$$\hat{f}(x) = \int_A f(x)\overline{\chi(x)}\,dx$$

Lemma: Let A be a compact abelian group. If we set Haar's integral, as

$$\int_A 1\,dx = 1$$

So, for any two characters $\chi, \eta \in \hat{A}$.

$$\int_A \chi(x)\overline{\eta(x)}\,dx = \begin{cases} 1 & \text{if } \chi = \eta \\ 0 & \text{if } \chi \neq \eta \end{cases}$$

Proof: If $\eta = \chi$, so $\chi(x)\overline{\eta(x)} = 1$, and it follows the affirmation.

Now, let's suppose that $\eta \neq \chi$. So $\alpha = \chi\overline{\eta} = \chi\eta^{-1} \neq 1$. Thus there is $a \in A$, such that $\alpha(a) \neq 1$.

$$\alpha(a)\int_A \alpha(x)\,dx = \int_A \alpha(ax)\,dx = \int_A \alpha(x)\,dx.$$

$$\longrightarrow (\alpha(a)-1)\int_A \alpha(x)\,dx = 0 \longrightarrow \int_A \alpha(x)\,dx = 0.$$

Theorem (Plancherel): Let A be a LCA group, there is a unique Haar's measure in $\hat{A}$ such that for all $f \in L^1_{bc}(A)$

$$\|f\|_2 = \|\hat{f}\|_2$$

i.e., for $f \in L^1_{bc}(A)$, the Fourier's transformed belongs in $L^1_{bc}(A)$, i.e., the Fourier transformation gives an isomorphism of $L^2(A) \longrightarrow L_2(\hat{A})$.

This is the key point of how the harmonic analysis generalizes the theory of Fourier.

In the case $\mathbb{R}\setminus\mathbb{Z}$, $f \in L^1_{bc}(\mathbb{R}\setminus\mathbb{Z})$

$$\int_0^1 |f(x)|^2\,dx = \|f\|_2^2 = \|\hat{f}\|_2^2 = \sum_{k\in\mathbb{Z}} |\hat{f}(L)|^2 = \sum_{k\in\mathbb{Z}} |C_k(f)|^2$$

$$\hat{f}(L) = \int_{\mathbb{R}\setminus\mathbb{Z}} f(y)\, e^{-2\pi S x y}\,dy$$

Riemann-Lebesgue Lemma

Proof: We will prove that in the special case of A is a discrete group

$$\int_A f(x)dx = \sum_{a\in A} f(a)$$

For $\chi \in \hat{A}$, $\hat{f}(\chi) = \sum_{a\in A} f(a)\overline{\chi(a)}$ and in $\hat{A}$ we choose the Haar's integral normalized.

$$\int_A 1\,da = 1.$$

$\blacksquare$

Lemma: for each $g \in L^1_{bc}(A)$, the Fourier transform $\hat{g}$ is in $C(\hat{A}) = L^1_{bc}(A)$ and we have that, for each $a \in A$

$$\hat{\hat{g}}(\delta_a) = g(a^{-1})$$

Proof:

$$\hat{\hat{g}}(\delta_a) = \int_{\hat{A}} \hat{g}(a)\,\overline{\delta_a(x)}\,dx.$$

$$= \int_{\hat{A}} \sum_{b\in A} g(b)\,\overline{\delta_b(x)}\,\overline{\delta_a(x)}\,dx$$

$$= \int_{\hat{A}} \sum_{b\in A} g(b^{-1})\,\delta_b(x)\,\overline{\delta_a(x)}\,dx$$

$$= \sum_{b\in A} g(b^{-1}) \int_{\hat{A}} \delta_b(x)\,\overline{\delta_a(x)}\,dx$$

$$= g(a^{-1}).$$

To prove the theorem, let $f \in L^1_{bc}(A)$ and

$$\tilde{f}(x) = \overline{f(x^{-1})}$$

Let $\gamma = \tilde{f} * f$

$$\gamma(x) = \int_{\hat{A}} \overline{f(yx^{-1})}\, f(y)\,dy$$

Thus $g(e)=\|f\|_2^2$, with e be the identity element in A.
By the convolution theorem,

$$\hat{g}(x)=\hat{\tilde{f}}(x)\,\hat{f}(x)=\overline{\hat{f}(x)}\,\hat{f}(x)=|\hat{f}(x)|^2$$

Therefore

$$\|f\|_2^2=g(e)=\hat{\hat{g}}(\delta e)=\int_{\hat{A}}|\hat{f}(x)|^2\,dx=\|\hat{f}\|_2^2 \qquad \blacksquare$$

Matrix groups

$GL_n(\mathbb{C})$, $U(n)$.

Let $n\in\mathbb{N}$, in the vector space of the complex matrices $n\times n$ $Mat_n(\mathbb{C})$, we define a norm

$$\|A\|_1=\sum_{i,j=1}^{n}|a_{ij}|\,,\quad \text{with}\quad A=(a_{ij})$$

This norm gives rise to the metric. $d_1(A,B)=\|A-B\|_1$.

