Examples (Green functions)

1) Consider $y' + p(x)y = f(x)$ for $x > a$ subject to the initial condition $y(a) = 0$.

Solution:

As $\mathcal{L}[y] = \frac{d}{dx} + p(x)$,

we want to find Green's as the solution to

$$\mathcal{L}[G(x,\xi)] = \delta(x-\xi)$$

subject to $G(a,\xi) = 0$.

$\Rightarrow$ $G' + p(x)G = \delta(x-\xi)$ $\longleftarrow$ As the operator is first order, we expect a discontinuity at $x = \xi$.

Integrating from ξ^- to ξ^+ the last differential equation, we have:

$$G(\xi^+,\xi) - G(\xi^-,\xi) + \cancel{\int_{\xi^-}^{\xi^+} p(x)\, G(x,\xi)\, dx}^{\to 0} = 1.$$

$$\Leftrightarrow G(\xi^+,\xi) - G(\xi^-,\xi) = 1.$$

As the homogeneous solution to the equation is given by

$$y_h = e^{-\int p(x)dx}.$$

thus, we propose

$$G(x,\xi) = \begin{cases} C_1 e^{-\int_a^{\xi} p(x)dx} & a < x < \xi. \\ C_2 e^{-\int_{\xi}^{\infty} p(x)dx} & \xi < x < \infty \end{cases}$$

As $G(a,\xi) = 0 \Rightarrow$ we choose $C_1 = 0$ on the interval (a,ξ)

$$\Rightarrow G(x,\xi) = \begin{cases} 0 & a < x < \xi \\ C_2 e^{-\int p(x)dx} & \xi < x < \infty \end{cases}$$

Further, the discontinuity fixes the constant C_2.

$\Rightarrow G(\xi^+,\xi) - \cancel{G(\xi^-,\xi)}^{\to 0} = 1 \quad \Rightarrow C_2 e^{-\int_x^{\xi^+} p(t)dt} = 1$

$$\Rightarrow C_2 = e^{\int_x^{\xi^+} p(t)dt} \qquad \therefore \quad G(x,\xi) = \begin{cases} 0 & , \quad a < x < \xi \\ e^{-\int_\xi^x p(t)dt} & , \quad \xi < x < \infty \end{cases}$$

$$\therefore \quad G(x,\xi) = H(x-\xi)\, e^{-\int_\xi^x p(t)dt}.$$

2) Solve the boundary value problem $y'' = f(x)$, $y(0) = y(1) = 0$. using the Green's function method.

Solution: Green's function most satisfy $G''(x,\xi) = \delta(x,\xi)$, $G(0,\xi) = G(1,\xi) = 0$.

Solution to the homogeneous equations are given by $y(x) = ax + b$.

$$\Rightarrow G(x,\xi) = \begin{cases} a_1 x + b_1, & 0 \le x < \xi \\ a_2 x + b_2, & \xi < x \le 1. \end{cases}$$

Applying boundary conditions

$$G(0,\xi) = 0 \longrightarrow b_1 = 0$$

$$G(1,\xi) = 0 \qquad a_2 + b_2 = 0$$

$$\Rightarrow G(x,\xi) = \begin{cases} a_1 x, & 0 \le x < \xi \\ a_2(x-1), & \xi < x \le 1 \end{cases}$$

Since Green's function must be continuous at $x = \xi$.

$$a_1 \xi = a_2(\xi - 1) \longrightarrow a_2 = \frac{a_1 \xi}{\xi - 1}.$$

And by the discontinuity of the first derivative

$$\left.\frac{d}{dx} G(x,\xi)\right|_{x=\xi^+} - \left.\frac{d}{dx} G(x,\xi)\right|_{x=\xi^-} = 1.$$

$$\Rightarrow \left. a_2 \right|_{x=\xi^+} - \left. a_1 \right|_{x=\xi^-} = 1. \Rightarrow a_2 - a_1 = 1$$

$$\Rightarrow \frac{a_1\xi}{\xi-1} - a_1 = 1 \Rightarrow a_1\left(\frac{\xi}{\xi-1} - 1\right) = 1$$

$$\Rightarrow a_1\left(\frac{1}{\xi-1}\right) = 1 \quad \Rightarrow \quad a_1 = \xi - 1.$$

And thus $a_2 = \xi$

$$\Rightarrow \boxed{G(x,\xi) = \begin{cases} (\xi-1)x & 0 \leq x < \xi \\ \xi(x-1) & \xi < x \leq 1. \end{cases}}$$

Thus, the general solution to the inhomogeneous equation is.

$$y(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi = \int_0^x (x-1)\xi f(\xi)\, d\xi + \int_x^1 (\xi-1)x f(\xi)\, d\xi$$

$$= (x-1)\int_0^x \xi f(\xi)\, d\xi + x\int_x^1 (\xi-1) f(\xi)\, d\xi.$$

by symmetries in x and ξ.

