

Symmetric and antisymmetric tensors.

Let t_{ab} be a symmetric tensor ($t_{ab} = t_{ba}$). Its spinorial equivalent

$$t_{A\dot{A}B\dot{B}} = t_{B\dot{B}A\dot{A}}$$

But, $t_{A\dot{A}B\dot{B}}$ not necessarily coincides with $t_{B\dot{A}A\dot{B}}$, that is equal to $t_{A\dot{B}B\dot{A}}$.

Using antisymmetry.

$$t_{A\dot{A}B\dot{B}} - t_{B\dot{A}A\dot{B}} = \epsilon_{AB} t^R_{\dot{A}\dot{B}}$$

then

$$t^R_{\dot{A}\dot{B}} = t_{R\dot{B}}{}^{\dot{A}} = -t^R_{\dot{B}\dot{A}}$$

and then $t^R_{\dot{A}\dot{B}}$ is proportional a $\epsilon_{\dot{A}\dot{B}}$, then.

$$t_{A\dot{A}B\dot{B}} - t_{B\dot{A}A\dot{B}} = \epsilon_{AB} \frac{1}{2} t^R_{\dot{A}\dot{B}} \epsilon_{\dot{A}\dot{B}}$$

$$= -\frac{1}{2} t^S_{AB} \epsilon_{AB} \epsilon_{\dot{A}\dot{B}}$$

which implies $t_{A\dot{A}B\dot{B}} = t_{B\dot{A}A\dot{B}} = t_{A\dot{B}B\dot{A}}$ if and only if, $t_{A\dot{A}B\dot{B}}$ is the equivalent spinor to a symmetric tensor without trace.

The equivalent tensor into symmetric tensor $t_{ab} = -t_{ba}$, satisfies $t_{A\dot{A}B\dot{B}} = -t_{B\dot{B}A\dot{A}}$, thus.

$$\begin{aligned} t_{A\dot{A}B\dot{B}} &= \frac{1}{2} (t_{A\dot{A}B\dot{B}} + t_{B\dot{A}A\dot{B}}) + \frac{1}{2} (t_{A\dot{A}B\dot{B}} - t_{B\dot{A}A\dot{B}}) \\ &= \frac{1}{2} (t_{A\dot{A}B\dot{B}} - t_{A\dot{B}B\dot{A}}) + \frac{1}{2} (t_{A\dot{A}B\dot{B}} - t_{B\dot{A}A\dot{B}}) \\ &= \frac{1}{2} t_A{}^{\dot{R}}{}_{\dot{B}\dot{B}} \epsilon_{\dot{A}\dot{B}} + \frac{1}{2} t^R_{\dot{A}\dot{B}} \epsilon_{AB} \\ &= T_{AB} \epsilon_{\dot{A}\dot{B}} + T_{\dot{A}\dot{B}} \epsilon_{AB}. \end{aligned}$$

with $T_{AB} = \frac{1}{2} t_A{}^{\dot{R}}{}_{\dot{B}\dot{B}}$, $T_{\dot{A}\dot{B}} = \frac{1}{2} t^R_{\dot{A}\dot{B}}$

Due to that $t_{A\dot{A}B\dot{B}} = -t_{B\dot{B}A\dot{A}}$ such that $\psi^A \phi_A = -\psi_A \phi^A$; $T_{AB}, T_{\dot{A}\dot{B}}$ are symmetric in its indices such that $t_B{}^{\dot{R}}{}_{\dot{A}\dot{B}} = -t_{\dot{A}\dot{B}}{}^{\dot{R}}$.

In particular the spinorial equivalent of an antisymmetric product.

$$\begin{aligned} \nabla_{\dot{A}\dot{B}} W_{\dot{B}\dot{B}} - \nabla_{\dot{B}\dot{B}} W_{\dot{A}\dot{A}} &= \frac{1}{2} (\nabla_A^{\dot{B}} W_{B\dot{B}} - \nabla_{B\dot{B}} W_A^{\dot{B}}) \epsilon_{\dot{A}\dot{B}} \\ &\quad + \frac{1}{2} (\nabla_{\dot{A}}^{\dot{B}} W_{B\dot{B}} - \nabla_{B\dot{B}} W_{\dot{A}}^{\dot{B}}) \epsilon_{AB} \\ &= V_A^{\dot{B}} (W_B)_{\dot{B}} \epsilon_{\dot{A}\dot{B}} + V^{\dot{B}} (\dot{A} (W_B)_{\dot{B}}) \epsilon_{AB}. \end{aligned}$$

where $\xi_{(AB)} = \frac{1}{2} (\xi_{AB} + \xi_{BA})$.

