

Example:

$$\varphi(z) = z - \frac{z^3}{3}$$

$$\varphi'(z) = 1 - z^2$$

$$\varphi'(z) = 0 \text{ at } z = \pm 1$$

$$\varphi''(z) = -2z$$

$$\varphi''(1) = -2 \quad \varphi''(-1) = 2.$$

Critical points at  $z_0 = \pm 1$  of order  $m=2$ .

Valleys  $\cos(\theta m + \psi) < 0$

$$\varphi''(z) = \alpha e^{iz\psi}$$

• For  $z_0 = 1 \rightarrow \varphi''(1) = -2 \rightarrow \psi = \pi$

• For  $z_0 = -1 \rightarrow \varphi''(-1) = 2 \rightarrow \psi = 0$ .

Steepest descent:  $\theta m + \psi = (2k+1)\pi$ ,  $k=0,1$ .

$$\theta = -\frac{\psi}{m} + \frac{(2k+1)\pi}{m}$$

• For  $z_0 = 1$

$$\theta = -\frac{\pi}{2} + \frac{(2k+1)\pi}{2} \quad \theta_0 = 0$$

$$\theta_1 = \pi$$

• For  $z_0 = -1$

$$\theta_0 = \frac{\pi}{2}, \quad \theta_1 = \frac{3\pi}{2}$$

$$\varphi(x, y) = \psi(x, y) + iV(x, y).$$

$$V(x, y) = \ln \left( z - \frac{z^3}{3} \right)$$

$$z = x + iy$$

$$\varphi(x, y) = \left( x + iy - \frac{(x+iy)^3}{3} \right)$$

$$= x + iy - \frac{(x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3)}{3}$$

$$= \left( x - \frac{x^3}{3} - xy^2 \right) + i \left( y - x^2y + \frac{y^3}{3} \right)$$

$$V(x,y) = \operatorname{Im} \left( z - \frac{z^3}{3} \right) = y \left( 1 - x^2 + \frac{y^2}{3} \right) = V(1,0) = 0$$

↑  
For  $z_0 = 1$ .

In a neighbourhood of  $z_0 = 1$ , the curves of steepest ascent / descent are

$$\text{a) } y=0 \quad \text{b) } 1 - x^2 + \frac{y^2}{3}$$

$$\text{as } V(x,y) = x \left( 1 - \frac{x^2}{3} + y^2 \right) \Rightarrow y=0 \quad \text{steepest descent.}$$

$$V(x,y) \Big|_{y=0} = x \left( 1 - \frac{x^2}{3} \right)$$

$$V(x,y) \Big|_{1-x^2+\frac{y^2}{3}} = x \left( 1 - \frac{x^2}{3} + 3x^2 - 3 \right) = x \left( -2 + \frac{8}{3}x^2 \right)$$

$$\text{Analogously, for } z_0 = -1, \quad V(x,y) = y \left( 1 - x^2 + \frac{y^2}{3} \right) = V(-1,0) = 0$$

$$V(x,y) \Big|_{y=0} = x \left( 1 - \frac{x^2}{3} \right)$$

$$V(x,y) \Big|_{1-x^2+\frac{y^2}{3}} = x \left( -2 + \frac{8}{3}x^2 \right)$$

Example:

$$\varphi(z) = \cosh(z) - \frac{z^2}{2}.$$

$$\varphi'(z) = \operatorname{sech}(z) - z$$

$$\varphi'(z) = 0 \text{ if } z=0.$$

$$\varphi''(z) = \cosh(z) - 1$$

$$\varphi''(0) = 0.$$

$$\varphi'''(z) = \sinh(z)$$

$$\varphi'''(0) = 0$$

$$\varphi^{(4)}(z) = \cosh(z)$$

$$\varphi^{(4)}(0) = 1 \neq 0.$$

$\therefore \varphi(z)$  has a critical point at  $z_0=0$  of order  $m=4$

$$\varphi^{(4)}(z) = \cosh(z) = \cosh(z) e^{iz} \quad \varphi = 0.$$

$$m\Theta + \psi = (2k+1)\pi \quad \text{steepest descent.}$$

$$k = 0, 1, 2, 3.$$

$$\Theta = -\frac{\psi}{m} + \frac{2k+1}{m}\pi$$

$$\Theta_0 = \frac{\pi}{4}, \quad \Theta_1 = \frac{3\pi}{4}, \quad \Theta_2 = \frac{5\pi}{4}, \quad \Theta_3 = \frac{7\pi}{4}$$

