Spinors and Twistors.

Orthogonal groups: Let V be a real vectorial space of 4-dimensions with a metric tensor g (g is non-degenerated and symmetrical). Always it is possible find an orthogonal base of V, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ such that $g(\vec{e}_a, \vec{e}_b) = 1$ or -1. In other words the 4x4 matrix g_{ab}; $a,b=1,2,3,4$. that represents the metric tensor g with respect to the base $\{\vec{e}_1', \vec{e}_2', \vec{e}_3', \vec{e}_4'\}$, defined by $g_{ab} = g(\vec{e}_a', \vec{e}_b)$ is diagonal with 1 or -1 along to its diagonal.

The numbers p, q do not depend of the orthogonal base chosen. The pair (p, q) define de signature of g.

Then (g_{ab}) must be $dig(1,1,1,1)$, $dig(1,1,1,-1)$, $dig(1,1,-1,-1)$, $dig(1,-1,-1,-1)$, $dig(-1,-1,-1,-1)$.

$$(g_{ab}) = \begin{cases} dig(1,1,1,1) & \text{Euclidean.} \\ dig(1,1,-1,-1) & \text{Lorentzian . or hiperbolic} \\ dig(1,1,-1,-1) & \text{Kleinean or oltrahiperbolic.} \end{cases}$$

Given a signature, there exist a infinite number of basis with respect to (g_{ab}) takes a diagonal form. If $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ and $\{\vec{e}_1', \vec{e}_2', \vec{e}_3', \vec{e}_4'\}$ two orthonormal bases of V.

$$g(\vec{e}_a', \vec{e}_b') = g(\vec{e}_a, \vec{e}_b)$$

A 4x4 real matrix $L^a{}_b$, such that

$$\vec{e}_a = L^b{}_a \vec{e}_b'$$

Substituying in the metric tensor

$$g_{ab} = L^c{}_a L^d{}_b g_{cd} \qquad *$$

Implies that $\det(L^a{}_b) = 1$ or -1, thus L is an invertible matrix.

The matrices $(L^a{}_b)$ that satisfy this relation, they are called orthogonals, and they form a group under the operation of product of matrices denoted by $O(p,q)$.

Those who $\det(L^a{}_b)=1$, $SO(p,q)\leq O(p,q)$. If $q=0$

$$SO(4,0)=SO(4)\leq O(4,0)=O(4)$$

If (g^{ab}) is the inverse matrix of (g_{ab}) i.e.,

$$g_{ab}\,g^{cd}=\delta^c_a$$

Then,

$$\delta^b_e=g^{eb}g_{ab}=g^{ea}L^c{}_a L^d{}_b\, g_{cd}=g^{ea}L^c{}_a\, g_{cd}\, L^d{}_b.$$

Which implies that the inverse matrix of $(L^a{}_b)$ with inputs $(L^{-1})^a{}_b$ its given by.

$$(L^{-1})^e{}_d=g^{ea}L^c{}_a\, g_{cd}.$$

The inverse matrix also belongs to $O(p,q)$ and therefore satisfy (*)

$$g_{ad}=L_a{}^c L_b{}^d g_{cd}.$$

where it's been used the role for up and down index.

$$t^a=g^{ab}t_b,\quad t_a=g_{ab}t^b,\quad (L^{-1})^e{}_d=L_d{}^e.$$

Let's consider the space $\mathbb{R}^{2\times 2}$, i.e., $\mathbb{R}^4$, with the ultrahiperbolic metric.

$$(g_{ab})=\mathrm{dig}(1,1,-1,-1).$$

The map.

$$(x,y,z,w)\longmapsto \frac{1}{\sqrt{2}}\begin{pmatrix} -x-y & -y-w \\ w-y & x-z \end{pmatrix}$$

its one to one between $\mathbb{R}^{2,2}$ and the matrices 2x2.

Denoting by P the matrix of the right side.

$$\det(P)=-\frac{1}{2}(x^2+y^2-z^2-w^2)$$

If $K,M\in \mathrm{Mat}(2,\mathbb{R})$ or $\mathrm{Mat}(2,\mathbb{C})$, both reals or pure imaginaries, the transformation

$$P\longrightarrow P'=KPM.$$

Is linear and equivalent to a transformation of $\mathbb{R}^{2,2}$ in Himself.

$$\det(P') = \det(KPM) = \det(K)\det(P)\det(M)$$

If $\det(K)\det(M) = 1$, then $\det(P) = \det(P')$ i.e., denoted by (x', y', z', w') the correspondent vector to P', the norm of (x, y, z, w) is equal to (x', y', z', w').

