Proposition: Let $\{\varphi_i(t)\}_{i=1}^{\infty}$ be a succession of the continuous successions that form an orthonormal base of $L_2[a,b]$. Then the system

$$\{\varphi_i(t)\,\varphi_j(\tau) = \psi_{ij}(t,\tau)\}_{i,j=1}^{\infty}$$

is an orthonormal base of $L_2([a,b]^2)$

Proof: Let's note that the system $\{\psi_{ij}(t)\}$ is orthogonal and let's define.

$$a_{ij} = \int_{I^2} f(t,\tau)\overline{\psi_i(t)\,\varphi_j(t)}\,dt\,d\tau$$

Is enough to prove (by the Parseval identity)

$$\int_{I^2} |f|^2\,dt\,d\tau = \sum_{i,j} |a_{ij}|^2$$

for all the continuous functions f in I^2, since this set is dense in $L_2[I^2]$. Let

$$a_j(t) = \int_I f(t,\tau)\overline{\varphi_j(\tau)}\,d\tau$$

Let's consider the function $f(t,\tau)$ as a function of one variable τ and t fixed.

By the Parseval's identity

$$\sum_{i=1}^{\infty} |a_j(t)|^2 = \int_I |f(t,\tau)|^2\,d\tau.$$

And again by Parseval's identity.

$$\sum_{i=1}^{\infty} |a_{ij}|^2 = \int_I |a_j(t)|^2\,dt$$

Mixing the results,

$$\sum_{i,j=1}^{\infty} |a_{ij}|^2 = \int_I \sum_j |a_j(t)|^2\,dt = \int_I \int_I |f(t,\tau)|^2\,dt\,d\tau.$$

Projections

Let L a closed subspace of $\mathcal{H}$, $L \subseteq \mathcal{H}$. Let's define the projection as follow:

Let's consider the distance

$$\rho(x,L) = \inf_{y \in L} \|x-y\|$$

If there is a $y \in L$, such that $\rho(x,L) = \|x-y\|$ i.e., the infimum is reached, then we write $y = P_L x$, and we call to y, the projection of L.

Separable case: If L is a separable subspace of $\mathcal{H}$. Let $\{e_i\}_{i\geq 1}$ an orthonormal base in L, and $x \in \mathcal{H}$. Taking

$$y = \sum_{i\geq 1} \langle x, e_i \rangle e_i \in L.$$

we see that $x - y \perp e_i$, for each i, which implies $x-y \perp L$. We see that for any $z \in L$.

$$x - z = (x-y) + (y-z)$$

and,

$$\|x-z\| = \|x-y\|^2 + \|y-z\|^2, \quad x-z \perp y-z$$

then,

$$\inf\{\|x-z\| : z \in L\} = \|x-y\|$$

and,

$$P_L x = y.$$

Now the general case (No separable).

Let M a convex set, closed in $\mathcal{H}$, let $\rho(x,M)$ the distance from x to the set M.

Proposition: There exists an only one $y \in M$, such that

$$\rho(x,M) = \|x-y\|.$$

Proof: Let $y_n \in M$, $\|x-y_n\| \longrightarrow \rho(x,M) = d$, such succession exists since

$$d = \inf_{w \in M} \|x-w\|$$

It can be seen that $\{y_n\}$ is a Cauchy's succession.

This is followed from the Parallelogram law.

$$2(\|x-y_n\|^2+\|x-y_m\|^2)=\|2x-(y_n+y_m)\|^2+\|y_n-y_m\|^2$$

as $(y_n+y_m)/2 \in M$, by the convexity of M, and

$$\left\| x - \frac{(y_n+y_m)}{2} \right\| \geq d$$

The left side of the equality tends to $4d^2$ if $n,m \to \infty$.
But the right side is not less than $4d^2 + \lim \|y_n - y_m\|$.

Then, $\|y_n - y_m\| \to 0$, if $n,m \to \infty$. After that, there is

$$y = \lim_{n\to\infty} y_n \in M$$

since M is closed.

