Penrose diagrams

- Kruskal are convenient for analyzing the "strong field" region but they are not very convenient for analyzing the asymtotically flat region $r \to \infty$.

- We want to study the conformal structure of infinity by an appropriate transformation. Penrose diagram!

Minkowski spacetime.

↳ Special relativity.

$\mathbb{R}^{1,3}$

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$
$$= -(dx^0)^2 + (d\vec{x})^2$$

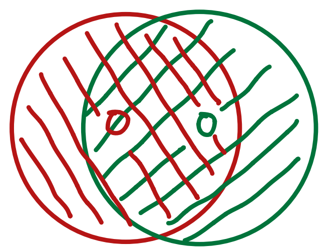
In spherical coordinates: (t, r, θ, φ)

$$x^0 = t$$
$$x^1 = r\sin\theta\sin\varphi$$
$$x^2 = r\sin\theta\cos\varphi$$
$$x^3 = r\cos\theta$$

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

"Apparently" singular at $r=0$.

$$g^{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^{-2} & \\ & & & r^{-2}\sin^{-2}\theta \end{pmatrix}$$



Coordinates are not admissible coordinates at this point.

Then, restrict the coordinates to

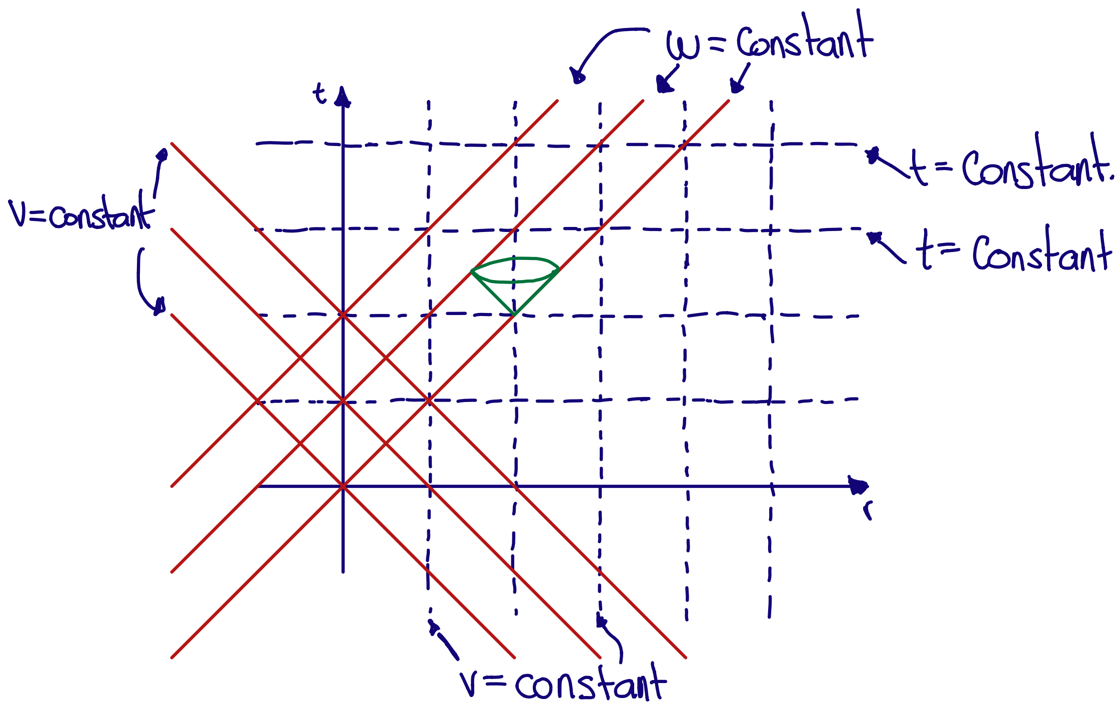
$$0 < r < \infty, \qquad 0 < \theta < \pi, \qquad 0 < \phi < 2\pi.$$

hence we need two copies of these restricted coordinates to cover all of Minkowski.

