Theorem (Ederlyi)

$$I(x)=\int_a^b f(t)e^{-x\phi(t)}dt.$$

$$I(x)\sim e^{-x\phi(a)}\sum_{n=0}^{\infty}\Gamma\left(\frac{n+\beta}{\alpha}\right)\frac{C_n}{x^{(n+\beta)/\alpha}}\quad \text{as}\quad x\longrightarrow\infty$$

$$C_n=\frac{1}{\alpha n!}\left[\frac{d^n}{dt^n}\left\{f(t)\left(\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right)^{\frac{n+\beta}{\alpha}}\right\}\right]_{t=a}.$$

$$\boxed{\begin{array}{l}\text{If}\quad \phi(t)=\phi(a)+\sum_{k=0}^{\infty}a_k(t-a)^{k+\alpha}\\ \text{and}\quad f(t)=\sum_{k=0}^{\infty}b_k(t-a)^{k+\beta-1}.\end{array}}$$

Example: Gamma function.

$$\Gamma(\lambda+1)=\int_0^\infty e^{-t}t^\lambda dt,\quad \text{for}\quad \lambda>0$$

Change $t=\lambda(1+x)\longrightarrow dt=\lambda dx$

$$\longrightarrow \Gamma(\lambda+1)=\int_0^\infty e^{-\lambda(x+1)}\lambda^\lambda(1+x)^\lambda\lambda dx.$$

$$=\lambda^{\lambda+1}e^{-\lambda}\int_{-1}^\infty e^{-\lambda x}(1+x)^\lambda dx.$$

$$=\lambda^{\lambda+1}e^{-\lambda}\int_{-1}^\infty e^{-\lambda(x-\log(1+x))}dx.$$

$$\frac{\Gamma(\lambda+1)}{\lambda^{\lambda+1}e^{-\lambda}}=\underbrace{\int_0^\infty e^{-\lambda(x-\log(1+x))}dx}_{:=I_1(\lambda)}+\underbrace{\int_0^1 e^{-\lambda(-x-\log(1-x))}dx}_{:=I_2(\lambda)}$$

(we changed the integration limits)

$$I_1(\lambda)=\int_0^\infty e^{-\lambda\phi(x)}\,dx \qquad \phi(x)=x-\log(1+x)$$

$$f(x)=1.$$

$$\log(1+x)\sim x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$$

$$\phi(x)=x-\log(1+x)\sim \frac{x^2}{2}-\frac{x^3}{3}+\dots$$

therefore $\alpha=2$ fixed around zero.

$$I_1(\lambda)\sim e^{-\lambda\phi(0)}\sum_{n=0}^{\infty}\Gamma\left(\frac{n+1}{2}\right)\frac{C_n}{\lambda^{(n+1)/2}}, \text{ as } \lambda\longrightarrow\infty$$

$$C_n=\frac{1}{2n!}\frac{d^n}{dx^n}\left[\frac{x^2}{x-\log(1+x)}\right]^{\frac{n+1}{2}}\Bigg|_{x=0}$$

$I_2(\lambda)$:

$$\phi(\lambda)=-x-\log(1-x)\sim -x-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right)$$

$$\sim \frac{x^2}{2}+\frac{x^3}{3}+\dots, \quad \text{for } x\longrightarrow 0^+$$

$$I_2(\lambda)\sim e^{-\lambda\phi(0)}\sum_{n=0}^{\infty}\Gamma\left(\frac{n+1}{2}\right)\frac{C_n(-1)^n}{\lambda^{(n+1)/2}}$$

same C_n as before.

