$$g(\vec{e}_{A\dot{B}}, \vec{e}_{C\dot{D}}) = -\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}} \qquad (*)$$

$$\vec{e}_{A\dot{B}} = \begin{pmatrix} \vec{E}_4 & \vec{E}_2 \\ \vec{E}_1 & -\vec{E}_3 \end{pmatrix}$$

$$\Rightarrow \vec{e}_{A\dot{B}} = \frac{1}{\sqrt{2}} \sigma^a{}_{A\dot{B}} \vec{e}_a$$

As consequence of $(*)$.

$$\sigma^a{}_{A\dot{B}}\, \sigma^b{}_{C\dot{D}}\, g_{ab} = -2\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}} \qquad (**)$$

The scalars are called conexion symbols or Infield.

The symbols of Levi-Cirita are used to raise and lower spinorial indices A,B. Using the conversion.

$$\psi_A = \varepsilon_{AB}\psi^B \quad , \quad \psi^A = \psi = \psi_B \varepsilon^{BA}$$

then

$$\psi^2 = \psi_1 \quad , \quad \psi^1 = -\psi_2$$

$$\psi^A = \psi_B \varepsilon^{BA}$$

$$\psi^2 = \psi_B \varepsilon^{B2} \qquad \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\psi^2 = \psi_1 \cancelto{1}{\varepsilon^{12}} + \psi_2 \cancelto{0}{\varepsilon^{22}}$$

$$\psi^2 = \psi_1$$

the antisymmetrie of ε_{AB}, implies

$$\psi^A \phi_A = \psi^A \varepsilon_{AB}\phi^B = -\psi^A \varepsilon_{BA}\phi^B = -\psi_B\phi^B = -\psi_A\phi^A.$$

$$\Rightarrow \psi^A\psi_A = 0.$$

Also,

$$\varepsilon^A{}_B = \delta^A{}_B \quad \text{and} \quad \varepsilon_A{}^B = -\delta^B{}_A.$$

$$\delta^A{}_C = \varepsilon_{BC}\varepsilon^{BA}$$

$$\delta^1{}_1 = \varepsilon_{B1}\varepsilon^{B1} = \varepsilon_{11}\varepsilon^{11} + \varepsilon_{21}\varepsilon^{21} = 1.$$

$$\delta^1{}_2 = \varepsilon_{B2}\varepsilon^{B1} = \varepsilon_{12}\varepsilon^{11} + \varepsilon_{22}\varepsilon^{21} = 0.$$

Proposition: If $\psi_{AB} = -\psi_{BA}$, then $\psi_{AB} = \frac{1}{2}\psi^R{}_R\,\varepsilon_{AB}$.

Proof: Any antisymmetric 2x2 matrix is of the form

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

which may be written as $a(\varepsilon_{AB})$. Then ψ_{AB} is antisymmetric and $\psi_{AB} = a\varepsilon_{AB}$, for some a. Contracting by ε^{AB}, then $\psi^A{}_A = 2a$.

$$\psi^B{}_B = \varepsilon^{AB}\psi_{AB} = a\varepsilon_{AB}\varepsilon^{AB} = a\varepsilon^B{}_B$$

$$\Longrightarrow \psi^B{}_B = a\delta^B{}_B = 2a.$$

$\blacksquare$

In a similar way

$$\psi^{AB} = \frac{1}{2}\psi^R{}_R\,\varepsilon^{AB}$$

if $(M^R{}_S)$ a 2x2 matrix (real or complex)

$$\varepsilon_{AC}M^A{}_B M^C{}_D = (\det(M^R{}_S))\,\varepsilon_{BD}$$

Proof: Let

$$\psi_{BD} = \varepsilon_{AC}M^A{}_B M^C{}_D$$

then

$$\psi_{DB} = \varepsilon_{AC}M^A{}_D M^C{}_B = -\varepsilon_{CA}M^C{}_B M^A{}_D = -\psi_{BD}.$$

as

$$\psi^R{}_R = \psi^1{}_1 + \psi^2{}_2 = -\psi_{21} + \psi_{12} = 2\psi_{12}$$

$$= 2\varepsilon_{AC}M^A{}_1 M^C{}_2 = 2(M^1{}_1M^2{}_2 - M^2{}_1M^1{}_2)$$

$$= 2\det(M^R{}_S)$$

$\blacksquare$

Choise of connection symbols

Comparing the

$$\vec{E}_a = (\vec{e}_{A\dot{B}}) = \begin{pmatrix} \vec{E}_4 & \vec{E}_2 \\ \vec{E}_1 & -\vec{E}_3 \end{pmatrix}$$

and

$$\vec{e}_{A\dot{B}} = \frac{1}{\sqrt{2}}\sigma^a{}_{A\dot{B}}\,\vec{E}_a$$

For

- $(g_{ab}) = \mathrm{diag}(1,1,1,1)$

$$(\sigma^1{}_{A\dot B}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2{}_{A\dot B}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\sigma^3{}_{A\dot B}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\sigma^4{}_{A\dot B}) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

