$$T : \Pi_r^s \longrightarrow \mathbb{R}$$

$$T = T_{\alpha_1 \cdots \alpha_r}{}^{\beta_1 \cdots \beta_s} dx^{\alpha_1} \cdots dx^{\alpha_r} \partial x_{\beta_1} \cdots \partial x_{\beta_s}$$

$$T = T_{\alpha_1 \cdots \alpha_r}{}^{\beta_1 \cdots \beta_s} dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_r} \otimes \partial x_{\beta_1} \otimes \cdots \otimes \partial x_{\beta_s}$$

Addition of tensor of the type:

$T + T' \longrightarrow$ Tensor of type (r,s) at p such that $\forall\, v^i \in T_pM,\ \omega^i \in T_p^*M$.

$$(T + T')(\omega_1, \dots, \omega_r, v^1, \dots, v^s) = T(\omega_1, \dots, \omega_r, v^1, \dots, v^s) + T'(\omega_1, \dots, \omega_r, v^1, \dots, v^s)$$

Multiplication by scalar $\alpha \in \mathbb{R}$.

$$(\alpha T)(\omega_1, \dots, \omega_r, v^1, \dots, v^s) = \alpha[T(\omega_1, \dots, \omega_r, v^1, \dots, v^s)]$$

T_r^s vector space of $\dim(r+s)$ over $\mathbb{R}$

<u>Symmetric part</u>

Consider for symplicity X_{ab}.

$$X_{(ab)} = \frac{1}{2!}(X_{ab} + X_{ba}) \qquad \text{Symmetric part.}$$

$$X_{[a,b]} = \frac{1}{2!}(X_{ab} - X_{ba}) \qquad \text{Skew-Symmetric part.}$$

In general,

$T_{(a_1,\dots,a_r)}{}^{b_1,\dots,b_s} = \frac{1}{r!} \longrightarrow$ Sum over all permutations of the indices a_1 to a_r for $T_{a_1 \cdots a_r}{}^{b_1 \cdots b_s}$.

$T_{[a_1,\dots,a_r]}{}^{b_1,\dots,b_s} = \frac{1}{r!} \longrightarrow$ Altering sum over all permutations of the indices a_1 to a_r for $T_{a_1 \cdots a_r}{}^{b_1 \cdots b_s}$.

Example:

$$K^a{}_{[bcd]} = \frac{1}{3!}(K^a{}_{bcd} + K^a{}_{dbc} + K^a{}_{cdb} - K^a{}_{bdc} - K^a{}_{cbd} - K^a{}_{dcb})$$

outer product: Let T be a tensor of type (r_1, s_1) and let K be a tensor of type (r_2, s_2), then

$$TK \in (\underbrace{\{T_p^*M\}}_{r_1\text{-copies}} \otimes \underbrace{\{T_pM\}}_{s_1\text{-Copies}}) \otimes (\underbrace{\{T_p^*M\}}_{r_2\text{-copies}} \otimes \underbrace{\{T_pM\}}_{s_2\text{-Copies}})$$

$$\in (\underbrace{\{T_p^*M\}}_{(r_1+r_2)\text{-copies}} \otimes \underbrace{\{T_pM\}}_{(s_1+s_2)\text{-copies}})$$

TK is a tensor of type (r_1+r_2, s_1+s_2)

Example: Y of type $(1,1)$

$$Y = Y_a{}^b dx^a \otimes \partial x_b, \quad Z = Z_{ab} dx^a \otimes dx^b$$

$$YZ = (Y_a{}^b dx^a \otimes \partial x_b)(Z_{cd} dx^c \otimes dx^d)$$

$$= Y_a{}^b Z_{cd} (dx^a \otimes dx^c \otimes dx^d \otimes \partial x_b)$$

$$= T_{acd}{}^b dx^a \otimes dx^c \otimes dx^d \otimes \partial x_b \in T^1_3$$

Contraction: Multiply the tensor components by a Kronecker delta δ^a_b

$$X_{cd} = X^a{}_{acd} = X^a{}_{bcd} \delta^b_a$$

$$\langle dx_a, \partial x_b \rangle = \delta^b_a$$

$$dx_{a_1} \otimes dx_{a_2} \otimes \ldots \otimes dx_{a_r} \otimes \partial x_{b_1} \otimes \partial x_{b_2} \otimes \ldots \otimes \partial x_{b_s}$$