$$\therefore \Gamma(\lambda+1)=\lambda^{\lambda+1}e^{-\lambda}\left(\sum_{n=0}^{\infty}\Gamma\left(\frac{n+1}{2}\right)\frac{C_n}{\lambda^{(n+1)/2}}\left(1+(-1)^n\right)\right)$$

$$=2\lambda^{\lambda+1}e^{-\lambda}\sum_{n=0}^{\infty}\Gamma\left(n+\frac{1}{2}\right)\frac{C_{2n}}{\lambda^{n+1/2}}$$

$$=\sqrt{2\pi}\,e^{-\lambda}\sum_{n=0}^{\infty}\frac{\gamma_n}{\lambda_n}$$

Where

$$\gamma_n=\sqrt{\frac{2}{\pi}}\,\Gamma\left(n+\frac{1}{2}\right)C_{2n}$$

Sterling coefficients.

$$\gamma_0 = 1, \quad \gamma_1 = \frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = -\frac{139}{51840}$$

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$$

$$\Gamma(\lambda) = \sqrt{2\pi}\,\lambda^{-1/2} e^{-\lambda}\left(1 + \frac{1}{12}x + \dots\right) \quad \text{as } x \to \infty.$$

Example: Legendre polynomials $P_m(x)$ $x > 1$.

$$P_m(x) = \frac{1}{\pi}\int_0^{\pi} \left(x - \cos(t)\sqrt{x^2-1}\right)^m dt.$$

Solution: Change $x = \cosh(\theta)$, $\theta > 0$

$$P_m(x) = \frac{1}{\pi}\int_0^{\pi} \left(\cosh(\theta) - \cos(t)\sinh(\theta)\right)^m dt.$$

$$\cosh\theta - \cos(t)\sinh(\theta) = \frac{e^{\theta}+e^{-\theta}}{2} - \cos(t)\frac{e^{\theta}-e^{-\theta}}{2}$$

$$= e^{\theta}\left(\frac{1+\cos(t)}{2}\right) + e^{-\theta}\left(\frac{1-\cos(t)}{2}\right)$$

$$= e^{\theta}\cos^2\left(\frac{t}{2}\right) + e^{-\theta}\sin^2\left(\frac{t}{2}\right)$$

$$= e^{\theta}\left(1 - \sin^2\left(\frac{t}{2}\right)\right) + e^{-\theta}\sin^2\left(\frac{t}{2}\right)$$

$$= e^{\theta}\left[1 - \sin^2\left(\frac{t}{2}\right)(1 - e^{-2\theta})\right]$$

$$P_m(x) = \frac{1}{\pi}e^{m\phi}\int_0^{\pi} e^{-m(-\log[1-\sin^2(\frac{t}{2})(1-e^{-2\theta})])}\,dt.$$

$$\phi(t) = -\log\left[1 - \sin^2\left(\frac{t}{2}\right)(1-e^{-2\theta})\right]$$

$$f(t) = 1, \quad \beta = 1.$$

$$\phi(t) = \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^k}{k} \sin^{2k}\left(\frac{t}{2}\right)$$

$$= \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^k}{k} \left(\frac{t}{2} - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right)^{2k}$$

$$= \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^k}{k} \left(\frac{t^2}{4} - \frac{2t^4}{12} - \theta(t^6)\right)^k$$

$$= \frac{1-e^{-2\theta}}{k}\left(\frac{t^2}{4} - \frac{2t^4}{12}\right) + \frac{(1-e^{-2\theta})^2}{2}\frac{t^4}{16} + \theta(t^6)$$

Thus $\alpha = 2$.

$$P_m(\cosh(\theta)) \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{m^{(n+1)/2}}$$

where

$$C_n = \frac{1}{2n!} \frac{d^n}{dt^n} \left[\frac{t^2}{-\log[1-\sin^2(t/2)(1-e^{-2\theta})]}\right]^{\frac{n+1}{2}} \Bigg|_{t=0}$$

$$C_0 = \frac{1}{2} \left[\frac{t^2}{-\log[1-\sin^2(t/2)(1-e^{-2\theta})]}\right]^{\frac{n+1}{2}} \Bigg|_{t=0}$$

$$= \frac{1}{\sqrt{1-e^{-2\theta}}}$$

$$f(t) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} = 1 = b_0 \;, \quad b_j = 0 \;, \forall j \neq 0.$$

we want $g(t)\phi'(t) = f(t)$, then.

