Covariant differentiation for tensors

Recall: Let A be a type $(1,2)$ tensor

$$A: \mathfrak{X}^*(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{F}(M).$$

A is multilinear, and for $\theta \in \mathfrak{X}^*(M)$.

$X, V \in \mathfrak{X}(M)$.

$$\Longrightarrow \quad \theta = \theta_k dx^k, \qquad X = X^i \partial_i, \qquad V = V^i \partial_i$$

$$A(\theta, X, V) = \sum_{i,j,k} A(\theta_k, X^i, V^j)\, dx^k \otimes \partial_i \otimes \partial_j$$

$$= \sum_{i,j,k} A^{ij}_k \, dx^k \otimes \partial_i \otimes \partial_j$$

$$A^{ij}_k := A(\theta_k, X^i, V^j)$$

Defenition: Let A be a tensor of type (r,s). The covariant derivative of A is a tensor of type $(r, s+1)$ given by ∇A defined by

$$\nabla : \mathfrak{X}(M) \times \mathcal{T}^s_r(M) \longrightarrow \mathcal{T}^{s+1}_r(M)$$

$$\nabla A =: \nabla_Z A.$$

In components:

1. First consider $\langle dx^l, \partial_j \rangle = \delta^l_j$.

$$0 = \nabla_{\partial_k} \langle dx^l, \partial_j \rangle = \partial_k \langle dx^l, \partial_j \rangle \overset{(v)}{=} \langle \nabla_{\partial_k} dx^l, \partial_j \rangle + \langle dx^l, \nabla_{\partial_k} \partial_j \rangle$$

then

$$\langle \nabla_{\partial_k} dx^l, \partial_j \rangle = -\langle dx^l, \nabla_{\partial_k} \partial_j \rangle = -\langle dx^l, \Gamma^m_{kj} \partial_m \rangle$$

$$= -\Gamma^m_{kj} \langle dx^l, \partial_m \rangle = -\Gamma^m_{kj} \delta^l_m = -\Gamma^l_{kj}.$$

therefore,

$$\nabla_{\partial_k} \partial_j = \Gamma^i_{kj} \partial_i$$

$$\nabla_{\partial_k} dx^l = -\Gamma^l_{ki} dx^i$$

II. $\nabla_Z A = \nabla_Z \left(A^{i_1 \dots i_s}_{j_1 \dots j_r} dx^{j_1} \otimes \dots \otimes dx^{j_r} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_s} \right)$

$$= (\nabla_Z A^{i_1 \dots i_s}_{j_1 \dots j_r})\, dx^{j_1} \otimes \dots \otimes dx^{j_r} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_s}$$

$$+ A^{i_1 \dots i_s}_{j_1 \dots j_r} (\nabla_Z dx^{j_1}) \otimes \dots \otimes dx^{j_r} \otimes \partial_{i_1} \otimes \dots \otimes \partial_{i_s}.$$

