Asymptotic approximation of integrals

Bibliography:

[1] Wong (2001) – Differential equations and Asymptotyc theory on mathematical physics.

[2] Bender and Orszag (1999) V_1 – Advanced mathematical methods for scientist.

Let f and g be two continuous complex functions defined on a subset of the complex functions defined on a subset of the complex plane, $H \subset \mathbb{C}$.

Let z_0 be a limit point of H.

Definition: $f(z) = O(g(z))$ as $z \longrightarrow z_0$ means that there is a constant $K>0$ and a neighbourhood U of z_0 such that

$$|f(z)| \leq K\,|g(z)| \qquad \text{for all } z \in U \cap H.$$

Definition: $f(z) = o(g(z))$ as $z \longrightarrow z_0$ means that for every $\varepsilon > 0$ there exist a neighbourhood U_ε of z_0 such that

$$|f(z)| \leq \varepsilon | \ (z)| \qquad \text{for all } z \in U_\varepsilon \cap H$$

Physics: $f(z) = O(g(z)) \sim$ f is big-oh g $\sim$ How fast a function grows or declines.

$f(z) = o(g(z)) \sim$ is smaller than.

f is of the same order of g.

We may consider.

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} < \infty \longrightarrow f(z) = O(g(z))$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0 \longrightarrow f(z) = o(g(z)).$$

Example:

- $f(x) = 4x^3 - 3x^2 + 2x - 1 \qquad \text{as} \quad x \longrightarrow \infty$

  $\hookrightarrow$ Highest grow term.

$$f(x) = O(x^3) \qquad \text{as} \quad x \longrightarrow \infty$$

Check:

$$|f(x)| = |4x^3 - 3x^2 + 2x - 1|$$
$$\leq |4x^3| + |3x^2| + |2x| + |1|$$
$$\leq 4x^3 + 3x^3 + 2x^3 + x^3$$
$$= 10x^3 = 10|x^3|$$

therefore
$$f(x) \leq 10x^3$$

other way is:

$$\lim_{x\to\infty} \frac{4x^3 - 3x^2 + 2x - 1}{x^3}$$
$$= \lim_{x\to\infty} \left(4 - \frac{3}{x} + \frac{2}{x^2} - \frac{1}{x^3}\right) = 4.$$

- $f(n) = 10\log(n) + 5(\log(n))^3 + 7n + 3n^2 + 6n^3$ as $n \to \infty$

then
$$f(n) = \Theta(n^3) \text{ as } n \to \infty$$
$$|f(n)| = |10\log(n) + 5(\log(n))^3 + 7n + 3n^2 + 6n^3|$$
$$\leq |10\log(n)| + |5(\log(n))^3| + |7n| + |3n^2| + |6n^3|$$
$$\leq 10n^3 + 5n^3 + 7n^3 + 3n^3 + 6n^3$$
$$= 31n^3 = 31|n^3|$$

therefore $|f(n)| = 31|n^3|$, by the other way
$$\lim_{n\to\infty} \frac{f(n)}{n^3} = 6$$

- $x^2 = o(x)$ as $x \to 0$

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

- $x - \sin(x) = o(x)$ as $x \to 0$.

$$\lim_{x\to 0} \left(\frac{x - \sin(x)}{x}\right) = \lim_{x\to 0} \left(1 - \frac{\sin(x)}{x}\right) = 0$$

- $x - \sin(x) \neq o(x^3)$

$$\lim_{x\to 0}\left(\frac{x-\sin(x)}{x^3}\right) = \lim_{x\to 0}\left(\frac{1-\cos(x)}{3x^2}\right)$$

$$= \lim_{x\to 0} \frac{\sin(x)}{6x} = \frac{1}{6}$$

therefore $x - \sin(x) = O(x^3)$

Definition: Let $\{\varphi_n\}_{n\geq 0}$ be a sequence of continuous complex functions defined on $H \subset \mathbb{C}$. We say $\{\varphi_n\}_{n\geq 0}$ is an asymptotic sequence as $z \to z_0$ in H, if we have

$$\varphi_{n+1}(z) = o(\varphi_n(z)) \quad , n \geq 0.$$

Definition: If $\{\varphi_n\}_{n\geq 0}$ is an asymptotic sequence as $z \to z_0$, we say that

$$\sum_{n=1}^{\infty} a_n \varphi_n(z) \quad , \quad a_n \text{ constant.}$$

is an asymptotic expansion of the function f if for each $N \geq 0$ we have

$$f(z) = \sum_{n=1}^{N} a_n \varphi_n(z) + o(\varphi_N(z)) \quad \text{as} \quad z \to z_0$$

In this case we write.

$$f(z) \sim \sum_{n=1}^{\infty} a_n \varphi_n(z) \quad \text{as} \quad z \to z_0$$

