# Symmetric and antisymmetric tensors.

Let $t_{ab}$ be a symmetric tensor ($t_{ab}=t_{ba}$). Its spinorial equivalent

$$t_{A\dot{A}B\dot{B}}=t_{B\dot{B}A\dot{A}}$$

But, $t_{A\dot{A}B\dot{B}}$ not necessarily coincides with $t_{B\dot{A}A\dot{B}}$, that is equal to $t_{A\dot{B}B\dot{A}}$.

Using antisymmetry.

$$t_{A\dot{A}B\dot{B}}-t_{B\dot{A}A\dot{B}}=\varepsilon_{AB}\,t^{R}{}_{\dot{A}R\dot{B}}$$

then

$$t^{R}{}_{\dot{A}R\dot{B}}=t_{R\dot{B}}{}^{R}{}_{\dot{A}}=-t^{R}{}_{\dot{B}R\dot{A}}$$

and then $t^{R}{}_{\dot{A}R\dot{B}}$ is proportional a $\varepsilon_{\dot{A}\dot{B}}$, then.

$$t_{A\dot{A}B\dot{B}}-t_{B\dot{A}A\dot{B}}=\varepsilon_{AB}\,\frac{1}{2}\,t^{R\dot{S}}{}_{R\dot{S}}\,\varepsilon_{\dot{A}\dot{B}}$$

$$=-\frac{1}{2}\,t^{S}{}_{S}\,\varepsilon_{AB}\,\varepsilon_{\dot{A}\dot{B}}$$

which implies $t_{A\dot{A}B\dot{B}}=t_{B\dot{A}A\dot{B}}=t_{A\dot{B}B\dot{A}}$ if and only if, $t_{A\dot{A}B\dot{B}}$ is the equivalent spinor to a symmetric tensor without trace.

The equivalent tensor into symmetric tensor $t_{ab}=-t_{ba}$, satisfies $t_{A\dot{A}B\dot{B}}=-t_{B\dot{B}A\dot{A}}$, thus.

$$t_{A\dot{A}B\dot{B}}=\frac{1}{2}\left(t_{A\dot{A}B\dot{B}}+t_{B\dot{A}A\dot{B}}\right)+\frac{1}{2}\left(t_{A\dot{A}B\dot{B}}-t_{B\dot{A}A\dot{B}}\right)$$

$$=\frac{1}{2}\left(t_{A\dot{A}B\dot{B}}-t_{A\dot{B}B\dot{A}}\right)+\frac{1}{2}\left(t_{A\dot{A}B\dot{B}}-t_{B\dot{A}A\dot{B}}\right)$$

$$=\frac{1}{2}\,t_{A}{}^{\dot{R}}{}_{B\dot{R}}\,\varepsilon_{\dot{A}\dot{B}}+\frac{1}{2}\,t^{R}{}_{\dot{A}R\dot{B}}\,\varepsilon_{AB}$$

$$=\tau_{AB}\,\varepsilon_{\dot{A}\dot{B}}+\tau_{\dot{A}\dot{B}}\,\varepsilon_{AB}.$$

with $\tau_{AB}=\frac{1}{2}\,t_{A}{}^{\dot{R}}{}_{B\dot{R}}$, $\tau_{\dot{A}\dot{B}}=\frac{1}{2}\,t^{R}{}_{\dot{A}R\dot{B}}$

Due to that $t_{A\dot{A}B\dot{B}}=-t_{B\dot{B}A\dot{A}}$ such that $\psi^{A}\phi_{A}=-\psi_{A}\phi^{A}$; $\tau_{AB}$, $\tau_{\dot{A}\dot{B}}$ are symmetric in its indices such that $t_{B}{}^{\dot{R}}{}_{A\dot{R}}=-t_{A\dot{R}B}{}^{\dot{R}}$.

In particular the spinorial equivalent of an antisymmetric product. $V_aW_b - V_bW_a$ is

$$V_{A\dot A}W_{B\dot B} - V_{B\dot B}W_{A\dot A} = \frac{1}{2}(V_A{}^{\dot R}W_{B\dot R} - V_{B\dot R}W_A{}^{R})\varepsilon_{\dot A\dot B}$$

$$+\frac{1}{2}(V^R{}_{\dot A}W_{R\dot B} - V_{R\dot B}W^R{}_{\dot A})\varepsilon_{AB}$$

$$= V_{(A}{}^{\dot R}W_{B)\dot R}\,\varepsilon_{\dot A\dot B} + V^R{}_{(\dot A}W_{|R|\dot B)}\,\varepsilon_{AB}.$$

where $\xi_{(AB)} = \frac{1}{2}(\xi_{AB} + \xi_{BA})$.

