Watson's Lemma:

Let $f(t)$ be a complex valued function of a real variable t, such that:

Check the extension of $f^*(z)$ to $\mathbb{C}^*$.

1. f is continuous on $(0,\infty)$.
2. As $t \to 0^+$, $f(t) \sim \sum_{k=0}^{\infty} a_k t^{\rho_k - 1}$ with $0 < Re(\rho_0) < Re(\rho_1) < \cdots < \lim_{k\to\infty} Re(\rho_k) = \infty$
3. For some fixed $c > 0$, $f(t) = O(e^{ct} t^{\rho_N - 1})$ as $t \to \infty$ then we have

$$I(x) := \int_0^{\infty} e^{-xt} f(t)\, dt \sim \sum_{k=0}^{\infty} \frac{a_k \Gamma(\rho_k)}{x^{\rho_k}} \quad \text{as } x \longrightarrow \infty$$

Proof:

The conditions (1-3) guarantee that $I(x)$ converges for $x > 0$, and the conditions (2-3) imply

$$\left| f(t) - \sum_{k=0}^{N-1} a_k t^{\rho_k - 1} \right| \leq K_N e^{ct} \left| t^{\rho_N - 1} \right| \quad \text{for } t > 0$$

then

$$e^{-xt} f(t) - \sum_{k=0}^{N-1} e^{-xt} a_k t^{\rho_k - 1} \leq K_N e^{-(x-c)t} t^{\rho_N - 1}$$

$$\left| \int_0^{\infty} e^{-xt} f(t) - \underbrace{\sum_{k=0}^{N-1} e^{-xt} a_k t^{\rho_k - 1}\, dt}_{I_1} \right| \leq K_N \underbrace{\int_0^{\infty} e^{-(x-c)t} |t^{\rho_N - 1}|\, dt}_{I_2}$$

With a change of variables in I_1 $u = xt$ we have

$$I_1 = \frac{1}{x^{\rho_k}} \int_0^{\infty} e^{-u} u^{\rho_k - 1}\, du = \frac{\Gamma(\rho_k)}{x^{\rho_k}},$$

and for I_2 with the substitution $u := (x-c)t$ and

$dt = \frac{du}{|x-c|}$

$$I_2 = \frac{1}{|x-c|^{\rho_N}} \int_0^{\infty} e^{-u} |t^{\rho_N - 1}| dt - \frac{1}{|x-c|^{\rho_N}} \Gamma(Re(\rho_N))$$

finally, $\left| I(x) - \sum_{k=0}^{N-1} a_k \frac{\Gamma(\rho_k)}{x^{\rho_k}} \right| \leq K_N \frac{\Gamma(Re(\rho_N))}{|x-c|^{\rho_N}}$

Therefore.

$$I(x) := \int_0^{\infty} e^{-xt} f(t) dt = \sum_{k=0}^{N-1} a_k \frac{\Gamma(\rho_k)}{x^{\rho_k}} + O\left(\frac{1}{x^{\rho_N}}\right)$$ (Laplace Transform)

Example

1. $I(x) = \int_0^5 \frac{e^{-xt}}{1+t^2} dt$ for large x.

$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \dots$ arround $t=0$,

$= \sum_{k=0}^{\infty} (-1)^k (t)^{(2k)}$

And by watson's: 1. Substitute this expansion into the integral.

2. Interchange integral and summation

3. Extend from 5 to ∞

So $a_k = (-1)^k$ and $\rho_k = 2k+1$, then

$$I(x) = \sum_{k=0}^{N-1} (-1)^k \frac{\Gamma(2k+1)}{x^{2k+1}} + O\left(\frac{1}{x^{2N+1}}\right)$$

as $x \to \infty$

$$I(x) = \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \frac{6!}{x^7} + \dots \quad \text{as } x \to \infty$$