Newtonian limit

Consider a slowly varying gravitational field

$$x^a = (x^0, x^i) = (ct, x^i)$$

Let η_{ab} be the Minkowskian metric and g_{ab} a general metric for the gravitational field

$$\epsilon := \text{Parameter of order } \frac{v}{c} \quad (v \ll c)$$

Assume:

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} + \mathcal{O}(\epsilon^2)$$

$$\delta x^a \sim v \delta t \sim \left(\frac{v}{c}\right) c \delta t \sim \epsilon \delta x^0$$

$$\frac{\delta}{\delta x^0} \sim \frac{\delta}{\delta x^a} \frac{\delta x^a}{\delta x^0} = \epsilon \frac{\delta}{\delta x^a}$$

$$\frac{\partial f}{\partial x^0} \sim \epsilon \frac{\partial f}{\partial x^a}$$

slow-motion approximation

Consider a free test particle moving with velocity $(v \ll c)$

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

$$c^2 d\tau = c^2 dt^2 - d\vec{x}^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$dt = \frac{d\tau}{\sqrt{1-\epsilon^2}} = d\tau\,(1 + \mathcal{O}(\epsilon^2))$$

$$t \longmapsto \tau$$

$$\frac{dx^i}{c\,dt} \sim \mathcal{O}(\epsilon).$$

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc})$$

$$= \frac{1}{2} (\eta^{ad} + \epsilon h^{ad})(\partial_c h_{bd} + \partial_b h_{cd} - \partial_d h_{bc})\,\epsilon$$

$$= \frac{\epsilon}{2} \eta^{ad} (\partial_c h_{bd} + \partial_b h_{cd} - \partial_d h_{bc}) = \mathcal{O}(\epsilon).$$

Examining spatial part of geodesics:

$$\frac{d^2x^i}{c^2dt^2} + \Gamma^i_{bc}\frac{dx^b}{cdt}\frac{dx^c}{cdt} = 0$$

$\Theta(\epsilon^2)$ $\qquad\qquad$ $\Theta(\epsilon^3)$

$$\frac{1}{c^2}\frac{d^2x^i}{dt^2} + \Gamma^i_{00}\frac{dx^0}{cdt}\frac{dx^0}{cdt} + \cancel{2\Gamma^i_{0j}\frac{dx^0}{cdt}\frac{dx^j}{cdt}} + \cancel{\Gamma^i_{jk}\frac{dx^j}{cdt}\frac{dx^k}{cdt}} = 0$$

$$\frac{1}{c^2}\frac{d^2x^i}{dt^2} + \Gamma^i_{00} = 0.$$

Also,

$$\Gamma^i_{00} = \frac{1}{2}\epsilon\eta^{id}(\partial_0 h_{0d} + \partial_0 h_{0d} - \partial_d h_{00})$$

$$= \frac{1}{2}\epsilon\eta^{id}(2\partial_0 h_{0d} - \partial_d h_{00})$$

slow-motion

$$= -\frac{1}{2}\epsilon\eta^{id}\partial_d h_{00} + \Theta(\epsilon^2)$$

$$= -\frac{1}{2}\epsilon\eta^{ij}\partial_j h_{00} + \Theta(\epsilon^2)$$

$$= \frac{1}{2}\epsilon\frac{\partial h_{00}}{\partial x^i} + \Theta(\epsilon^2)$$

therefore,

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}c^2\frac{\partial g_{00}}{\partial x^i} + \Theta(\epsilon^2)$$

but $F = -\nabla\phi$, then 2nd-law $\frac{d^2x^i}{dt^2} = \frac{-\partial\phi}{\partial x^i}$, $\phi :=$ Newtonian potential

$$\frac{d^2x^i}{dt^2} = \frac{-\partial\phi}{\partial x^i} = -\frac{1}{2}c^2\frac{\partial g_{00}}{\partial x^i} + \Theta(\epsilon^2)$$

$$\frac{\partial}{\partial x^i}\left(g_{00} - \frac{2\phi}{c^2}\right) = \Theta(\epsilon^2)$$

then

$$g_{00} = \beta + \frac{2\phi}{c^2} + \Theta(\epsilon^2) \qquad \beta = \text{constant.}$$

since $x \to \infty$, then $\phi \to 0$, $g_{00} \to 1$.

for $\beta = 1$.

$$\boxed{g_{00} = 1 + \frac{2\phi}{c^2} + \Theta(\epsilon^2)}$$ it embodies the correspondence between GR and Newton

Theorem: $R^i_{0j0} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} \longrightarrow R_{00} = 4\pi\rho \longrightarrow \nabla^2\phi = 4\pi\rho$,

in the Newtonian limit.

