Compact operators

A set $K \subseteq X$ is said compact if and only if for all succession $X_n \in K$, there exists $x \in K$, and a subsuccession X_{n_i} of X_n such that $X_{n_i} \longrightarrow X$.

K is relatively compact (or precompact) if for all succession $X_n \in K$, it has a Cauchy's subsuccession X_{n_i}. If X is complete and K relatively compact, then $\overline{K}$ is compact.

Example: Any bounded set in $\mathbb{R}^n$ is relatively compact (Heine-Borel).

Theorem: (Arzela-Ascoli) Let $M \subseteq C([a,b])$, M is relatively compact (on $C([a,b])$) if and only if M is:

I. Uniformly bounded (i.e., is a bounded set in $C([a,b])$). $(\exists\, C, \|f\| \leq C)$.

II. Equicontinuous: $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that,

$$|t_1 - t_2| < \delta \longrightarrow |f(t_1) - f(t_2)| < \varepsilon, \ \forall f \in M.$$

Example:

I. Let C_1 and C_2 two positive constants and let

$$M = \{ x(t) \in C([a,b]) : |x(t)| < C_1 \text{ and } |x'(t)| < C_2 \}$$

M is relatively compact. Using the Arzola's theorem, $|x(t)| < C_2$, implies uniform continuity.

$$\frac{x(t_1) - x(t_2)}{t_1 - t_2} = x'(\theta), \quad t_1 < \theta < t_2$$

therefore,

$$|x(t_1) - x(t_2)| \leq C_2 |t_1 - t_2|.$$

Definition: An operator $A: X \longrightarrow Y$ is compact if and only if the image of any bounded set of X (unitary ball on X) is a precompact set on Y

II. Let $C_1([a,b])$, and let

$$\|x(t)\|_{C_1} = \max_t |x(t)| + \max_t |x'(t)|.$$

consider $A: C_1([a,b]) \longrightarrow C([a,b])$, $Ax = x$. A is a compact operator, the unitary ball in $C_1([a,b])$ is a precompact set on $C([a,b])$ by the Arzela's theorem.

Definition: Let X be a metric space and let $A \subseteq X$. A set $\mathcal{A} \subseteq X$, is called ε-net of A if and only if, for all $x \in A$, there exists $y \in \mathcal{A}$ such that $d(x,y) < \varepsilon$.

Lemma: M is relatively compact if and only if for all $\varepsilon > 0$ there exists a finite ε-net in M.

Proof: $(\rightarrow)$ let's assume that there exists $\varepsilon_0 > 0$ such that there isn't a ε_0-net in M. Take $x_1 \in M$, and $x_2 \in M$, such that $\|x_1 - x_2\| \geq \varepsilon_0$. If $\{x_i\}_{i=1}^{n-1}$ are chosen, we can choose $x_n \in M$ such that $\|x_n - x_1\|, \|x_n - x_2\|, \dots, \|x_n - x_{n-1}\| \geq \varepsilon_0$.

such x_n exists, since if not, $\{x_1, x_2, \dots, x_{n-1}\}$ would be a ε_0-net of M. Therefore, there is no succession $\{x_n\}_{n=1}^{\infty}$ is of cauchy, it means that M is not precompact, this is a contradiction.

$(\leftarrow)$ For all $k \in \mathbb{N}$, let $\varepsilon_k \leq 1/2k$. M has a ε_k-net. There exists for each k (by superposition). Let $\{x_n\}_{n=1}^{\infty} \in M$ and consider ε_1-net.

There is atleast a ball which contains an infinite subsuccession of the original succession. Let's denote the subsuccesion contained by the ball with $\{x_n^{(1)}\}_{n=1}^{\infty}$. In a similar way a ε_k-net divides the succession $\{x_n^{(k-1)}\}_{n=1}^{\infty}$ and there is a ball that contains an infinite succession of the succession $\{x_n^{(k-1)}\}_{n=1}^{\infty}$, let's denote by $\{x_n^{(k)}\}_{n=1}^{\infty}$. We know that the radius of the ball is less than ε_k, thus for all $m, n \in \mathbb{N}$

$$\|x_m^{(k)} - x_n^{(k)}\| \leq 2\varepsilon_k \leq \frac{1}{k}$$

The subsuccession $\{x_n^{(k)}\}_{n=1}^{\infty}$ of the succession $\{x_n\}_{n=1}^{\infty}$ is from Cauchy. This implies that M is relatively complete.

