Renormalization from $\lambda\varphi^4$ to m-loop

Renormalization: Procedure that allows to remove infinite from the theory, by absorving them within the renormalization constants.

Initial Lagrangian:

$$\mathcal{L}_B = \frac{1}{2}(\partial\varphi_B)^2 - \frac{1}{2}m_B^2\varphi_B^2 - \frac{1}{4!}\lambda_B\varphi_B^4$$

Renormalization field:

$$\varphi_B(x) = Z^{1/2}\varphi(x)$$

$Z^{1/2}$ → Renormalization Constant.

$\varphi(x)$ → Renormalized field

$$Z = 1 + \delta Z \longrightarrow \text{Quantum correction}$$

Renormalization of parameters:

$$m_B^2\varphi_B^2 = Z\, m_B^2\,\varphi^2 \longrightarrow Z\, m_B^2 = Z_m m^2 = m^2 + \delta m^2$$

$$\lambda_B\varphi_B^4 = Z^2\lambda_B\varphi^4 \longrightarrow Z^2\lambda_B = Z_\lambda\lambda = \lambda + \delta\lambda$$

So that

$$\mathcal{L}_B = \mathcal{L} + \delta\mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \longrightarrow \text{finite theory}$$

$=: \mathcal{L}_0$

Counterterm Lagrangian:

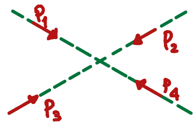
$$\delta\mathcal{L} = \frac{1}{2}\delta Z(\delta\varphi)^2 - \frac{1}{2}\delta m^2\varphi^2 - \frac{1}{4!}\delta\lambda\varphi^4$$

$$\mathcal{L}_I = -\frac{1}{4!}\lambda\varphi^4 + \delta\mathcal{L}$$: Interaction lagrangian

Feynman rules for $\mathcal{L}_I$:

$$\mathcal{L}_I = -\frac{1}{4!}\lambda\varphi^4 + \delta\mathcal{L}\,; \quad \delta\mathcal{L} = \frac{1}{2}\delta Z(\delta\varphi)^2 - \frac{1}{2}\delta m^2\varphi^2 - \frac{1}{4!}\delta\lambda\varphi^4$$

I. $\tilde{\Delta}_F(p)$:  ;

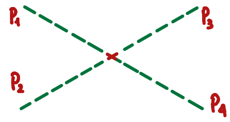
II. Vertex: 

; $-i\lambda \qquad \left(\sum p_i = 0\right)$

III. any loop: $\int \frac{d^4k}{(2\pi)^4}$

New terms (counterterms):

IV. $\frac{1}{2}\delta z(\varphi z)^2 - \frac{1}{2}\delta m^2 \varphi^2$:  : $i(\delta z p^2 - \delta m^2)$

V. $-\frac{1}{4!}\delta\lambda\varphi^4$: 

: $-i\delta\lambda \qquad \left(\sum p_i = 0\right)$

Propagator (zero order):

$$\tilde{G}_0^{(2)}(p_1 - p) = i\left[p^2(1+\delta z) - (m^2 + \delta m^2) + i\epsilon\right]^{-1}$$

finite

$\simeq$  +  + ...

$$= \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon} i(\delta z p^2 - \delta m^2) \frac{i}{p^2 - m^2 + i\epsilon} + \ldots$$

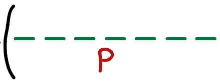
$\delta z(\lambda)$; $\delta m^2(\lambda)$; $\delta\lambda(\lambda)$

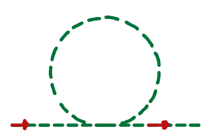
$$\delta z = \sum_{i=1}^{\infty} \delta z_i \ ; \qquad \delta m^2 = \sum_{i=1}^{\infty} \delta m_i^2 \ ; \qquad \delta\lambda = \sum_{i=2}^{\infty} \delta\lambda_i$$

$$\delta z_i \propto \lambda^i \qquad \delta m^2 \propto \lambda^i \qquad \delta\lambda \propto \lambda^i$$

then,
$$\tilde{\Gamma}^{(n)} = \tilde{\Gamma}^{(n)}(p_1, \ldots, p_n ; m, \lambda ; \delta z, \delta m^2, \delta\lambda)$$

example:

$i\tilde{\Gamma}^{(2)}(p,-p) = -\Big($ 

$\Big)^{-1} + \frac{1}{2}$  +  $+ \mathcal{O}(\lambda^2)$

$$= i(p^2 - m^2 + i\epsilon) + \frac{1}{2}(-i\lambda)\Delta_F(0) + i(\delta z_1 p^2 - \delta m_1^2) + \mathcal{O}(\lambda^2)$$

