The characters group.

The associated character to a matricial representation, is a function x

$$\chi(a) = \mathrm{Tr}\, D(a) = \sum_{i=1}^{n} D(a)_{ii} \qquad a \in G,\ \tau_{ij}(a) = D(a)_{ij}$$

If $x'(a)$ a character of $D'(a) = S D(a) S^{-1}$.

$$\chi'(a) = \mathrm{Tr}[S D(a) S^{-1}] = \mathrm{Tr}[S^{-1} S D(a)] = \mathrm{Tr}\, D(a) = \chi(a)$$

the characters satisfy the orthogonality property.

If $x^{\ell}(a)$ of the representation $D^{\ell}(a)$.

$$\int \overline{\chi^{i}(g)}\, \chi^{j}(g)\, dg = \Omega\, \delta_{ij} \qquad \Omega \to \mathrm{Vol}\, G = \int_G dg$$

$$\int \overline{\chi^{i}(g)}\, \chi^{j}(g)\, dg = \sum_{k,\ell} \int \overline{D^{i}(g)_{kk}}\, D^{j}(g)_{\ell\ell}\, dg = \sum_{k} \frac{\Omega}{n_i}\, \delta_{ij} = \Omega\, \delta_{ij}$$

is obtained from use

$$\int D^{\ell}(g)_{im}\, \overline{D^{k}(g)_{jn}}\, dg = \frac{\Omega}{n_\ell}\, \delta_{ij}\, \delta_{mn}\, \delta_{\ell k}\,, \quad n_\ell = \dim D^{\ell}(a).$$

Direct product of representations

Let's consider representations by the matrices $D^1(a)$ and $D^2(a)$ of dimensions m and n respectly. The direct product is defined as the space L, of all tensors A that has components a_{ij}, $1 \le i \le m$, $1 \le j \le n$.

A linear combination of tensors $\alpha A + \beta B$ where A and B are defined as a tensor with components $\alpha a_{ij} + \beta b_{ij}$ with a_{ij} and b_{ij} components of A and B.

The direct product of representations is the set of transformation $D(a)$ is define by

$$[D(a) A]_{ij} = \sum_{j=1}^{m} \sum_{\ell=1}^{n} D^1(a)_{ik}\, D^2(a)_{j\ell}\, a_{k\ell}.$$

$D(a)$ is a representation

$$[D(a) D(b) A]_{ij} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} \sum_{s=1}^{m} \sum_{t=1}^{n} D^1(a)_{ik}\, D^2(a)_{j\ell}\, D^1(b)_{ks}\, D^2(b)_{\ell t}\, a_{st}$$

$$= \sum_{s=1}^{m} \sum_{t=1}^{n} D^1(ab)_{is} D^2(ab)_{jt} a_{st} = [D(ab)A]_{ij}$$

$$D(a)_{ij,kl} = D^1(a)_{ik} D^2(a)_{jl} \qquad \begin{array}{l} i,j = 1,\dots,m. \\ k,l = 1,\dots,n. \end{array}$$

In a simplify way

$$D(a) = D^1(a) \times D^2(a)$$

$$\chi(a) = \sum_{ij} D(a)_{ij,ij} = \sum_{i} \sum_{j} D^1(a)_{ii} D^2(a)_{jj} = \chi^1(a)\chi^2(a).$$

Representations in the function space

If a G group can be considered as a set of transformations in a space S, is possible to build representations in the space of functions with domain S.

$$G \longrightarrow S,$$

$$x \longmapsto ax$$

Let be the set $H = C(S) = \{f \text{ continuous in } S\}$. $D(a)$ can be defined in H, of the following way. Let $f \in H$,

$$(D(a)f)(x) = f(a^{-1}x) \qquad \begin{array}{r} x_n \longrightarrow x \\ ax_n \longrightarrow ax \\ f(ax_n) \longrightarrow f(ax). \end{array}$$

$D(a)$ is linear $D(a)[\alpha f + \beta g](x) = \alpha f(a^{-1}x) + \beta g(a^{-1}x) = \alpha D(a)f + \beta D(a)g$

$D(a)$ is a representation.

Let the notation $D(b)f = f_b$, then $D(a)f_b(x) = f_b(a^{-1}x)$, by the other way

$$f_b(y) = f(b^{-1}y)$$

thus $y = a^{-1}x$, $f_b(a^{-1}x) = f(b^{-1}a^{-1}x) = f((ab)^{-1}x) = D((ab)f)(x)$

$$\Longrightarrow D(ab) = D(a)D(b)$$

Let H_ν, that are the homogeneous functions of grade ν, in S.

$$f(cx) = c^\nu f(x), \quad H_\nu \subseteq H.$$

We can generalize to more variables.

$$[D(a)f](x,y) = f(a^{-1}x, a^{-1}y)$$

$$h(x,y) = C_{ij}\, e^i(x) f^j(y).$$

with $C_{ij} \in \mathbb{C}$, the representation is equivalent to $D^1 \times D^2$.

