

Asymptotic approximation of integrals

Bibliography:

[1] Wong (2001) - Differential equations and Asymptotic theory on mathematical physics.

[2] Bender and Orszag (1999) V₁ - Advanced mathematical methods for scientist.

Let f and g be two continuous complex functions defined on a subset of the complex functions defined on a subset of the complex plane, $H \subset \mathbb{C}$.

Let z_0 be a limit point of H .

Definition: $f(z) = O(g(z))$ as $z \rightarrow z_0$ means that there is a constant $K > 0$ and a neighbourhood U of z_0 such that

$$|f(z)| \leq K |g(z)| \quad \text{for all } z \in U \cap H.$$

Definition: $f(z) = o(g(z))$ as $z \rightarrow z_0$ means that for every $\epsilon > 0$ there exist a neighbourhood U_ϵ of z_0 such that

$$|f(z)| \leq \epsilon |g(z)| \quad \text{for all } z \in U_\epsilon \cap H$$

Physics: $f(z) = O(g(z)) \sim f$ is big-oh $g \sim$ How fast a function grows or declines.

$f(z) = o(g(z)) \sim$ is smaller than,

f is of the same order of g .

We may consider.

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} < \infty \longrightarrow f(z) = O(g(z))$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = 0 \longrightarrow f(z) = o(g(z)).$$

Example:

- $f(x) = \underbrace{4x^3 - 3x^2 + 2x - 1}_{\text{Highest grow term.}} \quad \text{as } x \rightarrow \infty$

→ Highest grow term.

$$f(x) = O(x^3) \quad \text{as } x \rightarrow \infty$$

check:

$$\begin{aligned}|f(x)| &= |4x^3 - 3x^2 + 2x - 1| \\&\leq |4x^3| + |3x^2| + |2x| + |1| \\&\leq 4x^3 + 3x^3 + 2x^3 + x^3 \\&= 10x^3 = 10|x^3|\end{aligned}$$

therefore

$$f(x) \leq 10x^3$$

other way is:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{4x^3 - 3x^2 + 2x - 1}{x^3} \\= \lim_{x \rightarrow \infty} \left(4 - \frac{3}{x} + \frac{2}{x^2} - \frac{1}{x^3} \right) = 4.\end{aligned}$$

- $f(n) = 10\log(n) + 5(\log(n))^3 + 7n + 3n^2 + 6n^3$ as $n \rightarrow \infty$

then

$$f(n) = \Theta(n^3) \text{ as } n \rightarrow \infty$$

$$\begin{aligned}|f(n)| &= |10\log(n) + 5(\log(n))^3 + 7n + 3n^2 + 6n^3| \\&\leq |10\log(n)| + |5(\log(n))^3| + |7n| + |3n^2| + |6n^3| \\&\leq 10n^3 + 5n^3 + 7n^3 + 3n^3 + 6n^3 \\&= 31n^3 = 31|n^3|\end{aligned}$$

therefore $|f(n)| = 31|n^3|$, by the other way

$$\lim_{n \rightarrow \infty} \frac{f(n)}{n^3} = 6$$

- $x^2 = O(x)$ as $x \rightarrow 0$

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0$$

- $x - \sin(x) = O(x)$ as $x \rightarrow 0$.

$$\lim_{x \rightarrow 0} \left(\frac{x - \sin(x)}{x} \right) = \lim_{x \rightarrow 0} \left(1 - \frac{\sin(x)}{x} \right) = 0$$

- $x - \sin(x) \neq O(x^3)$

$$\lim_{x \rightarrow 0} \left(\frac{x - \sin(x)}{x^3} \right) = \lim_{x \rightarrow 0} \left(\frac{1 - \cos(x)}{3x^2} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\sin(x)}{6x} = \frac{1}{6}$$

therefore $x - \sin(x) = O(x^3)$

Definition: Let $\{\varphi_n\}_{n \geq 0}$ be a sequence of continuous complex functions defined on $H \subset \mathbb{C}$. We say $\{\varphi_n\}_{n \geq 0}$ is an asymptotic sequence as $z \rightarrow z_0$ in H , if we have

$$\varphi_{n+1}(z) = O(\varphi_n(z)) \quad , \text{ if } 0.$$

Definition: If $\{\varphi_n\}_{n \geq 0}$ is an asymptotic sequence as $z \rightarrow z_0$, we say that

$$\sum_{n=1}^{\infty} a_n \varphi_n(z) \quad , \quad a_n \text{ constant.}$$

is an asymptotic expansion of the function f if for each $N \geq 0$ we have

$$f(z) = \sum_{n=1}^N a_n \varphi_n(z) + O(\varphi_N(z)) \quad \text{as } z \rightarrow z_0.$$

In this case we write.

$$f(z) \sim \sum_{n=1}^{\infty} a_n \varphi_n(z) \quad \text{as } z \rightarrow z_0$$

