Operators in 2nd quantization

$$\hat{S}_+ \hat{T}_{tot} |\nu_{n_1}\rangle_1 |\nu_{n_2}\rangle_2 \dots |\nu_{n_N}\rangle_N$$

$$= \hat{S}_+ \sum_{j=1}^{N} \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} \delta_{\nu_a, \nu_{n_j}} |\nu_{n_1}\rangle_1 \cdots |\nu_b\rangle_j \cdots |\nu_{n_N}\rangle_N$$

$$= \sum_{j=1}^{N} \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} \delta_{\nu_a, \nu_{n_j}} \hat{S}_+ |\nu_{n_1}\rangle_1 \cdots |\nu_b\rangle_j \cdots |\nu_{n_N}\rangle_N$$

$$= \hat{T}_{tot} \hat{S}_+ |\nu_{n_1}\rangle_1 |\nu_{n_2}\rangle_2 \cdots |\nu_{n_N}\rangle_N$$

$$= \sum_{j=1}^{N} \sum_{\nu_a \nu_b} T_{\nu_b \nu_a} \delta_{\nu_a, \nu_{n_j}} \hat{b}^{+}_{\nu_{n_1}} \cdots \hat{b}^{+}_{\nu_b} \cdots \hat{b}^{+}_{\nu_{n_N}} |0\rangle$$

$$= \hat{T}_{tot} \hat{b}^{+}_{\nu_{n_1}} \cdots \hat{b}^{+}_{\nu_{n_N}} |0\rangle = \sum_{\nu_a, \nu_b} T_{\nu_b \nu_a} \sum_{j=1}^{N} \delta_{\nu_a, \nu_{n_j}} \hat{b}^{+}_{\nu_{n_1}} \cdots \hat{b}^{+}_{\nu_b} \cdots \hat{b}^{+}_{\nu_{n_N}} |0\rangle$$

n_j place

If $\nu := \nu_{n_j}$ appears n times, then

$$(\hat{b}^{+}_{\nu})^{p} |0\rangle \qquad \text{left hand side}$$

but,

$$\hat{b}^{+}_{\nu_b} (\hat{b}^{+}_{\nu})^{p-1} |0\rangle \qquad \text{right hand side}$$

therefore

$$\rightarrow (\hat{b}^{+}_{\nu})^{p-1} |0\rangle = \sqrt{(p-1)!}\, |p-1\rangle$$

$$\rightarrow \hat{b}^{+}_{\nu} \hat{b}^{+\,p-1} |0\rangle = \sqrt{p}\sqrt{(p-1)!}\, |p\rangle$$

$$\rightarrow \hat{b}_{\nu} \hat{b}^{+}_{\nu} \hat{b}^{+\,p-1} |0\rangle = p\sqrt{(p-1)!}\, |p-1\rangle$$

$$\rightarrow \frac{\hat{b}_{\nu} \hat{b}^{+}_{\nu}}{p} \hat{b}^{+\,p-1} |0\rangle = \sqrt{(p-1)!}\, |p-1\rangle = \hat{b}^{+\,p-1}_{\nu} |0\rangle$$

Finally,

$$\hat{b}^{+}_{\nu_b} (\hat{b}^{+}_{\nu})^{p-1} |0\rangle = \hat{b}^{+}_{\nu_b} \left(\frac{1}{p} \hat{b}_{\nu} \hat{b}^{+}_{\nu} \right) \hat{b}^{+\,p-1}_{\nu} |0\rangle$$

$$= \left(\frac{1}{p} \hat{b}^{+}_{\nu_b} \hat{b}_{\nu} \right) (\hat{b}^{+}_{\nu})^{p} |0\rangle$$

Then

$$\sum_{j=1}^{N} \delta_{\nu_a, \nu_{n_j}} \frac{1}{p} \hat{b}^{+}_{\nu_b} \hat{b}_{\nu_{n_j}} \hat{b}^{+\,p}_{\nu_{n_j}} |\cdots\rangle \longrightarrow \frac{p}{p} \hat{b}^{+}_{\nu_b} \hat{b}_{\nu_a} \hat{b}^{+\,p}_{\nu_b} |0\rangle.$$

Hence

$$\hat{T}_{tot}\left[\hat{b}^{+}_{\nu_{n_1}} \cdots \hat{b}^{+}_{\nu_{n_N}} |0\rangle\right] = \sum_{a,b} T_{\nu_a \nu_b} \hat{b}^{+}_{\nu_b} \hat{b}_{\nu_a} \left[\hat{b}^{+}_{\nu_{n_1}} \cdots \hat{b}_{\nu_{n_N}} |0\rangle\right]$$

$$\hat{T}_{tot} = \sum_{\nu_i, \nu_j} T_{\nu_i, \nu_j} \, \hat{a}^{+}_{\nu_i} \hat{a}_{\nu_j}$$

$$\hat{V}_{tot} = \frac{1}{2} \sum_{\substack{\nu_i, \nu_j \\ \nu_k, \nu_l}} V_{\nu_i \nu_j, \nu_k \nu_l} \, \hat{a}^{+}_{\nu_i} \hat{a}^{+}_{\nu_i} \hat{a}_{\nu_l} \hat{a}_{\nu_k}$$