The spin-half paramagnet

$$\mathcal{H} = -\mu_o H \sum_{j=1}^{N} \sigma_j \qquad ; \qquad \sigma_j = \pm 1.$$

$$Z = \sum_{\{\sigma_j\}} \exp(-\beta \mathcal{H}) = \sum_{\sigma_1,\dots,\sigma_N} \exp\left(\beta \mu_o \sum_{j=1}^{N} \sigma_j\right)$$

$$= \left[\sum_{\sigma_1} \exp(\beta \mu_o H \sigma_1)\right] \cdots \left[\sum_{\sigma_N} \exp(\beta \mu_o H \sigma_N)\right] = Z_1^N$$

$$Z_1 = \sum_{\sigma=\pm 1} \exp(\beta \mu_o H \sigma) = 2\cosh(\beta \mu_o H)$$

$$g(T,H) = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \ln(Z) = -\frac{1}{\beta} \lim_{N\to\infty} \frac{1}{N} \ln(Z_1^N)$$

$$= -k_B T \ln\left(2\cosh\left(\frac{\mu_o H}{k_B T}\right)\right)$$

$$s = -\left(\frac{\partial g}{\partial T}\right)_H = k_B \ln\left[2\cosh\left(\frac{\mu_o H}{k_B T}\right)\right] - k_B \left(\frac{\mu_o H}{k_B T}\right) \tanh\left(\frac{\mu_o H}{k_B T}\right)$$

$$m = -\left(\frac{\partial g}{\partial H}\right)_T = \mu_o \tanh\left(\frac{\mu_o H}{k_B T}\right)$$

In the other hand

$$\frac{1}{N}\left\langle \mu_o \sum_{j=1}^{N} \sigma_j \right\rangle = \frac{1}{N\beta} \frac{\partial}{\partial H} \ln(Z) = \frac{1}{\beta} \frac{\partial}{\partial H} \ln(Z_1)$$

$$= \frac{1}{\beta} \frac{\partial}{\partial H} \ln(2\cosh(\beta \mu_o H)) = \mu_o \tanh(\beta \mu_o H).$$

$$\chi(T,H) = \left(\frac{\partial m}{\partial H}\right)_T = \frac{\mu_o}{k_B T} \cosh^{-2}\left(\frac{\mu_o H}{k_B T}\right)$$

$$\chi(T,H=0) = \frac{\mu_o^2}{k_B T}$$ Curie's law

$$u = g + Ts = \cancel{-k_B T \ln\left(2\cosh\left(\frac{\mu_o H}{k_B T}\right)\right)} + \cancel{k_B T \ln\left(2\cosh\left(\frac{\mu_o H}{k_B T}\right)\right)}$$

$$- \cancel{k_B T}\left(\frac{\mu_o H}{\cancel{k_B T}}\right) \tanh\left(\frac{\mu_o H}{k_B T}\right) = -\mu_o H \tanh\left(\frac{\mu_o H}{k_B T}\right).$$

As $\langle \epsilon_j \rangle = -\frac{\partial}{\partial \beta} \ln(z)$, then

$$\frac{1}{N} \langle \epsilon \{\sigma_j\} \rangle = -\frac{1}{N} \frac{\partial}{\partial \beta} \ln(z) = \frac{-\mu_0 H 2}{z_1} \sinh(\beta \mu_0 H)$$

$$= -\mu_0 H \tanh\left(\frac{\mu_0 H}{K_B T}\right).$$

Finally, we get

$z = \sum_{\epsilon} \Omega(\epsilon) \exp(-\beta \epsilon)$

$$\epsilon(N_1) = -\mu_0 H N_1 + \mu_0 H (N - N_1)$$

$$z = \sum_{N_1=0}^{N} \frac{N!}{N_1! (N-N_1)!} \exp\left[\beta \mu_0 H N_1 - \beta \mu_0 H (N - N_1)\right]$$

$$= \sum_{N_1=0}^{N} \frac{N!}{N_1! (N-N_1)!} \left(\exp(\beta \mu_0 H N_1)\right) \left(\exp(-\beta \mu_0 H)\right)^{N-N_1}$$

$$= \left(\exp(\beta \mu_0 H) + \exp(-\beta \mu_0 H)\right)^N = \left(2 \cosh(\beta \mu_0 H)\right)^N.$$