Isometries

Definition: Let M and N be pseudo-Riemannian manifolds with metric tensors g_M and g_N, respectively. An isometry from M to N is a diffeomorphism $\phi: M \longrightarrow N$ that preserves metric tensors

$$\phi^*(g_N) = g_M$$

$$g_M(X, Y) = g_N(\phi_* X, \phi_* Y)$$

Isometries preserve the length of a vector.

In components.

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{N\,\alpha\beta}(\phi(p) = g_{M\,\mu\nu}(p)$$

y's in N, x's in M.

Example:

I. Identity map

II. Composition of isometries

III. Inverse of an isometry

Example: In $\mathbb{R}^n$, the set of maps $f: x \longmapsto Ax + T$, $A \in SO(n)$, $T \in \mathbb{R}^n$ is an isometry.

Note: locally isometric $\not\Rightarrow$ diffeomorphic.

Example: Euclidean plane is isometric to cylinder but not diffeomorphic

Definition: Let (M, g_M) and (N, g_N) be a couple of pseudo-Riemannian manifolds. A diffeomorphism $\phi: M \longrightarrow N$ is called a conformal transformation if it preserves the metric up to a scale

$$\phi^* g_N(f(p)) = e^{2\sigma} g_M(p), \quad \sigma \in \mathcal{F}(M)$$

or

$$g_N(\phi_* X, \phi_* Y) = e^{2\sigma} g_M(X, Y).$$

In components

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{N\,\alpha\beta}(f(p)) = e^{2\sigma(p)} g_{M\,\mu\nu}(p).$$

Isometries and conformal transformations form a group!

Algebraic structure G with operation $\circ$ such that.

I. $\forall\, a,b \in G,\quad a\circ b \in G$ — Closure

II. $\forall\, a,b,c \in G,\quad (a\circ b)\circ c = a\circ(b\circ c)$ — Associativity

III. $\exists\, e \in G$ such that $\quad e\circ a = a\circ e = a$ — Neutral

IV. $\forall\, a \in G,\ \exists\, a^{-1}\in G$ such that $a\circ a^{-1} = a^{-1}\circ a = e$ — Inverse

Theorem: Conformal transform preserve angles between vectors.

Proof:

$$\vec{A}\cdot\vec{B} = \|\vec{A}\|\|\vec{B}\|\cos\theta$$

$$g_p(X,Y) = \sqrt{g_p(X,X)}\sqrt{g_p(Y,Y)}\cos\theta$$

$$\cos\theta = \frac{g_p(X,Y)}{\sqrt{g_p(X,X)}\sqrt{g_p(Y,Y)}} = \frac{g_{\mu\nu}X^\mu Y^\nu}{\sqrt{(g_{\alpha\beta}X^\alpha X^\beta)(g_{\rho\lambda}Y^\rho Y^\lambda)}}$$

For a conformal transformation ϕ this goes to

$$\cos\theta' = \frac{e^{2\sigma}g_{\mu\nu}X^\mu Y^\nu}{\sqrt{e^{2\sigma}g_{\alpha\beta}X^\alpha X^\beta}\sqrt{e^{2\sigma}g_{\rho\lambda}Y^\rho Y^\lambda}} = \cos\theta$$

$\blacksquare$

Definition: If at every point p of a pseudo-Riemannian manifold M it has a local coordinate system such that

$$g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$$

then M is said to be conformally flat.

Theorem: Any two-dim Riemann manifold is conformally flat.

Proof: Let the metric

$$ds^2 = g_{\mu\nu}\,dx^\mu\otimes dx^\nu$$
$$= g_{xx}dx^2 + 2g_{xy}dx\,dy + g_{yy}dy^2$$

$$\det g_{\mu\nu} =: g = \begin{vmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy}\end{vmatrix} = g_{xx}g_{yy} - g_{xy}^2$$

$$ds^2 = \left(\sqrt{g_{xx}}\,dx + \frac{g_{xy}+i\sqrt{g}}{\sqrt{g_{xx}}}dy\right)\left(\sqrt{g_{xx}}\,dx + \frac{g_{xy}-i\sqrt{g}}{\sqrt{g_{xx}}}dy\right)$$

$$\sqrt{g_{xx}}\,dx + \frac{g_{xy}+i\sqrt{g}}{\sqrt{g_{xx}}} =: \frac{1}{\lambda}(du+i\,dv)$$

$$\sqrt{g_{xx}}\,dx + \frac{g_{xy}-i\sqrt{g}}{\sqrt{g_{xx}}} =: \frac{1}{\bar{\lambda}}(du-i\,dv)$$

