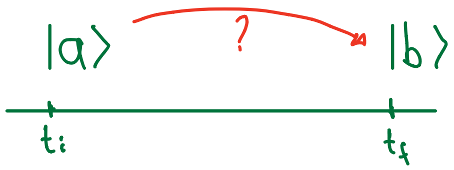
The S matrix

Dispersion problem:



$$|a(t)\rangle = \underbrace{e^{-iH(t-t_i)}}_{U(t_i,t)}|a(t_i)\rangle$$

amplitude: $\langle b| e^{-iH(t_f-t_i)}|a(t_i)\rangle$

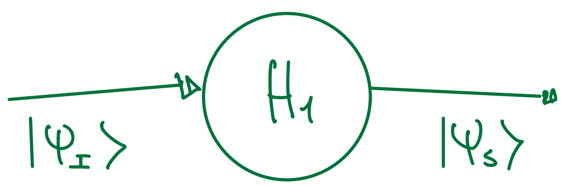
$$S_{ab} = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \langle b| e^{-iH(t_f-t_i)}|a\rangle$$

$$S = \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} U(t_f,t_i)$$

Unitary operator: $SS^\dagger = S^\dagger S = 1$

Define: $S = 1 + iT$

T- transition matrix



$$H(t) = H_0(t) + H_1$$

In general, the basis of

$$O(t_0): \{|O, t_0\rangle\}$$

no define an eigenstate of $O(t_n)$; $t_n \neq t$.

Then, the interaction scheme

$$\langle\psi_F|O(t)|\psi_I\rangle_S = \langle\psi_F(t)|O_I(t)|\psi_I(t)\rangle_I$$

where $|\psi(t)\rangle_I = e^{iH_0t}e^{-iHt}|\psi\rangle_S$, and

$$O_I(t) \equiv e^{iH_0t}O(0)e^{-iH_0t} \rightarrow \text{evolves with } H_0$$

therefore,

$$\frac{\partial}{\partial t}|\psi(t)\rangle_I = -i\left(e^{iH_0t}H_1e^{-iH_0t}\right)|\psi(t)\rangle_I$$

finally

$$\frac{\partial}{\partial t}|\psi(t)\rangle_I = -i\,H_I(t)|\psi(t)\rangle_I$$

Evolution operator:

$$|\Psi(t)\rangle_I = U(t,t_0)|\Psi(t_0)\rangle_I$$

then,

$$\frac{\partial}{\partial t} U(t,t_0) = -i H_I U(t,t_0).$$

Is clear that:

$$U(t_0,t_0) = \mathbb{1} \; ; \quad U(t,t')U(t',t_0) = U(t,t_0)$$

$$U^{\dagger}(t',t_0)U(t',t_0) = \mathbb{1} \; ; \quad U^{\dagger}(t,t') = U(t',t)$$

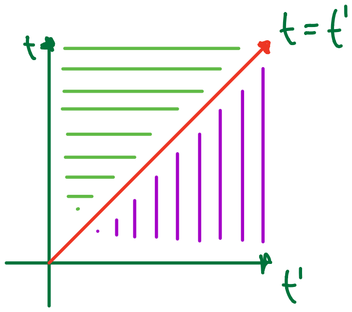
So

$$U(t_1,t_0) = \mathbb{1} - i\int_{t_0}^{t_1} dt \, H_I(t) U(t,t_0)$$

Iterating:

$$U(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t_1} dt \, H_I(t) + (-i)^2 \int_{t_0}^{t_1} dt \int_{t_0}^{t} dt' \, H_I(t) H_I(t')$$

$t' < t < t_1$



$$: \int_{t_0}^{t_1} dt \int_{t_0}^{t} dt' \, H_I(t) H_I(t') = \int_{0}^{t_1} dt' \int_{t_0}^{t'} dt \, H_I(t') H_I(t)$$

$t < t' < t_1$

In this way

$$\int_{0}^{t_1} dt \int_{t_0}^{t} dt' \, H_I(t) H_I(t') = \frac{1}{2}\int_{t_0}^{t_1} dt \int_{t_0}^{t_1} dt' \, T(H_I(t) H(t'))$$

In general

$$\int_{t_0}^{t_1} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{n-1}} d\tau_n \, H_I(\tau_1) \cdots H_I(\tau_n)$$

$$= \frac{1}{n!} \int_{t_0}^{t_1} d\tau_1 \cdots d\tau_n \, T(H_I(\tau_1) \cdots H_I(\tau_n))$$

