Self-Adjoint compact operators

Proposition: If A is a self-adjoint compact operator, then A has an eigenvalue λ, such that $|\lambda|=\|A\|$ and also, the maximum of $|\langle Ax,x\rangle|$ for $\|x\|=1$, is obtained in a eigenvector with eigenvalue λ.

Theorem: (Hilbert-Schmidt theorem) For all self-adjoint compact operator $T: H \longrightarrow H$ exists a non-zero set of eigenvalues $\{\lambda_i\}_{i\geq 1}$ in $\mathbb{R}$ such that

$$|\lambda_1| \geq \ldots \geq |\lambda_i| \geq |\lambda_{i+1}| \geq \ldots$$

are finites or converge to zero, and an orthogonal system of eigenvectors $\{e_i\}_{i\geq 1}$ with $Te_i=\lambda_i e_i$, such that

I. For all $x \in \mathcal{H}$, $x = y + \sum_{i\geq 1} \langle x, e_i\rangle e_i$, $\quad y \in \operatorname{Ker} T$

II. $Tx = \sum_{i\geq 1} \langle Tx, e_i\rangle e_i = \sum_{i\geq 1} \langle z, e_i\rangle e_i$ i.e., $\{e_i\}_{i\geq 1}$ is an orthonormal base of $\overline{\operatorname{Im} T}$.

Proof: we will construct a sequence $\{\lambda_i\}, \{e_i\}$ by induction. The vector e_1, is obtained by the last proposition, from the condition $\|e_1\|=1$, $Te_1=\lambda e_1$, $|\lambda_1|=\|T\|$.

Let $\{e_i\}_{i=1}^n$ be a sequence, by the induction process.

$$Te_i=\lambda_i e_i, \quad |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|, \quad \|e_i\|=1.$$

Let $E_n = \operatorname{span}\{e_i\}_{i=1}^n$. E_n an invariant subspace of T, therefore $E_n^\perp$ is also invariant. T is self-adjoint in each invariant subspace and in particular in $E_n^\perp$. let

$$T_{n+1} = T\big|_{E_n^\perp}$$

Applying the previous property to the operator T_{n+1}, we obtain a e_{n+1} vector, such that $\|e_{n+1}\|=1$, $Te_{n+1}=\lambda_{n+1}e_{n+1}$, and $|\lambda_{n+1}|=\|T_{n+1}\|$.

As $E_{n-1} \subseteq E_n$ and $E_n^\perp \subseteq E_{n-1}^\perp$, and

$$|\lambda_{n+1}| = \|T_{n+1}\| = \|T\big|_{E_n^\perp}\| \leq \|T\big|_{E_{n-1}^\perp}\| = \|T_n\| = \|\lambda_n\|$$

which implies that $|\lambda_{n+1}| \leq |\lambda_n|$ for all n.

Continuing with the process, it will end when $T_{n+1}=0$. or we will obtain an infinity sequence $\{\lambda_n\}$ of eigenvalues of T, and their correspondents eigenvectors $\{e_n\}$.

for any x in $\mathcal{H}$. Let

$$x_n = \sum_{i=1} \langle x, e_i \rangle e_i, \quad y_n = x - x_n$$

Is clear that $y \in E_n^\perp$. In the first case, $(T_{n+1} = 0)$

$$x = \sum_{i=1} \langle x, e_i \rangle e_i + y_n \quad ; \; y_n \in \operatorname{Ker} T$$

In the second case, the set of eigenvalues $\{\lambda_n\}$ is infinity and by a previous proposition $\lambda_n \longrightarrow 0$. Thus

$$\|Ty_n\| \leq \|T|_{E_n^\perp}\| \, \|y_n\| = |\lambda_{n+1}| \, \|y_n\| \leq |\lambda_{n+1}| \, \|x\| \longrightarrow 0$$

if $n \longrightarrow \infty$.

The limit $\lim_{n\to\infty} y_n$ exists, and from $\|y_n\| \to 0$, as T^* is bounded, we conclude that $Ty = 0$, then

$$x - \sum_{i=1} \langle x, e_i \rangle e_i = \lim_{n\to\infty} y_n = y \in \operatorname{Ker} T.$$

if $n \to \infty$, $\sum_{i=1} \langle x, e_i \rangle e_i$ exists.

