Diffeomorphism invariance of the matter action S_M.

If a theory describes nature in terms of a spacetime manifold M and tensor fields T^I ($I :=$ set indices) defined on the manifold.

If $\phi : M \longrightarrow N$ is a diffeomorphism then (M, T^I) and $(N, \phi^* T^I)$ have physical identical properties.

$\therefore$ Any physically meaningful statement about (M, T^I) will hold with equal validity for $(N, \phi^* T^I)$

$\longrightarrow$ General principle of relativity:

Laws of physics are the same for all observers.

Theorem: S_M must be invariant under diffeomorphisms, i.e., if $\phi_\lambda : M \longrightarrow N$ is a one-parameter group of diffeomorphisms

$$S_M[g^{ab}, \varphi] = S_M[\phi^*_\lambda g^{ab}, \phi^*_\lambda \varphi]$$

Proof:

$$0 = \delta S_M = \int \frac{\delta S_M}{\delta g^{ab}} \delta g^{ab} + \int \frac{\delta S_M}{\delta \varphi} \delta \varphi$$

suppose $\dfrac{\delta S_M}{\delta \varphi} = 0.$ (φ satisfies the field eq's.)

then $$\int \frac{\delta S_M}{\delta g^{ab}} \delta g^{ab} = 0$$

δg^{ab}

$$g^{ab}(x) = \frac{\partial x'_c}{\partial x_a} \frac{\partial x'_d}{\partial x_b} g'^{cd}(x')$$

$$x_a \longmapsto x'_a := x_a + \epsilon w_a(x)$$

ϵ small parameter

$w_a \longrightarrow$ Arbitrary vector field

$$\frac{\partial x'_a}{\partial x_b} = \delta^b_a + \epsilon \nabla^b w_a$$

$$g^{ab}(x) = [\delta^a_c + \epsilon \nabla^a w_c(x)][\delta^b_d + \epsilon \nabla^b w_d(x)]\, g'^{cd}(x')$$

$$= g'^{ab}(x') + \epsilon[\nabla^a w_c(x) g'^{cb}(x') + \nabla^b w_d(x) g'^{ad}(x') + \mathcal{O}(\epsilon^2)$$

$$= g'^{ab}(x') + \epsilon[\nabla^a w^b + \nabla^b w^a]$$

then

$$\delta^{ab}(x) := g^{ab}(x) - g'^{ab}(x')$$
$$= \epsilon(\nabla^a w^b + \nabla^b w^a)(x)$$

Define:

$$T_{ab} := \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}}$$

$$0 = \int \frac{\delta S_M}{\delta g^{ab}} \delta g^{ab}$$

$$= \epsilon \int \sqrt{-g}\, \frac{T_{ab}}{2} (\nabla^a w^b + \nabla^b w^a)$$

$$= \epsilon \int \sqrt{-g}\, T_{ab} \nabla^a w_b$$

$$= -\epsilon \int \sqrt{-g}\, (\nabla^a T_{ab}) w^b + \cancel{\text{Boundary terms}} \to 0$$

as w is arbitrary, then

$$\boxed{\nabla_a T^{ab} = 0}$$

$$T_{ab} := \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{ab}}$$ Gravitational stress-tensor

$$T_{ab} := \frac{\partial \mathcal{L}}{\partial \psi^{I,a}} \psi^I{}_{,b} - \mathcal{L} g_{ab}.$$ Canonical stress-tensor (Noether)

They are equivalent only for matter fields that do not couple to metric derivatives.

$\blacksquare$

(See leclerc's gr-qc/0510044v6 (2006))

Maxwell Field

$$\nabla\cdot\vec{E}=\rho \qquad \nabla\cdot\vec{B}=0$$
$$\nabla\times\vec{B}-\frac{\partial\vec{E}}{\partial t}=\vec{j} \qquad \nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

Maxwell equations.

$$\frac{\partial}{\partial t}\left(\nabla\cdot\vec{E}=\rho\right) \longrightarrow \nabla\cdot\frac{\partial\vec{E}}{\partial t}=\frac{\partial\rho}{\partial t}$$

$$\nabla\left(\nabla\times\vec{B}-\frac{\partial\vec{E}}{\partial t}=\vec{j}\right) \longrightarrow \cancel{\nabla\cdot(\nabla\times\vec{B})}^{\,0}-\nabla\cdot\frac{\partial\vec{E}}{\partial t}=\nabla\cdot\vec{j}$$

$$\therefore \quad -\frac{\partial\rho}{\partial t}=\nabla\cdot j \qquad \frac{\partial\rho}{\partial t}+\nabla\cdot j=0$$

Continuity equation

Let ϕ and $\vec{A}$ be the scalar and vector potentials such that

$$\vec{E}=-\nabla\phi-\frac{\partial\vec{A}}{\partial t} \quad ; \quad \vec{B}=\nabla\times\vec{A}$$

Define the potential $A^a:=(\phi,A)$, then

$$F_{ab}:=\partial_b A_a-\partial_a A_b=\nabla_b A_a-\nabla_a A_b$$

Electromagnetic tensor field.

$$F_{ab}=-F_{ba}$$

$$A_a\longmapsto A_a+\partial_a\psi \;\Rightarrow\; F_{ab}\longmapsto F_{ab}$$

Gauge Freedom

$$F^{ab}\doteq\begin{pmatrix}0 & E_x & E_y & E_z\\ & 0 & B_z & -B_y\\ & & 0 & B_x\\ & & & 0\end{pmatrix}$$

