Different convergences in $\mathcal{L}(X)=\mathcal{L}(X,X)$

Convergence in Norm (Uniform convergence): A sequence of operators A_n, converges in the norm to an operator A, $A_n \longrightarrow A$ if $\|A_n - A\| \longrightarrow 0$ if $n \longrightarrow \infty$.

The space $\mathcal{L}(X)$ is complete, thus that $\{A_n\}_{n=1}^{\infty}$ is Cauchy's with respect to the norm, converges to a bounded operator

Strong Convergence: $A_n \longrightarrow A$ strongly if $\forall\ x \in X$, we have that $A_n x \longrightarrow Ax$.

We write $A_n \xrightarrow{s} A$, a sequence $\{A_n\}_{n=1}^{\infty}$ is Cauchy in the strong sense if $\forall x \in X$ the sequence $\{A_n x\}_{n=1}^{\infty}$ is Cauchy in X.

The strong convergence is weaker that the norm convergence, because if $A_n \longrightarrow A$ (in norm) so $\forall\ x \in X$, $A_n x \longrightarrow Ax$,

$$\|A_n x - Ax\| = \|(A_n - A)x\|$$

$$\leq \|A_n - A\| \|x\| \longrightarrow 0 \text{ if } n \longrightarrow \infty.$$

The contrary claim is not true, since let $L_2([0,1])$ defined for $\varepsilon_n > 0$

$$P_{\varepsilon_n} f = \begin{cases} f(t) & , \text{if } t < \varepsilon_n \\ 0 & , \text{if } t \geq \varepsilon_n \end{cases}$$

We can see that $P_{\varepsilon_n} f \longrightarrow 0$ if $\varepsilon_n \longrightarrow 0$ $\forall f \in L^2([0,1])$ i.e., P_{ε_n} converges in the strong sence to 0.

But $\|P_{\varepsilon_n}\| = 1$ not converges to zero.

Weak convergence: A sequence of operators A_n converges weakly to A, $A_n \xrightarrow{w} A$, if for all $x \in X$ and $f \in X^*$, $f(A_n x) \longrightarrow f(Ax)$.

Example: Let the displacement operator A in ℓ_2.

For this operator $f(A^n x) \longrightarrow 0$ for all

$$f = \sum_{i=1}^{\infty} b_i e_i \quad \text{and} \quad x = \sum_{i=1}^{\infty} a_i e_i$$

$$f(A^n x) = \sum_{i=n+1}^{\infty} b_i a_{i-n}$$

then
$$|f(A^n x)| \leq \left(\sum_{j=1}^{\infty}|a_j|^2\right)^{1/2}\left(\sum_{j=n+1}^{\infty}|b_j|^2\right)^{1/2} \longrightarrow 0$$

but $\|Ax\|=\|x\|$, $\forall x \in l_2$. Thus $\|A^n x\|=\|x\|$, then A^n not converges to zero strongly

Invertible operators

Let $A \in \mathcal{L}(X)$, we will call to $B:=A^{-n}$, the inverse operator of A if and only if $BA=Id$ and $AB=Id$.

Properties:

I. If A and B are invertible operators, then AB is invertible and satisfies
$$(AB)^{-1}=B^{-1}A^{-1}$$

Proof:
$$AB(B^{-1}A^{-1})=A(BB^{-1})A^{-1}=A(Id)A^{-1}=Id$$
$$(B^{-1}A^{-1})AB=B^{-1}(A^{-1}A)B=B^{-1}(Id)B=Id.$$

II. If $\|A\|=q<1$, $(I-A)$ is invertible and
$$(I-A)^{-1}=\sum_{k=0}^{\infty}A^k$$
more over
$$\|(I-A)^{-1}\| \leq \frac{1}{(1-\|A\|)}$$

Proof: As $\|A^k\| \leq \|A\|^k = q^k \longrightarrow 0$ if $k \to \infty$.

let $S_n=\sum_{k=0}^{\infty}A^k$ is Cauchy and converges.

Moreover $(I-A)S_n = I - A^{n+1} \longrightarrow I$ if $n \to \infty$.

Finally $\lim_{n\to\infty} S_n$ is the inverse of $(I-A)$.

III. Let A invertible and B such that
$$\|A-B\| < \frac{1}{\|A^{-1}\|}$$
then B is invertible.

Proof: As $$B = A(I - A^{-1}(A-B))$$

The operator A is invertible and $I - A^{-1}(A-B)$ is invertible by **(II)**.

As $\|A^{-1}(A-B)\| < 1$, B is invertible by **(I)**.

Theorem (The open mapping): Let X, Y Banach spaces and let $A: X \longrightarrow Y$ a bound linear operator, If it is one-to-one **(**i.e., $\operatorname{Ker} A = 0$**)** and onto **(**$\operatorname{Im} A = Y$**)** the inverse operator A^{-1} is bounded.