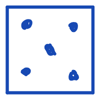
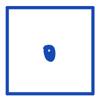
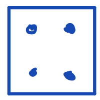
Random walk

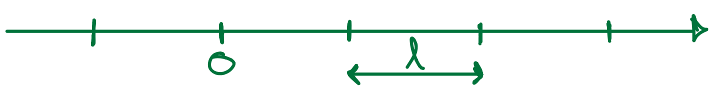
Probability, event and accessible states:

Coin $\longrightarrow$ probability $\frac{1}{2}$ 2 accessible states.

Dice $\longrightarrow$ probability $\frac{1}{6}$ 6 accessible states.

two dice?  and  ? $\frac{2}{6}\cdot\frac{1}{6}=\frac{2}{36}=\frac{1}{12}$

one dice two values?  or  ? $\frac{1}{6}+\frac{1}{6}=\frac{2}{6}=\frac{1}{3}$



$p \longrightarrow$ Probability of moving to the left.

$q=1-p \longrightarrow$ Probability of moving to the Right.

$P_N(m) \longrightarrow$ Probability of finding the particle at $x=ml$ after N steps.

$$-N \leq m < N$$

N_1: # Left steps. N_2: # Right steps.

$$N_2 = N - N_1$$

Probability of a given sequence

$$\underbrace{(p \cdots p)}_{N_1}\underbrace{(q \cdots q)}_{N_2} = p^{N_1} q^{N_2}$$

How many ways can I order p's and q's?

N_1 balls in N boxes $\qquad N > N_1$

$$\underset{\substack{1^{st}\\1-0}}{N}\;\underset{\substack{2^{nd}\\2-1}}{(N-1)}\;\underset{\substack{3^{rd}\\3-1}}{(N-2)}\cdots\underset{\substack{N_1^{th}\\N_1-1}}{(N-(N_1-1))}$$

$$= \frac{N(N-1)(N-2)\cdots(N-N_1+1)(N-N_1)(N-N_1-1)(N-N_1-2)\cdots(1)}{(N-N_1)(N-N_1-1)(N-N_1-2)\cdots(1)}$$

$$= \frac{N!}{(N-N_1)!}$$

So far each ball has a label.

$N_1!$ is the number of permutations of the balls.

Therefore

$$\frac{N!}{N_1!(N-N_1)!} = \frac{N!}{N_1!N_2!} \longrightarrow \text{\# of sequences of tipe } N_1N_2$$

then,

$$W_N(N_1) = \frac{N!}{N_1!\,N_2!} p^{N_1} q^{N_2} \longrightarrow \text{Probability of having } N_1 \text{ right displacements.}$$

For all possible N_i's we get

$$\sum_{N_1=0}^{N} W_N(N_1) = \sum_{N_1=0}^{N} \frac{N!}{N_1!\,N_2!} p^{N_1} q^{N_2} = (p+q)^N = 1$$

Binomio de Newton
$p = 1-q$

if $N=1 \longrightarrow q+p = (p+q)^1$

if $N=2 \longrightarrow q^2+2pq+p^2 = (p+q)^2$

$$0 \leq W_N(N_1) \leq 1 \quad \text{with} \quad 0 \leq N_1 \leq N \quad \text{as} \quad m = N_1 - N_2$$

$$P_N(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!} p^{\frac{N+m}{2}} q^{\frac{N-m}{2}} \quad \text{with} \quad p+q=1$$

Diffusion coefficient

let τ the time between successive steps,

then, $P_N(m) :=$ Probability of finding the particle at $x = ml$ in the time $t = N\tau$

$$P_{N+1}(m)? \longrightarrow t = (N+1)\tau$$

At time t, $x = (m+1)l$ or $x = (m-1)l$

$$\longrightarrow P_{N+1}(m) = p P_N(m-1) \quad q P_N(m+1)$$

If $p = q = \frac{1}{2}$ and τ, ℓ are small.

$$\longrightarrow \quad \frac{P_{N+1} - P_N}{\tau} \sim \frac{\partial P}{\partial t} \qquad (t \text{ small})$$

and

$$\frac{P(m\ell - \ell) + P(m\ell + \ell) - 2P(m\ell)}{\ell^2} \sim \frac{\partial^2 P}{\partial x^2} \qquad (\ell \text{ small}).$$

$$= \frac{2P_{N+1}(m) - 2P_N(m)}{\ell^\tau} = \frac{2\tau}{\ell^2} \frac{\partial P}{\partial t}$$

therefore,

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} \quad \text{with} \quad D := \frac{\ell^2}{2\tau}$$

diffusion Coefficient