$$\sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} = \left(\sum_{j=0}^{\infty} c_j (t-a)^j\right)\left(\sum_{l=0}^{\infty} a_l (l+\alpha)(t-a)^{l+\alpha-1}\right)$$

$$\rightarrow b_n = \sum_{k=0}^{n} c_k a_{n-k}^{1/\alpha} (n-k+\beta)^{1/\alpha}$$

for $n=0$.

$$b_0 = C_0 a_0^{1/2} \sqrt{2} \longrightarrow C_0 = \frac{b_0}{a_0^{1/2}\sqrt{2}} = \frac{1}{\sqrt{\frac{1-e^{-2\theta}}{\cancel{2}}}\,\cancel{\sqrt{2}}}$$

for $n=1$

$$0 = b_1 = C_0 a_1^{1/2}\sqrt{3} + C_1 a_0^{1/2}\sqrt{2} \longrightarrow C_1 = \cancelto{0}{-\frac{C_0 a_1^{1/2}\sqrt{3}}{a_0^{1/2}\sqrt{2}}} = 0$$

for $n=2$

$$0 = b_2 = C_0 a_2^{1/2}\sqrt{4} + \cancelto{0}{C_1 a_1^{1/2}\sqrt{3}} + C_2 a_0^{1/2}\sqrt{2}$$

$$C_2 = -\frac{C_0 a_2^{1/2}\sqrt{4}}{a_0^{1/2}\sqrt{2}}$$

for $n=3 \longrightarrow C_3 = 0$

Method of Stationary Phase

Let's consider $\phi(t)$ in Laplace Integral. as a pure imaginary function.

$\Rightarrow \phi(t) = i\psi(t)$, where $\psi(t)$ is real.

$$\int_a^b f(t)e^{-x\phi(t)}dt \longmapsto I(x) := \int_a^b f(t)e^{ix\psi(t)}dt$$

General Fourier Integral.

For $\psi(t) = t$,

$$I(x) = \int_a^b f(t)e^{ixt}dt$$

Ordinary Fourier Integral.

Example:

$$I(x) = \int_0^1 \sqrt{t}\,e^{ixt}dt = \int_0^1 \sqrt{t}\,\frac{1}{ix}\frac{d}{dt}(e^{ixt})\,dt \quad , x \to \infty$$

$$= \frac{1}{ix}\sqrt{t}\,e^{ixt}\Big|_0^1 - \frac{1}{2ix}\underbrace{\int_0^1 \frac{e^{ixt}}{\sqrt{t}}dt}_{=:I_1(x)} + O\left(\frac{1}{x^2}\right) = \frac{1}{ix}e^{ix}$$

$I_1(x)$ Change $s = xt$, $ds = xdt$

$$I_1(x) = \frac{i}{2x}\int_0^x \frac{e^{is}}{\sqrt{s/x}}\frac{ds}{x} = \frac{i}{2x^{3/2}}\int_0^x \frac{e^{ix}}{\sqrt{s}}ds, \quad x \to \infty.$$

Change $s \longmapsto is$ (wick rotation) $\longrightarrow$ it's a transformation that substitutes an imaginary-number variable for a real-number variable and vice versa.

$$I_1(x) \sim \frac{i}{2x^{3/2}} \int_0^\infty \frac{e^{-s}}{i^{1/2}} s^{1/2-1}\, ds$$

$$= \frac{i}{2x^{3/2}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \frac{i}{x^{3/2}} \left(e^{i\pi/2}\right)^{1/2}$$

$$= \frac{i\sqrt{\pi}}{2x^{3/2}} e^{i\pi/4}, \qquad \text{as} \quad x \longrightarrow \infty .$$