9) $|W(p,q)| \le 1$

Proof:

$$W(p,q) = \int dz\, \underbrace{\psi\left(q+\frac{z}{2}\right)}_{=:\,\phi_1(q,z)} \underbrace{\overline{\psi\left(q-\frac{z}{2}\right)}\, e^{-ipz/\hbar}}_{=:\,\overline{\phi_2(q,z)}}$$

$$= \int dz\, \phi_1(q,z)\,\overline{\phi_2(q,z)}$$

$$= (\phi_2,\phi_1) = \langle \phi_2,\phi_1\rangle$$

$$|W(p,q)|^2 = |\langle \phi_2,\phi_1\rangle|^2 \underset{\text{Cauchy-Schwartz}}{\le} \langle \phi_2,\phi_2\rangle\langle \phi_1,\phi_1\rangle \underset{\text{Normalized wave function.}}{=} 1$$

$\blacksquare$

10) Given $W(q,p)$ we may recover $\psi \in \mathcal{H}$.

Proof:

$$I := \int dp\, W(q,p)\, e^{ipz'/\hbar} = \int dp\, dz\, \psi\left(q+\frac{z}{2}\right)\overline{\psi\left(q-\frac{z}{2}\right)}\, e^{-ipz/\hbar}\, e^{ipz'/\hbar}$$

$$= \int dp\, dz\, \psi\left(q+\frac{z}{2}\right)\psi\left(q-\frac{z}{2}\right) e^{-ip(z-z')/\hbar}$$

Integrating in $p$ brings $\hbar\,\delta(z-z')$

$$I = \hbar \int dz\, \psi\left(q+\frac{z}{2}\right)\overline{\psi\left(q-\frac{z}{2}\right)}\,\delta(z-z')$$

$$= \hbar\, \psi\left(q+\frac{z'}{2}\right)\overline{\psi\left(q-\frac{z'}{2}\right)}$$

Fix $q := \frac{u}{2}$, $z' := u$.

$$I = \hbar\, \psi(u)\,\overline{\psi(0)} = \int dp\, W(q,p)\, e^{ipz'/\hbar}$$

$$= \int dp\, \omega\left(\frac{u}{2}, p\right) e^{ipu/\hbar}$$

$$\Longrightarrow \psi(u) = \frac{1}{N} \int dp\, W\left(\frac{u}{2}, p\right) e^{ipu/\hbar}$$

$\blacksquare$

$$\text{11)} \quad \lim_{\hbar \to 0} W(q,p) = |W(q)|^2 \, \delta(p)$$

Homework: Prove (11).

Now, let us define what we mean by a deformation

Definition: A formal deformation of the algebra $A_c = C^\infty(M)$ is define as a map.

No matter it converges.

$$* : A_c \times A_c \longrightarrow A[\hbar]$$

$$(f,g) \longmapsto f * g = \sum_{k=0}^{\infty} C_k(f,g)\hbar^k ; \quad C_k \neq C_n(\hbar).$$

such that.

1) Associativity: for all $p \geq 0$.

$$((f,g),\hbar) = (f,(g,h))$$

$$((f*g)*h) = (f*(g*h))$$

$$\sum_{k=0}^{\infty} (C_k(f,g),h)\hbar^k = \sum_{k=0}^{\infty} C_k(f,(g,h))\hbar^k$$

$$\Longleftrightarrow \sum_{k,l} C_k(C_l(f,g),h)\hbar^k = \sum_{k,l} C_k(f, C_l(g,h))\hbar^{k,l}$$

$$\sum_{k+l=p} \left[C_k(C_l(f,g),h) - C_k(f, C_l(g,h))\right] = 0$$

2) Point-product: $C_0(f,g) = fg.$

3) Poisson- Braket: $\frac{1}{2}\left[C_1(f,g) - C_1(g,f)\right] = \{f,g\}.$

4) Bidifferential: $C_K : A_c \times A_c \longrightarrow A_c$

Should be a bidifferential operator for each $K$.

Definition: A formal deformation of poisson bracket is a skew-symmetric map.

$$[\cdot,\cdot] : A_c \times A_c \longrightarrow A_c[\hbar]$$

$$[f,g] \longmapsto \sum_{K=0}^{\infty} T_K(f,g)\hbar^K; \quad T_K \neq T_K(\hbar).$$

such that

I) $[[f,g],h]+[[g,h],f]+[[h,f],g]=0.$ Jacobi Identity.

$$\Rightarrow \sum_{K+l=p} T_K(T_l(f,g),h)=0$$ over cyclic permutations of $f, g$ and $h$.

