Radiation Red Shift

Information about $a(t)$ comes throught the observation of shifts in frequency of light emitted by distant sources.

Consider an electromagnetic field wave traveling to us along r-direction (θ, ϕ fixed).

$$0 = ds^2 = dt^2 - a^2(t)\,\frac{dr^2}{1-Kr^2}$$

the wave leaves a galaxy located at $(t_1, r_1, \theta, \phi_1)$. Then it will reach us at a time t_0 given by

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{r_1}^{0} \frac{dr}{\sqrt{1-Kr^2}} = \begin{cases} \sin^{-1}(r_1) & ,K=1 \\ r_1 & ,K=0 \\ \sinh^{-1}(r_1) & ,K=-1 \end{cases} =: f(r_1).$$

Next wave leaves the galaxy at $t_1 + \delta t_1$, so

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = f(r_1)$$

therefore,

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)}$$

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} - \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = 0$$

and,

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = 0 \longleftrightarrow \frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

→ $a(t)$ changes very little for a light signal.

Frequency:

$$\frac{\nu_0}{\nu_1} = \frac{\delta t_1}{\delta t_0} = \frac{a(t_1)}{a(t_0)} = \frac{\lambda_1}{\lambda_0} \quad ; \quad \lambda := \text{Wavelet.}$$

Introduce a "red-shift" parameter

$$z := \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\lambda_0}{\lambda_1} - 1 = \frac{a(t_0)}{a(t_1)} - 1.$$

red-shift if $\lambda_0 > \lambda_1 \longrightarrow z>0 \longrightarrow a(t_0) > a(t_1)$ Expansion

No-shift if $\lambda_0 = \lambda_1 \longrightarrow z=0 \longrightarrow a(t_0) = a(t_1)$ Steady

Blue-shift if $\lambda_0 < \lambda_1 \longrightarrow z<0 \longrightarrow a(t_0) < a(t_1)$ Contraction

$$\rho = \frac{C}{a(t)^{3(1+\omega)}}$$

Dust $\omega = 0$; $\rho = \frac{C}{a^3}$; $\rho_a^3 = C$

Radiation $\omega = \frac{1}{3}$; $\rho = \frac{C}{a^4}$; $\rho_a^4 = C$

$(\rho_a^3)_a = C.$

Flat universes $(K=0)$

$$F \longmapsto \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho$$

$$S \longmapsto P = \omega\rho \ ; \quad \omega = \text{constant} \longrightarrow \rho = \frac{C}{a(t)^{3(1+\omega)}}$$

$$E \longmapsto \frac{\ddot{a}}{a} + \frac{4\pi}{3}\rho = -4\pi P = -4\pi\omega\rho$$

$$\frac{\ddot{a}}{a} = -4\pi\rho\left(\omega + \frac{1}{3}\right)$$

$$\dot{a}^2 = \frac{8\pi}{3}\rho a^2 = \frac{8\pi}{3} C a^{2-3(1+\omega)}$$

$$\gamma := -\left(\frac{2-3(1+\omega)}{2}\right) \qquad 1+\gamma = 1 - 1 + \frac{3}{2}(1+\omega)$$

$$\dot{a} = \pm \tilde{C} a^{[2-3(1+\omega)]/2} = \pm \tilde{C} a^{-\gamma} \qquad = \frac{3}{2}(1+\omega)$$

$$\left.\begin{aligned} a^\gamma da &= \pm \tilde{C} dt \\ \frac{a^{\gamma+1}}{\gamma+1} &= \pm \tilde{C} t \end{aligned}\right\} \gamma \neq -1.$$

$$\longrightarrow a(t) \propto t^{1/\gamma+1}$$

$$\longrightarrow a \propto t^{2/3(1+\omega)^{-1}}$$

choose,

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3 \, 1/(1+\omega)} \quad ; \text{ if } \gamma \neq -1.$$

$$a(t) \propto e^{\pm Ht} \quad ; \text{ if } \gamma \neq -1.$$

$$H^2 = \frac{8\pi C}{3} = \text{Constant}.$$

- $\omega \neq -1$ as $t \to 0$, then $a \to 0$ Big Bang Solution!

