Yang-Mills renormalization to a loop

Degree of superficial divergence:

QED

$$D = 4L - 2P_i - E_i$$

$L \to$ # loops.

$P_i \to$ # internal photon lines.

$E_i \to$ # internal electron lines.

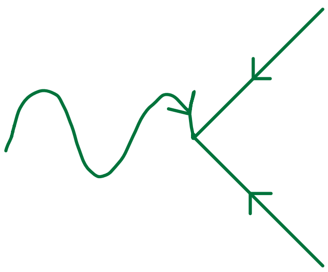
$$\int d^4K \; ; \; \frac{1}{K^2} \; ; \; \frac{1}{K-m}$$

Number integration internal moments:

$$L = E_i + P_i - (n-1)$$

$n \to$ # Vertex.

$\sum p = 0$ / vertex.



$\longrightarrow$ e exterior $\to$ Contributes once each 2 vertex

e interior $\to$ Contributes twice each 2 vertex.

$2n = E_e + 2E_i$; photons $\quad n = P_e + 2P_i$

then,

$$\boxed{D = 4 - \frac{3}{2} E_e - P_e}$$

$\leftarrow$ Does not depend in n

Yang-Mills

$$D = 4 - \frac{3}{2} N_F - N_B - N_D$$

$N_D \to$ # Derivatives in the vertex.

D decreases with the number of involved particles.

$\longrightarrow$ Suggest a renormalizable theory.

Parameters: g, m_f, ξ $\longrightarrow$ Reduced numbers of renormalization constants.

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a - \frac{1}{2\xi}(\partial A)^2 + \partial_\mu \eta_a^*(\partial^\mu \eta_a + g f_{abc} \eta_b A_c^\mu) + \bar{\psi}(i\not{\partial} - m)\psi$$

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f_{abc} A_b^\mu A_c^\nu$$

$$D_\mu \Psi = (\partial_\mu + i g T_a A_\mu^a)\Psi \quad ; \quad \Psi \in \text{irrep(dr)}.$$

Introducing the renormalization constants:

$$(A_a^\mu)_B := Z_A^{1/2} A_a^\mu ; \quad (\eta_a)_B := Z_\eta^{1/2} \eta ; \quad \Psi_B := Z_\Psi^{1/2} \Psi$$

define appropriate constants for the parameters

ξ : $\frac{1}{\xi_B}(\partial \cdot A_B)^2 \longrightarrow \left(\frac{Z_A}{\xi_B}\right)(\partial \cdot A)^2 \longrightarrow \boxed{\xi_B^{-1} = Z_A^{-1} Z_\xi \xi^{-1}}$

$Z_A / Z_B = Z_\xi / \xi$

m : $m_B \bar\Psi_B \Psi_B \longrightarrow \boxed{m_B = m Z_m Z_\Psi^{-1}}$

g : $g_B(\partial_\mu \eta_a^* \eta_b A_c^\mu)_B \longrightarrow \boxed{g_B = g Z_1 Z_\eta^{-1} Z_A^{-1/2}}$

$g_B(\bar\Psi \not{A} \Psi)_B \longrightarrow \boxed{g_B = g Z_2 Z_\Psi^{-1} Z_A^{-1/2}}$

F^2 $\left\{ \begin{array}{l} g_B(\partial_\mu A_\nu^a \cdot A_b^\mu A_a^\nu)_B \longrightarrow \boxed{g_B = g Z_3 Z_A^{-3/2}} \\ g_B^2 (A^4)_B \longrightarrow \boxed{g_B^2 = g^2 Z_4 Z_A^{-2}} \end{array} \right.$

$Z_1 Z_\eta^{-1} = Z_2 Z_\Psi^{-1}$

$= Z_3 Z_A^{-1}$

$= Z_4^{1/2} Z_A^{-1/2}$

$\downarrow$

Z_2 independent between Z_i

$\downarrow$

6-constants.

QED: $f_{abc} \to 0$; $\eta \to$ Uncouple.

Just $Z_A, Z_\Psi, Z_m, Z_\xi, Z_2 = Z_e \rightarrow$ 5-constants

Counterterms:

