$$I(\lambda) = \int_C f(z) e^{\lambda \varphi(z)} dz.$$

$$C \longrightarrow C^*$$

$$(x, y, u) \qquad u = u(x,y)$$

The steepness of the curve $(x(s), y(s), u(s))$ with the u-axis.

$$\cos(\alpha) = \frac{du/ds}{\sqrt{1 + \left(\frac{du}{ds}\right)^2}}$$

$$\frac{d}{d\theta}\cos(\alpha) = \frac{\frac{d}{d\theta}\left(\frac{du}{ds}\right)\sqrt{1+\left(\frac{du}{ds}\right)^2} - \frac{du}{ds}\,\frac{1}{2}\left(\sqrt{1+\left(\frac{du}{ds}\right)^2}\right)^{-1}\frac{du}{ds}\,\frac{d}{d\theta}\left(\frac{du}{ds}\right)}{\left(\sqrt{1+\left(\frac{du}{ds}\right)^2}\right)^2}$$

$$= \frac{\left(\sqrt{1+\left(\frac{du}{ds}\right)^2}\right)^2 - \left(\frac{du}{ds}\right)^2}{\left(\sqrt{1+\left(\frac{du}{ds}\right)^2}\right)^3} \qquad \frac{d}{d\theta}\left(\frac{du}{ds}\right)$$

$$= \frac{1}{\left(1+\left(\frac{du}{ds}\right)^2\right)^{3/2}} \frac{d}{d\theta}\left(U_x \cos(\theta) + U_y \sin(\theta)\right)$$

$$= \frac{1}{\left(1+\left(\frac{du}{ds}\right)^2\right)^{3/2}} \left(-U_x \sin(\theta) + U_y \cos(\theta)\right)$$

$$0 = \frac{d}{d\theta}\cos(\alpha) \rightleftarrows 0 = -U_x \sin(\theta) + U_y \cos(\theta) = -\left(V_y \sin(\theta) + V_x \cos(\theta)\right)$$

$$\frac{dv}{ds} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} = V_x \cos(\theta) + V_y \sin(\theta).$$

$$\frac{dv}{ds} = 0 \longrightarrow V = \text{Constant along } \gamma.$$

$$\frac{d^2}{d\theta^2}\cos(\alpha) = \frac{\frac{d^2}{d\theta^2}\left(\frac{du}{ds}\right)\left(1+\left(\frac{du}{ds}\right)^2\right)^{3/2} - \frac{d}{d\theta}\left(\frac{du}{ds}\right)\frac{3}{2}\left(1+\left(\frac{du}{ds}\right)^2\right)^{1/2} 2\left(\frac{du}{ds}\right)\frac{d}{d\theta}\left(\frac{du}{ds}\right)}{\left(1+\left(\frac{du}{ds}\right)^2\right)^3}$$

$$= \left[\frac{-\left(1+\left(\frac{du}{ds}\right)^2\right) - 3\left[\frac{d}{d\theta}\left(\frac{du}{ds}\right)\right]^2}{\left(1+\left(\frac{du}{ds}\right)^2\right)^{5/2}} \right] \frac{du}{ds}$$

$$= -\underbrace{\left[\frac{1+\left(\frac{du}{ds}\right)^2 + 3\left[\frac{d}{d\theta}\left(\frac{du}{ds}\right)\right]^2}{\left(1+\left(\frac{du}{ds}\right)^2\right)^{5/2}} \right]}_{>0} \frac{du}{ds}$$

Therefore, the sign of $\frac{d^2}{d\theta^2}\cos(\alpha)$ only depends on the sign of $\frac{du}{ds}$

- $\cos(\alpha)$ will have an absolute max when.

$$\frac{dv}{ds} = 0 \quad \text{and} \quad \frac{du}{ds} > 0$$

- $\cos(\alpha)$ will have an absolute min when

$$\frac{dv}{ds} = 0 \quad \text{and} \quad \frac{du}{ds} < 0.$$

Conclusion so far: For any point (x,y,z) a curve $v(x,y) = \text{constant}$ will have the property that the tangent vector to the curve $x = x(s)$, $y = y(s)$, $u = u(s)$ will have max and min inclination (steepness) with respect to the u-axis.

↳ Steepest ascent or descent.

Domains where $u(x,y) > u(x_0,y_0)$ are called hills and those, where $u(x,y) < u(x_0,y_0)$ are called valleys. Also, for saddle points $u(x,y) = u(x_0,y_0)$.

