Fermions Quantization

$$(i\not{\partial}-m)\Psi=0$$

$$\Psi(x)=\sum_s\int \tilde{d}^3p\left[a(p,s)u(p,s)e^{-ipx}+b^*(\bar{p},s)v(p,s)e^{ipx}\right]$$

Pauli principle: $\Psi \rightarrow$ anti-commutative

$\hookrightarrow$ a,b anti-commutative

Definition: Grassmann numbers

$$\theta / \{\theta,\theta\}=0 \longrightarrow \theta^2=0$$

- θ_1,θ_2 Grassmann: $\{\theta_1,\theta_2\}=0$
- $f(\theta_1,\ldots,\theta_n)=\alpha_0+\sum_i^n\alpha_i\theta_i+\sum_{i<j}\alpha_{ij}\theta_{ij}+\cdots+\alpha(\theta_1\cdots\theta_n)$

$\alpha\in\mathbb{C}$

example:

$$f(\theta_1,\theta_2)=\alpha_0+\alpha_1\theta_1+\alpha_2\theta_2+\alpha_{12}\theta_1\theta_2$$
$$=\alpha_0+\alpha_1\theta_1+\alpha_2\theta_2-\alpha_{12}\theta_2\theta_1$$

Definition: Differentiation by left-hand side

$$\frac{\partial f}{\partial\theta_1}=\alpha_1+\alpha_{12}\theta_2$$

$$\frac{\partial f}{\partial\theta_2}=\alpha_2-\alpha_{12}\theta_1$$

$$\frac{\partial^2 f}{\partial\theta_2\partial\theta_1}=-\frac{\partial^2 f}{\partial\theta_1\partial\theta_2}=-\alpha_{12}.$$

It is clear that, $\frac{\partial^2 f}{\partial\theta^2}=0 \longrightarrow$ Is not C^∞

Definition: Grassmann Integral $\longrightarrow$ As a measure concept

$d\theta$ is Grassmann $\longrightarrow \{d\theta,d\theta\}=0$

$$\int d\theta=0, \text{ because } \left(\int d\theta\right)^2=\int d\theta\, d\theta'=-\left(\int d\theta\right)^2$$

Multiple Integral: Ordered & Itered.

$$\int d\theta d\theta' f(\theta,\theta') = \int d\theta \left[\int d\theta' f \right]$$

Definition: $\int d\theta\, \theta = 1$

then, $\int d\theta_1 d\theta_2 f(\theta_1,\theta_2) = -\alpha_{12} = \frac{\partial^2 f}{\partial\theta_1 \partial\theta_2}$

Integration $\rightleftarrows$ differentiation

Ψ: Grassmann function $\longrightarrow$ also a, b.

Generating function:

$$W_0[\sigma,\bar\sigma] \propto \int D\bar\Psi D\Psi\, e^{i\int dx \left(\mathcal{L}_D + \bar\Psi\sigma + \Psi\bar\sigma \right)}$$

where $\mathcal{L}_D := \bar\Psi(i\not\partial - m)\Psi$

Writting the action with sources:

$$(i\not\partial - m)\Psi_0(x) = -\sigma(x) \quad \& \quad \bar\Psi_0(i\overleftarrow{\not\partial} + m) = \bar\sigma(x).$$

Fermionic Propagator:

$$(i\not\partial_x - m) S_F(x-y) = i\delta^4(x-y)$$

then

$$S_F(x-y) := \int \frac{d^4p}{(2\pi)^4} \left[\frac{-i}{\not p - m + i\epsilon} \right] e^{-ip(x-y)}$$

$$\tilde S_F(p) = \frac{-i(\not p + m)}{p^2 - m^2 + i\epsilon}$$

therefore, the general solution is

$$\Psi_0(x) = i\int dy\, S_F(x-y)\sigma$$

Functional form of $W_0[\sigma,\bar\sigma]$

$$\psi \longrightarrow \psi+\psi_0 \quad ; \quad \bar\psi \longrightarrow \bar\psi+\bar\psi_0$$

