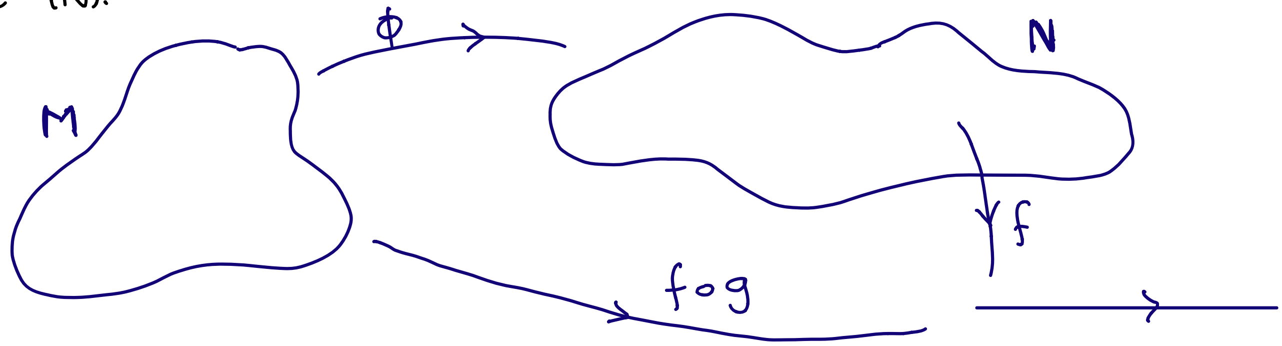
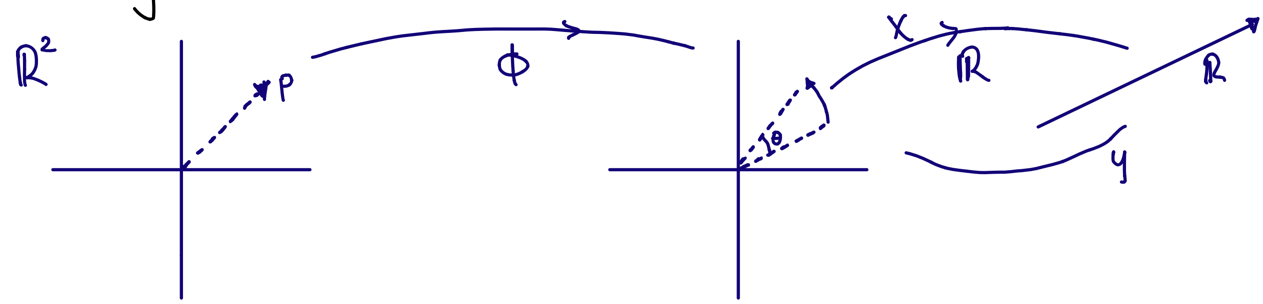
Let $\phi: M \to N$ from a manifold M to other N. If we get, $f: N \to \mathbb{R}$, $f \in C^\infty(N)$.



Let's call to this process the pullback of f from N to M by ϕ

$$\phi^* f \equiv f \circ \phi$$

Exercise: Let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$, be a relation in counterclock sense by an angle θ.



Let x, y coordinate functions of $\mathbb{R}^2$. Prove that

$$\phi^* x = (\cos\theta) x - (\sin\theta) y$$

$$\phi^* y = (\sin\theta) x + (\cos\theta) y$$

Answer: $\phi^* x(p) = x(\phi(p))$, the x coordinate, of the rotated point, if p is the point (s,t),