$$
\begin{aligned}
&+ \cdots \\
&+ A^{i_1\ldots i_s}_{j_1\ldots j_r}\, dx^{j_1}\otimes\ldots\otimes(\nabla_Z dx^{j_r})\otimes\partial_{i_1}\otimes\ldots\otimes\partial_{i_s}. \\
&+ A^{i_1\ldots i_s}_{j_1\ldots j_r}\, dx^{j_1}\otimes\ldots\otimes dx^{j_r}\otimes(\nabla_Z\partial_{i_1})\otimes\ldots\otimes\partial_{i_s}. \\
&+ \cdots \\
&+ A^{i_1\ldots i_s}_{j_1\ldots j_r}\, dx^{j_1}\otimes\ldots\otimes dx^{j_r}\otimes\partial_{i_1}\otimes\ldots\otimes(\nabla_Z\partial_{i_s}) \\
&= Z^{l}\Big(A^{i_1\ldots i_s}_{j_1\ldots j_r,l}\, dx^{j_1}\otimes\ldots\otimes dx^{j_r}\otimes\partial_{i_1}\otimes\ldots\otimes\partial_{i_s} \\
&\qquad - \Gamma^{j_1}_{lm} A^{i_1\ldots i_s}_{j_1\ldots j_r}\, dx^{m}\otimes\ldots\otimes dx^{j_r}\otimes\partial_{i_1}\otimes\ldots\otimes\partial_{i_s} \\
&\qquad - \cdots \\
&\qquad - \Gamma^{j_r}_{lm} A^{i_1\ldots i_s}_{j_1\ldots j_r}\, dx^{j_1}\otimes\ldots\otimes dx^{m}\otimes\partial_{i_1}\otimes\ldots\otimes\partial_{i_s} \\
&\qquad + \Gamma^{m}_{l i_1} A^{i_1\ldots i_s}_{j_1\ldots j_r}\, dx^{j_1}\otimes\ldots\otimes dx^{j_r}\otimes\partial_{m}\otimes\ldots\otimes\partial_{i_s} \\
&\qquad + \cdots \\
&\qquad + \Gamma^{m}_{l i_s} A^{i_1\ldots i_s}_{j_1\ldots j_r}\, dx^{j_1}\otimes\ldots\otimes dx^{j_r}\otimes\partial_{i_1}\otimes\ldots\otimes\partial_{m}\Big) \\
&= Z^{l}\Big\{A^{i_1\ldots i_r}_{j_1\ldots j_s,l} - A^{i_1\ldots i_r}_{m\ldots j_s}\Gamma^{m}_{l j_1} - \ldots - A^{i_1\ldots i_r}_{j_1\ldots m}\Gamma^{m}_{l j_s} + A^{m\ldots i_r}_{j_1\ldots j_s}\Gamma^{i_1}_{ml} + \ldots + A^{i_1\ldots m}_{j_1\ldots j_s}\Gamma^{i_r}_{ml}\Big\} \\
&\qquad \times dx^{j_1}\otimes\ldots\otimes dx^{j_s}\otimes\partial_{m}\otimes\ldots\otimes\partial_{i_s}
\end{aligned}
$$

Example:

I. $\nabla_U V = U^l (V^i_{,l} + V^m \Gamma^i_{lm})\partial_i$

$$U = U^i\partial_i \quad , \quad V = V^j\partial_j$$

$$
\begin{aligned}
\nabla_{U^i\partial_i}(V^j\partial_j) &= U^i\nabla_{\partial_i}(V^j\partial_j) \\
&= U^i\left[(\nabla_{\partial_i}V^j)\partial_j + V^j(\nabla_{\partial_i}\partial_j)\right] \\
&= U^i(V^j_{,i}\partial_j + V^j\Gamma^k_{ij}\partial_k) \\
&= U^l(V^i_{,l}\partial_i + V^m\Gamma^i_{lm}\partial_i) \\
&= U^l(V^i_{,l} + V^m\Gamma^i_{lm})\partial_i
\end{aligned}
$$

II. T of type $(1,1)$

$$\nabla_U T = U^l(T^j_{k,l} - T^j_m\Gamma^m_{lk} + T^m_k\Gamma^j_{ml})\, dx^k\otimes\partial_j$$

III. Let g be the metric tensor

Find ∇g $\qquad g = g_{\mu\nu} dx^\mu dx^\nu$

$$\nabla_z g = z^\lambda \underbrace{\left(g_{\mu\nu,\lambda} - g_{\kappa\nu}\Gamma^\kappa_{\mu\lambda} - g_{\mu\kappa}\Gamma^\kappa_{\nu\lambda}\right)}_{=0} dx^\mu \otimes dx^\nu$$

Metric compatibility

$$\nabla_z g = 0.$$

Transformation properties of Γ's

Let $\xi = (x^1, \dots, x^n)$ be a coordinate system and $\eta = (y^1, \dots, y^n)$ be a different coordinate system such that $\xi \cap \eta \neq 0$.

