Let's suppose that ω is a 1-form in $\mathbb{R}^n$, lets define the following functions

$$\omega(\partial_\mu) = \omega\left(\frac{\partial}{\partial x^\mu}\right) = \omega_\mu$$

i.e. $\omega = \omega^\mu dx_\mu$.

This imply that the 1-forms $\{dx^\mu\}$ generates the 1-forms in $\mathbb{R}^n$.

Let $v = v^\mu \partial_\mu$

$$\omega(v) = \omega(v^\mu \partial_\mu) = v^\mu \omega(\partial_\mu) = v^\mu \omega_\mu$$

$$\omega_\mu dx^\mu(v) = \omega_\mu dx^\mu(v^\nu \partial_\nu) = \omega_\mu v^\nu dx^\mu(\partial_\nu)$$

$$= \omega_\mu v^\nu \partial_\nu(x^\mu) = \omega_\mu v^\nu \delta^\mu_\nu = \omega_\mu v^\mu$$

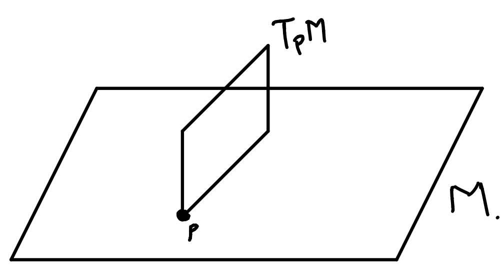
Exercise: Prove that the one-forms $\{dx^\mu\}$ are linearly independent i.e.,

$$\omega = \omega_\mu dx^\mu = 0 \Rightarrow \omega_\mu = 0.$$

$$v = v^\mu \partial_\mu$$

$$\omega = \omega_\mu dx^\mu$$

Cotangent vectors



Given a manifold M and $p \in M$, the cotangent vector ω in p, is defined as a linear map of $T_pM \longrightarrow \mathbb{R}$. We denote to T_p^*M the cotangent space

If we have a 1-form ω in M, we can define cotangent vector $\omega_p \in T_p^*M$ saying that for each $v \in T_pM$.

$$\omega_p(v_p) = \omega(v)(p)$$

Let V be a vector space, its dual space V^*, is define as the linear functional space $\omega: V \longrightarrow \mathbb{R}$, in particular the cotangent space T_p^*M is the dual of T_pM

If we have a linear map

$$f : V \longrightarrow W$$

Its dual f^*

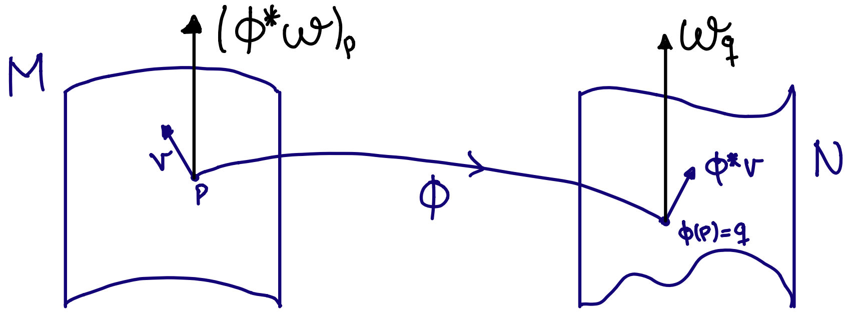
$$f^* : W^* \longrightarrow V^*$$

Is define, $\tilde{\omega} \in W^*$

$$(f^* \tilde{\omega})(v) = \tilde{\omega}(f(v)).$$

Exercise: let $f : V \to W$, $g : W \longrightarrow X$

$$(gf)^* = f^* g^*$$



$$\phi_* : T_pM \longrightarrow T_qN \ , \quad q = \phi(p)$$

The dual map

$$\phi^* : T_q^*N \longrightarrow T_p^*M$$

If ω is a cotagent vector in $\phi(p)$, we will call to $\phi^* \omega$ the pullback of ω by ϕ. Explicitely, if $v \in T_pM$ and $\omega \in T_q^*N$

$$(\phi^* \omega)(v) = \omega(\phi_* v)$$

Given a 1-form in N, ω, and we define a 1-form in M $\phi^* \omega$

$$(\phi^* \omega)_p \equiv \phi^*(\omega_q), \qquad \phi(p) = q$$

$$(\phi^* \omega_p)(v_p) = \phi^* \omega(v)(p) = \omega(\phi_* v)(\phi(p)) = \omega(\phi_* v)(q)$$

In the other hand

$$\phi^*(\omega_q)(v_q) = \omega_q(\phi_* v_q) = \omega(\phi_* v)(q)$$