Bibliography:

Kreyszig, Introductory Functional Analysis with Applications.

Linear Spaces (Vector spaces)

Let's study linear spaces E, over a field $\mathbb{R}$ or $\mathbb{C}$ or in a general way $\mathbb{F}$. The simplest example of linear spaces are polynomials of grade less or equal than n, etc. Other example is $C[a,b]$, the set of real continuous functions (or complex). that takes values in $[a,b]$

Definition: A map $A: E_1 \longrightarrow E_2$, between two linear spaces E_1 and E_2, is linear if and only if for each $x,y \in E_1$, there are two scalars a,b such that

$$A(ax+by) = aA(x) + bB(x) \quad ; \quad A(x) = Ax.$$

two important sets are:

$$\operatorname{Ker} A = \} x \in E_1 : Ax = 0 \{$$

$$\operatorname{Im} A = \} Ax : x \in E_1 \{$$

A linear map $A: E_1 \longrightarrow E_2$ it is said that is an isomorphism if $\operatorname{Ker} A = 0$ and $\operatorname{Im} A = E_2$. i.e., if A is one-to-one (injective) and surjective; consistently is reversible and we write as A^{-1}.

Example:

I. Let S^* be the set of successions with finite support i.e., successions whose elements are cero, except a finite number of them. In other words, the succesion

$$(a_i)_{i=1}^{\infty} \in S^*$$

if and only if there is $N \in \mathbb{N}$, such that $a_i = 0$, for all $i > N$.

II. The set C_0, of the successions that tend to cero.

III. The set C, of the convergent successions.

IV. The set l_∞, bounded successions.

V. The set S, of the successions.

$$S^* \subseteq C_0 \subseteq C \subseteq l_\infty \subseteq S$$

Definition: A subset E_1, of a linear space E, it is called subspace of E if it is closed with respect to the operations of E, and we write

$$E_1 \hookrightarrow E$$

Definition: We define the linear generator of a subset M, of a linear space E, as the intersection of all subspace of E that contains to M, i.e.,

$$\operatorname{span} M = \bigcap_{\alpha} \{ E_\alpha \hookrightarrow E \text{ and } M \subseteq E_\alpha \}$$

other way to say $\operatorname{span} M$, is as the set of all linear combinations of the vectors of M.

Vector space Quotient

Definition: For a subspace E_1 from E, we define a new linear spaco called quotient space of E with respect to E_1, in the following way

Theorem: Let's consider the colection of the following subsets

$$[x] = \{ x + E_1 : x \in E \}$$

the sets $[x]$ are called cosets of E. We see that two cosets $[x]$ and $[y]$ match or are disjuntist.

Proof: In fact

$$z \in [x] \cap [y],$$

then

$$z - x, z - y \in E_1$$

as E_1 is a linear space

$$y - x = (z - x) - (z - y) \in E_1$$

Thus if $a \in [x]$ we have that $a - x \in E_1$, and by linearity

$$a - y = (a - x) - (y - x) \in E_1$$

and $a \in [y]$. We have proved that $[x] \subseteq [y]$.

In a similar way $[y] \subseteq [x]$. Therefore $[x] = [y]$ $\blacksquare$

We denote E/E_1 to the colection of all the cosets $[x]$.
Let's introduce a linear structure in E/E_1.

$$[x]+[y]=[x+y]$$

$$a[x]=[ax]$$

Let's note that $[0]$ is the vector cero in the new space E/E_1.
It is said that E/E_1 is a quotient vector space.
the dimension of E/E_1 is called the codimension of E_1 with respect to E,

$$\operatorname{codim}_E E_1=\dim E/E_1$$

Normed spaces

Definition: A norm $\|x\|$, for $x\in E$, is a function of E in $\mathbb{R}$, such that satisfies

1. $\|x\|>0$ and $\|x\|=0$ if and only if $x=0$.
2. $\|\lambda x\|=|\lambda|\,\|x\|$
3. $\|x+y\|\leq\|x\|+\|y\|$ (Triangle inequality)

for all $x,y\in E$, $\lambda\in\mathbb{R}$ (or $\mathbb{C}$).

With this definition, the distance $d(x,y)$ between two dots $x,y\in E$, it is obtained by $\|x-y\|$, and the third property tell us that.

$$d(x,y)=\|x-y\|\leq\|x-z\|+\|z-y\|$$

$$=d(x,z)+d(z,y)$$

let's denote by $\underline{X}$, a vector space E, equiped by a norm $\|\cdot\|$.

$$\underline{X}=(E,\|\cdot\|)$$

and we will call normed space.

if $x\in\mathbb{R}^2$, $x=(x_1,x_2)$, then

$$\|x\|_1=\sqrt{x_1^2+x_2^2}$$

$$\|x\|_2=\sup(x_1,x_2)$$

Definition: The norm $\|\cdot\|_1$ is said that is "stronger" than the norm $\|\cdot\|_2$ if there exist a constan $c>0$, such that.

$$\|x\|_2 \leq c\|x\|_1, \quad \forall x \in E.$$

The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ is said that they are equivalents if there are constants $C, c > 0$, such that.

$$c\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2, \quad \forall x \in E$$

Examples:

I. In the spaces c_0, c, l_∞ we define the norm $\|\cdot\|$, as the supremum of the absolute value of the terms of the successions,

$$x = (a_i)_{i=1}^{\infty}, \quad \|x\| = \sup |a_i|$$

II. for $C[0,1]$ a norm

$$\|f\| = \max\{|f(t)| : t \in [0,1]\}$$

III. Let $l_1 = (\mathbb{R}^\infty, \|\cdot\|_1)$, the successions, $x = (a_i)_{i=1}^{\infty}$ that satisfies

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i| < \infty$$

In a similar way $l_p = (\mathbb{R}^\infty, \|x\|_p)$ con $1 \leq p \leq \infty$

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty$$

Theorem: (Hölder inequality).

For all successions of scalars (a_i) and (b_i) and $1<p<\infty$, is fulfilled

$$\left|\sum_k a_k b_k\right| \leq \sum_k |a_k b_k| \leq \left(\sum_k |a_k|^p\right)^{1/p} \left(\sum_k |b_k|^q\right)^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$

Proof: let's see the following relation

$$\frac{1}{p-1} = q-1 \rightarrow 1 = (q-1)(p-1) \rightarrow 0 = qp - p - q \rightarrow p + q = qp \rightarrow \frac{1}{q} + \frac{1}{q} = 1.$$

$$(p-1)q = p \rightarrow pq - q = p \rightarrow pq = p + q \rightarrow 1 = \frac{1}{q} + \frac{1}{p}$$