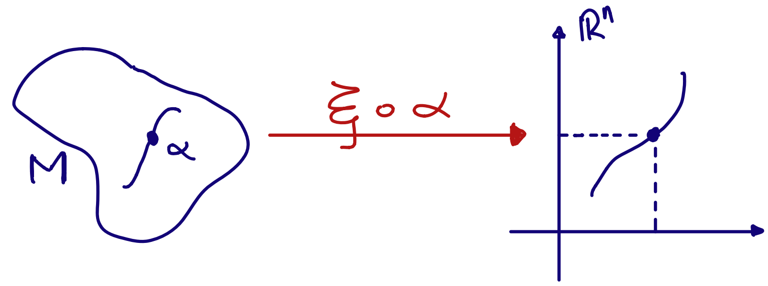
Curves

$$\alpha : I \longrightarrow M \;; \qquad I := \text{open interval in } \mathbb{R}.$$

Definition: A smooth curve in a manifold M is a map α of the open interval $I = (a,b) \in \mathbb{R} \longrightarrow M$, such that for any coordinate system ξ we have that

$$\xi \circ \alpha : I \longrightarrow \mathbb{R}^n \quad \text{is a smooth map.}$$



The curve is parametrised

$$\alpha(t) \;; \quad t \in (a,b)$$

Let $f \in \mathcal{F}(M)$, $f : M \to \mathbb{R}$, consider the map

$$f \circ \alpha : I \longrightarrow \mathbb{R}.$$
$$t \longmapsto f(\alpha(t))$$

Value of f on each point of the curve.

Derivative: Rate of change of f along the curve $\alpha(t)$ at $t = t_0$

$$\frac{d}{dt}(f \circ \alpha)(t_0), \quad \forall f \in \mathcal{F}(M).$$

Definition: The tangent vector

$$\dot{\alpha}_p := \frac{d}{dt}\bigg|_{\alpha(p)}$$

to a curve $\alpha(t)$ at a point p is the map

$$\dot{\alpha}(p) : \mathcal{F}(M) \longrightarrow \mathbb{R}$$

$$\dot{\alpha}_p(f) = \frac{d}{dt}(f \circ \alpha)\bigg|_p$$

Given a coordinate system $\xi = (x^1, \ldots, x^n)$ we have

$$\frac{d}{dt}(x^i \circ \alpha)\Big|_p = \frac{d}{dt} x^i(\alpha(t))\Big|_p$$

Now $f \circ \alpha = (f \circ \xi^{-1}) \circ (\xi \circ \alpha)$, then

$$f \circ \xi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$$ coordinate expression for f.

$$\xi \circ \alpha : I \longrightarrow \mathbb{R}^n$$

$$t \longmapsto \xi(\alpha(t))$$

$$(x^1(\alpha(t)), x^2(\alpha(t)), \dots, x^n(\alpha(t)))$$

$$\frac{d}{dt}(f \circ g) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i} f(x) \right) \frac{dx^i}{dt}(\alpha(t))$$

$$\therefore \ \dot{\alpha}_p(f) = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} f(x) \ \frac{dx^i(\alpha(t))}{dt}$$

$$= \sum_{i=1}^{n} \frac{dx^i(\alpha(t))}{dt} \frac{\partial}{\partial x^i}\Big|_p f$$

$$\dot{\alpha}_p \overset{n}{=} \sum_{i=1} \frac{dx^i}{dt}(\alpha(t))\, \partial_i\Big|_p$$

Vector fields

A vector field on a manifold M is a function that assings to each point $p \in M$ a tangent vector V_p to M at p. If $V \in T_pM$ and $f \in \mathfrak{F}(M)$, then

$$V(f)(p) =: V_p(f), \quad \forall p \in M$$

$V \in T_pM$ is smooth if and only if $V(f)$ is smooth $\forall f \in \mathfrak{F}(M)$

$\mathfrak{X}(M) :=$ set of all smooth vector fields on M.

