Koszul formula:

(Characterises ∇)

Sternberg "Curvature in Math and Physics"

$$2\langle \nabla_V W, X\rangle = V\langle W,X\rangle + W\langle X,V\rangle - X\langle V,W\rangle$$
$$- \langle V,[W,X]\rangle + \langle W,[X,V]\rangle + \langle X,[V,W]\rangle$$

Proof: Suppose ∇ is a connection on M satisfying (IV) and (V)

$$\left.\begin{aligned} V\langle W,X\rangle &= \langle \nabla_V W, X\rangle + \langle W, \nabla_V X\rangle \\ W\langle X,V\rangle &= \langle \nabla_W X, V\rangle + \langle X, \nabla_W V\rangle \\ -X\langle V,W\rangle &= -\langle \nabla_X V, W\rangle - \langle V, \nabla_X W\rangle \end{aligned}\right\} \text{(V)}$$

$$\text{(IV)}\left\{\begin{aligned} -\langle V,[W,X]\rangle &= -\langle V, \nabla_W X - \nabla_X W\rangle = -\langle V, \nabla_W X\rangle + \langle V, \nabla_X W\rangle \\ \langle W,[X,V]\rangle &= \langle W, \nabla_X V\rangle - \langle W, \nabla_V X\rangle \\ \langle X,[V,W]\rangle &= \langle X, \nabla_V W\rangle - \langle X, \nabla_W V\rangle \end{aligned}\right.$$

K $\longrightarrow$ (IV)

$$F(V,W,X) := 2(\nabla_V W, X)$$
$$\longrightarrow 2\langle \nabla_V W - \nabla_W V, X\rangle = F(V,W,X) - F(W,V,X)$$

Skew-symmetric in V,W.

$$2\langle \nabla_V W - \nabla_W V, W\rangle = \langle X,[V,W]\rangle - \langle X,[W,V]\rangle$$
$$= 2\langle [V,W], X\rangle$$

therefore, $[V,W] = \nabla_V W - \nabla_W V$. $\blacksquare$

$$2\langle \nabla_V fW, X\rangle = V\langle fW, X\rangle + fW\langle X,V\rangle - X\langle V, fW\rangle$$
$$- \langle V,[fW,X]\rangle + \langle fW,[X,V]\rangle + \langle X,[V,fW]\rangle$$
$$= V(f)\langle W,X\rangle + fV\langle W,X\rangle + fW\langle X,V\rangle$$
$$- X(f)\langle V,W\rangle - fX\langle V,W\rangle - \langle V, f[W,X] - X(f)W\rangle$$
$$+ f\langle W,[X,V]\rangle + V(f)\langle X,W\rangle + f\langle X,[V,W]\rangle$$

$$[fW,X] = fWX - X(fW) = fWX - X(f)W - fXW$$

$$[V, fW] = V(fW) - fWV = V(f)W - fVW - fWV$$

$$= f\{V\langle W,X\rangle + W\langle X,V\rangle - X\langle V,W\rangle - \langle V,[W,X]\rangle + \langle W,[X,V]\rangle + \langle X,[V,W]\rangle\}$$

$$+ 2V(f)\langle W,X\rangle$$

$$= f2\langle \nabla_V W, X\rangle + 2V(f)\langle W,X\rangle$$

$$\langle \nabla_V fW, X\rangle = \langle f\nabla_V W + V(f)W, X\rangle$$

Choosing a coordinate system $\xi = (x^1, \dots, x^n)$ on $U \subseteq M$.

$$X = X^i \partial_i \qquad Y = Y^j \partial_j$$

Definition: The christoffel symbols for a coordinate system ξ are the real valued functions Γ^k_{ij} on U such that

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$$

Properties:

1. $0 \overset{(IV)}{=} [\partial_i, \partial_j] = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i = \Gamma^k_{ij}\partial_k - \Gamma^k_{ji}\partial_k = (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k$

   Γ is symmetric in the lower indices!

