# An overview of deformation quantization.

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**Idea:** Alternative formulation of QM in phase space

$$\text{Phase space Quantum mechanics} \longleftrightarrow \text{Deformation Quantization.}$$

Point-wise product deformed into an appropriate non-commutative $*$-product.

Poisson bracket deformed into an appropriate Lie bracket. (Moyal).

$$\text{Standard QM} \xleftrightarrow[\text{Transformation.}]{\text{Wigner-wiel}} \text{Deformation Quantization.}$$

Ordinary QM is given by a set of axioms not necessarily connected to classical mechanics:

- QM generalizes CM in such a way that we recover CM from QM in the limit $\hbar \to 0$.

$\hbar :=$ Plank's constant.

Deformation Quantization is a natural formulation of QM in phase space.

I) Symplectic manifolds (Fedosov)

II) Poisson Manifolds. (Kontsevich).

III) Admissible states of classical Hamiltonian system are defined as (pseudo-) probability distributions on phase space.

## Classic Hamiltonian Mechanics.

**Definition:** A poisson manifold is a smooth manifold $M$ endowed with a real skew-symmetric tensor field of rank $(2,0)$ $P \in \Lambda^2 \Gamma(M)$ satisfying the relation.

$$\mathcal{L}_{X_f} P = 0.$$

for every vector field $X_f := P(df)$ and $f \in C^\infty(M)$.

$P \longrightarrow$ Poisson tensor

$X_f \longrightarrow$ Hamiltonian vector $\qquad P = P^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$

$\mathcal{L} \longrightarrow$ Lie derivative.

$\{X_f\} =: \mathrm{Ham}(M)$

Considering $P$ we define a bracket structure.

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

$$\{f,g\} = P(df,dg), \quad f,g \in C^\infty(M).$$

such that

I) $\{f,g\} = -\{g,f\}$ skew-symmetric.

II) $\{f,gh\} = \{f,g\}h + g\{f,h\}$ Leibniz.

III) $\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$. Jacobi.

- (I) & (III) algebra of $C^\infty(M)$, Poisson algebra (of classical observables.)
- (II) guarantees that for a given function $f \in C^\infty(M) \longrightarrow C^\infty(M)$ is a derivation of the point-product on $C^\infty(M)$, that is. $\{f,\cdot\} =: X_f$ is a vector field.
- Relation with a symplectic form $\omega$

$$\{f,g\} = \omega(P(df),P(dg)) = \omega(X_f, X_g).$$

•) This bracket is called a Poisson bracket.

Properties:

i) Follows from the skew-symmetry of $p$.

ii) Follows from $\{f, gh\} = p(df, d(gh))$.

$d(gh) = (dg)h + g\,dh$ and Linearity.

iii) Follows from the condition $[p, p] = 0$, where

$$[\cdot,\cdot] : \mathcal{X}^p(M) \times \mathcal{X}^q(M) \longrightarrow \mathcal{X}^{p+q-1}(M).$$

denotes the Schouten–Nijenhuis bracket on multivector fields

Let's define $A_c := C^\infty(M)$ the algebra of observables, and define and algebra $\hat{A}_c$ of all operators defined on $C^\infty(M)$ of the form $\hat{A} \in \hat{A}_c \longrightarrow \hat{A} = A$.

Where $A \in A_c$ and $\cdot$ denotes the point-wise product in $C^\infty(M)$.

We may induce a Lie algebra structure in $\hat{A}_c$ by the formula.

$$\{\hat{A}, \hat{B}\} := \{A, B\}\cdot \quad ; \quad \hat{A}, \hat{B} \in \hat{A}_c \;;\; A, B \in A_c$$

Note: A particular real observable $\hat{H}$ governs the time evolution of the system (as we will see below).

Definition: A triple $(M, p, \mathcal{H})$ is called a classic Hamiltonian system.

Definition: Canonical coordinates are local coordinates $q^i, p_i$ $(i = 1, \dots, N = \dim M)$ in which $p$ has the form

$$p = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i} = \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p^i} - \frac{\partial}{\partial p^i} \otimes \frac{\partial}{\partial q^i}$$

Or in components $p^{ij} = \begin{pmatrix} 0_N & \mathbb{I}_n \\ -\mathbb{I}_n & 0_N \end{pmatrix}$

In cardinal coordinates

$$X_f = p(df) = \left(\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}\right)\left(\frac{\partial f}{\partial q^i} dq^j + \frac{\partial f}{\partial p_j} \partial p_j\right)$$

$$= -\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}$$

and

Homework:

$$\{f,g\} = p(df, dg)$$

$$\{f,g\} = X_f(g) = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

Definition: Pure states of a classic Hamiltonian system are points in phase space $M$.

Pure states are caracterized by canonical coordinates $q^i, p_i$.

Not always is possible to Know the exact positions and momenta of a particle.

By thecnical issues on a lab or by considering an ensamble of particles, for example.

Generalize the concept of pure states to probability distributions on $M$, such that for $\xi = (q,p) \in M$. We may define a smooth function $\rho$ on $M$ satisfying:

i) $\rho(\xi) \geq 0$

ii) $\int_M \rho(\xi)\, d\xi = 1$

We will call these states "Mixed" states.

$\rho \to$ Classical distribution function.

from this point of view, pure states may identified with Dirac delta distributions $\rho = \delta(\xi - \xi_0)$.

Definition: A classical expectation value $\langle \hat{A} \rangle_\rho$ of an observable $\hat{A} \in \hat{A}_c$ in a state $\rho$ is defined by

$$\langle \hat{A} \rangle_\rho := \int_M (\hat{A}\rho)(\xi)\, d\xi = \int_M A(\xi)(\xi)\, d\xi.$$

For pure states $\langle \hat{A} \rangle_\delta = \int_M A(\xi) \cdot \delta(\xi - \xi_0)\, d\xi = A(\xi_0)$.

For an operator $\hat{A} \in \hat{A}_c$ we can associate a standard derivation.

$$\sigma_A := \sqrt{\langle \hat{A}^2 \rangle_\rho - \langle \hat{A} \rangle_\rho^2}$$

$$\Rightarrow \quad \sigma_{x_i} \sigma_{p_i} \geq$$

denotes the classical uncertainty relations.

It is possible to simoltaneously measure position and momentum with arbitrary precision.