Second Quantization

Representation of number of ocupation.

$\{|\nu_1\rangle, |\nu_2\rangle, \dots\}$ Complete base of state

$|n_{\nu_1}, n_{\nu_2}, \dots\rangle$ State of N-particles

$$\sum_j n_{\nu_j} = N$$

$$\hat{n}_{\nu_j} |n_{\nu_j}\rangle = n_{\nu_j} |n_{\nu_j}\rangle$$ Number of operator

$$n_{\nu_j} = \begin{cases} 0,1 & \text{Fermions} \\ 0,1,2,\dots & \text{Bosons} \end{cases}$$

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots$$ Fock space

$$\mathcal{F}_N = \text{span}\left\{ |n_{\nu_1}, n_{\nu_2}, \dots\rangle : \sum_j n_{\nu_j} = N \right\}$$

N	Fermions
0	$\|0,0,0,\dots\rangle$
1	$\|1,0,0,\dots\rangle, \|0,1,0\ 0,\dots\rangle, \|0,0,1,0,\dots\rangle, \dots$
2	$\|1,1,0,0,\dots\rangle, \|0,1,1,0,\dots\rangle, \|1,0,1,0,\dots\rangle, \dots$
⋮	

N	Bosons
0	$\|0,0,0,\dots\rangle$
1	$\|1,0,0,\dots\rangle, \|0,1,0\ 0,\dots\rangle$
2	$\|2,0,0,\dots\rangle, \|0,2,0,0,\dots\rangle, \|1,1,0,\dots\rangle$

Creation and annihilation operator for bosons

$$\hat{b}^{\dagger}_{\nu_j} |\dots, n_{\nu_{j-1}}, n_{\nu_j}, n_{\nu_{j+1}}, \dots\rangle$$ Creation operator

$$= B_+(n_{\nu_j})|\ldots, n_{\nu_{j-1}}, n_{\nu_j}+1, n_{\nu_{j+1}}, \ldots\rangle$$

$B_+(n_{\nu_j})$ normalization constant.

$\langle n_{\nu_j}+1|\hat{b}^+_{\nu_j}|n_{\nu_j}\rangle$ Elements $\neq 0$

$$\langle n_{\nu_j}+1|\hat{b}^+_{\nu_j}|n_{\nu_j}\rangle^* = \langle n_{\nu_j}|(\hat{b}^+_{\nu_j})^+|n_{\nu_j}+1\rangle$$

$\hat{b}_{\nu_j} := (b^+_{\nu_j})^+ \rightarrow$ Annihilation operator

$$\hat{b}_{\nu_j}|\ldots, n_{\nu_{j-1}}, n_{\nu_j}, n_{\nu_{j+1}}, \ldots\rangle$$
$$= B_-(n_{\nu_j})|\ldots, n_{\nu_{j-1}}, n_{\nu_j}-1, n_{\nu_{j+1}}, \ldots\rangle$$

$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ commotator.

Is clear that

$$\hat{b}^+_{\nu_j}\hat{b}^+_{\nu_k}|\rangle = \hat{b}^+_{\nu_k}\hat{b}^+_{\nu_j}|\rangle \rightarrow [\hat{b}^+_{\nu_j}, \hat{b}^+_{\nu_k}] = 0$$

and

$$[\hat{b}_{\nu_j}, \hat{b}_{\nu_k}] = 0 \quad \text{if} \quad j \neq k$$

but, if $k=j$ we have to be careful

$$\hat{b}_{\nu_j}|\ldots, 0, \ldots\rangle = 0 \rightarrow B_-(0) = 0$$

and, $\hat{b}^+_{\nu_j}|\ldots, 0, \ldots\rangle = |\ldots, 1, \ldots\rangle$

where we have taken $B_+(0) = 1$,

then,

$$\hat{b}_{\nu_j}\hat{b}^+_{\nu_j}|0\rangle = |0\rangle \neq \hat{b}^+_{\nu_j}\hat{b}_{\nu_j}|0\rangle = 0$$

Summarizing,

$$[\hat{b}^+_{\nu_j}, \hat{b}^+_{\nu_k}] = 0, \quad [\hat{b}_{\nu_j}, \hat{b}_{\nu_k}] = 0, \quad [\hat{b}_{\nu_j}, \hat{b}^+_{\nu_k}] = \delta_{\nu_j,\nu_k}$$

$\hat{b}$ and $\hat{b}^+$ are not hermitian.