II. By (III)

$$\begin{aligned}
\nabla_X Y &= X^i \nabla_{\partial_i}(Y^j \partial_j) \\
&= X^i((\partial_i Y^j)\partial_j + Y^j(\nabla_{\partial_i}\partial_j)) \\
&= (X^i \partial_i Y^k + X^i Y^j \Gamma^k_{ij})\partial_k \\
&= (X^k(Y) + X^i Y^j \Gamma^k_{ij})\partial_k.
\end{aligned}$$

III.

$$\begin{aligned}
\nabla_{\partial_i}(V) &= \nabla_{\partial_i}(V^j \partial_j) \\
&= (\partial_i V^j)\partial_j + V^j \nabla_{\partial_i}\partial_j \\
&= \left(\frac{\partial V^k}{\partial x^i} + V^j \Gamma^k_{ij}\right)\partial_k
\end{aligned}$$

Notes:

1. Γ_{ij} are differentiable.

II. As ∇ is not a tensor, then Γ's do not obey usual transformation rules.

Theorem: Christoffel symbols are explicitly given by

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{m} g^{km}\left(\frac{\partial g_{im}}{\partial x^{i}} + \frac{\partial g_{im}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{m}}\right)$$

Proof: Set $V = \partial_i$, $W = \partial_j$, $X = \partial_m$ in Koszul's

$$2\langle \nabla_{\partial_i}\partial_j, \partial_m\rangle = \partial_i\langle \partial_j, \partial_m\rangle + \partial_j\langle \partial_m, \partial_i\rangle - \partial_m\langle \partial_i, \partial_j\rangle$$

$$2\langle \Gamma_{ij}^{k}\partial_k, \partial_m\rangle = \frac{\partial}{\partial x^i} g_{jm} + \frac{\partial}{\partial x^j} g_{mi} - \frac{\partial}{\partial x^m} g_{ij}$$

$$2\,\Gamma_{ij}^{k}\langle \partial_k, \partial_m\rangle = \frac{\partial}{\partial x^i} g_{jm} + \frac{\partial}{\partial x^j} g_{mi} - \frac{\partial}{\partial x^m} g_{ij}$$

$$2\,\Gamma_{ij}^{k} g_{km} = \frac{\partial}{\partial x^i} g_{jm} + \frac{\partial}{\partial x^j} g_{mi} - \frac{\partial}{\partial x^m} g_{ij}$$

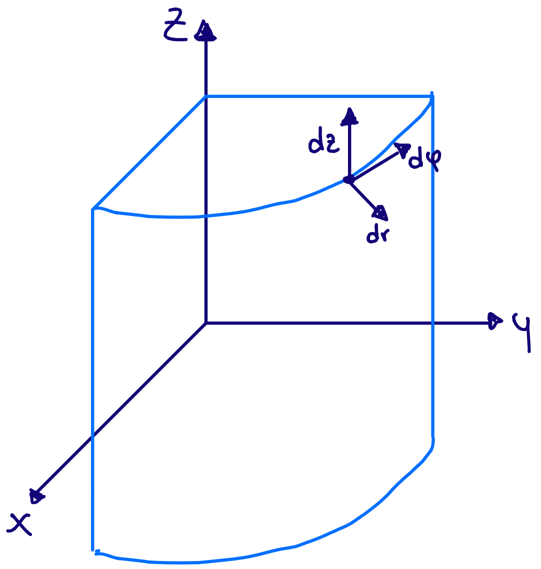
$$\Gamma_{ij}^{k} = \frac{1}{2} g^{km}\left(\frac{\partial}{\partial x^i} g_{jm} + \frac{\partial}{\partial x^j} g_{mi} - \frac{\partial}{\partial x^m} g_{ij}\right)$$

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{m} g^{km}\left(\frac{\partial g_{im}}{\partial x^{i}} + \frac{\partial g_{im}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{m}}\right)$$

