Free scalar field: Path integral

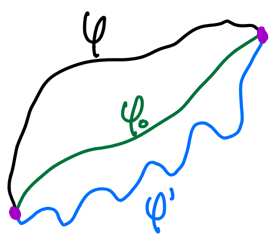
Generating function

$$W_0[J] \propto \int D\varphi\, e^{iS_0[\varphi, J]}$$

$$S_0[\varphi, J] = \int dx \left(\mathcal{L} + J(x)\varphi(x) \right); \quad \mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

denote: $d^4x \longrightarrow dx$

$$S_0[\varphi, J] = -\frac{1}{2}\int dx\, \varphi(\Box + m^2)\varphi + \int dx\, J\varphi$$



$$(\Box + m^2)\varphi_0 = J \longrightarrow \varphi = \varphi + \varphi_0$$

$$D\varphi \longrightarrow D\varphi$$

$$S_0[\varphi, J] \longrightarrow \int dx \left[-\frac{1}{2}(\varphi + \varphi_0)(\Box + m^2)(\varphi + \varphi_0) + J(\varphi + \varphi_0) \right]$$

$$S_0[\varphi, J] = \int dx \left[\mathcal{L}(\varphi) - \varphi(\Box + m^2)\varphi_0 - \frac{1}{2}\varphi_0(\Box + m^2)\varphi_0 \right.$$
$$\left. + J(\varphi + \varphi_0) \right]$$
$$= \int dx \left[\mathcal{L}(\varphi) \cancel{- \varphi J} - \frac{1}{2}\varphi_0 J + \cancel{J\varphi} + J\varphi_0 \right]$$

therefore,

$$S_0[\varphi, J] = \int dx \left[\mathcal{L}(\varphi) + \frac{1}{2}\varphi_0 J \right]$$

$$W_0[J] \propto \underbrace{\int D\varphi\, e^{iS_0[\varphi]}}_{=: Z} \cdot e^{i\frac{1}{2}\int dx\, \varphi_0 J}$$

$$W_0[J]\Big|_{J=0} = 1$$

$$\therefore\; W_0[J] = e^{i\frac{1}{2}\int dx\, J(x)\varphi(x)}.$$

but: $(\Box + m^2)\varphi_0(x) = J(x)$

then:

$$\varphi_0(x) = i\int dy\, \Delta_F(x-y)\, J(y)$$

therefore,

$$\boxed{W_0[J] = e^{-\frac{1}{2}\int dx dy\, J(x)\,\Delta_F(x-y)\,J(y)}}$$

Generating function for the free scalar field.

Green functions

$$G^{(n)}(x_1,\dots,x_n) = (-i)^n \frac{\delta^n W_0[J]}{\delta J(x_n)\cdots\delta J(x_1)}\Big|_{J=0}$$

1. $$G^{(1)}(x) = -i\frac{\delta W_0[J]}{\delta J(x)}\Big|_{J=0} = -i\frac{\delta}{\delta J(x)}\exp\left(-\frac{1}{2}\int dy dz\, J(y)\,\Delta_F(y-z)\,J(z)\right)\Big|_{J=0}$$

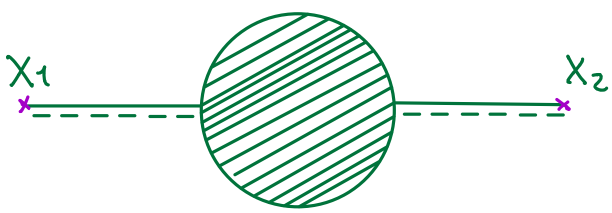
$$= -i\,\underbrace{W_0[J]}_{=1}\left(-\frac{1}{2}\,\frac{\delta}{\delta J(x)}\int dy dz\, J(y)\,\Delta_F(y-z)\,J(z)\right)\Big|_{J=0}$$

