# Peter-Weyl's theorem

For a compact group $K$, the irreducible representations form an orthonormal base of $L^2(K)$.

**Lemma:** Let $(\pi, V_\pi)$ be an unitary representation, of finite dimension of some $G$ group, locally compact. So, $\pi$, descomposes into the direct sums of irreducible representations.

**Proof:** By induction it's proved in the $V_\pi$ dimension.

If $V_\pi$ is onedimensional, the representation is irreducible, and the theorem is fulfilled.

generalized functions.
distribution space.

Now, let's suppose that it is valid for all spaces of smaller dimension than the $\dim(V_\pi)$. Then $V_\pi$ is irreducible, in such case its proved the lemma or it has an invariant representation $\omega$.

But, $\omega^\perp = \{ w \in V_\pi / \langle v, w \rangle = 0, \forall w \in W \}$, is also invariant.

Then $V_\pi = W \oplus W^\perp$ that are of smaller dimesion and these are descomposed in irreducibles and thus, $V_\pi$ also. ∎

Let $K$ be a compact matricial group i.e., a compact sobgroup of $GL_n(\mathbb{C})$. Let $\tau$ and $\gamma$ irreducible representations of finite dimension of $K$. Let $H$ the space of all linear maps from $V_\gamma$ to $V_\tau$. In this space let's define a new representation $\eta$ of $K$, by

$$\eta(K)T = \tau(K)T\gamma(K^{-1})$$

Let $Hom_K(V_\gamma, V_\tau)$ be the space of the $K$-homomorphisms i.e., the spaces of all linear maps $T: V_\gamma \longrightarrow V_\tau$, such that

$$T\gamma(K) = \tau(K)T, \quad \forall \kappa \in K.$$

**Lemma:** $Hom_K(V_\gamma, V_\tau)$ is atleast one dimensional.

Two unitary representations are called isomorphs $\gamma$ and $\tau$ of $K$, there is an unitary map $T: V_\gamma \longrightarrow V_\tau$ and satisfies $T\gamma(K) = \tau(K)T$, for all $\kappa \in K$. For each $(\tau, V_\tau)$, let be an ortonormalbase $e_1, \ldots, e_n$ of $V$ and let $\tau_{ij}(K) = \langle \tau(K)e_i, e_j \rangle$. The map $\tau_{ij}: K \longrightarrow \mathbb{C}$, it is called $(i,j)$-esim of the matrix $\tau$.

**Theorem (Peter–Weyl):** Let $\tau \neq \gamma$, then.

$$\int_K \tau_{ij}(K)\overline{\gamma_{rs}(K)}\,dK = 0 \qquad \text{for all index } i,j,r,s.$$

Moreover

$$\int_K \tau_{ij}(K)\overline{\tau_{rs}(K)}\,dK = 0 \qquad \text{unless } i=r,\ j=s.$$

in this case

$$\int_K \tau_{ij}(K)\overline{\tau_{ij}(K)}\,dK = \frac{1}{\dim(V_\tau)}$$

the family $(\sqrt{\dim(V_\tau)}\,\tau_{ij})$ form a base of $L^2(K)$

**Proof:** Let's assume that $\gamma \neq \tau$, then for each $T \in \mathrm{Hom}_K(V_\gamma, V_\tau)$ we have that

$$S = \int_K \tau(K)\,T\,\gamma(K^{-1})\,dK$$

satisfy $\tau(K)S = S\gamma(K)$, $S=0$. Let $e_1,\dots,e_n$ be a base of $V_\gamma$ and $f_1,\dots,f_m$ one of $V_\tau$.

