Finite groups theory

Group: $(G, \times)$; **Representations:** $D(g) \in GL(V_n)$; $\forall g \in G$

$$D(g_1 \times g_2) = D(g_1) \cdot D(g_2)$$

Definition: Subgroup $H \subseteq G$ / H is group.

Trivial: $\{e\}$; G.

Example:

$$Z_3 : \{e, a_1, a_2\} \subset S_3$$

subgroup.

Definition: right-coset: Set formed by the action of H over G (by left).

$$\{Hg\}; \; g \in G$$

Theorem: Each $g \in G$ belongs to one and only one coset

Corollary: $o(H)$ is a factor of $o(G)$

Definition: Quotient space: $G/H \equiv \{\{Hg\}\}$

Definition: If $H \subset G$ / $\forall g \in G : gH = Hg$. subgroup

$(\exists g_1, g_2 \in H / gg_1 = g_2 g)$

then, H is invariant or normal $\quad gHg^{-1} = H$

Theorem: $H \subset G$ is invariant, then G/H is a group. subgroup

under: $(Hg_1) \times (Hg_2) \equiv (Hg_1 Hg_1^{-1})(g_1 g_2)$

$$= H(g_1 g_2) \in G/H.$$

Definition: Conjugacy classes: $S \subset G$ / $\forall g \in G : g^{-1} S g = S$.

subset

Theorem: $\exists$ a correspondence one to one between the conjugacy classes and irrepresentations.

Definition: (Index) $H \subset G$ is the number of cosets of H that fills G

$$i \equiv o(G)/o(H)$$

Schur's lemma

Holi :)

Theorem: Let $D_1(g)$ and $D_2(g)$ be non-equivalent irreducible representations of G.

If $\forall g \in G: D_1(g)A = AD_2(g)$, with A a being a matrix, then $A=0$.

Theorem: If $D(g)A = AD(g)$, $\forall g \in G$, where D is an irreducible representation, then $A \propto \mathbb{I}$. $([D(g), A]=0)$.

Que feo escribes.

Homework: Prove the Schur's lemma.

Example: $\mathcal{O}$-observable $\to$ Invariant under a symmetry group.

let's label: $\mathcal{H}: \{|a,j,x\rangle\}$

- > Irreducible representation.
- > $j=1,\dots,n_a$ - dim (irrep).
- > Any other quantum #.

Let's fix $\langle a,j,x|\mathcal{O}|b,k,y\rangle = ?$

Consider:

$$D(g) = \begin{pmatrix} D_1 & 0 & 0 & \cdots \\ 0 & D_2 & 0 & \cdots \\ \vdots & & \ddots & \\ \vdots & & & \end{pmatrix} = \bigoplus_a D_a(g)$$

then,

$$\langle a,j,x|D(g)|b,k,y\rangle$$
$$= \langle a,j,x|\bigoplus_a D_a(g)|b,k,y\rangle$$
$$= \delta_{ab}\delta_{xy}[D_a(g)]_{jk}$$

G - Symmetry.

$$D(g) = \sum_{a,j,k,x} |a,j,x\rangle [D_a(g)]_{jk} \langle a,k,x|$$

under the action of the group:

$$|\mu\rangle \in \mathcal{H}: |\mu\rangle \longrightarrow D(g)|\mu\rangle$$
$$\langle\mu| \longrightarrow \langle\mu|D^\dagger(g).$$

while:

$$\mathcal{O} \longrightarrow D(g)\,\mathcal{O}\,D^\dagger(g).$$

If $\mathcal{O}$ is invariant:

$$\mathcal{O} = D(g)\,\mathcal{O}\,D^\dagger(g)$$

then

$$[\mathcal{O}, D(g)] = 0, \quad \forall g \in G.$$

Schur's lemma:

$$\langle a|\mathcal{O}|b\rangle \propto \delta_{ab}.$$

Explicitely:

$$\langle a,j,x|\mathcal{O}|b,k,y\rangle = ?$$

$$\begin{aligned}
0 &= \langle a,j,x|[\mathcal{O},D(g)]|b,k,y\rangle \\
&= \sum_{b',k',y'} \langle a,j,x|\mathcal{O}|b',k',y'\rangle\langle b',k',y'|D(g)|b,k,y\rangle \\
&- \sum_{b',k',y'} \langle a,j,x|D(g)|b',k',y'\rangle\langle b',k',y'|\mathcal{O}|b,k,y\rangle \\
&= \sum_{k'} \langle a,j,x|\mathcal{O}|b',k',y\rangle[D_b(g)]_{k'k} \\
&- \sum_{k'} [D_a(g)]_{jk'}\langle a,k',x|\mathcal{O}|b,k,y\rangle
\end{aligned}$$

$$\mathcal{O}^{ab}_{xy} D_b(g) = D_a(g)\mathcal{O}^{ab}_{xy} \quad \forall g \in G.$$

If $a \neq b$, $\mathcal{O}^{ab}_{xy} = 0$; $a=b$, $\mathcal{O}^{ab}_{xy} \propto \mathbb{1}$

$$\langle a,j,x|\mathcal{O}|b,k,y\rangle = f_a|x,y\rangle\,\delta_{ab}\,\delta_{jk}.$$

Theorem: The elements of the matrix of irreducible representations of G form an orthonormal complete set for the vector space of the regular representations, then

$$\sum_{g\in G} \frac{n_a}{N}[D_a(g^{-1})]_{kj}[D_b(g)]_{lm} = \delta_{ab}\,\delta_{jl}\,\delta_{km}$$

where:

$$N \rightarrow \mathcal{O}(G)$$

$$n_a = \dim(\text{irrep}).$$

either

$$\sum_{g\in G} \frac{n_a}{N}[D_a(g)]^*_{jk}[D_b(g)]_{lm} = \delta_{ab}\,\delta_{jl}\,\delta_{km}$$

then,

$$\sqrt{\frac{n_a}{N}}\,[D_a(g)]_{jk} \rightarrow \text{Orthogonal functions.}$$

corollary: $N = \sum_a n_a^2$