Hilbert Space's.

Definition: Let $\mathcal{H}$ a vectorial space on $\mathbb{C}$, together with a function of two variables

$$\langle x,y\rangle : H\times H \longrightarrow \mathbb{C}$$

with the following properties:

I. Linearity with respect to the first argument

$$\langle ax_1+bx_2,y\rangle = a\langle x_1,y\rangle + \langle x_2,y\rangle.$$

II. $\langle \overline{x,y}\rangle = \langle y,x\rangle$ the semilinearity in the second argument.

$$\langle x, ay_1+by_2\rangle = \bar{a}\langle x,y_1\rangle + \bar{b}\langle x,y_2\rangle$$

III. Non-negativity

$\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ if and only if $x=0$. Such function it is called inner product or scalar. The function $p(x)=\langle x,x\rangle^{1/2}$, is a norm and $p(x)=\|x\|$.

Example:

I. In $\mathbb{C}^n$, $\langle x,y\rangle = \sum_{i=1}^{n} a_i\bar{b}_i$ with $x=(a_i)_{i=1}^{n}$, $y=(b_i)_{i=1}^{n}$

II. In l_2, let $\langle x,y\rangle = \sum_{i=1}^{\infty} a_i\bar{b}_i$. by the Hölder's inequality

$$\left|\sum_{i=1}^{\infty} a_i\bar{b}_i\right| \leq \sqrt{\sum_{i=1}^{\infty}|a_i|^2}\sqrt{\sum_{i=1}^{\infty}|b_i|^2} < \infty$$

The product $\langle x,y\rangle$ is well defined in $l_2\times l_2$, also.

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|$$

III. In $C_2[a,b]$, let

$$\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt.$$

also, we have that $|\langle f,g\rangle| \leq \|f\|_2\|g\|_2$ by the Hölder's inequality for $p=q=2$.

Theorem: For all vectors $x, y \in H$, with inner product $\langle \cdot, \cdot \rangle$ is satisfied the Cauchy-Schwartz inequality.

$$|\langle x,y\rangle| \leq \langle x,x\rangle^{1/2} \langle y,y\rangle^{1/2}.$$

Proof: Remembering the notation $p(x) = \langle x,x\rangle^{1/2}$

$$0 \leq \langle x-\lambda x, x-\lambda y\rangle = p(x)^2 - 2\mathrm{Re}(\lambda\langle y,x\rangle) + |\lambda|^2 p(y)^2$$

If we assume that $\langle x,y\rangle \neq 0$, taking $\lambda = \dfrac{p(x)^2}{\langle y,x\rangle}$, we have.

$$0 \leq -p(x)^2 + \frac{p(x)^4 p(y)^2}{|\langle y,x\rangle|^2}$$

which implies the inequality $|\langle x,y\rangle| \leq p(x)p(y)$

Homework: $|\langle x,y\rangle| = p(x)p(y)$ if and only if $x = \lambda y$

Now, let's prove that $p(x) = \|x\|$ is a norm, actually $p(\lambda x) = |\lambda| p(x)$ and for the triangle inequality, we see that

$$\mathrm{Re}\langle x,y\rangle \leq |\langle x,y\rangle| \leq p(x)p(y).$$

and also

$$p(x+y)^2 = p(x)^2 + 2\mathrm{Re}(\langle x,y\rangle) + p(y)^2$$
$$\leq (p(x)+p(y))^2$$

finally

$$p(x+y) \leq p(x) + p(y).$$

Now, we will use $p(x)$ as $\|x\|$.

Then, H is a normed space with norm $\|x\|$, called Hilbertian Norm.

Definition: We say that H is a Hilbert space if it is a normed complete space, with this norm.

$\mathbb{C}^n, l_2$ are Hilbert spaces.

Let $x, y \in H$, $x \perp y$ if $\langle x,y\rangle = 0$.

I. The inner product is a continuous function with respect to both variables.

$$\langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle \quad \text{if} \quad x_n \to x, \quad y_n \to y \quad \text{since}$$

$$\langle x,y\rangle - \langle x_n, y_n\rangle = \langle x,y\rangle - \langle x, y_n\rangle + \langle x, y_n\rangle - \langle x_n, y_n\rangle$$

$$= \langle x, y-y_n\rangle + \langle x-x_n, y_n\rangle$$

Using Cauchy-Schartz

$$|\langle x,y\rangle - \langle x_n, y_n\rangle| \leq \|x\|\,\|y-y_n\| + \|x-x_n\|\,\|y_n\|$$

II. Parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

III. Pythagoras' theorem, if $x \perp y$.

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$\|x+y\|^2 = \langle x,x\rangle + \cancelto{0}{\langle x,y\rangle} + \cancelto{0}{\langle y,x\rangle} + \langle y,y\rangle$$

$$= \|x\|^2 + \|y\|^2$$

$$C_2[a,b] \longrightarrow L_2[a,b] \longrightarrow \text{Hilbert space}$$

$$\int_a^b f(t)\overline{g(t)}\,dt$$

Corollary: If $\{e_i\}_{i=1}^{\infty}$ orthogonal vectors by pairs and normalized i.e., $\|e_i\| = 1$, then

$$\left\|\sum_{i=1}^{n} \alpha_i e_i\right\| = \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2}$$

Proof: If $\{e_i\}_{i=1}^{\infty}$ is a set of orthonormal vectors i.e., are orthogonal by pairs, and normalized $\langle e_i, e_j\rangle = 1$, then.

$$\lim_{n\to\infty}\left\|\sum_{i=1}^{n} \alpha_i e_i\right\| = \left(\sum_{i=1}^{\infty} |\alpha_i|^2\right)^{1/2}.$$

Also $\left\|\sum_{i=n}^{m} \alpha_i e_i\right\| = \left(\sum_{i=n}^{m} |\alpha_i|^2\right)^{1/2} \to 0$ if $m > n \to \infty$

it means that $\sum_{i=1}^{m} \alpha_i e_i$ are Cauchy's. $\blacksquare$

Note: $\{x_i\}_{i\geq 1}$ both cases $\{x_i\}_{i=1}^{n}$ or $\{x_i\}_{i=1}^{\infty}$

Theorem: For any orthogonal system of vectors $\{e_i\}_{i\geq 1}$ in $\mathcal{H}$, and therefore for any $x \in \mathcal{H}$

$$\sum_{i\geq 1} |\langle x, e_i\rangle| \leq \|x\|^2$$

Bessel inequality.

Proof: Let

$$y_n = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$$

then

$$|\langle y_n, x\rangle| \leq \|x\|\|y_n\|, \quad \langle y_n, x\rangle = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$$

$$\|y_n\| = \left(\sum_{i=1}^{n} |\langle x, e_i\rangle|^2\right)^{1/2}$$

thus

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\| \left(\sum_{i=1}^{n} |\langle x, e_i\rangle|^2\right)^{1/2}$$

this implies that the inequality for a finite set $\{e_i\}_{i=1}^{n}$.
If the system is infinity, $n \to \infty$.

Corollary: for all $x \in \mathcal{H}$, and an orthogonal system $\{e_i\}_{i=1}^{\infty}$, there exists $y \in \mathcal{H}$ such that

$$y = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i$$

Examples:

I. In l_2, let the vectors $\{e_n = (0, 0, \ldots, \overbrace{1}^{n}, \ldots, 0, \ldots)\}_{n=1}^{\infty}$

II. In $L_2[-\pi, \pi]$, the vectors $\{1/\sqrt{2\pi}\, e^{int}\}_{n=-\infty}^{\infty}$

III. In $L_2[-\pi, \pi]$, the vectors $\{1/\sqrt{2\pi}, \sin(n\pi)/\sqrt{\pi}, \cos(n\pi)/\sqrt{\pi}\}_{n=1}^{\infty}$

Complete systems

A system $\{x_i\}_{i\geq 1}$ in a normed space $\underline{X}$, is called complete if the set generated by $\{x_i\}_{i\geq 1}$ or

$$\operatorname{span}\{x_i\}_{i\geq 1} = \left\{\sum_{i\geq 1} \alpha_i x_i, \forall n \in \mathbb{N}, \alpha_i \text{ scalars}\right\}$$

is dense in $\underline{X}$.

The theorems of calculus tells us that some systems of functions are complete. The theorem of Weierstrass approximation, $\{t^n\}_{n\geq 0}$ is complete in $C[0,1]$, also the polynomials are dense in $L_2[0,1]$ since

$$f_n \to f \quad \text{in} \quad C[0,1] \Longrightarrow f_n \to f \quad \text{in} \quad L_2[0,1]$$

Since

$$\|f\|_{L_2[0,1]} \leq \|f\|_{C_2[0,1]} \quad \forall \; f \in C[0,1]$$

i.e., the norm $C[0,1]$, is stronger than the $L_2[0,1]$.

Then $\{t^n\}_{n\geq 0}$ is complete in $L_2[0,1]$

other version of weierstrass:

The trigonometric Polynomials are dense in the periodic functions

$$\{1, \cos(nx), \sin(nx)\}_{n=1}^{\infty} \quad \text{in} \quad C[-\pi, \pi].$$