Poincaré group.

Let $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, with scalar product

$$xy = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3 = g_{\mu\nu} x^\mu y^\nu$$

this space is called the Minkowski space. The linear transforms in this space

$$x \longrightarrow x' = \Lambda x + y$$

$$x'^\mu = \Lambda^\mu_\nu x^\nu + y^\mu , \qquad \mu, \nu = 0, 1, 2, 3.$$

Translations.

which them satisfy.

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}$$

Lorentz transformation.

$$\begin{aligned} x' \cdot y' &= g_{\mu\nu} x'^\mu y'^\nu \\ &= g_{\mu\nu} \Lambda^\mu_\rho x^\rho \Lambda^\nu_\sigma y^\sigma \\ &= g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho y^\sigma \end{aligned}$$

Called inhomogeneous lorentz transformations and they are denoted by (Λ, y) and the homogeneous part $(\Lambda, 0)$ leaves invariant the scalar product.

$$x'y' = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\nu_\sigma x^\rho x^\sigma = xy.$$

this inhomogeneous transformations form a group, called Poincaré group $\mathbf{P}$.

$$(\Lambda_1, y_1)(\Lambda_2, y_2) = (\Lambda_1, \Lambda_2, y_1 + \Lambda_1 y_2)$$

$$x \longrightarrow x' = \Lambda_2 x + y_2$$

$$\begin{aligned} x' \longrightarrow x'' &= \Lambda_1(\Lambda_2 x + y_2) + y_1 \\ &= \Lambda_1\Lambda_2 x + \Lambda_1 y_2 + y_1 \\ &= (\Lambda_1\Lambda_2, \Lambda_1 y_2 + y_1) \end{aligned}$$

$$P = L \times T_4$$

L – Homogeneous part.

T_4 – Translations

In the natural topology of the matrices, L has fourth disconected parts.

$$(g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = g_{\rho\sigma}) \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x \cdot y = g_{\mu\nu} x^{\mu} y^{\nu} = \sum_{\mu,\nu=0}^{3} g_{\mu,\nu} x^{\mu} y^{\nu}$$

$$g_{\mu\nu} \Lambda^{\mu}_{0} \Lambda^{\nu}_{0} = 1$$

$$\rightarrow (\Lambda^{0}_{0})^2 = 1 \, , \; \sum_{K=1}^{3} (\Lambda^{K}_{0})^2 \geq 1.$$

$$\rightarrow \Lambda^{0}_{0} \geq 1 \quad \text{or} \quad \Lambda_{0} \leq -1.$$

the fourth parts of L are

- $L^{\uparrow}_{+} = \det \Lambda = +1$, $\quad \text{sign } \Lambda^{0}_{0} = +1.$

  This part contains to the identity matrix I and it is a group. It is called eigen lorentz group.

- $L^{\uparrow}_{-}$: $\det \Lambda = -1$, $\quad \text{sgn } \Lambda^{0}_{0} = +1$

  Contains to the element

$$I_s X = (X^{0}_{i} - X^{1}, -X^{2}, -X^{3})$$

- $L^{\downarrow}_{-}$: $\det \Lambda = -1$, $\quad \text{sign } \Lambda^{0}_{0} = -1$

$$I_t X = (-X^{0}, X^{1}, X^{2}, X^{3})$$

- $L^{\downarrow}_{+}$: $\det \Lambda = +1$, $\quad \text{sgn } \Lambda^{0}_{0} = -1$

$$I_s I_t$$

$$L^{\uparrow}_{+} \simeq SO(3,1) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$$

The group $SL(2,\mathbb{C})$

Consist in the complex matrices 2x2.