If t_{abc} , totally antisymmetric ($t_{abc} = -t_{acb} = t_{cba} \dots$) is antisymmetric in its two last indices.

$$t_{A\dot{B}B\dot{C}C} = \mu_{A\dot{A}B\dot{C}} \epsilon_{B\dot{C}} + \mu_{A\dot{A}\dot{B}\dot{C}} \epsilon_{BC} \quad (*)$$

with

$$\mu_{A\dot{A}B\dot{C}} = \mu_{A\dot{A}(B\dot{C})} \quad \mu_{A\dot{A}\dot{B}\dot{C}} = \mu_{A\dot{A}(\dot{B}\dot{C})}$$

using the antisymmetry in his pair of the indices.

$$\mu_{A\dot{A}B\dot{C}} \epsilon_{B\dot{C}} + \mu_{A\dot{A}\dot{B}\dot{C}} \epsilon_{BC} = -\mu_{B\dot{B}A\dot{C}} \epsilon_{A\dot{C}} - \mu_{B\dot{B}\dot{A}\dot{C}} \epsilon_{AC}.$$

Contracting with $\epsilon^{\dot{B}\dot{C}}$

$$\begin{aligned} \Rightarrow 2\mu_{A\dot{A}\dot{B}\dot{C}} &= -\mu_{B\dot{A}\dot{C}}^{\dot{B}} \epsilon_{\dot{A}\dot{B}} - \mu_{B\dot{A}\dot{B}}^{\dot{B}} \epsilon_{AC} \\ &= -\mu_{B\dot{A}\dot{C}} - \mu_{B\dot{A}\dot{B}}^{\dot{B}} \epsilon_{AC} \\ &= -\mu_{A\dot{A}B\dot{C}} + \mu_{A\dot{A}\dot{B}\dot{C}} - \mu_{B\dot{A}A\dot{C}} - \mu_{B\dot{A}\dot{B}}^{\dot{B}} \epsilon_{AC} \\ \Rightarrow 3\mu_{A\dot{A}\dot{B}\dot{C}} &= \mu_{\dot{A}\dot{B}\dot{C}}^R \epsilon_{AB} - \mu_{B\dot{A}\dot{B}}^{\dot{B}} \epsilon_{\dot{A}\dot{C}} \end{aligned}$$

Contracting with ϵ^{BC}

$$0 = -\mu_{\dot{A}\dot{B}\dot{C}}^R - \mu_{A\dot{B}}^{\dot{B}} \epsilon_{\dot{A}\dot{C}}$$

and defining

$$t_{B\dot{A}} = \frac{2}{3} \mu_{\dot{A}\dot{B}\dot{C}}^R$$

$$\Rightarrow 2\mu_{A\dot{A}B\dot{C}} = t_{C\dot{A}} \epsilon_{AB} + t_{B\dot{A}} \epsilon_{AC}$$

and changing in $(*)$

$$t_{A\bar{B}B\bar{C}\bar{C}} = t_{B\bar{A}} \epsilon_{AC} \epsilon_{\bar{B}\bar{C}} - t_{\bar{A}\bar{B}} \epsilon_{BC} \epsilon_{\bar{A}\bar{C}}$$

If t_{abcd} totally antisymmetric.

$$t_{A\bar{A}B\bar{B}C\bar{C}D\bar{D}} = \phi_{B\bar{A}D\bar{D}} \epsilon_{AC} \epsilon_{\bar{B}\bar{C}} - \phi_{A\bar{B}D\bar{D}} \epsilon_{\bar{A}\bar{C}} \epsilon_{BC} \quad (\text{**})$$

$$\Rightarrow \phi_{B\bar{A}D\bar{D}} \epsilon_{AC} \epsilon_{\bar{B}\bar{C}} - \phi_{A\bar{B}D\bar{D}} \epsilon_{\bar{A}\bar{C}} \epsilon_{BC} = -\phi_{B\bar{A}C\bar{C}} \epsilon_{AD} \epsilon_{\bar{B}\bar{D}} + \phi_{A\bar{B}C\bar{C}} \epsilon_{\bar{A}\bar{D}} \epsilon_{BD}$$

Contracting with ϵ^{AC} .