On the other hand, $Mat_n(\mathbb{C})\simeq\mathbb{C}^{n^2}$, lets take a natural interior product, given rise to another norm; the euclidean norm

$$\|A\|_2:=\sqrt{\sum_{i,j=1}^{n}|a_{ij}|^2}$$

and we obtain the following metric $d_2(A,B)=\|A-B\|_2$.

Lemma: A succesion of matrices $A^{(k)}=(a_{ij}^{(k)})$ converges if and only if, for two pair of indices (i,j) the succession of inputs converges in $\mathbb{C}$. The same happens to d_2, thus the metrics d_1 and d_2 are equivalents.

Proof: Let's suppose that succession $A^{(k)}=(a_{ij}^{(k)})$ converges in d_1 to $A=(a_{ij})\in Mat_n(\mathbb{C})$. So for each $\varepsilon>0$, there is $k_0\in\mathbb{N}$, such that for all $k>k_0$

$$\|A^{(k)}-A\|_1<\varepsilon$$

Let $i_n, j_n\in\{1,\dots,n\}$, then $k>k_0$.

$$|a_{i_0j_0}^{(k)}-a_{i_0j_0}|\leq\sum_{ij}|a_{ij}^{(k)}-a_{ij}|=\|A^{(k)}-A\|<\varepsilon$$

then each input converges. Reciprocally, let's assume $a^{(k)}_{ij} \longrightarrow a_{ij}$, for each pair of index (i,j). Given $\varepsilon > 0$, there is $K_0(i,j)$, such that for all $K \geq K_0(i,j)$

$$|a_{ij}^{(k)} - a_{ij}| < \frac{\varepsilon}{n^2}$$

Let $K_0 \in \mathbb{N}$, the maximum of the $K_0(i,j)$. Then for $K \geq K_0$

$$\|A^{(k)} - A\|_1 = \sum_{i,j} |a_{ij}^{(k)} - a_{ij}| < \sum_{i,j} \frac{\varepsilon}{n^2} = \varepsilon$$

Then $A^{(k)} \longrightarrow A$. And the same for d_2.

Proposition: With the generated topology by the defined metric previously, $GL_n(\mathbb{C})$ is a locally compact group, i.e., it's metrizable, σ-compact and locally compact.

Proof: We note that multiplication and inversion are given by rational functions in the inputs. As the polinomies are continuous, then $GL_n(\mathbb{C})$ is a topological group.

Since it's an open subset of a locally compact space $Mat_n(\mathbb{C})$, it's locally compact.

Moreover, it's σ-compact.

$$K_n = \{ a \in GL_n(G) / \|a\|_1 \leq n, \|a^{-1}\|_1 \leq n \}$$

For $A \in Mat_n(\mathbb{C})$, A^* is his adjoin matrix i.e., if $A = (a_{ij})$, then $A^* = (\overline{a_{ji}})$, thus $A^* = \overline{A^t}$.

$$U_n = \{ g \in Mat_n(\mathbb{C}) / g^* g = 1 \}$$

Lemma: $U(n)$ is a compact subgroup of $GL_n(\mathbb{C})$.

Proof: For $Mat_n(\mathbb{C})$ the equation $g^* g = 1$ implies that g is invertible and $g^* = g^{-1}$

$$U(n) \subseteq GL_n(\mathbb{C})$$

let $a, b \in U(n)$, to prove that $a, b \in U(n)$, $a^{-1} \in U(n)$. For this let's consider $(ab)^*(ab) = b^* a^* a b = b^* b = 1$, $a, b \in U(n)$.

On the other hand, as $a^* = a^{-1}$

$$1 = a a^* = a^* (a^*)^*$$

Therefore, $a^* = a^{-1}$ and it's in $U(n)$.

To see that it's compact, it's enough prove that the group $U(n)$ is closed and bounded in $Mat_n(\mathbb{C})$.

Let g_i be a succesion in $U(n)$, convergent to g in $Mat_n(\mathbb{C})$, then.

$$1 = \lim_{j\to\infty} g_j^* g_j = \left(\lim_{j\to\infty} g_j\right)^* \left(\lim_{j\to\infty} g_j\right) = g^* g$$

therefore, $U(n)$ is closed.

Moreover, it's bounded because for all $a \in Mat_n(\mathbb{C})$.

$$tr(a^* a) = \sum_{k}^{n} (a^* a)_{kk} = \sum_{k=1}^{n}\sum_{j=1}^{n} a_{kj}^* a_{jk} = \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{a_{jk}} a_{jk}$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n} |a_{jk}|^2 = \| a_{jk} \|_2^2$$

Then $$\|g\|_2 = \sqrt{tr\, 1} = \sqrt{n}$$

Thus $U(n)$ is bounded.

$\blacksquare$