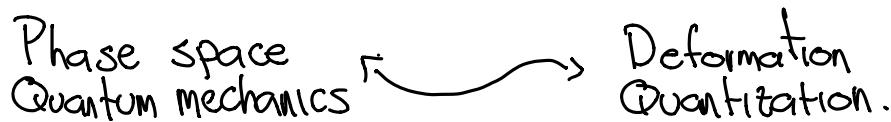


## An overview of deformation quantization.

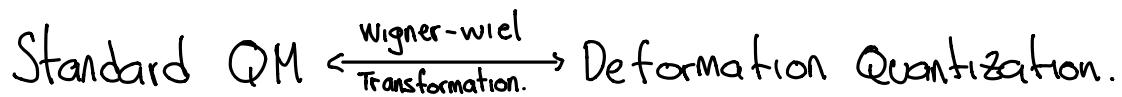
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Idea: Alternative formulation of QM in phase space



Point-wise product deformed into an appropriate non-commutative  $*$ -product.

Poisson bracket deformed into an appropriate Lie bracket.  
(Moyal).



Ordinary QM is given by a set of axioms not necessarily connected to classical mechanics:

- QM generalizes CM in such a way that we recover CM from QM in the limit  $\hbar \rightarrow 0$ .

$\hbar = \text{Plank's constant.}$

Deformation Quantization is a natural formulation of QM in phase space.

- 1) Symplectic manifolds (Fedosov)
- 2) Poisson Manifolds. (Kontsevich).

III) Admissible states of classical Hamiltonian system are defined as (pseudo-) probability distributions on phase space.

## Classic Hamiltonian Mechanics.

**Definition:** A Poisson manifold is a smooth manifold  $M$  endowed with a real skew-symmetric tensor field of rank  $(2,0)$   $P \in \Lambda^2 T(M)$  satisfying the relation.

$$\mathcal{L}_{x_f} P = 0.$$

for every vector field  $x_f := P(df)$  and  $f \in C^\infty(M)$ .

$P \rightarrow$  Poisson tensor

$x_f \rightarrow$  Hamiltonian vector

$$P = P^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

$\mathcal{L} \rightarrow$  Lie derivative.

$$\{x_f\} =: \text{Ham}(M)$$

Considering  $P$  we define a bracket structure.

$$\{\cdot, \cdot\}: C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

$$\{f, g\} = P(df, dg), \quad f, g \in C^\infty(M).$$

such that

$$I) \{f, g\} = -\{g, f\} \quad \text{skew-symmetric.}$$

$$II) \{f, gh\} = \{f, g\}h + g\{f, h\} \quad \text{Leibniz.}$$

$$III) \{f, \{g, h\}\} = \{f, g\}h + \{g, \{f, h\}\}. \quad \text{Jacobi.}$$

- I) & II) algebra of  $C^\infty(M)$ , Poisson algebra (of classical observables.)

- III guarantees that for a given function  $f \in C^\infty(M) \rightarrow C^\infty(M)$  is a derivation of the point-product on  $C^\infty(M)$ , that is.  $\{f, \cdot\} := x_f$  is a vector field.

- Relation with a symplectic form  $\omega$

$$\{f, g\} = \omega(P(df), P(dg)) = \omega(x_f, x_g).$$

•) This bracket is called a Poisson bracket.

Properties:

i) Follows from the skew-symmetry of  $p$ .

ii) Follows from  $\{f, gh\} = p(df, d(gh))$ .

$d(gh) = (dg)h + gdh$  and linearity.

iii) Follows from the condition  $[p, p] = 0$ , where

$$[\cdot, \cdot]: \mathcal{X}^p(M) \times \mathcal{X}^q(M) \longrightarrow \mathcal{X}^{p+q-1}(M).$$

denotes the Schouten-Nijenhuis bracket on multivector fields

Let's define  $A_c := C^\infty(M)$  the algebra of observables, and define an algebra  $\hat{A}_c$  of all operators defined on  $C^\infty(M)$  of the form  $\hat{A} \in \hat{A}_c \longrightarrow \hat{A} = A$ .

Where  $A \in A_c$  and  $\bullet$  denotes the point-wise product in  $C^\infty(M)$ .

We may induce a Lie algebra structure in  $\hat{A}_c$  by the formula.

$$\{\hat{A}, \hat{B}\} := \{A, B\} \bullet ; \hat{A}, \hat{B} \in \hat{A}_c ; A, B \in A_c$$

Note: A particular real observable  $\hat{H}$  governs the time evolution of the system (as we will see below).

Definition: A triple  $(M, P, \mathcal{L})$  is called a classic Hamiltonian system.

Definition: Canonical coordinates are local coordinates  $q^i, p_i$  ( $i = 1, \dots, N = \dim M$ ) in which  $P$  has the form

$$P = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} = \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p^i} - \frac{\partial}{\partial p^i} \otimes \frac{\partial}{\partial q^i}$$

Or in components  $P^{ij} = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$

In cardinal coordinates

$$x_f = P(df) = \left( \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} \right) \left( \frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_j} dp_j \right)$$

$$= -\frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} + \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i}$$

and

Homework:

$$\{f, g\} = p(df, dg)$$

$$\{f, g\} = X_f(g) = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}$$

**Definition:** Pure states of a classic Hamiltonian system are points in phase space  $M$ .

Pure states are characterized by canonical coordinates  $q^i, p_i$ .

Not always is possible to know the exact positions and momenta of a particle.

By theoretical issues on a lab or by considering an ensemble of particles, for example.

Generalize the concept of pure states to probability distributions on  $M$ , such that for  $\xi = (q, p) \in M$ . We may define a smooth function  $\rho$  on  $M$  satisfying.

$$\left. \begin{array}{l} \text{i)} \rho(\xi) \geq 0 \\ \text{ii)} \int_M \rho(\xi) d\xi = 1 \end{array} \right\} \text{We will call these states "Mixed" states.}$$

$\rho \rightarrow$  Classical distribution function.

from this point of view, pure states may identified with Dirac delta distributions  $\rho = \delta(\xi - \xi_0)$ .

**Definition:** A classical expectation value  $\langle \hat{A} \rangle_\rho$  of an observable  $\hat{A} \in \hat{A}^c$  in a state  $\rho$  is defined by

$$\langle \hat{A} \rangle_\rho := \int_M (\hat{A}_\rho)(\xi) d\xi = \int_M A(\xi)(\xi) d\xi.$$

For pure states

$$\langle \hat{A} \rangle_\delta = \int_M A(\xi) \cdot \delta(\xi - \xi_0) d\xi = A(\xi_0).$$

For an operator  $\hat{A} \in \hat{\mathcal{A}}_c$  we can associate a standard deviation.

$$\sigma_A := \sqrt{\langle \hat{A}^2 \rangle_p - \langle \hat{A} \rangle_p^2}$$

$$\Rightarrow \sigma_x \cdot \sigma_{p_i} \geq$$

denotes the classical uncertainty relations.

It is possible to simultaneously measure position and momentum with arbitrary precision.