Topics:

- Manifolds, vector fields, differential forms, Maxwell equations, cohomology of Maxwell equations.
- Gauge fields, symmetries (Lie algebras and groups), fibre boundles, conections, curvature. Yang-Mills and Chern-Simons.
- Gravitation, Riemannian geometry, Lagrangian of general relativity ($SU(2)$, $SO(3,1)$).

Bibliography:

- M. Nakahara, Geometry, topology and Physics, IOP, 2003.
- M.P. Do carmo, Differential forms and applications, springer, 2000.
- S.S. Chern, W.H. Chen, K.S. Lam, Lectures on differential geometry, 2000. (Modern differential geometry).

- The nature laws must be expressed by equations that are valid for all the coordinate systems, i.e., must be covariant.

Albert Einstein.

- For the definition of derivative in $\mathbb{R}$, is used from the fact that we can subtrack points by moving them so that no matter which way we go, but in a non-flat surface, it depends on the path.
- Let's study locally spaces that looks like $\mathbb{R}^n$ but not globally.

* The true size of the world.

To such spaces that locally are as $\mathbb{R}^n$ we will call it as manifolds

Example: The sphere $x^2+y^2+z^2=1$ locally look like $\mathbb{R}^2$ and we write S^2

In the same way, locally S^3 looks like $\mathbb{R}^3$, $x^2+y^2+z^2+w^2=1$.

Even, is possible that our space-time has more than 4-dim, as the Kaluza-Klein theory.

- The phase space of a pendulum is a cilinder, of a double pendulum is a donut.

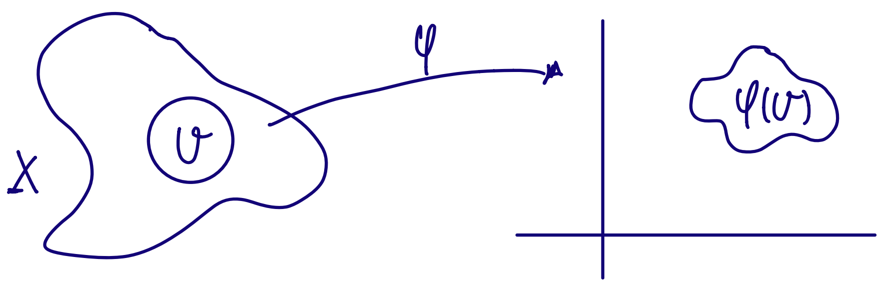
- A topological space is a set (X,τ) with a family of subsets of X called open.
- The collection of these subsets is called a topology of X
- An open set that contains to $x \in X$, is called a neighbourhood or enviroment of X.
- The use of the topology allow to define continuous functions. The trick is in make the notion of "close" using the opens.
- A function $f: X \to Y$ is continuous if given any open $U \subseteq Y$, its inverse image is also open in X, i.e., $f^{-1}(U) \subseteq X$

Homework: Prove that a function $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous in therms of last definition, if and only if $\forall x \in \mathbb{R}^n$ and $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\|y - x\| < \delta \to \|f(y) - f(x)\| < \varepsilon$.

- The idea of a manifold, is like a globe, we can get it using patches that look like $\mathbb{R}^n$. More precisely, we say that a collection of opens V_α cover a topological space X if

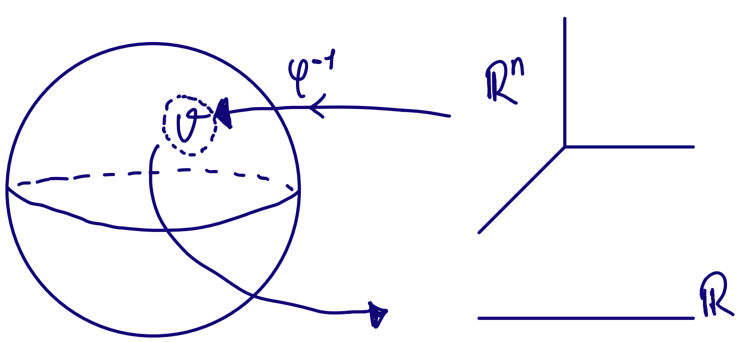
$$\bigcup V_\alpha = X$$

- Given a topological space X and an open $V \subseteq X$, we define a chart as a continuous function $\varphi: V \to \mathbb{R}^n$ with continuous inverse, defined on $\varphi(V)$

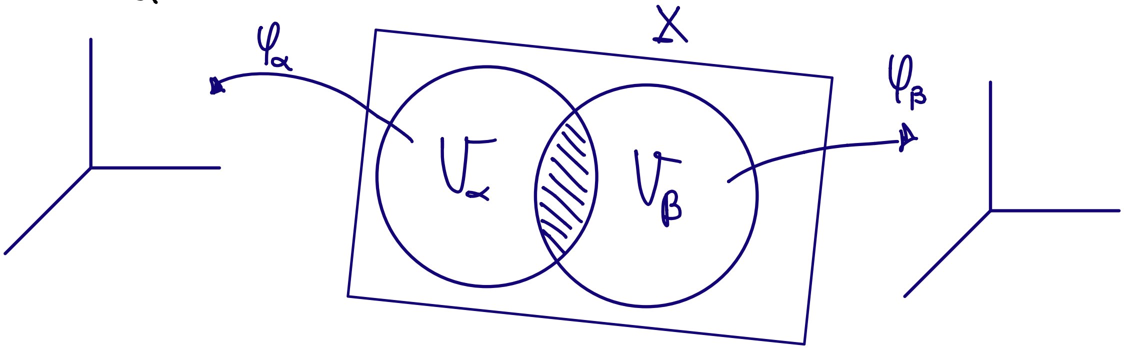


The chart is the map φ and the open is V.

