$$I(x)=\int_a^b e^{-x\phi(t)} f(t)\,dt \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right) C_n}{x^{(n+\beta)/\alpha}} \quad \text{as } x\to\infty$$

Theorem:

if f and g are C^n differentiable at z, then

$$\frac{d^n}{dz^n} f(z)g(z)=\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)$$

Proof:

For $n=1$ $\rightsquigarrow$ Standard Leibniz's rule

$$\frac{d}{dz} f(z)g(z)=f'(z)g(z)+f(z)g'(z)$$

Assume that it is valid for n and try for $n+1$

$$\Rightarrow \frac{d^{n+1}}{dz^{n+1}} f(z)g(z)=\frac{d}{dz}\left(\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(z)\, g^{(n-k)}(z)\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\frac{d}{dz} f^{(n)}(z)\, g^{(n-k)}(z)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left[f^{(k+1)}(z)\, g^{(n-k)}(z)+f^{(k)}(z)\, g^{(n-k+1)}(z)\right]$$

$$=f(z)g^{(n+1)}(z)+\sum_{k=1}^{n}\binom{n}{k} f^{(k)} g^{(n-k+1)}(z)$$

$$+\underbrace{\sum_{k=0}^{n-1}\binom{n}{k} f^{(k+1)}(z)\, g^{(n-k)}(z)}_{\text{Change } j=k+1}+f^{(n+1)}(z)g(z)$$

$$\sum_{j=1}^{n} \binom{n}{j-1} f^{(j)}(z)\, g^{(n-j+1)}(z)$$

then

$$\frac{d^{n+1}}{dz^{n+1}} f(z)g(z) = f(z)g^{(n+1)}(z) + \sum_{k=1}^{n}\left[\binom{n}{k}\binom{n}{k-1}\right] f^{(k)}(z)\, g^{(n-k+1)}(z) + f^{(n+1)}(z)g(z)$$

$$\binom{n}{k}\binom{n}{k-1} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!} = \frac{n!}{(k-1)!\,(n-k)!}\left[\frac{1}{k} + \frac{1}{n-k+1}\right]$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\;\frac{(n+1)}{k(n-k+1)} = \frac{(n+1)!}{k!\,(n-k+1)!} = \binom{n+1}{k}$$

then

$$\frac{d^{n+1}}{dz^{n+1}} f(z)g(z) = f(z)g^{(n+1)}(z) + \sum_{k=1}^{n}\binom{n+1}{k} f^{(k)}(z)\, g^{(n+1-k)}(z) + f^{(n+1)}(z)g(z)$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k} f^{(k)}(z)\, g^{(n+1-k)}(z)$$

Now, consider

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

therefore

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Where

$$c_n = \frac{1}{n!}\frac{d^n}{dz^n}\Big|_{z=z_0} f(z)g(z) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(z_0)\, g^{(n-k)}(z_0)$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}\, \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_n b_{n-k}.$$

Now,

$$g(y) = \frac{f(t)}{\phi'(t)} \implies g(y)\phi'(t) = f(t)$$

Using other notation

$$\frac{f(z)}{g(z)} = h(z) \implies f(z) = h(z)g(z).$$

therefore.

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = \left(\sum_{k=0}^{\infty} c_k (z-z_0)^k\right)\left(\sum_{j=0}^{\infty} b_j (z-z_0)^j\right)$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{n} c_k b_{n-k} (z-z_0)^{k+(n-k)}$$

$$a_n = \sum_{k=0}^{n} c_k b_{n-k}$$

for $n=0$

$$a_0 = c_0 b_0 \implies c_0 = \frac{a_0}{b_0}$$

for $n \geq 1$

$$a_n = \sum_{k=0}^{n} c_k b_{n-k} = c_n b_0 + \sum_{k=0}^{n-1} c_k b_{n-k}$$

$$C_n = \frac{a_n - \sum_{k=0}^{n-1} C_k b_{n-k}}{b_0}$$ Recurrence relation.

$$C_1 = \frac{a_1 - C_0 b_1}{b_0} = \frac{a_1 - \frac{a_0}{b_0} b_1}{b_0} = \frac{a_1 b_0 - a_0 b_1}{b_0^2}$$

$$C_2 = \frac{a_2 - C_0 b_2 - C_1 b_1}{b_0} = \frac{a_2 - \frac{a_0}{b_0} b_2 - \left(\frac{a_1 b_0 - a_0 b_1}{b_0^2}\right) b_1}{b_0}$$

To finish the proof of Erdelyi's theorem,

Check that the remaining part (c, b) is negligible

$$\varepsilon := \inf_{c \leq t \leq b} \phi(t) - \phi(a)$$

Assume x_0 such that $I(x_0)$ is absolutely convergent.

Assume $x \geqslant x_0$

$$x(\phi(t) - \phi(a)) = (x - x_0)(\phi(t) - \phi(a)) + x_0(\phi(t) - \phi(a))$$
$$\geqslant (x - x_0)\varepsilon + x_0(\phi(t) - \phi(a))$$

$$I(x) = \int_a^b e^{-x\phi(t)} f(t)\, dt$$

$$\int_a^b \longrightarrow \int_a^c + \int_c^b$$

then

$$\int_c^b e^{-x(\phi(t)-\phi(a))} f(t)\,dt \le \int_c^b e^{-(x-x_0)\varepsilon} f(t)\,dt + \int_c^b e^{-x_0(\phi(t)-\phi(a))} f(t)\,dt.$$

as the parts containing x_0 are convergent.

$$\left| e^{x\phi(a)} \int_c^b e^{-x\phi(t)} f(t)\,dt \right| \le K e^{-\varepsilon x}$$

Where K is given by,

$$K = e^{x_0(\varepsilon + \phi(a))} \int_c^b e^{-x_0\phi(t)} |f(t)|\,dt$$ is an appropriate constant.

