Creation and annihilation operator for Fermions

$$\hat{C}^{+}_{\nu_j}|\ldots,n_{\nu_{j-1}},n_{\nu_j},n_{\nu_{j+1}},\ldots\rangle = C_{+}(n_{\nu_j})|\ldots,n_{\nu_{j-1}},n_{\nu_j}+1,n_{\nu_{j+1}},\ldots\rangle$$
$$\hat{C}_{\nu_j}|\ldots,n_{\nu_{j-1}},n_{\nu_j},n_{\nu_{j+1}},\ldots\rangle = C_{-}(n_{\nu_j})|\ldots,n_{\nu_{j-1}},n_{\nu_j}-1,n_{\nu_{j+1}},\ldots\rangle$$

We have to ask for,

$$|\ldots,n_{\nu_j}=1,\ldots,n_{\nu_k}=1,\ldots\rangle = -|\ldots,n_{\nu_k}=1,\ldots,n_{\nu_j}=1,\ldots\rangle$$

$$\{\hat{A},\hat{B}\} := \hat{A}\hat{B}+\hat{B}\hat{A}$$ anticommutator

$$\{\hat{A},\hat{B}\}=0 \longrightarrow \hat{B}\hat{A} = -\hat{A}\hat{B}$$

$$\left.\begin{matrix} \hat{C}^{+}_{\nu_j} \text{ and } \hat{C}^{+}_{\nu_k} \\ \hat{C}_{\nu_j} \text{ and } \hat{C}_{\nu_k} \end{matrix}\right\} \text{must anticommute}$$

If $j\neq k$ we ask for $\hat{C}_{\nu_j}$ and $\hat{C}^{+}_{\nu_k}$ anticommute

If $j=k$

$$\hat{C}_{\nu_j}|\ldots,0,\ldots\rangle = 0 \longrightarrow C_{-}(0)=0$$
$$\hat{C}^{+}_{\nu_j}|\ldots,0,\ldots\rangle = |\ldots,1,\ldots\rangle \longrightarrow C_{+}(1)=1$$

Normality freedom

As $\langle 1|\hat{C}^{+}_{\nu_j}|0\rangle^{*} = \langle 0|\hat{C}_{\nu_j}|1\rangle$

then $\hat{C}_{\nu_j}|\ldots,1,\ldots\rangle = |\ldots,0,\ldots\rangle$,

thus $C_{-}(1)=1$

$$\hat{C}_{\nu_j}\hat{C}^{+}_{\nu_j}|0\rangle = |0\rangle \quad \text{but} \quad \hat{C}^{+}_{\nu_j}\hat{C}_{\nu_j}|0\rangle = 0$$

therefore

$$\{\hat{C}^{+}_{\nu_j},\hat{C}^{+}_{\nu_k}\}=0,\quad \{\hat{C}_{\nu_j},\hat{C}_{\nu_k}\}=0,\quad \{\hat{C}_{\nu_j},\hat{C}^{+}_{\nu_k}\}=\delta_{\nu_j,\nu_k}$$

As,
$$\hat{C}^{+}_{\nu_j}\hat{C}^{+}_{\nu_j} = -\hat{C}^{+}_{\nu_j}\hat{C}^{+}_{\nu_j} \longrightarrow (\hat{C}^{+}_{\nu_j})^2=0 \longrightarrow (\hat{C}_{\nu_j})^2=0,$$

let us analize $\hat{c}^+ \hat{c}$

$$[\hat{c}_\nu^+ \hat{c}_\nu, \hat{c}_\nu] = \hat{c}_\nu^+ \hat{c}_\nu \hat{c}_\nu - \hat{c}_\nu \hat{c}_\nu^+ \hat{c}_\nu = \hat{c}_\nu^+ \hat{c}_\nu \hat{c}_\nu - (1 - \hat{c}_\nu^+ \hat{c}_\nu)\hat{c}_\nu$$

$$= \cancel{2\hat{c}_\nu^+ \hat{c}_\nu \hat{c}_\nu} - \hat{c}_\nu = -\hat{c}_\nu$$

$$[\hat{c}_\nu^+ \hat{c}_\nu, \hat{c}_\nu^+] = \hat{c}_\nu^+ \hat{c}_\nu \hat{c}_\nu^+ - \cancel{\hat{c}_\nu^+ \hat{c}_\nu^+ \hat{c}_\nu} = \hat{c}_\nu^+ (1 - \hat{c}_\nu^+ \hat{c}_\nu)$$

$$= \hat{c}_\nu^+ - \cancel{\hat{c}_\nu^+ \hat{c}_\nu^+ \hat{c}_\nu} = \hat{c}_\nu^+$$

$$(\hat{c}_\nu^+ \hat{c}_\nu)^2 = \hat{c}_\nu^+ (\hat{c}_\nu \hat{c}_\nu^+) \hat{c}_\nu = \hat{c}_\nu^+ (1 - \hat{c}_\nu^+ \hat{c}_\nu) \hat{c}_\nu = \hat{c}_\nu^+ \hat{c}_\nu$$

then,

$$\hat{c}_\nu^+ \hat{c}_\nu (\hat{c}_\nu \hat{c}_\nu - 1) = \hat{c}_\nu^+ \hat{c}_\nu - \hat{c}_\nu^+ \hat{c}_\nu = 0$$

Summarizing,

$$\hat{c}_\nu^+ \hat{c}_\nu = \hat{n}_\nu, \qquad \hat{c}_\nu^+ \hat{c}_\nu |n_\nu\rangle = n_\nu |n_\nu\rangle \qquad n_\nu = 0, 1.$$

$$\hat{c}|0\rangle = 0, \quad \hat{c}_\nu^+ |0\rangle = |1\rangle, \quad \hat{c}_\nu |1\rangle = 0, \quad \hat{c}_\nu^+ |1\rangle = 0$$

Equivalence between states in first and second quantization.

$$\hat{S}_- |\nu_{n_1}\rangle_1 |\nu_{n_2}\rangle_2 \cdots |\nu_{n_N}\rangle_N \longleftrightarrow \hat{c}_{\nu_{n_1}}^+ \hat{c}_{\nu_{n_2}}^+ \cdots \hat{c}_{\nu_{n_N}}^+ |0\rangle$$