Let $e^i(x)$, $f^j(y)$ be a base of H.

$$e^m(a^{-1}x) f^n(a^{-1}y) = \sum_i D^1(a)_{im}\, e^i(x) \sum_j D^2(a)_{jn}\, f^j(y).$$

Representations in the plane

$R(\theta)$, a rotation by angle θ, $0 \leq \theta \leq 2\pi$, the plane

$$R(\theta) R(\phi) = R(\theta + \phi) \qquad 0 \leq \theta + \phi \leq 2\pi$$

$$R(\theta) R(\phi) = R(\theta + \phi - 2\pi) \qquad \theta + \phi \geq 2\pi.$$

a point $(x,y) \in \mathbb{R}^2$, is transformed by $R(\theta)$

$$(x,y) \longmapsto (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

$$D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

this is not an irreducible representation.

$$SDS^{-1} = D'(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$\int_0^{2\pi} e^{in\theta} e^{-im\theta}\, d\theta = 2\pi\, \delta_{mn}$$

$f \in L^2(R(\theta))$

$$f = \sum_{l=-\infty}^{\infty} b_l\, e^{il\theta} \longrightarrow \text{Fourier's theorem.}$$

SU(2)

$$SU(2) = \} AA^* = 1, \det(A) = 1 \{$$

let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we have for the groups characteristics that:

$$|a_{11}|^2 + |a_{12}|^2 = 1 \qquad \overline{a_{11}}\, a_{11} + \overline{a_{12}}\, a_{22} = 0$$

$$|a_{21}|^2 + |a_{22}|^2 = 1 \qquad a_{11}a_{22} - a_{12}a_{21} = 1$$

The equation it's satisfied if

$$a_{11} = \cos(\theta)e^{i\phi}, \quad a_{12} = i\sin(\theta)e^{i\varphi}.$$

with $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$, $0 \le \varphi \le 2\pi$, the two and three, $a_{21} = \overline{-a_{12}}\, e^{i\alpha}$, $a_{22} = \overline{a_{11}}\, e^{i\alpha}$, $\alpha \in \mathbb{R}$ arbitrary. The last $e^{i\alpha} = 1$.

$$A(\theta, \phi, \varphi) = \begin{pmatrix} \cos(\theta)e^{i\phi} & i\sin(\theta)e^{i\varphi} \\ i\sin(\theta)e^{-i\varphi} & \cos(\theta)e^{i\phi} \end{pmatrix}, \text{ if } \theta \neq 0 \text{ and } \frac{\pi}{2}$$

Let

$$\overline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and the set of functions

$$e^m(z) = \frac{z_1^{j+m}\, z_2^{j-m}}{[(j+m)!\,(j-m)!]^{1/2}}$$

$m = -j, -j+1, \dots, j$ and generates to the homogenious polynomials of grade $2j$, since there is $2j+1$ possibles relations, the representations indicates by m.

$$e^n(A^{-1}z) = (\text{Using } A^{-1} = A^*)$$

$$\frac{1}{[(j+n)!\,(j-n)!]^{1/2}} \left[(\cos(\theta)e^{-i\theta} z_1 - i\sin(\theta)e^{i\varphi} z_2)^{j+n} (-i\sin(\theta)e^{-i\varphi} z_1 + \cos(\theta)e^{i\phi} z_2)^{j-n} \right]$$

$$= \sum_{s,t} \frac{[(j+n)!(j-n)!]^{1/2}}{s!(j+n-s)!t!(j-n-t)!} (\cos(\theta) e^{-i\phi} z_1)^s (-i\sin(\theta) e^{i\psi} z_2)^{j+n-s} (-i\sin(\theta) e^{-i\psi} z_1)^t (\cos(\theta) e^{-i\phi} z_2)^{j-n-t}$$

$$= \sum_{s,t} (-i)^{j+n-s+t} \frac{[(j+n)!(j-n)!]^{1/2}}{s!(j+n-s)!t!(j-n-t)!} \cos(\theta)^{j-n+s-t} \sin(\theta)^{j-n+s-t} e^{i(j-n-s-t)\phi} e^{i(j+n-s-t)\psi} z_1^{s+t} z_2^{2j-s-t}$$

$0 \leq s \leq j+n,$

$0 \leq t \leq j-n.$

$$D(A)_{mn} = i^{m-n} \sum_t (-1)^t \frac{[(j+m)!(j-m)!(j+n)!(j-n)!]^{1/2}}{(j+m-n)!(t+n-m)!t!(j-n-t)!} \cos(\theta)^{2j+m-n-2t} \sin(\theta)^{2t+m-n} e^{-i(m-n)\phi} e^{i(n-m)\psi}.$$

coefficents.