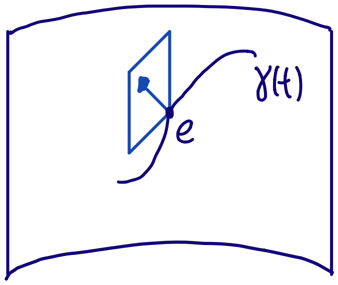
Lie algebras



G

$$\rho : G \to GL(n,\mathbb{R})$$

$$U(1) \to \rho_n$$

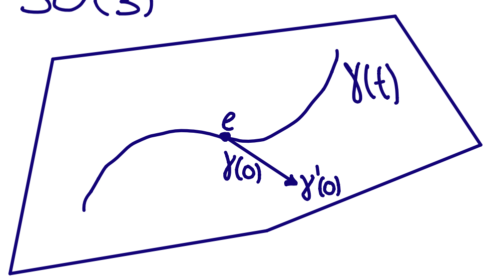
$$SU(2) \to \text{spin } 0, 1/2, 1, \dots, (2j+1).$$

$$SU(2) \underset{2-1}{\cong} SO(3)$$

Let γ be a curve in $SO(3)$, such that γ is a rotation by the angle t, around the z-axis.

$$\gamma(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

SO(3)



the tangent vector to $\gamma(t)$ in the identity

$$\gamma'(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the same way, the rotations arround the x,y-axis.

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} ; \quad J_y = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are in the Lie algebra of $SO(3)$, and its denoted by $\mathfrak{so}(3)$. The key is that lie algebra matrice's are infinitesimal rotations and we can describe finite rotations by exposing them.

The exponential of an nxn T-matrix, is defined

$$\exp(T) = 1 + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots$$

In order to obtain a matrix who describes a rotation by a T angle, around the z-axis, lets compute $\exp(tJ_z)$

$$J_z^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_z^3 = -J_z \ ; \ J_z^4 = -J_z^2 \ ; \ldots$$

$$\exp(tJ_z) = 1 + tJ_z + \frac{t^2}{2!}J_z^2 - \frac{t^3}{3!}J_z - \frac{t^4}{4!}J_z^2$$

$$= 1 + \left(-\frac{t^3}{3!} + \ldots\right)J_z + \left(\frac{t^2}{2!} - \frac{t^4}{4!}\ldots\right)J_z^2$$

$$= 1 + \sin(t)J_z + (1 - \cos(t))J_z^2$$

$$= \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotations in x no commute with the y. This is manifested in the Lie algebra from the fact that J_x and J_y not commute.

$$\exp(sJ_x)\exp(tJ_y) \neq \exp(tJ_y)\exp(sJ_x)$$

be the diference

$$\exp(sJ_x)\exp(tJ_y) - \exp(tJ_y)\exp(sJ_x)$$

expanding in series

$$(1 + sJ_x + \mathcal{O}(s^2))(1 + tJ_y + \mathcal{O}(t^2)) - (1 + tJ_y + \mathcal{O}(t^2))(1 + sJ_x + \mathcal{O}(s^2))$$

$$= st\,(J_xJ_y - J_yJ_x) + \mathcal{O}(s^2, t^2).$$

In general, given two $n\times n$ matrices S, T, we define the Lie commutator $[S,T]$ as

$$[S,T] = ST - TS$$

Homework: Show that

$$[J_x, J_y] = J_z \ , \ [J_y, J_z] = J_x \ , \ [J_z, J_x] = J_y.$$

Let's see the $so(n)$ Lie algebra of the group $SO(n)$.

Let γ be a curve in $SO(n)$, with $\gamma(0) = 1$, for any vectors $v, w \in \mathbb{R}^n$

$$\langle \gamma(t)v, \gamma(t)w\rangle = \langle v, w\rangle \ ; \ \forall t.$$

with $\langle\cdot,\cdot\rangle$ being the inner product of $\mathbb{R}^n$. Deriving and taking $t=0$.

$$\langle v, \gamma'(0)w\rangle + \langle \gamma'(0)v, w\rangle = 0$$

We might consider $\gamma(t)$ as a curve in the $n\times n$ matrix space, such that $\gamma'(0)=T$, this implies

$$T_{ij}+T_{ji}=0$$

$$\langle e_i, T_{jk}e_k\rangle + \langle T_{ik}e_k, e_j\rangle = 0$$

$$T_{jk}\langle e_i, e_k\rangle + T_{ik}\langle e_k, e_j\rangle = 0$$

$$T_{jk}\delta_{ik} + T_{ik}\delta_{kj} = 0$$

$$T_{ji}+T_{ij}=0 \longrightarrow \text{Skew-hermitian, Real (Skew-symmetric)}$$

Homework: For any matrix T $\det(\exp(T)) = e^{\mathrm{tr}(T)}$.