Sea $T \in \mathrm{Hom}_K(V_\gamma, V_\tau)$ given by the matrix $E_{ij}$, with ones in the position $(i,j)$ and zeros in the other positions. Then $\tau(K)\,T\,\gamma(K^{-1})$

$$\begin{pmatrix} \tau_{11} & \cdots & \tau_{1m} \\ \vdots & & \vdots \\ \tau_{m1} & \cdots & \tau_{mm} \end{pmatrix}\begin{pmatrix} 1 \end{pmatrix}\begin{pmatrix} \overline{\gamma_{11}} & \cdots & \overline{\gamma_{1n}} \\ \vdots & & \vdots \\ \overline{\gamma_{n1}} & \cdots & \overline{\gamma_{nn}} \end{pmatrix}$$

$$= \begin{pmatrix} \tau_{1i} \\ \vdots \\ \tau_{mi} \end{pmatrix}\begin{pmatrix} \overline{\gamma_{11}} & \cdots & \overline{\gamma_{1n}} \\ \vdots & & \vdots \\ \overline{\gamma_{n1}} & \cdots & \overline{\gamma_{nn}} \end{pmatrix} = \begin{pmatrix} \tau_{1i}\overline{\gamma_{ij}} & \cdots & \tau_{1i}\overline{\gamma_{nj}} \\ \vdots & & \vdots \\ \tau_{mi}\overline{\gamma_{ij}} & & \tau_{mi}\overline{\gamma_{nj}} \end{pmatrix} = 0$$

for $\gamma = \tau$

$$\int_K \tau(K)\,T\,\gamma(K^{-1})\,dK = \lambda\,\mathrm{Id}. \qquad \text{for some } \lambda \in \mathbb{C}$$

to see that $\displaystyle\int_K |\tau_{ij}(K)|^2\,dK = \frac{1}{\dim(V_\tau)}$

let $T = Id$, and remainding $\tau(K)\tau(K^*) = Id$, implies that

$$\sum_r \tau_{ir} \overline{\tau_{jr}(K)} = \delta_{ij}$$

then

$$\sum_r \int_K |\tau_{ij}(K)|^2 dK = 1$$

from the fact that

$$\int_K \tau(K) T \tau(K^{-1}) dK = \lambda Id.$$

in the case that $\tau = \epsilon_{ii}$, implies

$$\int_K |\tau_{ri}(K)|^2 dK = \int_K |\tau_{r'i}(K)|^2 dK \quad \text{for some } r \text{ and } r'.$$

Finally

$$\int_K |\tau_{ri}(K)|^2 dK = \int_K |\tau_{ri}(K^{-1})|^2 = \int_K |\tau_{ir}(K)|^2 dK$$

Homework:

$$\int_K f(K) dK = \int_K f(K^{-1}) dK.$$

$$\int_K |\tau_{ir}(K)|^2 dK = \int_K |\tau_{ir'}(K)|^2 dK$$

we will use the Stone-Weierstrass theorem.

Let $\underline{X}$ be a metrizable compact space. In $C(\underline{X})$ a norm

$$\|f\|_\infty = \sup_{x \in \underline{X}} |f(x)|$$

and a metric $d(f,g) = \|f-g\|_\infty$. $C(\underline{X})$ is an algebra.

Lemma (Stone-Weierstrass)

Let $\underline{X}$ be a compact metrizable space and $A$ be a subalgebra of $C(\underline{X})$, such that.

i) $A$ is closed under the complex conjugation i.e., $f \in A$, $\bar{f} \in A$.

ii) $A$ separates points i.e., $\forall x, y \in \underline{X}$, $x \neq y$ $\exists f \in A$ such that $f(x) \neq f(y)$

therefore $A$ is dense in $C(\underline{X})$.

Let $A$ be the space generates by $\tau_{ij} \in C(K)$.

$A$ is closed under conjugation, let $\bar{\tau}$ is a representation of $K$.

$A$ separates points, since $K$ is a matricial group and it has an unitary injective representation.

By the Stone-Weierstrass theorem $A$ is dense in $C(K)$, but $C(K)$ are dense in $L^2(K)$.

Therefore $A$ is dense in $L^2(K)$. ∎