$$S=\int dx\left(\bar\psi(i\not\partial-m)\psi+\bar\psi\sigma+\bar\sigma\psi\right)$$

$$\longrightarrow \int dx\left((\bar\psi+\underset{-\bar\sigma}{\bar\psi_0})(i\not\partial-m)(\psi+\underset{-\sigma}{\psi_0})+(\bar\psi+\bar\psi_0)\sigma+\bar\sigma(\psi+\psi_0)\right)$$

$$=\int dx\left(\mathcal{L}_D-\bar\sigma\psi-\bar\psi\sigma-\bar\sigma\psi_0+\bar\psi\sigma+\bar\psi_0\sigma+\bar\sigma\psi+\bar\sigma\psi_0\right)$$

therefore

$$S=\int dx\left(\mathcal{L}_D+\bar\psi_0\sigma\right)$$

Then,

$$W_0[\sigma,\bar\sigma]\propto\underbrace{\int D\psi D\bar\psi\, e^{i\int dx\mathcal{L}_0}}_{=:Z}\cdot e^{i\int dx\bar\psi_0\sigma}$$

$$W[\sigma,\bar\sigma]=\exp\left(i\int dx\,\bar\psi_0\sigma\right)$$

but,

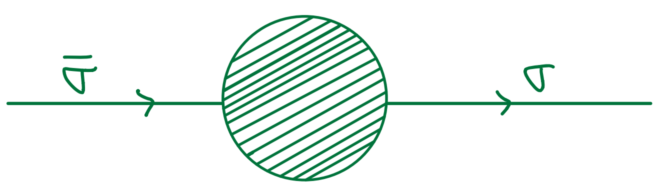
$$\psi_0(x)=i\int dy\, S_F(x-y)\sigma(y)$$

therefore

$$W_0[\sigma,\bar\sigma]=e^{-\int dx dy\,\bar\sigma(x)S_F(x-y)\sigma(y)}$$

Green functions:

$$(i)^{2n}G^{(2n)}(x_1,\dots,x_n;y_1,\dots,y_n)=\left.\frac{\delta^{2n}W_0[\sigma,\bar\sigma]}{\delta(x_1)\cdots\delta(x_n)\delta(y_1)\cdots\delta(y_n)}\right|_{\sigma=\bar\sigma=0}$$



$G^{(2n)}$ is skew-symmetric under $x_i\leftrightarrow x_j$: $\dfrac{\delta^2}{\delta\sigma(x_i)\delta\sigma(x_j)}=-\dfrac{\delta^2}{\delta\sigma(x_j)\delta\sigma(x_i)}$

$$-W_0[\sigma,\bar\sigma]=e^{-\int dx\,dy\,\bar\sigma(x)S_F(x-y)\sigma(y)}$$

$$\longrightarrow G^{(2)}(x-y)=S_F(x-y):$$

Feynman rule:

$$\tilde S_F(p)=\frac{i}{\not p-m+i\epsilon}:$$

Yukawa Interaction

$$\mathcal{L}_Y(\varphi,\psi,\bar\psi)=g\bar\psi(x)\Gamma\psi(x)\varphi(x)\;:\;\Gamma=1,\gamma_5$$

$$W_0[J,\sigma,\bar\sigma]=e^{-1/2\int dx\,dy\,J(x)\Delta_F(x-y)J(y)}\cdot e^{-\int dx'dy'\,\bar\sigma(x')S_F(x'-y')\sigma(y')}.$$

