Theory of distributions.

Distributions generalize the notion of functions by allowing derivatives of discontinuities.

Distributions $\leftrightarrow$ Generalized functions.

what we want to generalize?

1) Include delta "functions".

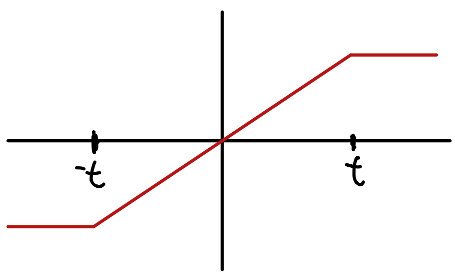
2) Allow non-differentiable functions.

3) Avoid discrepancies of what value to assign at a discontinuity.

4) Guarantee that limit and derivatives commute (in particular at isolate points)

Example:

$$f_t(x) = \begin{cases} -t & x < -t \\ x & -t \leq x \leq t \\ t & x > t \end{cases}$$



$$0 = \frac{d}{dx}\left(\lim_{t \to 0} f_t(x)\right)$$

$$\neq \lim_{t \to 0}\left(\frac{d}{dx} f_t(x)\right) =$$

5) Measures in the real world. $f(x)$ gives a value at every x but in the real world you only measure averages.

$$\int_{-\infty}^{\infty} f(x)\phi(x)dx \qquad \text{average with weight } \phi(x).$$

Definition: A test function $\phi(x)$ on $\mathbb{R}^n$ is a function that it is infinitely diffirentiable on $\mathbb{R}^n$, that is, $\phi(x) \in C^\infty(\mathbb{R}^n)$. In particular, we want best functions of compact support, then $\phi(x) \in C_c^\infty(\mathbb{R}^n)$.

$\phi(x) \neq 0$ only in a finite region.

Let $\phi^{(j)}(x) \in C_c^\infty(\mathbb{R}^n)$ denote the j-th derivative of $\phi(x)$.

Definition: We will say that a sequence $\{\phi_n\}$ converges to $\phi \in C_c^\infty(\mathbb{R}^n)$ as $n \to \infty$ if:

1) There is a compact subset $K \subset \mathbb{R}^n$ such that $\operatorname{supp} \phi_n \subset K$ for all n and $\operatorname{supp} \phi \subset K$.
2) $\phi_n^{(j)} \to \phi^{(j)}$ as $n \to \infty$ (Uniformly on K) for each $j \geq 0$

$$\sup_{x \in K} |\phi_n^{(j)}(x) - \phi(x)| \to 0 \quad \text{as } n \to \infty$$

Ex:
$$\phi(x) = \begin{cases} c e^{\frac{-1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

C is given by:
$$C = \frac{1}{\int_{|x|<1} e^{\frac{-1}{1-|x|^2}} dx}$$

Normalization constant such that.
$$\int_{\mathbb{R}^n} \phi(x) dx = 1$$

that is, C is a normaliza-tion construct.

Now, $\phi(x)$ clearly has compact support $\{x \mid |x| < 1\}$ and also it has continuous derivatives of all orders (even at $x = \pm 1$).

$\phi(x)$ is a test function.

Notes of test functions:

1) $C_c^\infty(\mathbb{R}^n)$ is a real linear space
$$\Rightarrow (a\phi_1(x) + b\phi_2(x)) \in C_c^\infty(\mathbb{R}^n) \text{ if } \phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^n),\ a, b \in \mathbb{R}.$$
2) If $\phi(x) \in C_c^\infty(\mathbb{R}^n) \to \phi\left(\frac{x - x_0}{t}\right) \in C_c^\infty(\mathbb{R}^n)$, where $\phi\left(\frac{x-x_0}{t}\right)$ vanishes outside a ball of ratios t with center at x_0.

3) If $\phi(x) \in C_c^{\infty}(\mathbb{R}^n) \Longrightarrow \phi^{j}(x) \in C_c^{\infty}(\mathbb{R}^n)$

4) if $\phi(x) \in C_c^{\infty}(\mathbb{R}^m)$ and $\psi(x) \in C_c^{\infty}(\mathbb{R}^{n-m}) \Longrightarrow \phi(x)\psi(x) \in C_c^{\infty}(\mathbb{R}^n)$

Definition: A function is a realvalued function on a vector space

$$F : V \longrightarrow \mathbb{R}.$$

In particular, V may be identified with the vector space of function

Example:

$$S[\mathcal{L}] = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

Variational calculus

Definition: The space of distribution D is defined to be the space of continuous linear functionals on the space $C_c^{\infty}(\mathbb{R}^n)$ of test functions, that is,

$$f \in D \text{ means } \quad f : C_c^{\infty}(\mathbb{R}^n) \longrightarrow \mathbb{R}.$$

1) f is linear $\quad f[a\phi_1 + b\phi_2] = af[\phi_1] + bf[\phi_2]$

2) f is continuous: $\phi_n \to \phi$ is $C_c^{\infty}(\mathbb{R}^n)$

implies

$$f[\phi_n] \longrightarrow f[\phi] \quad \text{as a sequence of numbers.}$$

Note: In topology, the space of continuous linear functionals on a given topological space X is called the dual space of X

- In our case, D is the dual of $C_c^{\infty}(\mathbb{R}^n)$
  - ↳ This satisfies the notation in some text books.

(f, ϕ) or $\langle f, \phi \rangle$ as an inner product

(Not to be confused with the L^2 inner product).

Example:

1) $f[\phi] = \int_{\mathbb{R}} \phi(x)\, dx$

2) $f[\phi] = \phi'(0)$

Homework.

$$\lim_{n\to\infty} |f[\phi] - f[\phi_n]|$$

Theorem: A locally integrable function on $\mathbb{R}^n$ defines a distribution f through the rule

$$f[\phi] = \int_{\mathbb{R}^n} f(x)\phi(x)dx$$

This is called the "regular" distribution.

Proof:

1) $$f[a\phi_1 + b\phi_2] = \int_{\mathbb{R}^n} f(x)(a\phi_1(x) + b\phi_2(x))dx$$

$$= a\int_{\mathbb{R}^n} f(x)\phi_1(x)dx + b\int_{\mathbb{R}^n} f(x)\phi_2(x)dx$$

$$= aF[\phi_1] + bF[\phi_2]$$

2) $$f[\phi] - f[\phi_m] = \int_{\Omega\in\mathbb{R}^n} f(x)(\phi(x) - \phi_m(x))dx$$

$$\leq \left(\int_\Omega |f(x)|dx\right) \max_{x\in\Omega} |\phi(x) - \phi_m(x)| \quad \lim_{m\to\infty} |f[\phi] - f[\phi_m]|$$

$$\leq \left(\int_\Omega |f(x)|dx \lim_{m\to\infty} \max_{x\in\Omega} |\phi(x) - \phi_m(x)|\right)$$

but

$$\lim_{m\to\infty} |\phi(x) - \phi_m(x)| \to 0 \quad \text{and} \quad \int_\Omega |f(x)|dx < \infty$$

then, f is locally integrable.

$$\Longrightarrow \lim_{m\to\infty} |f[\phi] - f[\phi_m]| \to 0.$$

Definition: A distribution f is called singular if it is not regular.

Definition: Define $f[\phi] = \phi[0]$, in general $\delta(x-x')[\phi] = \phi[x']$

This delta function is linear and continuous but does not correspond to any ordinary function!