$$\frac{da}{dt} dt = da$$

$$t_0 := \int_{BB}^{\text{Present}} dt = \int_{BB}^{0} \frac{da}{\dot{a}} = \int_{BB}^{0} \frac{da}{aH(a)}$$

$$H^2 = \left(\frac{8\pi}{3} C\right) \frac{1}{a^{3(1+\omega)}} \quad ; \quad aH(a) = \eta a^{1-3/2(1+\omega)}$$

$$t_0 = \eta \int_{BB}^{0} a^{-1+3/2(1+\omega)} da = \frac{\eta \, a^{3/2(1+\omega)}}{\frac{3}{2}(1+\omega)} \Bigg|_{BB}^{0}$$

$$t_0 = \frac{\eta \, a_0^{3/2(1+\omega)}}{\frac{3}{2}(1+\omega)} = \frac{2}{3(H\omega)} \frac{1}{H_0}$$ age of the universe

For $\omega = -1$ as $t \to 0$, then $a(t) \longrightarrow 1$ No Big bang solution

$$t_0 := \int \frac{da}{aH(a)} = \frac{1}{H} \ln(a_0)$$

Flat universes: ($\omega = 0$ matter dominated).

friedmann.

$$H^2 = \frac{\Lambda}{3} - \frac{K}{a^2} + 8\frac{\pi}{3}\rho.$$

$$\dot{a}^2 = \frac{C}{a} + \frac{1}{3}\Lambda a^2 - \cancel{K}^{\,0} \quad ; \quad \rho = \frac{C}{a^3}$$

Assume $\Lambda > 0$, and $u := \frac{2\Lambda}{3C} a^3$

$$\dot{u} = \frac{2\Lambda}{c} a^2 \dot{a}$$

$$\frac{\dot{a}^2 c^2}{4\Lambda^2 a^4} = \frac{c}{a}\left(1 + \frac{\Lambda}{3c} a^3\right)$$

$$\dot{u}^2 = 2\Lambda\left(\frac{2\Lambda a^3}{c}\right)\left(1 + \frac{u}{2}\right) = 2\Lambda(3u)\left(1 + \frac{u}{2}\right)$$

$$\dot{u}^2 = 3\Lambda(2u + u^2)$$

$$\dot{u} = \pm(3\Lambda)^{1/2}\sqrt{2u + u^2}$$

Assume "+" sign (Expansion) & Big Bang Solution ($t \to 0, a \to 0$).

$$\int_0^u \frac{du}{\sqrt{2u+u^2}} = (3\Lambda)^{1/2} \int_0^t dt' = (3\Lambda)^{1/2} t$$

$$t = \frac{1}{(3\Lambda)^{1/2}} \int_0^u \frac{du}{\sqrt{2u+u^2}} = \frac{1}{(3\Lambda)^{1/2}} \int_0^u \frac{du}{[(u+1)^2 - 1]^{1/2}}$$

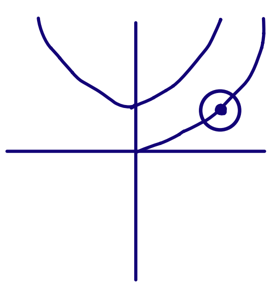
$$v := u + 1$$

$$t = \frac{1}{(3\Lambda)^{1/2}} \int_1^{u+1} \frac{dv}{(v^2+1)^{1/2}} \qquad v := \cosh(\omega)$$

$$t = \frac{1}{(3\Lambda)^{1/2}} \int_0^{\operatorname{arcosh}(v)} \frac{\sinh(\omega)}{[\sinh^2(\omega)]^{1/2}} d\omega$$

$$t = \frac{1}{(3\Lambda)^{1/2}} \int_0^{\operatorname{arcosh}(v)} d\omega = \frac{1}{(3\Lambda)^{1/2}} \operatorname{arcosh}(v)$$

