Riemann-Lebesgue Lemma.

let $f(t)$ be continuous in (a,b), then

$$I(x) = \int_a^b f(t)e^{ixt}\,dt = O\left(\frac{1}{x}\right), \quad \text{as } x \longrightarrow \infty$$

Provided that the integral.

$$\int_a^b |f(t)|\,dt$$

converges.

Proof:

$$I(x) = \int_a^{a+\pi/x} f(t)e^{ixt}\,dt + \int_{a+\pi/x}^{b} f(t)e^{ixt}\,dt.$$

$$I(x) = \int_a^{b-\pi/x} f(t)e^{ixt}\,dt + \int_{b-\pi/x}^{b} f(t)e^{ixt}\,dt$$

changing $t = t - \frac{\pi}{x}$

$$\int_a^{a+\pi/x} f(t)e^{ixt}\,dt = -\int_a^{b-\pi/x} f\left(t-\frac{\pi}{x}\right)e^{ix\left(t-\frac{\pi}{x}\right)}\,dt$$

then

$$I(x) = \frac{1}{2}\int_a^{a+\frac{\pi}{x}} f(t)e^{ixt}\,dt + \frac{1}{2}\int_{a+\frac{\pi}{x}}^{b} f(t)e^{ixt}\,dt$$

$$+\frac{1}{2}\int_a^{b+\frac{\pi}{x}} \left[f(t) - f\left(t+\frac{\pi}{x}\right)\right]e^{ixt}\,dt, \text{ we want } x \longrightarrow \infty.$$

Mean Value Theorem: $f(t)$ continuous and bounded on $[a,b]$.

$$\int_a^b f(t)\,dt = f'(c)(b-a), \text{ for some real-number } c \in [a,b]$$

the first two integrals

$$\int_a^{a+\frac{\pi}{x}} f(t)e^{ixt}\,dt = f'\left(\frac{\pi}{x}\right)\left(\frac{\pi}{x}\right) \sim O\left(\frac{1}{x}\right)$$

and

$$\int_{b-\frac{\pi}{x}}^{b} f(t)e^{ixt}\,dt = f'\left(\frac{\pi}{x}\right)\left(\frac{\pi}{x}\right) \sim O\left(\frac{1}{x}\right).$$

Finally as $f(t)$ is continuous $\forall\, t \in [a,b]$

$$\lim_{x\to\infty} \int_0^{b-\pi/x} \left[f(t) - f\left(t+\frac{\pi}{x}\right)\right] e^{ixt}\,dt = 0$$

$$\therefore \quad I(x) \sim O\left(\frac{1}{x}\right) \qquad \blacksquare$$

We may extend the Riemann-Lebesgue Lemma to generalized Fourier Integrals as long as $|f(t)|$ is integrable, $\varphi(t)$ is continuous differentiable and $\varphi'(t) \neq 0$. Take.

$$I(x) = \int_a^b f(t)\, e^{ix\varphi(t)}\,dt$$

$$= \frac{1}{xi} \int_a^b \frac{f(t)}{\varphi'(t)} \frac{d}{dt}\left(e^{ix\varphi(t)}\right)dt$$

$$I(x) = \frac{1}{xi} \frac{f(t)}{\varphi'(t)} e^{ix\varphi(t)}\Big|_a^b - \frac{1}{xi}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\varphi'(t)}\right) e^{ix\varphi(t)}\,dt$$

This vanishes more rapidly than $\frac{1}{x}$ as $x \to \infty$.

$$I(x) \sim \frac{1}{ix} \frac{f(t)}{\varphi'(t)} e^{ix\varphi(t)}\Big|_a^b \quad \text{as} \quad x \to \infty$$

This method does not work for stationary points $\varphi'(t) = 0$. The method of stationary phase will give the asymptotic behaviour of generalized Fourier Integrals with stationary points.

Choose the integral such that $\varphi'(a)=0$, and $\varphi'(t)=0$ for $a\leq t\leq b$.

$$I(x)=\int_a^b f(t)e^{ix\varphi(t)}dt = \underbrace{\int_a^{a+\varepsilon} f(t)e^{ix\varphi(t)}dt}_{=: I_1(x).} + \int_{a+\varepsilon}^b f(t)e^{ix\varphi(t)}dt$$

$\varepsilon>0$ is a small parameter

Note:

$$\int_{a+\varepsilon}^b f(t)e^{ix\varphi(t)}dt \sim \frac{1}{ix}\frac{f(t)}{\varphi'(t)}e^{ix\varphi(t)}\Big|_{a+\varepsilon}^b \sim \frac{1}{x} \quad \text{as } x\longrightarrow\infty.$$

In the first integral change

$$f(t) \longmapsto f(a)$$
$$\varphi(t) \longmapsto \varphi(a) + \frac{\varphi^{(p)}(a)(t-a)^p}{p!}$$

As before (for laplace) the leading contribution comes from a neighbourhood of the stationary point.

