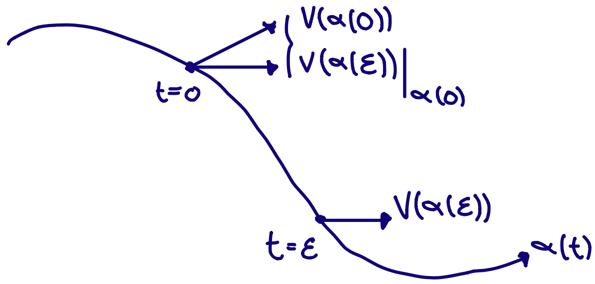
Definition: A vectorial field V is parallel if its covariant derivatives $\nabla_X V$ are zero for all $X \in \mathfrak{X}(M)$.



In particular

$$\nabla_{\alpha'(t)} V := \lim_{\varepsilon \to \infty} \frac{V(\alpha(\varepsilon)) \text{ Parallel transported to } \alpha(0) - V(\alpha(0))}{\varepsilon}$$

Definition: A connection ∇ on a pseudo Riemannian manifold M is compatible with metric if and only if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \quad \forall X, Y, Z \in \mathfrak{X}(M)$$

Take $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$.

Then,

$$\partial_i \langle \partial_j, \partial_k \rangle = \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_{\partial_i} \partial_k \rangle$$

$$\begin{aligned}
& \partial_i g_{jk} = \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl} \\
+ \quad & \partial_j g_{ki} = \Gamma^l_{jk} g_{li} + \Gamma^l_{ji} g_{kl} \\
- \quad & \partial_k g_{ij} = -\Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il} \\
\hline
& \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} = 2\Gamma^l_{ij} g_{kl}.
\end{aligned}$$

$$\Gamma^m_{ij} = \frac{1}{2} g^{km} (\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij})$$

$\Gamma \Longleftrightarrow \nabla$ is the levi-Civita condition

Geodesics

- Curve with zero acceleration
  - Generalise Euclidean notion of a straight line.

Objective: Show that geodesics minimize arc-length for sufficiently close points.

Definition: A parametrised curve $\gamma: I \longrightarrow M$ is a geodesic at $t = t_0 \in I$ if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)\bigg|_{t=t_0} = 0.$$

where

$$\frac{D}{dt} : \mathfrak{X}(\gamma) \longrightarrow \mathfrak{X}(\gamma)$$

such that

$$\frac{DV}{dt} = \nabla_{\frac{d\gamma}{dt}} V$$

therefore

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \nabla_{\frac{d\gamma}{dt}}\left(\frac{d\gamma}{dt}\right) = 0.$$

Definition: If γ is a geodesic at t, $\forall t \in I$, then γ is a geodesic on I

Definition: If $[a,b] \subset I$ and $\gamma: I \longrightarrow M$ is a geodesic, the restriction of γ to $[a,b]$ is called a geodesic segmen joining $\gamma(a)$ to $\gamma(b)$.

If γ is a geodesic, we have

$$\frac{D}{dt}\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = \left\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle + \left\langle \frac{d\gamma}{dt}, \frac{D}{dt}\frac{d\gamma}{dt} \right\rangle$$

$$= 2\left\langle \cancel{\frac{D}{dt}\frac{d\gamma}{dt}}^{\,0}, \frac{d\gamma}{dt} \right\rangle = 0$$

Length of the velocity vector is constant.

$$\langle X, Y \rangle = \langle X^i \partial_i, Y^j \partial_j \rangle$$
$$= X^i Y^j \langle \partial_i, \partial_j \rangle = X^i Y^j g_{ij}.$$

$$\left\| \frac{d\gamma}{dt} \right\| := \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle^{1/2}$$, length of the velocity vector.

then, arc-length s for γ is given by

$$s(t) = \int_{t_0}^{t} \left\| \frac{d\gamma}{dt'} \right\| dt' = c(t - t_0).$$

Local equation for a geodesic

let γ be written in system of coordinates ξ about $\gamma(t_0)$.