$$ds^2 = \frac{\varphi}{\|\lambda\|^2}(du^2+dv^2) = e^{2\sigma}(du^2+dv^2)$$

$\blacksquare$

Example: $$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 = \sin^2\theta\left(\frac{d\theta^2}{\sin^2\theta} + d\varphi^2\right)$$

$$\frac{d}{d\theta}\left(\log\left|\tan\frac{\theta}{2}\right|\right) = \frac{1}{\tan\frac{\theta}{2}}\left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{\frac{1+\cos\theta}{2}}} = \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\cdot\sqrt{\frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}} = \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{1}{2}\sec^2\theta = \frac{1}{2}\,\frac{1}{\cos^2\frac{\theta}{2}} = \frac{1}{\cancel{2}}\,\frac{1}{\frac{1+\cos\theta}{\cancel{2}}} = \frac{1}{1+\cos\theta}$$

$$= \frac{1}{\dfrac{\cancel{1+\cos\theta}}{\dfrac{\sin\theta}{\cancel{1+\cos\theta}}}} = \frac{1}{\sin\theta}$$

then, $$d\Big(\underbrace{\log\left|\tan\frac{\theta}{2}\right|}_{=:u}\Big) = \frac{d\theta}{\sin\theta} \qquad v := \varphi$$

$$ds^2 = \sin^2\theta\,(du^2+dv^2)\quad ; \quad \theta = \theta(u,v)$$

Homework: Let M be a two-dim Lorentz manifold with $ds^2 = -dt^2 + t^2dx^2$ (Milne universe). Use the transformation $|t| \longmapsto e^{\eta}$ to show that g is conformaly flat. Transform again

$$(\eta, x) \longmapsto u := e^{\eta}\sinh x \quad ; \quad v := e^{\eta}\cosh x$$

what is the resulting metric.

Lie derivative

Definition: Let $X \in \mathfrak{X}(M)$. An integral curve of X is a parametrized curve

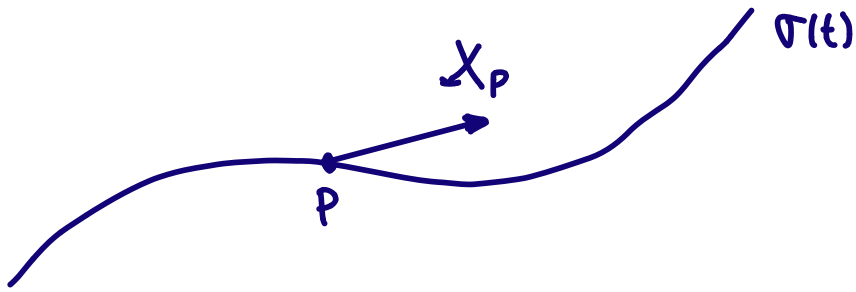
$$\sigma : (a,b) \longrightarrow M$$

whose tangent vector $\dot{\sigma}(t)$ at each point $p=\sigma(t)$ on the curve is equal to the tangent vector X_p assigned to p, that is

$$\dot{\sigma}(t) = X_{\sigma(t)}$$

or in local coordinates $x^i(t) = x^i(\sigma(t))$ and $X = X^k(x^1,\dots,x^n)\partial_{x_k}$ then

$$\frac{dx^i}{dt} = X^i(x^1(t), x^2(t), \dots, x^n(t))$$



Definition: A one parameter group of transformations on M is a map

$$\sigma : \mathbb{R} \times M \longrightarrow M$$

such that

I. For each $t \in \mathbb{R}$, the map

$$\sigma_t : M \longrightarrow M$$
$$\sigma_t(p) = \sigma(t,p)$$

is a transformation of M

II. For all $t,s \in \mathbb{R}$ we have the Abelian group property

$$\sigma_{t+s} = \sigma_t \circ \sigma_s$$

III. σ_t are bijective, then every $p \in M$ is the image of a unique point $q \in M$

$$p = \sigma_t(q) \iff q = \sigma_t^{-1}(p)$$

IV. $\sigma_0 = id_M$

V. $\sigma_t^{-1} = \sigma_{-t}$

The curve

$$\gamma_p : \mathbb{R} \longrightarrow M$$
$$\gamma_p(t) = \sigma_t(p)$$

Passes through p at $t=0$ and it is the orbit of p under the flow of σ, and defines a tangent vector X_p at p by

$$X_p(f) = \frac{df}{dt}(\gamma_p(t))\Big|_{t=0} = \frac{df(\sigma_t(p))}{dt}\Big|_{t=0}$$

then, X_p is the vector field induced by the flow of σ.