$$= -i\,(-)\int dy\,\Delta_F(x-y)\,J(y)\Big|_{J=0}$$

$\longrightarrow G^{(1)}(x) = 0$

$\rightarrow \langle 0|\varphi(x)|0\rangle = 0$

II. $$G^{(2)}(x_1,x_2) = (-i)^2\frac{\delta W_0[J]}{\delta J(x_2)\delta J(x_1)}\Big|_{J=0}$$



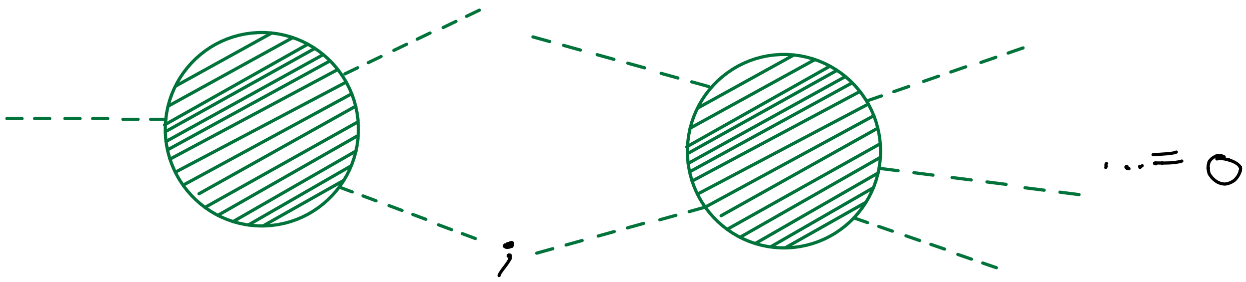
$$= (-i)^2\frac{\delta}{\delta J(x_2)}(-)\,W_0\int dy\,\Delta_F(x_1-y)\,J(y)\Big|_{J=0}$$

$$= (-i)^2(-)\left[\frac{\delta W_0}{\delta J(x_2)}\int dy\,\Delta_F(x_1-y)\,J(y) + W\cdot\Delta_F(x_1-x_2)\right]_{J=0}$$

$$= + \quad \hookrightarrow -W_0\int dy'\,\Delta_F J\cdot\int dy\,\Delta_F J \longrightarrow 0$$

therefore, $G^{(2)}(x_1,x_2)=\Delta_F(x_1-x_2)$:

III. $G^{(2n+1)}=0$



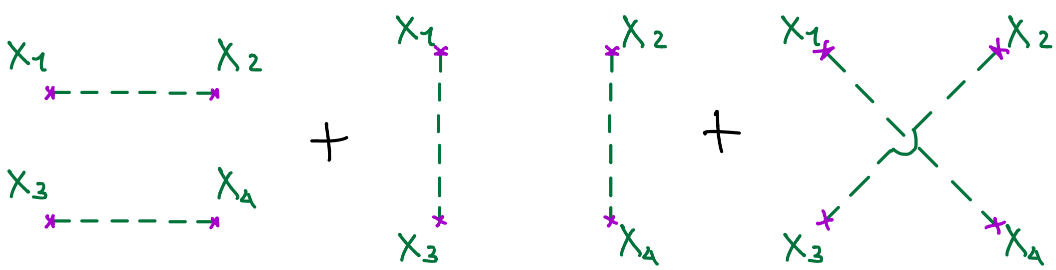
$$W[J]=e^{-\frac{1}{2}\iint J_x \Delta_F J_y} \longrightarrow \delta^{(2n+1)} : \quad (\quad)\cdot\int \Delta_F J(x).$$

IV. $$G^{(4)}(x_1,\dots,x_4)=(-i)^4(-)\frac{\delta^2}{\delta J(x_4)\delta J(x_3)} W_0\left[-\int dy\, J\Delta_F(y-x_1)\int dy'\, J\Delta_F(y'-x_2)+\Delta_F(x_1-x_2)\right]\Bigg|_{J=0}.$$

$$=(-)\left[\frac{\delta^2 W_0}{\delta J(x_4)\delta J(x_3)}\bigg|_{J=0}\left(-\cancelto{0}{\left(\iint J\Delta\right)^2_{J=0}}+\Delta_F(x_1-x_2)\right)-\cancelto{0}{\frac{\delta W_0}{\delta J}}\bigg|_{J=0}\frac{\delta}{\delta J}\left[\quad\right]_{J=0}\right.$$
$$\left.-W_0[0]\left(\Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)\right)\right]$$

therefore,

$$G^{(4)}(x_1,\dots,x_4)=$$
$$\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4)+\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)$$



In general,

$$G^{(2n)}(x_1,\dots,x_{2n})=\sum_{\text{Permutations}} G^{(2)}(x_{p_1},x_{p_2})\cdots G^{(2n)}(x_{p(2n-1)},x_{p(2n)})$$

Disconected, due to absence of interactions.

