The idea is to copy the structure of ΛV for the 1-forms $\Omega^1(M)$.

The differential forms in M, denoted by $\Omega(M)$ is the algebra generated by $\Omega^1(M)$, with the relations

$$\omega \wedge \mu = -\mu \wedge \omega, \qquad \forall \mu, \omega \in \Omega_1(M)$$

and it is generated on $C^\infty(M)$.

Let's define the 0-forms as functions, $\Omega^0(M) = C^\infty(M)$, and

$$f \wedge \omega = f\omega$$

elements that are the product p, 1-forms are called p-forms, and its denoted by $\Omega^p(M)$

$$\Omega(M) = \bigoplus_p \Omega^p(M)$$

Example: let $\mathbb{R}^n$, the:

- 0-forms, $f \in C^\infty(\mathbb{R}^n)$
- 1-forms, $\omega_\mu dx^\mu$, $\omega_\mu \in C^\infty(M)$.
- 2-forms, $\frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu$

We have added the $\frac{1}{2}$ since $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$.

Therefore, $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$$-\omega_{21} dx^1 \wedge dx^2 = \omega_{12} dx^1 \wedge dx^2$$

In $\mathbb{R}^3$:

$$\frac{1}{2}\omega_{12} dx^1 \wedge dx^2, \frac{1}{2}\omega_{21} dx^2 \wedge dx^1, \frac{1}{2}\omega_{23} dx^2 \wedge dx^3, \frac{1}{2}\omega_{32} dx^3 \wedge dx^2$$

$$\frac{1}{2}\omega_{13} dx^1 \wedge dx^3, \frac{1}{2}\omega_{31} dx^3 \wedge dx^1.$$

$$\omega = \omega_{12} dx^1 \wedge dx^2 + \omega_{23} dx^2 \wedge dx^3 + \omega_{31} dx^3 \wedge dx^1$$

- 3-forms, $\frac{1}{3!}\omega_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda$

$$dx^1 \wedge dx^2 \wedge dx^3 = - dx^1 \wedge dx^3 \wedge dx^2$$

$\omega_{\mu\nu\lambda}$ is totally antisymmetric in μ, ν, λ.

$$\omega_{\mu\nu\lambda} = -\omega_{\nu\mu\lambda} = \omega_{\nu\lambda\mu}$$

$$\omega = \omega_{123}\, dx^1 \wedge dx^2 \wedge dx^3$$

- There are no 4-forms, 5-forms, etc.

Homework: let $\omega \in \Omega^p(M)$ and $\mu \in \Omega^q(M)$, $p, q < n$.

$$\omega = \frac{1}{p!} \omega_{i_1} \cdots \omega_{i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$v = \frac{1}{q!} v_{i_1} \cdots v_{i_q}\, dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

Show that $\omega \wedge \mu = (-1)^{pq} \mu \wedge \omega$

$\Omega(M)$ is graded algebra.

Exterior derivative

$$d : C^\infty(M) \longrightarrow \Omega^1(M)$$

$$f \longmapsto df$$

$$d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$$

Hints:

$\nabla \times (\nabla f) = 0$, $d(df) = 0$, $\forall f \in C^\infty(M)$

- $\nabla(fg) = (\nabla f) g + f \nabla g$.
- $\nabla \times (fv) = \nabla f \times v + f \nabla \times v$
- $\nabla \cdot (fv) = \nabla f \cdot v + f \nabla \cdot v$
- $\nabla (v \times w) = (\nabla \cdot v) w - v \nabla \cdot w$

The exterior differential or diferential, is the map

$$d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$$

that satisfies:

1. $d : \Omega^0(M) \longrightarrow \Omega^1(M)$, Match with the definition.

II. $d(\omega+\mu)=d\omega+d\mu$ and $d(c\omega)=c\,d(\omega)$, $\forall\ \omega,\mu\in\Omega(M),\ c\in\mathbb{R}$.

III. $d(\omega\wedge\mu)=d\omega\wedge\mu+(-1)^p\,\omega\wedge d\mu \quad \forall\ \omega\in\Omega^p(M),\ \mu\in\Omega(M)$

IV. $d(d\omega)=0,\ \forall\ \omega\in\Omega(M)$.

let's suppose that we want calculate the differential

$$f\,dg\wedge dh,\quad f,g,h\in C^\infty(M).$$

$$
\begin{aligned}
d(f\,dg\wedge dh)&=df\wedge(dg\wedge dh)+(-1)^0 f\wedge d(dg\wedge dh)\\
&=df\wedge dg\wedge dh+\cancelto{0}{f\,d(dg)\wedge dh}+\cancelto{0}{(-1)^1 dg\wedge d(dh)}\\
&=df\wedge dg\wedge dh.
\end{aligned}
$$

As $\omega\wedge\mu=-\mu\wedge\omega$, $\omega,\mu\in\Omega^1(M)$

$$
\begin{aligned}
d(-\mu\wedge\omega)=-d(\mu\wedge\omega)&=-d\mu\wedge\omega+\mu\wedge d\omega\\
&=-\omega\wedge d\mu+d\omega\wedge\mu\\
&=d(\omega\wedge\mu)
\end{aligned}
$$

let $\mathbb{R}^3$, the 1-form.

