

$$q) |W(p, q)| \leq 1$$

Proof:

$$W(p, q) = \int dz \Psi\left(q + \frac{z}{2}\right) \overline{\Psi\left(q - \frac{z}{2}\right)} e^{-ipz/\hbar}$$

$$=: \phi_1(q, z) \quad =: \overline{\phi_2(q, z)}$$

$$= \int dz \phi_1(q, z) \overline{\phi_2(q, z)}$$

$$= (\phi_2, \phi_1) = \langle \phi_2, \phi_1 \rangle$$

$$|W(p, q)|^2 = |\langle \phi_2, \phi_1 \rangle|^2 \leq \langle \phi_2, \phi_2 \rangle \langle \phi_1, \phi_1 \rangle = 1$$

Cauchy-Schwarz

Normalized  
wave function.



10) Given  $W(q, p)$  we may recover  $\Psi \in \mathcal{H}$ .

Proof:

$$\mathcal{I} := \int dp W(q, p) e^{ipz/\hbar} = \int dp dz \Psi\left(q + \frac{z}{2}\right) \overline{\Psi\left(q - \frac{z}{2}\right)} e^{-ipz/\hbar} e^{ipz/\hbar}$$

$$= \int dp dz \Psi\left(q + \frac{z}{2}\right) \overline{\Psi\left(q - \frac{z}{2}\right)} e^{-ip(z-z')/\hbar}$$

Integrating in  $p$  brings  $\hbar \delta(z - z')$

$$\mathcal{I} = \hbar \int dz \Psi\left(q + \frac{z}{2}\right) \overline{\Psi\left(q - \frac{z}{2}\right)} \delta(z - z')$$

$$= \hbar \Psi\left(q + \frac{z'}{2}\right) \overline{\Psi\left(q - \frac{z'}{2}\right)}$$

$$\text{Fix } q := \frac{u}{2}, z' := u.$$

$$\mathcal{I} = \hbar \Psi(u) \overline{\Psi(0)} = \int dp W(q, p) e^{ipz'/\hbar}$$

$$= \int dp W\left(\frac{u}{2}, p\right) e^{ipu/\hbar}$$

$$\Rightarrow \Psi(u) = \frac{1}{N} \int dp W\left(\frac{u}{2}, p\right) e^{ipu/\hbar}$$

■

$$11) \lim_{\hbar \rightarrow 0} W(q, p) = |W(q)|^2 \delta(p)$$

**Homework:** Prove (11).

Now, let us define what we mean by a deformation

**Definition:** A **formal** deformation of the algebra  $A_c = C^\infty(M)$  is defined as a map. **No matter if converges.**

$$*: A_c \times A_c \longrightarrow A[\hbar]$$

$$(f, g) \longmapsto f * g = \sum_{k=0}^{\infty} C_k(f, g) \hbar^k; \quad C_k \neq C_n(\hbar).$$

such that.

1) **Associativity:** for all  $p \geq 0$ .

$$((f, g), \hbar) = (f, (g, \hbar))$$

$$((f * g) * h) = (f * (g * h))$$

$$\sum_{k=0}^{\infty} (C_k(f, g), h) \hbar^k = \sum_{k=0}^{\infty} C_k(f, (g, h)) \hbar^k$$

$$\iff \sum_{k,l} C_k(C_l(f, g), h) \hbar^k = \sum_{k,l} C_k(f, C_l(g, h)) \hbar^{k+l}$$

$$\sum_{k+l=p} [C_k(C_l(f, g), h) - C_k(f, C_l(g, h))] = 0$$

2) **Point-product:**  $C_0(f, g) = fg$ .

3) **Poisson-Braket:**  $\frac{1}{2} [C_1(f, g) - C_1(g, f)] = \{f, g\}$ .

4) Bidifferential:  $C_k : A_c \times A_c \rightarrow A_c$

Should be a bidifferential operator for each  $k$ .

**Definition:** A formal deformation of poisson bracket is a skew-symmetric map.

$$[\cdot, \cdot] : A_c \times A_c \rightarrow A_c[\hbar]$$

$$[f, g] \mapsto \sum_{k=0}^{\infty} T_k(f, g) \hbar^k; \quad T_k \neq T_{\ell}(\hbar).$$

such that

$$\text{I}) [[f, g], h] + [[g, h], f] + [[h, f], g] = 0.$$

Jacobi Identity.

$$\Rightarrow \sum_{k+\ell=p} T_k(T_\ell(f, g), h) = 0$$

over cyclic permutations  
of  $f, g$  and  $h$ .

$$\text{II}) T_0(f, g) = \{f, g\}$$

III) Bidifferential:

$$T_k : A_c \times A_c \rightarrow A_c$$

Should be differential operator for each  $k \in \mathbb{Z}^+$ .