$$\cosh(x+iy) = \cosh(x)\cos(y) + i \sinh(x)\sin(y)$$

$$\varphi(x,y) = \cosh(x+iy) - \frac{(x-iy)^2}{2}$$

$$= \left( \cosh(x)\cos(y) - \frac{x^2}{2} + \frac{y^2}{2} \right) + i \left( \sinh(x)\sin(y) - xy \right)$$

$=: U(x,y) \qquad \qquad \qquad =: V(x,y)$

$$U(x,y) = \operatorname{Im}(\varphi(z)) = \sinh(x)\sin(y) - xy = U(0,0) = 0$$

$$V(x,y) = \operatorname{Re}(\varphi(z)) = \cosh(x)\cos(y) - \frac{x^2}{2} + \frac{y^2}{2} = V(0,0) = 1.$$

Example:

$$I(\lambda) = \int_0^1 e^{i\lambda z} \log(z) dz.$$

$$\Rightarrow \varphi(z) = iz = i(x+iy) = -y+ix.$$

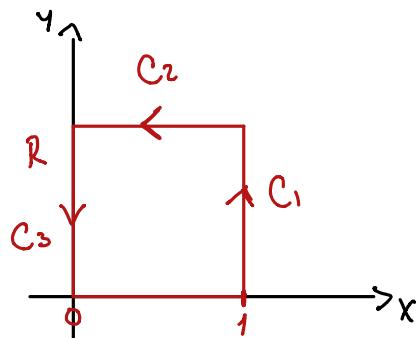
$$\varphi'(z) = i \neq 0 \quad \text{No saddle-points.}$$

steepest descent/ascent paths are given by

$$v = \operatorname{Im}(\varphi(z)) = x = \text{constant}.$$

$$u = \operatorname{Re}(\varphi(z)) = -y$$

For  $y > 0$  the curves  $x = \text{constant}$  are steepest descent paths



we deform the path as no continuous steepest descent path passes through the two end points.

$\Rightarrow$

$$\int_0^1 e^{i\lambda z} \log(z) dz = - \int_{C_1 + C_2 + C_3} e^{i\lambda z} \log(z) dz.$$

• for  $C_2$ :  $z = x + iR \Rightarrow \int_{C_2} = - \int_0^1 \log(x+iR) e^{i\lambda(x+iR)} dx$

and

$$\left| \int_{C_2} \right| \leq e^{-R\lambda} \int_0^1 |\log(x+iR)| dx$$

$$\Rightarrow \left| \int_{C_2} \right| \text{ goes to zero as } R \rightarrow \infty$$

• for  $C_1$ :

$$\begin{aligned} z &= 1+iy \quad dz = idy \\ \int_{C_1} &= i \int_0^R dy \log(1+iy) e^{i\lambda(1+iy)} \\ &= ie^{i\lambda} \int_0^R \log(1+iy) e^{-\lambda y} dy \end{aligned}$$

• for  $C_3$ :

$$\begin{aligned} z &= iy \quad dz = idy \\ \int_{C_3} &= -i \int_0^R dy \log(iy) e^{i\lambda iy} = -i \int_0^R (\log(iy)) e^{-\lambda y} dy \\ I(\lambda) &\stackrel{R \rightarrow \infty}{=} i \int_0^\infty \log(iy) e^{-\lambda y} dy - ie^{i\lambda} \int_0^\infty \log(1+iy) e^{-\lambda y} dy \\ &\qquad\qquad\qquad=: I_1 \\ &\qquad\qquad\qquad=: I_2 \end{aligned}$$

