Complete system

Definition: A system $\{x_i\}_{i\geq 1}$ is a normed space of X, it is said complete if

$$\operatorname{span}\left\{\sum_{i=1}^{n} \alpha_i x_i,\ \forall n \in \mathbb{N},\ \alpha_i \text{ scalars}\right\}$$

is dense in X

Lemma: If $\{f_i\}$ is a complete system in a Hilbert space $\mathcal{H}$, y $x \perp f_i$ then $x=0$.

Proof: $x \perp f_i$, implies $x \perp \operatorname{span}(f_i)$ and this implies that x is orthogonal to a dense set in $\mathcal{H}$.

from here we obtain that exist a succession $x_n \longrightarrow x$, and $x_n \perp x$ $\forall n$. Thus,

$$0=\langle x_n, x\rangle \longrightarrow \langle x,x\rangle = \|x\|^2, \Longrightarrow x=0. \qquad \blacksquare$$

Gram-Schmidt orthogonalization process

The following process transform a system linearly independent in an orthogonal one.

Let $\{x_i\}_{i=1}^{\infty}$ be a linearly independent system, in a Hilbert space $\mathcal{H}$.

Let's consider $e_1 = \frac{x_1}{\|x_1\|}$, we define inductively $e_n = \frac{(x_n - y_n)}{\|x_n - y_n\|}$, for

$$y_n = \sum_{i=1}^{n-1} \langle x_n, e_i\rangle e_i$$

we note that the definition is possible, since as $\|x_n - y_n\| \neq 0$, if $n=2$ and $x_2 - y_2 = 0$, implies that x_1 and x_2 are linearly dependend that contradicts the supposition and it continues the process by induction and it is obtained $\{e_i\}_{i=1}^{\infty}$.

Properties:

I. $\{e_i\}_{i=1}^{\infty}$ are orthonormals, if $m>n$, $\langle e_m, e_n\rangle = 0$ it follows from $\langle x_m, e_n\rangle = \langle y_m, e_n\rangle$.

II. $\operatorname{span}\{x_i\}_{i=1}^{n} = \operatorname{span}\{e_i\}_{i=1}^{n}$, for all $n=1,2,\ldots$

Proof: By induction, if we know that it is true for $n-1$, then

$$y_n \in \operatorname{span}\{x_i\}_{i=1}^{n} \quad \text{and} \quad x_n - y_n \neq 0.$$

As the x_i are linearly independent, then

$$\{e_i\}_{i=1}^{n} \subseteq \operatorname{span}\{x_i\}_{i=1}^{n}.$$

Also

$$x_n \in \operatorname{span}\{e_i\}_{i=1}^{n}$$

it means that $\{x_i\}_{i=1}^{n} \subseteq \operatorname{span}\{e_i\}_{i=1}^{n}$. ■

Definition: A normed space $\underline{X}$ is called separable, if there exist a numerable dense set.

Corollary: The Hilbert space $\mathcal{H}$, is separable if and only if there exists an orthonormal complete base $\{e_i\}_{i\geq 1}$.

Proof: $(\longrightarrow)$ Let $\mathcal{H}$ be separable, there is a numerable dense subset

$$\underline{Y} = \{y_i\}_{i\geq 1}.$$

let's choose in an inductive way, a linearly independent subset $\{x_i\}_{i=1}$ of $\underline{Y}$ such that for each n.

$$\operatorname{span}\{x_i\}_{i=1}^{n} = \operatorname{span}\{y_i\}_{i=1}^{N_n}, \quad \text{for some } N_n \in \mathbb{N}$$

this means that if $\{x_i\}_{i=1}^{n}$ are chosen, let's call x_{n+1}, the first y_i, $i > N$, that linearly independend of $\{x_i\}_{i=1}^{n}$, then

$$\operatorname{span}\{x_i\}_{i\geq 1}.$$

is dense, moreover, if $n = \dim \mathcal{H} < \infty$, the process finishes after of a finite number of steps producing a base $\{x_i\}_{i=1}^{n}$,

If not, it continues indefinitely, if $\dim \mathcal{H} = \infty$, then the Gram-Schmidt process is applied to the system $\{x_i\}_{i\geq 1}$ completing the proof.

$(\longleftarrow)$ If $\{e_i\}$ is complete, let all the finite sums $\sum \alpha_i e_i$ with rational coefficients, this is a numerable dense set in $\mathcal{H}$. ■

Definition: A succession $\{x_i\}_{i=1}^{\infty}$ is called a base of a normed space $\underline{X}$, if for all $x \in \underline{X}$ there exist a series $\sum_{i\geq 1} a_i x_i$ that converges to x.

Theorem: An orthonormal complete system $\{e_i\}_{i=1}^{\infty}$ in $\mathcal{H}$, is a base of $\mathcal{H}$.

Proof: for $x \in \mathcal{H}$, by Bessel's inequality

$$\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 < \infty$$

by the previous corollary

$$y = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i \in \mathcal{H}.$$

exists. This implies that $(y-x) \perp e_j$, $\forall j$.

$$\begin{aligned}
\langle y-x, e_j \rangle &= \langle \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i - x, e_j \rangle \\
&= \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle e_i, e_j \rangle - \langle x, e_j \rangle \\
&= \langle x, e_j \rangle - \langle x, e_j \rangle = 0.
\end{aligned}$$

by a previous Lemma, $y = x$, thus $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ ■

Corollary: All separable Hilbert space has an orthonormal base.