Also define a current density vector $j^a=(\rho,\vec{j})$

$$\nabla_b F^{ab} = j^a \qquad \oplus \qquad \nabla_a j^a = 0$$
$$\partial_{[a} F_{bc]} = 0$$

Maxwell's   Continuity

From maxwell's we have

$$\nabla_b F^{ab} = j^a$$
$$\nabla_b\left[g^{ac} g^{bd}\left(\nabla_d A_c - \nabla_c A_d\right)\right] = j^a$$
$$g^{ac} g^{bd}\left(\nabla_b \nabla_d A_c\right) - g^{ac} g^{bd} \nabla_b \nabla_c A_d = j^a$$
$$\Box A^a - \nabla^a \cancel{\left(\nabla^d A_d\right)}^{\to 0} = j^a$$

Lorentz-Gauss gauge.

$$\Box A^a = j^a$$

Maxwell Lagrangian

I. $\mathcal{L}_{EM}[A] = -\frac{\sqrt{-g}}{4} g^{ac} g^{bd} F_{ab} F_{cd}$ Standard

II. $\mathcal{L}_{EM}[A, F] = \frac{1}{4\pi}\left(-\frac{1}{2} F_{ab} F^{ab} + \left(A_{a,b} - A_{b,a}\right) F^{ab}\right)$

Non-standard.

I.

$$\partial_n\left(\frac{\partial \mathcal{L}_{EM}}{\partial A_{m,n}}\right) - \frac{\partial \mathcal{L}_{EM}}{\partial A_m} = 0$$

$$\frac{\partial \mathcal{L}_{EM}}{\partial A_\mu} = 0$$

$$\frac{\partial \mathcal{L}_{EM}}{\partial A_{m,n}} = \frac{-\sqrt{-g}}{4} g^{ac} g^{bd} \, 2 \, F_{ab} \frac{\partial F_{cd}}{\partial A_{m,n}}$$
$$= \frac{-\sqrt{-g}}{2} F^{cd}\left(\frac{\partial}{\partial A_{m,n}}\left(A_{c,d} - A_{d,c}\right)\right)$$
$$= \frac{1}{2} F^{cd}\left(\frac{\partial}{\partial A_{m,n}}\left(A_{c,d} - A_{d,c}\right)\right)$$

$$= \frac{1}{2}\left(F^{mn} - F^{nm}\right)$$

$$= F^{mn}$$

$$\boxed{\partial_n F^{mn} = 0}$$

II.

$\underline{A|}$

$$\cancel{\frac{\partial \mathcal{L}_{EM}}{\partial A_a}}^{\,0} - \partial_b \frac{\partial \mathcal{L}_{EM}}{\partial A_{a,b}} = 0$$

$\underline{F|}$

$$\frac{\partial \mathcal{L}_{EM}}{\partial F^{ab}} - \partial_c \cancel{\left(\frac{\partial \mathcal{L}_{EM}}{\partial F^{ab}{}_{,c}}\right)}^{\,0} = 0$$

$$\frac{\partial \mathcal{L}_{EM}}{\partial A_{a,b}} = F^{cd} \frac{\partial}{\partial A_{a,b}}\left(A_{c,d} - A_{d,c}\right) = F^{cd}\left(\delta^{ab}_{cd} - \delta^{ab}_{dc}\right)$$

$$= F^{ab} - F^{ba} = 2F^{ab}$$

$$\boxed{\partial_b F^{ab} = 0}$$

II.

$$\mathcal{L}_{EM}[A,F] = \frac{1}{4\pi}\left(-\frac{1}{2} F_{ab} F^{ab} + \left(A_{a,b} - A_{b,a}\right) F^{ab}\right)$$

$$\frac{\partial \mathcal{L}_{EM}}{\partial F^{ab}} = \frac{\partial}{\partial F^{ab}}\left[-\frac{1}{2} F_{cd} F^{cd} + \left(A_{c,d} - A_{d,c}\right) F^{cd}\right]$$

$$= -F_{cd} \frac{\partial F^{cd}}{\partial F^{ab}} + \left(A_{c,d} - A_{d,c}\right) \frac{\partial F^{cd}}{\partial F^{ab}}$$

$$= -F_{cd}\, \delta^{cd}_{ab} + \left(A_{c,d} - A_{d,c}\right) \delta^{cd}_{ab}$$

$$-F_{ab} + \left(A_{a,b} - A_{b,a}\right) = 0$$

$$F_{ab} = A_{a,b} - A_{b,a}$$

Maxwell energy-momentum tensor

Homework: Show

$$T_{ab} = \frac{1}{4\pi}\left(-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd}\right)$$

such that $\nabla_b T^{ab} = 0.$

$$T_{00} = \frac{1}{8\pi}(E^2 + B^2)$$ Energy density

$$T_{0i} = -\frac{1}{4\pi}(\vec{E} \times \vec{B})_i$$ Momentum density

$$\vec{E} \times \vec{B} =: $$ Potying vector.

Einstein - Maxwell

$$G_{ab} = -2 g^{cd} F_{ac} F_{bd} + \frac{1}{2} g_{ab} F_{cd} F^{cd}$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} S$$

Ricci

Scalar Curvature

Riemann

$$R = R(\partial\Gamma, \Gamma, g) = R(\partial^2 g, \partial g, g)$$