Let $\{e_\mu\} := \{\partial/\partial x^\mu\}$ and $\{f_\alpha\} := \{\partial/\partial y^\alpha\}$ be their respective basis, then

$$f_\alpha = \frac{\partial x^\mu}{\partial y^\alpha} e_\mu .$$

Define $\tilde{\Gamma}$ in the f_α basis as $\nabla_{f_\alpha} f_\beta = \tilde{\Gamma}^\gamma_{\alpha\beta} f_\gamma$

$$\tilde{\Gamma}^\gamma_{\alpha\beta} \frac{dx^\mu}{dy^\gamma} e_\mu = \tilde{\Gamma}^\gamma_{\alpha\beta} f_\gamma = \nabla_{f_\alpha} f_\beta = \nabla_{f_\alpha}\left(\frac{\partial x^\mu}{\partial y^\beta} e_\mu\right)$$

$$= \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} e_\mu + \frac{\partial x^\mu}{\partial y^\beta} \nabla_{f_\alpha} e_\mu$$

$$= \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} e_\mu + \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} \nabla_{e_\nu} e_\mu$$

$$= \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} e_\mu + \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} \Gamma^\rho_{\nu\mu} e_\rho .$$

$$= \left(\frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\rho}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} \Gamma^\mu_{\nu\rho}\right) e_\mu .$$

$$\tilde{\Gamma}^\gamma_{\alpha\beta} = \frac{\partial y^\gamma}{\partial x^\mu} \frac{\partial x^\rho}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} \Gamma^\mu_{\nu\rho} + \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^\mu}$$

∴ Not a tensor!

↳ It cannot have an intrisic geometrical meaning as a measure of how much a manifold is curved!

We need to build a tensor!

If $[X,Y]$ commute, $[\nabla_X, \nabla_Y]$ do not commute in general.

Theorem: Let M be a pseudo-Riemannian manifold with Levi-Civita connection ∇. The function

$$\text{Riemannian}: \mathfrak{X}(M)^4 \longrightarrow \mathbb{R}.$$

$$\text{Riemannian} := \langle R_{XY}, W \rangle$$

$$R(X,Y,Z)$$

$$R: \mathfrak{X}(M)^3 \longrightarrow \mathfrak{X}(M).$$

Notation: $R(X,Y,Z) = R(X,Y)Z = R_{XY}Z.$

$$R_{XY}Z := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

is a $(3,1)$-tensor field on M called the Riemann curvature tensor of M.

Proof: $\mathbb{R}$-linearity is obvious!

First, recall $\quad \nabla_V(fW) = V(f)W + f\nabla_V W$

$$\begin{aligned}
[fX,Y] &= fXY - YfX \\
&= fXY - Y(f)X - fYX \\
&= f[X,Y] - Y(f)X
\end{aligned}$$

Also

$$\begin{aligned}
[X,fY] &= XfY - fYX \\
&= X(f)Y + fXY - fYX \\
&= X(f)Y + f[X,Y]
\end{aligned}$$

a)
$$\begin{aligned}
R_{fX,Y}Z &= \nabla_{[fX,Y]}Z - [\nabla_{fX}, \nabla_Y]Z \\
&= \nabla_{f[X,Y]-Y(f)X}Z - \nabla_{fX}\nabla_Y Z + \nabla_Y \nabla_{fX} Z \\
&= f\nabla_{[X,Y]}Z - Y(f)\nabla_X Z - f\nabla_X\nabla_Y Z + \nabla_Y(f\nabla_X Z) \\
&= f\nabla_{[X,Y]}Z - \cancel{Y(f)\nabla_X Z} - f\nabla_X\nabla_Y Z + \cancel{Y(f)\nabla_X Z} + f\nabla_Y\nabla_X Z \\
&= f\nabla_{[X,Y]}Z - f[\nabla_X, \nabla_Y]Z \\
&= fR_{XY}Z
\end{aligned}$$

b) Analogously in the second entry

c) $R_{XY}(fZ)$

$$= \nabla_{[X,Y]} fZ - [\nabla_X, \nabla_Y](fZ)$$
$$= ([X,Y]f)Z + \nabla_{[X,Y]} Z - \nabla_X \nabla_Y (fZ) + \nabla_Y \nabla_X (fZ)$$
$$= [X,Y](f)Z + f\nabla_{[X,Y]} Z - \nabla_X (Y(f)Z + f\nabla_Y Z) + \nabla_Y (X(f)Z + f\nabla_X Z)$$
$$= [X,Y](f)Z + f\nabla_{[X,Y]} Z - (XY(f))Z - Y(f)\nabla_X Z - X(f)\nabla_Y Z$$
$$- f\nabla_X\nabla_Y Z + (YX(f))Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + f\nabla_Y\nabla_X Z$$
$$= f(\nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z)$$
$$= f R_{XY} Z$$

Note: Neither $[\,,\,]$ or ∇ are tensors but the combination $R_{XY}Z$ is!

