

$$\hat{f}(x) = \int_A f(x) \overline{\chi(x)} dx$$

Lemma: Let A be a compact abelian group. If we set Haar's integral, as

$$\int_A 1 dx = 1$$

So, for any two characters $x, n \in \widehat{A}$.

$$\int_A x(x) \overline{\eta(x)} dx = \begin{cases} 1 & \text{if } x = n \\ 0 & \text{if } x \neq n \end{cases}$$

Proof: If $n = x$, so $x(x) \overline{\eta(x)} = 1$, and it follows the affirmation.

Now, let's suppose that $n \neq x$. So $\alpha = x\bar{n} = xn' \neq 1$. Thus there is $a \in A$, such that $\alpha(a) \neq 1$.

$$\alpha(a) \int_A \alpha(x) dx = \int_A \alpha(ax) dx = \int_A \alpha(x) dx,$$

$$\rightarrow (\alpha(a) - 1) \int_A \alpha(x) dx = 0 \rightarrow \int_A \alpha(x) dx = 0.$$

Theorem (Plancherel): Let A be a LCA group, there is a unique Haar's measure in \widehat{A} such that for all $f \in L^1_{bc}(A)$

$$\|f\|_2 = \|\hat{f}\|_2$$

i.e., for $f \in L^1_{bc}(A)$, the Fourier's transformed belongs in $L^1_{bc}(\widehat{A})$, i.e., the Fourier transformation gives an isomorphism of $L^2(A) \rightarrow L^2(\widehat{A})$.

This is the key point of how the harmonic analysis generalizes the theory of Fourier.

In the case $\mathbb{R} \setminus \mathbb{Z}$, $f \in L^1_{bc}(\mathbb{R} \setminus \mathbb{Z})$

$$\int_0^1 |f(x)|^2 dx = \|f\|_2^2 = \|\hat{f}\|_2^2 = \sum_{k \in \mathbb{Z}} |f(k)|^2 = \sum_{k \in \mathbb{Z}} |C_k(f)|^2$$

Riemann-Lebesgue Lemma

$$\int_{\mathbb{R} \setminus \mathbb{Z}} f(y) e^{-2\pi sxy} dy$$

Proof: We will prove that in the special case of A is a discrete group

$$\int_A f(x) dx = \sum_{a \in A} f(a)$$

for $x \in \hat{A}$, $\hat{f}(x) = \sum_{a \in A} f(a) \overline{x(a)}$ and in \hat{A} we choose the Haar's integral normalized.

$$\int_A 1 da = 1.$$

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Lemma: for each $g \in L^1_{bc}(A)$, the Fourier transform \hat{g} is in $C(\hat{A}) = L^1_{bc}(\hat{A})$ and we have that, for each $a \in A$

$$\hat{\hat{g}}(\delta_a) = g(a^{-1})$$

Proof:

$$\hat{\hat{g}}(\delta_a) = \int_{\hat{A}} \hat{g}(a) \overline{\delta_a(x)} dx.$$

$$= \int_{\hat{A}} \sum_{b \in A} g(b) \overline{\delta_b(x)} \overline{\delta_a(x)} dx$$

$$= \int_{\hat{A}} \sum_{b \in A} g(b^{-1}) \delta_b(x) \overline{\delta_a(x)} dx$$

$$= \sum_{b \in A} g(b^{-1}) \int_{\hat{A}} \delta_b(x) \overline{\delta_a(x)} dx$$

$$= g(a^{-1}).$$

To prove the theorem, let $f \in L^1_{bc}(A)$ and

$$\tilde{f}(x) = \overline{f(x^{-1})}$$

Let $y = \tilde{f} * f$

$$y(x) = \int_{\hat{A}} \overline{f(yx^{-1})} f(y) dy$$

Thus $g(e) = \|f\|_2^2$, with e be the identity element in A .
 By the convolution theorem,

$$\hat{g}(x) = \hat{\bar{f}}(x) \hat{f}(x) = \overline{\hat{f}(x)} \hat{f}(x) = |\hat{f}(x)|^2$$

Therefore

$$\|f\|_2^2 = g(e) = \hat{\bar{g}}(\delta e) = \int_{\hat{A}} |\hat{f}(x)|^2 dx = \|\hat{f}\|_2^2$$

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Matrix groups

$GL_n(\mathbb{C}), U(n)$.

Let $n \in \mathbb{N}$, in the vector space of the complex matrices $n \times n$ $\text{Mat}_n(\mathbb{C})$, we define a norm

$$\|A\|_1 = \sum_{i,j=1}^n |a_{ij}|, \quad \text{with} \quad A = (a_{ij})$$

This norm gives rise to the metric $d_1(A, B) = \|A - B\|_1$.

