

$$I(\lambda) = \int_C f(z) e^{\lambda \psi(z)} dz.$$

$$C \longrightarrow C^*$$

$$(x, y, u) \quad u = u(x, y)$$

The steepness of the curve $(x(s), y(s), u(s))$ with the u -axis.

$$\cos(\alpha) = \frac{du/ds}{\sqrt{1 + (du/ds)^2}}$$

$$\begin{aligned} \frac{d}{d\theta} \cos(\alpha) &= \frac{d}{d\theta} \left(\frac{du}{ds} \right) \sqrt{1 + \left(\frac{du}{ds} \right)^2} - \frac{du}{ds} \frac{1}{2} \left(\sqrt{1 + \left(\frac{du}{ds} \right)^2} \right)^{-1} \frac{du}{ds} \frac{d}{d\theta} \left(\frac{du}{ds} \right) \\ &= \frac{\left(\sqrt{1 + \left(\frac{du}{ds} \right)^2} \right)^2 - \left(\frac{du}{ds} \right)^2}{\left(\sqrt{1 + \left(\frac{du}{ds} \right)^2} \right)^3} \frac{d}{d\theta} \left(\frac{du}{ds} \right) \\ &= \frac{1}{\left(1 + \left(\frac{du}{ds} \right)^2 \right)^{3/2}} \frac{d}{d\theta} (U_x \cos(\theta) + U_y \sin(\theta)) \\ &= \frac{1}{\left(1 + \left(\frac{du}{ds} \right)^2 \right)^{3/2}} (-U_x \sin(\theta) + U_y \cos(\theta)) \end{aligned}$$

$$\omega = \frac{d}{d\theta} \cos(\alpha) \Leftrightarrow \omega = -U_x \sin(\theta) + U_y \cos(\theta) = -(V_y \sin(\theta) + V_x \cos(\theta))$$

$$\frac{dv}{ds} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial s} = V_x \cos(\theta) + V_y \sin(\theta).$$

$$\frac{dv}{ds} = \omega \longrightarrow v = \text{constant along } x.$$

$$\frac{d^2}{d\theta^2} \cos(\alpha) = \frac{d^2}{d\theta^2} \left(\frac{du}{ds} \right) \left(1 + \left(\frac{du}{ds} \right)^2 \right)^{3/2} - \frac{d}{d\theta} \left(\frac{du}{ds} \right) \frac{3}{2} \left(1 + \left(\frac{du}{ds} \right)^2 \right)^{1/2} \frac{2}{\left(1 + \left(\frac{du}{ds} \right)^2 \right)^3} \frac{du}{ds} \frac{d}{d\theta} \left(\frac{du}{ds} \right)$$

$$\begin{aligned}
 &= \left[-\left(1 + \left(\frac{du}{ds} \right)^2 \right) - 3 \left[\frac{d}{d\theta} \left(\frac{du}{ds} \right) \right]^2 \right] \frac{du}{ds} \\
 &= - \left[\frac{1 + \left(\frac{du}{ds} \right)^2 + 3 \left[\frac{d}{d\theta} \left(\frac{du}{ds} \right) \right]^2}{\left(1 + \left(\frac{du}{ds} \right)^2 \right)^{5/2}} \right] \frac{du}{ds} \\
 &\quad \text{70}
 \end{aligned}$$

Therefore, the sign of $\frac{d^2}{d\theta^2} \cos(\alpha)$ only depends on the sign of $\frac{du}{ds}$

- $\cos(\alpha)$ will have an absolute max when.

$$\frac{dv}{ds} = 0 \text{ and } \frac{du}{ds} > 0$$

- $\cos(\alpha)$ will have an absolute min when

$$\frac{dv}{ds} = 0 \text{ and } \frac{du}{ds} < 0.$$

Conclusion so far: For any point (x, y, z) a curve $v(x, y) = \text{constant}$ will have the property that the tangent vector to the curve $x = x(s)$, $y = y(s)$, $z = z(s)$ will have Max and min inclination (steepness) with respect to the u -axis.

↳ Steepest ascent or descent.

Domains where $u(x, y) > u(x_0, y_0)$ are called hills and those, where $u(x, y) < u(x_0, y_0)$ are called valleys. Also, for saddle points $u(x, y) = u(x_0, y_0)$.