$\therefore$ Most of the contribution to the asymptotic behaviour of $I(x)$ comes from the interval $(a, c]$

Theorem (Perron's formula)

The coefficients c_n above are explicitly given by

$$c_n = \frac{1}{\alpha n!}\left[\frac{d^n}{dx^n}\left\{G(t)\left(\frac{(t-a)}{\phi(t)-\phi(a)}\right)^{(n+\beta)/\alpha}\right\}\right]_{t=a}$$

Where

$$G(t) \sim \sum_{k=0}^{\infty} b_k (t-a)^k$$

and

$$\phi(t) \sim \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$$

Proof :

Define l as the label of the first non-vanishing coefficient in this last expression (apart from a_0) and define.

$$\phi_l(t) := \frac{\phi(t) - \phi(a) - a_0(t-a)^\alpha}{(t-a)^{\alpha+l}} \sim \sum_{k=0}^{\infty} a_{k+l}(t-a)^k$$

as $t \longrightarrow a^+$

$$I(x) = \int_a^b e^{-x\phi(t)} f(t)\,dt = e^{-x\phi(a)} \int_a^b e^{-a_0 x(t-a)^\alpha} \underbrace{e^{-x(t-a)^{\alpha+l}\phi_l(t)} f(t)\,dt}_{=: h(x,t)}$$

$$= e^{-x\phi(a)} \int_a^b e^{-a_0 x(t-a)^\alpha} h(x,t)\,dt$$

Change $z := (t-a)x^{1/\alpha}$, $dz = x^{1/\alpha}dt$

$$I(x) = \frac{e^{-x\phi(a)}}{x^{1/\alpha}} \int_0^{x^{1/\alpha}(b-a)} e^{-a_0 z^\alpha} h(x, x^{-1/\alpha}z + a)\,dz$$

Set $s := x^{-1/\alpha}z$

$$h(x, s+a) = e^{-x(t-a)^{\alpha+l}\phi_l(t)} f(t) = \exp(-z^\alpha s^l \phi_l(s+a)) f(s+a)$$

$(t-a) = s$

$s + a = t$

$(t-a)^{\alpha+l} = s^{\alpha+l} = s^\alpha s^l = s^l(x^{-1}z^\alpha)$

Taylor-

$$\exp(-z^\alpha s^l \phi_l(s+a)) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dw^k}\left[e^{-z^\alpha w^l \phi_l(w+a)}\right]_{w=0} s^k \quad \text{as } s \to 0^+$$

and considering

$$f(x)=\sum_{k=0}^{\infty} b_k (x-a)^{k+\beta-1}$$

then

$$h(x,s+a)\sim\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\frac{d^k}{d\omega^k}\left[e^{-z^{\alpha}\omega^{l}\phi_l(\omega+a)}\right]_{\omega=0} s^{k+\beta-1}$$

$$I(x)\sim\frac{e^{-x\phi(a)}}{x^{1/\alpha}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{d^k}{d\omega^k}\int_0^{x^{1/\alpha}(b-a)} e^{-(a_0+\omega^l\phi_l(\omega+a))z^{\alpha}} s^{n+\beta-1}\right]_{\omega=0} dz$$

$$=e^{-x\phi(a)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{d^k}{d\omega^k}\int_0^{x^{1/\alpha}(b-a)} e^{-[(a_0+\omega^l\phi_l(\omega+a)]z^{\alpha}} z^{n+\beta-1}\right]_{\omega=0}\frac{dz}{x^{(n+\beta)/\alpha}}$$

$$x^{\left(\frac{-1}{\alpha}\right)(n+\beta-1)}=x^{\frac{-n+\beta}{\alpha}}\,x^{1/\alpha}$$

Change

$$\rho:=(a_0+\omega^l\phi_l(\omega-a))\,z^{\alpha}$$

then

$$dz=\frac{d\rho\;\rho^{\frac{1}{\alpha}-1}}{\alpha[a_0+\omega^l\phi_l(\omega+a)]^{1/\alpha}}$$

$$z^{n+\beta-1}=\frac{\rho^{(n+\beta-1)/\alpha}}{[a_0+\omega^l\phi_l(\omega+a)]^{(n+\beta-1)/\alpha}}$$

$$I(x)=e^{-x\phi(a)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\frac{d^k}{d\omega^k}\left\{\left(\int_0^{\infty}e^{-\rho}\rho^{\frac{n+\beta-1}{\alpha}}\,d\rho\right)\left(\frac{1}{\alpha[a_0+\omega^l\phi_l(\omega+a)]^{n+\beta/\alpha}}\right)\right\}_{\omega=0}$$

$$\frac{1}{x^{(n+\beta)/\alpha}}$$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k} \left[\frac{1}{[a_0 + \omega^1 \phi_1(\omega+a)]^{(n+\beta)/\alpha}} \right]_{\omega=0} \frac{1}{x^{(n+\beta)/\alpha}}$$

Using definition of ϕ_1

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k} \left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)} \right]^{\frac{n+\beta}{\alpha}} \Bigg|_{\omega=0} \frac{1}{x^{(n+\beta)/\alpha}}$$

Before we knew

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{C_n}{x^{(n+\beta)/\alpha}}$$

$\therefore$

$$C_n = \frac{1}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{dt^k} \left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)} \right]^{(n+\beta)/\alpha} \Bigg|_{t=a}$$

$$= \frac{1}{\alpha n!} \frac{d^n}{dt^n} \left\{ f(t) \left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)} \right]^{(n+\beta)/\alpha} \Bigg|_{t=a} \right\}$$