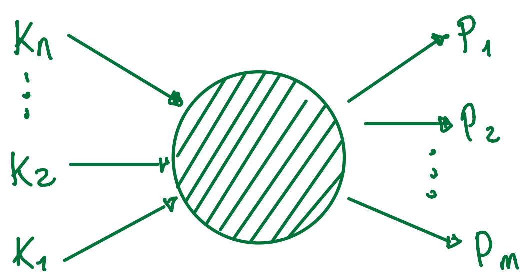
Therefore,

$$S \equiv \lim_{\substack{t_0 \to -\infty \\ t_1 \to \infty}} U(t_1, t_0) = T \exp\left(-i \int_{-\infty}^{\infty} dt \, H_I(t)\right)$$

The total dispersion amplitude is

$$S_{fi} \equiv \langle f | S | i \rangle$$

$\phi : n \longrightarrow m$



$$A_{fi} = \langle p_1, \ldots, p_m ; t_f | k_1, \ldots, k_n ; t_i \rangle$$

Heisenberg Picture

Free asymptotic states.

$$u_k = \frac{e^{-ikx}}{\sqrt{(2\pi)^3 2\omega_k}} \longrightarrow a_k^\dagger = -i \int d^3x \, e^{-ikx} \overleftrightarrow{\partial_0} \phi(x)$$

$$a_k^\dagger |0\rangle = |k\rangle$$

$$\phi(x) \to \begin{cases} \phi_{in} & t \to -\infty \\ \phi_{out} & t \to \infty \end{cases}$$

Define:

$$a_k^{\dagger(out, in)} = -i \lim_{t \to \pm\infty} \int d^3x \, e^{-ikx} \overleftrightarrow{\partial_0} \phi(x).$$

using

$$\int_{-\infty}^{\infty} \frac{\partial f(t)}{\partial t} = \lim_{t \to \infty} f(t) - \lim_{t \to -\infty} f(t) :$$

$$\left(a_k^{\dagger(out)} - a_k^{\dagger(in)}\right) = -i \int d^4x \, \partial_0 \left(e^{-ikx} \overleftrightarrow{\partial_0} \phi(x)\right)$$

algebra:

$$\int d^4x \, \partial_0 \left(e^{-ikx} \overleftrightarrow{\partial_0} \phi(x)\right) = \int d^4x \, \partial_0 \left[e^{-ikx} \partial_0 \phi - \phi \partial_0 e^{-ikx}\right]$$

$$= \int d^4x \left[(-i\omega_k) e^{-ikx} \partial_0 \phi - \partial_0 \phi (-i\omega_k) e^{-ikx} + e^{-ikx} \partial_0^2 \phi - \phi \partial_0^2 e^{-ikx}\right]$$

$$(\Box + m^2) e^{-ikx} = 0$$

$$= \int d^4x \left[e^{-ikx} \partial_0^2 \phi - \phi \left(\nabla^2 - m^2 \right) e^{-ikx} \right]$$

$$= \int d^4x \, e^{-ikx} \left(\Box + m^2 \right) \phi(x)$$

$\phi(x)\big|_{\bar{x}\to\infty} \to 0.$

therefore

$$a_k^{+(out)} - a_k^{+(in)} = \int d^4x \, e^{-ikx} \left(\Box + m^2 \right) \phi(x)$$

Suppose that $p_i \neq k_j$, $\forall\, i,j$.

(no-spectators)

$$|k_1, \dots, k_n; t_i\rangle = a_{k_1}^{+(in)} |k_2, \dots, k_n; t_i\rangle \, ; \; \langle p_1, \dots, p_m; t_f | a_{k_j}^{+(out)} = 0$$

then

$$A_{fi} = \langle p_1, \dots, p_m; t_f | \left(a_{k_1}^{+(in)} - a_{k_1}^{+(out)} \right) | k_2, \dots, k_n; t_i \rangle$$

$$= i \int d^4x \, e^{-ikx} \left(\Box + m^2 \right) \langle p_1, \dots, p_m; t_f | \phi(x) | k_2, \dots, k_n; t_i \rangle$$

Removing p_1:

$$\langle p_2, \dots, p_m; t_f | a_{p_1}^{(out)} \phi(x) | k_2, \dots, k_n; t_i \rangle$$

$$= \langle p_2, \dots, p_m; t_f | \left(a_{p_1}^{(out)} \phi(x) - \phi(x) a_{p_1}^{(in)} \right) | k_2, \dots, k_n; t_i \rangle$$

