Path integrals

- Ryder
- Zee
- Cacciatore
- Mathematical QFT

We may introduce first the path integral formulation of QM

Propagators

Given $\Psi(q_f, t_f)$ wave function at time t_i.

$\Longrightarrow$ The propagator gives the corresponding wave function at a later time t_f.

$\Longrightarrow$

$$\Psi(q_f, t_f) = \int \underbrace{K(q_f t_f ; q_i t_i)}_{\text{Propagator = Transition amplitude}} \Psi(q_i, t_i) dq_i$$

Divide $[t_i, t_f]$ into two subintervals

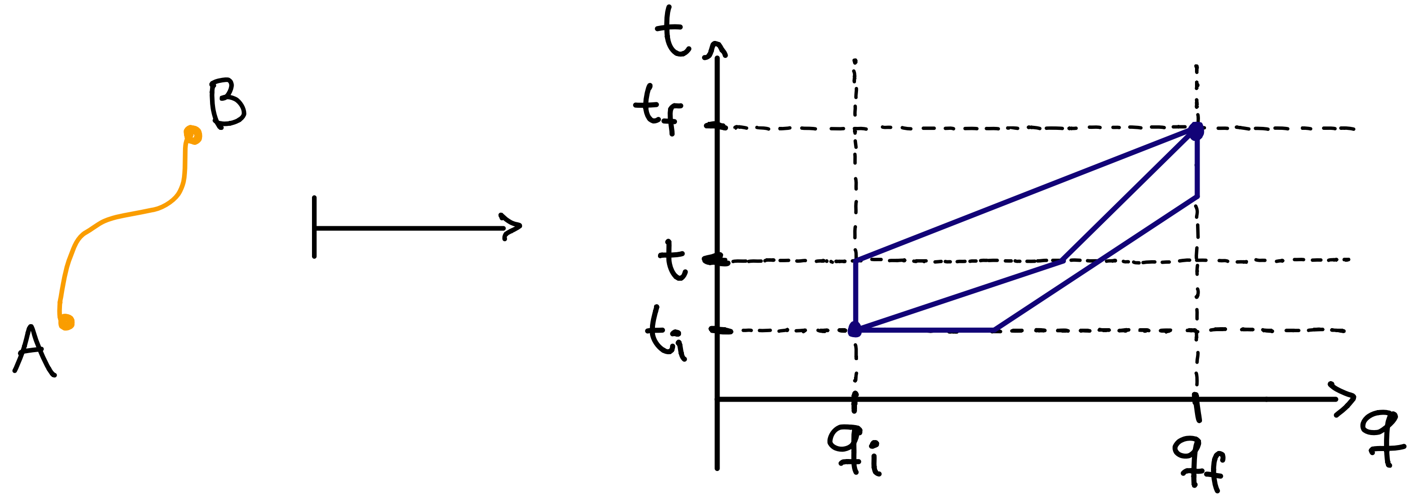
$$[t_i, t_f] = [t_i, t) \cup [t, t_f]$$

$$\Longrightarrow \quad \Psi(q_f, t_f) = \int K(q_f t_f, q t) \underbrace{K(q t, q_i t_i) \Psi(q_i t_i) dq_i}_{\Psi(q,t)} dq.$$

$$\Longrightarrow K(q_f t_f, q_i t_i) = \int K(q_f t_f ; q t) K(q t, q_i t_i) dq$$

$\therefore$ Transition from state i to state f may be thought of as the result of transition from state i to all intermediate points q followed by transition from q to state f.

This resembles a variational problem!



The integral over q intermediate step means a sum over all possible paths.

Theorem: Show that K is simply $\langle q_f t_f | q_i t_i \rangle$

Proof: first, note

$$\Psi(q,t) : \langle q | \Psi(t) \rangle_s$$

such that

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\Psi\rangle_H$$

Consider

$$|q(t)\rangle : |qt\rangle = e^{i\hat{H}t/\hbar} |q\rangle$$

$$\Longrightarrow \Psi(q,t) = \langle q | \Psi(t) \rangle_s = \left(e^{-i\hat{H}t/\hbar} \langle q(t)|\right)\left(e^{-i\hat{H}t/\hbar} |\Psi\rangle_H\right) = \langle qt | \Psi \rangle_H$$