For  $I_1$ :

$$\begin{aligned} \log(iy) &= \log(i) + \log(y) = \log(y) + \log e^{i\pi/2} \\ &= \log(y) + i \frac{\pi}{2} \quad \omega = \lambda y. \end{aligned}$$

$$\begin{aligned} I_1 &= i \int_0^\infty e^{-\lambda y} \left( \log(y) + i \frac{\pi}{2} \right) dy = -\frac{\pi}{2\lambda} + \frac{i}{\lambda} \int_0^\infty e^{-\omega} \log\left(\frac{\omega}{\lambda}\right) d\omega \\ &= -\frac{\pi}{2\lambda} + \frac{i}{\lambda} \int_0^\infty e^{-\omega} \log(\omega) d\omega - \frac{i}{\lambda} \log(\lambda) \int_0^\infty e^{-\omega} d\omega \\ &= -\frac{\pi}{2\lambda} - \frac{i}{\lambda} \log(\lambda). \end{aligned}$$

$$\int_0^\infty e^{-\omega} \log \omega d\omega = -\gamma$$

Euler-Mascheroni  
constant.

- for  $I_2$ : Use Watson's lemma.

$$I_2 = ie^{i\lambda} \int_0^\infty \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (iy)^n \right) e^{-\lambda y} dy.$$

$$\sim e^{i\lambda} \sum_{n=1}^{\infty} \frac{(-i)^{n+1}}{n \lambda^{n+1}} \Gamma(n+1)$$

$$\therefore I(\lambda) \sim -\frac{i \log(\lambda)}{\lambda} - \frac{2\pi i}{\lambda} - \frac{i\lambda}{\lambda} - e^{i\lambda} \sum_{n=1}^{\infty} \frac{(-i)^{n+1}}{n \lambda^{n+1}} \Gamma(n+1)$$

Example.

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{e^{i\lambda(z + z^3/3)}}{2z^2 + 1} dz$$

$$\varphi(z) = i \left( z + \frac{z^3}{3} \right)$$

$$\varphi'(z) = i(1 + z^2)$$

$$\varphi'(z) = 0 \quad \text{iff} \quad z_0 = \pm i$$

Saddle points at  $z_0 = \pm i$

$$\varphi''(z) = 2i z$$

$$\varphi''(i) = -2, \quad \varphi''(-i) = 2$$

order  $M=2$ .

Directions of steepest descent:

$$z_0 = i \quad \varphi''(i) = -2 \quad \rightarrow \quad \psi = \pi$$

$$2\theta + \pi = (2k+1)\pi, \quad k=0,1.$$

$$\theta = -\frac{\pi}{2} + \frac{2k+1\pi}{2}$$

$$\theta_0 = 0, \quad \theta_1 = \pi$$

$$z_0 = -i \rightarrow \varphi''(-i) = 2 \quad \varphi = 0.$$

$$\theta = (2k+1)\frac{\pi}{2}, \quad k=0,1.$$

$$\theta_0 = \frac{\pi}{2}, \quad \theta_1 = \frac{3\pi}{2}$$

$$\text{Now, set } z = Re^{i\phi} \rightarrow z + \frac{z^3}{3} = Re^{i\phi} + \frac{R^3 e^{3i\phi}}{3}$$

then, for large  $R$ .

$$e^{i\lambda(z+z^3/3)} \sim O(e^{-\frac{2R^3}{3}\sin(3\phi)}) \quad \text{that decays as long as } \sin(3\phi) > 0.$$

$$\sin(3\phi) > 0 \rightarrow 0 < 3\phi < \pi \quad \text{or} \quad 0 < \phi < \frac{\pi}{3}$$

