Let M be a manifold of dim n, with basis x^μ; vector field $\{\partial/\partial x^\mu \equiv \partial_\mu\}$

$$g(\partial_\mu, \partial_\nu) = g_{\mu\nu}$$

$$\text{vol} \equiv \sqrt{|\det g_{\mu\nu}|}\, dx^1 \wedge \cdots \wedge dx^n$$

Suppose other basis x'^μ, vector field $\{\partial'_\mu\}$

$$g(\partial'_\mu, \partial'_\nu) = g'_{\mu\nu}$$

$$\text{vol}' = \sqrt{|\det g'_{\mu\nu}|}\, dx'^1 \wedge \cdots \wedge dx'^n$$

$$dx'^\nu = T^\nu{}_\mu\, dx^\mu$$

$$T^\nu{}_\mu = \frac{dx'^\nu}{dx^\mu}$$

then,

$$dx'^1 \wedge dx'^2 \wedge \cdots \wedge dx'^n = (T^1_\mu dx^\mu) \wedge (T^2_\mu dx^\mu) \wedge \cdots \wedge (T^n_\mu dx^\mu)$$

$$= (\det T)\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

To show Vol = Vol', first show

$$\sqrt{|\det g'_{\mu\nu}|} = (\det T)^{-1} \sqrt{|\det g_{\mu\nu}|}$$

$$g'_{\mu\nu} = g(\partial'_\mu, \partial'_\nu) = g\left(\frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha, \frac{\partial x^\beta}{\partial x'^\nu}\partial_\beta\right)$$

$$= (T^{-1})^\alpha_\mu\, (T^{-1})^\beta_\nu\, g_{\alpha\beta}$$

then,

$$\det g'_{\mu\nu} = (\det T)^{-2} \det g_{\mu\nu}$$

$$\boxed{\det T > 0} \quad \text{orientable.}$$

therefore,

$$\sqrt{|\det g'_{\mu\nu}|} = (\det T)^{-1} \sqrt{|\det g_{\mu\nu}|} \qquad \blacksquare$$

$$\text{Vol} = \sqrt{g}\, d^n x$$

$$\int_V f\sqrt{g}\, d^n x$$

- If M is a Lorentzian manifold
  $\text{Vol} = \sqrt{-g}\, d^n x.$

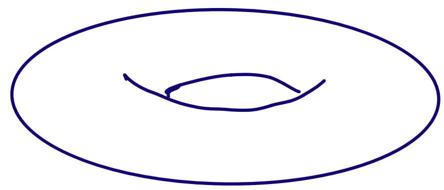
$$S[q(t)] = \int_{t_0}^{t_1} L\left(q(t), \frac{d}{dt}q(t)\right) dt \qquad q: \mathbb{R} \to \mathbb{R}$$

$$0 = \delta S \longrightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dq}{dt}\right)}\right) = 0$$

$$S = \int d^n x \, \mathcal{L}(\phi, \partial_\mu \phi)\sqrt{|\det g_{\mu\nu}|} \; ; \quad \phi: \mathbb{R}^n \to \mathbb{R}$$

$$\delta S = 0 \longrightarrow \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \phi)}\right) = 0$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi$$



$$\left.\begin{aligned} &\nabla \cdot \vec{B} = 0 \\ &\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{aligned}\right\} dF = 0 \longrightarrow \begin{aligned} &d_s B = 0 & B - \text{2-form.} \\ &\partial_t B + d_s E = 0 & E - \text{1-form.} \end{aligned}$$

$$\nabla \cdot \vec{E} = \rho$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$$

Let M be a semi-Riemannian manifold of dim n. The inner product of two p-forms ω, μ is a function $\langle \mu, \omega \rangle$ in M. Define the star Hodge operator (Hodge dual)

$$* : \Omega^p(M) \longrightarrow \Omega^{n-p}(M)$$

is a linear map from p-form to $(n-p)$-form. Such that $\forall\, \omega, \mu \in \Omega^p(M)$

$$\omega \wedge * \mu = \langle \omega, \mu \rangle \, \text{vol}.$$

(are n-form)

Let $e^1, \dots, e^n$ be an orthonormal basis of 1-forms

$$\langle e^\mu, e^\nu \rangle = 0 \quad \text{if} \quad \mu \neq \nu.$$

$$\langle e^\mu, e^\nu \rangle = \epsilon(\mu) \; ; \text{ with } \epsilon(\mu) = \pm 1 \text{ if } \mu = \nu.$$

Let be different $1 \leq i_1, \ldots, i_p \leq n$

$$*(e^{i_1} \wedge \cdots \wedge e^{i_p}) = \pm e^{i_{p+1}} \wedge \cdots \wedge e^{i_n}$$

with $\{i_{p+1}, \ldots, i_n\}$ the integers from 1 to n, not included in $\{i_1, \ldots, i_p\}$.
Where the sign $\pm$ is given by

$$\text{sign}(i_1, \ldots, i_n)\, \epsilon(i_1) \cdots \epsilon(i_p)$$

here, $\text{sign}(i_1, \ldots, i_n)$ is the permutation of $(1, \ldots, n)$.

