Schwarzschild $r \to 2M$ ~~"singularity"~~

Rindler original metric

$$ds^2 = -x^2 dt^2 + dx^2$$

$$x \longmapsto y = x^2$$

$$dy = 2x\,dx$$

$$(dy)^2 = 4x^2 dx^2$$

$$dx^2 = \frac{1}{4y^2}\,dy^2$$

$$ds^2 = -y\,dt^2 + \frac{1}{4y}\,dy^2$$

close analogous to Schwarzschild.

Schwarzschild $\longrightarrow$ 4-dim

Due to spherical symmetric, only interesting part is the rt-sector.

Analise 2dim metric.

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

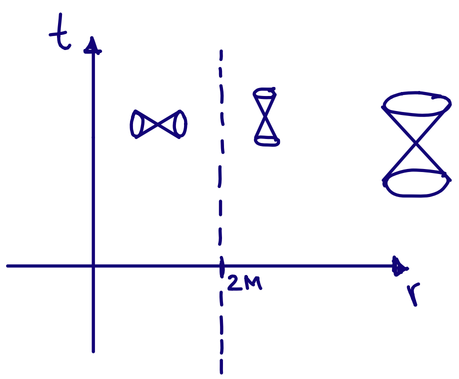
Null geodesics:

$$0 = g_{ab} K^a K^b = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2$$

$$\left(\frac{dt}{dr}\right)^2 = \left(\frac{r}{r-2M}\right)^2 = \left(\frac{1}{1-\frac{2M}{r}}\right)^2$$

$$\frac{dt}{dr} = \pm\left(\frac{1}{1-\frac{2M}{r}}\right)^2$$

measure the slope of the light cones in the tr-plane.



$r \longrightarrow \infty$ $\quad \frac{dt}{dr} = \pm 1$

Minkowski

$r \longrightarrow 2M$ cones colapse.

Define Regge-Wheeler coordinates.

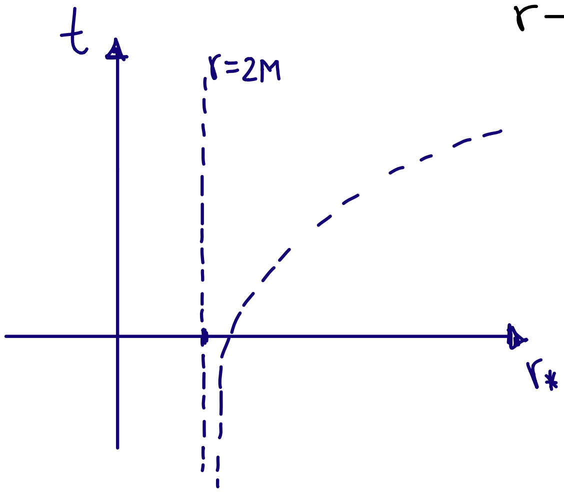
$$r_* := r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

$$\frac{dr_*}{dr} = 1 + \frac{2M}{\frac{r}{2M} - 1}\,\frac{1}{2M} = 1 + \frac{1}{\frac{r}{2M} - 1} = \frac{\frac{r}{2M} - 1 + 1}{\frac{r}{2M} - 1}$$

$$= \frac{\frac{r}{2M}}{\frac{r}{2M} - 1} = \frac{1}{1 - \frac{2M}{r}}$$

$$\frac{dt}{dr} = \pm \frac{dr_*}{dr} \qquad \frac{d}{dr}\left(t \pm r_*\right) = 0$$

$$\boxed{t \pm r_* = \alpha} \qquad \begin{array}{l} r \longrightarrow 2M \Rightarrow r_* \longrightarrow -\infty \\ r \longrightarrow \infty \Rightarrow r_* \longrightarrow \infty \end{array}$$



Define null coordinates.

