Lemma: The punctual product $\chi\eta(a)=\chi(a)\eta(a)$ makes to $\hat{A}$ an abelian group, called the dual group or pontryagin's dual of A.

Proof: Let $\chi,\eta\in\hat{A}$, as in the case of finite groops $\chi\eta$ and η^{-1} are homomorphisms.

To see that $\chi\eta$ is continuous, let a succession in A converges to $a\in A$. then

$$\chi\eta(a_n)=\chi(a_n)\eta(a_n)$$

and how $\chi(a_n)\longrightarrow\chi(a)$ and $\eta(a_n)\longrightarrow\eta(a)$, then $\chi\eta(a_n)\longrightarrow\chi(a)\eta(a)=\chi\eta(a)$, thus the multiplication is continuous, in the same way for the inverse.

Therefore, $\hat{A}$ is an abelian group.

$\hat{A}$ is a LCA group when $(K_n\subseteq K_{n+1},\ A=\bigcup_n K_n)$

Taking a compact cover $A=\bigcup_{n\in\mathbb{N}}K_n$, for $\eta,\chi\in\hat{A}$, $n\in\mathbb{N}$

Let $\hat{d}_n(\chi,\eta)=\sup_{x\in K_n}|\chi(x)-\eta(x)|$ and $\hat{d}(\chi,\eta)=\sum_{n=1}^{\infty}\frac{1}{2^n}\hat{d}_n(\chi,\eta)$

Lemma: The function $\hat{d}$, is a metric on $\hat{A}$.

Proof: For $\chi,\eta,\alpha\in\hat{A}$, we calculate

$$\hat{d}(\chi,\eta)=\sup_{x\in K_n}|\chi(x)-\eta(x)|=\sup_{x\in K_n}|\chi(x)-\alpha(x)+\alpha(x)-\eta(x)|$$
$$\leq\sup_{x\in K_n}|\chi(x)-\alpha(x)|+\sup_{x\in K_n}|\alpha(x)-\eta(x)|$$
$$=\hat{d}_n(\chi,\alpha)+\hat{d}_n(\alpha,\eta)$$

Then,

$$\hat{d}(\chi,\eta)=\sum_{n=1}^{\infty}\frac{1}{2^n}\hat{d}_n(\chi,\eta)\leq\sum_{n=1}^{\infty}\frac{1}{2^n}(\hat{d}_n(\chi,\alpha)+\hat{d}_n(\alpha,\eta))$$
$$=\hat{d}(\chi,\alpha)+\hat{d}(\alpha,\eta).$$

Theorem: With the define metric, the group $\hat{A}$ is an abelian group. A succession χ_n converges in this metric if and only if, converges locally uniform, with this topology makes that $\hat{A}$ be a LCA group.

Proof: First we have to prove that the group operations are continuous. For this, let χ_j and η_j be two convergent successions in $\hat{A}$ to χ and η.

Then, for all $n \in \mathbb{N}$

$$\hat{d}_n(\chi_j\eta_j, \chi\eta) = \sup_{x \in K_n} |\chi_j(x)\eta_j(x) - \chi(x)\eta(x)|$$

$$= \sup_{x \in K_n} |(\chi_j(x) - \chi(x))\eta_j(x) + \chi(x)(\eta_j(x) - \eta(x))|$$

$$\leq \sup_{x \in K_n} |\chi_j(x) - \chi(x)||\eta_j(x)| + \sup_{x \in K_n} |\eta_j(x) - \eta(x)||\chi(x)|$$

$$= \hat{d}_n(\chi_j, \chi)|\cancelto{1}{\eta_j(x)}| + \hat{d}_n(\eta_j, \eta)|\cancelto{1}{\chi(x)}|$$

Multiplying by $\frac{1}{2^n}$ and adding

$$\hat{d}(\chi_j\eta_j, \chi\eta) \leq \sum_n \frac{1}{2^n} (\hat{d}_n(\chi_j, \chi) + \hat{d}_n(\eta_j, \eta) \longrightarrow 0$$

Therefore, the multiplication is continuous, and the inverse is equal.

