$$ds^2 = a^2(\tau)\left[-d\tau + \frac{dr^2}{1-Kr^2} + r^2 d^2\Omega\right]$$

As in newtonian, define Hubble parameter

$$H := \frac{\dot{a}}{a}$$

$$V \sim Hd$$

Dynamics: H & I $\longrightarrow$ RW metric.

$G_{\mu\nu} = 8\pi T_{\mu\nu}$ & RW metric $\longrightarrow$ Friedmann equations

$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu}$ Perfect fluid.

$\rho :=$ Energy density.

$P :=$ Isotropic pressure.

Choose $U^\mu = (1,0,0,0)$ comoving in the cosmological rest frame, then

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & P g_{ij} \end{pmatrix}$$

Friedmann equation $:= F$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{K}{a^2}$$ $\rightarrow$ Only first-time derivatives.

$\rightarrow$ Integral of motion.

(associated to energy through T_{00})

Evolution equation $:= E$

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi P - \frac{K}{2a^2}$$ $\rightarrow$ 2nd-time derivative equation of motion.

Note:

I. Friedmann equation is a constraint, we are not allowed to specify $\dot{a}$ since it is dependent on ρ and K.

II. Friedmann equation relates the rate of increase of the scale of factor to the rate of increase of all matter in the universe.

III. F ⊕ E ⟶ $\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{8\pi}{3}\rho - \frac{K}{a^2}\right) = -4\pi P - \frac{K}{2a^2}$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P)$$

IV. If we include a cosmological constant

F:
$$H^2 = \frac{\Lambda}{3} - \frac{K}{a^2} + \frac{8}{3}\pi\rho.$$

E:
$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi P + \frac{\Lambda}{2} - \frac{K}{2a^2}$$

Differentiating Friedmann equation:

$$\left[2\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}a - \dot{a}^2}{a^2}\right) = \frac{8\pi}{3}\dot{\rho} + \frac{2K}{a^3}\dot{a}\right]\frac{3}{8\pi}$$

$$= \frac{3}{4\pi}\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}a - \dot{a}^2}{a^2}\right) = \dot{\rho} + \frac{3}{4\pi a^2}\left(\frac{\dot{a}}{a}\right) \qquad (I)$$

Multiply E by $-\frac{3}{4\pi}\left(\frac{\dot{a}}{a}\right)$

$$-\frac{3}{4\pi}\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a}\right) - \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}\right)^2 = 3P\left(\frac{\dot{a}}{a}\right) + \frac{3K}{8\pi a^2}\left(\frac{\dot{a}}{a}\right) \quad (II)$$

(I) + (II)

$$\frac{-9}{8\pi}\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}^2}{a^2}\right) - \frac{9}{8\pi}\frac{K}{a^2}\left(\frac{\dot{a}}{a}\right) = \dot{\rho} + 3P\left(\frac{\dot{a}}{a}\right)$$

$$-\frac{9}{8\pi}\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right)H = \dot{\rho} + 3PH$$

$$\frac{-9}{8\pi}\left(\frac{8}{3}\pi\rho\right)H = \dot{\rho} + 3PH.$$

$$\dot{\rho} + 3PH + 3\rho H = 0$$

$$\dot{\rho} + 3H(\rho + P) = 0$$

consistency condition.

fluid equation (see Liddle's modern cosmology).

$$\dot{\rho} a^3 + 3Ha^3(\rho + P) = 0$$

$$\frac{d}{dt}(\rho a^3) = \dot{\rho} a^3 + \rho 3a^2 \dot{a}$$

$$= \dot{\rho} a^3 + 3\frac{\dot{a}}{a} a^3 \rho$$

$$= \dot{\rho} a^3 + 3H\rho a^3$$

$$\boxed{\frac{d}{dt}(\rho a^3) + P\frac{d}{dt}a^3 = 0}$$

As before,

$$V \propto a^3, \quad E = \rho V$$

$$dE + P dV = 0$$

Thermodynamics 1st law

Pressure works on the expansion.

Conservation of energy

$$\nabla_\mu T^{\mu\nu} = 0$$

Define the critical energy density

$$\rho_c := \frac{3}{8\pi} H$$

(Spatial sections become flat $K=0$).

And the density parameter

$$\Omega_{total} := \frac{\rho}{\rho_c}$$

relating total energy density to its local geometry.

$$\Omega_{total} \begin{array}{lll} > 1 & ; & K = 1 \\ = 1 & ; & K = 0 \\ < 1 & ; & K = -1 \end{array}$$

Friedmann

$$\rho = \rho_c + \frac{3}{8\pi}\frac{K}{a^2}$$

$$\Omega_K := \frac{\rho_K}{\rho_c} \quad ; \quad \rho_K := \frac{3K}{8\pi a^2}$$

$$\Omega_{total} = 1 + \Omega_K$$

If we consider a cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi (T_{\mu\nu} + T'_{\mu\nu})$$

$$T'_{\mu\nu} := -\frac{\Lambda}{8\pi} g_{\mu\nu}$$

then,

$$\rho_\Lambda = \frac{\Lambda}{8\pi} \qquad \text{constant}$$

$$P_\Lambda = -\frac{\Lambda}{8\pi} = -\rho_\Lambda$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$$

So far, two equations (E & F) and three unknowns (a, ρ, P).

Equations of state

$$P = P(\rho) \qquad \text{(in general } P = P(\rho, T))$$

$P = \omega\rho$; ω parameter (constant)

$\omega = 0 \longrightarrow$ Dust (matter dominated)

$\omega = \frac{1}{3} \longrightarrow$ Gas of radiation.

$\omega = -1 \longrightarrow$ Cosmological constant.

$$0 = \dot{\rho} + 3H(\rho + P) = \dot{\rho} + 3H(\rho + \omega\rho)$$

$$= \dot{\rho} + 3H\rho(1 + \omega)$$

$$\frac{d\rho}{\rho} = -3(1+\omega)\frac{da}{a}$$

$$\ln(\rho) = -3(1+\omega)\ln(aC)$$

$$\rho(a) = \frac{C'}{a(t)^{3(1+\omega)}}$$

If $\omega = 0$; then $\rho a^3 = C'$ (Energy).