Functions in groups

Lets to denote the translations

$$f(t) \longmapsto T_{t_1} f(t-t_1)$$

and integrate them in an invariant way, under translations

$$f(t) \longmapsto I_f = \int f(t)\,dt \quad \text{is invariant}$$

A pair of integrable functions are in convolution

$$f_1 \cdot f_2(t) = \int f_1(t) f_2(t-t_1)\,dt_1$$

We can define real functions or complex in any group G.

If the group is topological, let's consider the continuous functions and if it is of Lie, the differentiables. We may have the operations.

i) Translations by the left

$$T^l_{g_1} f(g) = f(g_1^{-1} g)$$

ii) Translations by the right

$$T^r_{g_1} f(g) = f(g g_1)$$

as

$$T^l_{g_1} T^l_{g_2} = T^l_{g_1 g_2}, \quad T^r_{g_1} T^r_{g_2} = T^r_{g_1 g_2}$$

Invariant measures

In any Lie's group G, there are two invariant measures (Haar's Measure), that may be define, by asking for invariance by right.

$$\int f(g)\,d\mu_r(g) = \int f(g g_1)\,d\mu_r(g)$$

or by left

$$\int f(g)\,d\mu_l(g) = \int f(g_1^{-1} g)\,d\mu_l(g)$$

are unique until a constant factor. If the group is compact, both match.

for $SL(2,\mathbb{C})$, use the parametrization

$$a = a_0 e + \sum_{K=1}^{3} a_K \sigma_K$$

$Dz = dx\,dy$

$z = x + iy.$

then

$$d\mu(a) = C_0\, \delta\left(a_0^2 - \sum_{K=1}^{3} a_K - 1\right) \prod_{i=0}^{3} Da_i\,, \quad C_0^2 \text{ by convension.}$$

In a similar way for $SU(2), SU(1,1), SL(2,\mathbb{R})$.

- $SU(2)$: $$d\mu(u) = C_0\, \delta\left(\sum_{j=0}^{3} u_j^2 - 1\right) \prod_{i=0}^{3} du_i$$

- $SU(1,1)$: $$d\mu(v) = C_0\, \delta(\det(v) - 1) \prod_{i=0}^{3} dv_i$$

- $SL(2,\mathbb{R})$: $$d\mu(a) = C_0\, \delta(\det(a) - 1) \prod_{i=1}^{3} da_i$$

for $SU(2)$, the parameters

$$u_0^2 + \sum_{k=1}^{3} u_k^2 = 1$$

And therefore, thus

Homework:

$$\int_{SU(2)} d\mu(u) = \int_{SU(2)} C_0\, \delta\left(\sum_{j=0}^{3} u_j^2 - 1\right) du_0\, du_1\, du_2\, du_3 = \frac{1}{2} C_0\, \Omega_4.$$

where Ω_4, is the area of the unitary sphere in $\mathbb{R}^4$.

$$\Omega_4 = 2\pi^2 \longrightarrow C_0 = \frac{1}{\pi^2}$$

Let's consider the Lie's group, formed by a semidirect product $G = H \times T$, with multiplication

$$g = (h,t) \quad , \; h \in H,\; t \in T.$$

$$(h_1,t_1)(h_2,t_2) = (h_1 h_2 t_1,\; t_1 + A_{h_1}(t_2))$$

where A_{h_1} is a linear transformation on T.

The invariant measure of G is the product of the invariant measures if the $\det(A_n)=1$.

for the invariant case $SL(2,\mathbb{C})\times T_4$

$$d\mu = d\mu_{SL(2,\mathbb{C})}\, d\mu(T_4),\ \text{since } |\det(A)|=|\det(\Lambda)|=1$$

Unitary representations

A unitary representation is a homeomorphism of the G group in the set of unitary operators U, in the Hilbert space H.

$$g \longmapsto U_g \ , \quad U_{g_1}U_{g_2}=U_{g_1 g_2}, \quad U_e = I$$

In general, we assume that it is continuous.

$$\| U_g \xi - U_{g_0}\xi \| \longrightarrow 0 \quad \text{if } g\to g_0,\ \forall \xi \in H.$$

A representation is irreducible if and only if its uniques invariant subspaces of H, are H and the null space.

