Radially null geodesics and radially infalling particles

Definition: A radial null geodesic is defined by the conditions

$$ds^2 = 0 \quad \text{and} \quad \underline{d\theta} = 0 = \underline{d\phi}$$

Schwarzschild

$$0 = ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - \cancel{r^2 d\Omega^2}^{\,0}$$

$$\frac{dt}{dr} = \pm\left(\frac{1}{1 - 2m/r}\right) = \pm\left(\frac{r}{r-2m}\right)$$

$$t = \pm\left(r + 2m\ln|r-2m| + \text{constant}\right)$$

"+" sign (outgoing)

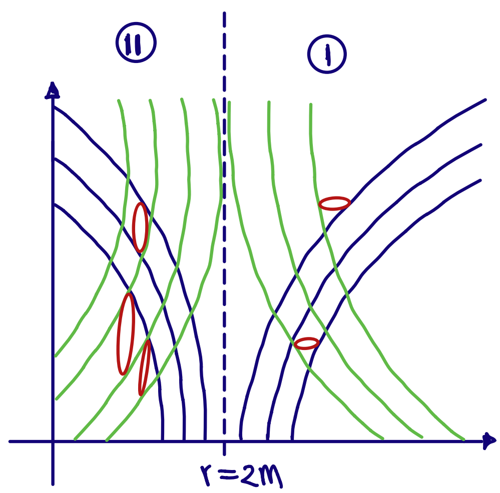
$$t = (r + 2m\ln|r-2m| + \alpha)$$

$$\frac{dt}{dr} = \frac{r}{r-2m}$$

"−" sign (ingoing)

$$t = -(r + 2m\ln|r-2m| + \alpha)$$

$$\frac{dt}{dr} = \frac{-r}{r-2m}$$



As $r \to \infty$ null geodesics make 45°.

a. Take t = constant. An observer in region II cannot stay at rest, that is, at a constant value of r. Thus it is forced to move towards the intrinsic singularity at $r=0$.

b. An observer in region I moving towards the origin would take an infinite amount of time to reach the Schwarzschild radius $r=2m$.

Consider the path of a radially infalling free particle.

Timelike geodesics
$$\begin{cases} \left(1 - \dfrac{2m}{r}\right)\dot{t} = K = \text{constant} \\[2ex] \left(1 - \dfrac{2m}{r}\right)\dot{t}^2 - \left(1 - \dfrac{2m}{r}\right)^{-1}\dot{r}^2 = 1. \end{cases}$$

Take $K=1$ so for large r, $t \to \tau$.

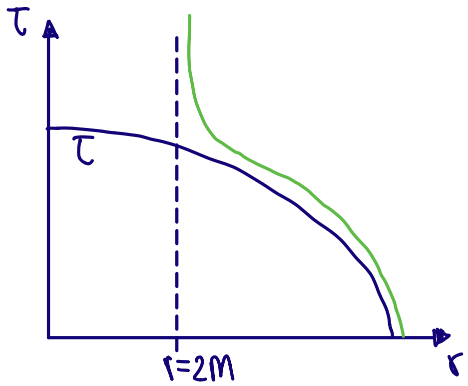
$$\left(1-\frac{2m}{r}\right)\left(\frac{K}{1-2m/r}\right)^2-\left(\frac{1}{1-2m/r}\right)\dot{r}^2=1$$

$$K^2-\dot{r}^2=1-\frac{2m}{r} \qquad \text{for} \quad K=1.$$

$$\dot{r}^2=\frac{2m}{r} \qquad \frac{d\tau}{dr}=\sqrt{\frac{r}{2m}}$$

$$\tau=\pm\frac{1}{\sqrt{2m}}\left(r^{3/2}+\text{constant}\right)\left(\frac{2}{3}\right)$$

Take negative solution as to guarantee the past of the particle is in spatial infinity, and hence we have an infalling particle



As $r \longrightarrow 2m$

$\tau < \infty$

The body falls continuosly to $r=0$

(Same result as Newton's)

Cosmology

Bibliography:

- G.F.R. Ellis, R. Maartens. & M.A.H. Mc Callum, Relativistic Cosmology.
- M. Ryan, Hamiltonian Cosmology.
- S. Carrol.

