Gauge theory

Quantum Electrodynamics (QED): Interaction e- γ

Lagrangian Theory

$$S = \int d^4x \, \mathcal{L} \qquad \mathcal{L}_{e\gamma} = \mathcal{L}_e + \mathcal{L}_{EM} + \mathcal{L}_{INT}$$

1. Classical theory:

$$\mathcal{L}_{EM} = -\frac{1}{4} F^2 \quad ; \quad F^2 = F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

1. Free theory:

$$\partial^\mu F_{\mu\nu} = 0 \quad \rightarrow \Box A_\mu = 0$$

Gauge transformations

$$A_\mu \longrightarrow A_\mu + \partial_\mu \alpha(x)$$

$$F_{\mu\nu} \longrightarrow F_{\mu\nu}$$

Symmetry

from $\delta A_\mu = \partial_\mu \alpha$:

$$J^\mu = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right] = \partial_\nu F^{\mu\nu} = 0$$

Problem: $\mathcal{L}_e + \mathcal{L}_{INT}$: Rule of minimal coupling

$$P_\mu \longrightarrow P_\mu - qA_\mu$$

$$\frac{1}{2m}(P_\mu - qA_\mu)^2 ; \qquad P_\mu = m\dot{x}^\mu : \text{ Gauge invariant.}$$

is not gauge invariant.

but $\partial_\mu j^\mu = 0$!

How we restart the Gauge symmetry?

The nature is Quantum-relativistic.

Duality: field-particle

Poincare: $e \to \psi$ spinor (Dirac)

$\gamma \to A_\mu$ vector

free-e: $$\mathcal{L}_e = \bar{\psi}(i\not{\partial} - m)\psi$$

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 \quad ; \quad \not{\partial} = \gamma^\mu \partial_\mu \quad ; \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Chiral representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \begin{aligned} \sigma^\mu &= (\mathbb{1}, \vec{\sigma}) \\ \bar{\sigma}^\mu &= (\mathbb{1}, -\vec{\sigma}) \end{aligned}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} ; \quad \{\gamma^\mu, \gamma_5\} = 0$$

$$P_{R,L} = \frac{1 \pm \gamma_5}{2} : \quad P_{R,L}\psi = \psi_{RL} \to \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Notice: $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R.$$

$\mathcal{L}_e = \bar{\psi}(i\not{\partial} - m)\psi$: has the global symmetry

$$\psi \to e^{i\lambda}\psi \quad ; \quad \bar{\psi} \to \bar{\psi}e^{-i\lambda}$$

The Noether's current.

$$-J^\mu_\lambda = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right]\delta\psi + \delta\bar{\psi}\cancel{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\right]}^{\,0}$$

$\delta\psi = i\lambda\psi$

$$= [i\bar{\psi}\gamma^\mu](i\lambda\psi) = -\bar{\psi}\gamma^\mu\psi\lambda$$

current probability.

Interaction: $P_\mu \to P_\mu - qA_\mu(x)$

then $i\partial_\mu \to i\partial_\mu - qA_\mu \to \mathcal{L}_\psi = \bar{\psi}[\gamma^\mu(i\partial_\mu - qA_\mu) - m]\psi$

thus, $\mathcal{L}_{INT} = -qA_\mu j^\mu$ ... Non-invariant under

$A_\mu \to A_\mu + \partial_\mu \alpha.$

... unless: $j^\mu = J^\mu$

note that: under $\psi \to e^{-iq\alpha(x)}\psi$; $j^\mu = \bar{\psi}\gamma^\mu\psi \to j^\mu$

but $i\bar{\psi}\partial\!\!\!/\psi \to i\bar{\psi}\partial\!\!\!/\psi + q(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi$

$A_\mu \to A_\mu + \partial_\mu\alpha$:

$-qA_\mu j^\mu \to -qA_\mu j^\mu - q(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi$

therefore $\mathcal{L}_{QED} = -\frac{1}{4}F^2 - qA_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}(i\partial\!\!\!/ - m)\psi$

is invariant under

$A_\mu \to A_\mu + \partial_\mu\alpha(x)$

Gauge transformation $\psi \to e^{iq\alpha(x)}\psi$

$Q = cte.$

Observation: We may build $\mathcal{L}_{QED}$ from $\mathcal{L}_\psi$

I. $\mathcal{L}_\psi$ has a global symmetry $U(1)$:

$\psi \to e^{-i\alpha}\psi$; $\alpha \in \mathbb{R}$

II. Let's build $\mathcal{L}$ with local symmetry:

$\psi \to e^{-iq\alpha(x)}\psi$; $\alpha(x) \in \mathcal{F}(\mathbb{R})$.

$\mathcal{L}_\psi \to \bar{\psi}[\gamma^\mu(i\partial_\mu + q(\partial_\mu\alpha)) - m]\psi$

for absorving this term, introduce $A_\mu(x)$

Covariant derivative: $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$

minimal couplin.

then $\oplus \mathcal{L}_{INT} = -qA_\mu j^\mu$: $\quad A_\mu \to A_\mu + \partial_\mu \alpha(x)$.

