Definition: $\varphi(A)$. Let $P_n(t) \searrow \varphi(t)$, for all $t \in [m, M]$. Then $P_n(A) \geq P_{n+1}(A) \geq \dots$ bounded.

The strong limit $\lim\limits_{n\to\infty} P_n(A)$ exists and we'll call it $\varphi(A)$.

Lemma: Let $Q_n(t)$ and $P_n(t)$ sequences of polynomials. Let's suppose that for all $t \in [m, M]$. $Q_n(t) \searrow \psi(t) \in K$ and $P_n(t) \searrow \varphi(t) \in K$. Let $\psi(t) \leq \varphi(t)$ $\forall\, t \in [m, M]$. Then

$$\lim_{n\to\infty} Q_n(t) = B_1 \leq B_2 = \lim_{n\to\infty} P_n(A)$$

If $\psi(t) = \varphi(t)$, then $B_1 = B_2 = \varphi(A)$ and the limit operator does not depend of the choosing of the polynomial.

Proof: For all $n \in \mathbb{Z}$, and $t \in [m, M]$, exists $N_0(t)$, such that $N \geq N_0(t)$, such that $Q_N(t) < \varphi(t) + 1/n$, and as

$$\psi(t) + \frac{1}{n} \leq \varphi(t) + \frac{1}{n} \leq P_n(t) + \frac{1}{n}$$

then

$$Q_N(t) < P_n(t) + \frac{1}{n}$$

this means that

$$Q_N(A) \leq P_n(A) + \frac{1}{n} I$$

taking $N \to \infty$ (n fixed)

$$B_1 \leq P_n(A) + \frac{1}{n} I.$$

and $n \to \infty$

$$B_1 \leq B_2$$

$\blacksquare$

For all self-adjoint operator A and $P_n(t)$ a real polynomial, the operator $P_n(A)$ is self-adjoint, then for all function $\varphi \in K$, the operator $\varphi(A)$ is also self-adjoint

1. $\varphi_1 + \varphi_2 \longrightarrow \varphi_1(A) + \varphi_2(A)$, i.e.

$$(\varphi_1 + \varphi_2)(A) = \varphi_1(A) + \varphi_2(A).$$

In fact, taking $P_n^{(i)} \searrow \varphi_i$, $i = 1, 2$. Then $P_n^{(1)} + P_n^{(2)} \searrow \varphi_1 + \varphi_2$.

II. For $c>0$, $(c\varphi_1)(A)=c_1\varphi(A)$

III. $(\varphi_1\cdot\varphi_2)(A)=\varphi_1(A)\,\varphi_2(A)$. (Where $\varphi_1\geq 0$, $\varphi_2\geq 0$, because $\varphi_1\cdot\varphi_2\in K$).

Spectral decomposition

Let's consider

$$e_\lambda(t)=\begin{cases}1, & \text{if } t\leq\lambda\\ 0, & \text{if } t\leq\lambda\end{cases}\qquad E_\lambda=\begin{cases}1, & \text{if } A\leq\lambda\\ 0, & \text{if } A>\lambda\end{cases}$$

$e_\lambda(t)\in K[a,b]$, and let's define $E_\lambda=e_\lambda(A)$. Then $E_\lambda^2=E_\lambda$, since $e_\lambda(t)\cdot e_\lambda(t)=e_\lambda(t)$, then E_λ is a projection.

I. E_λ is an orthogonal projection, $E_\lambda=0$ for $0<\lambda<m$ and $E_\lambda=I$, $\lambda\geq M$.

II. E_λ is continuous by the right with respect to λ in the strong sense.

If $\varphi_n(t)$ is a sequence of continuous functions such that

$$\varphi_n(t)\geq e_{\lambda+\frac{1}{n}}(t)$$

and $\varphi_n(t)\searrow e_\lambda(t)$, we obtain

$$\varphi_n(A)\geq E_{\lambda+\frac{1}{n}}\geq E_\lambda \quad \text{if } n\to\infty$$

then $\varphi_n(A)\searrow E_\lambda$.

III. $E_\lambda\cdot E_\mu=E_\lambda$, if $\lambda<\mu$. (because $e_\lambda(t)\cdot e_\mu(t)=e_\lambda(t)$)

A family $\{E_\lambda\}$ with this properties is called spectral family or the decomposition of the identity.

Let $\lambda_1<\lambda_2$. Then

$$\lambda_1(e_{\lambda_2}(t)-e_{\lambda_1}(t))\leq t(e_{\lambda_2}(t)-e_{\lambda_1}(t))\leq\lambda_2(e_{\lambda_2}(t)-e_{\lambda_1}(t))$$

Input A in place of t

$$\lambda_1(E_{\lambda_2}-E_{\lambda_1})\leq A(E_{\lambda_2}-E_{\lambda_1})\leq\lambda_2(E_{\lambda_2}-E_{\lambda_1}).$$

if $\lambda_1<m$, and $\lambda_2=\lambda$

$$AE_\lambda\leq\lambda E_\lambda.$$

If $\lambda_1 = \lambda$, and $\lambda_2 > M$.

$$\lambda(I - E_\lambda) \leq A(I - E_\lambda)$$

or that is the same

$$(A - \lambda I)E_\lambda \leq A - \lambda I$$

If $E_{\lambda_1\lambda_2} := E_{\lambda_2} - E_{\lambda_1}$, is an orthogonal projection. As $E_{\lambda_1\lambda_2}$ commute with A, and the subspace

$$H_{\lambda_1\lambda_2} = \operatorname{Im} E_{\lambda_1\lambda_2}.$$

is an invariant subspace of A, and for $x \in H_{\lambda_1\lambda_2}$ we have

$$\lambda_1 I_{H_{\lambda_1\lambda_2}} \leq A\big|_{H_{\lambda_1\lambda_2}} \leq \lambda_2 I_{H_{\lambda_1\lambda_2}}$$

then, for all $\lambda \in [\lambda_1, \lambda_2]$, we have

$$-\varepsilon I_{H_{\lambda_1\lambda_2}} \leq (A - \lambda I)\big|_{H_{\lambda_1\lambda_2}} \leq \varepsilon I_{H_{\lambda_1\lambda_2}}$$

with $\varepsilon = \lambda_2 - \lambda_1$.

This implies that

$$\|A - \lambda I\|\big|_{H_{\lambda_1\lambda_2}} \leq \varepsilon.$$