Corollary: for all $x_0 \in S(X) = \{x \in X : \|x\| = 1\}$ the unitary sphere of X, there exists $f_0 \in X^*$, such that

$$\|f_0\|_{X^*} = 1 \quad \text{and} \quad f_0(x_0) = 1$$

thus, the functional x_0 reach its supremum in the unitary ball in the vector x_0.

Proof: From the Hahn-Banach theorem, let the one-dimensional space $E_0 = \{\lambda x_0\}$ and we take the functional $\varphi(\lambda x_0) = \lambda$ then

$$\|\varphi\|_{E^*} = 1$$

By the Hahn-Banach theorem, there is an extension with the require properties. $\blacksquare$

Corollary: For all $x_0 \in X$, there is $f_0 \in X^* / \{0\}$, such that

$$f_0(x_0) \neq \|f_0\| \|x_0\|$$

Proof: By the last corollary: Using the $\frac{x_0}{\|x_0\|}$ vector, we get the proof. $\blacksquare$

Corollary: For all $x_1, x_2 \in X$ such that $x_1 \neq x_2$ there is $f \in X^*$ such that $f(x_1) \neq f(x_2)$

Proof: Using the previous, for $x_0 = x_1 - x_2$. $\blacksquare$

Corollary: X^* is a total set i.e., if $f(x) = 0$ for all f, then $x = 0$.

Corollary: Let $L \subseteq X$, a closed subspace of a normed space X, and let $x \in X$ such that

$$\text{dist}(x, L) = d > 0.$$

then, there is $f \in X^*$ such that $\|f\| = 1$, $f(L) = 0$ and $f(x) = d$

Proof: Let's consider $L_1 = \text{span}\{x, L\}$ i.e.,

$$L_1 = \{\lambda x + y : \lambda \in \mathbb{R}, y \in L\}$$

let $z = \lambda x + y$. Let's define the function $f_0(z) = \lambda d$, this it's well define, since z may be written of unique form

(If not, $z=\lambda_1 x+y_1=\lambda_2 x+y_2$, then $x(\lambda_1-\lambda_2)=y_1-y_2\in L$, then $x\in L$, $\lambda_1=\lambda_2$, then $y_1=y_2$)

Now, the linearity of f_0 is obvious. Thus, $f_0\in L^{\#}$. We see that: $f_0(L)=0$ and $f_0(x)=d$.

$$\|z\|=|\lambda|\left\|x+\frac{y}{\lambda}\right\|\geq|\lambda|d=|f_0(x)|$$

as $-y/\lambda\in L$ and d is the infimum.

Then $\|f_0\|\leq 1$, moreover, there is $y_n\in L$ such that $\|x_n-y_n\|\to d$.

$$d=|f_0(x+y_n)|\leq\|f_0\|\,\|x+y_n\|\longrightarrow d\|f_0\|\to\|f_0\|\geq 1.$$

Let the extension f of f_0, with $\|f\|_{X^*}=\|f\|_{L^*}$, whose existence is garantized by the Hahn-Banach theorem.

Finally $\|f\|=1$, and $f|_L=f_0$, it means that $f(L)=0$, and $f(x)=d$.

■

For any $L\subseteq X$, closed

$$L^{\perp}=\{f\in X^*,\ f(L)=0\},$$

$L^{\perp}$ is a closed subspace of X^* and $L^{\perp}=\overline{L}^{\perp}$, if we start from a subspace F of X^*, we can consider two diferent constructions.

The first is the same as before and $F^{\perp}$ is a closed subspace of $(X^*)^*:=X^{**}$, nevertheless, we can consider a subspace of X,

$$F_{\perp}=\{x\in X,\ f(x)=0,\ \forall f\in F\}$$

Corollary: Let L be a closed subspace. Consider $L^{\perp}\subseteq X^*$,

$$(L^{\perp})_{\perp}=\{x\in X,\ f(x)=0\ \ \forall f\in L^{\perp}\}$$

Proof: Let $L\subseteq(L^{\perp})_{\perp}$,

If $x\in L$, $f(x)=0\ \ \forall f\in L^{\perp}$, then $L\subseteq(L^{\perp})_{\perp}$

Let $(L^{\perp})_{\perp}\subseteq L$

for $x\notin L$, and L closed, $d(x,L)=d>0$, by the previous cases, there is f such $f(L)=0$, $(f\in L^{\perp})$ and $f(x)\neq 0$, then $x\notin(L^{\perp})_{\perp}$, finally $(L^{\perp})_{\perp}\subseteq L$.

■

Let X a Banach space and X^* its dual. let's consider $(X^*)^*$, the dual space to X^*, denoted by X^{**}.

X^{**} may be identified in X of the following way:

for $x \in X$ the functional $i(x) \in X^{**}$, such that for each $f \in X^*$, $i(x)f(x) := f(x)$, this defines map

$$i: X \longrightarrow X^{**}, \text{ which is injective.}$$

then,

$$|i(x)\, f(x)| \leq \|f\| \cdot \|x\|,$$

which means

$$\|i(x)\|^{**} = \sup_{f \neq 0} \frac{|i(x) f(x)|}{\|f\|} = \sup_{f \neq 0} \frac{|f(x)|}{\|f\|} \leq \|x\|.$$

by the Hahn-Banach theorem (Corollary), the supremum reach:

for any x, there exists $f \neq 0$ such that $|f(x)| = \|x\| \|f\|$. thus,

I. $$\sup_{f \neq 0} |f(x)| = \max_{f \neq 0} \frac{|f(x)|}{\|f\|} = \|x\|$$

II. The map $i: X \longrightarrow X^{**}$ is an isometry

If $i: X \longrightarrow X^{**}$ is onto, $X \simeq X^{**}$ are isometrically isomorphs.

Examples:

1. In the space C_0, with norm $\|x\| = \max |a_i|$, $x = (a_i)$, let's define the functional f, fixing a succession $(b_i) \in \ell_1$, taking as action over $x \in C_0$, by:

$$f(x) = \sum_{i=1}^{\infty} a_i b_i$$

$$|f(x)| \leq \sum_{i=1}^{\infty} |a_i b_i| \leq \max_{1 \leq i < \infty} |a_i| \sum_{i=1}^{\infty} |b_i| = \|x\|_{c_0} \|f\|_{\ell_1}$$

then

$$\|f\|_{c_0^*} \leq \|f\|_{\ell_1}$$

Now, let $f \in C_0^*$, let's define $f(e_n) = b_n$, donde

$$e_n = (0, \dots, 0, \underset{n\text{-th position}}{1}, 0 \dots, 0) \in C_0$$

Taking
$$y_n=\sum_{k=1}^{n} e^{-i\arg(b_k)} e_k,$$

for $\|y_n\|_{c_0}=1$, for each $n\in\mathbb{N}$, we have:
$$\|f\|_{c_0^*}\geq f(y_n)=\sum_{i=1}^{n}|b_i|$$
then, $\|f\|_{c_0^*}\geq\|(b_i)\|_{\ell_1}$, identifying $f\in C_0^*$, with the succession $f=(b_n)\in\ell_1$.
$$\|f\|_{c_0^*}\geq\|f\|_{\ell_1}$$
thus, $\|f\|_{c_0^*}=\|f\|_{\ell_1}$, finally $C_0^*=\ell_1$

II. $\ell_q=(\ell_p)^*$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$

III. $L_q=(L_p)^*$,
$$\left(\int_a^b|f(x)|^p\right)^{1/p}<\infty$$