Theorem Ederlyi

$$I(x) = \int_a^b f(t)\, e^{-x\phi(t)}\, dt$$

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{C_n}{x^{(n+\beta)/\alpha}} \quad \text{as } x \to \infty$$

$$C_n = \frac{1}{\alpha n!}\left[\frac{d^n}{dt^n}\left\{ f(t) \left(\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right)^{\frac{n+\beta}{\alpha}} \right\}\right]_{t=a}$$

(If $\phi(t) = \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$

and

$$f(t) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} \;)$$

Example

Gamma function

$$\Gamma(\lambda+1) = \int_0^\infty e^{-t}\, t^\lambda\, dt \qquad \text{for } \lambda > 0$$

Change $t = \lambda(1+x) \to dt = \lambda dx$

$$\Rightarrow \Gamma(\lambda+1) = \int_0^\infty e^{-\lambda(x+1)} \lambda^\lambda (1+x)^\lambda \lambda\, dx.$$

$$= \lambda^{\lambda+1} e^{-\lambda} \int_{-1}^{\infty} e^{-\lambda x} (1+x)^\lambda\, dx$$

$$= \lambda^{\lambda+1} e^{-\lambda} \int_{-1}^{\infty} e^{-\lambda(x - \log(1+x))}\, dx$$

$$\frac{\Gamma(\lambda+1)}{\lambda^{\lambda+1}e^{-\lambda}} = \underbrace{\int_0^\infty e^{-\lambda(x-\log(1+x))}\,dx}_{=:I_1(x)} + \underbrace{\int_0^1 e^{-\lambda(-x-\log(1-x))}\,dx}_{:=I_2(x)}$$ (Cambio los limites de integración)

$$I_1(\lambda) = \int_0^\infty e^{-\lambda\phi(x)}\,dx \qquad \phi(x) = x - \log(1+x)$$

$$f(x) = 1$$

$$\log(1+x) \sim 1 - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\phi(x) = x - \log(1+x) \sim \frac{x^2}{2} - \frac{x^3}{3} + \dots$$

therefore $\alpha = 2$ fixed around cero.

$$I_1(\lambda) \sim e^{-\lambda\phi(0)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{\lambda^{(n+1)/2}} \quad \text{as } \lambda \to \infty$$

$$C_n = \frac{1}{2n!} \frac{d^n}{dx^n}\left[\frac{x^2}{x-\log(1+x)}\right]^{(n+1)/2}\Bigg|_{x=0}$$

$I_2(\lambda)$:

$$\phi(x) = -x - \log(1-x) \sim \cancel{-x} - (\cancel{-x} - \frac{x^2}{2} - \frac{x^3}{3} - \dots)$$

$$\sim \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad \text{for } x \to 0^+$$

$$f(t) = 1 \quad \to \quad \beta = 1 \qquad \alpha = 2$$

$$I_2(\lambda) \sim e^{-\lambda\phi(0)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n(-1)^n}{\lambda^{(n+1)/2}}$$ (Same C_n as before)

$$\therefore \Gamma(\lambda+1) = \lambda^{\lambda+1} e^{-\lambda} \left(\sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{\lambda^{(n+1)/2}} \left(1+(-1)^n\right) \right)$$

$$= 2\lambda^{\lambda+1} e^{-\lambda} \sum_{n=0}^{\infty} \Gamma\left(n+\frac{1}{2}\right) \frac{C_{2n}}{\lambda^{n+1/2}}$$

$$= \sqrt{2\pi}\, e^{-\lambda} \sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda^n}$$

Where $\gamma_n = \sqrt{\frac{2}{\pi}}\, \Gamma\left(n+\frac{1}{2}\right) C_{2n}$ Sterling coefficients.

$$\gamma_0 = 1, \quad \gamma_1 = \frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = -\frac{139}{51840}$$

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$$

$$\Gamma(\lambda) = \sqrt{2\pi}\, \lambda^{\lambda-1/2} e^{-\lambda} \left(1 + \frac{1}{12}x + \cdots\right) \quad \text{as } \lambda \to \infty$$

Example

Legendre Polynomials $P_m(x)$ $x > 1$

$$P_m(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \cos t \sqrt{x^2-1}\right)^m dt$$

Solution

Change $x = \cosh\theta$ $\theta > 0$

$$P_m(x) = \frac{1}{\pi} \int_0^{\pi} (\cosh\theta + \cos t \sinh\theta)^m dt$$

$$\cosh\theta + \cos t \sinh\theta = \frac{e^{\theta}+e^{-\theta}}{2} + \cos t \, \frac{e^{\theta}-e^{-\theta}}{2}$$

$$= e^{\theta}\left(\frac{1+\cos t}{2}\right) + e^{-\theta}\left(\frac{1-\cos t}{2}\right)$$

$$= e^{\theta}\cos^2\frac{t}{2} + e^{-\theta}\sin^2\frac{t}{2}$$

$$= e^{\theta}\left(1-\sin^2\frac{t}{2}\right) + e^{-\theta}\sin^2\frac{t}{2}$$

$$= e^{\theta}\left[1-\sin^2\frac{t}{2}\left(1-e^{-2\theta}\right)\right]$$

$$\Rightarrow P_m(x) = \frac{1}{\pi}e^{m\theta}\int_0^{\pi} e^{-m\left(-\log\left[1-\sin^2\frac{t}{2}\left(1-e^{-2\theta}\right)\right]\right)}\,dt$$

$$\phi(t) = -\log\left[1-\sin^2\left(\frac{t}{2}\right)\left(1-e^{-2\theta}\right)\right]$$

