Example: Bessel

$$J_n(x) = \int_0^1 \cos(n\pi t - x\sin(\pi t))\,dt.$$

$$= \mathrm{Re}\left\{ \int_0^1 e^{n\pi i t} e^{-ix\sin(\pi t)} dt \right\}$$

$$\varphi(t) = -\sin(\pi t), \quad f(t) = e^{n\pi i t}$$

In $[0,1]$, $\varphi(t)$ is stationary only at $a = \frac{1}{2}$

$$\varphi'(t) = -\pi\cos(\pi t) \qquad \varphi'(a) = \cancel{-\pi\cos\left(\frac{\pi}{2}\right)} \to 0$$

$$\varphi''(t) = \pi^2 \sin(\pi t)$$

$$\varphi''(a) = \pi^2 \sin\left(\frac{\pi}{2}\right) = \pi^2 > 0$$

$$\Rightarrow \quad P = 2$$

$$I(x) \sim \mathrm{Re}\left\{ e^{n\pi i/2} e^{i(-x+\pi/4)} \left(\frac{2!}{x\pi^2}\right)^{1/2} \frac{\Gamma(1/2)}{2} \right\}$$

$$= \mathrm{Re}\left\{ \frac{1}{\sqrt{2\pi x}} \exp\left[-i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \qquad \text{as} \quad x \to \infty$$

Example: Waves of Klen-Gordon

$$\varepsilon^2 - p^2 = m^2$$

$$u(x,t) = a e^{i(kx - \omega t)} \qquad \text{Harmonic wave}$$

$k :=$ Wave number

$\omega :=$ frequency

For disspersive media we consider $\omega = \omega(k)$

A continuous supper-possition of waves brings

$$u(x,t) = \int_{-\infty}^{\infty} a(k) e^{i(kx - \omega(k)t)}\, dk$$

we want to apply the principle of stationary phase for the case $t \to \infty$ and fixed ratio.

$$V := \frac{x}{t}$$

First consider the "group velocity".

In the discrete case:

$$u(x,t) = a_0(x,t) e^{i(\bar{k}x - \bar{\omega}t)}$$

where

$$a_0(x,t) = 2a_{01} e^{i(k_m x - \omega_m t)}$$

is the resulting amplitude.

$$V_g := -\left(\frac{\partial a_0(x,t)/\partial t}{\partial a_0(x,t)/\partial x} \right) = \frac{\omega_m}{k_m} \frac{1/2\,(\omega_2 - \omega_1)}{1/2\,(k_2 - k_1)} = \frac{\Delta\omega}{\Delta k}$$

In the appropriate limit

$$V_g = \frac{d\omega}{dk}$$

we want to consider stationary points such that $\omega'(k) = 0$.

Label one such point ξ.

$$\rightarrow u(x,t) \sim (a(\xi) e^{ix\xi}) e^{-i\omega(\xi)t + i(\pi/4)\operatorname{sgn}(\omega''(\xi))} \sqrt{\frac{\pi}{2t\,|\omega''(t)|}}$$

Consider now the Klein-Gordon equation

$$u_{tt} - \gamma^2 u_{xx} + c^2 u = 0 \quad , \quad t > 0, \quad x \in \mathbb{R}$$

such that $u(x,0)$ and $u_t(x,0)$ are specified.

$$u(k,t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} u(x,t)\, dx$$

Fourier transform in x.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(k,t)\, dk$$

$$U_{tt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{\partial^2 u(k,t)}{\partial t^2} \, dk$$

$$U_{xx} = \frac{-k^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(k,t) \, dk.$$

$$\Rightarrow \frac{\partial^2 u(k,t)}{\partial t^2} + (\gamma^2 k^2 + c^2) \, u(k,t) = 0$$

$$\Rightarrow u(k,t) = a_+(k) e^{i\sqrt{\gamma^2 k^2 + c^2}\, t} + a_- e^{-i\sqrt{\gamma^2 k^2 + c^2}\, t}$$

$a_\pm(k)$ fixed by initial condiotnts.

Define $\omega_\pm := \pm\sqrt{\gamma^2 k^2 + c^2}$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_+(k) e^{i(\omega_+(k)t - kx)} \, dk.$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_-(k) e^{i(\omega_-(k)t - kx)} \, dk.$$

Both integrals are of the form previously seen, thus we already know its asymptotic behaviour!