Symmetries of Riemann

let $X, Y, V, W, Z \in \mathfrak{X}(M)$

I. $R(X,Y) = -R(Y,X)$ — Skew-Symmetry
$$R_{XY}Z = -R_{YX}Z$$

II. $\langle R(X,Y)V, W\rangle = -\langle R(Y,X)W, V\rangle$ — Skew-adjoint

III. $R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0$ — 1st-Bianchi Identity.

IV. $\langle R_{XY}V, W\rangle = \langle R_{VW}X, Y\rangle$ — symmetry in pairs.

V. $\nabla_Z R(X,Y) + \nabla_X R(Y,Z) + \nabla_Y R(Z,X) = 0$ — 2nd-Bianchi Identity

Proof:

I. Follows from the skew-symmetry of $[\,,\,]$

II. $\langle R(X,Y)V, W\rangle = -\langle R(X,Y)W, V\rangle$

$$\Longleftrightarrow \langle R(X,Y)V, V\rangle = 0.$$

$$\langle \nabla_{[X,Y]} V, V\rangle - \langle [\nabla_X, \nabla_Y]V, V\rangle$$

First note, $\nabla_X \langle \nabla_Y V, V\rangle = X\langle \nabla_Y V, V\rangle$

$$= \langle \nabla_X \nabla_Y V, V \rangle = \langle \nabla_Y V, \nabla_X V \rangle$$

Also

$$\langle \nabla_Y \nabla_X V, V \rangle = Y \langle \nabla_X V, V \rangle - \langle \nabla_X V, \nabla_Y V \rangle$$

then,

$$\langle [\nabla_X, \nabla_Y] V, V \rangle = X \langle \nabla_Y V, V \rangle - Y \langle \nabla_X V, V \rangle$$

Also,

$$\nabla_Y \langle V, V \rangle = Y \langle V, V \rangle = \langle \nabla_Y V, V \rangle + \langle V, \nabla_Y V \rangle$$

$$= 2 \langle \nabla_Y V, V \rangle$$

$$\langle \nabla_Y V, V \rangle = \frac{1}{2} Y \langle V, V \rangle$$

$$\langle [\nabla_X, \nabla_Y] V, V \rangle = \frac{1}{2} X Y \langle V, V \rangle - \frac{1}{2} Y X \langle V, V \rangle$$

Define $Z = [X, Y]$

$$\langle \nabla_Z V, V \rangle = \frac{1}{2} Z \langle V, V \rangle$$

$$= \frac{1}{2} [X, Y] \langle V, V \rangle.$$

III. $R_{XY} Z + R_{YZ} X + R_{ZX} Y$

$$= (\nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z) + (\nabla_{[Y,Z]} X - [\nabla_Y, \nabla_Z] X) + (\nabla_{[Z,X]} Y - [\nabla_Z, \nabla_X] Y)$$

$$= (\nabla_{[X,Y]} Z + \nabla_{[Y,Z]} X + \nabla_{[Z,X]} Y) - (\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z - (\nabla_Y \nabla_Z - \nabla_Z \nabla_Y) X - (\nabla_Z \nabla_X - \nabla_X \nabla_Z) Y$$

$$= \nabla_{[X,Y]} Z + \nabla_{[Y,Z]} X + \nabla_{[Z,X]} Y - \nabla_X (\nabla_Y Z - \nabla_Z Y) - \nabla_Y (\nabla_Z X - \nabla_X Z) - \nabla_Z (\nabla_X Y - \nabla_Y X)$$

For torsion free $\nabla_V W - \nabla_W V = [V, W]$

$$= \nabla_{[X,Y]} Z + \nabla_{[Y,Z]} X + \nabla_{[Z,X]} Y - \nabla_X [Y, Z] - \nabla_Y [Z, X] - \nabla_Z [X, Y]$$

$$= (\nabla_Z [X, Y] + [[X, Y], Z]) + (\nabla_X [Y, Z] + [[Y, Z], X]) + (\nabla_Y [Z, X] + [[Z, X], Y])$$

$$- \nabla_X [Y, Z] - \nabla_Y [Z, X] - \nabla_Z [X, Y]$$

Homework: Prove (IV) for Riemann

(See Schutz for a coordinate demostration or Nakahara or Wald for a coordinate independent proof).