$$f(t) = 1 \to \quad \beta = 1$$

$$\phi(t) = \sum_{k=1}^{\infty}\frac{\left(1-e^{-2\theta}\right)^k}{k}\sin^{2k}\left(\frac{t}{2}\right)$$

$$= \sum_{k=1}^{\infty}\frac{\left(1-e^{-2\theta}\right)}{k}\left(\frac{t}{2}-\frac{t^3}{3!}+\frac{t^5}{5!}+\dots\right)^{2k}$$

$$= \sum_{k=1}^{\infty}\frac{\left(1-e^{-2\theta}\right)}{k}\left(\frac{t^2}{4}-\frac{2t^4}{12}-\mathcal{O}(t^6)\right)^k$$

$$= \frac{1-e^{-2\theta}}{1}\left(\frac{t^2}{4}-\frac{2t^4}{12}\right) + \frac{\left(1-e^{-2\theta}\right)^2}{2}\frac{t^4}{16} + \mathcal{O}(t^6)$$

thus $\alpha = 2$

$$P_m(\cosh\theta) \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{m^{(n+1)/2}}$$

Where

$$C_n = \frac{1}{2n!} \frac{d^n}{dt^n} \left[\frac{t^2}{-\log[1-\sin^2\frac{t}{2}(1-e^{-2\theta})]} \right]^{\frac{n+1}{2}} \Bigg|_{t=0}$$

$$C_0 = \frac{1}{2} \left[\frac{t^2}{-\log[1-\sin^2\frac{t}{2}(1-e^{-2\theta})]} \right]^{1/2} \Bigg|_{t=0}$$

$$= \frac{1}{\sqrt{1-e^{-2\theta}}}$$

$$f(t) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} = 1 = b_0 \qquad b_j = 0 \quad \forall j \neq 0$$

We want $g(t)\,\phi'(t) = f(t)$

$$\Rightarrow \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} = \left(\sum_{j=0}^{\infty} c_j (t-a)^j\right)\left(\sum_{l=0}^{\infty} a_l (l+\alpha)(t-a)^{l+\alpha-1}\right)$$

$$\Rightarrow b_n = \sum_{k=0}^{n} c_k\, a_{n-k}^{1/\alpha} (n-k+\alpha)^{1/\alpha}$$

$n=0$ $\qquad b_0 = c_0 a_0^{1/2} \sqrt{2} \qquad c_0 = \frac{b_0}{a_0^{1/2}\sqrt{2}} = \frac{1}{\cancel{\sqrt{2}}\sqrt{\frac{1-e^{-2\theta}}{\cancel{2}}}}$ ✓

$n=1 \qquad 0 = b_1 = c_0 a_1^{1/2}\sqrt{3} + c_1 a_0^{1/2}\sqrt{2}$

$$c_1 = \frac{\cancel{-c_0 a_1^{1/2}\sqrt{3}}^{\to 0}}{c_1 a_0^{1/2}\sqrt{2}} = 0$$

$n=2 \qquad 0 = b_2 = c_0 a_2^{1/2}\sqrt{4} + \cancel{c_1 a_1^{1/2}\sqrt{3}}^{\to 0} + c_2 a_0^{1/2}\sqrt{2}$

$$c_2 = -\frac{c_0 a_2^{1/2}\sqrt{4}}{a_0^{1/2}\sqrt{2}}$$

$n=3$

$$c_3 = 0$$

Method of stationary phase

Consider $\phi(t)$ in Laplace integral as a pure imaginary function

$$\Rightarrow \phi(t) = i\psi(t) \quad ; \quad \psi(t) \text{ real}$$

$$\int_a^b f(t)\, e^{-x\phi(t)}\, dt \longmapsto I(x) := \int_a^b f(t)\, e^{ix\psi(t)}\, dt$$

Generalized Fourier Integral

for $\psi(t) = t \rightarrow I(x) = \int_a^b f(t)\, e^{ixt}\, dt$

Ordinary Fourier Integral

Example

$$I(x) = \int_0^1 \sqrt{t}\, e^{ixt}\, dt = \int_0^1 \sqrt{t}\, \frac{1}{ix}\frac{d}{dt}\left(e^{ixt}\right) dt$$

as $x \to \infty$

$$= \frac{1}{ix}\sqrt{t}\, e^{ixt}\Big|_0^1 - \frac{1}{2ix}\underbrace{\int_0^1 \frac{1}{\sqrt{t}} e^{ixt}\, dt}_{:= I_1(x)} + O\left(\frac{1}{x^2}\right) = \frac{1}{ix} e^{ix}$$

$I_1(x)$ change $s = xt$ $\quad ds = x\,dt$

$$I_1(x) = \frac{i}{2x}\int_0^x \frac{e^{is}}{\sqrt{\frac{s}{x}}}\frac{ds}{x} = \frac{i}{2x^{3/2}}\int_0^x \frac{e^{is}}{\sqrt{s}}\, ds \quad \text{as } x \longrightarrow \infty$$

Change $s \longmapsto is$ (Wick rotation)

Is a transformation that substitutes an imaginary-number variable for a real-number variable and vice versa.

$$I_1(x) \sim \frac{i}{2x^{3/2}}\int_0^\infty \frac{e^{-s}}{i^{1/2}}\, s^{1/2-1}\, i\, ds$$

$$= \frac{i^{3/2}}{2x^{3/2}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\frac{i}{x^{3/2}}\left(e^{i\pi/2}\right)^{1/2}$$

$$= \frac{i\sqrt{\pi}}{2x^{3/2}} e^{i\pi/4} \quad \text{as } x \to \infty$$