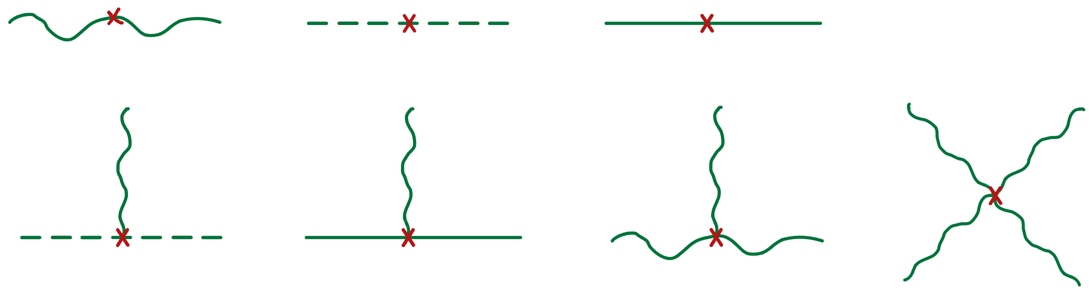
$$Z_\varphi = 1 + \Delta Z_\varphi \quad \varphi = A, \eta, \Psi. \quad ; \quad Z_i = 1 + K_i \quad i = \xi, m, 1, \dots, 4.$$

$$\Delta\mathcal{L} = -\frac{1}{4}\Delta Z_A (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{K_\xi}{2\xi}(\partial A)^2 + \Delta Z_\eta \partial_\mu \eta_a^* \cdot \partial^\mu \eta_a$$

$$+ \bar\Psi(i \Delta Z_A \not\partial - m K_m)\Psi + g K_1 f_{abc}(\partial_\mu \eta_a^*)\eta_b A_c^\mu - g K_2 A_{a\mu} \bar\Psi \gamma^\mu T_a \Psi$$

$$+ g K_3 f_{abc} A_b^\mu A_c^\nu \partial_\mu A_\nu^a - \frac{g^2}{4} K_4 f_{abc} f_{ade} A_b^\mu A_c^\nu A_\mu^b A_\nu^c$$

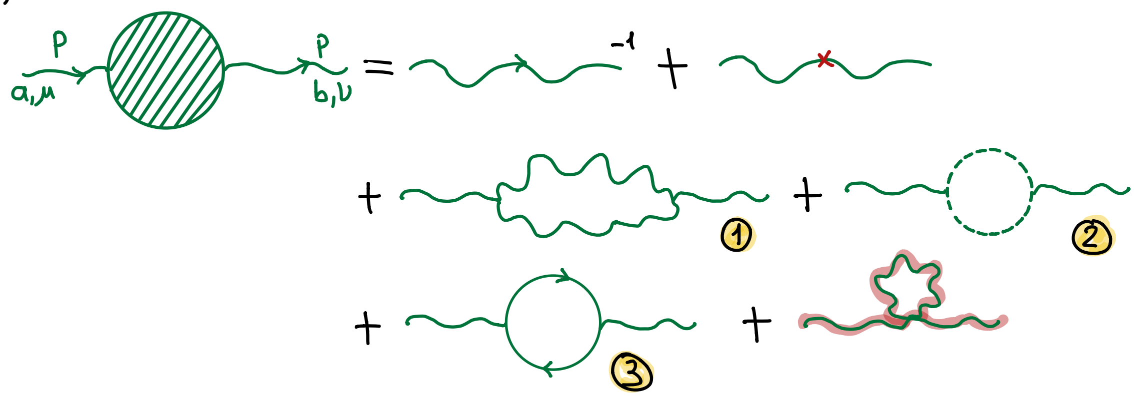
$\longrightarrow \oplus$ vertex.



Computing the renormalization constants

$D=2 \qquad (N_B=2)$

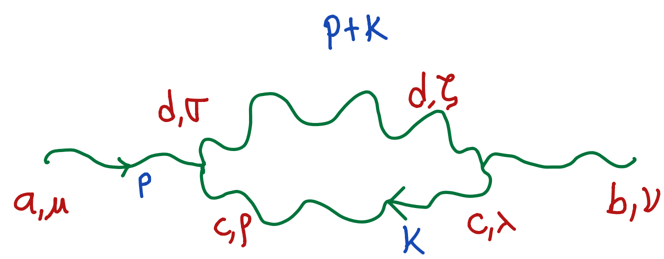
$\tilde{\Gamma}^{(2)\,ab}_{\mu\nu}$:



P-independent $\propto m_A^2 = 0 \longleftrightarrow$ Gauge symmetry

$\longrightarrow$ Zero in Regular Dimensionalization.

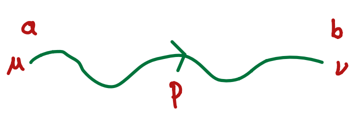
① =



RD

$$= -\frac{1}{2}\hat{g}^2 M^{\varepsilon} f_{acd} f_{cbd} \int \frac{d^{2\omega}K}{(2\pi)^{2\omega}} J_{\mu\nu}(p,K) \qquad \varepsilon = 4-2\omega$$

$$J_{\mu\nu}(p,K) = \left[-(p-2K)_\mu \eta_{\rho\sigma} + (K-p)_\sigma \eta_{\rho\mu} + (2p+K)_\rho \eta_{\mu\sigma}\right]$$
$$\times\left[-(p+2K)_\nu \eta_{\lambda\tau} + (K-p)_\tau \eta_{\lambda\nu} + (2p+K)_\lambda \eta_{\tau\nu}\right]$$
$$\times D_F^{\sigma\tau}(p+K) \times D_F^{\lambda\rho}(p)$$



