Spinorial Tensors

$$t_{A\dot{A}B\dot{B}} = \tau_{AB}\,\varepsilon_{\dot{A}\dot{B}} + \tau_{\dot{A}\dot{B}}\,\varepsilon_{AB}$$

If we substitute in

$$t_{ab} = \left(-\frac{1}{\sqrt{2}}\sigma_a{}^{A\dot{A}}\right)\left(-\frac{1}{\sqrt{2}}\sigma_b{}^{B\dot{B}}\right)t_{A\dot{A}B\dot{B}}$$

$$= \frac{1}{2}\sigma_a{}^{A\dot{A}}\sigma_b{}^{B\dot{B}}\left(\tau_{AB}\varepsilon_{\dot{A}\dot{B}} + \tau_{\dot{A}\dot{B}}\varepsilon_{AB}\right)$$

$$= \frac{1}{2}\tau_{AB}\,\sigma_a{}^{(A|\dot{R}|}\,\sigma_b{}^{\dot{B})}{}_{\dot{R}} + \frac{1}{2}\tau_{\dot{A}\dot{B}}\,\sigma_a{}^{R(\dot{A}}\,\sigma_{bR}{}^{\dot{B})}$$

$$= \frac{1}{2}\tau_{AB}\,S_{ab}{}^{AB} + \frac{1}{2}\tau_{\dot{A}\dot{B}}\,S_{ab}{}^{\dot{A}\dot{B}}$$

$$S_{ab}{}^{AB} \equiv \sigma_a{}^{(A|\dot{R}|}\,\sigma_b{}^{\dot{B})}{}_{\dot{R}}\ , \qquad S_{ab}{}^{\dot{A}\dot{B}} \equiv \sigma_a{}^{R(\dot{A}}\,\sigma_{bR}{}^{\dot{B})}$$

$$\frac{\partial}{\partial x^a} = \partial_a \longmapsto \partial_{A\dot{B}} = \frac{1}{\sqrt{2}}\sigma^a{}_{A\dot{B}}\,\partial_a.$$

Maxwell's equations

Homogeneous: $\nabla\cdot\vec{B} = 0, \quad \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0, \quad \vec{B}\in\mathbb{R}^3, \ \vec{E}:\mathbb{R}^4 \longrightarrow \mathbb{R}^3.$

Inhomogeneous: $\nabla\cdot\vec{E} = \rho, \quad \nabla\times\vec{B} + \frac{\partial\vec{E}}{\partial t} = \vec{J}$

Tensorial form:

$$F^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & -E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

$$\partial_b F^{ab} = J^a, \quad \partial_b {}^{*}F^{ab} = 0.$$

Homework: Prove that from the tensorial form we can get the normal form of maxwell equations.

The spinorial equivalent of F_{ab}

$$F_{A\dot{A}B\dot{B}} = f_{AB}\varepsilon_{\dot{A}\dot{B}} + f_{\dot{A}\dot{B}}\varepsilon_{AB}$$

with $f_{AB} = \frac{1}{2}F_{A}{}^{\dot{R}}{}_{B\dot{R}}$, $f_{\dot{A}\dot{B}} = \frac{1}{2}F^{R}{}_{\dot{A}R\dot{B}}$ symmetrics.

As F_{ab} is real, $\overline{F_{A\dot{B}C\dot{D}}} = F_{B\dot{A}D\dot{C}}$, then

$$f_{\dot{A}\dot{B}} = \overline{f_{AB}}$$

Homework: Prove that efectively $f_{\dot{A}\dot{B}} = \overline{f_{AB}}$.

So the spinorial equivalent to the maxwell equations are

$$-\partial_{B\dot{B}}(f^{AB}\varepsilon^{\dot{A}\dot{B}} + f^{\dot{A}\dot{B}}\varepsilon^{AB}) = J^{A\dot{A}} \quad \text{and} \quad -i\partial_{B\dot{B}}(f^{AB}\varepsilon^{\dot{A}\dot{B}} + f^{\dot{A}\dot{B}}\varepsilon^{AB} = 0$$

Combining them

$$\partial_{B}{}^{\dot{A}} f^{AB} = 2J^{A\dot{A}}, \quad \partial^{A}{}_{\dot{B}} f^{\dot{A}\dot{B}} = 2J^{A\dot{A}}$$

Due to the signature, the second equation is the complex conjugate of the first.

Using the spinorial tensors

$$f^{A}{}_{B} = \frac{1}{4}F^{ab}S_{ab}{}^{A}{}_{B}.$$

$$= \frac{1}{2}\left[(-F^{14} + iF^{23})\sigma_1 + (F^{24} - iF^{31})\sigma_2 + (-F^{34} + iF^{12})\sigma_3\right]$$

$$\Longrightarrow \quad f_{AB} = \frac{1}{2}\begin{pmatrix} -F^{14} + iF^{23} + i(F^{24} - iF^{31}) & F^{34} - iF^{12} \\ F^{34} - iF^{12} & F^{14} - iF^{23} + i(F^{24} - iF^{31}) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} E_x + iB_x - i(E_y + iB_y) & -(E_z + iB_z) \\ -(E_z + iB_z) & -(E_x + iB_x) - i(E_y + iB_y) \end{pmatrix}$$

from $\partial_b {}^{*}F^{ab} = 0$, then $\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$, implies the existence of a vector field A_a, such that

$$F_{ab} = \partial_a A_b - \partial_b A_a$$

If $A_{B\dot{B}}$ is the spinorial equivalent to A_a

$$f_{BC} = \partial_{(B}{}^{\dot{B}} A_{C)\dot{B}}, \quad f_{\dot{B}\dot{C}} = \partial^{B}{}_{(\dot{B}} A_{|B|\dot{C})}$$

Analogously, from $\partial_b {}^*F^{ab}=0$, exist a vector field $\check{A}_a$.

$$ {}^*F_{ab} = \partial_a \check{A}_b - \partial_b \check{A}_a $$

or

$$ f_{BC} = i\,\partial_{(B}{}^{\dot{B}} \check{A}_{C)\dot{B}}, \quad f_{\dot{B}\dot{C}} = i\,\partial^{B}{}_{(\dot{B}} \check{A}_{|B|\dot{C})} $$

the complex vector field.

$$ \phi_b = \frac{1}{2}(A_a - i\check{A}_a) $$

It follows that

$$ \partial_{(B}{}^{\dot{B}} \phi_{C)\dot{B}} = \frac{1}{2}\left(\partial_{(B}{}^{\dot{B}} A_{C)\dot{B}} - i\,\partial_{(B}{}^{\dot{B}} \check{A}_{C)\dot{B}}\right) = 0. $$

i.e.,

$$ \partial_{(B}{}^{\dot{B}} \phi_{C)\dot{B}} = 0. $$

Maxwell equation in spinorial form.