Canonical Quantization I

What is quantization?

- Duality → wave / particle
- Uncertainty principle
- Pauli exclusion principle

Problem ($e-\gamma$):

Quantum Mechanics:

e^- – quantized

$\psi(x)$: Wave function.

Electromagnetic:

Classic field (exterior)

$A_\mu(x)$.

Then we may say that:

$$\mathcal{L}_{QED}(\psi(x), A_\mu(x)) \to \text{Semiclassical approximation}$$

Main goal ⟶ Complete the quantization.

"Canonical" quantization:

Classical Mechanics: state: $(q, \dot{q}) \longrightarrow (q, p)$

$$S = \int dt\, L(q,\dot{q}) \longrightarrow H(q,p) \longrightarrow \dot{A}(p,q;t) = \{A, H\} + \frac{\partial A}{\partial t}$$

Quantum Mechanics: state: $|\psi\rangle \in \mathcal{H}$

$q \longrightarrow \hat{q} = q$

$p \longrightarrow \hat{p} = -i\frac{\partial}{\partial q}$

$\psi(q) \equiv \langle q | \psi \rangle$

$\to P(q) = |\psi(q)|^2$ Probability Density

$A \longrightarrow \hat{A}$ $\hbar = 1$

$\{A,B\} \longrightarrow \frac{[\hat{A},\hat{B}]}{i} \longrightarrow$ Uncertainty

Spin? $\longrightarrow$ Relativistic Quantum Mechanics: ψ – Dirac.

Pauli? $\rightarrow$ by hand. $\gamma\gamma \rightarrow e^+e^-$?

Classical field theory:

$q \longrightarrow \phi(\vec{x},t)$

$L(q,\dot{q}) \longrightarrow \mathcal{L}(\phi,\dot{\phi})$

$S = \int dt\, L \longrightarrow S[\phi] = \int d^4x\, \mathcal{L}$

$\frac{\delta F}{\delta \phi}$ =: functional derivative

Hamiltonian formalism

$\pi(\vec{x},t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$

$\mathcal{H}(\phi,\pi) = \pi\dot{\phi} - \mathcal{L}$.

$\{F,G\} = \int d^3x \left(\frac{\delta F}{\delta \phi}\frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi}\frac{\delta G}{\delta \phi} \right)$

Basic brackets:

$$\{\phi(\vec{x},t),\phi(\vec{x}',t)\} = \{\pi(\vec{x},t),\pi(\vec{x}',t)\} = 0.$$

$$\{\phi(\vec{x},t),\pi(\vec{x}',t)\} = \delta^{(3)}(\vec{x}-\vec{x}')$$

Canonical quantization of fields:

$\phi(\vec{x},t) \longrightarrow \hat{\phi}(\vec{x},t)$ over an abstract space

$\pi(\vec{x},t) \longrightarrow \hat{\pi}(\vec{x},t)$ (to specify).

$\{\,,\,\} \longrightarrow [\,,\,]/i$

$$[\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{x}',t)] = i\delta^{(3)}(\vec{x}-\vec{x}')$$

$$[\hat{\phi}(\vec{x},t),\hat{\phi}(\vec{x}',t)] = 0 = [\hat{\pi}(\vec{x},t),\hat{\pi}(\vec{x}',t)]$$

Scalar field

Classical Theory:

$$(\Box + m^2)\phi(x) = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2$$

Solutions:

$$e^{\pm ikx} \; ; \; k^2 = m^2 \; ; \; k^\mu = (\omega_k, \vec{k}) \to \omega_k \equiv \sqrt{\vec{k}^2 + m^2}$$

Solution for a fixed ω_k

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[a_k e^{-ikx} + a_k^* e^{ikx}]$$

$$\phi^* = \phi$$

$$\hat{\pi}(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial(\dot{\phi}(\vec{x}, t))} = \dot{\phi}(\vec{x}, t) = \int \widetilde{d^3k}\,(-i\omega_k)(a_k e^{-ikx} - a_k^* e^{ikx})$$

$$\widetilde{d^3k} = \frac{d^3k}{(2\pi)^3 2\omega_k}$$

$$\widetilde{d^3k} = \frac{d^4k}{(2\pi)^4} 2\pi\,\delta(k^2 - m^2)\,\Theta(\omega_k)$$

Lorentz invariant

$$H = \int d^3x \, \frac{1}{2}(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2)$$

$$\frac{dF}{dt} = \{F, H\}$$

Canonical conjugacy:

$$\phi \to \hat{\phi} \; ; \; \pi \to \hat{\pi} \; ; \; H \to \hat{H}$$

$$\dot{F} = -i[F, H] : \quad \dot{\hat{\phi}}(\vec{x}, t) = -i[\hat{\phi}(\vec{x}, t), H] = \hat{\pi}(\vec{x}, t)$$

$$\dot{\hat{\pi}}(\vec{x}, t) = -i[\hat{\pi}(\vec{x}, t), H] = (\vec{\nabla}^2 - m^2)\hat{\phi}(\vec{x}, t).$$

Then, $\quad (\Box + m^2)\hat{\phi}(\vec{x}, t) = 0 : \quad a_k \to \hat{a}_k$

$$a_k^* \to \hat{a}_k^\dagger$$

Therefore

$$\phi(\vec{x},t) = \int \widetilde{d^3k}\,[\hat{a}_k e^{-ikx} + \hat{a}_k^\dagger e^{ikx}]$$

Inverting

$$\hat{a}_k = \int d^3x\, e^{ikx}\, \omega_k \underline{\left(\hat{\phi}(\vec{x},t) + \frac{i}{\omega_k}\hat{\pi}(\vec{x},t)\right)}$$

similarly for $\hat{a}_k^\dagger$

$\sim(\vec{x} + i\,\hat{P}/m\omega) \sim a$ oscillator.