$$\phi(p) = (s', t') = (s\cos\theta - t\sin\theta, s\sin\theta + t\cos\theta)$$

then,

$$\phi^* x(p) = x(\phi(p)) = s\cos(\theta) - t\sin(\theta)$$

$$= x(p)\cos\theta - y(p)\sin\theta$$

therefore

$$\phi^*(x(p)) = x\cos\theta - y\sin\theta$$

Now,

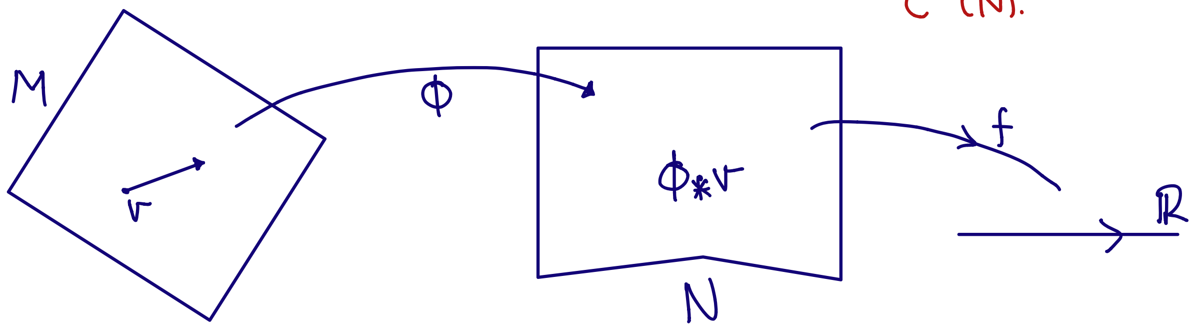
$$\phi^* y(p) = y(\phi(p)) = s\sin(\theta) + t\cos\theta = x(p)\sin\theta + y(p)\cos\theta$$

therefore, $\phi^*(y(p)) = x\sin\theta + y\cos\theta$

• The tangent vectors by the other hand are covariants a vector $v \in T_pM$ and a function $\varphi: M \longrightarrow N$ gives a tangent vector $\phi_* v \in T_{\phi(p)}N$, called pushforward of v by ϕ. Defined by

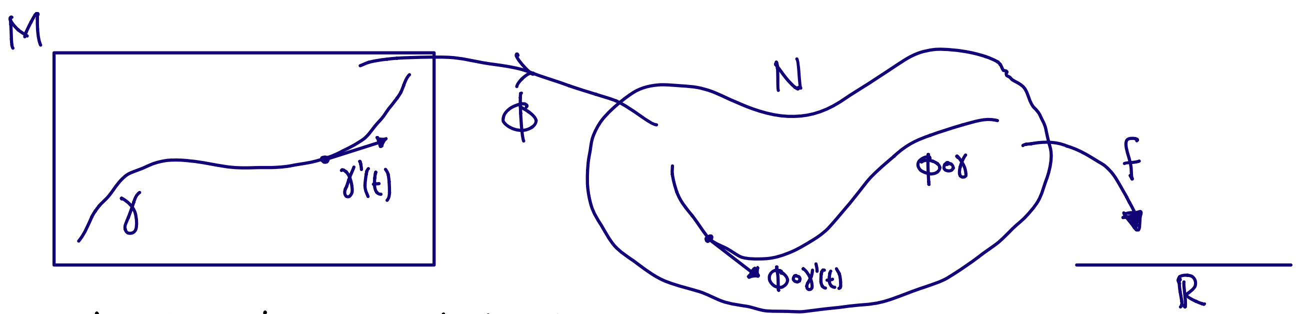
$$\phi_* v(f) \equiv v(\phi^* f) = v(f \circ \phi)$$

$C^\infty(N)$.



Note: The pullback

$$\phi^*: C^\infty(N) \longrightarrow C^\infty(M)$$



$$\mathrm{Vect}(M) \ni \gamma'(t)[f] \equiv \frac{df(\gamma(t))}{dt}$$

Exercise: Show that

$$(\phi \circ \gamma)'(t) = \phi_*(\gamma'(t)) \in T_{\phi(p)}N.$$

Answer:

$$\begin{aligned}
\phi_*(\gamma'(t))[f] &= \gamma'(t)(\phi^* f) \\
&= \gamma'(t)(f \circ \phi) \qquad C^\infty(M) \\
&= \frac{d}{dt}(f \circ \phi)[\gamma(t)] \\
&= \frac{d}{dt} f(\phi(\gamma(t))) \\
&= \frac{d}{dt} f((\phi \circ \gamma)(t)) \\
&= (\phi \circ \gamma)'(t)[f]
\end{aligned}$$