Suppose z_0 is a saddle point of order $m-1$, $m \geq 2$.

$$\varphi'(z_0) = \varphi''(z_0) = \ldots = \varphi^{(m-1)}(z_0) = 0.$$

and take $\varphi^{(m)}(z_0) = ae^{i\psi}$; $a > 0$; ψ real-valued. Also $z = z_0 + re^{i\theta}$

$$\varphi(z) = \varphi(z_0) + \frac{1}{m!} ae^{i\psi} r^m e^{im\theta} + \ldots$$

$$= \varphi(z_0) + \frac{a}{m!} r^m e^{i(m\theta+\psi)} + \ldots$$

In consequence, near to $\varphi(z) = u(x,y) + iv(x,y)$

$$u(x,y) = u(x_0,y_0) + \frac{r^m}{m!} a \cos(m\theta + \varphi) + \dots$$

$$v(x,y) = v(x_0,y_0) + \frac{r^m}{m!} a \sin(m\theta + \varphi) + \dots$$

Directions of level curves where u is constant are given by $\cos(m\theta + \varphi) = 0$.

$$m\theta + \varphi = (2k+1)\frac{\pi}{2} \quad , \quad k = 0, 1, \dots, m-1.$$

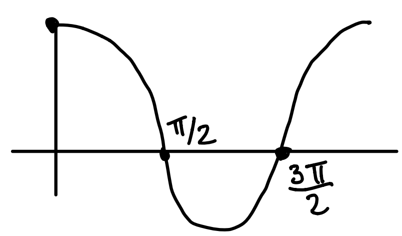
$$\theta = -\frac{\varphi}{m} + \frac{2k+1}{m}\frac{\pi}{2}$$

Analogously, for $v =$ constant $\longrightarrow \sin(m\theta + \varphi) = 0$.

$$\left.\begin{aligned} m\theta + \varphi &= k\pi \\ \theta &= -\frac{\varphi}{m} + \frac{k\pi}{m} \end{aligned}\right\} k = 0, 1, \dots, m-1.$$

Valleys: $\cos(\theta m + \varphi) < 0$

Hills: $\cos(\theta m + \varphi) > 0$.



Take, for example $m=2$ (Saddle point 1)

For u constant: $\theta = -\frac{\varphi}{2} + \left(k + \frac{1}{2}\right)\frac{\pi}{2} \quad , \quad k = 0, 1$

$$\theta_0 = -\frac{\varphi}{2} + \frac{\pi}{4} \quad , \quad \theta_1 = -\frac{\varphi}{2} + \frac{3\pi}{4}$$

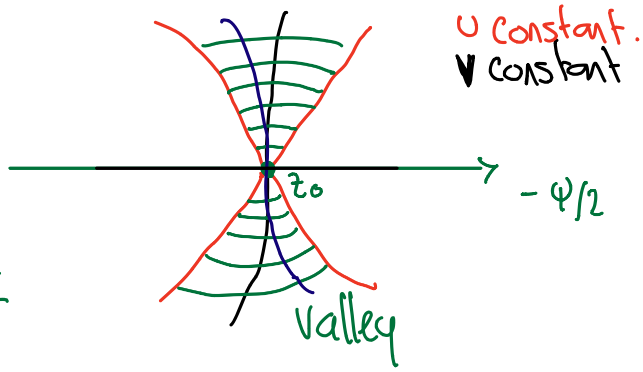
For v constant: $\theta = -\frac{\varphi}{2} + \frac{k\pi}{2}$

$$\theta_0 = -\frac{\varphi}{2} \quad , \quad \theta_1 = -\frac{\varphi}{2} + \frac{\pi}{2}$$

$\cos(2\theta+\varphi) < 0$

$\frac{\pi}{2} < 2\theta + \varphi < \frac{3\pi}{2}$

$-\frac{\varphi}{2} + \frac{\pi}{4} < \varphi < -\frac{\varphi}{2} + \frac{3\pi}{4}$



$\cos(2\theta+\varphi) = -1$

$2\theta + \varphi = (2k+1)\pi$

$\theta = -\frac{\varphi}{2} + \frac{(2k+1)\pi}{2}$

$\theta_0 = -\frac{\varphi}{2} + \frac{\pi}{2}$

$\theta_1 = -\frac{\varphi}{2} + \frac{3\pi}{2}$

Now, we consider a path where $v(x,y) = \text{constant}$

$$\Rightarrow \varphi(z) = \varphi(z_0) + \frac{r^m}{m!} a e^{i(m\theta+\varphi)}$$

$$\longmapsto \varphi(z) = \varphi(z_0) + \frac{r^m}{m!} a e^{ik\pi} + \dots$$

$$= \varphi(z_0) + \underbrace{\frac{r^m}{m!} a (-1)^k}_{} + \dots$$

(k even) either monotonically increasing.

