac{1}{2} \nabla_u (-1) = 0$$

$$\boxed{\nabla_u \rho = -(\rho + P) \nabla \cdot u} \qquad \text{(II)}$$

from (I) & (II)

$$\boxed{(\rho + P) \nabla_u u = -\nabla P - (\nabla_u P) u}$$

Euler equation for relativistic hydrodynamics.

Only radial part has content since pressure only depends on r.

$$(\rho + P) u_{r;\nu} u^\nu = -\frac{dP}{dr}$$

$$\nabla_u P \longrightarrow 0 \quad \text{since} \quad P = P(r)$$

$$\text{and} \quad u = e^{-\nu/2}\left(\frac{\partial}{\partial t}, 0, 0, 0\right)$$

$$U_{r;\nu} U^\nu = U_{r,\nu} U^\nu - \Gamma^\alpha_{r\nu} U_\alpha U^\nu$$
$$= -\Gamma^0_{r0} U_0 U^0 = \Gamma^0_{r0}$$

It only depends on r and u
$\hookrightarrow$ time translation.

For diagonal matrices

$$\Gamma^0_{r0} = \frac{\partial}{\partial r}\left(\log\sqrt{|g_{00}|}\right)$$
$$= \frac{\partial}{\partial r} \log\sqrt{|e^\nu|}$$
$$= \frac{d}{dr} \log(e^{\nu/2})$$
$$= \frac{1}{e^{\nu/2}} \frac{d}{dr} e^{\nu/2} = \frac{1}{2} \frac{d\nu}{dr}$$
$$(\rho + P) \frac{1}{2} \frac{d\nu}{dr} = -\frac{dP}{dr}$$

$$\boxed{\frac{dP}{dr} = -(\rho + P)\left[\frac{m(r) + 4\pi r^3 P}{r[r - 2m(r)]}\right]}$$

Tolman-Oppenheimer-Volkof equation for hydrostatic equilibrium.

In the Newtonian limit

$$P \ll \rho, \quad 2m(r) \ll r$$

$$\frac{dP}{dr} \approx -\frac{\rho m(r)}{r^2}$$ } Newtonian hydrostatic equilibrium equation.

Schwarzschild interior solution

$$ds^2 = e^\nu dt^2 + \left[1 - \frac{2m(r)}{r}\right]^{-1} dr^2 + r^2 d\Omega^2$$

subject to equation for ν and to TOV equation

$$m(r) := 4\pi \int_0^r \rho(r') r'^2 dr'$$

Algorithm for finding solutions:

I. Find a suitable equation of state $P = P(\rho)$

II. Prescribe a central density $\rho_c \rightarrow P_c$ at $r = 0$

III. Integrate TOV and $m(r)$ equations outwards until we reach the surface of the star $(P = \rho = 0)$

IV. On this solution we join them with the vacuum schwarzchild solution.

V. Integrate $\frac{\partial \nu}{\partial r}$ subject to condition $\nu \longrightarrow 0$ as $r \longrightarrow \infty$

Example: Incompressible fluid of density ρ_0

$$\rho(r) = \begin{cases} \rho_0 , & r \leq R \\ 0 , & r > R \end{cases}$$

then,

$$m(r) = \frac{4}{3} \pi r^3 \rho_0$$

$$M = \frac{4}{3} \pi R^3 \rho_0 \longleftrightarrow \left(\frac{3M}{4\pi \rho_0} \right)^{1/3} = R.$$

Newtonian hydrostatic equilibrium

$$\frac{dP}{dr} = - \frac{\rho_0 \left(\frac{4}{3} \pi r^3 \rho_0 \right)}{r} = -\frac{4}{3} \pi \rho_0^2 r$$

then,

$$P(r) = \int -\frac{4}{3} \pi \rho_0^2 r' dr' = -\frac{2}{3} \pi \rho_0^2 r^2 - A$$

Take $P(R) = 0$

$$P(r) = \frac{2}{3} \pi \rho_0^2 (R^2 - r^2)$$

$$P_c = \frac{2}{3} \pi \rho_0^2 R^2$$

$$P_c = \frac{2}{3}\pi \rho_0^2 \left(\frac{3M}{4\pi\rho_0}\right)^{2/3} = \left(\frac{\pi}{6}\right)^{1/6} \rho_0^{4/3} M^{2/3} < \infty$$

TOV equation:

$$\frac{dP}{dr} = -(P+\rho)\left(\frac{m(r)+4\pi r^2 P}{r[r-2m(r)]}\right)$$

Idea: Write

$$\frac{dP}{dr} = \frac{-\rho m(r)}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi r^3 P}{m(r)}\right)\left[1-\frac{2m(r)}{r}\right]^{-1}$$

$$P(r) = \rho_0 \left[\frac{\left(1-\frac{2M}{R}\right) - \left(1-\frac{2Mr^2}{R^3}\right)^{1/2}}{\left(1-\frac{2Mr^2}{R^3}\right)^{1/3} - 3\left(1-\frac{2M}{R}\right)^{1/2}}\right]$$

then

$$P_c = \rho_0\left[\frac{\left(1-\frac{2M}{R}\right)-1}{1-3\left(1-\frac{2M}{R}\right)^{1/2}}\right]$$

$P_c \longrightarrow \infty$ as $3\left(1-\frac{2M}{R}\right)^{1/2} \longrightarrow 1$

then

$$\boxed{R \longrightarrow \frac{9}{4}M}$$

therefore, uniform density stars with $M > \frac{4R}{9}$ cannot exist!

This conclusion may be extended to non-negative monotonic decreasing density functions $\rho(r)$