Steepest descent method.

$$I(\lambda) = \int_C f(z) e^{\lambda \varphi(z)} \, dz$$ Complex integral.

$f(z)$ and $\varphi(z)$ are analytic, λ large positive parameter, and c contour of integration.

Idea: Deform the contour of integration c into a new contour C^* such that.

i) C^* passes through one more zeros of $\varphi'(z)$

ii) The imaginary part of $\varphi(z)$ is constant on C^*

Write $z := x+iy$, $\varphi(z) = u(x,y) + iv(x,y)$.

First "recall", $\varphi(z)$ is analytic.

- Cauchy-Riemann:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

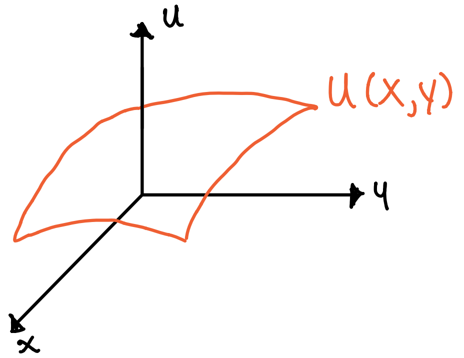
- $\nabla^2 u = 0 \longrightarrow u$ is harmonic.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$= -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

$$\Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$

Suppose we consider a 3D space coordinates (x,y,u), the $u = u(x,y)$ defines a surface. S



Suppose that $z_0 = x_0 + iy_0$ is a zero for $\varphi'(z)$, that is, $\varphi'(z_0) = 0$.

Following Cauchy-Riemann

$$\varphi(z) = u_x + iv_x = u_x - iu_y$$

at z_0

$$\varphi(z_0) = 0 = u_x - iu_y$$

$$\Longrightarrow u_x(x_0, y_0) = 0 = u_y(x_0, y_0)$$

That is, (x_0, y_0) is a critical point for the real function $u(x_0, y_0)$.

Theorem: Let $u(x,y)$ be an harmonic function on a bounded region Ω, then an isolated critical point of u in Ω cannot give a relative max or min of u

Proof: Let p be an isolated critical point and suppose p gives a local max for u. Thus $u(q) \leq u(p)$ $\forall q \in \Omega$ in a neighbourhood of p. Now, for $\varepsilon > 0$, we may consider the level surface

$$S = \{ q \in B \,/\, u(q) = u(p) - \varepsilon \}$$

where B is a closed ball about p such that $\nabla u(q) \neq 0$ and $u(q) \geq u(p) - \varepsilon$, $q \neq p$.

Thus the outer unit normal to S is given by

$$\hat{n} = -\frac{\vec{\nabla} u}{|\vec{\nabla} u|} \quad , \text{ at each point of } S$$

minus sign needed to denote the direction of greatest increase.

$$\Rightarrow \iint_S \nabla u \cdot n \, dS < 0 \text{, but the divergence theorem.}$$

$$\iint_S \nabla u \cdot n \, dS = \iiint_E \cancelto{0}{\nabla^2 u} \, dV = 0$$

E region bounded by S

$\Rightarrow$ p cannot denote a local max. Similarly, it cannot denote a local min.

$\therefore$ p must be a saddle point. ∎

In hour notation $p = (x_0, y_0)$.

In order to get the condition $V(x,y) = \text{constant}$ on C^*, we will consider a curve γ such that points in γ are parametrized by a parameter $s \in \mathbb{R}$ as

$$x = x(s) \;,\; y = y(s) \;,\; u = u(s) = u(x(s), y(s))$$

In particular, one may consider s as the arc-length of the curve γ.

$$\Rightarrow dx^2 + dy^2 = ds^2$$

$$\Rightarrow \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$$

Define a parameter θ by introducing.

$$\cos(\theta) = \frac{dx}{ds} \quad , \quad \sin(\theta) = \frac{dy}{ds}.$$

and thus

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dy}\frac{dy}{ds} = u_x \cos(\theta) + u_y \sin(\theta).$$

The steepness of the curve $(x(s), y(s), u(s))$ is measured with the u-axis by introducing an angle α.

$$\cos\alpha = \frac{du/ds}{\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{du}{ds}\right)^2}} = \frac{du/ds}{\sqrt{1 + \left(\frac{du}{ds}\right)^2}}.$$

Note: If we have tricky functions, we need to approach by all directions just to prove that this function is differentiable in one point.

we need to find two directions to prove this.