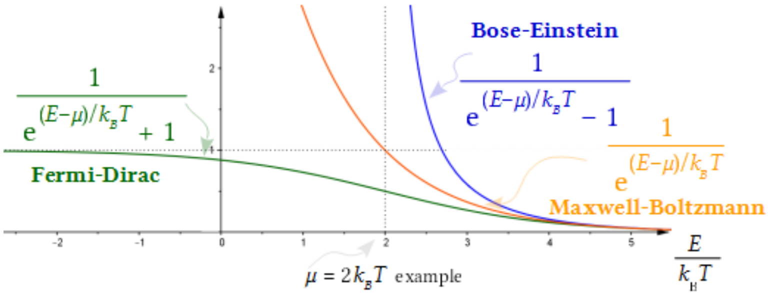
Classical limit



I. Fermions

$\langle n_j \rangle \approx 1$ if $\epsilon_j < \mu$

$\langle n_j \rangle \approx 0$ if $\epsilon_j > \mu$

II. Bosons

$\langle n_j \rangle \gg 1$ low energies.

$\langle n_j \rangle \approx 0$ Most states.

$$\exp(\beta(\epsilon_j - \mu)) \gg 1 \quad \text{classical limit}$$

This should happen for any ϵ_j, then

$$z = \exp(\beta\mu) \ll 1$$

since,

$$\frac{\exp(\beta\epsilon_j)}{\exp(\beta\mu)} \gg 1$$

Also

$$\ln(\Xi)_{FD,BE} = \sum_j \exp(-\beta(\epsilon_j - \mu)) \pm \frac{1}{2}\sum_j \exp(-2\beta(\epsilon_j - \mu)) + \dots$$

Remember

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

and

$$\langle n_j \rangle_{FD,BE} = \exp(-\beta(\epsilon_j - \mu))\} 1 \pm \exp(-\beta(\epsilon_j - \mu)) + \dots \{$$

$$\ln(\Xi)_{cl} = \sum_j \exp(-\beta(\epsilon_j - \mu))$$

$$\langle n_j \rangle_{cl} = \exp(-\beta(\epsilon_j - \mu))$$

Example:

$$\varepsilon_j = \varepsilon_{\vec{k},\sigma} := \frac{\hbar^2 k^2}{2m} \qquad \vec{p} = \hbar\vec{k}$$

$$\ln(\Xi)_{cl} = \sum_{\vec{k},\sigma} \exp\left(-\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)\right)$$

$$= \gamma \frac{V}{(2\pi)^3} \int d^3\vec{k}\, \exp\left(-\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)\right)$$

$$= \gamma \frac{V}{(2\pi)^3} \left(\frac{2\pi m}{\beta \hbar^2}\right)^{3/2}$$

where $\gamma = 2s+1$ Degeneration

$$\phi_{cl} = -\gamma V \left(\frac{2\pi m}{h^2}\right)^{3/2} (k_B T)^{5/2} \exp\left(\frac{\mu}{k_B T}\right)$$

has no classical analog.

As same as when we put $1/N!h^{3N}$ for the classical case.

We may write

$$\ln(\Xi)_{cl} = \gamma V z \left(\frac{2\pi m}{\beta h^2}\right)^{3/2}$$

then

$$\langle \sum_j n_j \rangle = z \frac{\partial}{\partial z} \ln(\Xi)_{cl} = \gamma V z \left(\frac{2\pi m}{\beta h^2}\right)^{3/2} \longrightarrow N$$

Also,

$$z = \exp(\beta\mu) = \frac{N}{V} \frac{h^3}{\gamma (2\pi m k_B T)^{3/2}}$$

Hence, as $z \ll 1$ in the classical limit, then

$$\frac{N}{V} \frac{h^3}{\gamma (2\pi m k_B T)^{3/2}} \ll 1$$

Notes:

I. $\left(\frac{V}{N}\right)^{1/3} = a$ Interatomic distance.

II. $\lambda = \frac{h}{(2\pi m K_B T)^{1/2}}$ Thermal wavelength.

$$\lambda_T = \frac{h}{p} = \frac{h}{(2mE)^{1/2}} = \frac{h}{\left(2m\frac{3K_BT}{2}\right)^{1/2}} = \frac{h}{(3mK_BT)^{1/2}}$$

of Broglie.

then, $\lambda_T \approx \lambda$

As.

$$\left(\frac{V}{N}\right)^{1/3} \gg \left(\frac{h^3}{\gamma(2\pi m K_B T)^{3/2}}\right)^{1/3} \approx \lambda_T$$

then, $a \gg \lambda_T$ classical limit.