Suppose z_0 is a saddle point of order $m-1, m \geq 2$.

$$\varphi'(z_0) = \varphi''(z_0) = \dots = \varphi^{(m-1)}(z_0) = 0.$$

and take $\varphi^{(m)}(z_0) = \alpha e^{i\psi}$; $\alpha > 0$; ψ real-valued. Also $z = z_0 + r e^{i\theta}$

$$\varphi(z) = \varphi(z_0) + \frac{1}{m!} \alpha e^{i\psi} r^m e^{im\theta} + \dots$$

$$= \varphi(z_0) + \frac{\alpha}{m!} r^m e^{i(m\theta + \psi)} + \dots$$

In consequence, near to $\varphi(z) = u(x,y) + i v(x,y)$

$$u(x,y) = u(x_0, y_0) + \frac{r^m}{m!} a \cos(m\theta + \psi) + \dots$$

$$v(x,y) = v(x_0, y_0) + \frac{r^m}{m!} a \sin(m\theta + \psi) + \dots$$

Directions of level curves where u is constant are given by $\cos(m\theta + \psi) = 0$.

$$m\theta + \psi = (2k+1)\frac{\pi}{2}, \quad k=0, 1, \dots, m-1.$$

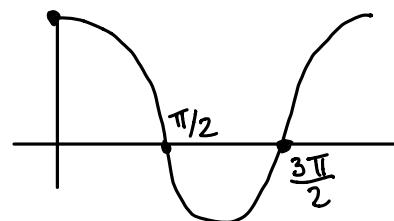
$$\theta = -\frac{\psi}{m} + \frac{2k+1}{m}\frac{\pi}{2}$$

Analogously, for $v = \text{constant} \rightarrow \sin(m\theta + \psi) = 0$.

$$\left. \begin{aligned} m\theta + \psi &= k\pi \\ \theta &= -\frac{\psi}{m} + \frac{k\pi}{m} \end{aligned} \right\} k=0, 1, \dots, m-1.$$

Valleys: $\cos(\theta m + \psi) < 0$

Hills: $\cos(\theta m + \psi) > 0$.



Take, for example $m=2$ (Saddle point 1)

For u constant: $\theta = -\frac{\psi}{2} + \left(k + \frac{1}{2}\right)\frac{\pi}{2}, \quad k=0, 1$

$$\theta_0 = -\frac{\psi}{2} + \frac{\pi}{4}, \quad \theta_1 = -\frac{\psi}{2} + \frac{3\pi}{4}$$

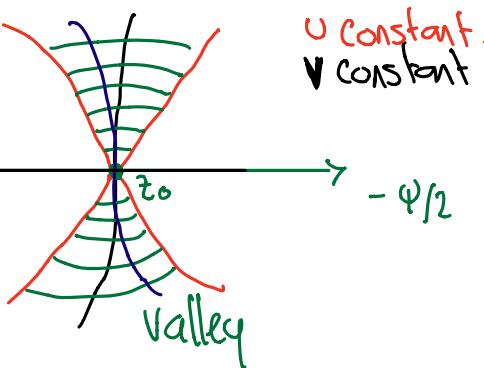
For v constant: $\theta = -\frac{\psi}{2} + \frac{k\pi}{2}$

$$\theta_0 = -\frac{\psi}{2}, \quad \theta_1 = -\frac{\psi}{2} + \frac{\pi}{2}$$

$$\cos(2\theta + \psi) < 0$$

$$\frac{\pi}{2} < 2\theta + \psi < \frac{3\pi}{2}$$

$$-\frac{\psi}{2} + \frac{\pi}{4} < \psi < -\frac{\psi}{2} + \frac{3\pi}{4}$$



$$\cos(2\theta + \psi) = -1$$

$$2\theta + \psi = (2K+1)\pi$$

$$\theta = -\frac{\psi}{2} + (2K+1)\pi$$

$$\theta_0 = -\frac{\psi}{2} + \frac{\pi}{2}$$

$$\theta_1 = -\frac{\psi}{2} + \frac{3\pi}{2}$$

Now, we consider a path where $v(x, y) = \text{constant}$

$$\Rightarrow \varphi(z) = \varphi(z_0) + \sum_{m=1}^{\infty} \frac{a e^{i(m\theta + \psi)}}{m!}$$

$$\hookrightarrow \varphi(z) = \varphi(z_0) + \frac{r^m}{m!} a e^{ik\pi} + \dots$$

$$= \varphi(z_0) + \underbrace{\frac{r^m}{m!} a (-1)^k}_{\sim} + \dots$$

(K even) either monotonically increasing.