$$= \frac{1}{(3\Lambda)^{1/2}} \operatorname{arcosh}(u+1)$$



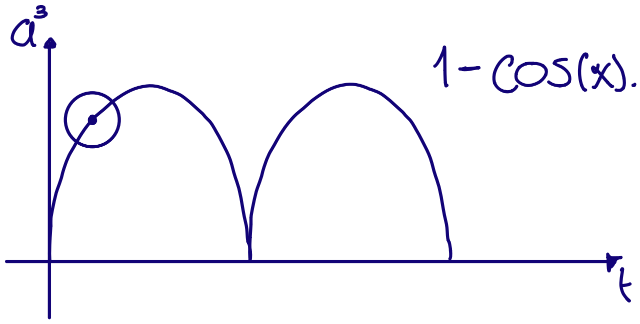
$$u + 1 = \cosh((3\Lambda)^{1/2} t)$$

$$u = \cosh((3\Lambda)^{1/2} t) - 1$$

$$a^3 = \frac{3c}{2\Lambda}\left[\cosh((3\Lambda)^{1/2} t) - 1\right]$$

For $\Lambda<0$

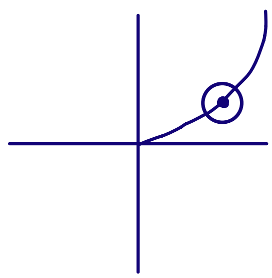
$$a^3 = \frac{3c}{2\|\Lambda\|}\left[1-\cos\left((3\|\Lambda\|)^{1/2}\right)t\right]$$



For $\Lambda=0$

take $\cosh(x) \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

$$a^3 = \frac{9}{4}ct^2 \longrightarrow \text{Einstein-de Sitter Universe}$$



Models with cosmological vanishing constant:

$$\dot{a}^2 = \frac{c}{a} - k, \quad \text{consider } k=1.$$

Define $u^2 := \frac{a}{c}$

$$2u\dot{u} = \frac{\dot{a}}{c}$$

$$\dot{a}^2 = 4c^2u^2\dot{u}^2 = \frac{1}{u^2} - 1$$

$$\dot{u}^2 = \frac{1}{4c^2u^2}\left(\frac{1}{u^2} - 1\right)$$

$$\dot{u} = \pm\frac{1}{2cu}\left(\frac{1}{u^2} - 1\right)^{1/2}$$

$$2c\int_0^u \frac{u^2\,du}{(1-u^2)^{1/2}} = t$$

$$u =: \sin(\theta)$$

$$t = 2c\int_0^{\arcsin(u)} \frac{\sin^2(\theta)\cancel{\cos(\theta)}\,d\theta}{\cancel{\cos(\theta)}}$$

$$= 2c\int_0^{\arcsin(u)} \sin^2(\theta)\,d\theta$$

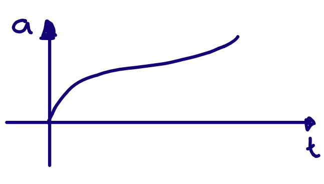
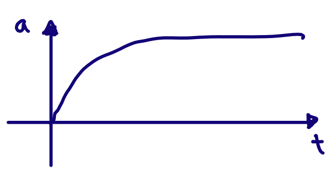
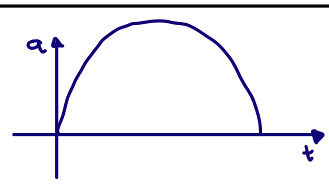
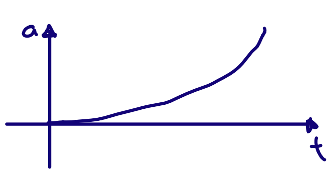
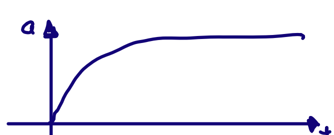
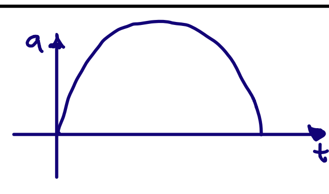
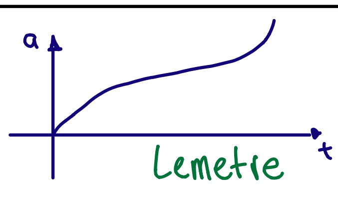
$$t = c\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\arcsin(u)}$$

$$t = c\left[\arcsin(u) - u\,(1-u^2)^{1/2}\right]$$

$$t = c\left[\arcsin\left(\frac{a}{c}\right)^{1/2} - \left(\frac{a}{c}\right)^{1/2}\left(1-\frac{a}{c}\right)^{1/2}\right]$$

Analogously, for $K=-1$

$$t = c\left[\left(\frac{a}{c}\right)^{1/2}\left(1+\frac{a}{c}\right)^{1/2} - \operatorname{arcsinh}\left(\frac{a}{c}\right)^{1/2}\right]$$

	$\Lambda > 0$	$\Lambda = 0$	$\Lambda < 0$
$K=-1$			
$K=0$		 Einstein-de Sitter	
$K=1$	 Lemetre