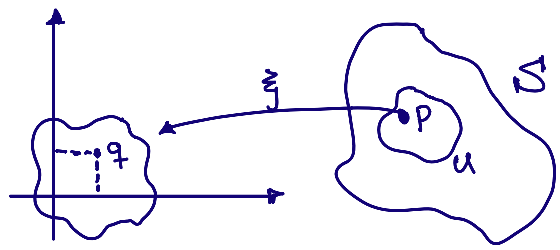
For $1 \le i \le n$, let

$$u^i : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$u^i(q) = q^i$$

Then $u^1, u^2, \dots, u^n$, are the natural coordinate functions of $\mathbb{R}^n$.

Definition: A coordinate system (or chart) in a topological space S is a homeomorphism ξ of an open set $U \in S$ onto an open set $\xi(U)$ of $\mathbb{R}^n$.

$$\xi : U \longrightarrow D \subseteq \mathbb{R}^n$$



$$\xi(p) := (x^1(p), \dots, x^n(p)) \quad \forall\, p \in U$$

x^i are called the coordinate functions of ξ.

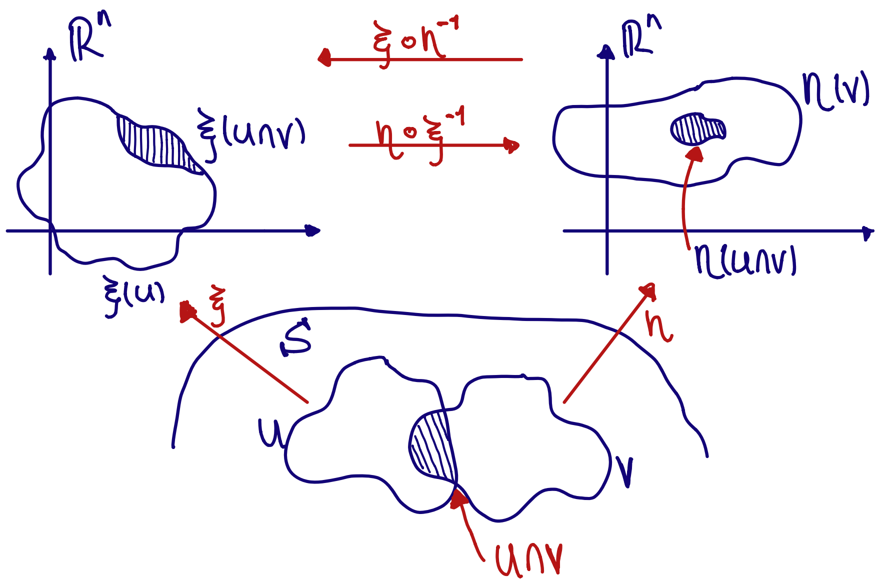
$$x^i(p) = (u^i \circ \xi)(p) = u^i(\xi(p)) = u^i(q) = q^i$$

Suppose

$$\xi : U \longrightarrow \mathbb{R}^n \quad \text{and} \quad \eta : V \longrightarrow \mathbb{R}^n$$

are two different coordinate systems.

Definition: 2 n-dim coordinate systems ξ and η in S overlap smoothly provided $\xi \circ \eta^{-1}$ are both smooth.



$$\xi \circ \eta^{-1} : \eta(U\cap V) \subseteq \mathbb{R}^n \longrightarrow U\cap V \longrightarrow \xi(U\cap V) \subseteq \mathbb{R}^n$$
$$: \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Definition: An Atlas A of dimension n on a manifold M is a collection of n-dim coordinate systems in M such that:

I. Each point of M is contained in the domain of some coordinate system for an open $U \subseteq M$.

II. Any two coordinate systems overlap smoothly.

Smooth mappings

Let f be a real-valued function defined on a manifold M, that is, if $U \subseteq M$

$$f : M \longrightarrow \mathbb{R}.$$

if $\xi : U \longrightarrow \mathbb{R}^n$ is a coordinate system in M.

$$f \circ \xi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

is called the coordinate expression for f in terms of ξ.

Definition: $f : M \longrightarrow \mathbb{R}$ is smooth if and only if $f \circ \xi^{-1}$ is smooth $\forall\, \xi \in M$.

$\mathcal{F}(M) :=$ Set of all smooth real-valued functions on M.

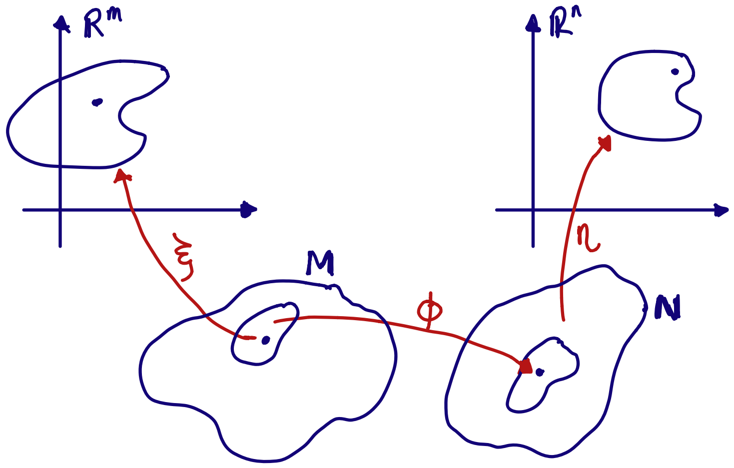
If f and $g \in \mathcal{F}(M)$.

$$\left.\begin{matrix} f+g \\ f\cdot g \end{matrix}\right\} \text{are smooth.}$$

$$(f+g)(x) = f(x) + g(x)$$
$$(f\cdot g)(x) = f(x)g(x).$$

Also, if $f \in \mathcal{F}(M)$ is never zero, then $\frac{1}{f} \in \mathcal{F}(M)$.

Definition: Let M and N be m-dim and n-dim manifolds, respectively. A mapping $\phi : M \longrightarrow N$ is smooth provided that for every coordinate system ξ in M and $\eta \in N$, the coordinate expression $\eta \circ \phi \circ \xi^{-1}$ is Euclidean smooth and defined on an open set $\mathbb{R}^m$.



$$\eta \circ \phi \circ \xi^{-1} : \mathbb{R}^m \longrightarrow M \longrightarrow N \longrightarrow \mathbb{R}^n$$
$$: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

Notes:

I. Smootness is a local property.

II. Identity map is smooth.

III. Composition of smooth mappings is smooth

IV. Smooth mappings are continuous.

Definition: A diffeomorphism $\phi : M \longrightarrow N$ is a smooth mapping that has an inverse mapping which is also smooth.

$$\mathrm{Diff}(M) := \text{set of diffeomorphisms.}$$
$$\phi : M \longrightarrow M.$$

In physics, the set $\mathrm{Diff}(M)$ is related to the group of reparametrizationz.

A diffeomorphism is a homeomorphism since smooth functions are continuous.

$$\text{diffeomorphism} \;\substack{\longrightarrow \\ \not\longleftarrow}\; \text{homeomorphism}$$

Examples:

I. (a,b) is diffeomorphic to $(-1,1)$:

$\longrightarrow$ linear map $\qquad \phi(t) = \dfrac{2t-(a+b)}{b-a}$

II. $(-1,1)$ is diffeomorphic to $\mathbb{R}$

$$\phi(t) = \frac{t}{1-t}$$