Then,

$$\tilde{\Gamma}^{(2)}(p,-p) = p^2(1+\delta Z_1) - (m^2 + \frac{1}{2}\lambda\Delta_F(0) + \delta m_1^2) + \mathcal{O}(\lambda^2)$$

Dimensional regularization:

$$\lambda\Delta_F(0) = -\hat{\lambda}\,\frac{m^2}{16\pi^2}\left[\frac{1}{2-\omega} + \Gamma'(1) + 1 - \ln\left(\frac{m^2}{4\pi M^2}\right) + \mathcal{O}(\omega-2)\right]$$

$$\hat{\lambda} = \lambda M^{2\omega-4}.$$

Therefore,

$$\tilde{\Gamma}^{(2)}(p,-p) = p^2(1+\delta Z_1) - m^2 - \delta m_1^2 + \hat{\lambda}\,\frac{m^2}{32\pi^2}\left[\frac{1}{2-\omega} + \Gamma'(1) + 1 - \ln\left(\frac{m^2}{4\pi M^2}\right)\right]$$

finite $\longrightarrow$ **Renormalization schemes:**

Agreement to fix δZ_1, δm_1^2, ...

Regularization + Boundary conditions in $\tilde{\Gamma}^{(n)}$

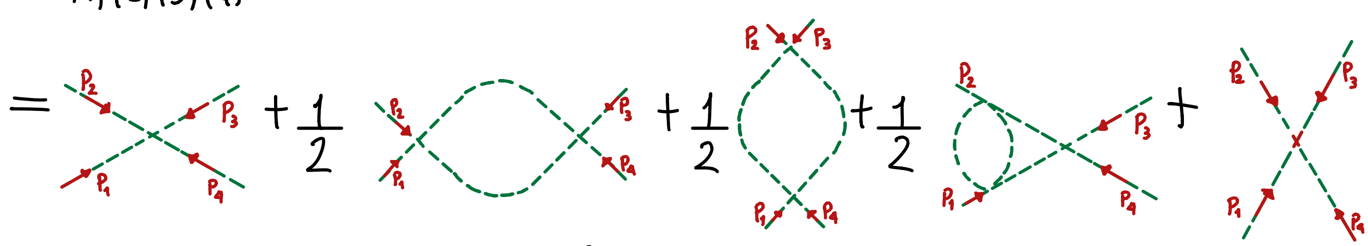
$$\lim_{\omega\to 2}\left[\delta m_1^2 - \frac{\hat{\lambda} m^2}{32\pi^2}\,\frac{1}{2-\omega}\right] = \text{constant}.$$

$$\lim_{\omega\to 2}(\delta Z_1) = \text{constant}.$$

Different renormalization schemes mean, different choise of the finite parts of the counterterms.

The scheme is set to fix the renormalization parameters.

$$i\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) =$$



$$s = (p_1+p_2)^2 \qquad t = (p_3+p_2)^2 \qquad u = (p_4+p_2)^2$$

$$= -i\lambda + \frac{\lambda^2}{2} J(s,m^2) + \frac{\lambda^2}{2} J(t,m^2) + \frac{\lambda^2}{2} J(u,m^2) + (-i\delta\lambda_2)$$

$$J(s,m^2) = \frac{i}{16\pi^2} M^{2\omega-4}\left[\frac{1}{2-\omega} + \Gamma'(1) - F(s,m^2,M^2)\right]$$

$$F(s,m^2,M^2) = \int_0^1 dx \ln\left(\frac{m^2 - s x(1-x)}{4\pi M^2}\right)$$

then,

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) = -\lambda + \lambda \frac{\hat{\lambda}}{32\pi^2}\left[\frac{3}{2-\omega} + 3\Gamma'(1) - F(s,m^2,M^2) - F(t,m^2,M^2) - F(t,m^2,M^2)\right]$$

$$-\delta\lambda_2 + \mathcal{O}(\lambda^3)$$

Renormalization scheme must define

$$\lim_{\omega\to 2}\left[\frac{3\lambda\hat{\lambda}}{32\pi^2}\frac{1}{2-\omega} - \delta\lambda_2\right] = \text{constant.}$$

In general $\delta\lambda, \delta m^2, \delta z$ should contain poles:

$$\delta\lambda = \lambda M^{4-2\omega}\left[a_0(\hat{\lambda}, m/M, \omega) + \sum_{\nu=1}^{\infty}\frac{a_\nu(\hat{\lambda}, m/M)}{(2-\omega)^\nu}\right]$$

$$\delta m^2 = m^2\left[b_0(\hat{\lambda}, m/M, \omega) + \sum_{\nu=1}^{\infty}\frac{b_\nu(\hat{\lambda}, m/M)}{(2-\omega)^\nu}\right]$$

$$\delta z = c_0(\hat{\lambda}, m/M, \omega) + \sum_{\nu=1}^{\infty}\frac{c_\nu(\hat{\lambda}, m/M)}{(2-\omega)^\nu}$$

