Schwarzschild solution

(See Hawking & Ellis, in particular Appendix A for Laplace's treatise on BH)

Gravitational fields occurring in our solar system

↳ Precise measurements can be done.

Exterior gravitational field of a static spherically symmetric body

Schwarzschild solution:

I. Deviations from Newtonian theory

II. Bending of light

III. Gravitational redshift of light

IV. Time delay effects.

Vacuum Schwarzschild solution contains a BH solution

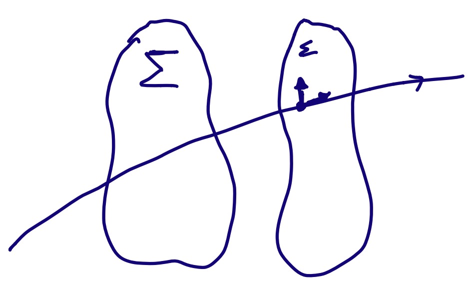
Solutions static, spherically symmetric

↳ all 4-dim Lorentz signature whose Ricci tensor vanishes and which are static and possess spherical symmetry.

Definition: A spacetime is said to be stationary if there exists a 1-parameter group of isometries Φ_t whose orbit are timelike curves (Time translation symmetry).

Definition: A spacetime is said to be static if it is stationary and if, in addition, there exists a (spacelike) hypersurface Σ which is orthogonal to the orbits of the isometry

$$\dot{F} = d_t F = \{F, H\}$$



Stationary ⟶ It admits a timelike Killing vector field ξ^a

Definition: Let $\phi_t : M \to M$ is a 1-parameter group of isometries, $\phi^* g_{ab} = g_{ab}$, the vector field ξ which generates ϕ_b is called a Killing vector field

$$\mathcal{L}_\xi(f) = \xi(f)$$

where ξ^a is tangent to the integral curves of ϕ_t

Killing $\mathcal{L}_\xi(g) = 0.$

$\longrightarrow$ ϕ_t is a group of isometries.

$$0 = \mathcal{L}_\xi g = \cancelto{0}{\nabla_\xi g_{ab}} + g_{cb}\nabla_a \xi^c + g_{ac}\nabla_b \xi^c$$
$$= \nabla_a \xi_b + \nabla_b \xi_a$$

Then, ξ_a is a Killing field if

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \qquad \text{Killing equation}$$

Theorem: Let ξ^a be a Killing vector field and let γ be a geodesic with tangent u^a. Then $\xi_a u^a$ is constant along γ.

Proof:

$$\nabla_u(\xi_a u^a) = u^b \nabla_b(\xi_a u^a)$$
$$= u^b u^a \nabla_b \xi_a + \xi_a u^b \nabla_b u^a$$
$$= \frac{1}{2} u^b u^a \cancelto{0}{(\nabla_b \xi_a + \nabla_a \xi_b)} + \cancelto{0}{\xi_a \nabla_b u^a} = 0.$$

Stationary $\longrightarrow$ ξ^a Killing vector field

Static $\longrightarrow$ $\xi_{[a}\nabla_b \xi_{c]} = 0$

$\longrightarrow$ Frobenious theorem.

Consider a particular coordinate system where $\xi^a \overset{*}{=} \delta^a_0$

$$\mathcal{L}_\xi g_{ab} = \xi^c g_{ab,c} + g_{ac}\xi^c{}_{,b} + g_{bc}\xi^c{}_{,a}$$
$$\overset{*}{=} \delta^c_0 g_{ab,c} + 0 + 0$$
$$\overset{*}{=} g_{ab,0} \overset{*}{=} 0$$

As $\mathcal{L}_\xi g$ is a tensor, then $\mathcal{L}_\xi g = 0$ in every coordinate system, therefore ξ is a Killing!

