Notes:

I. If $\beta = c\beta$

$$\Rightarrow \beta \wedge \alpha = c\alpha \wedge \alpha = c(\cancel{\alpha \wedge \alpha}) = 0$$

II. $\alpha \wedge \beta = \alpha^i e_i \wedge \beta^j e_j = \alpha^i \beta^j e_i \wedge e_j = \sum_{i,j} \alpha^i \beta^j (e_i \wedge e_j)$

as $e_i \wedge e_j = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$

$$\Rightarrow \alpha \wedge \beta = \sum_{i<j} (\alpha^i \beta^j - \alpha^j \beta^i)(e_i \wedge e_j)$$

$$= \alpha \otimes \beta - \beta \otimes \alpha.$$

III. If $\gamma \in \Lambda^2(V)$ $\quad \gamma = \gamma^{ij}(e_i \wedge e_j)$

$e^i \wedge e^j$ is the basis for $\Lambda^2(V)$.

IV. $\dim(\Lambda^2(V)) = \dfrac{n(n-1)}{2} = \binom{n}{2}$

V. Elements of $\Lambda^2(V)$ will be called 2-vectors or bivectors.

Now, we can extend the definition of exterior product for vectors and 2-vectors to obtain 3-vectors and so on.

Then, the exterior product of K vectors $\alpha_1, \dots, \alpha_n$

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_K \; ; \qquad K \leq n$$

will be termed a K-vector.

Definition: The exterior product of a p-vector and a q-vector is a mapping

$$\wedge : \Lambda^p(V) \times \Lambda^q(V) \longrightarrow \Lambda^{p+q}(V); \quad p+q \leq n$$

$$(\alpha, \beta) \longmapsto \alpha \wedge \beta.$$

Defined by:

I. Associativity, bilineal.

II. $e_1 \wedge \dots \wedge e_{p+q} = 0$ if for $i \neq j$, $e_i \neq e_j$.

III. $e_1 \wedge \cdots \wedge e_{p+q}$ changes sign if any two adjacent e_i are interchanged

$$e_i \wedge e_j = -e_j \wedge e_i.$$

$$\alpha \wedge \beta = \frac{1}{p!\,q!} \sum_{\pi} (\operatorname{sgn} \pi)\, \pi(\alpha \otimes \beta);$$

$\pi :=$ Permutation of $(1,2,\dots,p+q)$

- Exterior product in general is non-commutative.

$$\alpha \wedge \beta = (-1)^{pq}\, \beta \wedge \alpha.$$

- $\dim \Lambda^p(V) = \binom{n}{p}$; $p \leq n$ — Combination of n things taken p at a time.

In particular $\binom{n}{0} = \binom{n}{n}$ and $\binom{n}{k} = \binom{n}{n-k}$; $k \leq n$.

$$\binom{n}{k} = 0, \quad k > n.$$

- If $\alpha^1, \alpha^2, \dots, \alpha^p$ are vectors

$$\Longrightarrow \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p = \epsilon^{1,2,\dots,p}_{i_1,i_2,\dots,i_p}\, \alpha^{i_1} \otimes \alpha^{i_2} \otimes \cdots \otimes \alpha^{i_p}.$$

Definition: A multivector is formally defined

$$A \in \Lambda(V) = \underbrace{\Lambda^0(V)}_{\text{scalars}} \oplus \underbrace{\Lambda^1(V)}_{\text{vectors}} \oplus \underbrace{\Lambda^2(V)}_{\text{bivectors}} \oplus \cdots \oplus \underbrace{\Lambda^n(V)}_{\text{Pseudo-scalars.}}$$

$(\Lambda(V), \wedge) :=$ Exterior algebra. (Grassmann).

$$\dim(\Lambda(V)) = \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Notation: Let $H = \{h_1, h_2, \dots, h_p\}$

$$1 \leq h_1 < h_2 < \dots < h_p \leq n$$

we set $e^H = e^{h_1} \wedge e^{h_2} \wedge \cdots \wedge e^{h_p}.$

$\Longrightarrow e^H$ is a basis of $\Lambda^p(V)$.