Proof: Examine relative acceleration of two test particles, one at $x^i + \xi^i$ and the other at x^i

Newton:

$$\frac{d^2\xi^i}{dt^2} = \frac{d^2}{dt^2}(x^i + \xi^i) - \frac{d^2}{dt^2}(x^i)$$

$$= -\frac{\partial \phi}{\partial x^i}\Big|_{x^j+\xi^j} + \frac{\partial \phi}{\partial x^i}\Big|_{x^j}$$

$$= -\frac{\partial^2 \phi}{\partial x^i \partial x^j}\xi^j$$

Einstein:

$$\frac{D^2\xi^i}{dt^2} = \frac{d^2\xi^i}{dt^2} = -R^i_{0j0}\xi^j$$

Newtonian limit.

therefore $R^i_{0j0} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$

$$\frac{d^2\xi^a}{dt^2} = R^a_{\ 00d} V^0 V^0 \xi^d = R^a_{\ 00d}\xi^d$$

$$= -R^a_{\ 0d0}\xi^d$$

$v^i \sim \Theta(\epsilon)$

only relevant $v^0 \sim 1$.

From Einstein's

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S = K T_{\mu\nu}$$

Its trace is

$$S - \frac{1}{2} 4 S = KT$$

$$-S = KT$$

and

$$R_{00} = \frac{1}{2} g_{00} S + K T_{00}$$

$$= \frac{1}{2} K T + K T_{00} = \frac{1}{2} K (2 T_{00} + T)$$

$$= \frac{1}{2} K (2 T_{00} + T^{0}{}_{0} + T^{i}{}_{i})$$

$$= \frac{1}{2} K (2 T_{00} - T_{00} + T_{ii})$$

$$= \frac{1}{2} K T_{00} \left(1 + \frac{T_{ii}}{T_{00}} \right)$$

For Earth $\left\{ \begin{array}{l} T_{ii} \rightarrow P \\ T_{00} \rightarrow \rho \end{array} \right.$ $\left| \frac{T_{ii}}{T_{00}} \right| \sim \frac{\text{Pressure}}{\text{density}} \sim \frac{dP}{d\rho} \sim \begin{pmatrix} \text{Velocity} \\ \text{of sound} \end{pmatrix}^2 \ll c^2$

Earth's measures

$$\left. \begin{array}{l} P \sim 10^{-12} \\ \rho \sim 10^{1} \end{array} \right\}$$ V.G Kirtskhalia, Open J. Acoustics 2, 80 (2012).

therefore

$$R_{00} = \frac{1}{2} K \rho = 4\pi\rho \longrightarrow K = 8\pi$$

$$R_{00} = R^{a}{}_{0a0} = \underbrace{R^{0}{}_{000}}_{\mathcal{O}(\epsilon^2)} + R^{i}{}_{0i0}$$

$$R^{0}{}_{000} = \frac{\partial^2 \phi}{\partial t^2} \sim \mathcal{O}(\epsilon^2)$$

Moreover

$$R^{i}{}_{0i0} = \sum_i \frac{\partial^2 \phi}{\partial x^i \partial x^i}$$

$$= \nabla^2 \phi$$

$$= 4\pi\rho.$$

∎

Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH} := (-g)^{1/2} R \qquad \text{(before } R \text{ was } S\text{)}$$

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\sigma}_{\mu\sigma\nu} = g^{\mu\nu}\left[\Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}\right]$$

but

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\beta}\left[g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right] \iff \nabla g = 0$$

$$\mathcal{L}_{EH} = \mathcal{L}_{EH}(g_{\mu\nu}, g_{\mu\nu,\sigma}, g_{\mu\nu,\sigma\rho})$$

(Euler-Lagrange) Field equations

$$\frac{\partial \mathcal{L}_{EH}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^{\sigma}} \frac{\partial \mathcal{L}_{EH}}{\partial g_{\mu\nu,\sigma}} + \frac{\partial^2}{\partial x^{\rho}\partial x^{\sigma}} \frac{\partial \mathcal{L}_{EH}}{\partial g_{\mu\nu,\sigma\rho}} = 0$$

$$\mathcal{L}_{EH} =: \mathcal{L}_{P}$$

$$\mathcal{L}_{P} = \mathcal{L}_{P}(g, \partial g, \Gamma, \partial \Gamma).$$