

$$I(x) = \int_a^b e^{-x\phi(t)} f(t) dt \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n+\beta}{\alpha}) C_n}{x^{(n+\beta)/\alpha}} \quad \text{as } x \rightarrow \infty$$

Theorem: If  $f$  and  $g$  are  $C^n$  differentiable at  $z$ , then.

$$\frac{d^n}{dz^n} f(z)g(z) = \sum_{k=0}^n \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z)$$

Proof: For  $n=1 \leadsto$  standard Leibniz's rule.

$$\frac{d}{dz} f(z)g(z) = f'(z)g(z) + f(z)g'(z)$$

Assume that it is valid for  $n$  and try for  $n+1$ .

$$\frac{d^{n+1}}{dz^{n+1}} f(z)g(z) = \frac{d}{dz} \left( \sum_{k=0}^n \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z) \right)$$

$$= \sum_{k=0}^n \binom{n}{k} \left[ f^{(k+1)}(z) g^{(n-k)}(z) f^{(k)}(z) g^{(n-k+1)}(z) \right]$$

$$= f(z)g^{(n+1)}(z) + \sum_{k=1}^n \binom{n}{k} f^{(k)} g^{(n-k+1)}(z)$$

$$+ \sum_{k=0}^{n-1} \binom{n}{k} f^{(k+1)}(z) g^{(n-k)}(z) + f^{(n+1)}(z)g(z)$$

Change  $j=k+1$

$$\rightarrow \sum_{j=1}^n \binom{n}{j-1} f^{(j)}(z) g^{(n-j+1)}(z)$$

then

$$\frac{d^{n+1}}{dz^{n+1}} f(z)g(z) = f(z)g^{(n+1)}(z) + \sum_{k=1}^n \left[ \binom{n}{k} \binom{n}{k-1} \right] f^{(k)}(z) g^{(n-k+1)}(z) + f^{(n+1)}(z)g(z)$$

$$\binom{n}{k} \binom{n}{k-1} = \frac{n!}{(n-k)! k!} + \frac{n!}{(n-k+1)! (k-1)!} = \frac{n!}{(k-1)! (n-k)!} \left[ \frac{1}{k} + \frac{1}{n-k+1} \right]$$

$$= \frac{n!}{(k-1)! (n-k)!} \cdot \frac{(n+1)}{k(n-k+1)} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

then

$$\begin{aligned} \frac{d^{n+1}}{dz^{n+1}} f(z)g(z) &= f(z)g^{(n+1)}(z) + \sum_{k=1}^n \binom{n+1}{k} f^{(k)}(z)g^{(n+1-k)}(z) + f^{(n+1)}(z)g(z) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(z)g^{(n+1-k)}(z) \end{aligned}$$

Now, consider

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

forefore

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Where

$$c_n = \frac{1}{n!} \left. \frac{d^n}{dz^n} \right|_{z=z_0}$$

$$f(z)g(z) = \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0) = \sum_{k=0}^n \frac{f^{(k)}(z_0)}{k!} \frac{g^{(n-k)}(z_0)}{(n-k)!}$$

$$= \sum_{k=0}^n a_k b_{n-k},$$

now

$$g(\psi) = \frac{f(t)}{\phi'(t)} \longrightarrow g(\psi)\phi'(t) = f(t)$$

using another notation

$$\frac{f(z)}{g(z)} = h(z) \longrightarrow f(z) = h(z)g(z)$$

therefore

$$\begin{aligned} \sum_{n=0}^{\infty} a_n (z-z_0)^n &= \left( \sum_{k=0}^{\infty} c_k (z-z_0)^k \right) \left( \sum_{j=0}^{\infty} b_j (z-z_0)^j \right) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^n c_k b_{n-k} (z-z_0)^{k+(n-k)} \end{aligned}$$

$$a_n = \sum_{k=0}^n c_k b_{n-k}$$

for  $n=0$ :  $a_0 = b_0 c_0 \longrightarrow c_0 = \frac{a_0}{b_0}$

for  $n \geq 1$ :

$$a_n = \sum_{k=0}^n c_k b_{n-k} = c_n b_0 + \sum_{k=0}^{n-1} c_k b_{n-k}$$

$$\frac{c_n = a_n - \sum_{k=0}^{n-1} c_k b_{n-k}}{b_0}$$

Recurrence relation.

$$c_1 = \frac{a_1 - c_0 b_1}{b_0} = \frac{a_1 - \frac{a_0}{b_0} b_1}{\frac{b_0}{b_0}} = \frac{a_1 b_0 - a_0 b_1}{b_0^2}$$

$$c_2 = \frac{a_2 - c_0 b_2 - c_1 b_1}{b_0} = \frac{a_2 - \frac{a_0}{b_0} b_2 - \left( \frac{a_1 b_0 - a_0 b_1}{b_0^2} \right) b_1}{b_0}$$

to finish the proof of Ederlyi's theorem. Check that the remaining part  $(c_1 b_1)$  is negligible

$$\varepsilon := \inf_{a \leq t \leq b} \phi(t) - \phi(a)$$

Assume  $x_0$  such that  $I(x_0)$  is absolutely convergent.