Hodge star operator (Hodge dual)

Definition: Let V have an inner product (α, β) such that

$$(e_i, e_j) = \delta_{ij}$$

Given that $\binom{n}{k} = \binom{n}{n-k}$, then $\dim \Lambda^k(V) = \dim \Lambda^{n-k}(V)$,

$$*: \Lambda^p(V) \longrightarrow \Lambda^{n-p}(V).$$

Consider $\sigma \in \Lambda^n(V)$, $\lambda \in \Lambda^p(V)$, $\mu \in \Lambda^{n-p}(V)$

$$\Rightarrow \mu \longrightarrow \lambda \wedge \mu$$

is a linear transformation on $\Lambda^{n-p}(V)$ into $\Lambda^n(V)$

Then there is a unique $(n-p)$-vector, denoted by $*\lambda$.

$$\lambda \wedge \mu = (*\lambda, \mu)\sigma.$$

$\lambda = e_1 \wedge e_2 \in \Lambda^2(V)$	$(*\lambda, \mu) = 1$
$\mu = e_3 \in \Lambda^1(V)$	$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_3 (e_3, e_3)$
$\sigma = e_1 \wedge e_2 \wedge e_3 \in \Lambda^3(V)$	$= \lambda_3$
$\lambda \wedge \mu = (e_1 \wedge e_2) \wedge_3 = \sigma = (*\lambda, \mu)\sigma$	$\lambda_3 = 1$
	$\therefore *\lambda = e_3$

In order to explicitly compute $*\lambda$ we will consider for simplicity

$$\lambda = e^1 \wedge e^2 \wedge \cdots \wedge e^p = e^H.$$

where $e^1, e^2, \ldots, e^p$ is an orthonormal basis for V.

Let K runs over sets of $q = n-p$ indices.

$$\Rightarrow \lambda \wedge e^K = (*\lambda, e^K)\sigma.$$

but <u>lhs</u> vanishes unless

$$K = \{p+1, p+2, \ldots, n\}$$

$$\Rightarrow \pm\sigma = \lambda \wedge e^K = (*\lambda, e^K)\sigma$$

$$(*\lambda, e^K) = \pm 1.$$

$$\Rightarrow *\lambda = C_0 e^{p+1} \wedge e^{p+2} \wedge \cdots \wedge e^n$$

$$= C_0 e^K$$

$$C_0(e^k,e^k)=\pm 1$$

$$C_0=(e^k,e^k)$$

$$\Longrightarrow *\lambda=(e^k,e^k)\,e^{p+1}\wedge e^{p+2}\wedge\cdots\wedge e^n$$

Homework:

I. Prove that for a p-vector α we have

$$**\alpha=(-1)^{p(n+p)+(n-t)/2}\,\alpha$$

where t is the signature of the metric.

II. $\alpha\wedge *\beta=\beta\wedge *\alpha$.

$*$

P-vectors	Basis	Dimensions
$\Lambda^0(V)=:\mathbb{F}(V)$	$\{1\}$	1
$\Lambda^1(V)=:V$	$\{e_i\}$	n
$\Lambda^2(V)$	$\{e_i\wedge e_j\}_{i\neq j}$	$n(n-1)/2$
$\Lambda^3(V)$	$\{e_i\wedge e_j\wedge e_k\}_{i\neq j\neq k}$	$n(n-1)(n-2)/6$
$\vdots$	$\vdots$	$\vdots$
$\Lambda^n(V)$	$\{e_1\wedge e_2\wedge\ldots\wedge e_n\}$	1

Examples: Clifford algebras

For any n-dim vector space V, it corresponds a Clifford algebra Cl(V) by defining an associative bilinear product.

