If X, Y are Banach spaces, and $D(X)$ is the unitary ball in X, an operator $T: X \longrightarrow Y$ is compact if $T(D(X))$ is precompact.

Theorem: let X, Y two Banach spaces, if $T: X \longrightarrow Y$ is linear, T is compact if and only if $(x_n) \in X$ a bounded sequence implies that there is a subsequence x_{n_i}, such that Tx_{n_i} converges in Y.

Proof: $(\longrightarrow)$ Let's suppose that T is compact, and let (x_n) be a bounded in X, with

$$M = \sup_n \|x_n\| < \infty$$

let V be a closed ball in X of radius M, and center in zero. V is bounded in X, thus the $N = \overline{T(V)}$ is compact in Y.

As $Tx_n \in N$, there exists a convergent subsequence Tx_{n_i}, that converges to $y \in N$.

$(\longleftarrow)$ Let's suppose that if (x_n) is a bounded sequence in X, then, there exists a subsequence such that Tx_{n_i} is convergent.

let U the unitary ball in X, and let $y_n \in T(U)$ a subsequence. As a subset in a metric space is precompact if and only if all sequence has a convergent subsequence.

As y_n is in the image of T, there is a subsequence such that y_{n_i} is convergent and this implies that $T(U)$ is precompact. Then T is complete. $\blacksquare$

Dual operators

$$A: X \longrightarrow Y, \qquad A^*: Y^* \longrightarrow X^*$$

$$\varphi(Ax) \equiv (A^*\varphi)(x)$$

Let's introduce a notation more suitable, for caracterizing the duality of operators.

let X be a normed space and X^* its dual. For $f \in X^*$, $x \in X$, let's write the action of $f(x)$ as $\langle x, f \rangle$.

Applying this notation for the case of X and X^*, we can write a dual operator A^* of an operator A, of the following way:

For all $x \in X$ and $f \in Y^*$

$$\langle Ax, f \rangle = \langle x, A^* f \rangle$$

let's see that if $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$, then,

$$(BA)^* = A^* B^* : Z^* \longrightarrow X^*$$

In the case of Hilbert spaces $A \in \mathcal{L}(H_1, H_2)$ the dual operator is called adjoint $A \in \mathcal{L}(H_1, H_2)$ is defined as

$$\langle Ax, y \rangle = \langle x, A^* y \rangle, \quad x \in H_1, \quad y \in H_2.$$

Example: let $K(t,\tau) \in L_2(I^2)$, $I = [0,1]$, let's define the operator K in $L_2([0,1])$ by

$$(K,x)(t) = \int_0^1 K(t,\tau) x(\tau) d\tau.$$

show that the adjoint K^*, is

$$(K^* y)(t) = \int_0^1 \overline{K(t,\tau)} y(\tau) d\tau$$

$$\langle Kx, y \rangle = \int_0^1 \left(\int_0^1 K(t,\tau) x(\tau) d\tau \right) \overline{y}(t) dt$$

$$= \int_0^1 x(\tau) \overline{\left(\int_0^1 \overline{K(t,\tau)}\, y(t) dt \right)} d\tau$$

$$= \langle x, y' \rangle$$

with

$$y' = \underbrace{\int_0^1 \overline{K(t,\tau)}\, y(t) dt}_{=: K^* y}$$

Theorem: If $A : X \longrightarrow Y$ compact, $A^* : Y^* \longrightarrow X^*$ is compact.

Proof: We will show that, $A^* D(Y^*) = K \subseteq X^*$ is precompact

Let $f \in A^* D(Y^*)$, then $f(x) = (A^*\varphi)(x)$ for each $x \in (X)$ and some $\varphi \in D(Y^*)$, this is

$$f(x) = (A^*\varphi)(x) = \varphi(Ax)$$

then

$$\|f\|_{X^*} = \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\| \leq 1} |\varphi(Ax)| = \sup_{y \in T} |\varphi(y)|$$

for $T = AD(x)$, then T is precompact in Y, as A is compact.

Consider the collection of continuous functions in T, $M = \{\varphi \in D(Y^*)\}$. for any $\varphi \in D(Y^*)$, we have that

$$|\varphi(y)| \leq \|A\|, \quad y \in T.$$

and

$$|\varphi(y_1) - \varphi(y_2)| \leq \|y_1 - y_2\| < \varepsilon, \text{ for } \|y_1 - y_2\| < \varepsilon.$$

by the Arzela's theorem, M is precompact.

Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $A^* D(Y^*)$ of the relations of norms, it follows that the sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ is also bounded.

Then there is a sequence φ_n' such that $\|\varphi_n' - \varphi_n''\|_{C(T)} \to 0$ if $n', n'' \to \infty$.

Which means that $\|f_{n'} - f_{n''}\|_{X^*} \longrightarrow 0$, if $n', n'' \longrightarrow \infty$.

Then $A \in K(X, Y)$ implies that $A^* \in K(X^*, Y^*)$ i.e., A^* is compact.