If $t_{abc}$, totally antisymmetric ($t_{abc} = -t_{acb} = t_{cba}$ ...) is antisymmetric in its two last indices.

$$t_{A\dot AB\dot BC\dot C} = \mu_{A\dot ABC}\,\varepsilon_{\dot B\dot C} + \mu_{A\dot A\dot B\dot C}\,\varepsilon_{BC} \qquad (*)$$

with

$$\mu_{A\dot ABC} = \mu_{A\dot A(BC)} \qquad\qquad \mu_{A\dot A\dot B\dot C} = \mu_{A\dot A(\dot B\dot C)}$$

using the antisymmetry in his pair of the indices.

$$\mu_{A\dot ABC}\,\varepsilon_{\dot B\dot C} + \mu_{A\dot A\dot B\dot C}\,\varepsilon_{BC} = -\mu_{B\dot BAC}\,\varepsilon_{\dot A\dot C} - \mu_{B\dot B\dot A\dot C}\,\varepsilon_{AC}.$$

Contracting with $\varepsilon^{\dot B\dot C}$

$$\Rightarrow 2\mu_{A\dot AB\dot C} = -\mu_B{}^{\dot B}{}_{A\dot C}\,\varepsilon_{\dot A\dot B} - \mu_B{}^{\dot B}{}_{A\dot B}\,\varepsilon_{AC}$$

$$= -\mu_{B\dot AA\dot C} - \mu_B{}^{\dot B}{}_{A\dot B}\,\varepsilon_{AC}$$

$$= -\mu_{A\dot ABC} + \mu_{A\dot ABC} - \mu_{B\dot AAC} - \mu_B{}^{\dot R}{}_{\dot AR}\,\varepsilon_{AC}$$

$$\Rightarrow 3\mu_{A\dot AB\dot C} = \mu^R{}_{\dot AR\dot C}\,\varepsilon_{AB} - \mu_B{}^{\dot R}{}_{A\dot R}\,\varepsilon_{\dot A\dot C}$$

Contracting with $\varepsilon^{BC}$

$$0 = -\mu^R{}_{\dot ARA} - \mu_A{}^{\dot R}{}_{\dot A\dot R}.$$

and defining

$$t_{B\dot A} = \frac{2}{3}\mu^R{}_{\dot ARB}$$

$$\Rightarrow 2\mu_{A\dot ABC} = t_{C\dot A}\,\varepsilon_{AB} + t_{B\dot A}\,\varepsilon_{AC}$$

and changing in (*)

$$t_{A\dot{A}B\dot{B}C\dot{C}} = t_{B\dot{A}}\,\varepsilon_{AC}\,\varepsilon_{\dot{B}\dot{C}} - t_{A\dot{B}}\,\varepsilon_{BC}\,\varepsilon_{\dot{A}\dot{C}}$$

If $t_{abcd}$ totally antisymmetric.

$$t_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = \Phi_{B\dot{A}D\dot{D}}\,\varepsilon_{AC}\,\varepsilon_{\dot{B}\dot{C}} - \Phi_{A\dot{B}D\dot{D}}\,\varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{BC} \quad (**)$$

$\Rightarrow$

$$\Phi_{B\dot{A}D\dot{D}}\,\varepsilon_{AC}\,\varepsilon_{\dot{B}\dot{C}} - \Phi_{A\dot{B}D\dot{D}}\,\varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{BC} = -\Phi_{B\dot{A}C\dot{C}}\,\varepsilon_{AD}\,\varepsilon_{\dot{B}\dot{D}} + \Phi_{A\dot{B}C\dot{C}}\,\varepsilon_{\dot{A}\dot{D}}\,\varepsilon_{BD}$$

Contracting with $\varepsilon^{AC}$.