(k odd) or " decreasing.

$$\tau := \frac{a r^m}{m!} + \left(\text{higher order terms}\right)$$

$$\varphi(z) = \varphi(z_0) + (-1)^k \tau$$

$$I(\lambda) = \int_C f(z) e^{\lambda \varphi(z)} dz \qquad \text{as } \lambda \to \infty$$

$$\Rightarrow I(\lambda) = \int_{C^*} f(z) e^{\lambda\varphi(z_0) + \lambda(-1)^k \tau} dz$$

In order to guarantee convergence we may chosse k odd.

$$I(\lambda) = \int_{C^*} f(z)\, e^{\lambda\varphi(z_0) - \lambda\tau}\, dz$$

$$= e^{\lambda\varphi(z_0)} \int_{C^*} f(z)\, e^{-\lambda\tau}\, dz$$

$$= e^{\lambda\varphi(z_0)} \int_{C^*} \left(f(z) \frac{dz}{d\tau} \right) e^{-\lambda\tau}\, d\tau$$

We may apply Watson's lemma to this integral.

Watson's:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

and

$$f(t) = t^\alpha \left[\sum_{k=0}^{n} a_k t^k + R_{n+1}(t) \right]$$

$$\Rightarrow F(s) \sim \sum_{k=0}^{n} a_k \frac{\Gamma(\alpha+k+1)}{s^{\alpha+k+1}} + O\left(\frac{1}{s^{\alpha+k+1}}\right) \qquad \text{as} \quad s \to \infty$$

Thus, we may require $f(z) = \frac{dz}{d\tau}$ to be expanded in powers of τ

Consider again that z_0 is a saddle point of order $m-1$

$$\Rightarrow \varphi(z) = \varphi(z_0) - \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} \qquad a_0 \neq 0.$$

$$=: \varphi(z_0) - (z-z_0)^m \varphi_1(z)$$

$$\Rightarrow \quad \tau = (z-z_0)^m \varphi_1(z)$$

Define $\nu^m := \tau$

$$\nu^m = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

$$\nu^m = (z-z_0)^m \sum_{n=0}^{\infty} a_n (z-z_0)^n = (z-z_0)(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots)$$

$$\nu = a_0^{1/m}(z-z_0)\left(1 + \frac{a_1}{a_0}(z-z_0) + \frac{a_2}{a_0}(z-z_0)^2 + \dots\right)^{1/m}$$

$$\nu = a_0^{1/m}(z-z_0)\left(1 + \frac{a_1}{m a_0}(z-z_0) + \dots\right)$$

by the inverse series theorem

$$z - z_0 = \sum_{s=1}^{\infty} \alpha_s \nu^s = \sum_{s=1}^{\infty} \alpha_s \tau^{s/m}$$

where $\alpha_1 = \frac{1}{a_0^{1/m}}$, $\alpha_2 = \frac{-a_1}{m a_0^{1+2/m}}$,

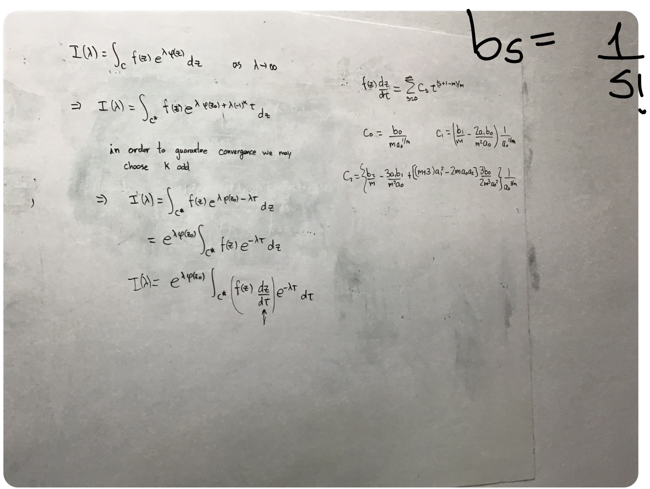
$$\alpha_3 = \frac{(m+3)a_1^2 - 2m a_0 a_2}{2m^2 a_0^{2+3/m}}$$

$$\frac{dz}{d\tau} = \sum_{s=1}^{\infty} \alpha_s \left(\frac{s}{m}\right) \tau^{\frac{s}{m}-1}$$

Now, as $f(z)$ is analytic.

$$f(z) = \sum_{s=1}^{\infty} b_s (z-z_0)^s$$

where



$$b_s = \frac{1}{s!} f^{(s)}(z_0)$$