Projection operators

Let E be a linear space, a linear operator $P: E \longrightarrow E$ is a projection if and only if $P^2 = P$

Definition: $E_1 = \operatorname{Im} P$, $E_2 = \operatorname{Ker} P$.

Proposition: $P|_{E_1} = \operatorname{Id} E_1$ (i.e. $\forall x \in E_1$, $Px = x$)

Proof: $\forall x \in E_1$, $\exists y \in E$, $Py = x$. Thus, $P^2 y = Px$, then $x = Py = P^2 y = Px$, therefore $x = Px$.

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Proposition: The operator $Q: I - P$ is a projection, and

$$\operatorname{Im} P = \operatorname{Ker} Q \quad , \quad \operatorname{Ker} P = \operatorname{Im} Q .$$

Proof:

$$P(I-P) = 0 \longrightarrow \operatorname{Im}(I-P) = \operatorname{Ker} P,$$

moreover, if $x \in \operatorname{Ker} P$, $(I-P)x = x$, this means $\operatorname{Ker} P \subseteq \operatorname{Im}(I-P)$. The other way is similar.

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Proposition: Let $E_1 = \operatorname{Im} P$ and $E_2 = \operatorname{Ker} P$, then $E_1 + E_2 = E$ and $E_1 \cap E_2 = \Phi$

Proof:

$$PE + (I-P)E = E \quad \text{and} \quad \{Px = 0, (I-P)x = 0\}$$

then $x = 0$.

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Orthogonal projections

A projection in a Hilbert space is orthogonal if $P = P^*$, i.e.,

$$\langle Px, y \rangle = \langle x, Py \rangle \quad , \quad \forall x, y \in \mathcal{H}.$$

Proposition: If P is an orthogonal projection, then $\operatorname{Im} P \perp \operatorname{Ker} P$.

Proof: $\forall x, y \in \mathcal{H}$,

$$\langle Px, (I-P)y \rangle = \langle x, (P-P^2)y \rangle = 0$$

i.e., $\operatorname{Im} P \perp \operatorname{Ker} P$.

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Proposition: All orthonormal projection satisfy: $0 \le P \le I$.

Proof:

$$\langle Px, x\rangle = \langle P^2x, x\rangle = \|P\|^2 \geq 0$$

by the previous prepositions.

$$\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2,$$

thus

$$\|Px\|^2 \leq \|x\|^2$$

therefore

$$0 \leq P \leq I.$$

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Let $a, b, m, M \in \mathbb{R}$, such that $a < m \leq M < b$ and A and operator that satisfies $mI \leq A \leq MI$. Let $K(a,b]$ the set of continuous functions by parts, bounded that are decreasing limit of continuous functions.

For a decreasing sequence f_n that converges to f, we write $f_n \searrow f$.

Lemma: Let $\varphi(t) \in K[a,b]$. Exists a sequence of polynomials $P_n(t) \searrow \varphi(t)$ if $n \to \infty$, $\forall t \in (a,b]$

Proof: By definition of $K \in [a,b]$, exists $\varphi_n(t) \in C[a,b]$ such that $\varphi_n(t) \searrow \varphi(t)$. Applying the Weierstrass to the continuous functions $\varphi_n(t) + \frac{3}{2^{n+2}}$, we obtain that $\forall n$, exists $P_n(t)$ such that

$$\left| P_n(t) - \left(\varphi_n(t) + \frac{3}{2^{n+2}} \right) \right| \leq \frac{1}{2^{n+2}}$$

thus,

$$P_{n+1}(t) \leq \varphi_{n+1}(t) + \frac{1}{2^{n+2}} \leq \varphi_n(t) + \frac{1}{2^{n+2}} \leq P_n(t)$$

then, P_n is not-increasing and converges to $\varphi(t)$ since φ_n also does it.

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Definition: Let $P_n(t) \searrow \varphi(t)$ $\forall t \in [m, M]$. So the decreasing sequence $P_n(A) \geq P_{n+1}(A) \geq \dots$ is bounded (because φ is), then by the previous theorem, the strong limit of

$$\lim_{n \to \infty} P_n(A)$$

exists and we will call it $\varphi(A)$.