Static

Consider the family of hypersurfaces

$$\Sigma(x^a)=\mu$$

Consider on Σ_t the points P and Q with coordinates x^a and x^a+dx^a, respectively.

$$\mu=\Sigma(x^a+dx^a)=\Sigma(x^a)+\frac{\partial\Sigma}{\partial x^a}dx^a+\ldots \quad \text{first order}$$

As

$$\Sigma(x^a)=\mu=\Sigma(x^a+dx^a)$$

then

$$\frac{\partial\Sigma}{\partial x^a}dx^a=0 \qquad \text{at } P$$

Define the 1-form n to Σ_t by

$$n:=\frac{\partial\Sigma}{\partial x^a}dx^a \quad , \quad n_a=\frac{\partial\Sigma}{\partial x^a}$$

$$n_a dx^a=0=g_{ab}n^a dx^b=0 \quad \text{at } P$$

then

$$n^a\perp dx^a$$

as dx^a lies Σ_t.

Therefore n^a is a normal vector field to Σ_t at P

Define $\xi^a:=\lambda(x)n^a$ everywhere.

$$\xi^a=\lambda\frac{\partial\Sigma}{\partial x_a}$$

$$\xi_a\partial_b\xi_c=\xi_a\partial_b\lambda\Sigma_{,c}+\xi_a\lambda\Sigma_{,cb}$$

$$=\lambda\Sigma_{,a}\Sigma_{,c}\partial_b\lambda+\lambda^2\Sigma_{,a}\Sigma_{,cb}$$

symmetric in a,c. symmetric in b,c.

then

$$\xi_{[a}\partial_b\xi_{c]}=0$$

total antisymmetric part must vanish

$$\partial \longmapsto \nabla \qquad \text{remains unchanged.}$$

So

$$\xi_{[a}\nabla_b\xi_{c]} = 0 \qquad \text{Frobenius.}$$

So far

$$\mathcal{L}_\xi g_{ab} = \nabla_b\xi_a + \nabla_a\xi_b = 0$$

$$\xi_{[a}\nabla_b\xi_{c]} = 0 \qquad\qquad \xi^2 = \xi_a\xi^a$$

only 3-terms

$$\xi^c(\xi_a\nabla_b\xi_c + \xi_c\nabla_a\xi_b + \xi_b\nabla_c\xi_a = 0)$$

$$\xi_a\xi^c\nabla_b\xi_c + \xi^2\nabla_a\xi_b + \xi_b\xi^c\nabla_c\xi_a = 0$$

(I) $$\xi_a\xi^c\nabla_b\xi_c + \xi^2\nabla_a\xi_b - \xi_b\xi^c\nabla_a\xi_c = 0$$

$$\xi_a\xi_c\nabla_b\xi^c + \xi^2\nabla_a\xi_b - \xi_b\xi_c\nabla_a\xi^c = 0.$$

(II) $$\xi_a\xi_c\nabla_b\xi^c - \xi^2\nabla_b\xi_a - \xi_b\xi_c\nabla_a\xi^c = 0$$

$$\xi_a\nabla_b\xi^2 - \xi_b\nabla_a\xi^2 + \xi^2(\nabla_a\xi_b - \nabla_b\xi_a) = 0$$

$$\xi_a\partial_b\xi^2 - \xi_b\partial_a\xi^2 + \xi^2(\partial_a\xi_b - \partial_b\xi_a) = 0$$

$$\xi^2\partial_a\xi_b - \xi_b\partial_a\xi^2 = \xi^2\partial_b\xi_a - \xi_a\partial_b\xi^2$$

$$\partial_a\left(\frac{\xi_b}{\xi^2}\right) = \partial_b\left(\frac{\xi_a}{\xi^2}\right)$$

therefore

$$\frac{\xi_a}{\xi^2} = \frac{\partial f}{\partial x^a} \qquad\qquad \xi^a = n^a\lambda$$

$$\xi_a = \frac{\partial f}{\partial x^a}\xi^2 \qquad\qquad n^a = \frac{\partial\Sigma}{\partial x_a}$$

$$ds^2 = g_{00}\,dx^0 + 2g_{0\alpha}\,dx^0dx^\alpha + g_{\alpha\beta}\,dx^\alpha dx^\beta$$

under time reversol.

$$x^0 \longmapsto x^{0'} = -x^0$$

$$ds^2 = g_{00}\,dx^0dx^0 - 2g_{0\alpha}\,dx^0dx^\alpha + g_{\alpha\beta}\,dx^\alpha dx^\beta$$

Then $g_{0\alpha} = 0.$ Abscence of cross terms express the orthogonality of ξ and Σ.