Homogeneous function space.

The representations of $SL(2,\mathbb{C})$, may constructed using the homogeneous function space. Which is a generalization of the polynomial space of grade $2s$ and $2s'$ in the variables (ξ^1,ξ^2) and (η_1,η_2), and then we are going to denoting them by (z_1,z_2) and $(\bar{z}_1,\bar{z}_2)$

We say that $F(z_1,z_2)$ is homogeneous of grade λ and μ in z_1,z_2 if for all $\alpha\in\mathbb{C}$, $\alpha\neq 0$.

$$F(\alpha z_1,\alpha z_2)=\alpha^{\lambda}\bar{\alpha}^{\mu}F(z_1,z_2)$$

we can change the condition of homogenous

$$F(e^{i\omega}z_1,e^{i\omega}z_2)=e^{i(\lambda-\mu)\omega}F(z_1,z_2)$$

It is reduced to the identity if $\omega=2\pi n$, $n\in\mathbb{Z}$. This requires that

$$\mu-\lambda=m\in\mathbb{Z}$$

Instead of the grades μ and λ. Characterizing the homogeneous functions by the tags $\chi\}n_1,n_2)=(m,\rho)$

$$n_1=\lambda+1=-\frac{1}{2}m+\frac{i}{2}\rho$$

$$n_2=\lambda+1=\frac{1}{2}m+\frac{i}{2}\rho$$

The space of homogeneous functions D_x, it's define as:

I) D_x is a vectorial space of the homogeneous functions of grade x.

II) Any element infinitely differentiable in $z_1, \bar{z}_1, z_2, \bar{z}_2$

The importance of D_x, is that we may define an operador T_a^x, for $a \in SL(2,\mathbb{C})$ such that

$$T_{a_1}^x T_{a_2}^x = T^x_{a_1 a_2}$$

Group operations:

We define an operator T_a^x, for $a \in SL(2,\mathbb{C})$ in D_x

$$T_a^x F(z_1, z_2) = F(z_1', z_2') = F(z_1 a_{11} + z_2 a_{21} + z_1 a_{12} + z_2 a_{22})$$

as the origin $z_1 = z_2 = 0$, map itself $z_1' = z_2' = 0$.

$$T_a^x F \in D_x \text{ , if } F \in D$$

It's is very useful to use other form D_x

$$F(z) = F(z, 1)$$

Because of the homogeneity of $F(z_1, z_2)$, know $F(z)$ is sufficient to construct $F(z_1, z_2)$

$$F(z_1, z_2) = z_2^{n_1 - 1} z_2^{n_2 - 1} f\left(\frac{z_1}{z_2}\right)$$

In this space T_a^x looks like

$$T_a^x f(z) = \alpha(z, a) f(z_a)$$

$$z_a = \frac{a_{11} z + a_{21}}{a_{12} z + a_{22}} \quad , \quad \alpha(z,a) = (a_{12} z + a_{12})^{n_1 - 1} (a_{12} z + a_{22})^{n_2 - 1}$$

The transformation $T_a^x F(z_1, z_2)$ may be written by multiplication

$$\begin{pmatrix} \cdots & \cdots \\ z_1 & z_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cdots & \cdots \\ z_1' & z_2' \end{pmatrix}$$

The matrix consisting of z_1 and z_2

$$\begin{pmatrix} \cdots & \cdots \\ z_1 & z_2 \end{pmatrix} = \begin{pmatrix} z_2^{-1} & \cdots \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{z_1}{z_2} & 1 \end{pmatrix}$$

as any matrix of $SL(2,\mathbb{C})$, may decomposed as a product

$$a = k \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \text{ if } a_{22} \neq 0 \qquad k = \begin{pmatrix} z_2^{-1} & 0 \\ 0 & z_2 \end{pmatrix}$$

$SL(2,\mathbb{C})/K$ maps to $z \in \mathbb{C}$.