Steepest descent paths satisfy

$$V(x,y) = \ln(\varphi(z)) = x \left( 1 + \frac{x^2}{3} - y^2 \right)$$

$$U(x,y) = \Re(\varphi(z)) = y \left( \frac{y^2}{3} - 1 - x^2 \right)$$

Steepest descent are either  $x=0$  or  $1 + \frac{x^2}{3} - y^2 = 0$

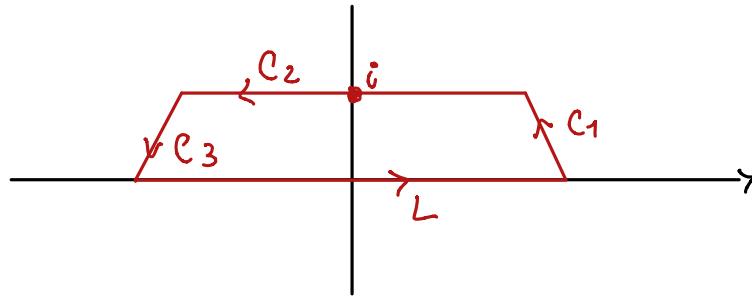
$$U_{x=0} = y \left( \frac{y^2}{3} - 1 \right)$$

$$U_{1+\frac{x^2}{3}-y^2=0} = y \left( -\frac{8y^2}{3} + 2 \right)$$

for  $z_0 = i$  ( $x_0 = 0, y_0 = 1$ ) steepest descent  $1 + \frac{x^2}{3} - y^2 = 0$

for  $z_0 = -i$  ( $x_0 = 0, y_0 = -1$ ) steepest descent  $x = 0$ .

We deform the original contour to the upper-half plane on the steepest descent path through  $t = i$



$$\oint_{L+C_1+C_2+C_3} \frac{e^{i\lambda(z+\frac{z^3}{3})}}{2z^2+1} dz = 2\pi i \operatorname{Res}\left(z = \frac{i}{\sqrt{2}}\right)$$

$$2\pi i \operatorname{Res}\left(z = \frac{i}{\sqrt{2}}\right) = 2\pi i \left( \frac{e^{i\lambda(\frac{i}{\sqrt{2}} + (\frac{i}{\sqrt{2}})^3 \frac{1}{3})}}{4(\frac{i}{\sqrt{2}})} \right) = \frac{\sqrt{2}}{2} \pi e^{i\lambda(\frac{i}{\sqrt{2}} - \frac{i}{6\sqrt{2}})}$$

$$\int_L = \int_{C_1} - \int_{C_2} - \int_{C_3} + \frac{\sqrt{2}}{2} \pi e^{-5\lambda/6\sqrt{2}} \quad \text{as } R \rightarrow \infty$$

for  $C_2$ :

$$z = i + t, \quad t \in \mathbb{R}$$

$$z + \frac{z^3}{3} = (i+t) + \frac{(i+t)^3}{3}$$

$$= i + t + \frac{i^3 + 3i^2t + 3it^2 + t^3}{3}$$

$$= \left(t - i + \frac{t^3}{3}\right) + i \left(1 - \frac{1}{3} + t^2\right)$$

$$= \frac{t^3}{3} + i \left(\frac{2}{3} + t^2\right)$$

$$i\lambda \left(z + \frac{z^3}{3}\right) = \frac{it^3\lambda}{3} - \lambda \left(\frac{2}{3} + t^2\right)$$

This term oscillates.

$$\text{Also, as } t \rightarrow 0 \quad \frac{1}{1+2t^2} \xrightarrow{} \frac{1}{-2+1} = -1$$

and (up to leading order) the oscillates term goes to 1.

$$\rightarrow \int_{-\infty}^{\infty} \frac{e^{i\lambda(z+z^3/3)}}{2z^2+1} dz \sim \int_{-\infty}^{\infty} \frac{e^{-2\lambda/3} e^{-\lambda t^2}}{(-1)} dt + \frac{\sqrt{2}}{2} \pi e^{-5\lambda/\pi^2 6}$$

$$\sim -\frac{\sqrt{\pi}}{\sqrt{\lambda}} e^{-2\lambda/3} + \frac{\sqrt{2}}{2} \pi e^{-5\lambda/\pi^2 6}, \quad \text{as } \lambda \rightarrow \infty$$