$$2\Phi_{B\dot{A}D\dot{D}}\,\varepsilon_{\dot{B}\dot{C}} - \Phi_{B\dot{B}D\dot{D}}\,\varepsilon_{\dot{A}\dot{C}} = -\Phi_{B\dot{A}D\dot{C}}\,\varepsilon_{\dot{B}\dot{D}} + \Phi^{R}{}_{\dot{B}R\dot{C}}\,\varepsilon_{\dot{A}\dot{D}}\,\varepsilon_{BD}$$

and contracting with $\varepsilon^{\dot{A}\dot{C}}$

$$0 = -\Phi_{B}{}^{\dot{R}}{}_{D\dot{R}}\,\varepsilon_{\dot{B}\dot{D}} + \Phi^{R}{}_{\dot{B}R\dot{D}}\,\varepsilon_{BD}$$

$$\Rightarrow \qquad \Phi_{B}{}^{\dot{R}}{}_{D\dot{R}} = 2a\,\varepsilon_{BD}\ , \quad \Phi^{R}{}_{\dot{B}R\dot{D}} = 2a\,\varepsilon_{\dot{B}\dot{D}}$$

for some scalar $a$.

In the other hand, contracting with $\varepsilon^{\dot{B}\dot{D}}$

$$4\Phi_{B\dot{A}D\dot{C}} = \Phi_{B}{}^{\dot{B}}{}_{D\dot{B}}\,\varepsilon_{\dot{A}\dot{C}} + \Phi^{R\dot{B}}{}_{R\dot{C}}\,\varepsilon_{\dot{A}\dot{B}}\,\varepsilon_{BD}.$$

$$= 2a(\varepsilon_{BD}\,\varepsilon_{\dot{A}\dot{C}} + \varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{BD})$$

$$= 4a\,\varepsilon_{BD}\,\varepsilon_{\dot{A}\dot{C}}$$

changing in $(**)$

$$t_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = a(\varepsilon_{BD}\,\varepsilon_{\dot{A}\dot{D}}\,\varepsilon_{AC}\,\varepsilon_{\dot{B}\dot{C}} - \varepsilon_{AD}\,\varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{BC}\,\varepsilon_{\dot{B}\dot{D}})$$

The equivalent tensor to the Levi-Civita's symbol $\varepsilon_{abcd}$, is $\varepsilon_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}}$, $\varepsilon_{1234}=1$, must be proportional to.

$$\varepsilon_{AC}\,\varepsilon_{BD}\,\varepsilon_{\dot{A}\dot{D}}\,\varepsilon_{\dot{B}\dot{C}} - \varepsilon_{AD}\,\varepsilon_{BC}\,\varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{\dot{B}\dot{D}}$$

as $\varepsilon^{abcd}\varepsilon_{abcd} = (-1)^{q}\ 24$ where $q$ is the signature

$$\varepsilon_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = i^{q}(\varepsilon_{AC}\,\varepsilon_{BD}\,\varepsilon_{\dot{A}\dot{D}}\,\varepsilon_{\dot{B}\dot{C}} - \varepsilon_{AD}\,\varepsilon_{BC}\,\varepsilon_{\dot{A}\dot{C}}\,\varepsilon_{\dot{B}\dot{D}})$$

The spinorial equivalent of the dual of an antisymmetric tensor, defined.

$$^{*}t_{ab} \equiv \frac{1}{2}\,\varepsilon_{abcd}\,t^{cd}$$

$$*t_{A\dot{A}B\dot{B}} = i^{q}(-\tau_{AB}\varepsilon_{\dot{A}\dot{B}} + \tau_{\dot{A}\dot{B}}\varepsilon_{AB})$$

Parte Autodual. Parte Dual.

$$*t_{ab}t^{ab} = -i^{q}(\tau_{AB}\varepsilon_{\dot{A}\dot{B}} - \tau_{\dot{A}\dot{B}}\varepsilon_{AB})(\tau^{AB}\varepsilon^{\dot{A}\dot{B}} - \tau^{\dot{A}\dot{B}}\varepsilon^{AB})$$

$$= -2i^{q}(\tau_{AB}\tau^{AB} - \tau_{\dot{A}\dot{B}}\tau^{\dot{A}\dot{B}})$$

A bivector its called simple if there are $v_a, w_a$, such that

$$t_{ab} = v_a w_b - v_b w_a$$

Then, for a simple bivector.

$$*t_{ab}t^{ab} = \frac{1}{2}\varepsilon_{abcd}t^{cd}t^{ab}$$

$$= 2\,\varepsilon_{abcd}v^{c}w^{d}v^{a}w^{b} = 0$$

Proposition: If the bivector $t_{ab}$ is simple

$$\tau_{AB}\tau^{AB} = \tau_{\dot{A}\dot{B}}\tau^{\dot{A}\dot{B}}$$