On the other hand, $\text{Mat}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, lets take a natural interior product, given rise to another norm, the euclidean norm

$$\|A\|_2 := \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}$$

and we obtain the following metric $d_2(A, B) = \|A - B\|_2$.

Lemma: A succession of matrices $A^{(k)} = (a_{ij}^{(k)})$ converges if and only if for two pair of indices (i, j) the succession of inputs converges in \mathbb{C} . The same happens to d_2 , thus the metrics d_1 and d_2 are equivalents.

Proof: Let's suppose that succession $A^{(k)} = (a_{ij}^{(k)})$ converges in d_1 to $A = (a_{ij}) \in \text{Mat}_n(\mathbb{C})$. So for each $\epsilon > 0$, there is $k_0 \in \mathbb{N}$, such that for all $k > k_0$

$$\|A^{(k)} - A\|_1 < \epsilon$$

Let $i_n, j_n \in \{1, \dots, n\}$, then $k > k_0$.

$$|a_{i_n j_n}^{(k)} - a_{i_n j_n}| \leq \sum_{ij} |a_{ij}^{(k)} - a_{ij}| = \|A^{(k)} - A\|_1 < \epsilon$$

then each input converges. Reciprocally, let's assume there is $a_{ij}^{(k)} \xrightarrow{k \rightarrow \infty} a_{ij}$, for each pair of index (i, j) . Given $\epsilon > 0$, there is $K_0(i, j)$, such that for all $K \geq K_0(i, j)$

$$|a_{ij}^{(k)} - a_{ij}| < \frac{\epsilon}{n^2}$$

Let $K_0 \in \mathbb{N}$, the maximum of the $K_0(i, j)$. Then for $K \geq K_0$

$$\|A^{(k)} - A\|_1 = \sum_{i,j} |a_{ij}^{(k)} - a_{ij}| < \sum_{i,j} \frac{\epsilon}{n^2} = \epsilon$$

Then $A^{(k)} \xrightarrow{k \rightarrow \infty} A$. And the same for d_2 .

Proposition: With the generated topology by the defined metric previously, $GL_n(\mathbb{C})$ is a locally compact group, i.e., it's metrizable, σ -compact and locally compact.

Proof: We note that multiplication and inversion are given by rational functions in the inputs. As the polynomials are continuous, then $GL_n(\mathbb{C})$ is a topological group.

Since it's an open subset of a locally compact space $Mat_n(\mathbb{C})$, it's locally compact.

Moreover, it's σ -compact.

$$K_n = \{ a \in GL_n(\mathbb{C}) / \|a\|_1 \leq n, \|a^{-1}\|_1 \leq n \}$$

For $A \in Mat_n(\mathbb{C})$, A^* is his adjoint matrix i.e., if $A = (a_{ij})$, then $A^* = (\overline{a_{ji}})$, thus $A^* = \overline{A^t}$.

$$U_n = \{ g \in Mat_n(\mathbb{C}) / g^* g = 1 \}$$

Lemma: $U(n)$ is a compact subgroup of $GL_n(\mathbb{C})$.

Proof: For $Mat_n(\mathbb{C})$ the equation $g^* g = 1$ implies that g is invertible and $g^* = g^{-1}$

$$U(n) \subseteq GL_n(\mathbb{C})$$

let $a, b \in U(n)$, to prove that $a, b \in U(n)$, $a^{-1} \in U(n)$. For this let's consider $(ab)^*(ab) = b^* a^* ab = b^* b = 1$, $a, b \in U(n)$.

On the other hand, as $a^* = a^{-1}$

$$1 = a a^* = a^* (a^*)^*$$

Therefore, $a^* = a^{-1}$ and it's in $U(n)$.

To see that it's compact, it's enough prove that the group $U(n)$ is closed and bounded in $\text{Mat}_n(\mathbb{C})$.

Let g_i be a succession in $U(n)$, convergent to g in $\text{Mat}_n(\mathbb{C})$, then.

$$1 = \lim_{j \rightarrow \infty} g_j^* g_j = \left(\lim_{j \rightarrow \infty} g_j \right)^* \left(\lim_{j \rightarrow \infty} g_j \right) = g^* g$$

therefore, $U(n)$ is closed.

Moreover, it's bounded because for all $a \in \text{Mat}_n(\mathbb{C})$.

$$\begin{aligned} \text{tr}(a^* a) &= \sum_k^n (a^* a)_{kk} = \sum_{k=1}^n \sum_{j=1}^n a_{kj}^* a_{jk} = \sum_{k=1}^n \sum_{j=1}^n \overline{a_{jk}} a_{jk} \\ &= \sum_{k=1}^n \sum_{j=1}^n |a_{jk}|^2 = \|a_{jk}\|_2^2 \end{aligned}$$

Then

$$\|g\|_2 = \sqrt{\text{tr} 1} = \sqrt{n}$$

Thus $U(n)$ is bounded.