$\hat{b}_\nu^+ \hat{b}_\nu$ Operator:

$$[\hat{b}_\nu^+\hat{b}_\nu, \hat{b}_\nu] = \hat{b}_\nu^+\hat{b}_\nu\hat{b}_\nu - \hat{b}_\nu\hat{b}_\nu^+\hat{b}_\nu = \hat{b}_\nu^+\hat{b}_\nu\hat{b}_\nu - (1 + \hat{b}_\nu^+\hat{b}_\nu)\hat{b}_\nu = -\hat{b}_\nu$$

$$[\hat{b}_\nu^+\hat{b}_\nu, \hat{b}_\nu^+] = \hat{b}_\nu^+\hat{b}_\nu\hat{b}_\nu^+ - \hat{b}_\nu^+\hat{b}_\nu^+\hat{b}_\nu = \hat{b}_\nu^+\hat{b}_\nu\hat{b}_\nu^+ - \hat{b}_\nu^+(\hat{b}_\nu\hat{b}_\nu^+ - 1) = \hat{b}_\nu^+$$

$\langle\phi|\hat{b}_\nu^+\hat{b}_\nu|\phi\rangle \geq 0$ Norm of $\hat{b}_\nu|\phi\rangle$

let

$$\hat{b}_\nu^+\hat{b}_\nu|\phi_\lambda\rangle = \lambda|\phi_\lambda\rangle \quad \text{with } \lambda > 0$$

let us take $\lambda = \lambda_0$ particular

$$(\hat{b}_\nu^+ \hat{b}_\nu^+)\hat{b}_\nu|\phi_{\lambda_0}\rangle = (\hat{b}_\nu\hat{b}_\nu^+ - 1)\hat{b}_\nu|\phi_{\lambda_0}\rangle$$

$$= \hat{b}_\nu(\hat{b}_\nu^+\hat{b}_\nu - 1)|\phi_{\lambda_0}\rangle = \hat{b}_\nu(\lambda_0 - 1)|\phi_{\lambda_0}\rangle = (\lambda_0 - 1)\hat{b}_\nu|\phi_{\lambda_0}\rangle$$

therefore,

$$(\hat{b}_\nu^+\hat{b}_\nu)\hat{b}_\nu|\phi_{\lambda_0}\rangle = (\lambda_0 - 1)\hat{b}_\nu|\phi_{\lambda_0}\rangle$$

Eigenvalues equation

$$(\hat{b}_\nu^+\hat{b}_\nu)\hat{b}_\nu\hat{b}_\nu|\phi_{\lambda_0}\rangle = (\hat{b}_\nu\hat{b}_\nu^+ - 1)\hat{b}_\nu\hat{b}_\nu|\phi_{\lambda_0}\rangle$$

$$= \hat{b}_\nu(\hat{b}_\nu^+\hat{b} - 1)\hat{b}_\nu|\phi_{\lambda_0}\rangle$$

$$= \hat{b}_\nu(\lambda_0 - 2)\hat{b}_\nu|\phi_{\lambda_0}\rangle = (\lambda_0 - 2)\hat{b}_\nu\hat{b}_\nu|\phi_{\lambda_0}\rangle$$

then

$$\hat{b}_\nu\hat{b}_\nu(\lambda_0 - 2)|\phi_{\lambda_0}\rangle$$

$$\rightarrow (\hat{b}_\nu^+\hat{b}_\nu)\hat{b}_\nu^2|\phi_{\lambda_0}\rangle = (\lambda_0 - 2)\hat{b}_\nu^2|\phi_{\lambda_0}\rangle$$

$$\rightarrow \hat{b}_\nu^+\hat{b}_\nu(\hat{b}_\nu^m|\phi_{\lambda_0}\rangle) = (\lambda_0 - m)(\hat{b}_\nu^m|\phi_{\lambda_0}\rangle)$$

If $\lambda_0 \in \mathbb{Z}^+ \rightarrow \exists\, m = \lambda_0$ such that $\hat{b}_\nu^+\hat{b}_\nu(\hat{b}_\nu^m|\phi_{\lambda_0}\rangle) = 0$

The process ends.