In order to construct these deformations first consider a map.

$$*: A_c \times A_c \rightarrow A_c$$

$$(f, g) \mapsto f * g = (Q_h^\omega)^{-1} [Q_h^\omega(f) Q_h^\omega(g)]$$

$$\Rightarrow Q_h^\omega(f * g) = Q_h^\omega(f) \circ Q_h^\omega(g) \quad \text{Non-commutative convolution.}$$

Theorem:

$$f * g = \frac{1}{2\pi} \int_{\mathbb{R}^2} du' dv' \tilde{f}((u-u'), (v-v')) \tilde{g}(u', v') e^{i\hbar/2(uv' - u'v)}$$

Proof:

$$Q_h^\omega(f)(p, q) = \frac{1}{2\pi} \int_{\mathbb{R}^2} du'' dv'' \tilde{f}(u'', v'') e^{-i\hbar(u''\hat{p} + v''\hat{q})}$$

$$Q_h^\omega(g)(p,q) = \frac{1}{2\pi} \int_{\mathbb{R}^2} du' dv' \tilde{g}(u', v') e^{-i/h(u' \hat{p} + v' \hat{q})}$$

$$Q_h^\omega(f)(p,q) \cdot Q_h^\omega(g)(p,q) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du' dv' du'' dv'' \tilde{f}(u'', v'') \tilde{g}(u', v')$$

$$x e^{-i/h(u'' \hat{p} + v'' \hat{q})} e^{-i/h(u' \hat{p} + v' \hat{q})}$$

Convolution of inverse  
Fourier

$$\mathcal{F}^{-1}(f \circ g) = \mathcal{F}^{-1}(f) \mathcal{F}^{-1}(g)$$

By Baker-Campbell-Hausdorff. formula

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]}$$

$$e^{-i/h(u'' \hat{p} + v'' \hat{q})} e^{-i/h(u' \hat{p} + v' \hat{q})} = e^{-i/h((u'+u'')\hat{p} + (v'+v'')\hat{q})} e^{i/h/2(u''v' - v''u')}$$

$$I = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du' dv' du'' dv'' \tilde{f}(u'', v'') \tilde{g}(u', v') e^{i/h/2(u''v' - v''u')} e^{-i/h((u'+u'')\hat{p} + (v'+v'')\hat{q})}$$

Change

$$u' + u'' =: u, \quad v' + v'' =: v, \quad du'' = du$$

$$u'' = u - u', \quad v'' = v - v' \quad dv'' = dv$$

$$u''v' - v''u' = (u - u')v'$$

$$= -(v - v')u''$$

$$= uv' - vu'$$

$$I = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv du' dv' \tilde{f}(u-u', v-v') \tilde{g}(u', v') e^{i/h/2(uv' - vu')} e^{-i/h(u \hat{p} + v \hat{q})}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} du dv \left[ \frac{1}{2\pi} \int_{\mathbb{R}^2} du' dv' \tilde{f}(u-u', v-v') \tilde{g}(u', v') e^{i/h/2(uv' - vu')} \right] e^{-i/h(u \hat{p} + v \hat{q})}$$

$$= (Q_h^\omega(f * g))(q, p)$$



## Star-product properties

$$1) (f * g) * h = f * (g * h) \quad \text{Associativity}$$

$$2) f * 1 = 1 * f$$

$$3) \int_{\mathbb{R}^2} dp dq f * g = \int_{\mathbb{R}^2} dp dq fg = \int_{\mathbb{R}^2} dp dq g * f$$

**Homework:** Prove the star-product properties.

**Theorem:** for any two observables  $f, g \in A_c$ , the star-product introduced, allows an exponential representation given by

$$(f * g)(p, q) = f(p, q) \exp \left[ \left( \frac{i\hbar}{2} \right) \left( \frac{\vec{\delta}}{\partial q^i} \vec{\hat{p}}_i - \frac{\vec{\delta}}{\partial p_i} \vec{\hat{q}}_i \right) \right] g(p, q)$$

