Penrose diagram for Schwarzschild

Heinicke & Hehl, arXiv: 1503.02172 v1.

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

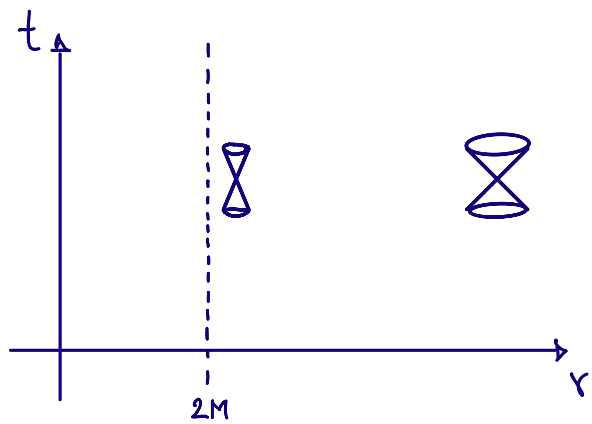
$M \longrightarrow 0$ Minkowski

$r \longrightarrow \infty$ $\parallel$

Consider radial null coordinates $ds^2=0$

$$\frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$

Measure the slope of the light cones in the rt-plane.



Before we introduced Kruskal.

$$(t,r) \longmapsto (T,X)$$

$$T := \sqrt{\left|\frac{r}{2M}-1\right|}\, e^{r/2M}\sinh\left(\frac{t}{4M}\right)$$

$$X := \sqrt{\left|\frac{r}{2M}-1\right|}\, e^{r/2M}\cosh\left(\frac{t}{4M}\right)$$

such that

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}\left(-dT^2+dX^2\right) + r^2 d\Omega^2$$

$$r = r(X,T)$$

Now: We will introduce essentialy the same transformation as use in flat Minkowski spacetime, in order to bring infinity into finite coordinate values.

$$T+X := \tan\left(\frac{T'+X'}{2}\right)$$

$$T-X := \tan\left(\frac{-T'+X'}{2}\right)$$

Null coordinates

$$-\pi < T' \pm X' < \pi.$$

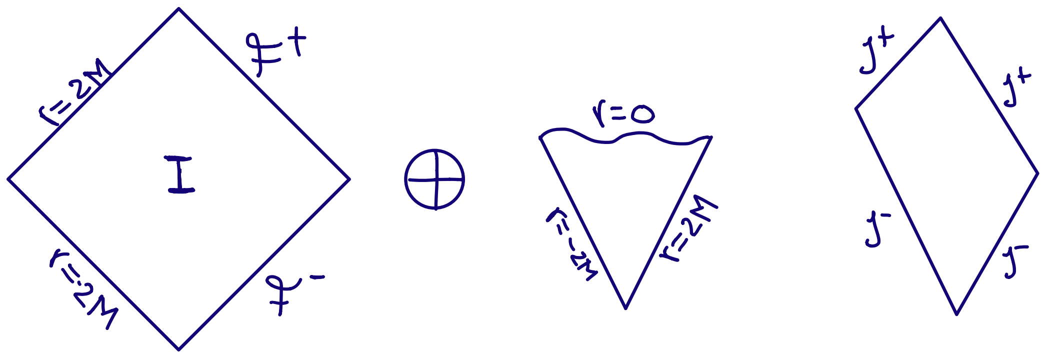
Schwarzschild:

$$ds^2 = \frac{M^3 e^{-r/2M}}{r} \sec^2\left(\frac{T'+X'}{2}\right) \sec^2\left(\frac{-T'+X'}{2}\right)(-dT'^2 + dX'^2)$$
$$+ r^2 d\Omega^2$$
$$r = r(T', X')$$