Also, using

$$\int |qt\rangle\langle qt| \, dq = 1$$

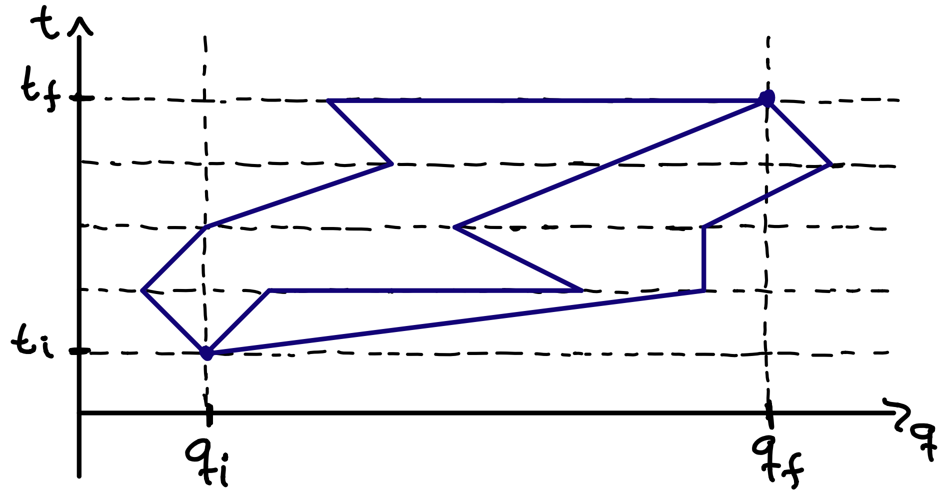
$$\Longrightarrow \Psi(q_f, t_f) = \langle q_f t_f | \Psi \rangle = \int \langle q_f t_f | q_i t_i \rangle \langle q_i t_i | \Psi \rangle \, dq_i$$

$$= \int \langle q_f t_f | q_i t_i \rangle \Psi(q_i, t_i) \, dq_i$$

$$\therefore \; K(q_f t_f ; q_i t_i) = \langle q_f t_f | q_i t_i \rangle$$

We want to express K as path integral.

Let us divide the interval $[t_i, t_f]$ into $n+1$ subintervals of size τ



$$\langle q_f t_f | q_i t_i \rangle = \int dq_1 \ldots dq_n \big(\langle q_f t_f | q_n t_n \rangle \langle q_n t_n | q_{n-1} t_{n-1} \rangle \times \cdots \times \langle q_2 t_2 | q_1 t_1 \rangle \langle q_1 t_1 | q_i t_i \rangle \big)$$

All possible trajectories

Consider now

$$\langle q_{j+1}, t_{j+1} | q_j t_j \rangle = \left(e^{-i\hat{H} t_{j+1}/\hbar} \langle q_{j+1} | \right) \left(e^{-i\hat{H} t_j/\hbar} | q_j \rangle \right)$$

$$= \langle q_{j+1} | \left(e^{-i\hat{H}(t_j - t_{j+1})/\hbar} | q_j \rangle \right)$$

$$= \langle q_{j+1} | \left(e^{-i\hat{H}\tau/\hbar} | q_j \rangle \right)$$

$$\simeq \langle q_{j+1} | \left(\left[1 - \frac{i}{\hbar} \hat{H} \tau + \mathcal{O}(\tau^2) \right] | q_j \rangle \right)$$

$$\simeq \langle q_{j+1} | q_j \rangle - \frac{i\tau}{\hbar} \langle q_{j+1} | \left(\hat{H} | q_j \rangle \right)$$

$$\delta(q_j - q_{j+1}) = \frac{1}{2\pi\hbar} \int dp \exp\left[-\frac{i}{\hbar} p (q_j - q_{j+1}) \right]$$

Fourier

$$\langle q_f t_f | q_i t_i \rangle = \int \underbrace{dq_1 \cdots dq_n}_{\text{All possible trajectories.}} \Big(\langle q_f t_f | q_n t_n \rangle \langle q_n t_n | q_{n-1} t_{n-1} \rangle \times \cdots \times \langle q_2 t_2 | q_1 t_1 \rangle \langle q_1 t_1 | q_i t_i \rangle \Big)$$

Consider now,

$$\langle q_{j+1} t_{j+1} | q_j t_j \rangle = \left(e^{-i\hat{H} t_{j+1}/\hbar} \langle q_{j+1}| \right) \left(e^{-i\hat{H} t_j/\hbar} | q_j \rangle \right)$$

$$= \langle q_{j+1} | \left(e^{-i\hat{H}(t_j - t_{j+1})/\hbar} | q_j \rangle \right)$$

$$= \langle q_{j+1} | \left(e^{-i\hat{H}\tau/\hbar} | q_j \rangle \right)$$

$$\simeq \langle q_{j+1} | \left(\left[1 - \frac{i}{\hbar} \hat{H}\tau + \mathcal{O}(\tau^2) \right] | q_j \rangle \right)$$

$$\simeq \langle q_{j+1} | q_j \rangle - \frac{i\tau}{\hbar} \langle q_{j+1} | \left(\hat{H} | q_j \rangle \right)$$

$$= \delta(q_j - q_{j+1}) - \frac{i\tau}{\hbar} \langle q_{j+1} | \left(\hat{H} | q_j \rangle \right)$$

$$\delta(q_j - q_{j+1}) \underset{\text{Fourier.}}{=} \frac{1}{2\pi\hbar} \int dp \exp\left[-\frac{i}{\hbar} p (q_j - q_{j+1}) \right]$$