$$u := t - r_* \qquad r \longrightarrow 2M \qquad u \longrightarrow \infty$$

$$v := t + r_* \qquad\qquad\qquad v \longrightarrow -\infty$$

$$du = dt - dr_* = dt - \left(\frac{r}{r - 2M}\right) dr$$

$$dv = dt + dr_* = dt + \left(\frac{r}{r - 2M}\right) dr$$

$$r_* = \frac{v - u}{2} = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

r must be thought of as function of u and v.

$$2M\ln\left(\frac{r}{2M}-1\right)=\frac{v-u}{2}-r$$

$$\ln\left(\frac{r}{2M}-1\right)=\frac{v-u}{4M}-\frac{r}{2M}$$

$$\frac{r}{2M}-1=e^{\frac{v-u}{4M}}e^{-r/2M}$$

$$\frac{r}{2M}\left(1-\frac{2M}{r}\right)=e^{\frac{v-u}{4M}}e^{-r/2M}$$

$$1-\frac{2M}{r}=\frac{2M}{r}e^{\frac{v-u}{4M}}e^{-r/2M}$$

$$du\,dv=dt^2-\frac{r^2}{(r-2M)^2}dr^2$$

$$=dt^2-\frac{1}{\left(1-\frac{2M}{r}\right)^2}dr^2$$

$$ds^2=-\frac{2M}{r}e^{-r/2M}e^{(v-u)/4M}du\,dv \quad \Big\} \text{ Non-singular as } r\to 2M.$$

Define now

$$U:=-e^{-u/4M}$$

$$V:=e^{v/4M}$$

$$dU=\frac{1}{4M}e^{-u/4M}du \quad , \quad dV=\frac{1}{4M}e^{v/4M}dv.$$

$$dU\,dV=\frac{1}{16M^2}e^{(v-u)/4M}du\,dv$$

$$ds^2=\frac{-32M^3}{r}e^{-r/2M}dU\,dV \to \text{Non-singular.}$$

We may extend to $U,V\in\mathbb{R}$ (Compatible with $r>0$)

Define now

$$U:=-e^{-u/4M}$$

$$V:=e^{v/4M}$$

$$\ln U := \frac{-u}{4m} \longrightarrow u = -4m \ln -U$$

$$\ln V := \frac{v}{4m} \longrightarrow v = 4m \ln - V$$

$$X = \frac{-U+V}{2} = \frac{e^{v/4M} + e^{-u/4M}}{2} = \frac{e^{\frac{t+r_*}{4M}} + e^{\frac{-r_*+t}{4M}}}{2}$$

$$= \frac{e^{(t+r_*)/4M} - e^{-(t-r_*)/4M}}{2}$$

$$= e^{r_*/4M} \frac{(e^{t/4M} - e^{-t/4M})}{2} = e^{r_*/4M} \sinh\left(\frac{t}{4M}\right)$$

last transform

$$T := \frac{U+V}{2}, \qquad X := \frac{V-U}{2}$$

$$dT = \frac{dV + dU}{2} \qquad dX = \frac{dV - dU}{2}$$

$$-(dT)^2 + dX^2 = -dU\,dV$$

Therefore,

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dT^2 + dX^2)$$ Minkowski. $r = r(T, X)$

$$\phi_* g = e^{2\sigma} g$$ Conformal transformation.

$$u = t - r_*$$

$$v = t + r_*$$

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

$$e^{r_*/4M} = e^{[r/4M + 1/2 \ln(r/2M - 1)]}$$

$$= e^{r/4M} e^{1/2 \ln(r/2M-1)}$$

$$= e^{r/4M} \sqrt{\frac{r}{2M} - 1}$$

$$\boxed{T = \sqrt{\left(\frac{r}{2M} - 1\right) e^{r/2M}} \sinh\left(\frac{t}{4M}\right)}$$

$$\left(\frac{r}{2M}-1\right)e^{r/2M} = X^2 - T^2$$

$$t = 4M\,\mathrm{arctanh}\left(\frac{T}{X}\right)$$

$$\tanh\frac{t}{4M} = \frac{T}{X}$$

$$T = \alpha X$$

$$\frac{dT}{dX} = \alpha = \tan\left(\frac{t}{4M}\right) < 1.$$