$$K(q_f t_f; q_i t_i) := \langle q_f t_f | q_i t_i \rangle = \int \frac{Dq\,Dp}{2\pi\hbar} e^{i/\hbar S} \quad ; \quad S := \text{classical action.}$$

$$\exp\left(\frac{iS}{\hbar}\right) = \exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f} (K - V)\,dt\right)$$

$$\exp\left[-\frac{i}{\hbar}\int_{t_i}^{t_f} V(x,t)\,dt\right] = 1 - \frac{i}{\hbar}\int_{t_i} V(x,t)\,dt - \frac{1}{2!\hbar}\left[\int_{t_i}^{t_f} V(x,t)\,dt\right]^2 + \dots$$

Born series

Zeidler.

$$\Rightarrow K = K_0 + K_1 + \dots$$

$K_0 :=$ Free propagator

$$K_0 = N \int \exp\left(\frac{i}{\hbar}\int \frac{1}{2} m\dot{x}\,dt\right) Dx$$

$$\boxed{N = \int \frac{Dp}{2\pi\hbar} \text{ momentum value}}$$

$$K_0 = \lim_{n\to\infty}\left(\frac{m}{i\hbar\tau}\right)^{(n+1)/2} \int_{-\infty}^{\infty} \prod_{j=1}^{n} dx_j \exp\left[\frac{im}{2\pi\tau}\left[\sum_{j=0}^{n}(X_{j+1} - X_j)^2\right]\right]$$

Discrete form

Theorem:

$$\int_{-\infty}^{\infty} \exp[i\lambda[(X-a)^2 + (X_2 - X_1)^2 + \dots + (b - X_n)^2]]\,dx_1 \cdots dx_n = \left(\frac{i^n \pi^n}{(n+1)\lambda}\right)^{1/2} \exp\left[\frac{i\lambda}{n+1}(b-a)^2\right]$$

$$\therefore \quad K_0 = \frac{1}{(n+1)^{1/2}}\left(\frac{i\hbar\tau}{m}\right)^{-1/2} \exp\left[\frac{im}{2\hbar(n+1)\tau}(X_f - X_i)^2\right]$$

$$(n+1)\tau = t_f - t_i$$

$$K_0 = \left(\frac{m}{i\hbar(t_f - t_i)}\right)^{1/2} \exp\left[\frac{im\,(X_f - X_i)^2}{2\hbar\,(t_f - X_i)}\right] \qquad t_f > t_i \rightarrow \text{Causality}$$

$$K_0 = \Theta(t_f - t_i)\left(\frac{m}{i\hbar(t_f - t_i)}\right)^{1/2} \exp\left[\frac{im}{2\hbar}\frac{(X_f - X_i)^2}{(t_f - t_i)}\right]$$

$\theta :=$ Heaviside step function.

$$\theta(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

For K_1 discrete $\int dt \longrightarrow \sum_i t_i$.

$$K_1 = -\frac{i}{\hbar} \lim_{n\to\infty} N^{(n+1)/2} \sum_{i=1}^{n} \tau \int \exp\left[\frac{im}{2\hbar\tau} \sum_{j=0}^{n} (x_{j+1} - x_j)^2\right] V(x_i, t_i)\, dx_1 \ldots dx_n.$$

$$N = \frac{m}{i\hbar\tau}$$

$$\sum_{j=0}^{n} \longrightarrow \sum_{j=0}^{i-1} + \sum_{j=i}^{n}$$

$$K_1 = \lim_{n\to\infty} \frac{-i}{\hbar} \sum_{i=1}^{n} \tau \int dx_i \left\{ N^{(n-i+1)/2} \int \exp\left[\frac{im}{2\hbar\tau} \sum_{j=i}^{n} (x_{j+1} - x_j)^2\right] dx_{i+1} \ldots dx_n \right\}$$

$$\times V(x_i, t_i) \left\{ N^{i/2} \int \exp\left[\frac{im}{2\hbar\tau} \sum_{j=0}^{i-1} (x_{j+1} - x_j)^2\right] dx_1 \cdots dx_{i-1} \right\}$$

but,

$$K_0(x_f t_f, x_i t_i) = \lim_{n\to\infty} (N)^{(n+1)/2} \int \prod_{j=1}^{n} dx_j \exp\left[\frac{im}{2\hbar\tau} \sum_{j=0}^{n} (x_{j+1} - x_j)^2\right]$$

$$x_i t_i \longmapsto x t$$

then, the first term in $\} \{$ is $K_0(x_f t_f, x t)$ and 2nd term in $\} \{$ is $K_0(x t, x_i t_i)$.

$$\Rightarrow K_1(x_f t_f, x_i t_i) = \frac{-i}{\hbar} \int_{t_i}^{t_f} dt \int K_0(x_f t_f; x t) V(x t) K_0(x t; x_i t_i)\, dx$$