Differential calculus in Banach spaces

Gateaux derivative: Let X, Y be Banach spaces on $\mathbb{R}$ and let $S: X \to Y$ be a operator.

Definition: (Gateaux derivative) Let $x, t \in X$

$$\lim_{\eta \to 0} \left\| \frac{S(x+\eta t) - S(t)}{\eta} \right\| = 0$$

$\forall t \in X$, with $\eta \to 0$ in $\mathbb{R}$. $DS(x)t \in Y$ is called the derivative of Gateaux of S, in x, in the t direction.

Properties:

I. If S is linear

$$DS(x)t = S(t) \quad \text{i.e.,} \quad DS(x) = S \quad , \forall x \in X$$

II. If F is a function in $\mathbb{R}$, that acts on X, i.e., $S: X \to \mathbb{R}$, then

$$DS(x)t = \left[\frac{d}{d\eta} F(x+\eta t) \right]_{\eta=0}$$

Theorem: The Gateaux derivative of an operator S is unique, if this exists.

Lemma: Let $S_1(t)$ and $S_2(t)$ satisfying the derivative definition. Then for all $t \in X$, and $\eta > 0$.

$$\|S_1(t) - S_2(t)\| = \left\| \frac{S(x+\eta t) - S(x)}{\eta} - S_1(t) - \left(\frac{S(x+\eta t) - S(x)}{\eta} - S_2(t) \right) \right\|$$

$$\leq \left\| \frac{S(x+\eta t) - S(x)}{\eta} - S_1(t) \right\| + \left\| \frac{S(x+\eta t) - S(x)}{\eta} - S_2(t) \right\|$$

$\to 0$ if $\eta \to 0$.

$\blacksquare$

Example: Let $X = \mathbb{R}^n$, $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$

$$DF(x)e_i = \frac{\partial F}{\partial x_i} \quad ; \quad i = 1, \ldots, n.$$

Definition: (Fréchet derivative): Let x be a point in a Banach space X, and Y other Banach space. A linear continuous operator $S: X \to Y$ in x if

$$S(x+t) - S(t) = T(t) + \varphi(x, t)$$

and

$$\lim_{\|t\| \to 0} \frac{\|\varphi(x,t)\|}{\|t\|} = 0$$

or equivalently

$$\lim_{\|t\| \to 0} \left\| \frac{S(x+t) - S(x)}{\|t\|} - T(t) \right\| = 0$$

Theorem: If an operator has a Fréchet derivative in a point, then has a Gateaux derivative in that point and both are equals.

Proof: Let's $S : X \longrightarrow Y$ and suppose that S has a Fréchet derivative in X, then

$$\lim_{\|t\| \to 0} \left\| \frac{S(x+t) - S(x)}{\|t\|} - T(t) \right\| = 0$$

for some value $t \in X$, non-zero

$$\lim_{\eta \to 0} \left\| \frac{S(x+\eta t) - S(x)}{\eta} - T(t) \right\| = \lim_{\eta \to 0} \left\| \frac{S(x+\eta t) - S(x)}{\|\eta t\|} - T(\eta t) \right\| \|t\|$$

So S has Gateaux derivative in x and is T. ∎

Theorem: Let Ω be an open in X, and $S : \Omega \longrightarrow Y$ with Fréchet derivative in any point of Ω. Then, S is continuous in Ω. This means that any operator Fréchet differentiable defined in an open from a Banach space, is continuous.

Proof: For $a \in \Omega$, let $\varepsilon > 0$ such that $a + t \in \Omega$, if $\|t\| < \varepsilon$

$$\|S(a+t) - S(a)\| = \|T(t) + \varphi(a,t)\| \longrightarrow 0 \quad \text{if} \quad \|t\| \longrightarrow 0$$

So, S is continuous in a. ∎

Homework: $S : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$S(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if} \quad x \neq 0,\ y \neq 0. \\ 0 & \text{if} \quad x = 0,\ y = 0. \end{cases}$$

S has Gateaux derivative in $(0,0)$ and is zero, but it has not Fréchet derivative.