in particular basis $\{\partial_i\}$

$$R(\partial_i,\partial_j)\partial_k = \nabla_{[\partial_i,\partial_j]}\partial_k - [\nabla_{\partial_i},\nabla_{\partial_j}]\partial_k.$$

$[\partial_i,\partial_j]=0$

$$= -(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i})\partial_k$$

$\therefore$ Curvature measures the non-commutativity of the covariant derivative.

Lemma: $R_{\partial_k\partial_l}(\partial_j) = \sum_i R^i_{klj}\,\partial_i$

where $R^i_{jkl} = \frac{\partial}{\partial x^l}\Gamma^i_{kj} - \frac{\partial}{\partial x^k}\Gamma^i_{lj} + \Gamma^i_{lm}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{lj}$

Proof: $[\partial_i,\partial_j]=0$

$$R_{\partial_k\partial_l}(\partial_j) = \nabla_{\partial_l}\nabla_{\partial_k}\partial_j - \nabla_{\partial_k}\nabla_{\partial_l}\partial_j$$

$$= \nabla_{\partial_l}(\Gamma^i_{kj}\partial_i) - \nabla_{\partial_k}(\Gamma^i_{lj}\partial_i)$$

$$= (\nabla_{\partial_l}\Gamma^i_{kj})\partial_i - (\nabla_{\partial_k}\Gamma^i_{lj})\partial_i$$

continue...

Symmetries of Riemann components

I. $R_{ijkm} = -R_{jikm}$ skew-symmetry

II. $R_{ijkm} = -R_{ijmk}$ skew-adjoint

III. $R_{ijkm} + R_{jkim} + R_{kijm} = 0$ 1st-Bianchi

IV. $R_{ijkm} = R_{kmij}$ Symmetry in pairs

V. $\nabla_i R^m_{jkl} + \nabla_j R^m_{kil} + \nabla_k R^m_{ijl} = 0$ 2nd-Bianchi

Homework: Show

$$[\nabla_\alpha,\nabla_\beta]F^\mu_\nu = R^\mu_{\sigma\alpha\beta}F^\sigma_\nu + R^\sigma_{\nu\alpha\beta}F^\mu_\sigma.$$

Let M be a n-dim R m^4 components.

skew-symmetry $\longmapsto \binom{m}{2} =: N$

skew-adjoint $\longmapsto \binom{m}{2}$

Symmetry by pairs $\longmapsto$ $N:=\binom{m}{2}$ $\quad \frac{N(N+1)}{2}$

Bianchi $\longmapsto$ • If $m<4$ trivially satisfied

$\rightsquigarrow$ No further restrictions.

• If $m\geq 4$ impose restrictions only when all the indices are different.

$\rightsquigarrow$ Number of possible ways to choose four different indices out of m

$$\binom{m}{4}$$

Linear independent components of Riemann

$$F(m):=\frac{N(N+1)}{2}-\binom{m}{4}=\frac{1}{2}\binom{m}{2}\left(\binom{m}{2}+1\right)-\binom{m}{4}$$

$$=\frac{1}{2}\frac{m!}{(m-2)!2!}\left(\frac{m!}{(m-2)!2!}+1\right)-\frac{m!}{(m-4)!4!}$$

$$=\frac{1}{2}\frac{m(m-1)}{2}\left(\frac{m(m-1)}{2}+1\right)-\frac{m(m-1)(m-2)(m-3)}{24}$$

$$=\frac{m(m-1)}{8}\left[(m(m-1)+2)-\frac{(m-2)(m-3)}{3}\right]$$

$$=\frac{m(m-1)}{24}\left(3m^2-3m\cancel{+6}-m^2+5m\cancel{-6}\right)$$

$$=\frac{m(m-1)}{24}(2m^2+2m)=\frac{m^2(m^2-1)}{12}$$

$F(0)=0$

$F(1)=0$

$F(2)=1$ $\qquad R_{1212}\neq 0.$

$F(3)=6$

$F(4)=20$