Potryagin Duality

Proposition: If A is compact, $\hat{A}$ is discrete. If A is discrete, $\hat{A}$ is compact

Proof: Let's suppose that A is compact. Let's choose a compact cover $K_1 = K_2 = \dots = A$, and the metric of $\hat{A}$ given by

$$d(\chi, \eta) = \sup_{x \in A} |\chi(x) - \eta(x)|.$$

To prove that $\hat{A}$ is discrete, it's enough to prove that for any two characters χ, η, if

$$d(\chi, \eta) \leq \sqrt{2} \quad , \quad \chi = \eta$$

For this, let $\alpha = \chi^{-1}\eta$ and let's assume that $d(\alpha, 1) \leq \sqrt{2}$, it means

$$\alpha(A) \subseteq \}Re(z) \geq 0\{$$

$$|z-1|^2=(z-1)(\bar{z}-1)\leq 2$$

$$2-2\mathrm{Re}(z)\leq 2$$

$$\mathrm{Re}(z)\geq 0$$

As $\alpha(A)$ is a subgroup of $\mathbb{T}$, then $\alpha(A)=\{1\}$, Thus $\alpha=1$, then $x=n$.

Let A be discrete. As A is σ-compact, is numerable.

Let $(a_k)_{k\in\mathbb{N}}$, numerable of A, x_j a succession in A, there is a subsuccession x_j° of x_j that converges locally uniform and these limits are characters and $\hat{A}$ is compact.

Example: Let $G=GL_n(\mathbb{R})$ invertible matrices of $n\times n$ over $\mathbb{R}$. As $GL_n(\mathbb{R})\subseteq \mathrm{Mat}_n(\mathbb{R})\simeq\mathbb{R}^{n^2}$, taking the usual the usual topology, it is locally compact.

Let f a valued continuous function in the reals with compact support in $\mathbb{R}$, which is non-negative, i.e., $f(x)\geq 0$, $\forall x\in\mathbb{R}$.

$$\mathrm{supp}(f)=\overline{\{x\in X \,/\, f(x)\neq 0\}}$$

The Riemann integral of f, is given by the infimum of the Riemann integrals of the step functions that domains to f.

For $n\in\mathbb{N}$, let 1_n the characteristic function of the interval $\left[\frac{-1}{2^n},\frac{1}{2^n}\right]$

$$1_n(x)=\begin{cases}1 & \text{if } x\in\left[\frac{-1}{2^n},\frac{1}{2^n}\right]\\ 0 & \text{in other way}\end{cases}$$

There is $x_1,\dots,x_n\in\mathbb{R}$, and $c_1,\dots,c_m\geq 0$, such that

$$f(x)=\sum_{j=1}^{m}c_j\,1(x-x_j)$$

Let $(f:1_n)$ the

$$\inf\left\{\sum_{j=1}^{m}c_j \;\middle|\; \begin{array}{l} c_1,\dots,c_m\geq 0 \text{ and there are } x_1,\dots,x_n\in\mathbb{R} \\ \text{such that } f(x)=\sum_{j=1}^{m}c_j\,1_n(x-x_j)\end{array}\right\}$$

The Riemann's integral is:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} \frac{(f:1_n)}{n}$$

In a general group G, we can replace $[-1/2n, 1/2n]$ by a neighbourhood around the identity. But we don't know what means $1/n$. We have to modify the definition of integral.

Let f_0, the characteristic function of the interval $[0,1]$

$$f_0(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases}$$

Then $(f_0 : 1_n) = n$,

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} \frac{(f:1_n)}{(f_0:1_n)}$$

$$\int_G f(x)\,dx = \lim_{U\to\{e\}} \frac{(f:1_n)}{(f_0:1_n)} \longrightarrow \text{Haar's Measure.}$$