Stirling approximation

$$n! \approx n^n e^{-n} (2\pi n)^{1/2} \quad \text{for} \quad n \gg 1$$

Theorem:

$$n! = \int_0^\infty x^n e^{-x} dx = \Gamma(n+1)$$

Proof:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = \cancelto{0}{-x^n e^{-x} \Big|_0^\infty} + \int_0^\infty n x^{n-1} e^{-x} dx = n\Gamma(n)$$

$$= n\int_0^\infty x^{n-1} e^{-x} dx = n\left[\cancelto{0}{-x^{n-1} e^{-x} \Big|_0^\infty} + \int_0^\infty (n-1) x^{n-2} e^{-x} dx\right]$$

$$= n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = \cancelto{0}{-e^{-\infty}} + \cancelto{1}{e^{-0}} = 1$$

therefore, $n! = \Gamma(n+1)$ $\blacksquare$

Let $x = ny$, then

$$n! = \int_0^\infty (ny)^n e^{-ny} n\, dy = \int_0^\infty n^{n+1} e^{\ln(y)} e^{-ny} dy$$

$$= n^{n+1} \int_0^\infty \exp[n(\ln(y) - y)]\, dy$$

$$= n^{n+1} \int_0^\infty \exp[n f(y)]\, dy = n^{n+1} I(n)$$

$f(y)$ has a maximum at $y = y_0 = 1$.

$$f(y) = -1 - \frac{1}{2}(y-1)^2 + \ldots \longrightarrow I(n) \approx \int_{-\infty}^{\infty} \exp\left[-n - \frac{n}{2}(y-1)^2\right] dy$$

$$= \exp(-n) \int_{-\infty}^{\infty} \exp\left(\frac{-n}{2} z^2\right) dz = \sqrt{\frac{2\pi}{n}} \exp(-n)$$

finally,

$$n! = n^{n+1} \sqrt{\frac{2\pi}{n}} \exp(-n) = n^n e^{-n} \sqrt{2\pi n}$$

Now,

$$\ln(n) n! = n \ln(n) - n + \frac{1}{2} \ln(2\pi) + \frac{1}{2} \ln(n)$$

$$\ln(n!) \approx n \ln(n) - n \quad \text{for } n \gg 1.$$