II) $T_0(f,g)=\{f,g\}$

III) Bidifferential:

$$T_K : A_c \times A_c \longrightarrow A_c$$

Should be differential operator for each $K \in \mathbb{Z}^+$.

In order to construct these deformations first consider a map.

$$* : A_c \times A_c \longrightarrow A_c$$

$$(f,g) \longmapsto f*g = (Q_\hbar^\omega)^{-1}[Q_\hbar^\omega(f)\, Q_\hbar^\omega(g)]$$

$$\Rightarrow Q_\hbar^\omega(f*g) = Q_\hbar^\omega(f)\cdot Q_\hbar^\omega(g)$$ Non-Commotative convolution.

Theorem:

$$f*g = \frac{1}{2\pi}\int_{\mathbb{R}^2} du'dv'\, \tilde{f}((u-u'),(v-v'))\,\tilde{g}(u',v')\, e^{i\hbar/2(uv'-u'v)}$$

Proof:

$$Q_\hbar^\omega(f)(p,q) = \frac{1}{2\pi}\int_{\mathbb{R}^2} du''dv''\, \tilde{f}(u'',v'')\, e^{-i/\hbar(u''\hat{p}+v''\hat{q})}$$

$$Q_\hbar^w(g)(p,q) = \frac{1}{2\pi}\int_{\mathbb{R}^2} du'dv'\, \tilde{g}(u',v')\, e^{-i/\hbar(u'\hat{p}+v'\hat{q})}$$

$$Q_\hbar^w(f)(p,q)\cdot Q_\hbar^w(g)(p,q) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du'dv'du''dv''\, f(u'',v'')\tilde{g}(u',v')$$

$$\times e^{-i/\hbar(u''\hat{p}+v''\hat{q})}\, e^{-i/\hbar(u'\hat{p}+v'\hat{q})}$$

Convolution of inverse Fourier

$$\mathcal{F}^{-1}(f\circ g) = \mathcal{F}^{-1}(f)\,\mathcal{F}^{-1}(g)$$

By Baker-Campbell-Hausdorff. formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$$

$$e^{-i/\hbar(u''\hat{p}+v''\hat{q})}\, e^{-i/\hbar(u'\hat{p}+v'\hat{q})} = e^{-i/\hbar((u'+u'')\hat{p}+(v'+v'')\hat{q})}\, e^{i\hbar/2(u''v'-v''u')}$$

$$I = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du'dv'du''dv''\, \tilde{f}(u'',v'')\tilde{g}(u',v')\, e^{i\hbar/2(u''v'-v''u')}\, e^{-i/\hbar((u'+u'')\hat{p}+(v'+v'')\hat{q})}$$

Change

$$u'+u'' =: u, \quad v'+v'' =: v, \qquad du'' = du$$

$$u'' = u-u', \quad v'' = v-v' \qquad dv'' = dv$$

$$\begin{aligned} u''v' - v''u' &= (u-u')v' \\ &= -(v-v')u'' \\ &= uv' - vu' \end{aligned}$$

$$I = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} dudvdu'dv'\, \tilde{f}(u-u',v-v')\tilde{g}(u',v')\, e^{i\hbar/2(uv'-vu')}\, e^{-i/\hbar(u\hat{p}+v\hat{q})}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2} dudv \left[\frac{1}{2\pi}\int_{\mathbb{R}^2} du'dv'\, \tilde{f}(u-u',v-v')\tilde{g}(u',v')\, e^{i\hbar/2(uv'-vu')}\right] e^{-i/\hbar(u\hat{p}+v\hat{q})}$$

$$= (Q_\hbar^w(f * g))(q,p)$$

$\blacksquare$

## Star-product properties

1) $(f*g)*h = f*(g*h)$ Associativity

2) $f*1 = 1*f$

3) $\int_{\mathbb{R}^2} dp\,dq\, f*g = \int_{\mathbb{R}^2} dp\,dq\, fg = \int_{\mathbb{R}^2} dp\,dq\, g*f$

**Homework:** Prove the star-product properties.

**Theorem:** for any two observables $f,g \in A_c$, the star-product introduced, allows an exponential representation given by

$$(f*g)(p,q) = f(p,q)\exp\left[\left(\frac{i\hbar}{2}\right)\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)\right] g(p,q)$$

