to prove unitarity, we show that the function

$$f(\omega, z) = \sum_{m=-j}^{j} \overline{e^m}(\omega) e^m(z)$$

is invariant under $f(A\omega, Az)$

$$f(\omega, z) = \sum_{m=-j}^{j} \frac{\overline{\omega_1}^{j+m}\,\overline{\omega_2}^{j-m}\, z_1^{j+m} z_2^{j-m}}{(j+m)!(j-m)!}$$

$$= \frac{[\overline{\omega_1} z_1 + \overline{\omega_2} z_2]^{2j}}{(2j)!} = \frac{(\omega, z)^{2j}}{(2j)!}$$

it follows the unitarity since

$$f(A\omega, Az) = \frac{(A\omega, Az)^{2j}}{(2j)!} = \frac{(\omega, z)^{2j}}{(2j)!} = f(\omega, z)$$

for irreducibility (idea)

$$MD^j(A) = D^j(A)M$$

$\longrightarrow M = \lambda I$, A is irreducible.

Reduction of a direct product.

The direct product of representations $D^j(A)$ and $D^k(A)$, will be denoted by $D(A)$, we assume that $j \geq k$

we will prove first that $D(A)$ is reduced, contains to the representation $D^l(A)$ exactly once if $j-k \leq l \leq j+k$, and $j+k-l$ is an integer.

It's may be prove using the characters.

$$A \in SU(2) \longrightarrow \begin{pmatrix} e^{i\rho} & 0 \\ 0 & e^{-i\rho} \end{pmatrix}, \quad 0 \leq \rho \leq \pi.$$

Let $\omega = e^{2i\rho}$, as $D^j(0, \phi, \varphi)_{mn} = e^{-2im\phi} \delta_{mn}$.

In aditional,

$$\chi^j = \omega^{2j} + \omega^{2j-2} + \dots + \omega^{-2j+2} + \omega^{-2j}$$

So

$$\chi^l(\rho) = \sum_{m=-l}^{l} \omega^m = \omega^{-l} \sum_{n=0}^{2l} \omega^n$$

$$= (1-\omega)^{-1}(\omega^{-l} - \omega^{l+1})$$

$$= (1-\omega)^{-2}(\omega^{-j} - \omega^{j+1})(\omega^{-k} - \omega^{k+1}) = \chi^{j}(\rho)\,\chi^{k}(\rho)$$

The direct product may be build in the function space of two variables, generate by $e^m(x)e^n(y)$; $-j \leq m \leq j$, $-k \leq n \leq k$.

$$f_{l,p}(x,y) = \sum_{m,n} [l]^{1/2} \underbrace{\begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix}}_{\text{3j-symbols.}} e^m(x)\, e^n(y)$$

Where $[l] = 2l+1$.

If x, y are replaced by $A^{-1}x$ and $A^{-1}y$.

$$\sum_{m'n'} D^j(A)_{m'm}\, D^k(A)_{n'n}\, e^{m'}(x)\, e^{n'}(y) = \sum_{lpp'} [e]^{1/2} \begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix} D^l_{p'p}(A)\, f_{lp'}(x,y).$$

$$\rightarrow [l]\,\overline{\begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix}} \begin{pmatrix} j & k & l \\ m' & n' & p' \end{pmatrix} = \frac{[l]}{\Omega} \int D^l(A)_{p'p}\, D^j(A)_{m'm}\, D^k(A)_{n'n}\, dA\,.$$

$$\begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix} = (-1)^{2j-k+n} \left[\frac{(j+k-l)!\,(k+l-j)!\,(l+j-k)!\,(l+p)!\,(l-p)!}{(j+k+l+1)!\,(j+m)!\,(j-m)!\,(k+n)!\,(k-n)!} \right]^{1/2}$$

$$\times \sum_t (-1)^k \frac{(l+j-n-t)!\,(k+n+t)!}{(l+p-k)!\,(t+k-j-p)!\,t!\,(l-k+j-t)!}$$

where t is the greatest non-negative factorial