$$2\phi_{B\bar{A}D\bar{D}} \epsilon_{\bar{B}\bar{C}} - \phi_{B\bar{B}D\bar{D}} \epsilon_{\bar{A}\bar{C}} = -\phi_{B\bar{A}D\bar{C}} \epsilon_{\bar{B}\bar{D}} + \phi^R_{\bar{B}\bar{C}\bar{D}} \epsilon_{\bar{A}\bar{D}} \epsilon_{BD}$$

and contracting with $\epsilon^{\bar{A}\bar{C}}$

$$0 = -\phi^R_{B\bar{D}\bar{R}} \epsilon_{\bar{B}\bar{D}} + \phi^R_{\bar{B}\bar{R}\bar{D}} \epsilon_{BD}$$

$$\Rightarrow \phi^R_{B\bar{D}\bar{R}} = 2a \epsilon_{BD}, \quad \phi^R_{\bar{B}\bar{R}\bar{D}} = 2a \epsilon_{\bar{B}\bar{D}}$$

for some scalar a .

In the other hand, contracting with $\epsilon^{\bar{B}\bar{D}}$

$$4\phi_{B\bar{A}\bar{D}\bar{C}} = \phi^R_{B\bar{D}\bar{B}} \epsilon_{\bar{A}\bar{C}} + \phi^R_{\bar{B}\bar{C}\bar{B}} \epsilon_{\bar{A}\bar{C}} \epsilon_{BD}$$

$$= 2a(\epsilon_{BD} \epsilon_{AC} + \epsilon_{\bar{A}\bar{C}} \epsilon_{BD})$$

$$= 4a \epsilon_{BD} \epsilon_{AC}$$

changing in (**)

$$t_{A\bar{A}B\bar{B}C\bar{C}D\bar{D}} = a(\epsilon_{BD} \epsilon_{\bar{A}\bar{D}} \epsilon_{AC} \epsilon_{\bar{B}\bar{C}} - \epsilon_{AD} \epsilon_{\bar{A}\bar{C}} \epsilon_{BC} \epsilon_{\bar{B}\bar{D}})$$

The equivalent tensor to the Levi-Civita's symbol ϵ_{abcd} , is

$\epsilon_{A\bar{A}B\bar{B}C\bar{C}D\bar{D}}$, $\epsilon_{1234}=1$, must be proportional to.

$$\epsilon_{AC} \epsilon_{BD} \epsilon_{\bar{A}\bar{D}} \epsilon_{\bar{B}\bar{C}} - \epsilon_{AD} \epsilon_{BC} \epsilon_{\bar{A}\bar{C}} \epsilon_{\bar{B}\bar{D}}$$

as $\epsilon^{abcd} \epsilon_{abcd} = (-1)^q 24$ where q is the signature

$$\epsilon_{A\bar{A}B\bar{B}C\bar{C}D\bar{D}} = i^q (\epsilon_{AC} \epsilon_{BD} \epsilon_{\bar{A}\bar{D}} \epsilon_{\bar{B}\bar{C}} - \epsilon_{AD} \epsilon_{BC} \epsilon_{\bar{A}\bar{C}} \epsilon_{\bar{B}\bar{D}})$$

The spinorial equivalent of the dual of an antisymmetric tensor, defined.

$$t_{ab} = \frac{1}{2} \epsilon_{abcd} t^{cd}$$

$$* t_{A\bar{B}\bar{B}} = i^q (-T_{AB} \epsilon_{\bar{A}\bar{B}} + T_{\bar{A}\bar{B}} \epsilon_{AB})$$

Parte
Autodual.

Parte
Dual.

$$* t_{ab}^{ab} = -i^q (T_{AB} \epsilon_{\bar{A}\bar{B}} - T_{\bar{A}\bar{B}} \epsilon_{AB}) (T^{AB} \epsilon^{\bar{A}\bar{B}} - T^{\bar{A}\bar{B}} \epsilon^{AB}) \\ = -2i^q (T_{AB} T^{AB} - T_{\bar{A}\bar{B}} T^{\bar{A}\bar{B}})$$

A bivector its called simple if there are v_a, w_a , such that

$$t_{ab} = v_a w_b - v_b w_a$$

Then, for a simple bivector.

$$* t_{ab}^{ab} = \frac{1}{2} \epsilon_{abcd} t^{cd} t^{ab} \\ = 2 \epsilon_{abcd} v^c w^d v^a w^b = 0$$

Proposition: If the bivector t_{ab} is simple

$$T_{AB} T^{AB} = T_{\bar{A}\bar{B}} T^{\bar{A}\bar{B}}$$