Examples:

I. The system $\{1/\sqrt{2\pi}\, e^{int}\}_{n=-\infty}^{\infty}$ is an orthonormal base in $L_2([-\pi, \pi])$.

II. In a similar way $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(nt)}{\sqrt{\pi}}, \frac{\sin(nt)}{\sqrt{\pi}} \right\}_{n \in \mathbb{N}}$ an orthonormal base of $L_2([-\pi, \pi])$

Theorem: Let $\{e_i\}_{i \geq 1}$ be an orthonormal system. So $\{e_i\}_{i \geq 1}$ is a base in $\mathcal{H}$, if and only if for all $x \in \mathcal{H}$.

$$\|x\|^2 = \sum_{i \geq 1} |\langle x, e_i \rangle|^2$$

Proof: if $x = \sum_{i \geq 1} \langle x, e_i \rangle e_i$, then

$$\begin{aligned}
\|x\|^2 = \langle x, x \rangle &= \langle \sum_{i \geq 1} \langle x, e_i \rangle e_i, \sum_{j \geq 1} \langle x, e_j \rangle e_j \rangle \\
&= \sum_{\substack{i \geq 1 \\ j \geq 1}} \langle x, e_i \rangle \overline{\langle x, e_j \rangle} \langle e_i, e_j \rangle \\
&= \sum_{i \geq 1} \langle x, e_i \rangle \overline{\langle x, e_i \rangle} = \sum_{i \geq 1} |\langle x, e_i \rangle|^2
\end{aligned}$$

now, let

$$\|x-\sum_{i=1}^{n}\langle x,e_i\rangle e_i\|^2$$

$$=\langle\sum_{i=1}\langle x,e_i\rangle e_i-\sum_{i=1}^{n}\langle x,e_i\rangle e_i,\sum_{j=1}^{n}\langle x,e_j\rangle e_j-\sum_{j=1}^{n}\langle x,e_j\rangle e_j\rangle$$

$$=\langle\sum_{i=1}\langle x,e_i\rangle e_i,\sum_{j=1}\langle x,e_j\rangle e_j\rangle-\sum_{\substack{i=1\\j=1}}\overline{\langle x,e_i\rangle}\langle x,e_j\rangle\langle e_i,e_j\rangle$$

$$+\sum_{\substack{i\geq 1\\j\geq 1}}\langle x,e_i\rangle\overline{\langle x,e_j\rangle}\langle e_i,e_j\rangle-\sum_{\substack{i\geq 1\\j\geq 1}}\langle x,e_i\rangle\overline{\langle x,e_j\rangle}\langle e_i,e_j\rangle$$

$$=\|x\|^2-\sum_{i=1}^{n}|\langle x,e_i\rangle|^2\longrightarrow 0 \quad \text{if} \quad n\longrightarrow\infty.$$

Since we haved used

$$\|x\|^2=\sum_{i=1}^{\infty}|\langle x,e_i\rangle|^2$$

then

$$x=\sum_{i=1}^{\infty}\langle x,e_i\rangle e_i.$$

$\blacksquare$

Theorem: Any two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ separables and of finite dimension, are isometric equivalents i.e., there is an isomorphism $T:\mathcal{H}_1\longrightarrow\mathcal{H}_2$, such that $\|Tx\|=\|x\|$ for all $x\in\mathcal{H}_1$ and also

$$\langle Tx,Ty\rangle_{\mathcal{H}_2}=\langle x,y\rangle_{\mathcal{H}_1}$$

Proof: Is enough with construct T for a given $\mathcal{H}$ and l_2 instead of $\mathcal{H}_1$ and $\mathcal{H}_2$.

let an orthonormal base $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$. For all $x\in\mathcal{H}$.

$$x=\sum_{i=1}^{\infty}\langle x_i,f_i\rangle f_i$$

and

$$\|x\|^2=\sum_{i=1}^{\infty}|\langle x_i,f_i\rangle|^2$$

let $\{e_i\}_{i=1}^{\infty}$ a base of l_2

$$Tx:=\sum_{i=1}^{\infty}\langle x_i,f_i\rangle e_i\in l_2$$

$$\|Tx\|^2=\sum_{i=1}^{\infty}|\langle x_i,f_i\rangle|^2$$

therefore,

$$\|Tx\|^2 = \langle \sum_{i=1}^{\infty} \langle x_i, f_i\rangle e_i, \sum_{j=1}^{\infty} \langle x_j, f_j\rangle e_j \rangle$$

$$= \sum_{\substack{i=1\\ j=1}}^{\infty} \langle x_i, f_i\rangle \overline{\langle x_j, f_j\rangle} \langle e_i, e_j\rangle$$

$$= \sum_{i=1}^{\infty} |\langle x_i, f_i\rangle|^2 = \|x\|^2$$

T is surjective and also satisfies

$$\langle Tx, Ty\rangle_{l_2} = \langle x, y\rangle_H$$

$$\langle Tx, Ty\rangle = \langle \sum_{i=1}^{\infty} \langle x_i, f_i\rangle e_i, \sum_{j=1}^{\infty} \langle y_j, f_j\rangle e_j \rangle$$

$$= \sum_{\substack{i=1\\ j=1}}^{\infty} \langle x_i, f_i\rangle \overline{\langle y_j, f_j\rangle} \langle e_i, e_j\rangle$$

$$= \sum_{i=1}^{\infty} \langle x_i, f_i\rangle \langle y_i, f_i\rangle = \langle x, y\rangle$$

let $I = [a,b]$, and let's denote $L_2(I^2)$ the completion of the space of the functions square integrables. in two variables, with the norm

$$\|f(t,\tau)\| = \sqrt{\int_I \int_I |f(t,\tau)|^2 \, dt \, d\tau}$$