Is easy to verify:

$$[\hat{a}_k, \hat{a}_{k'}] = 0 \quad [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger]$$

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = 2\omega_k (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \rightarrow \text{Lorentz invariant}$$

Define $\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k$

$$[\hat{N}, \hat{a}_k^\dagger] = \hat{a}_k [\hat{a}_k, \hat{a}_k^\dagger]$$

$$[\hat{N}, \hat{a}_k] = -[\hat{a}_k, \hat{a}_k^\dagger]\hat{a}_k$$

$\longrightarrow$ in the box:

$$2\pi\delta(\vec{k} - \vec{k}') \rightarrow \delta_{kk'}$$

normalize $\hat{N} \rightarrow 2\omega_k \hat{N}$

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$$

$$[\hat{N}, \hat{a}] = -\hat{a}$$

$\hat{N}_k \rightarrow$ number operator

$$\Downarrow$$

$$\hat{N}\hat{a}|n\rangle = (\hat{a}\hat{N} - \hat{a})|n\rangle$$

$$= |n-1\rangle \hat{a}|n\rangle$$

$$\Downarrow$$

$$\hat{a}|n\rangle \propto |n-1\rangle$$

$$\hat{a}^\dagger|n\rangle \propto |n+1\rangle$$

$\{|n_{k_1}, n_{k_2}, \ldots\rangle\}$: Fock space

$\hat{a}_k, \hat{a}_k^\dagger \rightarrow$ ladder operators over $|n_k\rangle$

Vacuum: $\hat{a}_k|0\rangle = 0$.

In this terms

$$\hat{H} = \frac{1}{2}\int d^3x\left[\hat{\dot{\phi}}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right]$$

then,

$$\hat{H} = \frac{1}{2}\int \widetilde{d^3k}\, \omega_k\left[\hat{a}_k^{\dagger}\hat{a}_k + \hat{a}_k\hat{a}_k^{\dagger}\right]$$

but,

$$\hat{H} = \int \widetilde{d^3k}\, \omega_k\Big(\underbrace{\hat{a}_k^{\dagger}\hat{a}_k}_{\hat{N}_k} + \frac{1}{2}\underbrace{[\hat{a}_k, \hat{a}_k^{\dagger}]}_{\propto\, \delta^{(3)}(0)\,!!}\Big) \longrightarrow \infty$$

Point zero of energy

V.S. Quantum Mechanics

$$H = \hbar\omega\left(n + \frac{1}{2}\right)$$

Redefine:

$$:\hat{H}: = \hat{H} - \langle 0|H|0\rangle$$

Normal order (Wick): annihilation operator by right-hand only.

$$:\hat{H}: = \int \widetilde{d^3k}\, \omega_k :\hat{a}_k^{\dagger}\hat{a}_k: \qquad :p: = \int \widetilde{d^3k}\, \vec{k}\, :\hat{a}_k^{\dagger}\hat{a}_k:$$

then

$$[\hat{H}, \hat{a}_p^{\dagger}] = \omega_p\hat{a}_p^{\dagger}\,; \quad [\hat{H}, \hat{a}_p] = -\omega_p\hat{a}_p.$$

Charged Scalar field (Complex):

$$\phi^* \neq \phi \longrightarrow \mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \longrightarrow (\Box + m^2)\varphi^{(*)} = 0$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*\;;\quad \pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi}$$

$$H = \int d^3x\left(\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi\right)$$

Cuantization:

$$\phi \to \hat{\phi} \qquad [\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t)] = i\delta^3(x-\vec{x}') = [\hat{\phi}^\dagger(\vec{x},t), \hat{\pi}^\dagger(\vec{x}',t)]$$

$$\phi^* \to \hat{\phi}^\dagger$$

other null

$$\phi(x) = \int \widetilde{d^3k}[a_k e^{-ikx} + b_k^* e^{ikx}]$$

$$a_k \to \hat{a}_k$$

$$b_k^* \to \hat{b}_k^\dagger$$

$$\hat{\phi}(x) = \int \widetilde{d^3k}[\hat{a}_k e^{-ikx} + \hat{b}_k^\dagger e^{ikx}]$$

then,

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}') = [\hat{b}_k, \hat{b}_{k'}^\dagger]$$

other null

Identify:

$$\hat{a}^\dagger \longrightarrow \text{particles} \qquad q=1$$

$$\hat{b}^\dagger \longrightarrow \text{anti-particles} \qquad q=-1 \qquad \omega_k > 0.$$

$$\hat{N}_{ak} = \hat{a}_k^\dagger \hat{a}_k \,, \qquad \hat{N}_{bk} = \hat{b}_k^\dagger \hat{b}_k$$

$$\{ |n_a(\vec{k}), n_b(\vec{k})\rangle \} \;;\; \hat{a}_k|0\rangle = \hat{b}_k|0\rangle = 0$$

$$:H: = \int \widetilde{d^3k}\, \omega_k (\hat{N}_{ak} + \hat{N}_{bk}) \;; \qquad :P: = \int \widetilde{d^3k}\, \vec{k}(\hat{N}_{ak} + \hat{N}_{bk})$$

Charge operator: from $\phi \to e^{i\alpha}\phi$: $j_\mu = i\phi^* \overleftrightarrow{\partial_\mu} \phi$

$$\longrightarrow Q = i\int d^3x\, \phi^* \overleftrightarrow{\partial_0} \phi = -i\int d^3x (\pi\phi - \phi^*\pi^*)$$

therefore

$$:Q: = \int \widetilde{d^3k} (\hat{N}_{ak} - \hat{N}_{bk})$$