Heisenberg's representations.

$$U(t,t_0) = e^{itH/\hbar}$$

$$\langle f \rangle_\psi = \langle \psi, f\psi \rangle \quad \text{as} \quad \psi(t) = U(t,t_0)\psi(t_0).$$

where f is $f(t_0)$.

$$= \frac{\langle U(t,t_0)\psi(t_0), f\,U(t,t_0)\psi(t_0)\rangle}{\langle U(t,t_0)\psi(t_0), U(t,t_0)\psi(t_0)\rangle}$$

$$= \frac{\langle \psi(t_0), U^{-1}(t,t_0) f\, U(t,t_0)\psi(t_0)\rangle}{\langle \psi(t_0), U^{-1}(t,t_0)U(t,t_0)\psi(t_0)\rangle}$$

$$= \frac{\langle \psi(t_0), f(t)\psi(t_0)\rangle}{\langle \psi(t_0), \psi(t_0)\rangle}$$

where

$$f(t) := U^{-1}(t,t_0) f(t_0) U(t,t_0)$$

Also, taking the conjugate to schrödinger

$$-i\hbar \frac{\partial}{\partial t} U^{\dagger}(t,t_0) = U^{\dagger}(t,t_0) H$$

for an arbitrary $f \in \hat{A}_Q$

$$i\hbar \frac{\partial}{\partial t} f(t) = i\hbar\left[\left(\frac{\partial}{\partial t} U^{-1}(t,t_0)\right) f(t_0) U(t,t_0) + U^{-1}(t,t_0)\frac{\partial}{\partial t}U(t,t_0)\right]$$

$$= i\hbar\left[-\frac{1}{i\hbar} U^{\dagger}(t,t_0) H f(t_0) U(t,t_0) + U^{-1}(t,t_0) f(t_0)\frac{1}{i\hbar} H U(t,t_0)\right]$$

$$= U^{\dagger}(t,t_0)\left[-H f(t_0) + f(t_0) H\right] U(t,t_0)$$

$$= U^{\dagger}(t,t_0)\left[H, f(t_0)\right] U(t,t_0)$$

$$= [H, f](t) \qquad \text{equivalent to liouville's.}$$

Now, in our case we want to reproduce something similar.

Definition: The $*$-exponential is defined as

$$\mathrm{Exp}_*(a) := \sum_{n=0}^{\infty} \frac{1}{n!}(a*)^n$$

Where $(a_*)^n := a * a * \ldots * a$ (n-times)

In particular, we may propose an $*$-unitary evolution operator.

$$U_*(q,p,t) := \mathrm{Exp}_*\left(\frac{itH}{\hbar}\right) = 1 + \left(\frac{it}{\hbar}\right)H + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 H * H + \ldots$$

for an arbitrary Hamiltonian $H \in A_c$.

As in quantum mechanics, one may show that a solution to Moyal's equation.

$$d_t f = \}f, H\{_M$$

is given by

$$W(q,p,t) = U_*^{-1}(q,p,t) * W(q,p,t) * U_*(q,p,t).$$

Homework: Verify the solution to the Moyal's equation.

Example: Harmonic oscillator.

$$H = \frac{1}{2}(p^2 + q^2)$$

1

$$H * W = \mathcal{E} W$$

$$H * W = H(p,q)\exp\left[\left(\frac{i\hbar}{2}\right)\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}\right)\right]W(p,q)$$

$$= H(p,q)\left[1 + \frac{i\hbar}{2}\overleftrightarrow{P} + \left(\frac{i\hbar}{2}\right)^2\frac{1}{2!}\overleftrightarrow{P}^2 + \ldots\right]W(p,q)$$

$$H\overleftrightarrow{P}W = \frac{\partial H}{\partial q}\frac{\partial W}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial W}{\partial q} = q\frac{\partial W}{\partial p} - p\frac{\partial W}{\partial q}$$

$$H\overleftrightarrow{P}^2 W = H\left(\frac{\overleftarrow{\partial^2}}{\partial q^2}\frac{\overrightarrow{\partial^2}}{\partial p^2} - 2\frac{\overleftarrow{\partial^2}}{\partial q\partial p}\frac{\overrightarrow{\partial^2}}{\partial p\partial q} + \frac{\overleftarrow{\partial^2}}{\partial p^2}\frac{\overrightarrow{\partial^2}}{\partial q^2}\right)W$$

$$= \frac{\partial^2 W}{\partial p^2} + \frac{\partial^2 W}{\partial q^2}$$

$$H * W = (HW)(p,q) + \frac{i\hbar}{2}\left(q\frac{\partial W}{\partial p} - p\frac{\partial W}{\partial q}\right) - \frac{\hbar^2}{8}\left(\frac{\partial^2 W}{\partial p^2} + \frac{\partial^2 W}{\partial q^2}\right) = \mathcal{E}W$$

$$\mathrm{Re}(H*\omega) = H\omega - \frac{\hbar^2}{8}\left(\frac{\partial^2\omega}{\partial p^2} + \frac{\partial^2\omega}{\partial q^2}\right) = \epsilon\omega$$

$$\mathrm{Im}(H*W) = q\frac{\partial\omega}{\partial p} - p\frac{\partial\omega}{\partial q} = 0$$