While we work in the chart φ, we can pretend that we work in $\mathbb{R}^n$. For example, if we have a function $f: V \to \mathbb{R}$, we can convert it in a function of $\mathbb{R}^n$, using $f \circ \varphi^{-1}: \mathbb{R}^n \to \mathbb{R}$.



- We say that a n-dim manifold is a topological space M equiped with charts $U_\alpha : V_\alpha \to \mathbb{R}^n$ where V_α, are opens of M and they cover M, such that the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are smooth where they are defined. To such charts colection we call atlas.



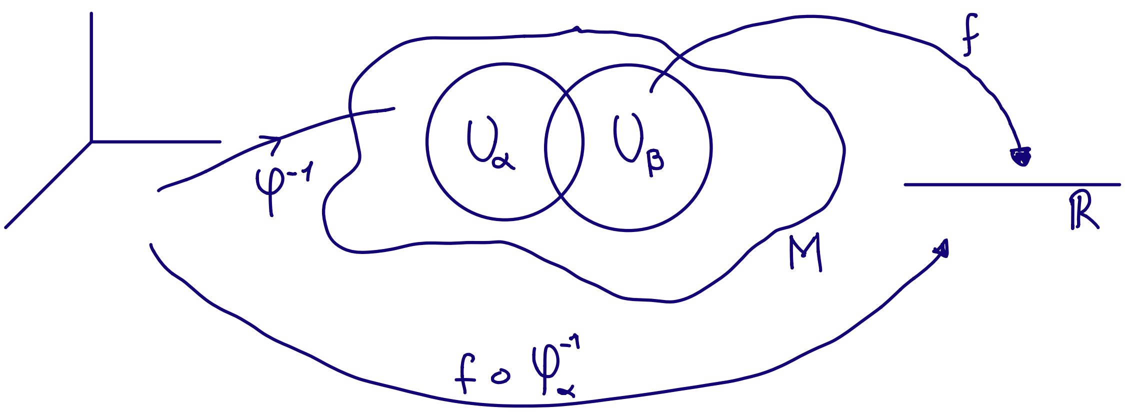
The transition functions goes from one chart to other chart.

What means this?

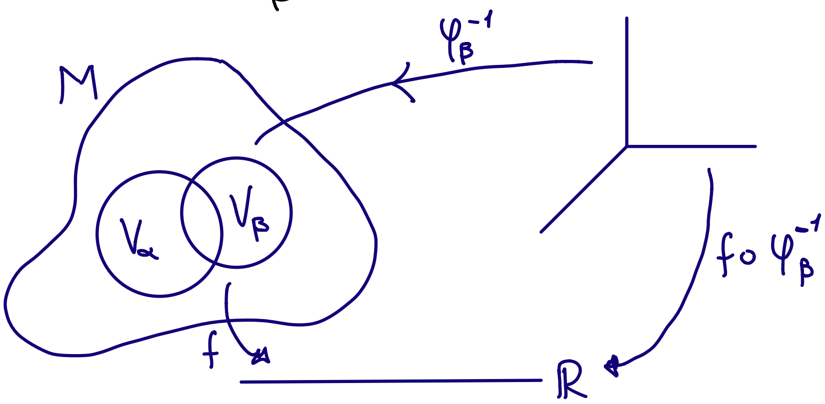
A point of M, lives in some open V_α, that looks like $\mathbb{R}^n$, i.e., we can patch the manifold with peaces that look like $\mathbb{R}^n$.

Using charts, we can Know if a function on M is smooth, $f : M \to \mathbb{R}$ is smooth if for all α, $f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth.

- lets suppose that you use the chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, and I use the chart $\varphi_\beta : V_\beta \to \mathbb{R}^n$, and let $V = U_\alpha \cap U_\beta$



And its function is smooth, i.e., $f \circ \varphi_\alpha^{-1}$ is smooth in $\varphi_\alpha(V)$. Then I will in agree with that f is smooth, i.e., $f \circ \varphi_\beta^{-1}$ is smooth in $\varphi_\beta(V)$



Since we can express my function in terms of your function.

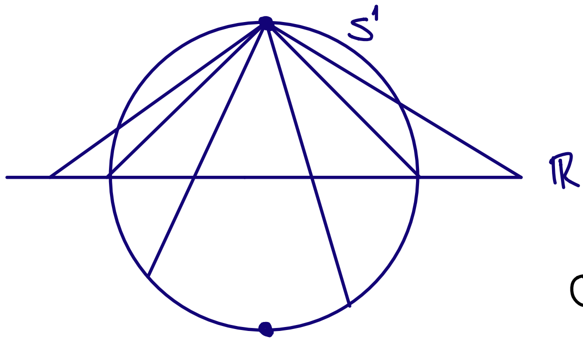
$$f \circ \varphi_\beta^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})$$

for that, in the definition of manifold we ask that the transition functions be smooth. It is said, that it is a smooth manifold. If $\varphi_\alpha \circ \varphi_\beta^{-1}$ is continuous, so we say that it has a topological manifold.

- A Hausdorff space is needed for that the convergent sequences converges to a single point.

- An example of manifold is $\mathbb{R}^n$ where the charts would be the identity.

Other example is S^1 that may map to $\mathbb{R}$.



If I take lines from the north pole and then lines go from the south pole, the intersection of this charts would be S^1 with out the poles.

Other chart would be $e^{i\theta}$.