

Theorem (Euler's)

$$I(x) = \int_a^b f(t) e^{-x\phi(t)} dt.$$

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{C_n}{x^{(n+\beta)/\alpha}} \quad \text{as } x \rightarrow \infty$$

$$C_n = \frac{1}{\alpha n!} \left[\left. \frac{d^n}{dt^n} \right\} f(t) \left\{ \frac{(t-a)^\alpha}{\phi(t)-\phi(a)} \right\}^{\frac{n+\beta}{\alpha}} \right]_{t=a}$$

If $\phi(t) = \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$

and $f(t) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$

Example: Gamma function.

$$\Gamma(\lambda+1) = \int_0^\infty e^{-t} t^\lambda dt, \quad \text{for } \lambda > 0$$

$$\text{Change } t = \lambda(1+x) \longrightarrow dt = \lambda dx$$

$$\rightarrow \Gamma(\lambda+1) = \int_0^\infty e^{-\lambda(1+x)} \lambda^x (1+x)^\lambda \lambda dx.$$

$$= \lambda^{\lambda+1} e^{-\lambda} \int_{-1}^{\infty} e^{-\lambda x} (1+x)^\lambda dx.$$

$$= \lambda^{\lambda+1} e^{-\lambda} \int_{-1}^{\infty} e^{-\lambda(x-\log(1+x))} dx.$$

$$\frac{\Gamma(\lambda+1)}{\lambda^{\lambda+1} e^{-\lambda}} = \int_0^\infty e^{-\lambda(x-\log(1+x))} dx + \int_0^1 e^{-\lambda(x-\log(1-x))} dx$$

$\therefore I_1(\lambda) \qquad \qquad \qquad \therefore I_2(\lambda)$

(we changed
the integration
limits)

$$I_1(\lambda) = \int_0^\infty e^{-\lambda \phi(x)} dx$$

$$\phi(x) = x - \log(1+x)$$

$$f(x) = 1.$$

$$\log(1+x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\phi(x) = x - \log(1+x) \sim \frac{x^2}{2} - \frac{x^3}{3} + \dots$$

therefore $\alpha = 2$

fixed around zero.

$$I_1(\lambda) \sim e^{-\lambda \phi(0)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{\lambda^{\frac{(n+1)/2}{2}}}, \text{ as } \lambda \rightarrow \infty$$

$$C_n = \frac{1}{2n!} \left. \frac{d^n}{dx^n} \left[\frac{x^2}{x - \log(1+x)} \right]^{\frac{n+1}{2}} \right|_{x=0}$$

$I_2(\lambda)$:

$$\begin{aligned} \phi(\lambda) &= -x - \log(1-x) \sim -x - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right) \\ &\sim \frac{x^2}{2} + \frac{x^3}{3} + \dots, \text{ for } x \rightarrow 0^+ \end{aligned}$$

$$I_2(\lambda) \sim e^{-\lambda \phi(0)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n (-1)^n}{\lambda^{\frac{(n+1)/2}{2}}} \quad \text{same } C_n \text{ as before.}$$

$$\because \Gamma(\lambda+1) = \lambda^{\lambda+1} e^{-\lambda} \left(\sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{\lambda^{\frac{(n+1)/2}{2}}} (1 + (-1)^n) \right)$$

$$= 2\lambda^{\lambda+1} e^{-\lambda} \sum_{n=0}^{\infty} \Gamma\left(n + \frac{1}{2}\right) \frac{C_{2n}}{\lambda^{\frac{(n+1)/2}{2}}}$$

$$= \sqrt{2\pi} e^{-\lambda} \sum_{n=0}^{\infty} \frac{x_n}{\lambda^n}$$

Where

$$x_n = \sqrt{\frac{2}{\pi}} \Gamma\left(n + \frac{1}{2}\right) C_{2n}$$

Sterling coefficients.

$$\gamma_0 = 1, \quad \gamma_1 = \frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = -\frac{139}{51840}$$

$$\Gamma(\lambda+1) = \lambda \Gamma(\lambda)$$

$$\Gamma(\lambda) = \sqrt{2\pi} \lambda^{1/2} e^{-\lambda} \left(1 + \frac{1}{12} x + \dots \right) \text{ as } x \rightarrow \infty.$$

Example: Legendre polynomials $P_m(x) \quad x > 1$.

$$P_m(x) = \frac{1}{\pi} \int_0^{\pi} (x - \cos(t) \sqrt{x^2 - 1})^m dt.$$

Solution: Change $x = \cosh(\theta), \quad \theta > 0$

$$P_m(x) = \frac{1}{\pi} \int_0^{\pi} (\cosh(\theta) - \cos(t) \sinh(\theta))^m dt.$$

$$\begin{aligned} \cosh \theta - \cos(t) \sinh(\theta) &= \frac{e^\theta + e^{-\theta}}{2} - \cos(t) \frac{e^\theta - e^{-\theta}}{2} \\ &= e^\theta \left(\frac{1 + \cos(t)}{2} \right) + e^{-\theta} \left(\frac{1 - \cos(t)}{2} \right) \\ &= e^\theta \cos^2\left(\frac{t}{2}\right) + e^{-\theta} \sin^2\left(\frac{t}{2}\right) \\ &= e^\theta \left(1 - \sin^2\left(\frac{t}{2}\right) \right) + e^{-\theta} \sin^2\left(\frac{t}{2}\right) \\ &= e^\theta \left[1 - \sin^2\left(\frac{t}{2}\right) (1 - e^{-2\theta}) \right] \end{aligned}$$