Distributions Basic Theory

$$\delta(x-x_0)=\begin{cases}\infty & \text{if} \quad x=x_0\\ 0 & \text{if} \quad x\neq x_0\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(x-x_0)\,dx=1,\quad \int_{-\infty}^{\infty}f(x)\,\delta(x-x_0)\,dx=f(x_0)$$

1950 - Laurent Schwartz

Distributions

Let n be a positive integer, a vector of n-tuples $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ where α_i, $i=1,\ldots,n$ are non-negative integers, is called multi-index, of dimension n, such that

$$|\alpha|=\sum_{i=1}^{n}\alpha_i$$

is called the magnitude of the multi-index.

$$x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},\quad \text{with} \quad x=(x_1,\ldots,x_n)\in\mathbb{R}^n$$

$$D^\alpha f=\frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}}f \quad \text{with order } |\alpha|.$$

If $n=2$, $\alpha_1=1$, $\alpha_2=1$.

$$D^\alpha f=\frac{\partial^2}{\partial x_1\partial x_2}=D^{(1,1)}f$$

All functions will be define in a bounded subset $\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ with boundary $\partial\Omega$, thus $\overline{\Omega}=\Omega+\partial\Omega$

$$K=\operatorname{supp} f=\{x\in\Omega: f(x)\neq 0\}$$

called support of f. If K is compact, it is said that f has compact support.

The space $C_0^\infty(\Omega)$, of functions with compact support and it derivatives of all order be continuous in a vector space. A sequence $\{\varphi_n\}\in C_0^\infty$, is said that converges to $\varphi\in C_0^\infty(\Omega)$, $\varphi_n\longrightarrow\varphi$ if

1. There exists a fixed compact $K\subseteq\Omega$, such that

$$\operatorname{supp}\varphi_n\subseteq K,\quad \forall n.$$

II. φ_n and all it derivatives converging uniformly to $\varphi(K)$ and it derivatives i.e.,

$$D^\alpha \varphi_n \longrightarrow D^\alpha \varphi \qquad \forall \alpha \text{ in a uniformly.}$$

Definition (Test functions): Let $C_0^\infty(\Omega)$ equiped with the induced topology by the convergence, is called the space of test functions and they are denoted by $D(\Omega)$.

Definition: A linear functional defined in $D(\Omega)$ is Known as a distribution $D^*(\Omega)$.

1. $F(\varphi+\psi)=F(\varphi)+F(\psi)$

II. $F(\alpha\varphi)=\alpha F(\varphi), \quad \alpha\in\mathbb{R}.$

III. If $\varphi_n\longrightarrow\varphi$, $F(\varphi_n)\longrightarrow F(\varphi)$.

Definition (distributional derivative): The derivative of a distribution is a continuous linear functional, denotes by $D^\alpha f$ is define as:

$$\text{If } F\in D^*(\Omega), \quad \langle D^\alpha F,\varphi\rangle=(-1)^{|\alpha|}\langle F, D^\alpha\varphi\rangle \quad \forall\, \varphi\in D(\Omega)$$

Definition: A function $f:\Omega\longrightarrow\mathbb{R}$ is said that is locally integrable if $\forall$ compact $K\subseteq\Omega$,

$$\int_K |f|\,dx<\infty$$

the functions f locally integrables, may be identified with distibutions F_f, as follows

$$\langle F_f,\varphi\rangle=\int_\Omega f\varphi\,dx.$$

regular functions

Examples:

1.
$$\varphi(x)=\begin{cases} e^{-x^2}, & x>0\\ 0, & x\leq 0.\end{cases}$$

II.

$$\varphi(x)=\begin{cases} \exp\left[-\frac{1}{(x-1)(x-3)}\right], & 1<x<3 \\ 0 & , \quad x\notin(1,3). \end{cases}$$

Example: Delta Dirac.

$$\langle \delta, \varphi\rangle = \varphi(0) \qquad \forall\ \varphi\in D(\Omega).$$

singular distributions

Example: $|x|$ is not differentiable in $x=0$ in the usual sense $|x|$ is locally integrable, and defines a distribution and in the distributional sense is differentiable in $x=0$.

$$\langle |x|', \varphi\rangle = (-1)\langle |x|, \varphi'\rangle$$

$$=-\int_{-\infty}^{\infty} |x|\,\varphi'(x)\,dx = \int_{-\infty}^{0} x\varphi'(x)\,dx - \int_{0}^{\infty} x\varphi'(x)\,dx$$

$$=-\int_{-\infty}^{0} \varphi(x)\,dx + \int_{0}^{\infty} \varphi(x)\,dx, \qquad \varphi(x)\in D(\Omega).$$

$$\operatorname{sgn} x = \begin{cases} -1 & , \ x<0 \\ 0 & , \ x=0 \\ 1 & , \ x>0 \end{cases}$$

$$\langle |x|', \varphi\rangle = \int_{-\infty}^{\infty} \operatorname{sgn}(x)\,\varphi(x)\,dx, \qquad \forall\ \varphi\in D(\Omega)$$

then

$$|x|' = \operatorname{sgn}(x)$$