And it is unique since, if it would exist two points y, z with the distance d, we could choose a succession $y_{2n}=y$, $y_{2n+1}=z$, but this succession is not of Cauchy. ∎

Orthogonal decomposition

Let's apply the previous projection to a closed subspace L, instead of M.

We will see that there exist an unique projection $P_L x = y$. Also $\rho(x,L)=\|x-y\|$, for some $y \in L$, if and only if $x-y \perp L$.

Proof: $(\leftarrow)$ Let $y \in L$, such that $x-y \perp L$, we proved before that this y, gives the minimum distance and it is unique:

For all $z \in L$, we have

$$\|x-z\|^2 = \|x-y\|^2 + \|y-z\|^2 \geq \|x-y\|^2.$$

$(\rightarrow)$ If $y \in L$, is the projection $P_L x$, let $z \in L$, any. Then

$$\|x-y\|^2 \leq \|x-(y+\lambda z)\|^2$$

$$= \|x-y\| - 2\operatorname{Re}\lambda\langle z, x-y\rangle + |\lambda|^2 \|z\|^2$$

$$\rightarrow 2\operatorname{Re}\lambda\langle z,x-y\rangle \leq |\lambda|^2\|z\|^2$$

Taking $\lambda = t\,\overline{\langle z, x-y\rangle}$, $t \in \mathbb{R}$.

$$2t|\langle z, x-y\rangle|^2 \leq t^2 |\langle z, x-y\rangle|^2 \|z\|^2, \quad \forall\, t \in \mathbb{R}$$

if $t \to 0$

$$\langle z, x-y\rangle = 0$$

thus all $z \in L$, is orthogonal to $x-y$. ∎

In short...

Proposition: For all $x \in \mathcal{H}$, there exists an unique $y \in L$, such that $x-y$ is orthogonal to L, and $y = P_L x$, then

$$x = (x-y)+y \quad ; \quad \|x\|^2 = \|x-y\|^2 + \|y\|^2$$

Definition: For $L \subseteq \mathcal{H}$, closed, the set $L^\perp = \{x \in \mathcal{H} : x \perp L\}$, this is a closed subspace, we will call to $L^\perp$ the orthogonal complement of L in $\mathcal{H}$.

Theorem: For each $L \subseteq \mathcal{H}$, closed

$$L \oplus L^\perp = 0.$$

Proof: For each $x \in \mathcal{H}$ and $L \subseteq \mathcal{H}$, there is y such that $x = (x-y)+y$, with $x-y \in L^\perp$, and $y \in L$ (i.e., $y = P_L x$), also

$$L \cap L^\perp = 0$$

therefore, if $x = z_1 + y_1$, $z_1 \in L^\perp$, $y_1 \in L$, $y_1 = y$, $z_1 = x-y$. ∎

Corollary: If L is a closed subspace of $\mathcal{H}$, then

$$(L^\perp)^\perp = L.$$

Linear functions

Definitions: Let E be a vectorial space on $\mathbb{R}$ (or $\mathbb{C}$), the linear functions $f : E \to \mathbb{R}$ (or $\mathbb{C}$), such that

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

for all $x, y \in E$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

Examples:

1. Let c_0, let the functional f, defined by

$$f(x)=\sum_{i=1}^{\infty} a_i b_i$$

with b_i satisfying

$$\sum_{i=1}^{\infty} |b_i| < \infty$$

i.e., $b_i \in \ell_1$

2. Let ℓ_p, $1<p<\infty$, and let f the functional

$$f(x)=\sum_i a_i b_i$$

with $(b_i) \in \ell_q$, we see that

$$|f(x)| \leq \sum_i |a_i||b_i| \leq \|x\|_{\ell_p} \|f\|_{\ell_q}$$

the functional defined over all ℓ_p, we can identify with an element of ℓ_q

3. Let be the space $c[0,1]$

a) $$F(x)=\int_0^1 x(t) f(t)\, dt, \quad f \text{ integrable}$$

b) $\delta_\alpha(x) = x(\alpha)$, $\alpha \in [0,1]$

$$|\delta_\alpha(x)| \leq \|x\|_{c[0,1]}$$

Let $E^{\#}$ the space of linear funtionals on E.