$$\gamma(t) = (x^1(t), \dots, x^n(t))$$

$\hookrightarrow$ Notation before: $\xi^i \circ \gamma : I \longrightarrow M \longrightarrow \mathbb{R}^n$

$$\frac{D}{dt} : \mathfrak{X}(\gamma) \longrightarrow \mathfrak{X}(\gamma)$$

$$\frac{DV}{dt} := \nabla_{\frac{d\gamma}{dt}} V$$

$$V = V^i \partial_i$$

$$\frac{d\gamma}{dt} = \frac{dx^j}{dt} \partial_j = \frac{d}{dt}$$

$$\nabla_{\frac{dx^j}{dt}\partial_j}(V^i \partial_i) = \frac{dV^i}{dt} \partial_i + V^i \nabla_{\frac{dx^j}{dt}\partial_j}(\partial_j)$$

$$= \frac{dV^i}{dt} \partial_i + \frac{dx^j}{dt} V^i \nabla_{\partial_j} \partial_i + \frac{dx^j}{dt} V^i \Gamma^k_{ji} \partial_k .$$

$$\frac{DV}{dt} = \frac{dV^i}{dt} \partial_i + \Gamma^k_{ij} \frac{dx^j}{dt} V^i \partial_k$$

$$= \left(\frac{dV^k}{dt} + \Gamma^k_{ij} \frac{dx^j}{dt} V^i \right) \partial_k$$

$$0 = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \left(\frac{d}{dt}\left(\frac{d\gamma^k}{dt}\right) + \Gamma^k_{ij} \frac{dx^j}{dt} \frac{dx^i}{dt} \right) \partial_k$$

$$= \left(\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^j}{dt} \frac{dx^i}{dt} \right) \partial_k$$

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^j}{dt} \frac{dx^i}{dt} = 0$$ Geodesic equation

$$\frac{d^2 (x^k \circ \gamma)}{dt^2} + \Gamma^k_{ij} \frac{d(x^j \circ \gamma)}{dt} \frac{d(x^i \circ \gamma)}{dt} = 0 .$$

Geodesic from a variational principal

$$s = \int_a^b \left[g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right]^{1/2} dt$$

a, b endpoints of the curve γ, δx vanishes on the boundary

$$\delta s = \int_a^b \delta\left[g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]^{1/2} dt$$

$$= \int_a^b \frac{1}{2}\left[g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]^{-1/2}\left(\delta g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + 2g_{\mu\nu}\frac{dx^\mu}{dt}\delta\left(\frac{dx^\nu}{dt}\right)\right) dt$$

$$= \int_a^b \left(g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{-1/2}\left[g_{\mu\nu}\frac{dx^\mu}{dt}\delta\left(\frac{dx^\nu}{dt}\right) + \frac{1}{2}\delta g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right] dt$$

$$\delta\left(\frac{dx^\nu}{dt}\right) = \frac{d}{dt}(\delta x^\nu)$$

$$\delta g_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\delta x^\sigma$$

Also, since length is parametrization invariant, we can take $g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 1$

$$0 = ds = \int_a^b \left[g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{d}{dt}(\delta x^\beta) + \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\delta x^\sigma \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right] dt$$

$$= \int_a^b \left[\cancelto{0}{\frac{d}{dt}\left(g_{\alpha\beta}\frac{dx^\alpha}{dt}\delta x^\beta\right)} - \frac{d}{dt}\left(g_{\alpha\beta}\frac{dx^\alpha}{dt}\right)\delta x^\beta + \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\delta x^\sigma\right] dt$$

$$= \int_a^b \left[-\frac{d}{dt}\left(g_{\alpha\sigma}\frac{dx^\alpha}{dt}\right) + \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right]\delta x^\sigma dt.$$

then

$$\frac{d}{dt}\left(g_{\alpha\sigma}\frac{dx^\alpha}{dt}\right) - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0$$

$$\frac{d^2x^\alpha}{dt^2}g_{\alpha\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta}\frac{dx^\beta}{dt}\frac{dx^\alpha}{dt} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0$$

$$g^{\sigma\mu}\left[\frac{d^2x^\alpha}{dt^2}g_{\alpha\sigma} + \frac{1}{2}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \frac{\partial g_{\beta\sigma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0\right]$$

$$\frac{d^2x^\alpha}{dt^2}\delta^\mu_\alpha + \frac{1}{2}g^{\sigma\mu}(\partial_\beta g_{\alpha\sigma} + \partial_\alpha g_{\beta\sigma} - \partial_\sigma g_{\alpha\beta})\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0$$

$$\frac{d^2x^\alpha}{dt^2}\delta^\mu_\alpha + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0$$

Example: Sheet of paper in polar coordinates (r,ϕ)

2D-metric $\qquad ds^2 = dr^2 + r^2 d\phi^2$

connection coefficients

$$\Gamma^r_{\phi\phi} = -r$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}$$

Geodesic equations:

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 = 0$$

$$\frac{d^2\phi}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\phi}{dt} = 0$$