Choose (advanced & retracted) coordinates.

$$v = t + r$$
$$w = t - r \qquad v \geq w$$
$$-\infty < v < \infty$$
$$-\infty < w < \infty$$

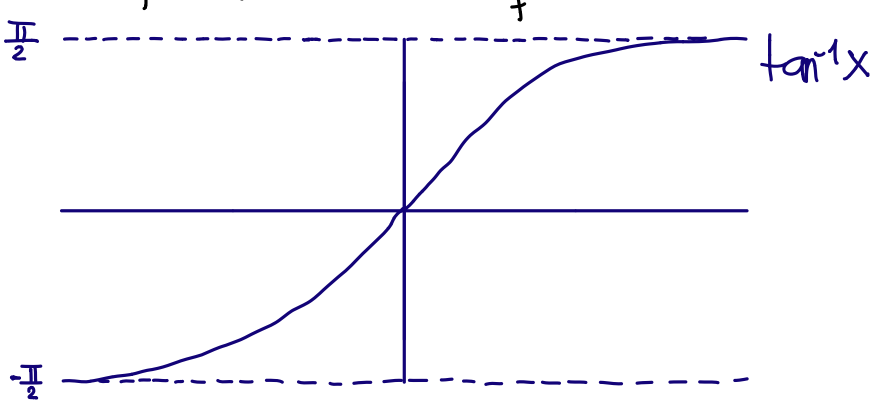
Surfaces $\begin{matrix} w = \text{Constant} \\ v = \text{constant} \end{matrix}$ are null.



Define new null coordinates in which the infinities of v and w become finite.

Penrose's idea.

$$p := \tan^{-1} v \qquad q := \tan^{-1} w$$



$$-\frac{1}{2}\pi < p, q < \frac{1}{2}\pi.$$

$$ds^2 = -dv\,dw + \frac{1}{4}(v-w)^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$ds^2 = \sec^2 p \sec^2 q\left[-dp\,dq + \frac{1}{4}\sin^2(p-q)(d\Omega^2)\right]$$

$$v = \tan p \qquad w = \tan q$$

$$dv = \sec^2 p\, dp \qquad dw = \sec^2 q\, dq$$

$$(v-w)^2 = (\tan p - \tan q)^2 = \left(\frac{\sin p}{\cos p} - \frac{\sin q}{\cos q}\right)^2$$

$$= \left(\frac{\sin p \cos q - \sin q \cos p}{\cos p \cos q}\right)^2 = \sec^2 p \sec^2 q \sin^2(p-q)$$

Now change $t' := p+q$, $r' := p-q$

$$dp\,dq = \frac{1}{4}(dt'^2 - dr'^2)$$

η' is conformal to g such that

$$d\bar{s}^2 = -4\,dp\,dq + \sin^2(p-q)(d\Omega^2)$$

$$d\bar{s}^2 = -dt'^2 + dr'^2 + \sin^2 r'(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Again Minkowski.

but, with restricted coordinates.

$$ds^2 = \frac{1}{4}\sec^2\left(\frac{t'+r'}{2}\right)\sec^2\left(\frac{t'-r'}{2}\right)d\bar{s}^2$$

total transform

$$\begin{pmatrix} t \\ r \end{pmatrix} \longmapsto \begin{pmatrix} t' \\ r' \end{pmatrix}$$

explicitely given by

$$t = \frac{1}{2}\left[\tan\left(\frac{t'+r'}{2}\right) + \tan\left(\frac{t'-r'}{2}\right)\right]$$

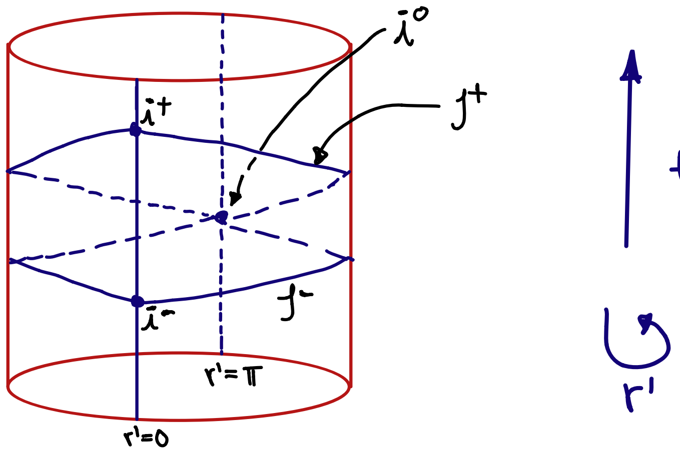
$$r = \frac{1}{2}\left[\tan\left(\frac{t'+r'}{2}\right) + \tan\left(\frac{t'-r'}{2}\right)\right]$$

Therefore, the whole of Minkowski spacetime is conformal to the region.

$$-\pi < t' - r' < \pi, \qquad -\pi < t' + r' < \pi.$$

Idea: We can see $\mathbb{R}^{1,3}$ as imbedded in a five-dim flat space with metric.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2$$



a. Boundary of this region represents the conformal structure of infinity.

$$\left.\begin{aligned} p &= \tfrac{1}{2}\pi \\ q &= -\tfrac{1}{2}\pi \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} &\mathcal{J}^+ \text{ "Scri plus"} \\ &\mathcal{J}^- \text{ "Scri minus"} \end{aligned}\right.$$

Together with the points

$$(p,q) = \left\{\begin{aligned} &\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \\ &\left(\tfrac{\pi}{2}, -\tfrac{\pi}{2}\right) \\ &\left(-\tfrac{\pi}{2}, -\tfrac{\pi}{2}\right) \end{aligned}\right. \quad \text{denoted} \left\{\begin{aligned} &i^+ \\ &i^0 \\ &i^- \end{aligned}\right.$$

Note that point $(p,q) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is not an interesting region since it will violate condition $p \geq q$.