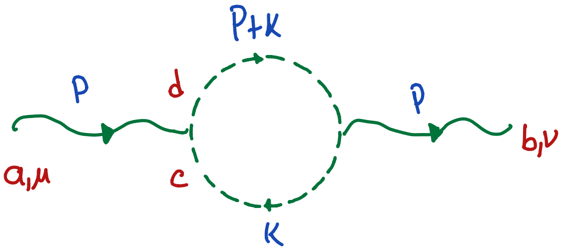
$$: \; D^{\mu\nu}_{F\,ab}(p) = \frac{i\delta_{ab}}{p^2+i\epsilon}\left[-\eta^{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]$$

After algebra:

$$① = \frac{i\hat{g}^2 C_1 \delta_{ab}}{16\pi^2}\frac{1}{\varepsilon}\left[\left(-\frac{11}{3} - 2\xi\right)p_\mu p_\nu + \left(\frac{19}{6}+\xi\right)p^2\eta_{\mu\nu}\right] + (\text{finite})$$

$C_1\delta_{ab} = f_{acd}f_{bcd}$ $\qquad \bar{\xi} = 1-\xi$

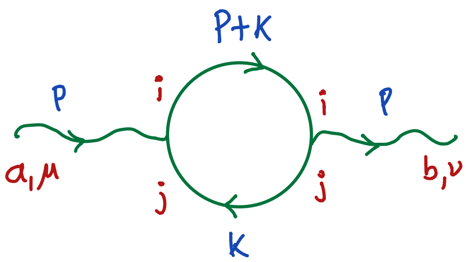
↳ Adjoint index

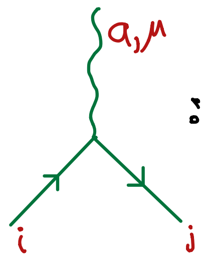
② = 

$$= -M^{\varepsilon}\hat{g}^2 C_1\delta_{ab}\int\frac{d^{2\omega}K}{(2\pi)^{2\omega}}\frac{(P+K)_\mu K_\nu}{[(P+K)^2+i\epsilon](K^2+i\epsilon)}$$

Grassmann

$$= \frac{i\hat{g}^2 C_1\delta_{ab}}{16\pi^2}\frac{1}{\varepsilon}\left(\frac{1}{3}P_\mu P_\nu + \frac{1}{6}P^2\eta_{\mu\nu}\right) + (\text{finite})$$

③ = 



$: -ig\gamma_\mu(T_a)_{ij}$

$\downarrow$

$\mathrm{Tr}(T_aT_b) = C_r\delta_{ab}$

$$= -\hat{g}^2 M^{\varepsilon} C_r\delta_{ab}\int\frac{d^{2\omega}K}{(2\pi)^{2\omega}}\mathrm{Tr}(\gamma_\mu\tilde{S}_F(K)\gamma_\nu\tilde{S}(P+K))$$

$$= -\hat{g}^2 M^{\varepsilon} C_r\delta_{ab}\int\frac{d^{2\omega}K}{(2\pi)^{2\omega}}\frac{K_\mu(P+K)_\nu + (P+q)_\mu K_\nu + (m^2-K^2-PK)\eta_{\mu\nu}}{(K^2-m^2+i\varepsilon)[(P+K)^2-m^2+i\varepsilon]}$$

The pole is independent of m:

$$③ = -\frac{i\hat{g}^2 C_r\delta_{ab}}{16\pi^2}\frac{1}{\varepsilon}\left(-\frac{8}{3}P_\mu P_\nu + \frac{8}{3}P^2\eta_{\mu\nu}\right) + (\text{finite})$$

finally,  $= i\Delta Z_A(-P^2\eta_{\mu\nu}+P_\mu P_\nu) - i\xi^{-1}K_\xi P_\mu P_\nu$

In MS

$$= \frac{-i\hat{g}^2}{16\pi^2}\frac{1}{\varepsilon}\left\{(-P^2\eta_{\mu\nu}+P_\mu P_\nu)\left[\left(-\frac{10}{3}-\bar{\xi}\right)C_1+\frac{8}{3}C_2\right]-\bar{\xi}C_1P_\mu P_\nu\right\}$$

Comparing terms,

$$\Delta Z_A = -\frac{\hat{g}^2}{16\pi^2}\frac{1}{\varepsilon}\left[\left(-\frac{13}{3}+\xi\right)C_1+\frac{8}{3}C_r\right]$$

$$\xi^{-1}K_\xi = -\frac{\hat{g}^2}{16\pi^2}\frac{1}{\varepsilon}(1-\xi)C_1.$$