Cosmological principle:

- The universe is spatialy homogeneous and isotropic.

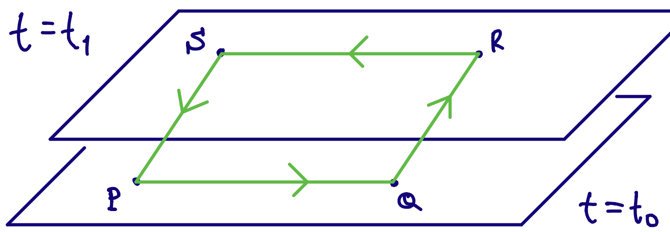
Think of an ideal gas.

Isotropic = Does not exists a prefeered direction.

Homogeneously = The universe is the same in any point and it has the same properties.

- At each epoch, the Universe presents the same aspect from every point.

(Except maybe for local irregularities)



$$g(P) \overset{H}{=} g(Q) \overset{I}{=} g(R) \overset{H}{=} g(S) \overset{I}{=} g(P)$$

Problems:

I. Statical fluctuation suggest that the models based on the cosmological principle cannot collapse fast enough to form observed galaxies.

II. Original Big Bang singularity may not have spherically symmetric pointlike structure.

III. The universe may have been anisotropic and inhomogeneous in the past.

We will consider the Cosmological principle since it allow us to make use of the extremely limited data provided by observational astronomical.

Definition: A substractum is a subspace of space where the galaxies can be considered as fundamental particles.

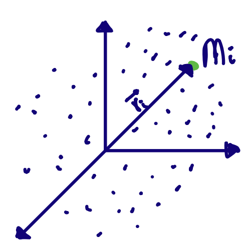
Weyl's postulate: The particles of the substractum lie in spacetime on a congruence of timelike geodesics diverging from a point in the finite or infinite past.

$$\text{Weyl's} \longrightarrow \text{perfect fluid.}$$

Newtonian cosmology

Consider a Universe which consists of a finite number of galaxies.

$$\left.\begin{array}{ll} m_i & \text{mass} \\ \vec{r_i}(t) & \text{position} \end{array}\right\} \text{i-th particle.}$$



Assumption: The motion about O must be spherically symmetric central forces

then, $\vec{r}_1(t) = r_1(t)\,\hat{r}$

Kinetic energy:

$$K = \frac{1}{2}\sum_{i=1}^{m} m_i \dot{r}_i^2$$

Potential energy:

$$V_G := -G \sum_{\substack{i,j=1\\ i>j}}^{n} \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

Cosmological energy:

$$\vec{F}_c = \frac{1}{3}\Lambda m_i \vec{r}_i \qquad \Lambda := \text{Cosmological force.}$$

As $\vec{F}_c = \vec{\nabla} U_c \longrightarrow U_c = -\frac{\Lambda}{6}\sum_{i=1}^{n} m_i r_i^2$

F_c is repulsive or atractive depending on the sign of Λ

$$\text{Energy} := E = \frac{1}{2}\sum_{i=1}^{m} m_i \dot{r}_i^2 - G \sum_{\substack{i,j=1\\ i>j}}^{n} \frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|} - \frac{\Lambda}{6}\sum_{i=1}^{n} m_i \vec{r}_i^{\,2} = \text{constant.}$$

Initial conditions: Distribution and motion are Known at some fixed point t.

Cosmological principle implies:

$$r_i(t) = a(t)\, r_i(t_0) \longleftarrow \text{Uniform expansion or contraction}$$

$a(t) :=$ Scale factor $\longrightarrow$ Independent of the particle

$$\dot{r}_i = \dot{a}(t)\, r_i(t_0) = \dot{a}(t)\frac{r_i(t)}{a(t)} = \frac{\dot{a}(t)}{a(t)}\, r_i(t).$$

$$=: H(t)\, r_i(t) \longrightarrow \text{Hubble's parameter}$$

$$\frac{\dot{r}_i(t)}{r_i(t)} = \frac{\dot{a}(t)}{a(t)} = H(t).$$

Read: A. Kaya, Am. J. Phys. 79. 1151 (2011).