??

$$\phi(t \to t_i) : a^{(in)} \qquad \longrightarrow \qquad T\left(a_p^{(in)} \phi(x)\right) = \phi(x) a_p^{(in)}$$

$$\phi(t \to t_f) : a^{(out)} \qquad \qquad \qquad T\left(a_p^{(out)} \phi(x)\right) = a_p^{(out)} \phi(x)$$

therefore

$$\langle p_1, \dots, p_m; t_f | \phi(x) | k_2, \dots, k_n; t_i \rangle$$

$$= \langle p_2, \dots, p_m; t_f | T\left[\left(a_{p_1}^{(out)} - a_{p_1}^{(in)} \right) \phi(x) \right] | k_2, \dots, k_n; t_i \rangle$$

$$= i \int d^4y \, e^{ip_1 y} \left(\Box_y + m^2 \right) \langle p_2, \dots, p_m; t_f | T\left(\phi(y) \phi(x) \right) | k_2, \dots, k_n; t_i \rangle$$

therefore.

$$S_{fi} \equiv \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} A_{fi} = \langle p_1, \dots, p_m | \; iT \; | k_1, \dots, k_n \rangle$$

$$= i^{(n+m)} \int \prod_{i=1}^{n} d^4x_i \prod_{j=1}^{m} d^4y_j \, e^{-i\sum k_i x_i} e^{-i\sum k_j y_j}$$

$$\times \prod_{i=1}^{n} (\Box_{x_i} + m^2) \prod_{j=1}^{m} (\Box_{y_j} + m^2) \langle 0 | T(\phi(x_1)\cdots\phi(y_m)) | 0 \rangle$$

by parts:

$$\prod_{i=1}^{n} \int d^4x_i \, e^{-ik_i x_i} \prod_{j=1}^{m} \int d^4y_j \, e^{-ik_j y_j} \langle 0 | T(\phi(x_1)\cdots\phi(y_m)) | 0 \rangle$$

$$= \left(\prod_{i=1}^{n} \frac{i}{k_i^2 - m^2} \right) \left(\prod_{j=1}^{m} \frac{i}{k_j^2 - m^2} \right) \langle p_1, \dots, p_m | \, iT \, | k_1, \dots, k_n \rangle$$

LSZ reduction formula.

Definition: n-points Green function.

$$G^{(n)}(x_1,\dots,x_n) = \langle 0 | T(\phi(x_1)\cdots\phi(x_n)) | 0 \rangle$$

Using the interaction scheme, is possible to show that

$$G^{(n)}(x_1,\dots,x_n) = \frac{\langle 0 | T\left[\phi_I(x_1)\cdots\phi_I(x_n) \exp\left(-i\int d^4x \, \mathcal{H}_I\right)\right] | 0 \rangle}{\langle 0 | T\left(\exp\left(-i\int d^4x \, \mathcal{H}_I\right)\right) | 0 \rangle}$$

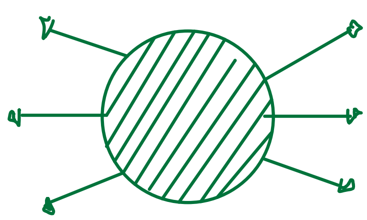
where $\mathcal{H}_I = e^{iH_0\tau} \mathcal{H}' e^{-iH_0\tau}$

Definition:

$$G^{(n)}(x_1,\dots,x_n) = \int \prod_{i=1}^{n} \frac{d^4 q_i}{(2\pi)^4} e^{-iq_i x_i} \tilde{G}^{(n)}(q_1,\dots,q_n) (2\pi)^4 \delta\left(\sum q_i\right)$$

LSZ formula:

$$\langle p_n, \dots | iT | -p_1, \dots \rangle = i^n \int \prod \left[dx_i \, e^{ip_i x_i} (\Box_i - m^2) G^{(n)}(x_1, \dots, x_n) \right.$$



$$= (2\pi)^4 \delta\left(\sum p\right) \cdot \left[\prod \tilde{\Delta}_F(p_i)^{-1} \right] \tilde{G}^{(n)}(p_1, \dots, p_n)$$

correlation function without external propagators.

Definition: Invariant amplitude:

$$\mathcal{M}_{fi} = \left[\hat{\prod} \tilde{\Delta}_F (p_i)^{-1} \right] \tilde{G}^{(n)}(p_1, \dots, p_n) :$$