Definition: If $V, W \in \mathfrak{X}(M)$, let the bracket of V, W be defined as

$$[V,W] = VW - W - V : \mathfrak{F}(M) \longrightarrow \mathfrak{F}(M)$$

$$[V,W]_p(f) = V_p(W(f)) - W_p(V(f))$$

Proof: Let $v = v^i \partial_i$, $\omega = \omega^i \partial_i$

$$V(\omega(f)) = V^i \partial_i(\omega^j \partial_j(f)) = V^i \partial_i \omega^j \partial_j(f) + \cancel{V^i W^j \partial^2_{ij} f}$$

$$\omega(V(f)) = \omega^j \partial_j (V^i \partial_i(f)) = \omega^j \partial_j V^i \partial_i(f) + \cancel{\omega^j V^i \partial^2_{ji} f}$$

$$[V,W]_p(f) = V_p(W(f)) - W_p(V(f)) = v^i\partial_i w^j\partial_j(f) - w^j\partial_j v^i\partial_i(f)$$
$$= (v^i\partial_i w^j - w^i\partial_i v^j)\partial_j f$$
$$= \Omega^j\partial_j$$

Properties of the bracket:

I. $\mathbb{R}$-linearity

$$[av+bw, X] = a[v,X] + b[w,X]$$

II. Skew-symmetry

$$[W,V] = -[V,W]$$

III. Jacobi Identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$
$$[X,[Y,Z]] = [Z,[Y,X]] + [Y,[X,Z]]$$

Notes:

I. The bracket operation on $\mathfrak{X}(M)$ though $\mathbb{R}$-bilinear is not $\mathcal{F}(M)$-bilinear.

II. $[V,V] = 0$

III. $[\partial_i, \partial_j] = 0$

Homework:

I. Prove the Jacobi Identity.

II. Let x,y be the natural coordinates of $\mathbb{R}^n$ and consider the vector fields.

$$V = y\partial_y \qquad W = x\partial_y$$

show that $[V,W] = -W$.

Tensor algebra

So far,

$$V \in T_pM \longrightarrow V = v^i\partial_i|_p$$
$$W \in T_p^*M \longrightarrow W = W_i dx^i$$

Consider a coordinate transformation $x^i \longmapsto y^i(x)$.

$$dx^i \longrightarrow dy^i = A^i{}_m dx^m$$

where $A^i{}_m := \left.\frac{\partial y^i}{\partial x^m}\right|_p$

$$\left.\frac{\partial}{\partial x^i}\right|_p \longmapsto \left.\frac{\partial}{\partial y^i}\right|_p = \frac{\partial x^n}{\partial y^i}\frac{\partial}{\partial x^m}$$

$$=: A_i{}^m \frac{\partial}{\partial x^m} = (A_i{}^m)^{-1}\frac{\partial}{\partial x^m}$$

$$\omega = \omega_i dx^i = \hat{\omega}_i dy^i = \hat{\omega}_i A_i{}^m dx^m$$

Components must transform as

$$\omega_i = \hat{\omega}_i A_i{}^m$$

$$\hat{\omega}_i = (A_i{}^m)^{-1}\omega_i$$

$$V = V^i \partial x_i = \hat{V}^i \partial y_i = \hat{V}^i (A^i{}_m)^{-1} \partial x_m$$

$$\hat{V}^i (A_i{}^m)^{-1} = V^i$$

$$\hat{V}^i = (A_i{}^m) V^m$$

Covariant vector: Transforms as the basis of T_pM

$$\omega_i$$

Contravariant vector: Transforms as the basis of T_p^*M

$$V^i$$

then

$$Y_i = \left(\frac{\partial x^j}{\partial y^i}\right)_p X_j \qquad \text{Covariant vector}$$

$$Y^i = \left(\frac{\partial y^i}{\partial x^j}\right)_p X_j \qquad \text{Contravariant vector}$$

From the space T_pM of tangent vectors at p and the space T_p^*M of one forms at p, we can define the cartesian product

$$\Pi_r^s = \underbrace{T_p^*M \times T_p^*M \times \cdots \times T_p^*M}_{r\text{-factors}} \times \underbrace{T_pM \times T_pM \times \ldots \times T_pM}_{s\text{-factors}}$$

i.e., the ordered set of vectors and one-forms

$$(\omega_1, \dots, \omega_r, V^1, \dots, V^s)$$

Definition: A tensor of type (r,s) T at p is a function on Π_r^s which is linear in each argument

$$T : \Pi_r^s \longrightarrow \mathbb{R}$$

$$T(\omega_1, \dots, \omega_r, V^1, \dots, V^s) \ni \mathbb{R}$$

such that

$$T(\omega_1, \dots, \omega_r, \alpha X + \beta Y, V^2, \dots, V^s)$$

$$= \alpha T(\omega_1, \dots, \omega_r, X, V^2, \dots, V^s)$$

$$+ \beta T(\omega_1, \dots, \omega_r, X, V^2, \dots, V^s)$$

for any entry on T. It only holds if $\alpha, \beta \in \mathbb{R}$

In particular

$$T_0^1(p) = T_pM \quad , \quad T_1^0(p) = T_p^*M.$$