$\blacksquare$

Example: Cylindrical coordinates in $\mathbb{R}^3$

Let (r, φ, z) be the cylindrical coordinates in $\mathbb{R}^3$



$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$z = z$$

Basis theorem

$$\partial r = \cos\varphi\,\partial x + \sin\varphi\,\partial y$$

$$\partial\varphi = -r\sin\varphi\,\partial x + r\cos\varphi\,\partial y$$

$$\partial z = \partial z$$

$$g_{ij} = \langle \partial_i, \partial_j\rangle$$

- $g_{rr} = \langle \partial_r, \partial_r \rangle = \langle \cos\varphi\, \partial x + \sin\varphi\, \partial \varphi, \cos\varphi\, \partial x + \sin\varphi\, \partial \varphi \rangle$

$$= \cos^2\varphi \langle \partial x, \partial x \rangle + \cancel{\cos\varphi \sin\varphi \langle \partial x, \partial y \rangle} \to 0$$

$$+ \cancel{\sin\varphi \cos\varphi \langle \partial y, \partial x \rangle} \to 0 + \sin^2\varphi \langle \partial y, \partial y \rangle$$

$$= \cos^2\varphi + \sin^2\varphi = 1$$

- $g_{\varphi\varphi} = \langle \partial\varphi, \partial\varphi \rangle = r^2$
- $g_{zz} = \langle \partial_z, \partial_z \rangle = 1$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g_{ij}\, dx^i \otimes dx^j = dr^2 + r^2 d\varphi + dz^2$$

Christoffel's:

- $$\Gamma^1_{22} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{2m}}{\partial x^2} + \frac{\partial g_{2m}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^m} \right)$$

$$= \frac{1}{2} g^{11} \left(\cancel{\frac{\partial g_{21}}{\partial x^2}} \to 0 + \cancel{\frac{\partial g_{21}}{\partial x^2}} \to 0 - \frac{\partial g_{12}}{\partial x^1} \right)$$

$$= \frac{1}{2}\, 1 \left(-\frac{\partial r^2}{\partial r} \right) = \frac{-2r}{2} = -r.$$

- $$\Gamma^2_{21} = \frac{1}{2} g^{2m} \left(\frac{\partial g_{2m}}{\partial x^1} + \cancel{\frac{\partial g_{1m}}{\partial x^2}} \to 0 - \cancel{\frac{\partial g_{21}}{\partial x^m}} \to 0 \right)$$

$$= \frac{1}{2} g^{22} \frac{\partial g_{22}}{\partial x^1} = \frac{1}{2} \left(\frac{1}{r^2} \right) \left(\frac{\partial r^2}{\partial r} \right) = \frac{2r}{2r^2} = \frac{1}{r} = \Gamma^2_{12}$$

The rest of the Γ's are vanishing!

Covariant derivatives: (non-vanishing)

- $\nabla_{\partial\varphi} (\partial\varphi) = \Gamma^k_{22}\, \partial k = \Gamma^1_{22}\, \partial r = -r\, \partial r$
- $\nabla_{\partial\varphi} (\partial r) = \Gamma^k_{21}\, \partial k = \Gamma^2_{21}\, \partial\varphi = \frac{1}{r}\, \partial\varphi$
- $\nabla_{\partial z} (\partial\varphi) = \Gamma^k_{32}\, \partial k = 0$
- $\nabla_{\partial z} (\partial r) = \Gamma^k_{31}\, \partial k = 0$

∂r and $\partial\varphi$ remain parallel as the point p moves in the z-direction!

Homework: For a diagonal metric, prove that Christoffel symbols are given by

- $\Gamma^{\mu}_{\nu\lambda} = 0$
- $\Gamma^{\mu}_{\lambda\lambda} = \dfrac{-1}{2g_{\mu\mu}} \dfrac{\partial g_{\lambda\lambda}}{\partial x^{\mu}}$
- $\Gamma^{\mu}_{\mu\lambda} = \dfrac{\partial}{\partial x^{\lambda}} \log(|g_{\mu\mu}|)^{1/2}$
- $\Gamma^{\mu}_{\mu\mu} = \dfrac{\partial}{\partial x^{\mu}} \log(|g_{\mu\mu}|)^{1/2}$

$\mu \neq \nu \neq \lambda$ and no sum over repeated indices!