Where $\varphi'(a) = \varphi''(a) = \ldots = \varphi^{(p-1)}(a) = 0$

$$I_1(x) = \int_a^{a+\varepsilon} f(a) \exp\left\{ ix\left[\varphi(a) + \frac{1}{p!}\varphi^{(p)}(a)(t-a)^p\right]\right\} dt$$

Next, replace $\varepsilon \longrightarrow \infty$

(this will introduce error terms $\mathcal{O}(1/x)$)

Let $s = t-a$, $ds = dt$

$$I_1(x) = f(a)\, e^{ix\varphi(a)} \int^{\infty} \exp\left(\frac{ix}{\ } \varphi^{(p)}(a) s^p \right) ds$$

$$\qquad\qquad {}_0 \qquad\qquad \Big(\ \frac{}{p!} \qquad\qquad \Big)$$

Define $\quad \mp \dfrac{i s^p x \, \varphi^{(p)}(a)}{p!} =: u$

upper sign if $\varphi^{(p)}(a) > 0$

lower sign if $\varphi^{(p)}(a) < 0$

$$s = \left\{ e^{\pm i\pi/2} \left[\frac{p!\,u}{x\,|\varphi^{(p)}(a)|} \right] \right\}^{1/p}$$

$$ds = e^{\pm i\pi/2p} \left(\frac{p!}{x\,|\varphi^{(p)}(a)|} \right)^{1/p} \frac{1}{p} \, u^{\frac{1}{p}-1} \, du$$

$$\Rightarrow I_1(x) = f(a)\, e^{ix\varphi(a)} \int_0^\infty e^{-u} e^{\pm i\pi/2p} \left(\frac{p!}{x\,|\varphi^{(p)}(a)|} \right)^{1/p} \frac{1}{p} \, u^{\frac{1}{p}-1} \, du$$

$$= f(a)\, e^{i(x\varphi(a) \pm \pi/2p)} \left(\frac{p!}{x\,|\varphi^{(p)}(a)|} \right)^{1/p} \frac{1}{p} \int_0^\infty e^{-u} u^{\frac{1}{p}-1} \, du$$

$$I_1(x) \sim f(a)\, e^{i(x\varphi(a) \pm \pi/2p)} \left(\frac{p!}{x\,|\varphi^{(p)}(a)|} \right)^{1/p} \frac{\Gamma(1/p)}{p} \qquad \text{as } x \to \infty$$

Example

$$I(x) = \int_0^{\pi/2} e^{ix\cos t} \, dt$$

$$f(t) = 1 \qquad \varphi(t) = \cos t$$

$0 = \varphi'(t) = -\sin t \quad \Rightarrow \quad t = 0$ stationary point.

$$\varphi''(t) = -\cos t \quad \Rightarrow \quad \varphi''(0) = -1 < 0$$

$$\Rightarrow p = 2$$

$$\Rightarrow I(x) \sim e^{i(x-\pi/4)} \left(\frac{2}{x}\right)^{1/2} \frac{\Gamma(1/2)}{2}$$

$$= e^{i(x-\pi/4)} \sqrt{\frac{\pi}{2x}} \quad \text{as} \quad x \longrightarrow \infty$$

Example (Bessel functions)

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin(t) - nt)\,dt$$

We want the leading behaviour of $J_n(n)$ as $n \to \infty$

$$\Rightarrow J_n(n) = \mathrm{Re}\left(\frac{1}{\pi}\int_0^{\pi} e^{in(\sin t - t)}\,dt\right)$$

$$\psi(t) = \sin t - t \qquad f(t) = 1$$

$$0 = \psi'(t) = \cos t - 1 \qquad t = 0 \quad \text{stationary point.}$$

$$\psi''(t) = -\sin t \qquad \psi''(0) = 0$$

$$\psi'''(t) = -\cos t \qquad \psi'''(0) = -1 < 0$$

$$P = 3$$

$$J_n(n) \sim \frac{1}{\pi}\,\mathrm{Re}\left(e^{i(n(0)-\pi/6)}\left(\frac{3!}{n}\right)^{1/3}\frac{\Gamma(1/3)}{3}\right)$$

$$= \frac{1}{\pi}\frac{\sqrt{3}}{2}\left(\frac{6}{n}\right)^{1/3}\frac{\Gamma(1/3)}{3} = \frac{\Gamma(1/3)}{\pi}\,\frac{3^{1/2}\cdot 3^{1/3}\cdot 3^{-1}}{2^{1}\,2^{-1/3}}\,\frac{1}{n^{1/3}}$$

$$= \frac{\Gamma(1/3)}{\pi}\,\frac{1}{3^{1/6}\,2^{2/3}}\,\frac{1}{n^{1/3}}$$

steepest descend.