3) Harmonic oscillator:

Find the green's function for the boundary value problem.

$$y''(x) + y(x) = f(x); \qquad y(0) = 0, \quad y'(1) = 0.$$

Solution: The homogeneous equation has the solution

$$y(x) = C_1 \sin(x) + C_2 \cos(x).$$

then propose

$$G(x,t) = \begin{cases} A\sin(x) + B\cos(x) & ; 0 \leq x < t \\ C\sin(x) + D\cos(x) & ; t < x \leq 1 \end{cases}$$

fixing the boundary conditions on $G(x,t)$

$$G(0,t)=0 \longrightarrow B=0.$$

$$d_x G(1,t)=0 \longrightarrow C\cos(1)-D\sin(1)=0$$

$$\rightarrow G(x,t)=\begin{cases} A\sin(x) \; ; & 0\leq x<t \\ D\frac{\sin(1)}{\cos(1)}\sin(x)+D\cos(x) \; ; & t<x\leq 1. \end{cases}$$

$$=\begin{cases} A\sin(x) \; ; & 0\leq x<t \\ \frac{D}{\cos(1)}\left(\sin(1)\sin(x)+\cos(x)\cos(1)\right) \; ; & t<x\leq 1. \end{cases}$$

$$=\begin{cases} A\sin(x) \; ; & 0\leq x<t \\ \frac{D}{\cos(1)}\cos(x-1) \; ; & t<x\leq 1 \end{cases}$$

Imposing continuity conditions, we obtain.

$$G(t^+,t)-G(t^-,t)=0$$

$$\frac{D}{\cos(1)}\cos(t-1)-A\sin(t)=0$$

And

$$\left.\frac{dG}{dx}\right|_{x=t^+}-\left.\frac{dG}{dx}\right|_{x=t^-}=1$$

$$\frac{-D}{\cos(1)}\sin(t-1)-A\cos(t)=1$$

solving for A and D we get

$$\begin{cases} \dfrac{D}{\cos(1)}\cos(t-1) - A\sin(t) = 0 & (1) \\ \dfrac{-D}{\cos(1)}\sin(t-1) - A\cos(t) = 1 & (2) \end{cases}$$

by substitution of D from (1) into (2).

$$-\frac{A\sin(t)\sin(t-1)}{\cos(t-1)} - A\cos(t) = 1$$

$$-A\left(\frac{\sin(t)\sin(t-1) + \cos(t)\cos(t-1)}{\cos(t-1)}\right) = 1$$

$$-A\frac{\cos(1)}{\cos(t-1)} = 1$$

$$A = -\frac{\cos(t-1)}{\cos(1)}$$

substituying A in (1)

$$\frac{D}{\cos(1)}\cos(t-1) + \frac{\cos(t-1)\sin(t)}{\cos(1)} = 0$$

$$\frac{D}{\cancel{\cos(1)}}\cancel{\cos(t-1)} = -\frac{\cancel{\cos(t-1)}\sin(t)}{\cancel{\cos(1)}}$$

$$D = -\sin(t)$$

therefore.

$$G(x,t) = \begin{cases} -\dfrac{\cos(t-1)}{\cos(1)}\sin(x) \;; & 0 \le x < t \\ -\dfrac{\sin(t)}{\cos(1)}\cos(x-1) \;; & t < x \le 1. \end{cases}$$

4) Harmonic oscillator again:

Change of boundary condition's

$$y''(x) + y(x) = x \qquad y=0 = y(\pi/2) = 0.$$

Construct $G(x,t)$ and find $y(x)$

Solution: As before

$$G(x,t)=\begin{cases} A\sin(x)+B\cos(x) & ;\ 0\le x<t \\ C\sin(x)+D\cos(x) & ;\ t<x\le \pi/2. \end{cases}$$

Imposing boundary conditions

$$G(0,t)=0 \longrightarrow B=0$$
$$G(\pi/2,t)=0 \longrightarrow C=0.$$

$$G(x,t)=\begin{cases} A\sin(x) & ;\ 0\le x<t \\ D\cos(x) & ;\ t<x\le \pi/2. \end{cases}$$

In order to fix A and D consider

Continuity in **G**:

$$G(t^+,t)-G(t^-,t)=0.$$
$$D\cos(t)-A\sin(t)=0$$

Discontinuty in dx G:

$$\left.\frac{dG}{dx}\right|_{x=t^+}-\left.\frac{dG}{dx}\right|_{x=t^-}=1$$
$$-D\sin(t)-A\cos(t)=1$$

solving for A and D we get

$$A=-\cos(t)\quad,\quad D=-\sin(t).$$

thus,

$$G(x,t)=\begin{cases} -\cos(t)\sin(t) & 0\le x<t \\ -\sin(t)\cos(t) & t<x\le \pi/2 \end{cases}$$

In order to obtain the solution $y(x)$ to the inhomogeneous function, use.

$$y(x) = \int_0^{\pi/2} t\, G(x,t)\, dt = \int_0^{x} (-\cos(t)\sin(t))\, t\, dt + \int_x^{\pi/2} t(-\sin(t)\cos(t))\, dt$$

Change $t \longleftrightarrow x$ in order to obtain a solution in x.

$$y(x) = -\cos(x)\int_0^{x} t\sin(t)\, dt - \sin(x)\int_x^{\pi/2} t\cos(t)\, dt$$

$$= -\cos(x)\left(\sin t - t\cos(t)\right)\Big|_0^{x} - \sin(x)\left(\cos(t) + t\sin(t)\right)\Big|_x^{\pi/2}$$

$$= -\cos(x)\left(\sin(x) - x\cos(x)\right) - \sin(x)\left(\frac{\pi}{2} - \cos(x) - x\sin(x)\right)$$

$$= \cancel{-\cos(x)\sin(x)} + x\cos^2(x) - \frac{\pi}{2}\sin(x) + \cancel{\sin(x)\cos(x)} + x\sin^2(x)$$

$$= x(\sin^2(x) + \cos^2(x)) - \frac{\pi}{2}\sin(x).$$

$$y(x) = x - \frac{\pi}{2}\sin(x).$$