Spectrum theory (Compact operators)

let $A: X \longrightarrow X$ be a bounded operator. $\lambda \in \mathbb{C}$ is called a regular point of A if and only if the inverse operator $(A-\lambda I)^{-1}: X \longrightarrow X$ exists and is a bounded operator. If x is not regular, we will call a point of the spectrum.

To the set of all points of this form we will call as the spectrum of A and it will be denoted by $\sigma(A)$. Then $\sigma(A) \subseteq \mathbb{C}$.

Proposition: All $\lambda \in \mathbb{C}$, with $|\lambda| > \|A\|$ is a regular point.

Proof:

$$A - \lambda I = -\lambda\left(I - \frac{1}{\lambda} A\right) \quad \text{and} \quad \left\|\frac{A}{\lambda}\right\| < 1.$$

by the property (II), the operator $A - \lambda I$ is invertible.

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Proposition: $\sigma(A)$ is closed.

Proof: $\forall \lambda$ regular, exists $\varepsilon > 0$ such that, $\forall \mu, |\lambda - \mu| < \varepsilon$, μ is regular.

let
$$A - \mu I = A - \mu I + (\lambda - \mu) I$$
$$= (A - \lambda I)(I + (\lambda - \mu)(A - \lambda I)^{-1})$$

which is invertible, since $A - \lambda I$ is invertible and $I + (\lambda - \mu)(A - \lambda I)^{-1}$ is invertible by

$$\|(\lambda - \mu)(A - \lambda I)^{-1}\| = |\lambda - \mu| \, \|(A - \lambda I)^{-1}\| < 1$$

if
$$|\lambda - \mu| < \|(A - \lambda I)^{-1}\|^{-1}$$

ε wanted.

$\blacksquare$

Spectrum clasification

1. The point spectrum $\sigma_p(A)$, is the set of eigenvalues of an operator A, i.e., $\lambda \in \sigma_p$ if and only if there exists $x \in X \setminus \{0\}$ and $Ax = \lambda x$.

   This is equivalent to say that $\operatorname{Ker}(A - \lambda I) \neq 0$ and the dimension of $\operatorname{Ker}(A - \lambda I)$ is called multiplicity of the eigenvalue λ.

Now, let's assume that $Ker(A-\lambda I)=0$, means that $A-\lambda I: X \to X$ is one-to-one, between X and $Im(A-\lambda I)$.

By the open mapping theorem, if $Im(A-\lambda I)=X$, then exists an inverse bounded operator $(A-\lambda I)^{-1}$ such that λ is regular.

In our classification of $\sigma(A)$, if $\lambda \notin \sigma_p(A)$ but $\lambda \in \sigma(A)$, then $Im(A-\lambda I) \neq X$.

Lemma: Let $A: X \longrightarrow X$ be a bounded operator, and let $(\lambda)_{i=1}^{n}$ different eigenvalues of A.

Let $x_i = 0$ and $Ax_i = \lambda_i x_i$ (eigenvectors of different eigenvalues). Then $\{x_i\}_{i=1}^{n}$ are linearly independents.

II. The continuous spectrum $\sigma_c(A)$, $\lambda \in \sigma_c(A)$ if and only if $\lambda \in \sigma(A) \setminus \sigma_p(A)$ and $Im(A-\lambda I)$ is dense in X.

Example: In $L_2([0,1])$, $A: L_2([0,1]) \longrightarrow L_2([0,1])$

III. The residual spectrum $\sigma_r(A) = \sigma(A) \setminus (\sigma_p(A) \cup \sigma_c(A))$, for $\lambda \in \sigma_r(A)$, we have $Im(A-\lambda I) \neq X$, and $Ker(A-\lambda I)=0$.

Example: Let the traslation operator in l_2

$$A e_i = e_{i+1}$$

$$0 \in \sigma_r(A).$$

Fredholm Theory

Let $T: X \longrightarrow X$ be compact. Let T_λ the operator $T-\lambda I$ and $\Delta_K = Im\, T_\lambda$

Lemma: Let E_1 be a closed subspace such that $E \neq E_1 \subseteq E \subseteq X$. Exists $y_0 \in E$ with $\|y_0\|=1$ such that the distance from y_0 to E_1 is

$$d(y_0, E) \geq \frac{1}{2}.$$

Proof: Let $y \in E \setminus E_1$ and $d(y, E_1) = a > 0$ (such y exists since E_1 is closed).

Let $x_0 \in E_1$, such that $\|y - x_0\| < 2a$. Then

$$y_0 = \frac{(y-x_0)}{\|y-x_0\|}$$

satisfy the properties, since $\|y_0\|=1$ and for all $x \in E_1$

$$\|y_0 - x\| = \left\| \frac{(y - x_0)}{\|y - x_0\|} - x \right\|$$

$$= \frac{\|y - (x_0 - \|y - x_0\| x)\|}{\|y - x_0\|} \geq \frac{a}{2a} = \frac{1}{2}$$

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Corollary: If $\dim X = \infty$, the identity operator $I : X \to X$ is not compact

Proof: We have to prove that the unitary ball

$$D(X) = \} x : \|x\| \leq 1 \{$$

is not relatively compact.

Let n be a family of subspaces $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \ldots$ with $\dim E_n = n$. Are closed subspaces. By the above lemma there is a sequence $y_i \in E_i$ with $\|y_i\|=1$ such that

$$d(y_i, E_{i-1}) \geq \frac{1}{2}$$

Then $i \neq j$, $\|y_i - y_j\| \geq 1/2$, which is not Cauchy, therefore it not has a Cauchy subsequence, thus, I is not compact.

Proposition: For all compact operators T, $0 \in \sigma(T)$

Proof: Let $T - 0I = T$, then $T^{-1} \cdot T = I$.