If $\lambda_0 \notin \mathbb{Z}^+$ $\exists\, m$ such that $\hat{b}_\nu^+\hat{b}_\nu(\hat{b}_\nu^m|\phi_{\lambda_0}\rangle) < 0$

then $\lambda < 0$! $\rightarrow$ Contradiction therefore

$$\lambda := n = 0, 1, 2, \ldots$$

$$|\phi_\lambda\rangle := |n_\nu\rangle \rightarrow \hat{b}_\nu^\dagger \hat{b}_\nu |n_\nu\rangle = n_\nu |n_\nu\rangle$$ number operator

and

$$\hat{b}_\nu |n_\nu\rangle \sim |n_\nu - 1\rangle$$

Now,

$$\hat{b}_\nu^\dagger \hat{b}_\nu \hat{b}_\nu^\dagger |\phi_{\lambda_0}\rangle = \hat{b}_\nu^\dagger (1 + \hat{b}_\nu^\dagger \hat{b}_\nu) |\phi_{\lambda_0}\rangle$$
$$= \hat{b}_\nu (1 + \lambda_0) |\phi_{\lambda_0}\rangle$$
$$= (\lambda_0 + 1) \hat{b}_\nu^\dagger |\phi_{\lambda_0}\rangle$$

then $$\hat{b}_\nu^\dagger \hat{b}_\nu \hat{b}_\nu^{\dagger n} |\phi_{\lambda_0}\rangle = (\lambda_0 + n) \hat{b}_\nu^{\dagger n} |\phi_{\lambda_0}\rangle$$

thus, $$(\hat{b}_\nu^\dagger \hat{b}_\nu) \hat{b}_\nu^\dagger |n_\nu\rangle = (n+1) \hat{b}_\nu^\dagger |n_\nu\rangle$$

Finally $$\hat{b}_\nu^\dagger |n_\nu\rangle \sim |n_\nu + 1\rangle$$

So, we have to

$$\| \hat{b}_\nu |n_\nu\rangle \|^2 = (\hat{b}_\nu |n_\nu\rangle)^\dagger (\hat{b}_\nu |n_\nu\rangle) = \langle n_\nu | \hat{b}_\nu^\dagger \hat{b}_\nu | n_\nu\rangle = n_\nu$$
$$\| \hat{b}_\nu^\dagger |n_\nu\rangle \|^2 = (\hat{b}_\nu^\dagger |n_\nu\rangle)^\dagger (\hat{b}_\nu^\dagger |n_\nu\rangle) = \langle n_\nu | \hat{b}_\nu \hat{b}_\nu^\dagger | n_\nu\rangle = n_\nu + 1$$

$\hat{b}_\nu^\dagger \hat{b}_\nu = \hat{n}_\nu$ $\quad\quad$ $\hat{b}_\nu^\dagger \hat{b}_\nu |n_\nu\rangle = n_\nu |n_\nu\rangle$ $\quad\quad$ $n_\nu = 0$

$$\hat{b}_\nu |n_\nu\rangle = \sqrt{n_\nu}\, |n_\nu - 1\rangle$$
$$\hat{b}_\nu^\dagger |n_\nu\rangle = \sqrt{n_\nu + 1}\, |n_\nu + 1\rangle$$
$$(\hat{b}_\nu^\dagger)^{n_\nu} |0\rangle = \sqrt{0+1}\sqrt{1+1}\sqrt{2+1} \cdots \sqrt{n_\nu - 1 + 1}\, |n_\nu\rangle$$
$$= \sqrt{1 \cdot 2 \cdot 3 \cdots n_\nu}\, |n_\nu\rangle = \sqrt{n_\nu !}\, |n_\nu\rangle$$

Equivalence between states in first and second quantization.

$$\hat{S}_+ |\nu_{n_1}\rangle_1 |\nu_{n_2}\rangle \cdots |\nu_{n_N}\rangle_N \longleftrightarrow \hat{b}_{\nu_{n_1}}^\dagger \hat{b}_{\nu_{n_2}}^\dagger \cdots \hat{b}_{\nu_{n_N}}^\dagger |0\rangle$$