$\vec{P}$ , Poisson tensor in local coordinates

**Proof:** As we seen before

$$\Omega_{\hbar}^{\omega}(f * g)(p, q) = I = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du^i dv^i du^m dv^m \tilde{f}(u^i, v^i) \tilde{g}(u^m, v^m) e^{i\hbar/2(u^m v^i - v^m u^i)} e^{-i\hbar((u^i + u^m)\hat{p}_i + (v^i + v^m)\hat{q}_i)}$$

$$\begin{aligned} e^{i\hbar/2(u^m v^i - v^m u^i)} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k (u^m v^i - u^i v^m)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \sum_{m=0}^k \binom{k}{m} (u^m v^i)^{k-m} (u^i v^m)^m (-1)^m \end{aligned}$$

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(2\pi)^2} \int du^i dv^i du^m dv^m (u^m v^i)^{k-m} (u^i v^m)^m \tilde{f}(u^i, v^i) \tilde{g}(u^m, v^m) \\ &\quad \times e^{-i\hbar((u^i + u^m)\hat{p}_i + (v^i + v^m)\hat{q}_i)} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(2\pi)^2} I_1 I_2.$$

$$I_1 := \int du^i dv^i (v^i)^{k-m} (u^i)^m \tilde{g}(u^i, v^i) e^{-i\hbar(u^i \hat{p}_i + v^i \hat{q}_i)}$$

$$I_2 := \int du^m dv^m (u^m)^{k-m} (v^m)^m \tilde{f}(u^m, v^m) e^{-i\hbar(u^m \hat{p}_i + v^m \hat{q}_i)}$$

$I_1$  and  $I_2$  are of the form

$$Q_n^\omega(\hat{q}^{k-m}\hat{p}^m g(q,p)) \text{ and } Q_n^\omega(\hat{p}^{k-m}\hat{q}^m f(q,p)) \text{ respectively.}$$

where

$$\begin{array}{c} \hat{q}^{k-m}\hat{p}^m g(q,p) \\ \hat{p}^{k-m}\hat{q}^m f(q,p) \end{array} \left\{ \begin{array}{l} \text{are Fourier} \\ \text{transforms} \\ \text{of} \end{array} \right\} \begin{array}{c} v'^{k-m} u'^m \\ v''^{k-m} u''^{k-m} \end{array} \begin{array}{c} \tilde{g}(u',v') \\ \tilde{f}(u'',v'') \end{array}$$

By the Fourier property

$$\mathcal{F}[u \tilde{h}(u,v)] =: \frac{\partial}{\partial p} h(p,q)$$

$$\Rightarrow I_1 = i^k \left( \frac{\partial}{\partial p_j} \right)^m \left( \frac{\partial}{\partial q^j} \right)^{k-m} g(p,q).$$

$$I_2 = i^k \left( \frac{\partial}{\partial p_j} \right)^{k-m} \left( \frac{\partial}{\partial q^j} \right)^m f(p,q).$$

$$I = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \sum_{m=0}^k \binom{k}{m} (-1)^m \left[ \left( \frac{\partial}{\partial p_j} \right)^m \left( \frac{\partial}{\partial q^j} \right)^{k-m} g(q,p) \right] \left[ \left( \frac{\partial}{\partial p_j} \right)^{k-m} \left( \frac{\partial}{\partial q^j} \right)^m f(q,p) \right]$$

$$(f*g)(p,q) = f(p,q) \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{i\hbar}{2} \right)^k \sum_{m=0}^k \binom{k}{m} (-1)^m \left[ \left( \frac{\partial}{\partial p_j} \frac{\partial}{\partial q^j} \right)^m \left( \frac{\partial}{\partial q^j} \frac{\partial}{\partial p_j} \right)^{k-m} \right] g(p,q) \right]$$

$$= f(p,q) \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{i\hbar}{2} \right)^k \left( \frac{\partial}{\partial p_j} \frac{\partial}{\partial q^j} - \frac{\partial}{\partial q^j} \frac{\partial}{\partial p_j} \right)^k \right] g(p,q)$$

$$= f(p,q) \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k \left( \frac{\partial}{\partial q^j} \frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j} \frac{\partial}{\partial q^j} \right)^k \right] g(p,q)$$

$$= f(p,q) \exp \left[ \left( \frac{i\hbar}{2} \right) \left( \frac{\partial}{\partial q^j} \frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j} \frac{\partial}{\partial q^j} \right) \right] g(p,q)$$

$$= f(p,q) e^{i\hbar/2 \frac{\partial}{\partial p_j}} g(p,q)$$

$$f*g = fg + \frac{i\hbar}{2} + \vec{p}g + \left( \frac{i\hbar}{2} \right)^2 + \vec{p}^2 g + \dots$$