$$\omega=\omega_x\,dx+\omega_y\,dy+\omega_z\,dz$$

$$d\omega=d\omega_x\wedge dx+d\omega_y\wedge dy+d\omega_z\wedge dz$$

$$\frac{\partial\omega_x}{\partial x}dx+\frac{\partial\omega_x}{\partial y}dy+\frac{\partial\omega_x}{\partial z}dz$$

$$
\begin{aligned}
d\omega=&\ \frac{\partial\omega_x}{\partial y}dy\wedge dx+\frac{\partial\omega_x}{\partial z}dz\wedge dx\\
&+\frac{\partial\omega_y}{\partial x}dx\wedge dy+\frac{\partial\omega_y}{\partial z}dz\wedge dy\\
&+\frac{\partial\omega_z}{\partial x}dx\wedge dz+\frac{\partial\omega_z}{\partial y}dy\wedge dz
\end{aligned}
$$

$$d\omega=\left(\frac{\partial\omega_y}{\partial x}-\frac{\partial\omega_x}{\partial y}\right)dx\wedge dy+\left(\frac{\partial\omega_z}{\partial y}-\frac{\partial\omega_y}{\partial z}\right)dy\wedge dz$$

$$+\left(\frac{\partial \omega_z}{\partial x}-\frac{\partial \omega_x}{\partial z}\right) dz\wedge dx$$

i.e., the exterior derivative of an 1-form in $\mathbb{R}^3$ is the rotational.

Now, let be 2-form in $\mathbb{R}^3$

$$\omega=\omega_{xy}\, dx\wedge dy+\omega_{yz}\, dy\wedge dz+\omega_{zx}\, dz\wedge dx.$$

$$\begin{aligned} d\omega &= d\omega_{xy}\wedge dx\wedge dy+d\omega_{yz}\wedge dy\wedge dz+d\omega_{zx}\wedge dz\wedge dx \\ &=\partial_z\omega_{xy}\, dz\wedge dx\wedge dy+\partial_x\omega_{yz}\, dx\wedge dy\wedge dz+\partial_y\omega_{zx}\, dy\wedge dz\wedge dx \\ &=(\partial_z\omega_{xy}+\partial_x\omega_{yz}+\partial_y\omega_{zx})\, dx\wedge dy\wedge dz \end{aligned}$$

is the divergence.

So, in $\mathbb{R}^3$:

- Gradient, $d:\Omega^0(\mathbb{R}^3)\longrightarrow\Omega^1(\mathbb{R}^3)$
- Curl, $d:\Omega^1(\mathbb{R}^3)\longrightarrow\Omega^2(\mathbb{R}^3)$
- Divergence, $d:\Omega^2(\mathbb{R}^3)\longrightarrow\Omega^3(\mathbb{R}^3)$

Notation: In $\mathbb{R}^n$ let I a multi-index i.e., a p-tuple $(i_1,\dots,i_p)$ of an integer different from n, let dx^I be a p-form.

$$dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_p}$$

A p-form in $\mathbb{R}^n$, may be expressed.

$$\omega=\omega_I\, d\omega^I$$

$$d\omega=d\omega_I\wedge d\omega^I$$

$$d(d\omega)=0$$

$$d\omega=(\partial_\mu\omega_I)\, dx^\mu\wedge dx^I$$

$$d(d\omega)=\partial_\nu\partial_\mu\omega_I\, dx^\nu\wedge dx^\mu\wedge dx^I=0$$

symmetric in μ,ν symmetric in μ,ν.

Maxwell equations (Any manifold)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

let's consider the static case

$$\nabla \cdot \vec{B} = 0 \ , \ \nabla \times \vec{E} = 0$$

let's take the vector of the magnetic field.

$$\vec{B} = (B_x, B_y, B_z)$$

as a 2-form

$$B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy.$$

now, the electric field

$$\vec{E} = (E_x, E_y, E_z)$$

as a 1-form

$$E = E_x dx + E_y dy + E_z dz.$$

Thus the first maxwell equations (static)

$$dE = 0 \ , \ dB = 0.$$

For depending time case, we must put E and B in the space-time.

Working on $\mathbb{R}^4$ (Minkowski) using coordinates (x^0, x^1, x^2, x^3), where $x^0 := t$.

The electric and magnetic field are 1-forms and 2-forms in $\mathbb{R}^4$

$$E = E_x dx + E_y dy + E_z dz.$$

$$B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy.$$

We may combine then in a electromagnetic field.

$$F = B + E \wedge dt$$

$$F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu = F_{01} \, dx^0 \wedge dx^1 + F_{02} \, dx^0 \wedge dx^2$$

$$+ F_{03} \, dx^0 \wedge dx^3 + F_{12} \, dx^1 \wedge dx^2 + \dots$$

$$\frac{1}{2} F_{00}\, dx^0 \wedge dx^0 = 0$$

$$\frac{1}{2} F_{01}\, dx^0 \wedge dx^1 = \frac{1}{2} F_{01}\, dt \wedge dx = \frac{1}{2} F_{10}\, dt \wedge dx$$

$$F_{01}\, dt \wedge dx$$

$$F_{01} = -E_x$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Thus, the maxwell equations

$$dF = 0.$$