Consider a coordinate system $\xi^a \stackrel{*}{=} \delta^a_0$

$$\xi_a = g_{ab}\xi^b \stackrel{*}{=} g_{ab}\delta^b_0 = g_{0a}$$

$$\xi^2 = \xi_a \xi^a = g_{0a}\delta^a_0 = g_{00}$$

then

$$g_{0a} = g_{00}\frac{\partial \Sigma}{\partial x^a}$$

$$ds^2 = V^2(x^\alpha)dt^2 - h_{\mu\nu}(x^\alpha)dx^\mu dx^\nu$$

$$V^2 = \xi^a \xi_a$$

Definition: Spherically symmetric: A spacetime is said to be spherically if its isometry group contains a subgroup isomorphic to the group SO(3), and the orbits of this subgroup are two dimensional spheres.

SO(3) rotations $\longrightarrow$ implies metric remains invariant under rotations.

$$(+ - - -)$$

$$ds^2 = f(r)dt^2 - h(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$r :=$ "radial coordinate".

Define $e^\nu := f(r)$ $\quad e^\lambda := h(r)$

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$g_{ab} = \mathrm{diag}(e^\nu, -e^\lambda, -r^2, -r^2\sin^2\theta$$

$$g^{ab} = \mathrm{diag}\left(e^{-\nu}, -e^{-\lambda}, -\frac{1}{r^2}, -\frac{1}{r^2\sin^2\theta}\right)$$

Einstein tensor (non-vanishing components) $\qquad \lambda' = \frac{\partial \lambda}{\partial r}$

$$\dot{\lambda} = \frac{\partial \lambda}{\partial t}$$

$$G_0^{\ 0} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$G_0^{\ 1} = -e^{-\lambda}\,\cancel{r^{-1}\dot{\lambda}}^{\to 0} = -e^{\lambda-\nu}G_1^{\ 0}$$

$$G_1{}^1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$G_2{}^2 = G_3{}^3 = \frac{1}{2} e^{-\lambda}\left(\frac{\nu'\lambda'}{2} + \frac{\lambda'}{r} - \frac{\nu'}{r} - \frac{\nu'^2}{2} - \nu''\right)$$

Exterior solution: No matter content

Vacuum Einstein equations.

$$G_{ab} = 0$$

$$G_0{}^0 = 0 = G_1{}^1$$

$$G_0{}^0 - G_1{}^1 = 0$$

$$e^{-\lambda}\left(\frac{\lambda' - \nu'}{r}\right) = 0$$

$$\lambda' + \nu' = 0$$

$$\frac{d}{dr}(\lambda + \nu) = 0 \qquad ; \lambda + \nu = \text{constant} =: A.$$

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0$$

$$e^{-\lambda} - re^{-\lambda}\lambda' = 1$$

$$\frac{d}{dr}(re^{-\lambda}) = 1$$

$$re^{-\lambda} = r + \text{constant} =: r - 2m$$

$$e^{-\lambda} = 1 - \frac{2m}{r}$$

$$e^{\lambda} = \frac{1}{1 - \frac{2m}{r}}$$

$$\nu = A - \lambda$$

$$e^{\nu} = e^{A} e^{-\lambda} = e^{A}\left(1 - \frac{2m}{r}\right)$$

$$\boxed{ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{1 - 2m/r} - r^2 d\Omega^2}$$

Schwarzschild line element

As $re^{-\lambda} > 0$

$$re^{-\lambda} = r - 2m > 0$$

$$r > 2m$$

$$g_{00} \simeq 1 + \frac{2\phi}{c^2}$$

$$\phi = \frac{-GM}{r}$$

$$g_{00} \simeq 1 - \frac{2GM}{rc^2} = 1 - \frac{2m}{r}$$

$$m := \frac{GM}{c^2}$$

$M :=$ Newtonian mass

$m :=$ Geometric mass.