Theorem: δ is a singular distribution.

$$\delta[\phi] = \phi(0)$$

Proof: Suppose δ is regular, that is a locally integrable function g such that

$$\delta[\phi] = \int g(x)\phi(x)dx \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n)$$

Now, define a t-parameter family of test function.

$$\phi^t(x) = \begin{cases} e^{-\frac{1}{1-(|x|/t)^2}} & , \text{if } |x| < t \\ 0 & , \text{if } |x| \geq t \end{cases}$$

Note $f[\phi^t] = \phi^t(0) = e^{-1}$, and also

$$\delta[\phi^t] = \int_{\mathbb{R}^n} g(x)\phi^t(x)dx$$

$$\Rightarrow e^{-1} = \left| \int_{\mathbb{R}^n} g(x)\phi^t(x)dx \right| \leq \left| \int_{|x|\leq t} g(x)dx \right| e^{-1}$$

as $t \to 0$ $\left| \int_{|x|\leq t} g(x)dx \right|$

As $t \to 0$ the integral goes to zero

we may choose $t > 0$ small enough in order to obtain an integral less than 1!

Therefore the rhs will be less than e^{-1}, contradicting the inequality

$\therefore$ There is no such g!

Derivative of a distribution: Let f be a distribution, then we define.

$$f'[\phi] = \int_{\mathbb{R}^n} f'(x)\phi(x)dx$$

Theorem: $F'[\phi] = F[-\phi']$

Proof:

compact support.

$$F'[\phi] = \int_{\mathbb{R}^n} f'(x)\phi(x)dx = \cancel{f(x)\phi(x)\Big|_{\mathbb{R}^n}} - \int f(x)\phi'(x)dx$$

$$= \int_{\mathbb{R}^n} f(x)(-\phi'(x))dx = f[-\phi']$$

In particular $\delta[\phi] = \phi(0)$.

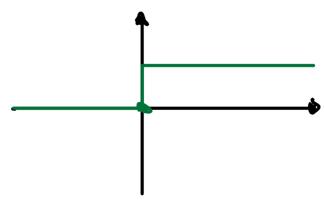
$$g'[\phi] = g[-\phi'] = -\phi'(0)$$

Note: As $\phi(x) \in C_c^\infty(\mathbb{R}^n)$, then every distribution is infinitely differentiable (in the distribution sense).

Theorem: Every discontinuous functions are differentiable (as distribution), and their derivatives give delta functions for each discontinuity.

Proof: Consider the heaviside step-function.

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



$$H[\phi] = \int_{\mathbb{R}^n} H(x)\phi(x)dx = \int_0^\infty \phi(x)dx.$$

$$H'[\phi] = H[-\phi'] = -\int_0^\infty \phi'(x)dx = -\phi'(x)\Big|_0^\infty = -\cancel{\phi'(\phi)} + \phi^*(0) - \phi(0)$$

$$= \delta[\phi]$$

As distribution $H' = \delta$.

As any function with jump discontinuities may be written in terms of $H(x)$.

$\rightarrow$ the derivative of any jump discontinuity brings a delta function.

Suppose $f(x)$ is infinitely differentiable on the real line except at some points $a_1, \dots, a_k$.

Define $g: f - \sum_{i=1}^{k} \Delta_{f_i} H(x-a_i)$.

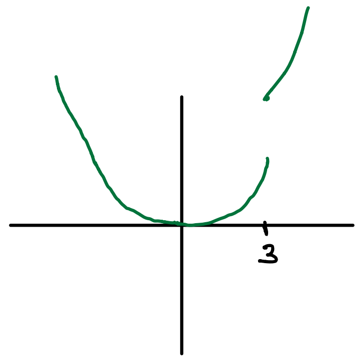
Δ_{f_i} amount of the i-th jump of point a_i

$\rightarrow g' = f' - \sum_{i=1}^{k} \Delta_{f_i} \delta(x-a_i)$. (as distribution).

$\therefore f' = g' + \sum_{i=1}^{k} \Delta_{f_i} \delta(x-a_i)$.

Example:

$$f(x) = \begin{cases} x^2, & x<3 \\ x^3, & x \geq 3 \end{cases}$$



$f(x) = x^2 + (x^3 - x^2) H(x-3)$

think on f as a distribution.

$$f' = 2x - (3x^2 - 2x) H(x-3) + (x^3 - x^2)\delta(x-3)$$

$$f'[\phi] = \int_{\mathbb{R}} \left(2x - (3x^2 - 2x) H(x-3) + (x^3 - x^2)\delta(x-3)\right) \phi(x) dx.$$

$$= \int_{\mathbb{R}} \begin{pmatrix} 2x, & x<3 \\ 3x^2, & x>3 \end{pmatrix} \phi(x) dx + 18\phi(3).$$

Example: $f(x) = |x|$

as distribution

$$f'(x) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$$

$f'(x)$ is a locally integrable function and $f'(x) = 2H(x) - 1$

In the distribution sense

$$f'' = 2H' = 2\delta \longrightarrow f'[\phi] = 2\delta[\phi] = 2\phi(0).$$