Let's assume  $x \geq x_0$

$$\begin{aligned} x(\phi(t) - \phi(a)) &= (x - x_0)(\phi(t) - \phi(a)) + x_0(\phi(t) - \phi(a)) \\ &\geq (x - x_0)\varepsilon + x_0(\phi(t) - \phi(a)) \end{aligned}$$

$$I(x) = \int_a^b e^{-x\phi(t)} f(t) dt \quad \xrightarrow{\text{split}} \int_a^b + \int_c^b$$

then

$$\int_c^b e^{-x(\phi(t) - \phi(a))} f(t) dt \leq \int_c^b e^{-(x-x_0)\varepsilon} f(t) dt + \int_c^b e^{-x(\phi(t) - \phi(a))} f(t) dt$$

as the parts containing  $x_0$  are convergent.

$$\left| e^{x\phi(a)} \int_a^b e^{-x\phi(t)} f(t) dt \right| \leq K e^{-\varepsilon x}$$

where  $K$  is given by

$$K = e^{x_0(\varepsilon + \phi(a))} \int_a^{x_0} e^{-x_0\phi(t)} |f(t)|$$

is an appropriate constant.

∴ Most of the contribution to the asymptotic behaviour of  $I(x)$  comes from the interval  $(a, c]$



Theorem (Perron's formula): The coefficients above are explicitly given by

$$C_n = \frac{1}{\alpha n!} \left[ \frac{d^n}{dx^n} \right] G(t) \left( \frac{t-a}{\phi(t)-\phi(a)} \right)^{\frac{n+\beta}{\alpha}} \Big|_{t=a}$$

where

$$G(t) \sim \sum_{k=0}^{\infty} b_k (t-a)^k$$

and

$$\phi(t) \sim \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$$

Proof: Let's define  $\ell$  as the label of the first non-vanishing coefficient in this last expression (apart from  $a_0$ ) and define.

$$\phi_\ell(t) := \frac{\phi(t) - \phi(a) - a_0(t-a)^\alpha}{(t-a)^{\alpha+\ell}} \sim \sum_{k=0}^{\infty} a_{k+\ell} (t-a)^k \text{ as } t \rightarrow a^+$$

$$\begin{aligned} I(x) &= \int_a^b e^{-x\phi(t)} f(t) dt = e^{-x\phi(a)} \int_a^b e^{-a_0 x (t-a)^\alpha} \underbrace{e^{-x(t-a)^{\alpha+1}} \phi_\ell(t)}_{:= h(x,t)} f(t) dt \\ &= e^{-x\phi(a)} \int_a^b e^{-a_0 x (t-a)^\alpha} h(x,t) dt \end{aligned}$$

change  $z := (t-a)x^{1/\alpha}$ ,  $dz = x^{1/\alpha} dt$

$$I(x) = \frac{e^{-x\phi(a)}}{x^{1/\alpha}} \int_0^{1/\alpha(b-a)} e^{-a_0 z^\alpha} h(x, x^{1/\alpha}t+a) dz$$

set  $s := x^{-1/\alpha} z$

$$h(x, s+a) = e^{-x(t-a)^{\alpha+1} \phi_1(t)} f(t) = \exp(-z^\alpha s^{\alpha+1} \phi_1(s+a)) f(s+a)$$

$$t-a=s$$

$$t=s+a.$$

$$(t-a)^{\alpha+1} = s^{\alpha+1} = s^\alpha s^1 = s^1 (x^{1/\alpha})$$

using Taylor-series.

$$\exp(-z^\alpha s^1 \phi_1(s-a)) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{dw^k} \left[ e^{-z^\alpha w^1 \phi_1(w+a)} \right]_{w=0} s^k, \text{ as } s \rightarrow 0^+$$

and considering

$$f(x) = \sum_{k=0}^{\infty} b_k (x-a)^{k+\beta-1}$$

then

$$h(x, s+a) \sim \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{dw^k} \left[ e^{-z^\alpha w^1 \phi_1(w+a)} \right]_{w=0} s^{k+\beta-1}$$

$$\begin{aligned} I(x) &\sim \frac{e^{-x\phi(a)}}{x^{1/\alpha}} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{b_{n-k}}{k!} \left[ \frac{d^k}{dw^k} \int_0^{1/\alpha(b-a)} e^{-(a_0 + w^1 \phi_1(w+a)) z^\alpha} s^{k+\beta-1} dz \right]_{w=0} \\ &= e^{-x\phi(a)} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{b_{n-k}}{k!} \left[ \frac{d^k}{dw^k} \int_0^{1/\alpha(b-a)} e^{-(a_0 + w^1 \phi_1(w+a)) z^\alpha} s^{k+\beta-1} dz \right]_{w=0} \frac{d^k}{X^{(n+\beta)/\alpha}} \end{aligned}$$