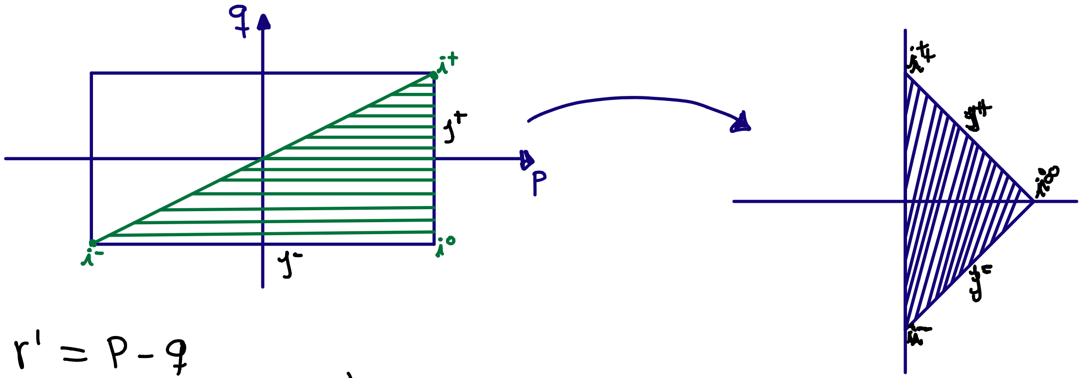
$$0 < r < \infty \qquad v = t + r$$

$$-\infty < t < \infty \qquad w = t - r.$$

$$-\infty < v, w < \infty \qquad v \geq w$$

$$p = \tan^{-1} v$$

$$q = \tan^{-1} w \qquad -\frac{\pi}{2} < (p,q) < \frac{\pi}{2} \qquad \boxed{p \geq q}$$



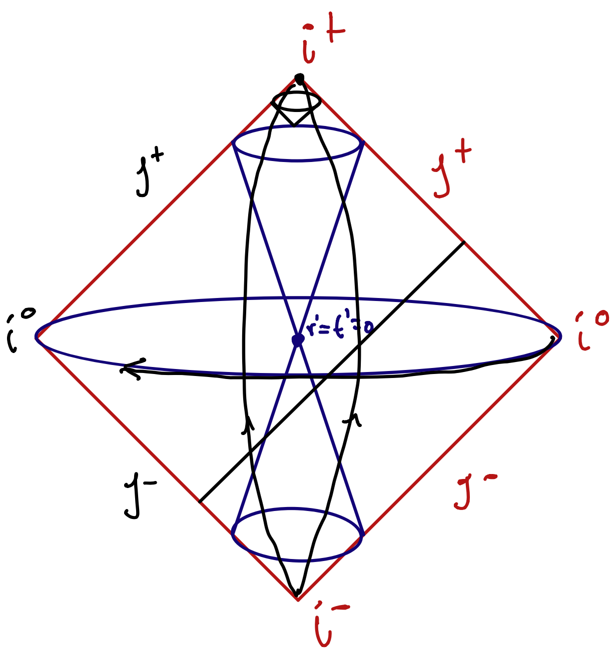
$$r' = p - q$$

$$r' \geq 0$$

$$t' = p + q$$

$$-\pi < t' < \pi$$

$$0 \leq r' < \pi$$



- Any future-directed timelike geodesic in Minkowski approaches i^+ for sufficiently large value of its evolution parameter.
- Timelike geodesics originate at i^- and finish at i^+.
- Null geodesics originate at $\mathscr{J}^-$ and finish at $\mathscr{J}^+$
- Spacelike geodesics originate and end at i^0
- One may regard i^+ and i^- as representing future and past timelike infinity, $\mathscr{J}^+$ and $\mathscr{J}^-$ represents future and past null infinity and finally i^0 represents spacelike infinity.

Penrose diagram

The structure of infinity in any spherically symmetric spacetime can be represented by a Penrose diagram with the characteristic drawings:

I. Infinity is represented by solid lines.

II. Origin of coordinates is represented by dotted lines.

III. Irremomable singularities is represented by double lines.