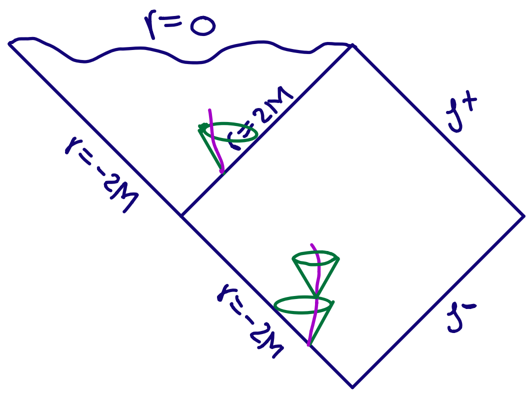
given by

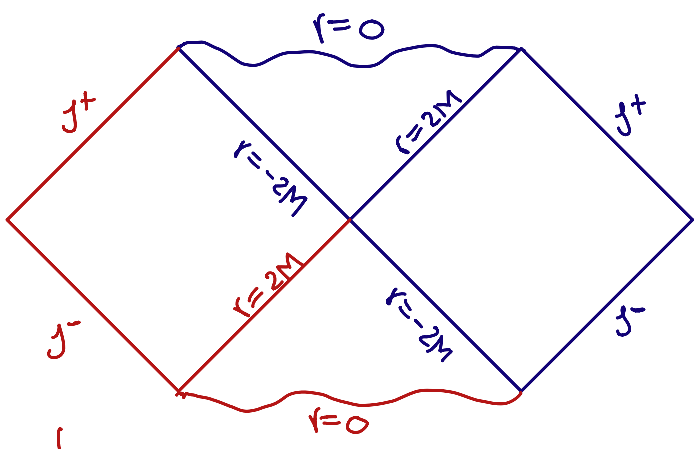
$$\left(\frac{r}{2M} - 1\right) \exp\left(\frac{r}{2M}\right) = \tan\left(\frac{T'+X'}{2}\right) \tan\left(\frac{-T+X'}{2}\right)$$

Two blacks: Asymptotically flat



Joining:





Kruskal extension.

Geodesics (Timelike & Null)

We may restrict to the equatorial plane due to rotational symmetry.

$$u^\mu := \frac{dx^\mu}{d\tau}$$ tangent to a curved parametrised by τ.

$\tau \longrightarrow$ timelike geodesic "proper time"

$\tau \longrightarrow$ null geodesic. "affine parameter"

$$-K = g_{\alpha\beta} u^\alpha u^\beta = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)\dot{r}^2 + r^2\dot{\phi}^2$$

$$K := \begin{cases} 0, & \text{null} \\ 1, & \text{timelike.} \end{cases}$$

- Consider $E = -g_{ab}\xi^a u^b = \left(1 - \frac{2M}{r}\right)\dot{t}$

$$\xi^a := \left(\frac{\partial}{\partial t}\right)^a$$ static Killing field.

$E = \text{constant}$ along a geodesic γ

First show that $\xi_a u^a$ is constant along γ

Proof:

$$\begin{aligned}
\nabla_\gamma(\xi_a u^a) &= u^b \nabla_b(\xi_a u^a) \\
&= u^b(\nabla_b \xi_a)u^a + u^b \xi_a \nabla_b u^a \\
&= u^a u^b(\nabla_b \xi_a) + \xi_a(u^b \nabla_b u^a) \\
&= \frac{u^a u^b}{2}\underbrace{\cancel{(\nabla_a \xi_b + \nabla_b \xi_a)}}_{\text{Killing equation}}^{\to 0} + \xi_a \underbrace{\cancel{(\nabla_\gamma u^a)}}_{\text{Geodesic equation.}}^{\to 0} = 0
\end{aligned}$$

$\blacksquare$

E can be interpreted for timelike geodesics as total energy and KE the total energy for a photon in the null case.

- Consider $L = g_{ab}\psi^a u^b = r^2\dot{\phi}$

$$\psi^a := \left(\frac{\partial}{\partial\phi}\right)^a$$ Angular Killing field.

L may be interpreted as the angular momentum.

In Newtonian limit.

$L \rightarrow$ 2nd Kepler's law.

$\hookrightarrow$ In general, this interpretation is missing as spacetime is non-flat!

Substitution of E and L in geodesic equation

$$-K = -\left(1-\frac{2M}{r}\right)\frac{E^2}{\left(1-\frac{2M}{r}\right)^2} + \frac{\dot{r}^2}{1-\frac{2M}{r}} + r^2\left(\frac{L^2}{r^4}\right)$$

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+K\right) = \frac{1}{2}E^2$$

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2$$

1D non-relativistic "mass" particle of energy $\frac{E^2}{2}$

$$V(r) = \frac{1}{2}K - \frac{KM}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$ Effective Potential.

- Newtonian term
- Centrifugal term
- General Relativity term.