$$x^{(-1/\alpha)(n+\beta-1)} = x^{-\frac{n+\beta}{\alpha}} x^{1/\alpha}$$

Change

$$P := (a_0 + w^1 \phi_1(w+a)) z^\alpha$$

then

$$dz = \frac{dp}{\alpha [a_0 + \omega^1 \phi_1(\omega + a)]^{1/\alpha}} p^{1/\alpha - 1}$$

$$z^{n+\beta-1} = \frac{p^{(n+\beta-1)/\alpha}}{(\alpha [a_0 + \omega^1 \phi_1(\omega + a)]^{1/\alpha})^{(n+\beta-1)/\alpha}}$$

$$\begin{aligned} I(x) &= e^{-x\phi(a)} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k} \left[ \left( \int_0^\infty e^{-p} p^{(n+\beta-1)/\alpha} dp \right) \left( \frac{1}{\alpha [a_0 + \omega^1 \phi_1(\omega + a)]^{(n+\beta)/\alpha}} \right) \right]_{\omega=0} \frac{1}{X^{(n+\beta)/\alpha}} \\ &= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{\alpha} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k} \left[ \frac{1}{\alpha [a_0 + \omega^1 \phi_1(\omega + a)]^{(n+\beta)/\alpha}} \right]_{\omega=0} \frac{1}{X^{(n+\beta)/\alpha}} \end{aligned}$$

using definition of  $\phi$ ,

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{\alpha} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k} \left[ \frac{(t-a)^\alpha}{\phi(t) - \phi(a)} \right]^{\frac{n+\beta}{\alpha}} \Big|_{\omega=0} \frac{1}{X^{(n+\beta)/\alpha}}$$

Before, we knew.

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{C_n}{X^{(n+\beta)/\alpha}}$$

$$\therefore C_n = \frac{1}{\alpha} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{dt^k} \left[ \frac{(t-a)^\alpha}{\phi(t) - \phi(a)} \right]^{\frac{n+\beta}{\alpha}} \Big|_{t=a}$$

$$= \frac{1}{\alpha n!} \frac{d^n}{dt^n} \left\{ f(t) \left[ \frac{(t-a)^\alpha}{\phi(t) - \phi(a)} \right]^{\frac{n+\beta}{\alpha}} \Big|_{t=a} \right\}$$



$$x^{\left(\frac{-1}{\alpha}\right)(n+p-1)} = x^{-\frac{n+p}{\alpha}} x^{1/\alpha}$$

Change

$$f := (a_0 + \omega^\ell \phi_\ell(\omega - a)) z^\alpha$$

then

$$dz = \frac{df}{\alpha(a_0 + \omega^\ell \phi_\ell(\omega - a))} z^{1/\alpha - 1}$$

$$z^{n+p-1} = \frac{f^{(n+p-1)/\alpha}}{(a_0 + \omega^\ell \phi_\ell(\omega - a))^{(n+p-1)/\alpha}}$$

$$I(x) = e^{-x\phi(a)} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k} \left[ \left( \int_0^{\infty} e^{-p} p^{\frac{n+p-1}{\alpha}} dp \right) \left( \frac{1}{\alpha(a_0 + \omega^\ell \phi_\ell(\omega - a))^{(n+p)/\alpha}} \right) \right]_{\omega=0}$$

$$\cancel{\frac{1}{X^{(n+p)/\alpha}}}$$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{dw^k} \left[ \frac{1}{[\alpha_0 + w^{\frac{1}{\alpha}} \phi'(w+a)]^{(n+1)/\alpha}} \right] \Big|_{w=0} \frac{1}{x^{(n+\beta)/\alpha}}$$

Using definition of  $\phi$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{dw^k} \left[ \frac{(t-a)^{\frac{n+\beta}{\alpha}}}{\phi(t) - \phi(a)} \right] \Big|_{w=0} \frac{1}{x^{(n+\beta)/\alpha}}$$

Before we knew

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{C_n}{x^{(n+\beta)/\alpha}}$$

∴

$$C_n = \frac{1}{\alpha} \sum_{k=0}^n \frac{b_{n-k}}{k!} \frac{d^k}{dt^k} \left[ \frac{(t-a)^{\frac{n+\beta}{\alpha}}}{\phi(t) - \phi(a)} \right] \Big|_{t=a}^{(n+\beta)/\alpha}$$

$$= \frac{1}{\alpha n!} \frac{d^n}{dt^n} \left\{ f(t) \left[ \frac{(t-a)^{\frac{n+\beta}{\alpha}}}{\phi(t) - \phi(a)} \right] \right\} \Big|_{t=a}^{(n+\beta)/\alpha}$$