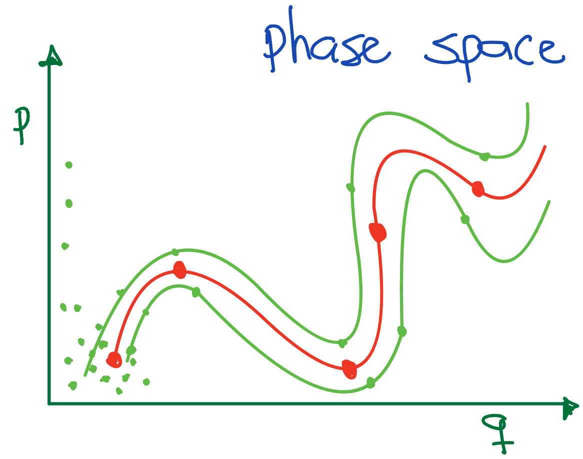
Ergodic Hypothesis



Hamilton equations:

$$\frac{\partial H}{\partial p} = \dot{q} \qquad \frac{\partial H}{\partial p} = -\dot{p}$$

Point density:

$$\rho = \rho(q,p,t)$$

then,

$\rho(q,p,t)\,dq\,dp \longrightarrow$ Number of points at time t between q and $q+dq$ also p and $p+dp$.

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial q}\dot{q} + \frac{\partial \rho}{\partial p}\dot{p} + \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial \rho}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial \rho}{\partial t}$$

Poisson bracket.

thereafter,

$$\frac{d\rho}{dt} = \{\rho, H\} + \frac{\partial \rho}{\partial t}$$

let $\vec{J} := \rho\vec{v}$ be the flux, with $v := (\dot{q},\dot{p})$ generalized velocity

$S :=$ hypersurface

$V :=$ hypervolume

$$\oint_S \vec{J}\cdot d\vec{S} = -\frac{d}{dt}\int_{V(S)} \rho\, dV$$

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$$\int_{V(S)} \nabla\cdot\vec{J}\,dV \longrightarrow \nabla\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\longrightarrow \nabla\cdot(\rho\vec{v}) = -\frac{\partial \rho}{\partial t} \quad \text{with} \quad \nabla := \left(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right)$$

$$\longrightarrow \frac{\partial}{\partial q}(\rho \dot{q}) + \frac{\partial}{\partial p}(\rho \dot{p}) = -\frac{\partial \rho}{\partial t}$$

$$\longrightarrow \frac{\partial \rho}{\partial q}\dot{q} + \rho\frac{\partial \dot{q}}{\partial q} + \frac{\partial \rho}{\partial p}\dot{p} + \rho\frac{\partial \dot{p}}{\partial p} = -\frac{\partial \rho}{\partial t}$$

$$\longrightarrow \frac{\partial \rho}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial \rho}{\partial p}\frac{\partial \mathcal{H}}{\partial q} + \rho\left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right) = -\frac{\partial \rho}{\partial t}$$

$$\longrightarrow \{\rho, \mathcal{H}\} + \rho\cancelto{0}{\left(\frac{\partial}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial}{\partial p}\frac{\partial \mathcal{H}}{\partial q}\right)} = -\frac{\partial \rho}{\partial t}$$

$$\longrightarrow \{\rho, \mathcal{H}\} = -\frac{\partial \rho}{\partial t}$$

therefore,

$$\frac{d\rho}{dt} = 0 \longrightarrow \rho = \text{constant}$$ Liouville theorem

If $\rho \neq \rho(t) \longrightarrow \dfrac{\partial \rho}{\partial t} = 0$ stationary case

System in balance

$$\longrightarrow \rho = \rho(q, p)$$

$$\langle f \rangle_{lab} = \lim_{\tau \to \infty} \frac{1}{\tau}\int_0^{\tau} f(t)\,dt$$ Temporal average

$$\langle f \rangle_{est} = \frac{\int f(q,p)\,\rho(q,p)\,dq\,dp}{\int \rho(q,p)\,dq\,dp}$$ phase-space average.

$$\langle f \rangle_{lab} = \langle f \rangle_{est}$$ Ergodic hypothesis.