Homework: Show that $f(z)$ may be written as

$$f(z) = \sum_{n=1}^{N-1} a_n \varphi_n(z) + O(\varphi_N(z))$$

Integration by parts

(Easy, but not general method)

Example:

$$I(x) = \int_x^\infty e^{-4t}\,dt \quad , \quad \text{as} \quad x \to \infty$$

First note that we do not that $x \to 0$. In the case $x \to 0$, Taylor e^{-4t} gives an (apparent) divergent result.

$$e^{-4t} = 1 - t^4 + \frac{t^8}{2!} - \frac{t^{12}}{3!} + \dots \longleftarrow \text{Not.}$$

So we write

$$I(x) = \underbrace{\int_0^\infty e^{-4t}\,dt}_{:= I_1} - \underbrace{\int_0^x e^{-4t}\,dt}_{:= I_2}$$

$$I_1 = \int_0^\infty e^{-4t}\,dt \qquad\qquad s = t^4$$

$$ds = 4t^3 dt$$

$$I_1 = \frac{1}{4}\int_0^\infty e^{-s} s^{-3/4}\,ds = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \qquad dt = \frac{1}{4} s^{-3/4} ds$$

$$\Gamma(n) = \int e^{-z} z^{n-1}\,dz$$

$$z\,\Gamma(z) = \Gamma(z+1)$$

$$I_1 = \Gamma\left(\frac{5}{4}\right) \quad \text{as} \quad x \longrightarrow 0$$

$$I_2 = \int_0^x e^{-4t}\,dt = \int_0^x \left(1 - t^4 + \frac{t^8}{2!} - \frac{t^{12}}{3!} + \dots\right) dt$$

$$= \left[t - \frac{1}{5}t^5 + \frac{t^9}{18} - \frac{t^{13}}{78} + \dots\right]_0^x$$

$$= x - \frac{x^5}{5} + \frac{x^9}{18} - \frac{x^{13}}{78} + \dots$$

then

$$I(x) \sim \Gamma\left(\frac{5}{4}\right) \quad \text{as} \quad x \longrightarrow 0.$$

However, in the case $x \longrightarrow \infty$, we need to develop $I(x)$ in inverse powers of x.

$$I(x) = -\frac{1}{4}\int_x^\infty \frac{1}{t^3} \frac{d}{dt}\left(e^{-t^4}\right) dt$$

$$= -\frac{1}{4}\frac{e^{-t^4}}{t^3}\Big|_x^\infty - \left(-\frac{1}{4}\right)\int_x^\infty \frac{d}{dt}\left(\frac{1}{t^3}\right) dt$$

$$= \frac{-e^{-x^4}}{4x^3} - \frac{3}{4}\int_x^\infty \frac{e^{-4t}}{t^4}\,dt,$$

but $$\int_x^\infty \frac{1}{t^4} e^{-4t}\,dt < \frac{1}{x^4}\int_x^\infty e^{-t^4}\,dt = \frac{1}{x^4} I(x) \ll I(x)$$

leading behaviour of $I(x)$.

$$I(x) \sim \frac{1}{4x^3} e^{-x^4}.$$

Let

$$f(z) = \sum_{n=1}^{N} a_n \varphi_n(z) + o(\varphi_N(z))$$

$$= \sum_{n=1}^{N-1} a_n \varphi_n(z) + a_N \varphi_N(z) + o(\varphi_N(z))$$

$$= \sum_{n=1}^{N-1} a_n \varphi_n(z) + a_N \varphi_N(z) + \varphi_{N+1}(z)$$

$$f(z) - \sum_{n=1}^{N-1} a_n \varphi_n(z) = a_N \varphi_N(z) + \varphi_{N+1}(z)$$

$$\hat{f}(z) = a_N \varphi_N(z) + \varphi_{N+1}(z)$$

$$\longrightarrow \quad \hat{f}(z) = O(\underbrace{\varphi_N(z)}_{g(z)})$$

$$\lim_{z \to z_0} \frac{\hat{f}(z)}{g(z)} < \infty$$

$$\lim_{z \to z_0} \frac{a_N \varphi_N(z) + \varphi_{N+1}(z)}{\varphi_N(z)} = a_n + \cancelto{0}{\lim_{z \to z_0} \frac{\varphi_{N+1}(z)}{\varphi_N(z)}}$$

$$\lim_{z \to z_0} \frac{a_N \varphi_N(z) + \varphi_{N+1}(z)}{\varphi_N(z)} = a_n$$

$$f(z) - \sum_{n=1}^{N-1} a_n \varphi_n(z) = O(\varphi_N(z))$$

$$f(z) = \sum_{n=1}^{N-1} a_n \varphi_n(z) + O(\varphi_N(z)). \qquad \blacksquare$$