Lie algebras

Definition: Invariant subalgebra:

$$X \in \text{SubAlg-inv}, \ \forall\, Y \in \text{alg}$$
$$[X, Y] \in \text{SubAlg-inv.}$$

Simple algebra: Does not have non-trivial invariant subalgebras.

$\rightarrow$ Generates simple group.

Semi-simple algebras: Algebras without abelian invariant subalgebras

Casimir operator:

$$\mathcal{O}(T_a) / [\mathcal{O}(T_a), T_b] = 0$$

Schur's lemma:

$\mathcal{O} \propto \mathbb{1}$ in each irreducible representation.

Cartan's Subalgebra:

$$\{H_i \in \text{alg} \,/\, H_i = H_i^\dagger, [H_i, H_j] = 0\}$$

Cartan Generators

$i_{max} \equiv \text{range} \equiv l.$

Is escentially unique.

Generates a linear space. $\longrightarrow$ $T_r(H_i H_j) = K\,\delta_{ij}$; $i, j = 1, \ldots, l.$

$$H_i \,|\vec{\mu}, x, D\rangle = \mu_i |\vec{\mu}, x, D\rangle$$
$$\mu_i \equiv \text{weight} \ (\in \mathbb{R}) \quad , \ \vec{\mu} = (\mu_1, \ldots, \mu_l) \rightarrow \text{Weight vectors}$$

Example: SU(2)

$$[J_i, J_j] = i\,\epsilon_{ijn} J_n \ ; \quad i, j, n = 1, 2, 3.$$

$\rightarrow$ Cartan SubAlg. : $\{J_j\}$

Cuadratic Casimir : $\vec{J}^2$

Roots: Weights of the adjoint representation.

Due to dim alg = dim adj. rep.

$\rightarrow$ Basis of the adjoin representation $\stackrel{\text{iso}}{\equiv}$ X_a generators.

$$X_a \longleftrightarrow |X_a\rangle$$

therefore

$$\alpha|X_a\rangle + \beta|X_b\rangle = |\alpha X_a + \beta X_b\rangle$$

and

$$\langle X_a|X_b\rangle = \lambda^{-1}\,\mathrm{Tr}(X_a^\dagger X_b)$$

Hence

$$X_a|X_b\rangle = |X_c\rangle\langle X_c|X_a|X_b\rangle = |X_c\rangle[T_a]_{cb}$$

$$= -i f_{acb}|X_c\rangle = |i f_{abc} X_c\rangle$$

$$= |[X_a, X_b]\rangle$$

finally, the Cartan subalgebra

$$H_i|H_j\rangle = |[H_i, H_j]\rangle = 0.$$

The states with vector of null weight $\longleftrightarrow$ Cartan generators.

Orthogonality:

$$\langle H_i, H_j\rangle = \lambda^{-1}\,\mathrm{Tr}(H_i, H_j) = \delta_{ij}$$

$\lambda\delta_{ij} \rightarrow$ adjont

The other states of the adjoint representation.

We have non-null weight vectors: $(d-l)$-elements.

$\rightarrow H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle$; $\alpha_i \neq 0$, $\alpha_i \in \mathbb{R}$.

Root

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

Root vector: $\vec{\alpha}$

$\rightarrow$ Especifies in a unique way the states

Notice:

$$[H_i, E_\alpha^\dagger] = -[H_i, E_\alpha]^\dagger = -\alpha_i E_\alpha^\dagger$$

then, $E_\alpha^\dagger = E_{-\alpha}$, $-\alpha$ is root.

also,

$$\langle E_\alpha|E_\beta\rangle = \delta_{\alpha\beta} \longrightarrow \lambda^{-1}\,\mathrm{Tr}(E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta}$$

In $SU(2)$: Cartan: $\{J_3\}$; J_+, J_-

Effectively:

$$H_i E_{\pm\alpha}|\mu, D\rangle = [H_i, E_{\pm\alpha}]|\mu, D\rangle + E_{\pm\alpha} H_i |\mu, D\rangle$$
$$= (\mu \pm \alpha)_i E_{\pm\alpha}|\mu, D\rangle$$

$E_{\pm\alpha}$ is the up and down operator.

