Proof: To prove the inequality, let

$$C_i = \frac{|a_i|}{\left(\sum |a_j|^p\right)^{1/p}} \quad , \quad d_i = \frac{|b_i|}{\left(\sum |b_j|^q\right)^{1/q}}$$

then, $\sum_i C_i^p = 1$, $\sum_i d_i^q = 1$ and we have to prove that

$$\sum C_i d_i \leq 1$$

Let's see that $C_i d_i \leq \frac{1}{p} C_i^p + \frac{1}{q} d_i^q$

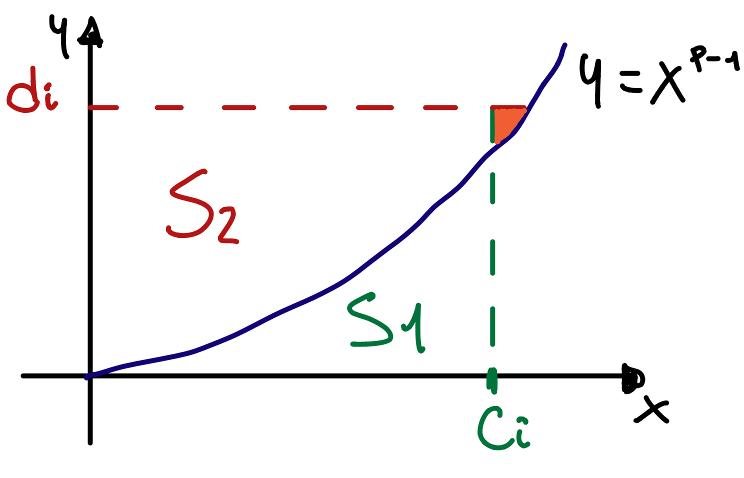
this is fulfilled for the following argument:

let the function $y = x^{p-1}$, integrating with respect to x from 0 to C_i, then the area S_1, that is the integral is

$$S_1 = \frac{1}{p} C_i^p$$

In a similar way, integrating the inverse $x = y^{1/(1-p)} = y^{q-1}$ with respect to y, from 0 to d_i. We have that the sum $S_1 + S_2$ always is greater than the product of $C_i d_i$. Then

$$C_i d_i \leq \frac{1}{p} C_i^q + \frac{1}{q} d_i^q$$



Summing with respect to i, we get

$$\sum_i C_i d_i \leq \frac{1}{p} + \frac{1}{q} \leq 1$$

Since $\sum_i C_i^p = \sum_i d_i^q = 1$ ■

We can define in a formal way, $\left(\sum_k |b_k|^q\right)^{1/q}$ as $\sup|b_k|$ for $q=\infty$, and thus we have the inequality for $1\leq p\leq\infty$

Theorem: Let $1<p<\infty$ and q, such that $1/p+1/q=1$. The for all function $f,g\in[a,b]$ such that the integrals

$$\int_a^b |f(t)|^p dt,\ \int_a^b |g(t)|^q dt \text{ and } \int_a^b |f(t)g(t)|\,dt.$$

exist, then

$$\int_a^b |f(t)g(t)|dt \leq \left(\int_a^b |f(t)|^p dt\right)^{1/p}\left(\int_a^b |g(t)|^q dt\right)^{1/q}$$

let's see that if the integral

$$\int_a^b f(t)g(t)dt$$

exists, then the inequality

$$\left|\int_a^b f(t)g(t)\,dt\right| \leq \int_a^b |f(t)g(t)|dt$$

it follows that

$$\left|\int_a^b f(t)g(t)\,dt\right| \leq \left(\int_a^b |f(t)|^p dt\right)^{1/p}\left(\int_a^b |g(t)|^q dt\right)^{1/q}$$

Proof: Let's consider a partision of the interval $[a,b]$, in n equal subintervals with $a=x_0<x_1<\ldots<x_n=b$.

Let $\Delta x=\frac{(b-a)}{n}$, we have

$$\begin{aligned}\sum_{i=1}^n |f(x_i)g(x_i)|\Delta x &= \sum_{i=1}^n |f(x_i)g(x_i)|(\Delta x)^{1/p+1/q}\\ &\leq \sum_{i=1}^n (|f(x_i)|^p\Delta x)^{1/p}(|g(x_i)|^q\Delta x)^{1/q}\\ &\leq \left(\sum_{i=1}^n |f(x_i)|^p\Delta x\right)^{1/p}\left(\sum_{i=1}^n |g(x_i)|^q\Delta x\right)^{1/q}\end{aligned}$$

Combining

$$\sum_{i=1}^{n} |f(x_i)g(x_i)|\Delta x \leq \left(\sum_{i=1}^{n} |f(x_i)|^p \Delta x\right)^{1/p} \left(\sum_{i=1}^{n} |g(x_i)|^q \Delta x\right)^{1/q}$$

If $n \to \infty$, we get integrals.

$\blacksquare$

We can use the Hölder inequality to prove the Minkowski inequality, that actually is the triangle inequality for ℓ_p.

$$(a_i) \in \ell_p, \quad \sum_{i=1}^{\infty} |a_i|^p < \infty$$

Theorem: For each succession $a=(a_i)$ and $b=(b_i)$ and $1 \leq p \leq \infty$, we have

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p$$

Proof: For $p=1$ is the triangle inequality for the absolute value in $\mathbb{R}$ (or $\mathbb{C}$).