Remember:

$$S_0[\varphi]=\int dx\left(\mathcal{L}(\varphi)+J\varphi\right)$$

$$\frac{\delta}{\delta J(x)}e^{iS_0[\varphi]}=i\varphi(x)e^{iS_0'[\varphi]}\quad:\;\varphi(x)\longrightarrow -i\frac{\delta}{\delta J(x)}$$

then,

$$S_0[\psi,\bar\psi]=\int dx\left(\mathcal{L}_0+\bar\sigma\psi+\bar\psi\sigma\right)\quad\in\mathbb{R}$$

$$\frac{\delta}{\delta\sigma(x)}e^{iS_0[\psi,\bar\psi]}=-i\bar\psi(x)e^{iS_0[\psi,\bar\psi]}:\;\begin{cases}\bar\psi(x)\rightarrow i\dfrac{\delta}{\delta\sigma(x)}\\[2mm]\psi(x)\rightarrow -i\dfrac{\delta}{\delta\bar\sigma(x)}\end{cases}$$

$$W[J,\sigma,\bar\sigma]=e^{i\int dx\,\mathcal{L}_Y\left(-i\frac{\delta}{\delta J(x)},-i\frac{\delta}{\delta\bar\sigma(x)},i\frac{\delta}{\delta\sigma(x)}\right)}W_0[J,\sigma,\bar\sigma]$$

then $g\ll 1$: Perturbative development

$$\frac{\delta W_0}{\delta\bar\sigma(x)}=-\int dy\,S_F(x-y)\sigma(y)\cdot W_0$$

$$\frac{\delta W_0}{\delta\sigma(x)}=\int dy\,\bar\sigma(y)S_F(x-y)\cdot W_0$$

Feynman rule $\times(-1)$ for each fermions loop.

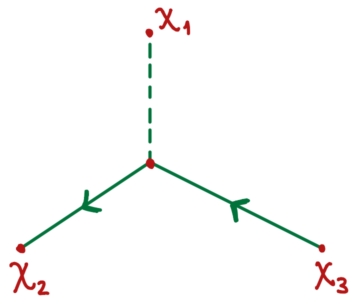
$$\frac{\delta W_0}{\delta J(x)}=-\int dy\,\Delta_F(x-y)\,J(y)\cdot W_0$$

then, we obtain

$$W[J,\sigma,\bar\sigma] \approx \left(1 + \int dx \frac{\delta}{\delta\sigma(x)} (-ig)\Gamma \frac{\delta}{\delta\bar\sigma(x)} \cdot \frac{\delta}{\delta J(x)}\right) W_0$$

the new Green function is

$$G^{(3)}(x_1,x_2,x_3) =$$



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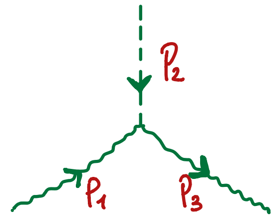
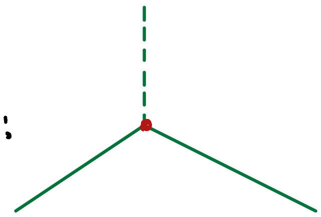


$$= i \frac{\delta^3 W[J,\sigma,\bar\sigma]}{\delta J(x_1)\delta\bar\sigma(x_2)\delta\sigma(x_3)}$$

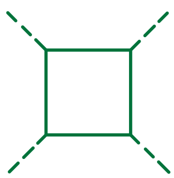
$$p_3 = p_1 + p_2$$

Feynman rule:



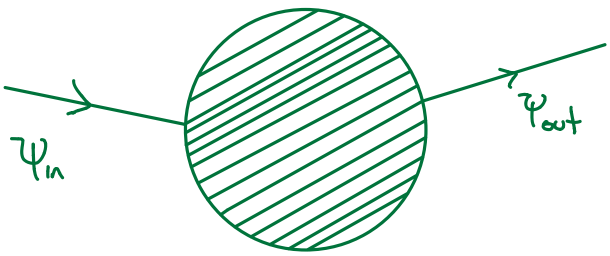
$: -ig\Gamma$

Effective Interactions



$\sim \lambda\varphi^4$; $\lambda \sim g^4$

Dispersion amplitude



$$(i\not\partial - m)\psi_0 = 0$$

$$\psi \longrightarrow \psi + \psi_0$$

$$\bar\psi \longrightarrow \bar\psi + \bar\psi_0$$

$$S_0 = \int dx \left[\bar\psi(i\not\partial - m)\psi + \bar\psi\sigma + \bar\sigma\psi\right]$$