$$P_m(x) = \frac{1}{\pi} e^{m\phi} \int_0^{\pi} e^{-m(-\log[1 - \sin^2\left(\frac{t}{2}\right)(1 - e^{-2\theta})])} dt.$$

$$\phi(t) = -\log[1 - \sin^2\left(\frac{t}{2}\right)(1 - e^{-2\theta})]$$

$$f(t) = 1, \quad \beta = 1.$$

$$\begin{aligned}
\phi(t) &= \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^k}{k} \leq (1-e^{-2\theta}) \left(\frac{t}{2}\right)^2 \\
&= \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^k}{k} \left(\frac{t}{2} - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right)^2 \\
&= \sum_{k=1}^{\infty} \frac{(1-e^{-2\theta})^k}{k} \left(\frac{t^2}{4} - \frac{2t^4}{12} - O(t^6) \right)^2 \\
&= \frac{1-e^{-2\theta}}{2} \left(\frac{t^2}{4} - \frac{2t^4}{12} \right) + \frac{(1-e^{-2\theta})^2}{2} \frac{t^4}{16} + O(t^6)
\end{aligned}$$

Thus $\alpha = 2$.

$$P_m(\cosh(\theta)) \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{C_n}{m^{(n+1)/2}}$$

where

$$C_n = \frac{1}{2n!} \left. \frac{d^n}{dt^n} \left[\frac{t^2}{-\log[1 - \sin^2(t/2)(1 - e^{-2\theta})]} \right]^{\frac{n+1}{2}} \right|_{t=0}$$

$$C_0 = \left. \frac{1}{2} \left[\frac{t^2}{-\log[1 - \sin^2(t/2)(1 - e^{-2\theta})]} \right]^{\frac{n+1}{2}} \right|_{t=0}$$

$$= \frac{1}{\sqrt{1 - e^{-2\theta}}}$$

$$f(t) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} = 1 = b_0, \quad b_j = 0, \quad \forall j \neq 0.$$

We want $g(t)\phi(t) = f(t)$, then.

$$\sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1} = \left(\sum_{j=0}^{\infty} C_j (t-a)^j \right) \left(\sum_{l=0}^{\infty} \alpha_l (l+\alpha) (t-a)^{l+\alpha-1} \right)$$

$$\rightarrow b_n = \sum_{k=0}^n C_k \alpha_{n-k}^{\alpha} (n-k+\beta)^{\alpha}$$

for $n=0$.

$$b_0 = C_0 \alpha_0^{\alpha} \sqrt{2} \quad \rightarrow \quad C_0 = \frac{b_0}{\alpha_0^{\alpha} \sqrt{2}} = \frac{1}{\sqrt{\frac{1-e^{-2\theta}}{2} \sqrt{2}}}$$

for $n=1$

$$0 = b_1 = C_0 A_1^{1/2} \sqrt{3} + C_1 A_0^{1/2} \sqrt{2} \longrightarrow C_1 = -\frac{C_0 A_1^{1/2} \sqrt{3}}{A_0^{1/2} \sqrt{2}} = 0$$

for $n=2$

$$0 = b_2 = C_0 A_2^{1/2} \sqrt{4} + C_1 A_1^{1/2} \sqrt{3} + C_2 A_0^{1/2} \sqrt{2}$$

$$C_2 = -\frac{C_0 A_2^{1/2} \sqrt{4}}{A_0^{1/2} \sqrt{2}}$$

for $n=3 \longrightarrow C_3 = 0$

Method of Stationary Phase.

Let's consider $\phi(t)$ in Laplace Integral as a pure imaginary function.

$\Rightarrow \phi(t) = i\psi(t)$, where $\psi(t)$ is real.

$$\int_a^b f(t) e^{-xt\phi(t)} dt \longmapsto I(x) := \int_a^b f(t) e^{ix\psi(t)} dt$$

General Fourier Integral.

For $\psi(t) = t$,

$$I(x) = \int_a^b f(t) e^{ixt} dt$$

Ordinary Fourier Integral.

Example:

$$I(x) = \int_0^1 \sqrt{t} e^{ixt} dt = \int_0^1 \sqrt{t} \frac{1}{ix} \frac{d}{dt}(e^{ixt}) dt, x \rightarrow \infty$$

$$= \frac{1}{ix} \sqrt{t} e^{ixt} \Big|_0^1 - \frac{1}{2ix} \int_0^1 \frac{e^{ixt}}{\sqrt{t}} dt + \theta\left(\frac{1}{x^2}\right) = \frac{1}{ix} e^{ix}$$

$$=: I_1(x)$$

$I_1(x)$ Change $s = xt$, $ds = xdt$

$$I_1(x) = \frac{i}{2x} \int_0^x \frac{e^{is}}{\sqrt{s/x}} \frac{ds}{x} = \frac{i}{2x^{3/2}} \int_0^x \frac{e^{is}}{\sqrt{s}} ds, x \rightarrow \infty$$

Change $s \mapsto is$ (Wick rotation) \rightarrow it's a transformation that substitutes an imaginary-number variable for a real-number variable and vice versa.

$$I_1(x) \sim \frac{i}{2x^{3/2}} \int_0^\infty \frac{e^{-s}}{s^{1/2}} s^{1/2-1} ds$$

$$\begin{aligned} &= \frac{i}{2x^{3/2}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \frac{i}{x^{3/2}} (e^{i\pi/4})^{1/2} \\ &= \frac{i\sqrt{\pi}}{2x^{3/2}} e^{i\pi/4}, \quad \text{as } x \rightarrow \infty. \end{aligned}$$