

Peter-Weyl's theorem

For a compact group K , the irreducible representations form an orthonormal base of $L^2(K)$.

Lemma: Let (π, V_π) be an unitary representation of finite dimension of some G group, locally compact. So π , decomposes into the direct sums of irreducible representations.

Proof: By induction it's proved in the V_π dimension.

If V_π is onedimensional, the representation is irreducible, and the theorem is fulfilled.

generalized functions.
distribution space.

Now, let's suppose that it is valid for all spaces of smaller dimension than the $\dim(V_\pi)$. Then V_π is irreducible, in such case its proved the lemma or it has an invariant representation w .

But, $w^\perp = \{w \in V_\pi / \langle v, w \rangle = 0, \forall w \in W\}$, is also invariant.

Then $V_\pi = W \oplus W^\perp$ that are of smaller dimension and these are decomposed in irreducibles and thus, V_π also. ■

Let K be a compact matricial group i.e., a compact subgroup of $GL_n(\mathbb{C})$. Let τ and γ irreducible representations of finite dimension of K . Let H the space of all linear maps from V_γ to V_τ . In this space let's define a new representation η of K , by

$$\eta(K)T = \tau(K)T\gamma(K^{-1})$$

Let $\text{Hom}_K(V_\gamma, V_\tau)$ be the space of the K -homomorphisms i.e., the spaces of all linear maps $T: V_\gamma \rightarrow V_\tau$, such that

$$T_\gamma(K) = \tau(K)T, \quad \forall K \in K.$$

Lemma: $\text{Hom}_K(V_\gamma, V_\tau)$ is atleast one dimensional.

Two unitary representations are called isomorphs if and τ of K , there is an unitary map $T: V_\gamma \rightarrow V_\tau$ and satisfies $T_\gamma(K) = \tau(K)T$, for all $K \in K$. For each (τ, V_τ) , let be an orthonormal base e_1, \dots, e_n of V and let $\tau_{ij}(K) = \langle \tau(K)e_i, e_j \rangle$. The map $\tau_{ij}: K \rightarrow \mathbb{C}$, it is called (i,j) -eslm of the matrix τ .

Theorem (Peter-Weyl): Let $\tau \neq \tau$, then.

$$\int_K \tau_{ij}(K) \overline{\gamma_{rs}(K)} dK = 0 \quad \text{for all index } i,j,r,s.$$

Moreover

$$\int_K \tau_{ij}(K) \overline{\tau_{rs}(K)} dK = 0 \quad \text{unless } i=r, j=s.$$

In this case

$$\int_K \tau_{ij}(K) \overline{\tau_{ij}(K)} dK = \frac{1}{\dim(V_\tau)}$$

the family $(\sqrt{\dim(V_\tau)}' \tau_{ij})$ form a base of $L^2(K)$

Proof: let's assume that $\gamma \neq \tau$, then for each $T \in \text{Hom}_K(V_\gamma, V_\tau)$ we have that

$$S = \int_K T(K) T \gamma(K') dK$$

satisfy $T(K)S = S \gamma(K)$, $S = 0$. Let e_1, \dots, e_n be a base of V_γ and f_1, \dots, f_m one of V_τ .

Sea $T \in \text{Hom}_K(V_\gamma, V_\tau)$ given by the matrix E_{ij} , with ones in the position (i, j) and zeros in the other positions. Then

$$\begin{aligned} & \begin{pmatrix} \tau_{11} & \cdots & \tau_{1m} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \tau_{m1} & \cdots & \tau_{mm} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ \tau_{nn} \end{pmatrix} \begin{pmatrix} \overline{\gamma_{11}} & \cdots & \overline{\gamma_{1n}} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \overline{\gamma_{n1}} & \cdots & \overline{\gamma_{nn}} \end{pmatrix} \\ &= \begin{pmatrix} \tau_{1i} \\ \vdots \\ \vdots \\ \vdots \\ \tau_{ni} \end{pmatrix} \begin{pmatrix} \overline{\gamma_{11}} & \cdots & \overline{\gamma_{1n}} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \overline{\gamma_{n1}} & \cdots & \overline{\gamma_{nn}} \end{pmatrix} = \begin{pmatrix} \tau_{1i} \overline{\gamma_{1j}} & \cdots & \tau_{1i} \overline{\gamma_{nj}} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \tau_{ni} \overline{\gamma_{1j}} & \cdots & \tau_{ni} \overline{\gamma_{nj}} \end{pmatrix} = 0 \end{aligned}$$

for $\gamma = \tau$

$$\int_K T(K) T \gamma(K') dK = \lambda \text{Id.} \quad \text{for some } \lambda \in \mathbb{C}$$

to see that $\int_K |\tau_{ij}(K)|^2 dK = \frac{1}{\dim(V_\tau)}$

let $T = \text{Id}$, and remaining $T(K)T(K^*) = \text{Id}$, implies that

$$\sum_r T_{ir} \overline{T_{jr}(K)} = \delta_{ij}$$

then

$$\sum_r \int_K |T_{ij}(K)|^2 dK = 1$$

from the fact that

$$\int_K T(K) T(K^{-1}) dK = \lambda \text{Id}.$$

In the case that $T = E_{ii}$, implies

$$\int_K |T_{ri}(K)|^2 dK = \int_K |T_{r'i}(K)|^2 dK \quad \text{for some } r \text{ and } r'.$$

Finally

$$\int_K |T_{ri}(K)|^2 dK = \int_K |T_{ri}(K^{-1})|^2 = \int_K |T_{ir}(K)|^2 dK$$

Homework:

$$\int_K f(K) dK = \int_K f(K^{-1}) dK.$$

$$\int_K |T_{ir}(K)|^2 dK = \int_K |T_{ir}(K)|^2 dK$$

we will use the Stone-Weierstrass theorem.

Let X be a metrizable compact space. In $C(X)$ a norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

and a metric $d(f, g) = \|f - g\|_\infty$. $C(X)$ is an algebra.

Lemma (Stone-Weierstrass)

Let X be a compact metrizable space and A be a subalgebra of $C(X)$, such that.

i) A is closed under the complex conjugation i.e., $f \in A, \bar{f} \in A$.

ii) A separates points i.e., $\forall x, y \in X, x \neq y \exists f \in A$ such that $f(x) \neq f(y)$

therefore A is dense in $C(X)$.

Let A be the space generated by $\tilde{\tau}_i$ in $C(K)$.

A is closed under conjugation, let $\bar{\tau}$ is a representation of K .

A separates points, since K is a matricial group and it has an unitary injective representation.

By the Stone-Weierstrass theorem A is dense in $C(K)$, but $C(K)$ are dense in $L^2(K)$.

Therefore A is dense in $L^2(K)$. ■