$$\longrightarrow S_0 + \int dx \left(\bar\psi_0\sigma + \bar\sigma\psi_0\right)$$

$$(i\not\partial_x - m)\frac{\delta}{\delta\bar\sigma(x)} W_0 = -\int dy \underbrace{(i\not\partial_x - m)S_F(x-y)}_{i\delta(x-y)}\sigma(y) W_0$$

$$= -i\sigma(y) W_0$$

$$W[J,\sigma,\bar\sigma,\psi_0] = \exp\left(\int dx \left(\bar\psi_0 D_x \frac{\delta}{\delta\bar\sigma(x)} + \text{h.c.}\right)\right) W[J,\sigma,\bar\sigma]$$

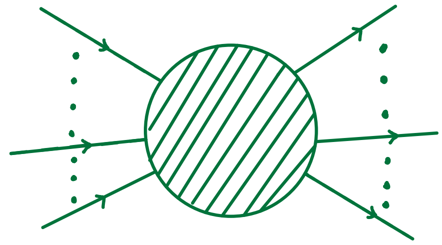
hermitian conjugate.

$$S[\Psi_0] = \exp\left(\int dx \left(\bar{\Psi}_0 D_x \frac{\delta}{\delta \bar{\sigma}(x)} + h.c. \right) \right) W[J,\sigma,\bar{\sigma}] \Big|_0$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n!} \int dx_1 \cdots dx_n \, dy_1 \cdots dy_n \, \bar{\Psi}_0(y_1) \cdots \bar{\Psi}_0(y_n) D_{y_1} \cdots D_{y_n} G^{(2)}(x_1, \dots, x_n, y_1, \dots, y_n)$$

$$\times \overleftarrow{D}_{x_1} \cdots D_{x_n} \Psi_0(x_1) \cdots \Psi_0(x_n)$$

Ψ: n-legs Ingoing

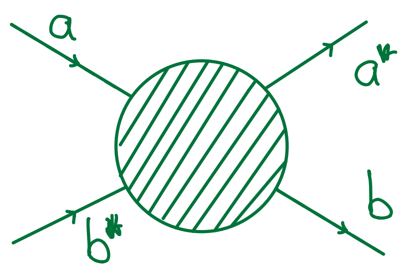


n-legs outgoing : $\bar{\Psi}$

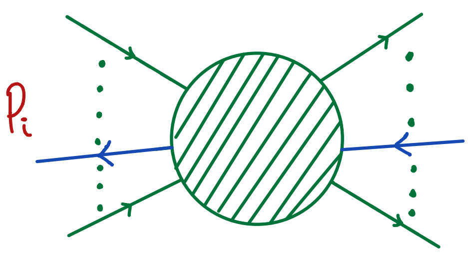
$D_x \longrightarrow \tilde{D}(p) = \not{p} - m$: will cut the external properties

$$\Psi_0(x) = \sum_s \int \tilde{d}^3 p \left[a(p,s) u(p,s) e^{-ipx} + b^*(p,s) v(p,s) e^{ipx} \right]$$

S_{fi} :
- $\delta / \delta a(p,s)$: Ingoing fermion
- $\delta / \delta a^*(p,s)$: outgoing fermion
- $\delta b^*(p,s)$: outgoing anti-fermion
- $\delta / \delta b(p,s)$: Ingoing anti-fermion



Feynman rules for m:



$\tilde{\Gamma}^{(2n)}(p_1, \dots, p_n, q_1, \dots, q_n)$

Initial fermion	:	$\times \bar{u}(p,s)$
final fermion	:	$\times u(p,s)$
Initial anti-fermion	:	$\times \bar{v}(p,s)$
final anti-fermion	:	$\times v(p,s)$