Constants

Minimal substraction: the counteterms remove only divergences.

$$a_0^{MS} = b_0^{MS} = c_0^{MS} = 0; \text{ moreover } a_\nu^{MS} = a_\nu^{MS}(\hat{\lambda})$$

as well as b_ν and c_ν

Explicitly, to the lowest order:

$$\delta\lambda^{MS} = \frac{3\lambda\hat{\lambda}}{32\pi^2}\frac{1}{2-\omega} + \mathcal{O}(\lambda^3) \longrightarrow a_1^{MS} = \frac{3\hat{\lambda}}{32\pi^2} + \mathcal{O}(\lambda^3)$$

$$\delta m^2_{MS} = \frac{m^2\hat{\lambda}}{32\pi^2}\frac{1}{2-\omega} + \mathcal{O}(\lambda^2) \longrightarrow b_1^{MS} = \frac{\hat{\lambda}}{32\pi^2} + \mathcal{O}(\lambda^3)$$

$$\delta z^{MS} = \mathcal{O}(\lambda^2) \longrightarrow c_1^{MS} = \mathcal{O}(\lambda^2)$$

Having stablished the counterterms:

$$\tilde{\Gamma}^{(2)}(p,-p) = p^2 - m^2 + \frac{\lambda m^2}{32\pi^2}\left(\Gamma'(1) + 1 - \ln\left(\frac{m^2}{4\pi m^2}\right)\right) + \mathcal{O}(\lambda^2)$$

$$m^2_{physical} = -\tilde{\Gamma}^{(2)}(0)$$

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) = -\lambda + \frac{\lambda^2}{32\pi^2}\left[3\tilde{\Gamma}'(1) - F(s,m^2,M^2) - F(t,m^2,M^2) - F(u,m^2,M^2)\right]$$

$$+\mathcal{O}(\lambda^3)$$

$$\lambda_{physical} = -\tilde{\Gamma}^{(4)}(0)$$

both $f(\lambda, m^2, \ln(m^2/M^2))$

Other schemes

$\overline{MS}$: The terms $\sim \Gamma'(1)$, $\ln(4\pi)$ also are removed.

examples:

$$\delta\lambda^{\overline{MS}} = \frac{3\lambda\hat{\lambda}}{32\pi^2}\left[\frac{1}{2-\omega} + \Gamma'(1) + \ln(4\pi)\right] + \mathcal{O}(\lambda^3)$$

then,

$$a_0^{\overline{MS}} = a_1^{MS}\left[\Gamma'(1) + \ln(4\pi)\right]$$

$$a_1^{\overline{MS}} = a_1^{MS}$$

...

therefore,

$$\tilde{\Gamma}^{(2)}(p,-p) = p^2 - m^2 + \frac{\lambda m^2}{32\pi^2}\left(1 - \ln(m^2/M^2)\right)$$

$$\tilde{\Gamma}^{(4)} = -\lambda + \frac{\lambda^2}{32\pi^2}\left(-3\ln\left(\frac{m^2}{M^2}\right) + A(s,m^2) + A(t,m^2) + A(u,m^2)\right)$$

$$= -\int_0^1 dx \ln\left(1 - \frac{sx(1-x)}{m^2}\right)$$

Relation between schemes:

Green functions: Physical objects $\longrightarrow$ has a unique value

Does not depend from the renormalization scheme

the parameters differ in each scheme in a finite quantity.

$$\varphi_B = Z^{1/2}\varphi \longrightarrow J_B(x)\varphi_B(x) = Z^{1/2} J_B(x)\varphi(x)$$

$$W[J] = W_B[J_B] \longrightarrow J_B(x) = Z^{-1/2} J(x)$$

then

$$\tilde{G}(p_1,\dots,p_n) = (Z^{-1/2})^n \, \tilde{G}_B^{(n)}(p_1,\dots,p_n)$$

Similarly,

$$X[J] = X_B[J_B]$$

So,

$$\varphi_c(x) = \frac{\delta X}{\delta J(x)} = Z^{-1/2}\frac{\delta X_B}{\delta J_B(x)} = Z^{-1/2}\varphi_{cB}(x)$$

$$\Gamma[\varphi_c] = \Gamma_B[\varphi_{cB}] \longrightarrow \tilde{\Gamma}^{(n)}(p_1,\dots,p_n) = Z^{n/2}\,\tilde{\Gamma}_B^{(n)}(p_1,\dots,p_n)$$