Let's assume v an integrable vector in M. We can thingk of them as the velocity vector of a water molecule.

Let $\phi_t(p)$ the integral curve of v, what goes through $p \in M$. For each t, the map

$$\phi_t : M \longrightarrow M$$

is smooth, since the solution of a differential equation.

The molecule that was in p, at $t=0$, will be now $\phi_t(p)$, in the time t.

Thus, we will call to ϕ_t, the flow generated by v. The equation for the flow is

$$\frac{d\phi_t}{dt}(p) = V_{\phi(t)}P.$$

We can obtain new vector fields using the flow concept.

This form is called the Lie's bracket, or vector fields conmutator.

Given $v, w \in \mathrm{Vect}(M)$, we define

$$[v,w](f) = v(w(f)) - w(v(f)) \quad , \quad \forall f \in C^\infty(M)$$

or

$$[v,w] = vw - wv$$

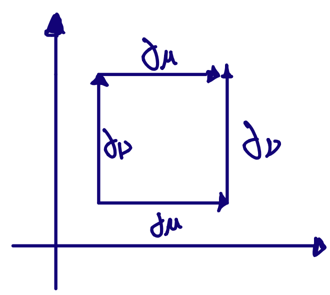
To prove that is a vector field, we must show the vector field properties. Let $u = [v,w]$

$$\begin{aligned} u(fg) &= (vw - wv)(fg) \\ &= v[w(f)g + fw(g)] - w[v(f)g + fv(g)] \\ &= vw(f)g + fvw(g) - wv(f)g - fwv(g) \\ &= u(f)g + fu(g). \end{aligned}$$

The Lie's bracket, measure the failure in which the directional derivatives not commute.

For the case of the ordinary derivatives.

$$[\partial_\mu, \partial_\nu] = 0$$



More precisely, let v generates the flow ϕ_t, and let w the ψ_t. For all $f \in C^\infty(M)$,

$$v(f)(p) = \frac{d}{dt} f(\phi_t(p))\Big|_{t=0}$$

$$w(f)(p) = \frac{d}{ds} f(\psi_s(p))\Big|_{t=0}$$

Now

$$[v,w](f)(p) = \frac{\partial^2}{\partial t \partial s}\left[f(\phi_t(\psi_s(p))) - f(\psi_s(\phi_t(p)))\right]_{s=t=0}$$

Exercise: Prove that for all vector field u, v, w, and $\alpha, \beta \in \mathbb{R}$

1. $[v,w] = -[w,v]$
II. $[u, \alpha v + \beta w] = \alpha[u,v] + \beta[u,w]$
III. Jacobi Identity:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

Answer:

1. $[v,w](f) = v(w(f)) - w(v(f))$

$$= -w(v(f)) + v(w(f))$$

$$= -(w(v(f)) - v(w(f)))$$

$$= -[w,v](f) \quad ; \qquad \forall f \in C^\infty(M).$$

II. $[u, \alpha v + \beta w](f) = u(\alpha v(f) + \beta w(f)) - (\alpha v(u(f)) + \beta w(u(f)))$

$$= \alpha uv(f) + \beta uw(f) - \alpha vu(f) - \beta wu(f)$$

$$= \alpha(uv(f) - vu(f)) + \beta(uw(f) - wu(f))$$

$$= \alpha[u,v] + \beta[u,w]$$

III. $[u,[v,w]](f) = u([v,w](f)) - [v,w](u(f))$

$$= uvw(f) - uwv(f) - vwu(f) + wvu(f)$$

Now,

$$[u,[v,w]](f) + [v,[w,u]](f) + [w,[u,v]](f)$$

$$= uvw(f) - uwv(f) - vwu(f) + wvu(f)$$

$$+vwu(f)-vuw(f)-wuv(f)+uwv(f)$$
$$+wuv(f)-wvu(f)-uvw(f)+vuw(f)$$
$$=0$$

$\blacksquare$

Differential forms

The idea is to generalise the gradient of a function in arbitrary manifolds concept: for each $f \in C^\infty(M)$ we will call df, the analogous of ∇f in $\mathbb{R}^n$.