This imply that the map $P \longmapsto P'$ is orthogonal in $\mathbb{R}^{2,2}$, is element of $O(2,2)$.

Assuming that $\det(K)\det(M) = 1$, if we take.

$$\tilde{K} = \frac{K}{(\det(K))^{1/2}} \quad , \quad \tilde{M} = M(\det(M))^{1/2}.$$

$$P' = KPM = \tilde{K}P\tilde{M} \qquad \text{and} \qquad \det(\tilde{K}) = \det(\tilde{M}) = 1.$$

Thus we can assume $\det(K) = \det(M) = 1$.

Denoting the inputs of P and $P^i{}_j$ where the superindex by the row and the subindex by the column, then

$$P'^i{}_j = K^i{}_k P^k{}_l M^l{}_j$$
$$= K^i{}_k M^l{}_j P^k{}_l$$

The inputs of the K matrix, will be denoted by

$$A, B = 1, 2 \qquad K = (K^A{}_B)$$

For M, by

$$\dot{A}, \dot{B} = 1, 2 \qquad M = (M^{\dot{A}}{}_{\dot{B}})$$

For P, by

$$P = (P^A{}_{\dot{B}})$$

$$P'^A{}_{\dot{B}} = K^A{}_C P^C{}_{\dot{D}} M^{\dot{D}}{}_{\dot{B}} = K^A{}_C M^{\dot{D}}{}_{\dot{B}} P^C{}_{\dot{D}}$$

Spinorial equivalent of a tensor.

Consider a base $\{\vec{E}_1, \vec{E}_2, \vec{E}_3, \vec{E}_4\}$ with respect to the metric tensor is represented by the matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This base is called null tetrad, $g(\vec{E}_a, \vec{E}_a) = 0$.

The vectors $\vec{E}_a$, belong to the complexification of V.

If g is euclidean and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ an orthonormal base of V:

$$\vec{E}_1 = \frac{1}{\sqrt{2}}(\vec{e}_1 + i\vec{e}_2) \qquad \vec{E}_2 = \frac{1}{\sqrt{2}}(\vec{e}_1 - i\vec{e}_2)$$

$$\vec{E}_3 = \frac{1}{\sqrt{2}}(\vec{e}_3 - i\vec{e}_4) \qquad \vec{E}_4 = \frac{1}{\sqrt{2}}(\vec{e}_3 + i\vec{e}_4)$$

If $(g_{ab}) = \mathrm{dig}(1,1,1,-1)$

$$\vec{E}_1 = \frac{1}{\sqrt{2}}(\vec{e}_1 + i\vec{e}_2) \qquad \vec{E}_2 = \frac{1}{\sqrt{2}}(\vec{e}_1 - i\vec{e}_2)$$

$$\vec{E}_3 = \frac{1}{\sqrt{2}}(\vec{e}_3 + i\vec{e}_4) \qquad \vec{E}_4 = \frac{1}{\sqrt{2}}(\vec{e}_3 - \vec{e}_4)$$

If $(g_{ab}) = \mathrm{dig}(1,1,-1,-1)$

$$\vec{E}_1 = \frac{1}{\sqrt{2}}(\vec{e}_1 - \vec{e}_3) \qquad \vec{E}_2 = \frac{1}{\sqrt{2}}(\vec{e}_1 + \vec{e}_3)$$

$$\vec{E}_3 = \frac{1}{\sqrt{2}}(-\vec{e}_2 + \vec{e}_4) \qquad \vec{E}_4 = \frac{1}{\sqrt{2}}(-\vec{e}_2 - \vec{e}_4)$$

Instead of use index $a = 1,2,3,4$ for tag to

$$(e_{A\dot{B}}) = \begin{pmatrix} \vec{E}_4 & \vec{E}_2 \\ \vec{E}_1 & -\vec{E}_3 \end{pmatrix} \qquad \begin{array}{l} A, B = 1,2 \\ \dot{A}, \dot{B} = 1,2. \end{array}$$

So the null metric is equivalent to $g(\vec{e}_{A\dot{B}}, \vec{e}_{C\dot{D}}) = -\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}$, with

$$(\varepsilon_{AB}) = (\varepsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\varepsilon_{\dot{A}\dot{B}}) = (\varepsilon^{\dot{A}\dot{B}})$$

As $\vec{e}_{A\dot{B}}$ belongs to V, there is scalars $\sigma^a{}_{A\dot{B}}$, such that.

$$\vec{e}_{A\dot{B}} = \frac{1}{\sqrt{2}}\sigma^a{}_{A\dot{B}}\,\vec{e}_a$$