$$E_1 \subseteq E$$

$$[x] = \{ x + E_1, x \in E \}$$

$$\left.\begin{aligned} [x]+[y] &= [x+y] \\ \lambda[x] &= [\lambda x] \end{aligned}\right\} E/E_1$$

Quotient normed space

Lemma: If $X_0 \hookrightarrow X$, and X_0 is a closed subspace, so the quotient space X/X_0 can be equiped with a norm given by:

$$\|[x]\| = \inf_{y \in X_0} \|x-y\|, \text{ for } [x] \in X/X_0.$$

Proof: If $\|[x]\| = 0$, so there is a succession $y_n \in X_0$, such that X_0 is closed, and then $[x]=0$.

The multiplication by scalar, result that X_0 is a linear space.

$$\|\lambda[x]\| = \inf_{y \in X_0} \|\lambda(x-y)\|$$

$$= \inf_{y \in X_0} |\lambda| \|x-y\|$$

$$= |\lambda| \inf_{y \in X_0} \|x-y\| = |\lambda| \|[x]\|.$$

Left to prove that the triangle inequality. For all $\varepsilon > 0$, we take $z_1, z_2 \in X_0$, such that

$$\|x+z_1\| \leq \|[x]\| + \varepsilon \quad \text{and} \quad \|y+z_2\| \leq \|[y]\| + \varepsilon.$$

then, for each $\varepsilon > 0$.

$$\|[x]+[y]\| = \inf_{z \in X_0} \|x+y+z\|$$

$$\leq \|x+y+z_1+z_2\|$$

$$\leq \|x+z_1\| + \|y+z_2\|$$

$$\leq \|[x]\| + \|[y]\| + 2\varepsilon$$

finally, $\|[x]+[y]\| \leq \|[x]\| + [y]\|$. $\blacksquare$

There are examples where the norm not fulfills the first property i.e., are non-negatives but it may be zero, for elements diferents than zero, to this norms we call seminorms.

Definition: A function $p(x)$ in a vectorial space E, is a seminorm if it satisfies the norm properties except that may be zero, for non-zero vectors, $p: E \longrightarrow \mathbb{R}^+$ ($\mathbb{R}$ positives $\cup \{0\}$).

I. $P(0) = 0$

II. $p(\lambda x) = |\lambda| \, p(x)$. $\qquad \forall x, y \in E, \ \lambda \in \mathbb{R}$ (or $\mathbb{C}$).

III. $p(x+y) \leq p(x) + p(y)$.

If p is a seminorm and E_0 is the Kernel of p i.e.,

$$E_0 = \{ x \in E : p(x) = 0 \}$$

then

I. E_0 is a linear subspace.

if $x, y \in E_0$, $p(x+y) = 0$

$$p(x+y) \leq p(x) + p(y) = 0.$$

II. $p(x+y)$ is independent of $y \in E_0$.

$$p(x+y_1) \leq p(x+y_2) + \underbrace{\cancel{p(y_1 - y_2)}}_{\to 0.}$$
$$= p(x+y_2)$$

and similarly

$$p(x+y_2) \leq p(x+y_1) + \underbrace{\cancel{p(y_2 - y_1)}}_{\to 0.}$$
$$= p(x+y_1)$$

finally

$$p(x+y_1) = p(x+y_2).$$

$$p(x+y) \leq p(x) + p(y) = 0.$$

therefore, p may express as a function of cosets.

$$P([x]) = p(x)$$

and it no depends of the represent. Then the seminorm $p(x)$ in E defines from natural way a norm in the quotient E/E_0.

Example: let the space $C_p[a,b]$, of the continuous function, with norm

$$\|f\| = \left(\int_a^b |f(x)|^p dx \right)^{1/p} < \infty$$

the condition that f be continuous in $[a,b]$ is overdone (in pieces with a finite number of discontinuities in $[a,b]$) whose integral it is well define.

Let's consider $R_p[a,b]$ functions f in $[a,b]$ such that

$$|f(x)|^p$$

is Riemann integrable.

Then $\|f\|_p$, is no longer a norm but a seminorm since there is $f \neq 0$; $\|f\|_p = 0$.