$P$, Poisson tensor in local coordinates

**Proof:** As we seen before

$$Q_\hbar^\omega(f*g)(p,q) = I = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du'dv'du''dv''\,\tilde{f}(u'',v'')\tilde{g}(u',v')\,e^{i\hbar/2(u''v'-v''u')}\,e^{-i/\hbar((u'+u'')\hat{p}+(v'+v'')\hat{q})}$$

$$e^{i\hbar/2(u''v'-v''u')} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k (u''v'-u'v'')^k$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \sum_{m=0}^{k}\binom{k}{m}(u''v')^{k-m}(u'v'')^m(-1)^m$$

$$I = \sum_{k=0}^{\infty}\frac{1}{k}\left(\frac{i\hbar}{2}\right)^k\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m}{(2\pi)^2}\int du''dv''du'dv'\,(u''v')^{k-m}(u'v'')^m\tilde{f}(u'',v'')\tilde{g}(u',v')$$
$$\times\, e^{-i/\hbar((u'+u'')\hat{p}+(v'+v'')\hat{q})}$$

$$= \sum_{k=0}^{\infty}\frac{1}{k}\left(\frac{i\hbar}{2}\right)^k\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m}{(2\pi)^2}\, I_1 I_2.$$

$$I_1 := \int du'dv'\,(v')^{k-m}(u')^m\,\tilde{g}(u',v')\,e^{-i/\hbar(u'\hat{p}+v'\hat{q})}$$

$$I_2 := \int du''dv''\,(u'')^{k-m}(v'')^m\,\tilde{f}(u'',v'')\,e^{-i/\hbar(u''\hat{p}+v''\hat{q})}$$

$I_1$ and $I_2$ are of the form

$$Q_\hbar^\omega(\hat{q}^{k-m}\hat{p}^m g(q,p)) \quad \text{and} \quad Q_\hbar^\omega(\hat{p}^{k-m}\hat{q}^m f(q,p)) \text{ respectively.}$$

Where

$$\left.\begin{matrix} \tilde{q}^{k-m}\tilde{p}^m g(q,p) \\ \tilde{p}^{k-m}\tilde{q}^m f(q,p) \end{matrix}\right\} \begin{matrix}\text{are Fourier}\\ \text{transforms}\\ \text{of}\end{matrix} \left\{\begin{matrix} v'^{k-m} u'^m \tilde{g}(u',v') \\ v''^m u''^{k-m} \tilde{f}(u'',v'') \end{matrix}\right.$$

By the Fourier property

$$\mathcal{F}[u\tilde{h}(u,v)] =: \frac{\partial}{\partial p} h(p,q)$$

$$\Rightarrow I_1 = i^k \left(\frac{\partial}{\partial p_j}\right)^m \left(\frac{\partial}{\partial q^j}\right)^{k-m} g(p,q).$$

$$I_2 = i^k \left(\frac{\partial}{\partial p_j}\right)^{k-m} \left(\frac{\partial}{\partial q^j}\right)^m f(p,q).$$

$$I = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \sum_{m=0}^{k} \binom{k}{m} \frac{(-1)^m}{(2\pi)^2} i^{-2k} \left[\left(\frac{\partial}{\partial p_j}\right)^m \left(\frac{\partial}{\partial q^j}\right)^{k-m} g(q,p)\right]\left[\left(\frac{\partial}{\partial p_j}\right)^{k-m}\left(\frac{\partial}{\partial q^j}\right)^m f(q,p)\right]$$

$$(f*g)(p,q) = f(p,q)\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-i\hbar}{2}\right)^k \sum_{m=0}^{k}\binom{k}{m}(-1)^m \left[\left(\frac{\overleftarrow{\partial}}{\partial p_j}\frac{\overrightarrow{\partial}}{\partial q^j}\right)^m \left(\frac{\overleftarrow{\partial}}{\partial q^j}\frac{\overrightarrow{\partial}}{\partial p_j}\right)^{k-m}\right]\right] g(p,q)$$

$$= f(p,q)\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-i\hbar}{2}\right)^k \left(\frac{\overleftarrow{\partial}}{\partial p_j}\frac{\overrightarrow{\partial}}{\partial q^j} - \frac{\overleftarrow{\partial}}{\partial q^j}\frac{\overrightarrow{\partial}}{\partial p_j}\right)^k\right] g(p,q)$$

$$= f(p,q)\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \left(\frac{\overleftarrow{\partial}}{\partial q^j}\frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j}\frac{\overrightarrow{\partial}}{\partial q^j}\right)^k\right] g(p,q)$$

$$= f(p,q)\exp\left[\left(\frac{i\hbar}{2}\right)\left(\frac{\overleftarrow{\partial}}{\partial q^j}\frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j}\frac{\overrightarrow{\partial}}{\partial q^j}\right)\right] g(p,q)$$

$$= f(p,q)e^{i\hbar/2\overleftrightarrow{P}} g(p,q)$$

$$f*g = fg + \frac{i\hbar}{2} + \overleftrightarrow{P}g + \left(\frac{i\hbar}{2}\right)^2 + \overleftrightarrow{P}^2 g + \ldots$$