In the adjoint representation

$$H_i E_\alpha |E_{-\alpha}\rangle = 0$$

weight $= -\alpha + \alpha$

then, $E_\alpha |E_{-\alpha}\rangle = \beta_i |H_i\rangle$

$$[E_\alpha, E_{-\alpha}] = \beta_i H_i$$

Its clear

$$\beta_i = \langle H_i | E_\alpha | E_{-\alpha}\rangle = \lambda^{-1} \mathrm{Tr}(H_i [E_\alpha, E_{-\alpha}])$$
$$= \lambda^{-1} \mathrm{Tr}(E_{-\alpha} \underbrace{[H_i, E_\alpha]}_{\alpha_i E_i}) = \alpha_i \lambda^{-1} \mathrm{Tr}(\cancel{E_\alpha^\dagger E_\alpha})^{1}$$

therefore, $\beta_i = \alpha_i$,

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i$$

$SU(2): [J_+, J_-] = J_3$: Each pair $E_{\pm\alpha}$ generates subalg $SU(2)$.

for each pair $\pm\vec{\alpha}$:

$$E^\pm = |\vec{\alpha}|^{-1} E_{\pm\alpha}$$
$$E_3 = |\vec{\alpha}|^{-2} \vec{\alpha} \cdot \vec{H}$$

Because,

$$[E_3, E^\pm] = \pm E^\pm$$
$$= |\vec{\alpha}|^{-3} [\alpha_i H_i, E_{\pm\alpha}]$$
$$= |\vec{\alpha}|^{-3} \alpha_i \underbrace{[H_i, E_{\pm\alpha}]}_{\pm\alpha_i E_{\pm\alpha}}$$

And

$$[E^+, E^-] = E_3 \qquad = |\vec{\alpha}|^{-2} \underbrace{[E_\alpha, E_{-\alpha}]}_{\alpha_i H_i}$$

All irreducible representations of the algebra, may be decompouse in irreducible representations of $SU(2)$.

Due that E_3 has eigenvalues: n, $n/2$, $n \in \mathbb{N}$.

In any irreducible representation.

$$E_3 |\mu, D\rangle = \frac{\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} |\mu, D\rangle$$

then,
$$\frac{2\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} = \text{integer!}$$

also, $\exists\, p > 0 \,/\, (E^+)^p |\mu, D\rangle \neq 0$; $(E^+)^{p+1} |\mu, D\rangle = 0$, then

$$\frac{\vec{\alpha} \cdot (\vec{\mu} + p\vec{\alpha})}{\alpha^2} = \frac{\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} + p = j \to \text{higher ``spin''}$$

In the other hand, for E^-, $\exists\, q > 0 \,/\, \frac{\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} - q = -j$

then,
$$\frac{\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} = -\frac{1}{2}(p - q)$$

General topics

Theorem: Range algebra: l has l casimir operators

Definition: In a representation r: $Tr(T_r^a, T_r^b = C_r \delta^{ab}$

Dynkin index.

Cuadratic Casimir:
$$T^2 = \sum_a T^a T^a = C_r^{(2)} \mathbb{1}$$

then
$$\dim alg \times C_r = C_r^{(2)} \times \dim rep.$$

for rep. $s \otimes r$:
$$C_{s\otimes r} = \dim r \cdot C_s + \dim s \cdot C_r$$

for the adjoint:
$$Tr([T_a, T_b] T_c) \equiv d^{abc}$$

for any r:
$$Tr([T_r^a, T_r^b] T_r^c) = A_r d^{abc}$$

$A_r \in \mathbb{N}$ - anomality, $\neq 0$ just for $SU(n)$, $n \geq 3$.