III. Add a Kinetic term for A_μ

$\rightarrow \mathcal{L}_A((\partial A)^2)$

analise $\quad \mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$; $F_{\mu\nu} : \partial_\mu A_\nu - \partial_\nu A_\mu$

Notice:

a. $[D_\mu, D_\nu] = [\partial_\mu + iqA_\mu , \partial_\nu + iqA_\nu]$

$$= iq[\partial_\mu , A_\nu] + iq[A_\mu , \partial_\nu]$$

$$= iq(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

then $\quad F_{\mu\nu} \equiv \frac{1}{iq}[D_\mu , D_\nu]$

b. D_μ is covariant: $\quad D_\mu \varphi \longrightarrow e^{-iq\alpha}(D_\mu \varphi)$

$$\to D_\mu \to e^{-iq\alpha} D_\mu e^{iq\alpha}$$

c. $F_{\mu\nu}$ is covariant:

$$U(\alpha) = e^{-iq\alpha}$$

$$U^\dagger = U^{-1}$$

$$iqF_{\mu\nu} = [D_\mu , D_\nu] \longrightarrow [UD_\mu U^\dagger , UD_\nu U^\dagger]$$

$$= U[D_\mu , D_\nu]U^\dagger$$

$$iqF_{\mu\nu} \longrightarrow U(\alpha) F_{\mu\nu} U^\dagger(\alpha)$$

Obviously, as $[F_{\mu\nu}, U(\alpha)] = 0$, then $F_{\mu\nu}$ is covariant.

therefore, $\quad \mathcal{L}_A \propto F_{\mu\nu}F^{\mu\nu}$: Lorentz invariant

$$\to \mathcal{L}_{QED} = \bar{\varphi}(i\not{\partial} - m) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

Yang-Mills Theory

Consider a theory with symmetry $SU(n)$

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} : \text{fundamental representation}, \quad \dim = n.$$

$$SU(n) = \{ U(\vec{\alpha}) = \exp(-ig\vec{\alpha}\cdot\vec{T}) / \vec{\alpha} \in \mathbb{R}^n ; [T_a, T_b] = i f_{abc} T_c ; \operatorname{Tr} T_a T_b = \frac{1}{2}\delta_{ab} \}$$

$$\mathcal{L}_\phi = \partial_\mu \Phi^\dagger \partial^\mu \Phi = \sum_{i=1}^{n} (\partial_\mu \phi_i)^* (\partial^\mu \phi_i)$$

$\partial_\mu \equiv \mathbb{1}\cdot\partial_\mu$

$\rightarrow$ Globally invariant

$\Phi \rightarrow U(\alpha)\Phi$

local Theory:

$\alpha \rightarrow \alpha(x)$

$$D_\mu = \partial_\mu + ig \underset{n\times n}{A_\mu(x)} = \partial_\mu + ig T_a \underset{\text{Yang-Mills}}{A^a_\mu(x)}$$

Covariant:

$$D\Phi \longrightarrow U(\alpha(x))[D\Phi] = D'_\mu U(\alpha(x))\Phi$$

$$D'_\mu = U(\alpha) D_\mu U^\dagger(\alpha)$$

$$\rightarrow \boxed{A'_\mu = U(\alpha) A_\mu U^\dagger(\alpha) - \frac{i}{g} U(\alpha)\partial_\mu U^\dagger(\alpha)}$$

Infinitesimally:

$$U(\alpha) \simeq 1 + ig\vec{\alpha}\cdot\vec{T} \qquad A^a_\mu \rightarrow A^a_\mu + \partial_\mu \alpha^a + g f^{abc} \alpha^b A^c_\mu$$

Due to

$$(D\Phi)^\dagger \rightarrow (D\Phi)^\dagger U^\dagger \Rightarrow \mathcal{L}_\phi = (D\Phi)^\dagger (D\Phi)$$

Fermions

$$\mathcal{L}_\psi = i\bar{\psi}_i \not{D} \psi \rightarrow \mathcal{L}_{INT} = -g A^a_\mu \bar{\psi}\gamma^\mu T_a \psi.$$

$\mathcal{L}_{YM}$:

$$F_{\mu\nu} \equiv \frac{1}{ig}[D_\mu, D_\nu]$$

then

$$igF'_{\mu\nu} \equiv [D'_\mu, D'_\nu] = U(\alpha)[D_\mu, D_\nu]U^\dagger(\alpha)$$

$\rightarrow \mathbb{F}'_{\mu\nu} = U(\alpha)\,\mathbb{F}\,U^{\dagger}(\alpha).$ covariant

therefore.

$$\mathcal{L}_{YM} \propto \mathrm{Tr}\,\mathbb{F}^{\mu\nu}\mathbb{F}_{\mu\nu}$$

$$\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + ig[\mathbb{A}_\mu, \mathbb{A}_\nu] \equiv F^a_{\mu\nu} T_a$$

$$\mathrm{Tr}\,T_a T_b = \frac{1}{2}\delta_{ab}$$

$$[T_a, T_b] = i f_{abc}$$

then

$$\boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu}$$

$$\mathrm{Tr}\,\mathbb{F}^{\mu\nu}\mathbb{F}_{\mu\nu} = F^a_{\mu\nu} F^{\mu\nu}_b \,\mathrm{Tr}\,T_a T_b$$

finally

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}\,\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$