contraction gives a tensor of type $(r-1, s-1)$

Example:

$$\underset{(3,1)}{X^a{}_{bcd}} \xrightarrow[\text{a and c indices.}]{\text{contraction on}} \underset{(2,0)}{Y_{bd}.}$$

$$X = X^a_{bcd} \otimes dx^b \otimes dx^c \otimes dx^d \otimes \partial x_a$$

$$Y = Y_{bd} \otimes dx^b \otimes dx^d$$

Inner product theorem: An inner product of two tensors of rank (r_1, s_1) and (r_2, s_2) is a tensor of rank (r_1+r_2-1, s_1+s_2-1) (provided that the contraction is over a pair of indices and covariant and one contravariant).

$$T : T_{ab}{}^{cd} dx^a dx^b \partial x_c \partial x_d, \quad R : R^a_b dx^b \partial x_a.$$

Quotient theorem: If the product (outer or inner) of something with components $X^{ij\dots n}{}_{pq\dots t}$ with an arbitrary tensor yields a non-zero tensor of the appropriate rank, then the components $X^{ij\dots n}{}_{pq\dots t}$ are the components of a tensor

$$X = X_{abc\dots}{}^{pq\dots r}\, dx_a\, dx_b \dots \partial x_p \partial x_q \cdots \partial x_r$$

$$\underset{\substack{\downarrow \\ \text{Tensor}}}{X}\, Y = Z \rightarrow \text{Tensor}$$

Let $A : (T_p M)^r * (T_p M)^s \longrightarrow \mathbb{R}$

In order to check if A is a tensor we must show that A is $\mathcal{F}(M)$ is linear in each slot

$$\omega \longmapsto f_1\omega_1 + f_2\omega_2$$

Additivity is trivial in most cases so, we want

$$A(\omega_1, \dots, \omega_r, fV^1, V^2, \dots, V^s) = fA(\omega_1, \dots, \omega_r, V^1, V^2, \dots, V^s)$$

Examples: I. $\mathcal{E} = T_p^*M \times T_pM \longrightarrow \mathbb{R}$

$$\mathcal{E}(\omega, V) = \omega V$$

$$\mathcal{E}(f\omega, V) = (f\omega)V = f(\omega V) = f\mathcal{E}(\omega, V)$$

$$\mathcal{E}(\omega, fV) = \omega(fV) = f(V)\omega = f(V(\omega)) = f(\omega(V))$$
$$= f\mathcal{E}(\omega, V).$$

II. Take a one form $\omega \neq 0$ and define

$$\mathcal{F} : T_pM \times T_pM \longrightarrow \mathbb{R}$$

such that

$$\mathcal{F}(X, Y) = X(\omega Y) \qquad \forall\, X, Y \in T_pM,\ \omega \in T_p^*M.$$

$$\mathcal{F}(fX, Y) = fX(\omega Y) = f\mathcal{F}(X, Y)$$

$$\mathcal{F}(X, fY) = X(\omega(fY)) = X((fY)\omega) = X(fY(\omega)) = X(f\omega Y)$$

$$= Xf(\omega Y) + f(X(\omega Y)) = Xf(\omega Y) + f\mathcal{F}(X, Y)$$

$\therefore \mathcal{F}$ is not a tensor.

III. $g: T_pM \times T_pM \longrightarrow \mathbb{R}$, such that $g(X,Y) = \langle \omega_X, Y \rangle$ where $\omega_X \in T_p^*M$ associated to $X \in T_pM$ through duality.

Examples of inner product:

a. Real numbers

$$\langle x, y \rangle = xy$$

b. Euclidean space $\mathbb{R}^n$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_i x_i y_i$$

c. $\mathbb{C}^n$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \sum_i x_i^* y_i$$

d. Real functions on $[a,b]$

$$\langle f, g \rangle := \int_a^b f(x) g(x) dx$$

e. Complex function

$$\langle f, g \rangle := \int_a^b \overline{f(x)} g(x) dx$$

Definition: For a real vector space V, an inner product $\langle \cdot, \cdot \rangle_p$ satisfies the following properties.

Let $u, v \in V$, $\alpha \in \mathbb{R}$, then:

I. $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ linear

II. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ multiplication by scalar.

III. $\langle u, v \rangle = \langle v, u \rangle$ symmetric

IV. $\langle u, u \rangle \geq 0$ positive-definite.

$\langle u, u \rangle = 0$ if and only if $u = 0$.