Example: In $\mathbb{R}^3$, we have 1-forms $\{dx, dy, dz\}$

$$*dx = dy \wedge dz$$

$$\text{sign}(1,2,3)\,\epsilon(x)$$

$$= (+1)(+1).$$

$$*dy = dz \wedge dx$$

$$\text{sign}(2,3,1)\, 1.$$

$$*dz = dx \wedge dy.$$

$$*(dx \wedge dy) = dz$$

$$\text{sign}(1\ 2\ 3)\,\epsilon(x)\epsilon(y)$$

$$*(dy \wedge dz) = dx$$

$$*(dz \wedge dx) = dy$$

$$*1 = dx \wedge dy \wedge dz$$

$$\text{sign}(1,2,3).$$

$$*(dx \wedge dy \wedge dz) = 1.$$

Homework: let ω be a 1-form in $\mathbb{R}^3$ $(\omega = \omega_\mu dx^\mu)$.

$*d\omega$ (rotational)

$*d*\omega$ (divergence)

Second pair of ecuations.

$$F = B + E \wedge dt = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Now, the metric is Minkowski.

$$\eta(v,w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3.$$

$$*F = \frac{1}{2} (*F)_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$*\left(\frac{1}{2} F_{01} dx^0 \wedge dx^1\right) = \frac{1}{2} F_{01} *(dx^0 \wedge dx^1)$$

$$= \frac{1}{2} F_{01} (-)(dx^2 \wedge dx^3)$$

$$= \frac{1}{2} E_x dx^2 \wedge dx^3$$

$$\mathrm{sign}(0,1,2,3)\, \epsilon(X_0)\, \epsilon(X_1)$$
$$= (+1)(-1)(+1)$$

$$(*F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

$dF = 0 \qquad F = dA$

$d*F = 0 \qquad dF = d^2 A = 0$

$$\downarrow$$

$d * dA = 0.$

In other words, the dual of F means replace

$$E_i \mapsto -B_i, \qquad B_i \mapsto E_i$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \longrightarrow dF = 0$$

$$\nabla\cdot\vec{E}=0, \quad \nabla\times\vec{B}-\frac{\partial\vec{E}}{\partial t}=0 \longrightarrow d{*}F=0$$

ρ $\quad\quad$ $\vec{J}$

$${*}d{*}F=J$$

$$j=j_1dx+j_2dy+j_3dz\ (j_1,j_2,j_3)$$

$$J=j-\rho dt.$$

Lie (algebras & groups)

Definition: A group G, is a set equiped with a binary operation

$$\bullet: G\times G\longrightarrow G,$$

called product, an operation

$$^{-1}: G\longrightarrow G$$

called inverse and an element $1\in G$, called identity, such that $\forall\, g,h,K\in G$

I. $(g\cdot h)K=g\cdot(h\cdot K)$

II. $g\cdot 1=1\cdot g=g$

III. $g\cdot g^{-1}=g^{-1}\cdot g=1$

Notation $g\cdot h=gh$

Most of the groups with applications, are matrix groups, ex.

$GL(n,\mathbb{R})$ - General linear group, invertible $n\times n$ matrices, $GL(n,\mathbb{C})$.

A subgroup is a subset of a group such that is closed under the multiplication and the inverse

$$GL(n,\mathbb{R})\subseteq GL(n,\mathbb{C})$$

The spetial linear group, is the set of $n\times n$ matrices, with $\det=1$, $SL(n,\mathbb{R})$. (Preserves Volume).

If p,q are non-negative integers such that $p+q=n$. Let g be a metric in $\mathbb{R}^n$ with signature (p,q).

$$g(v,w)=v^1w^1+v^2w^2+\dots v^pw^p+v^{p+1}w^{p+1}+\dots+v^{p+q}w^{p+q}.$$

$$g(T_v, T_w) = g(v, w)$$

The orthogonal group $O(p,q)$, in the set of matrices $n \times n$, such that,

$$g(T_v, T_w) = g(v, w) \quad , \forall \; v, w \in \mathbb{R}^n$$

$$SO(p,q) \subseteq O(p,q)$$
$$\parallel$$
$$\{ T \in O(p,q) \, , \det T = 1 \}$$

If $p = n$, g is the euclidean metric, write $O(n)$ and $SO(n)$.

The group $SO(3,1)$, are the orthogonal matrices in the Minkowski metric and it is labeled the Lorentz group.

In a space $\mathbb{C}^n$

$$\langle v, w \rangle = \sum_{i=1}^{n} \bar{v}_i w_i$$

the group $U(n)$ unitary and saves the inner product, then

$$SU(n) \subseteq U(n).$$