- $(g_{ab}) = \mathrm{diag}(1,1,1,-1)$

$$(\sigma^1{}_{A\dot B}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2{}_{A\dot B}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\sigma^3{}_{A\dot B}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\sigma^4{}_{A\dot B}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $(g_{ab}) = \mathrm{diag}(1,1,-1,-1)$

$$(\sigma^1{}_{A\dot B}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2{}_{A\dot B}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\sigma^3{}_{A\dot B}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\sigma^4{}_{A\dot B}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\overline{(\sigma^a{}_{A\dot B})} = \begin{cases} \sigma^{aA\dot B} & \text{Euclidean} \\ \sigma^a{}_{A\dot B} & \text{Lorentzian.} \\ \sigma^a{}_{A\dot B} & \text{Ultrahiperbolic} \end{cases}$$

$$\sigma^a{}_{A\dot B}\,\sigma^b{}_{C\dot D}\,g_{ab} = -2\varepsilon_{AC}\varepsilon_{\dot B\dot D}$$

Contracting with $\sigma_c{}^{C\dot D}$

$$\sigma^a{}_{A\dot B}\,\sigma^b{}_{C\dot D}\,\sigma^{C\dot D}{}_c\,g_{ab} = -2\varepsilon_{AC}\varepsilon_{\dot B\dot D}\,\sigma^{C\dot D}{}_c = -2\sigma_{cA\dot B}$$

$$\rightarrow \sigma^{a}{}_{A\dot{B}}\left(\sigma_{ac\dot{D}}\,\sigma_{c}{}^{C\dot{D}} + 2g_{ac}\right) = 0 \qquad \left(\sigma_{cA\dot{B}}\right) = \sigma^{a}{}_{A\dot{B}}\,g_{ac}$$

As the σ symbols are invertibles.

$$\sigma_{aA\dot{B}}\,\sigma_{b}{}^{A\dot{B}} = -2g_{ab}$$

If $t_{ab\ldots d}$ the component of a tensor with respect to the orthonormal base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$, the tensorial equivalent.

$$t_{A\dot{B}C\dot{D}\ldots G\dot{H}} = \frac{1}{\sqrt{2}}\sigma^{a}{}_{A\dot{B}}\,\frac{1}{\sqrt{2}}\sigma^{b}{}_{C\dot{D}}\ldots\frac{1}{\sqrt{2}}\sigma^{d}{}_{G\dot{H}}\,t_{ab\ldots d}.$$

is convenient

$$t_{A\dot{A}B\dot{B}\cdots D\dot{D}} = \frac{1}{\sqrt{2}}\sigma^{a}{}_{A\dot{A}}\,\frac{1}{\sqrt{2}}\sigma^{b}{}_{B\dot{B}}\ldots\frac{1}{\sqrt{2}}\sigma^{d}{}_{D\dot{D}}\,t_{ab\cdots d}.$$

also

Homework:

$$t_{ab\cdots d} = \left(-\frac{1}{\sqrt{2}}\sigma^{a}{}_{A\dot{A}}\right)\left(-\frac{1}{\sqrt{2}}\sigma^{b}{}_{B\dot{B}}\right)\ldots\left(-\frac{1}{\sqrt{2}}\sigma^{d}{}_{D\dot{D}}\right)t_{A\dot{A}B\dot{B}\cdots D\dot{D}}$$

$$V^{a}\vec{e}_a = -\frac{1}{\sqrt{2}}\sigma^{a}{}_{A\dot{A}}\,V^{A\dot{A}}\,\vec{e}_a$$

$$= -V^{A\dot{A}}\,\vec{e}_{A\dot{A}}$$

where $\vec{e}_{A\dot{A}} \simeq \vec{e}_A \otimes \vec{e}_{\dot{A}}$

$\hookrightarrow$ Spin space.