Definition: (Character): $\chi_D(g) \equiv \mathrm{Tr}\, D(g)$

Representation: $D(g) \in GL(V_n)$

Theorem:

$$\frac{1}{N}\sum_{g\in G} \chi^*_{D_a}(g)\, \chi_{D_b}(g) = \delta_{ab}.$$

are constants in the conjugacy classes.

$$\forall g_1 \in S: \mathrm{Tr}\, D(g^{-1}\cdot g_1 \cdot g) = \mathrm{Tr}\, D(g^{-1}) D(g_1) D(g) = \mathrm{Tr}\, D(g_1)$$

$$D(g^{-1}) = D^{-1}(g)$$

Theorem: The number of irreducible representations is equal to the number of conjugacy classes.

S_n Group

$$M \in S_n : \left\{ \begin{array}{l} 1 \longrightarrow P_1 \\ 2 \longrightarrow P_2 \\ \vdots \\ n \longrightarrow P_n \end{array} \right\} P_i \in \mathbb{N} \qquad \sigma(S_n) = n!$$

Alternatively:

$$M = \begin{pmatrix} 1 & 2 & \cdots & n \\ P_1 & P_2 & \cdots & P_n \end{pmatrix}$$

Example: S_8:

$$M = \left(\begin{array}{ccc|cc|cc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 5 & 4 & 7 & 6 & 8 \end{array}\right)$$

K-Cycles.

Particularly: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \longrightarrow$ 3-cycle

$\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} \longrightarrow$ 2-cycle : transposition.

$M: (1\,2\,3)(4\,5)(6\,7)(8).$

Notice:

$$(1\ 2\ 3) = \left\{ \begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 3 \\ 3 \rightarrow 1 \end{array} \right\} = \left\{ \begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 1 \rightarrow 3 \\ 3 \rightarrow 3 \rightarrow 1 \end{array} \right\}$$

Factorization:

$$(1\ 2\ 3) = (2\ 3) = (1\ 2)$$

In general $M \in S_n$ will have K_j, j-cycles where

$$\sum_{j=1}^{n} j K_j = n.$$

Definition: Representation which defines S_n:

Let's take V_n, $n =$ no of elements to permute,

then,

$$V_n = \}|1\rangle, |2\rangle, \ldots, |n\rangle\{$$

If $g : x_i \rightarrow x_j$, then the representation $D(g)|i\rangle \rightarrow |j\rangle$.

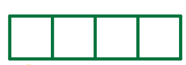
therefore, $\langle j | D(g) | i \rangle = \delta_{ij}$

Conjugacy classes: The cyclic structure: Labeled by K_j.

Young tableau

j-cycle $\longrightarrow$ Horizontal array of boxes

$\rightarrow$ Arranged in decreasing order in j

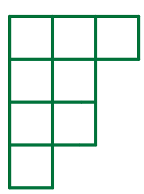
Example: $e \in S_4 \longrightarrow$ 

1-cycle

$$e \left\{ \begin{array}{l} 1 \rightarrow 1 \\ 2 \rightarrow 2 \\ 3 - 3 \\ 4 \rightarrow 4 \end{array} \right\} = (1)(2)(3)(4).$$

Example: S_8 8 boxes

M: (4-cycle)(3-cycle)(1-cycle)

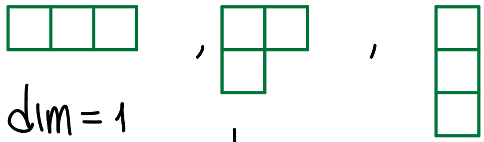


Theorem:

$\{\text{Young tableaux}\} \xrightarrow{\text{one to one}}$ Irreducible representations of S_n

Conjugacy classes

S_3:



dim = 1 , dim = 3 , dim = 2

Irreducible representations:

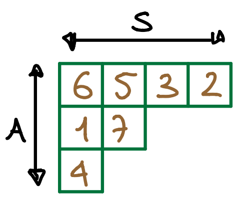
I. Assign 1 to n in all possible forms

- n! forms to do it.

II. Identify each assignment of integers with a state in the regular representation.

- Define an "standard order"
  - > Read from left to right, from top to bottom.

S_7:



$\longrightarrow |6\ 5\ 3\ 2\ 1\ 7\ 4\rangle$

Permutation:

$$(1\ 2\ 3\ 4\ 5\ 6\ 7) \longrightarrow (6\ 5\ 3\ 2\ 1\ 7\ 4)$$

III. Symmetrization:

- We symmetrize the states based on each row and we skew-symmetrize in each column.
  - > The O.D.E's built in this way expand a sub invariance.

Example: $S_2 = Z_2$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$; $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \longrightarrow |2\ 1\rangle$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \longrightarrow |1\ 2\rangle$

$\xrightarrow{S} |1\ 2\rangle + |2\ 1\rangle \longrightarrow \dim = 1$

S_3

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longrightarrow |1\ 2\ 3\rangle + |2\,1\,3\rangle - |3\,2\,1\rangle - |2\,3\,1\rangle$

Irreducible representations:

$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array} = 1_S \qquad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} = 1_A \qquad \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 2.$

Lie group

$\{g(\vec{\alpha}) / \vec{\alpha} \in \mathbb{R}\} \longrightarrow$ Continuous group

n real parameters.

$g(\vec{\alpha}) \longrightarrow$ Smooth function.

Use: $g(\vec{\alpha})\big|_{\vec{\alpha}=0} = e$

Representation:

$$D(\vec{\alpha})\big|_{\vec{\alpha}=0} = \mathbb{1}_{n\times n}$$

$$D(\vec{\alpha}) \in GL(V_n)$$

Taylor:

$$D(d\vec{\alpha}) = \mathbb{1} + i\,d\vec{\alpha}\cdot\vec{X} + \cdots$$

$d\alpha_a X_a$

$$X_a = -i \frac{\partial D(\alpha)}{\partial \alpha_a}\bigg|_{\alpha=0} \longrightarrow \text{Group generator}$$

Constants

Base $\longleftarrow$ $\{X_a\}$ are linear independent.

$\longrightarrow$ form a vector space

In the finite limit (compact)

$$D(\vec{\alpha}) = \lim_{K\to\infty}\left(1 + i\frac{\vec{\alpha}\cdot X}{K}\right)^K$$

$$D(\vec{\alpha}) = \exp[i\vec{\alpha}X]$$

$\downarrow$

Exponential representation.