Self-Adjoint Operators

A bounded operator $A:\mathcal{H}\to\mathcal{H}$, in a Hilbert space $\mathcal{H}$ is called self-adjoint or symmetric if and only if for all $x,y\in\mathcal{H}$,

$$\langle Ax,y\rangle=\langle x,Ay\rangle$$

Proposition: The point spectrum σ_p of a self-adjoint operator A satisfies

$$\sigma_p(A)\subseteq\mathbb{R}$$

Proof: Let $\lambda\in\sigma_p$ (eigenvalues), $Ax=\lambda x$ $(x\neq 0)$

$$\lambda\|x\|^2=\langle Ax,x\rangle=\langle x,Ax\rangle=\langle x,\lambda x\rangle$$
$$=\overline{\lambda}\,\|x\|^2$$

then $\lambda=\overline{\lambda}$, therefore $\lambda\in\mathbb{R}$.

$\blacksquare$

Proposition: If $\lambda_1,\lambda_2\in\sigma_p$ and $Ax_1=\lambda_1x_1$, $Ax_2=\lambda_2x_2$, then $x_1\perp x_2$.

Proof: Let

$$\lambda_1\langle x_1,x_2\rangle=\langle Ax_1,x_2\rangle=\langle x_1,Ax_2\rangle=\overline{\lambda_2}\langle x_1,x_2\rangle$$

but $\lambda_1\neq\lambda_2$

$$\langle x_1,x_2\rangle=0.$$

$\blacksquare$

A subspace L is invariant with respect to an operator A if and only if $A(L)\subseteq L$

Proposition: If L is invariant with respect to a self-adjoint operator A_r then $L^\perp$ is invariant.

Proposition: If A and B are self-adjoint and $AB=BA$, AB is self-adjoint

Proposition: Let $C=\sup\frac{|\langle Ax,x\rangle|}{\|x\|^2}$, then if A is self-adjoint, $C=\|A\|$.

Proof: first we will show $C\leq\|A\|$.

$$|\langle Ax,x\rangle|\leq\|Ax\|\,\|x\|\leq\|A\|\,\|x\|^2\rightarrow\frac{|\langle Ax,x\rangle|}{\|x\|^2}\leq\|A\|,\ \forall x\neq 0$$

then $C\leq\|A\|$.

The other direction

$$\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle = 2(\langle Ax, y\rangle + \langle Ay, x\rangle)$$

by the triangle inequality

$$2|\langle Ax,y\rangle + \langle Ay,x\rangle| \leq |\langle A(x+y), x+y\rangle| + |\langle A(x-y), x-y\rangle|$$

from the definition of C and by the parallelogram law

$$|\langle Ax,y\rangle + \langle Ay,x\rangle| \leq \frac{1}{2} C(\|x+y\|^2 + \|x-y\|^2)$$

$$= \frac{1}{2} C(\|x\|^2 + \|y\|^2)$$

taking any $x \in \mathcal{H}$, $\|x\|=1$, $y = \frac{Ax}{\|Ax\|} \leq C$, $\forall x \in \mathcal{H}$, $\|x\|=1$.

Therefore $\|A\| \leq C$.

$\blacksquare$

Proposition: $\langle Ax, x\rangle \in \mathbb{R}$, $\forall x \in \mathcal{H}$ if and only if A is self-adjoint.

Proof: Let

$$4\langle Ax, y\rangle = \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle$$
$$+ i\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle$$
$$= 4\langle x, Ay\rangle$$

changing positions of x, y and taking the complex conjugate the right hand side does not change.

$$\overline{\langle Ay, x\rangle} = \langle x, Ay\rangle$$

therefore

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

$\blacksquare$

Proposition: Let A be a self-adjoint operator, and let

$$\|A\| = \mu = \sup\{|\langle Ax, x\rangle| : \|x\|=1\}$$

then μ or $-\mu$ is an element of $\sigma(A)$.

Proof: let x_n, $\|x_n\|=1$ and $|\langle Ax_n, x_n\rangle| \longrightarrow \mu$ (which is possible because $c=\|A\|$).

let

$$\langle Ax_n, x_n\rangle \longrightarrow \lambda \quad , \quad \lambda = \pm\mu$$

$$0 \geqslant \|Ax_n - \lambda x_n\|^2 = \|Ax_n\|^2 - 2\lambda\langle Ax_n, x_n\rangle + \lambda^2\|x_n\|^2 \qquad (\leq \lambda^2)$$

$$\leq 2\lambda^2 - 2\lambda\langle Ax_n, x_n\rangle$$

taking $n \longrightarrow \infty$, the right hand side converge to zero, then $\lambda \in \sigma(A)$.

Self-Adjoint Compact operators

Proposition: If A is compact, self-adjoint, then A has a eigenvalue λ, such that $|\lambda| = \|A\|$, moreover, the maximum of $|\langle Ax, x\rangle|$ for $\|x\|=1$, is obtained in the eigenvector with eigenvalue λ.

Proof: the existence of $\lambda \in \sigma_p(A)$, such that $|\lambda| = \|A\|$, it follows the above proposition, but $\sigma(A) = \sigma_p(A) \cup \{0\}$.

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