If $p=\infty$, is the property of the sum for the supremum.

Now, the cases $1 < p < \infty$

$$\|a+b\|_p^p = \sum_k |a_k+b_k|^p \leq \sum_k (|a_k+b_k|)^{p-1}(|a_k|+|b_k|)$$

$$\leq \sum_k (|a_k+b_k|)^{p-1}|a_k| + \sum_k (|a_k+b_k|)^{p-1}|b_k|$$

we apply the Hölder's inequality for each sum and we use the fact that $(p-1)q = p$.

$$\|a+b\|_p^p \leq \sum_k (|a_k+b_k|^p)^{1/q} \left(\sum_k |a_k|^p\right)^{1/p} + \sum_k (|a_k+b_k|^p)^{1/q} \left(\sum_k |b_k|^p\right)^{1/p}$$

Thus

$$\|a+b\|_p^p \leq \sum_k (|a_k+b_k|^p)^{1/p} (\|a\|_p + \|b\|_p)$$

Dividing both sides by $\sum_k (|a_k+b_k|^p)^{1/q}$, we get

$$\|a+b\|_p = \|a\|_p + \|b\|_p$$

Since, $1 - 1/q = 1/p$.

$\blacksquare$

Theorem: For all $1<p<\infty$ and all functions f,g defined in an interval $[a,b]$ for which the integrals

$$\int_a^b |f(x)|^p dx \quad \text{and}$$

exist, the integral

$$\int_a^b |f(x)+g(x)|^p dx$$

exists and satisfies

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}.$$

Proof: Let's consider a partition of the interval $[a,b]$ in n equal subintervals, with length

$$\Delta x = \frac{b-a}{n},$$

using the Minkowski's inequality

$$\left(\sum_{i=1}^{n} |f(x_i)+g(x_i)|^p \Delta x\right)^{1/p} \leq \left(\sum_{i=1}^{n} |f(x_i)|^p \Delta x\right)^{1/p} + \left(\sum_{i=1}^{n} |g(x_i)|^p \Delta x\right)^{1/p}$$

if $n \longrightarrow \infty$ we get the result. $\blacksquare$

Topology notation

Definition: We say that a succession (x_n) of a normed space X converges if and only if $\|x_n - x\| \to 0$.

Definition: An open ball of radius $r>0$, with center in x_0 it is defined as the set

$$D_r(x_0) = \{x : \|x - x_0\| < r\}$$

Definition: A set $O \subseteq X$, it is said that is open if and only if for each $x \in O$, there is $r>0$, such that $D_r(x) \subseteq O$.

Definition: A set $F \subseteq X$, it is said that is closed, if for each succession $x_n \in F$, that converges to some $x \in X$, it follows that $x \in F$.

Lemma: If $\mathcal{O}$ is an open set, so $F=\mathcal{O}^c$ is closed. Reciprocally, if F is closed, $\mathcal{O}=F^c$ is open.

Proof: If $\mathcal{O}$ is open, $F=\mathcal{O}^c$. Let $x_n \in F$, $x_n \longrightarrow x$. Then $x \in F$ since if $x \notin F$, for some $r>0$. The open ball $D_r(x)$ is totally content in $\mathcal{O}$.

Also for some n large, $d(x_n, x) < r$, that implies $x_n \in \mathcal{O}$ and not in F, this is a contradiction.

For the other implication. Let F closed, and $\mathcal{O}=F^c$ we prove that for each $x \in \mathcal{O}$ there is $\varepsilon>0$, such that $D_\varepsilon(x) \subseteq \mathcal{O}$. If it not, for each decrecent succession to cero. $\varepsilon_n>0$, there would be a $x_n \in F$, such that $d(x_n, x) < \varepsilon_n$, which would imply that $x_n \to x$. Then $x \in F$, since is closed.

$\blacksquare$

Let's remember that the union of open sets is open and the intersection of closed sets is closed. X, ϕ are closed and open.

The closure $\bar{A}$ of a set A, is the set of all $x \in X$ such that there is a succession $x_n \in A$, with $x_n \longrightarrow x$. We say that the set $A \subseteq X$. is dense if $\bar{A}=X$.

by the other way, a consequence of the triangle inequality, the norm is a continuous function i.e., if $x_n \to x$ in $(X, \|\cdot\|)$, then $\|x_n\| \longrightarrow \|x\|$, for verifying we use, $\forall x,y \in X$

$$\big|\|x\|-\|y\|\big| \leq \|x-y\|$$

Definition: for both points x, y the set $\{\lambda x+(1-\lambda)y\}$ for $0 \leq \lambda \leq 1$, is the line segment that join both points and we call it interval $I[x,y]$.

Exercise: If $z \in I[x,y]$; $\|x-y\|=\|x-z\|+\|z-y\|$.

Definition: A subset M, of a linear space E, it is called convex if and only if, for whatever two points $x, y \in M$, the interval $I[x,y]$ it is contained in M.

If E is a normed space

$$D(E)=\{x : \|x\| \leq 1\}$$

it is call unitary ball in E, and it is convex

$$S(E) = \{ x : \|x\| < 1 \}$$

unitary sphere. (Prove)

Definition: A subset M, of a normed space X it is called bounded if there is $C>0$, such that for all $x \in M$, we have that $\|x\| < C$.