The directional derivative of f in the v direction.

$$\nabla f \cdot v = v(f).$$

The gradient of f in $\mathbb{R}^n$, is a vector. The problem is the inner product. $\mathbb{R}^n$ has inner product of vectors, but manifolds not.

In geometry, a way to obtain inner products is with a metric.

So we can't think of df as a vector field.

from the expression $\nabla f \cdot v = v(f)$.

For each $v \in \mathrm{Vect}(M)$, $v(f)$ is a function, that is the directional derivative of f along of v, in other words, ∇f acts as an operator

$$v \longmapsto \nabla f \cdot v \quad \text{or} \quad v \longmapsto v(f)$$

The properties of this map are:

I. $\nabla f(v+w) = \nabla f \cdot v + \nabla f \cdot w$; $\forall\, v, w \in \mathrm{Vect}(\mathbb{R}^n)$

II. $\nabla f(gv) = g(\nabla f \cdot v)$; $g \in C^\infty(\mathbb{R}^n)$

linearity in $C^\infty(\mathbb{R}^n)$

Let's define a 1-form from a manifold M, as a map from $\mathrm{Vect}(M)$ to $C^\infty(M)$ and is linear in $C^\infty(M)$.

In other words, if $v \in \mathrm{Vect}(M)$, ω is a 1-form, if $\omega(v) \in C^\infty(M)$ and

$$\omega(v+u) = \omega(v) + \omega(u)$$

$$\omega(gu) = g\,\omega(u).$$

Using $\Omega^1(M)$, to denote the 1-forms space in M.
If $f \in C^\infty(M)$, there exists a 1-form df, defined by
$$df(v) = v(f)$$
is other form
$\nabla f \cdot v = v(f)$

To prove that df is a 1-form, it must satisfy linearity.
$$df(v+w) = (v+w)(f) = v(f) + w(f)$$
$$= df(v) + df(w).$$
$$df(gv) = gv(f) = g\,df(v).$$
df is called the differential of f or the exterior derivative of f

The 1-forms may be summed and multiplied by functions.
let ω, μ be 1-forms, lets define the sum and multiplication.
$$(\omega + \mu)(v) = \omega(v) + \mu(v)$$
$$(f\omega)(v) = f\omega(v).$$
The map $d : C^\infty(M) \longrightarrow \Omega^1(M)$ and is called differential or exterior derivative
$$f \longmapsto df.$$
Properties:

I. $d(f+g) = df + dg$

II. $d(\alpha g) = \alpha dg$

III. $(f+g)dh = fdh + gdh.$

IV. $d(fg) = fdg + dfg.$

Proof:

I. $d(f+g)(v) = v(f+g) = v(f) + v(g) = df(v) + dg(v)$
then
$$d(f+g) = df + dg.$$
IV. $d(fg)(v) = v(fg) = v(f)g + fv(g) = df(v)g + f\,dg(v)$

then,

$$d(fg) = df\,g + f\,dg.$$

Usually, $\frac{df}{dx}$, $d\sin x = \cos x\,dx$, $\int f(x)dx$.

Lets prove that $d\sin x = \cos x\,dx$ as 1-form.

Any vector in $\mathbb{R}$, has the form $v = f(x)\partial x$.

$$(d\sin x)(v) = v(\sin x) = f(x)\partial x(\sin x) = f(x)\cos x.$$

$$(\cos x\,dx)(v) = \cos x\, v(x) = \cos x f(x)\,\partial x(x) = \cos(x) f(x).$$

Theorem: let $f(x^1, \dots, x^n) \in C^\infty(\mathbb{R}^n)$, prove that

$$df = \frac{\partial f}{\partial x^\mu}\partial x^\mu = \partial_\mu f\, dx^\mu$$

Proof: let $v \in \text{Vect}(\mathbb{R}^n)$, $v = v^\mu \partial_\mu$

$$df(v) = v(f) = v^\mu \partial_\mu f$$

$$\partial_\mu f\, dx^\mu(v) = \partial_\mu f\; v(x^\mu)$$

$$= \partial_\mu f(v^\nu \cancel{\partial_\nu(x^\mu)})$$

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

$$= \partial_\mu f\, v^\mu$$

The exterior derivative of a function in $\mathbb{R}^n$, is other way to think the gradient

$$\nabla f = (\partial_1 f, \dots, \partial_n f)$$