Any group of G, not only the matrices, they have an exponential mapping, i.e.,

$$\exp: \mathfrak{g} \longrightarrow G$$

I. $\exp(0)=1$ of the group G

II. $\exp(sX)\exp(tX)=\exp((s+t)X)$; $\forall X\in\mathfrak{g}$, $s,t\in\mathbb{R}$.

III. $\frac{d}{dt}\exp(tX)\Big|_{t=0} = X\in\mathfrak{g}$

$$\exp(sv)\exp(tw)-\exp(tw)\exp(sv) = st[v,w]+\mathcal{O}(s^2,t^2)$$

$$[v,w] = \frac{d^2}{dsdt}\Big(\exp(sv)\exp(tw)-\exp(tw)\exp(sv)\Big)\Big|_{s=t=0}$$

I. $[v,w]=-[w,v]$; $\forall v,w\in\mathfrak{g}$

II. $[u,\alpha v+\beta w]=\alpha[u,v]+\beta[u,w]$; $\forall u,v,w\in\mathfrak{g}$; $\alpha,\beta\in\mathbb{R}$

III. $[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$; $\forall u,v,w\in\mathfrak{g}$

$$\rho: G\longrightarrow H$$

$$\rho(gg')=\rho(g)\rho(g')$$

A Lie algebra homomorphism

$$f:\mathfrak{g}\longrightarrow \mathfrak{h}$$

$$f([v,w])=[f(v),f(w)]$$

Homework: Show that Lie algebras $SU(2)$ and $SO(3)$ are isomorphism.

First prove that $S(U)$ are the quaternions I, J, K $(-i\sigma_1, -i\sigma_2, -i\sigma_3)$, then prove that the map

$$f : SU(2) \longrightarrow SO(3)$$

$$-\frac{1}{2}\sigma_j \longmapsto J_j$$

is a Lie algebra isomorphism.

Let $g \in G$, there exists a map from G to G through

$$h \longmapsto gh$$

multiplication by the left-hand side by g, and its denoted $L_g : G \longrightarrow G$. The map has inverse, by g^{-1}.

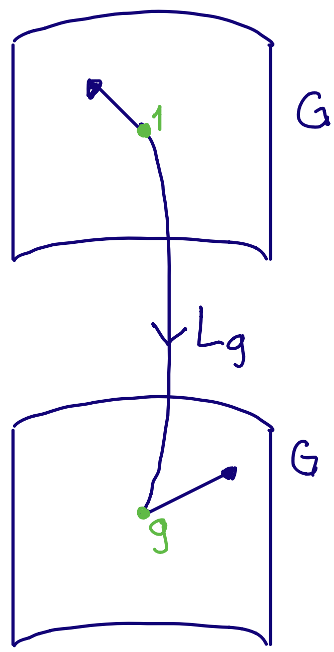
Then L_g is a diffeomorphism.

$$(L_g)_* : Vect(G) \longrightarrow Vect(G)$$

We say that a vector field v in G is invariant by the left if

$$(L_g)_* v = v$$

are the Lie algebra



Given $v_1 \in \mathfrak{g}$ being tangent to G in 1, and

$$(L_g)_* v_1 = v_g$$

The vector v_1 is left invariant.

We need to prove that $(L_g)_* v = v$, i.e., for all $h \in G$.

$$(L_g)_* v_* = v_{L_g h} = v_{gh}$$

$$\begin{aligned} (L_g)_* v_h &= (L_g)_* (L_h)_* v_1 \\ &= (L_g L_h)_* v_1 \\ &= (L_{gh})_* v_1 \\ &= v_{gh}. \end{aligned}$$

$\blacksquare$