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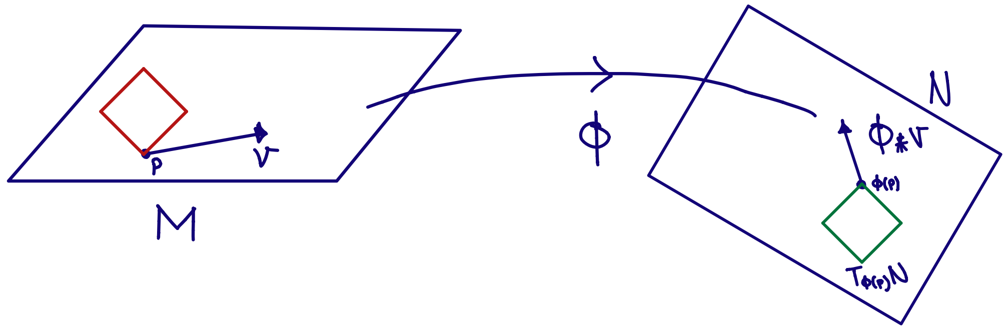


Exercise: Prove that the pushforward is linear

$$\phi_* : T_pM \longrightarrow T_{\phi(p)}N$$

If $v, w \in TM$, $\phi_*(v+w) = \phi_*(v) + \phi_*(w)$

Answer: $f \in C^\infty(N)\ C^\infty(M)$

$$\phi_*(v+w)(f) = \phi_*(v(f) + w(f))$$

$$(v+w)(\phi^* f) = v(\phi^* f) + w(\phi^* f)$$

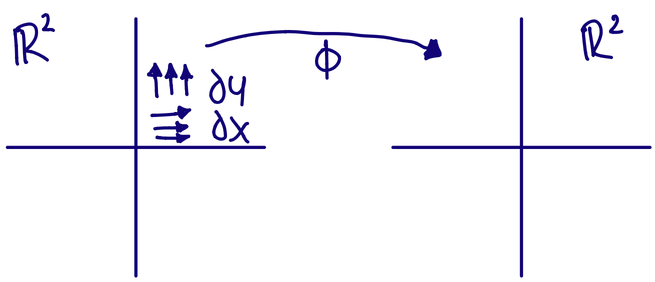
$$= \phi_* v(f) + \phi_* w(f)$$

$$= \phi_* v + \phi_* w \qquad \forall f \in C^\infty(N).$$

Homework: let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a counterclock rotation by θ, let $\partial x, \partial y$ vector fields in $\mathbb{R}^2$. Prove that

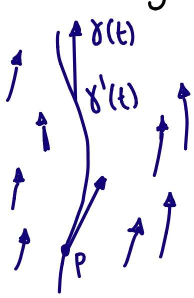
$$\phi_* \partial x = (\cos\theta)\partial x + (\sin\theta)\partial y$$

$$\phi_* \partial y = (-\sin\theta)\partial x + (\cos\theta)\partial y$$



Lie's bracket

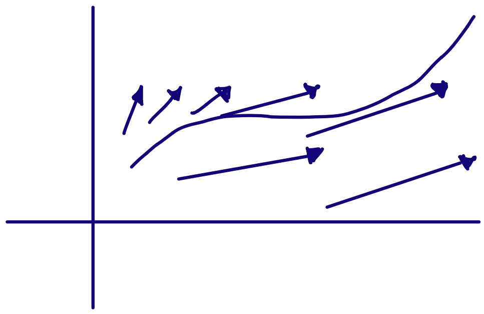
Let's imagine a fluid, let's trace the curve of a water particle



$$\gamma'(t) = v_{\gamma(t)} \quad ; \forall t$$

If the curve starts in $p \in M$ such that $\gamma(0) = p$, we will call to γ the integral curve through p for the vector field v

Exercise: Let v the vector field $x^2\partial x + y\partial y$ in $\mathbb{R}^2$, find the integral curve $\gamma(t)$.



$$\gamma'(t)[f] = \frac{d}{dt}(f(\gamma(t)) = \frac{\partial f}{\partial x^\mu}\frac{\partial \gamma^\mu}{\partial t} = \partial_\mu f \frac{\partial \gamma^\mu}{\partial t}$$

then

$$\gamma'(t) = \frac{d\gamma^\mu}{dt}\partial_\mu$$

if

$$\gamma(t) = (x(t), y(t))$$

thus

$$\gamma'(t) = (\dot{x}(t)\partial x + \dot{y}(t)\partial y).$$

Luego $\dot{x} = x^2$ y $\dot{y} = y$

$$r(t) = \left(\frac{x(0)}{1 - x(0)t} , y(0) e^{t} \right)$$ definido $1 - x(0)t \neq 0$.

Def: Un campo vectorial v es integrable en M, si sus curvas integrales están bien definidas para todo t.