$$\Longrightarrow W = W(z) \quad ; \qquad z := \frac{2(p^2+q^2)}{\hbar} = \left(\frac{4}{\hbar}p\right)\frac{d\omega}{dz}$$

$$\frac{\partial\omega}{\partial p} = \frac{\partial z}{\partial p}\frac{\partial\omega}{\partial z} = \frac{4p}{\hbar} \qquad\qquad \frac{\partial\omega}{\partial q} = \frac{\partial z}{\partial q}\frac{\partial\omega}{\partial z} = \left(\frac{4}{\hbar}q\right)\frac{d\omega}{dz}$$

$$\frac{d^2\omega}{dp^2} = \frac{\partial}{\partial p}\left(\frac{\partial\omega}{\partial p}\right) = \frac{\partial}{\partial p}\left(\frac{4p}{\hbar}\frac{d\omega}{dz}\right) = \frac{4}{\hbar}\frac{d\omega}{dz} + \left(\frac{4q}{\hbar}\right)^2\frac{d^2\omega}{dz^2}$$

Real:

$$\frac{\hbar z}{4}\omega(z) - \frac{\hbar^2}{8}\left(\frac{4}{\hbar}\frac{d\omega}{dz} + 16\frac{p^2}{\hbar^2}\frac{d^2\omega}{dz^2} + \frac{4}{\hbar}\frac{d\omega}{dz} + \frac{16q^2}{\hbar^2}\frac{d^2\omega}{dz^2}\right) = \epsilon\omega(z).$$

$$2(p^2+q^2)\frac{d^2\omega}{dz^2} + \hbar\frac{d\omega}{dz} - \frac{\hbar z}{4}\omega + \epsilon\omega = 0.$$

$$\hbar z\frac{d^2\omega}{dz^2} + \hbar\frac{d\omega}{dz} + \left(\epsilon - \frac{\hbar z}{4}\right)\omega = 0.$$

$$\left[z\partial z^2 + \partial z + \left(\frac{\epsilon}{\hbar} - \frac{z}{4}\right)\right]\omega = 0 \qquad\qquad (\#)$$

$$\omega(z) = e^{-z/2}L(z).$$

$$\frac{d\omega}{dz} = -\frac{1}{2}e^{-z/2}L + e^{-z/2}\frac{dL}{dz} = \left(-\frac{1}{2}L + \frac{dL}{dz}\right)e^{-z/2}$$

$$\frac{d^2\omega}{dz^2} = \left(-\frac{1}{2}\frac{dL}{dz} + \frac{d^2L}{dz^2}\right)e^{-z/2} + \left(-\frac{1}{2}L + \frac{dL}{dz}\right)\left(-\frac{1}{2}e^{-z/2}\right)$$

$$= \left(\frac{d^2L}{dz^2} - \frac{dL}{dz} + \frac{1}{4}L\right)e^{-z/2}$$

from (#), then

$$\left\{z\left(\frac{d^2L}{dz^2} - \frac{dL}{dz} + \frac{1}{4}L\right) + \left(-\frac{1}{2}L + \frac{dL}{dz}\right) + \left(\frac{\epsilon}{\hbar} - \frac{z}{4}\right)L\right\}e^{-z/2} = 0$$

$$z\frac{d^2L}{dz^2} + (1-z)\frac{dL}{dz} + \underbrace{\left(\frac{\epsilon}{\hbar} - \cancel{\frac{z}{4}} + \cancel{\frac{z}{4}} - \frac{1}{2}\right)}_{=:\,n\,\in\,\mathbb{Z}^+}L = 0$$

$$\therefore L = L_n(z) = L_n\left(\frac{4H}{\hbar}\right) \quad \text{Laguerre equation.}$$

$$n = \frac{E}{\hbar} - \frac{1}{2}$$

$$\Rightarrow E_n = \hbar(\omega)\left(n + \frac{1}{2}\right)$$

$$W(p,q) = \alpha_n e^{-2H/\hbar} L\left(\frac{4H}{\hbar}\right)$$

$$W_n(p,q) = \alpha_n e^{-(p^2+q^2)/\hbar} L_n\left(\frac{2(p^2+q^2)}{\hbar}\right)$$

Change $\quad Q := \operatorname{arccot}\left(\frac{p}{q}\right)$

$$H := \frac{p^2+q^2}{2}$$

such that

$$dp \wedge dq = dH \wedge dQ$$

$$\Rightarrow \quad 1 = \int dH\, dQ\, \alpha_n e^{-2H/\hbar} L_n\left(\frac{4H}{\hbar}\right)$$

$$= \alpha_n \int_0^{\pi} dQ \int_0^{\infty} dH e^{-2H/\hbar} L_n\left(\frac{4H}{\hbar}\right) = \alpha_n \int dx \left(\frac{\hbar}{4}\right) e^{-x/2} L_n(x)$$

$$x = \frac{4H}{\hbar}$$

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$$\int_0^{\infty} dx\, e^{-ax} L_n(x) = (a-1)^n a^{-n-1}.$$

$$= \alpha_n \pi \left(\frac{\hbar}{4}\right)\left[\left(\frac{1}{2} - 1\right)^n \left(\frac{1}{2}\right)^{-n-1}\right] = \frac{\alpha_n \pi \hbar (-1)^n}{2}$$