$$\alpha\circ\beta=(\alpha,\beta)+\alpha\wedge\beta=:\alpha\cdot\beta+\alpha\wedge\beta$$

$$\alpha\in\Lambda^1(V),\quad \beta\in\Lambda^2(V)$$

$$\alpha\circ\beta=\alpha\lrcorner\beta+\alpha\wedge\beta$$

$$\alpha\in\Lambda^p(V),\quad \beta\in\Lambda^q(V)$$

$$\alpha\circ\beta=\alpha\lrcorner\beta+\alpha\wedge\beta\in\Lambda^{|p-q|}+\Lambda^{|p-q+2|}+\cdots+\Lambda^{p+q}$$

1. $Cl(2) \longrightarrow V = \mathbb{R}^2$

basis for $V = \{e_1, e_2\}$

Consider an arbitrary vector

$$X = x^1 e_1 + x^2 e_2$$

$$\begin{aligned}
\|X\|^2 = X \circ X &= (x^1 e_1 + x^2 e_2) + (x^1 e_1 + x^2 e_2) \\
&= (x^1)^2 (e_1 \circ e_1) + (x^2)^2 (e_2 \circ e_2) \\
&\quad + x^1 x^2 (e_1 \circ e_2) + x^2 x^1 (e_2 \circ e_1) \\
&= (x^1)^2 (e_1 \circ e_1) + (x^2)^2 (e_2 \circ e_2) \\
&\quad + x^1 x^2 (e_1 \circ e_2 + e_2 \circ e_1)
\end{aligned}$$

As $\|X\|^2$ represents the norm we choose:

$$e_1 \circ e_1 = 1 = e_2 \circ e_2$$

$$e_1 \circ e_2 = 0 \qquad \text{Gibbs}$$

$$e_1 \circ e_2 + e_2 \circ e_1 = 0 \qquad \text{Clifford.}$$

$$e_1 \circ e_2 = -e_2 \circ e_1$$

We want to show that $e_1 \circ e_2$ is neither a scalar nor a vector

a. $e_1 \circ (e_1 \circ e_2) = (e_1 \circ e_1) \circ e_2 = e_2$

$$\begin{aligned}
(e_1 \circ e_2) \circ e_1 &= e_1 \circ (e_2 \circ e_1) \\
&= -e_1 \circ (e_1 \circ e_2) \\
&= -(e_1 \circ e_1) \circ e_2 \\
&= -e_2
\end{aligned}$$

Therefore, $e_1 \circ e_2$ is not an scalar.

b. $\|e_1 \circ e_2\|^2 = (e_1 \circ e_2) \circ (e_1 \circ e_2) = e_1 \circ (e_2 \circ e_1) \circ e_2$

$$\begin{aligned}
&= -e_1 \circ (e_1 \circ e_2) \circ e_2 = -(e_1 \circ e_1) \circ (e_2 \circ e_2) \\
&= -1
\end{aligned}$$

Therefore $e_1 \circ e_2$ is not a vector.

c. $e_1 \circ e_2$ is a bivector that is

$$e_1 \circ e_2 = \cancel{(e_1, e_2)}^{\,0} + e_1 \wedge e_2$$

Bivectors may be thought of as an oriented area segment.

$$\Lambda(\mathbb{R}^2) = \Lambda^0(\mathbb{R}^2) \oplus \Lambda^1(\mathbb{R}^2) \oplus \Lambda^2(\mathbb{R}^2)$$

$$1 \qquad e_1, e_2 \qquad e_1 \wedge e_2$$

As $e_1 \circ e_2$ is of norm -1, we may think of it as an imaginary unit

$$\Lambda^0(\mathbb{R}^2) + \Lambda^2(\mathbb{R}^2) \ni \alpha = \alpha_1 + i\alpha_2$$

$i = e_1 \circ e_2$ pseudo-scalar unit.

then, the even part of $Cl(2)\ \Lambda^+(\mathbb{R}^2) := \Lambda^0(\mathbb{R}^2) \oplus \Lambda^2(\mathbb{R}^2)$ is associated to relations on the plane $\mathbb{R}^2$.