$\blacksquare$

Proposition: The set $K(X,Y)$ of compact operators from X to Y satisfy:

I. $K(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$.

II. If $A \in K(X,Y)$, $B \in \mathcal{L}(Z,X)$, $C \in \mathcal{L}(Y,Z)$, then $AB \in K(Z,Y)$, $CA \in K(X,Z)$. The case $X = Y$, $K(X) \equiv K(X,X)$.

III. $K(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$.

Proof:

II. For the AB operator, let $\{z_n\}_{n=1}^{\infty}$ be a bounded succession on Z. B is bounded, then $\{B(z_n)\}_{n=1}^{\infty}$ is bounded. A is compact, thus $\{A(Bz_n)\}_{n=1}^{\infty}$, has a convergent subsuccession and thus AB is compact.

For the operator BA, let $\{x_n\}_{n=1}^{\infty}$ be a bounded succession on X. A is compact, then it has a convergent succession $\{A(x_n)\}_{n=1}^{\infty}$, i.e., $\{A(x_{n_k})\}_{n=1}^{\infty}$.

B is bounded, then $\{B(A(x_{n_k}))\}_{n=1}^{\infty}$ is also convergent and thus BA is compact.

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III. Let $A_n \longrightarrow A$, i.e. $\|A_n - A\| \longrightarrow 0$, and let $A_n \in K(X, Y)$. Prove that A is compact, is enough to prove that $AD(X)$, the image of the unitary ball is precompact. Then, we have to find a ε-net, for all $\varepsilon > 0$ for $AD(X)$

for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$, such that $\|A_n - A\| < \varepsilon/2$ and $A_n D(X)$ is precompact (A_n is complete). Let $\varepsilon/2$-net, $\{y_i\}_{i=1}^{N}$ of $A_n D(X)$.

Then $\{y_i\}_{i=1}^{N}$ is a ε-net of $AD(X)$. In fact, for any $y \in AD(X)$, there is a $x \in D(X)$, such that $y = Ax$.

As $\{y_i\}_{i=1}^{N}$ is a $\varepsilon/2$-net, of $A_n D(X)$, let y_{i_0} one of the elements of the net, such that

$$\|y_{i_0} - A_n x\| \leq \frac{\varepsilon}{2}$$

then,

$$\|y - y_{i_0}\| \leq \|Ax - A_n x\| + \|A_n x - y_{i_0}\| \leq \varepsilon.$$

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Dual operators

Let $A : X \longrightarrow Y$ be a bounded operator and let $\varphi \in Y^*$, then $f(x) = \varphi(Ax)$, for $x \in X$, defines a linear functional in X. Also

$$|f(x)| \leq \|\varphi\|_{Y^*} \cdot \|A\| \cdot \|x\|$$

$$\leq \|\varphi\|_{Y^*} \cdot \|Ax\| \leq \|\varphi\|_{Y^*} \|A\| \cdot \|x\|$$

Then $f \in X^*$, thus the map $\varphi \longmapsto f := A^* \varphi$.

The map $A^*: Y^* \longrightarrow X^*$, moreover, for all $x \in X$ and $\varphi \in Y^*$, we have

$$\varphi(Ax) \equiv (A^*\varphi)x.$$

A^* is bounded.

$$\begin{aligned}
\|A\| &= \sup_{\|x\|=1} \|A(x)\|_Y \\
&= \sup_{\|x\|=1} \sup_{\|\varphi\|=1} |\varphi(A(x))| \\
&= \sup_{\|\varphi\|=1} \sup_{\|x\|=1} |\varphi(A(x))| \\
&= \sup_{\|\varphi\|=1} \sup_{\|x\|=1} |(A^*\varphi)(x)| \\
&= \sup_{\|\varphi\|=1} \|A^*\varphi\|_{X^*} = \|A^*\|
\end{aligned}$$

then $\|A\| = \|A^*\|$ and A^* is called the dual operator of A.

$$A^*: Y^* \longrightarrow X^*$$