Homework: Show that $f(z)$ may be written as

$$f(z) = \sum_{n=1}^{N-1} a_n \varphi_n(z) + O(\varphi_N(z))$$

Integration by parts

(Easy, but not general method)

Example:

$$I(x) = \int_x^{\infty} e^{-4t} dt, \quad \text{as } x \rightarrow \infty$$

First note that we do not have $x \rightarrow 0$. In the case $x \rightarrow 0$, Taylor e^{-4t} gives an (apparent) divergent result.

$$e^{-at} = 1 - t^4 + \frac{t^8}{2!} - \frac{t^{12}}{3!} + \dots \leftarrow \text{Not.}$$

So we write

$$I(x) = \underbrace{\int_0^\infty e^{-at} dt}_{:= I_1} - \underbrace{\int_0^x e^{-at} dt}_{:= I_2}$$

$$I_1 = \int_0^\infty e^{-at} dt$$

$$s = t^4$$

$$I_1 = \frac{1}{4} \int_0^\infty e^{-s} s^{-3/4} ds = \frac{1}{4} \Gamma\left(\frac{1}{4}\right)$$

$$ds = 4t^3 dt$$

$$dt = \frac{1}{4} s^{-3/4} ds$$

$$\Gamma(n) = \int_0^\infty e^{-z} z^{n-1} dz$$

$$z \Gamma(z) = \Gamma(z+1)$$

$$I_1 = \Gamma\left(\frac{5}{4}\right) \quad \text{as } x \rightarrow 0$$

$$I_2 = \int_0^x e^{-at} dt = \int_0^x \left(1 - t^4 + \frac{t^8}{2!} - \frac{t^{12}}{3!} + \dots\right) dt$$

$$= \left[t - \frac{1}{5} t^5 + \frac{t^9}{18} - \frac{t^{13}}{78} + \dots \right]_0^x$$

$$= x - \frac{x^5}{5} + \frac{x^9}{18} - \frac{x^{13}}{78} + \dots$$

then

$$I(x) \sim \Gamma\left(\frac{5}{4}\right) \quad \text{as } x \rightarrow 0.$$

However, in the case $x \rightarrow \infty$, we need to develop $I(x)$ in inverse powers of x .

$$I(x) = -\frac{1}{4} \int_x^\infty \frac{1}{t^3} \frac{d}{dt} (e^{-t^4}) dt$$

$$= -\frac{1}{4} \frac{e^{-t^4}}{t^3} \Big|_x^\infty - \left(-\frac{1}{4}\right) \int_x^\infty \frac{d}{dt} \left(\frac{1}{t^3}\right) dt$$

$$= -\frac{e^{-x^4}}{4x^3} - \frac{3}{4} \int_x^\infty \frac{e^{-4t}}{t^4} dt,$$

but

$$\int_x^\infty \frac{1}{t^4} e^{-4t} dt < \frac{1}{x^4} \int_x^\infty e^{-t^4} dt = \frac{1}{x^4} I(x) \ll I(x)$$

leading behaviour of $I(x)$.

$$I(x) \sim \frac{1}{4x^3} e^{-x^4}.$$

Let

$$\begin{aligned}
 f(z) &= \sum_{n=1}^N a_n \varphi_n(z) + O(\varphi_N(z)) \\
 &= \sum_{n=1}^{N-1} a_n \varphi_n(z) + a_N \varphi_N(z) + O(\varphi_N(z)) \\
 &= \sum_{n=1}^{N-1} a_n \varphi_n(z) + a_N \varphi_N(z) + \varphi_{N+1}(z)
 \end{aligned}$$

$$f(z) - \sum_{n=1}^{N-1} a_n \varphi_n(z) = a_N \varphi_N(z) + \varphi_{N+1}(z)$$

$$\hat{f}(z) = a_N \varphi_N(z) + \varphi_{N+1}(z)$$

$$\hat{f}(z) = O(\underbrace{\varphi_N(z)}_{})$$

$$\lim_{z \rightarrow z_0} \frac{\hat{f}(z)}{g(z)} < \infty$$

$$\lim_{z \rightarrow z_0} \frac{a_N \varphi_N(z) + \varphi_{N+1}(z)}{\varphi_N(z)} = a_N + \lim_{z \rightarrow z_0} \frac{\varphi_{N+1}(z)}{\varphi_N(z)}$$

$$\lim_{z \rightarrow z_0} \frac{a_N \varphi_N(z) + \varphi_{N+1}(z)}{\varphi_N(z)} = a_N$$

$$f(z) - \sum_{n=1}^{N-1} a_n \varphi_n(z) = O(\varphi_N(z))$$

$$f(z) = \sum_{n=1}^{N-1} a_n \varphi_n(z) + O(\varphi_N(z)).$$

□