Definition: Generating function of conected Green functions.

$$W[J] = e^{iX[J]}$$

Definition: Conected Green function.

$$(i)^n g^{(n)}(x_1, \dots, x_n) = i \left. \frac{\delta^n X[J]}{\delta J(x_n) \cdots \delta J(x_1)} \right|_{J=0}$$

$$X[J] = \sum_{n=1}^{\infty} \frac{(i)^{n-1}}{n!} \int \left(\prod_{i=1}^{n} dx_i \right) g^{(n)}(x_1, \dots, x_n) J(x_1) \cdots J(x_n)$$

$$X[0] = 0$$

Free scalar field:

$$X_0[J] = \frac{i}{2} \int dx\, dy\, J(x) \Delta_F (x-y) J(y)$$

therefore

$$g^{(2)}(x_1, x_2) = \Delta_F (x_1, x_2) \quad \text{is non-null.}$$

Definition: Irreducible Green functions for a particle (OPI).

→ Conected diagrams that not may be disconected by "cutting" just one intern line.

observe,

$$X_0[J] = \frac{i}{2} \int dx\, dy\, J(x) \Delta_F (x-y) J(y)$$

then,

$$\frac{\delta X_0[J]}{\delta J(x)} = i \int dy\, \Delta (x-y) J(y) = \varphi_c(x)$$

classic solution

Definition: effective action:

$$\Gamma(\varphi_c) = X[J] - \int dx\, J(x) \varphi_c(x)$$

$$\Gamma[\varphi_c] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \left(\prod_{i=1}^{n} dx_i \right) \Gamma^{(n)}(x_1, \dots, x_n) \varphi(x_1) \cdots \varphi(x_n)$$

OPI function

Free theory: $\Gamma_0[\varphi_c] = -\frac{1}{2}\int dx\, \varphi_c(x) J(x) = \frac{-1}{2}\int dx\, \varphi_c(\Box + m^2)\varphi_c$

then,
$$\Gamma_0[\varphi_c] = S_0[\varphi_c]$$

Unique non-null function:

$$\Gamma^{(2)}(x,y) = (\Box_x + m^2)\delta(x-y)$$

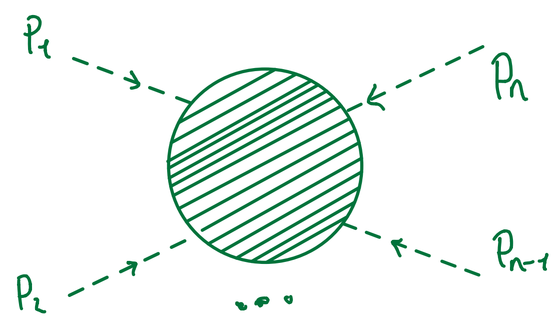
Green function in the momentum space

Traslational invariance $\longrightarrow$ Green function: $\mathcal{F}(x_i - x_j)$

$\downarrow$

momentum conservation.

$$\int dx_1 \cdots dx_n \mathcal{F}^{(n)}(x_1,\ldots,x_n) e^{i\sum_i p_i x_i} = \tilde{\mathcal{F}}^{(n)}(p_1,\ldots,p_n)(2\pi)^4 \delta(\sum_i p_i)$$



clearly,
$$\tilde{g}^{(2)}(p,-p) = \tilde{\Delta}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

while
$$\tilde{\Gamma}^{(2)}(p) = -(p^2 - m^2) \rightarrow i\tilde{\Gamma}(p) = \left[\frac{i}{p^2-m^2}\right]^{-1} = \tilde{\Delta}_F^{-1}(p)$$

Notice,
$$i\tilde{\Gamma}^{(2)}(p) = \tilde{\Delta}_F^{-1}(p)\Delta_F(p)\Delta_F^{-1}(p)$$