(K odd) " decreasing.

$$\tau := \frac{ar^m}{m!} + (\text{higher order terms})$$

$$\varphi(z) = \varphi(z_0) + (-1)^k \tau$$

$$I(\lambda) = \int_C f(z) e^{\lambda \varphi(z)} dz \quad \text{as } \lambda \rightarrow \infty$$

$$\Rightarrow I(\lambda) = \int_{C^*} f(z) e^{\lambda \varphi(z_0) + \lambda (-1)^k \tau} dz$$

In order to guarantee convergence we may choose K odd.

$$\begin{aligned}
 I(\lambda) &= \int_{C^*} f(z) e^{\lambda\varphi(z_0) - \lambda z} dz \\
 &= e^{\lambda\varphi(z_0)} \int_{C^*} f(z) e^{-\lambda z} dz \\
 &= e^{\lambda\varphi(z_0)} \int_{C^*} \left(f(z) \frac{dz}{d\tau} e^{-\lambda z} \right) e^{-\lambda\tau} d\tau
 \end{aligned}$$

We may apply Watson's lemma to this integral.

Watson's:

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

and

$$f(t) = t^\alpha \left[\sum_{k=0}^n a_k t^k + R_{n+1}(t) \right]$$

$$\Rightarrow F(s) \sim \sum_{k=0}^n a_k \frac{\Gamma(\alpha+k+1)}{s^{\alpha+k+1}} + O\left(\frac{1}{s^{\alpha+n+1}}\right) \quad \text{as } s \rightarrow \infty$$

Thus, we may require $f(z) = \frac{dz}{d\tau}$ to be expanded in powers of τ .

Consider again that z_0 is a saddle point of order $m-1$

$$\Rightarrow \varphi(z) = \varphi(z_0) - \sum_{n=0}^{\infty} a_n (z - z_0)^{m+n} \quad a_0 \neq 0.$$

$$=: \varphi(z_0) - (z - z_0)^m \varphi_1(z)$$

$$\Rightarrow \tau = (z - z_0)^m \varphi_1(z)$$

$$\text{Define } \gamma^m := \tau \quad \gamma^m = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m}$$

$$y^m = (z - z_0)^m \sum_{n=0}^{\infty} a_n (z - z_0)^n = (z - z_0) (a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots)$$

$$y = a_0(z - z_0) \left(1 + \frac{a_1}{a_0}(z - z_0) + \frac{a_2}{a_0}(z - z_0)^2 + \dots \right)^{1/m}$$

$$y = a_0^{1/m} (z - z_0) \left(1 + \frac{a_1}{ma_0} (z - z_0) + \dots \right)$$

by the inverse series theorem

$$z - z_0 = \sum_{s=1}^{\infty} \alpha_s y^s = \sum_{s=1}^{\infty} \alpha_s \bar{z}^{s/m}$$

where $\alpha_1 = \frac{1}{a_0^{1/m}}$, $\alpha_2 = \frac{-a_1}{ma_0^{1+2/m}}$,

$$\alpha_3 = \frac{(m+3)a_1^2 - 2ma_0a_2}{2m^2a_0^{2+3/m}}$$

$$\frac{dz}{dc} = \sum_{s=1}^{\infty} \alpha_s \left(\frac{s}{m}\right) \bar{z}^{\frac{s}{m}-1}$$

Now, as $f(z)$ is analytic.

$$f(z) = \sum_{s=1}^{\infty} b_s (z - z_0)^s$$

where

$$b_s = \frac{1}{s!} f^{(s)}(z_0)$$

$$\begin{aligned} I(\lambda) &= \int_C f(z) e^{\lambda z} dz \quad \text{as } \lambda \rightarrow \infty \\ \Rightarrow I(\lambda) &= \int_C f(z) e^{\lambda z} e^{\lambda z_0 + \lambda z_0 \tau} dz \\ &\quad \text{in order to guarantee convergence we may choose } K \text{ odd} \\ \Rightarrow I(\lambda) &= \int_C f(z) e^{\lambda z_0 - \lambda \tau} dz \\ &= e^{\lambda z_0} \int_C f(z) e^{-\lambda \tau} dz \\ I(\lambda) &= e^{\lambda z_0} \int_C \left(f(z) \frac{dz}{d\tau} \right) e^{-\lambda \tau} d\tau \end{aligned}$$