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{ik} \in \mathbb{C}, \quad i,k = 1,2.$$

This group is conected $[\pi : SL(2,\mathbb{C}) \longrightarrow SU(2)]$. We may parametrize it by

$$a = a_0 e + \sum_{k=1}^{3} a_k \sigma_k, \qquad a_0, a_k \in \mathbb{C}, \quad e \text{ is the identity matrix } I_{2\times 2}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli's matrices

any point of the Minkowski space may be represent as a matrix.

$$x \longmapsto x = x^0 e + \sum_{k=1}^{3} x^k \sigma_k$$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = x \longmapsto \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} = a$$

$$\|x\|^2 = g_{\mu\nu} x^\mu x^\nu \qquad \det(a) = {x^0}^2 - {x^3}^2 - {x^1}^2 - {x^2}^2$$

Using $a \in SL(2,\mathbb{C})$, we may define a linear map of $SL(2,\sigma)$ in itself.

$$x' = a x a^\dagger$$

Preserves the inner product.

$$(x')^2 = \det(x') = \det(a x a^\dagger) = \det(x) = (x)^2$$

Let's write this correspondence as.

$$x' = a x a^\dagger \longrightarrow x' = \Lambda(a) x$$

$$\Lambda^0_0 = |a_0|^2 + \sum_{k=1}^{3} |a_k|^2$$

$$\Lambda^k_0 = a_0 \overline{a_k} + \overline{a_0} a_k + i \varepsilon^{klm} a_l \overline{a_m}$$

$$\Lambda^l_k = \delta^l_k \left(|a_0|^2 - \sum_{t=1}^{3} |a_t|^2 \right) a_k \overline{a_l} + \overline{a_k} a_l + i \varepsilon^{klm} (\overline{a_0} a_m - a_0 \overline{a_m})$$

$$\varepsilon^{klm} \begin{cases} 1 , & klm \text{ is permutation} \\ 0 & \text{is odd.} \\ -1 & \text{there are representations.} \end{cases}$$

moreover.

$$\Lambda^\nu_\nu (a) = Tr(\Lambda(a)) = Tr(a)^2 = 4|a_0|^2$$

Since $SL(2,\mathbb{C})$ is connected, the map on L gives that the image of $SL(2,\mathbb{C})$ is $L^\uparrow_+$.

We can invert

$$a_0 e + \sum_{k=1}^{3} a_k \sigma_k = D^{-1} \left\{ \Lambda^\nu_\nu e + \sum_{k=1}^{3} (\Lambda^k_0 + \Lambda^0_k + i \varepsilon^{0k\varphi\tau} \Lambda^\tau_\varphi) \sigma_k \right\}$$

$$D^2 = 4 - Tr(\Lambda\Lambda) + (Tr(N^2) + i \varepsilon^{\mu\nu\rho\tau} \Lambda^\rho_\mu \Lambda^\tau_l$$

In other words

$$SL(2,\mathbb{C}) \ni \pm e \longrightarrow I_{4\times4} \in L^\uparrow_+$$

$$L^\uparrow_+ \simeq SL(2,\mathbb{C}) / \mathbb{Z}_2$$

Subgroups of $SL(2,\mathbb{C})$

Given the equivalence between $L^\uparrow_+$ and $SL(2,\mathbb{C})$ is easier to study $SL(2,\mathbb{C}) \times T_4$

$SU(2) \subseteq SL(2,\mathbb{C})$: which is formed by

$$u^\dagger = u^{-1}$$

and its convenient to parametrize $SU(2)$ by

$$u = u_0 e + i \sum_{k=1}^{3} u_k \sigma_k , \quad u_0, u_k \in \mathbb{R}$$

$SU(1,1)$, $v \in SL(2,\mathbb{C})$: formed by

$$v^{\dagger}\sigma_3 = \sigma v^{-1}$$

its parametrized by

$$v = v_0 e + v_1\sigma_1 + v_2\sigma_2 + i v_3\sigma_3$$

with

$$v_0^2 - v_1^2 - v_2^2 + v_3^2 = 1 \ , \quad v_0, v_k \in \mathbb{R}$$

$SL(2,\mathbb{R}) \subseteq SL(2,\mathbb{C})$: formed by

$$a^{\dagger}\sigma_2 = \sigma_2 a^{-1}$$

and its parametrized by

$$a = a_0 e + a_1\sigma_1 + i a_2\sigma_2 + a_3\sigma_3,$$

$$a_0, a_k \in \mathbb{R}.$$