Thus,

$$x = \sum_{i=1} \langle x, e_i \rangle e_i + y \; ; \quad y \in \operatorname{Ker} T.$$

As e_i, are eigenvectors of T with eigenvalues

$$Tx = \sum_{i=1} \lambda_i \langle x, e_i \rangle e_i$$

Then $\operatorname{Im} T \subseteq \overline{\operatorname{span}\{e_i\}_{i\geq 1}}$ and as

$$e_i = \frac{1}{\lambda_i} T e_i \in \operatorname{Im} T, \quad \overline{\operatorname{Im} T} = \overline{\operatorname{span}\{e_i\}_{i\geq 1}}$$

As $\{e_i\}_{i\geq 1}$ is an orthonormal system, then $\{e_i\}$ is a base of $\overline{\operatorname{Im} T}$. ■

Order in the self-adjoint operators space

Definition: An operator A is said that is non-negative $(A \geq 0)$ if and only if $\langle Ax, x \rangle \geq 0 \quad \forall x \in \mathcal{H}$.

This implies that A is self-adjoint. Moreover, $A \leq B$ means that:

I. A and B are self-adjoint.

II. $B - A \geq 0$.

Order properties:

I. $-I \leq A \leq I$ implies that $\|A\| \leq 1$. In fact, this inequalities means that

$$\sup_{\|x\| \leq 1} |\langle Ax, x\rangle| \leq 1$$

this implies that $\|A\| \leq 1$.

II. $A \geq 0$ and $A \leq 0$, implies $A = 0$.

III. Cauchy-Schwartz generalised inequality,

Let $A \geq 0$, then

$$|\langle Ax, y\rangle| \leq \sqrt{\langle Ax, x\rangle}\sqrt{\langle Ay, y\rangle}$$

IV. If C is self-adjoint and $A \leq B$, then $A + C \leq B + C$.

V. If A is self-adjoint, $A^{2n} \geq 0$. In fact

$$\langle A^{2n}x, x\rangle = \langle A^n x, A^n x\rangle = \|A^n x\|^2 \geq 0.$$

Proof: If $A \geq 0$, then $A^{2n+1} \geq 0$, since

$$\langle A^{2n+1}x, x\rangle = \langle AA^n x, A^n x\rangle \geq 0.$$

then if $A \geq 0$, for any polynomial $P(\lambda)$ with negative coefficients $P(A) \geq 0$ $\blacksquare$

$$P(A) = a_n A^n + a_{n-1}A^{n-1} + \dots + A^0 \geq 0.$$

Theorem: Let $A_0 \leq A_1 \leq \dots \leq A_n \leq \dots \leq A$. Then exists a strong limit of $(A_n)_n$, i.e., exists a bounded operator B and $A_n x \longrightarrow Bx$, $\forall x \in \mathcal{H}$.

Proof: For a self-adjoint operator A, there exists a number c_1 such that $A \leq cI$. Changing the sequence to

$$0 \leq \frac{(A_n - A_0)}{c_1} \leq I$$

such that $A_n - A_0 \leq C_1 I$. Without loss of generality, the sequence satisfies $0 \leq A_n \leq I$.

for $n > m$, let's define $A_{mn} = A_n - A_m \geq 0$.

We have that $\|A_{mn}\| \leq 1$, since $0 \leq A_{mn} \leq I$. Using the Cauchy-Schartz inequality, for each $x, y = A_{mn}x$

$$\|A_m x - A_n x\|^2 = \langle A_{mn} x, A_{mn} x\rangle = \langle A_{mn} x, y\rangle$$

$$\leq \sqrt{\langle A_{mn} x, x\rangle}\sqrt{\langle A_{mn} y, y\rangle}$$

$$= \sqrt{\langle A_{mn} x, x\rangle}\sqrt{\langle A_{mn}^2 x, A_{mn} x\rangle}$$

$$\leq \sqrt{\langle A_{mn} x, x\rangle}\,\|x\|^2$$

Since $\|A_{mn} x\| \leq \|x\|$ and $\|A_{mn}^2 x\| \leq \|x\|$. Thus

$$\|A_m x - A_n x\|^2 \leq \sqrt{|\langle A_n x, x\rangle - \langle A_m x, x\rangle|}\cdot\|x\| \longrightarrow 0$$

if $n > m \longrightarrow \infty$ $\forall x \in \mathcal{H}$. As the sequence $\langle A_n x, x\rangle$ is monotonic increasing and bounded, therefore converges

Then $\{A_n x\}$ is Cauchy and $\lim\limits_{n\to\infty} A_n x = Bx$ exists.

moreover $0 \leq \langle A_n x, x\rangle \leq \langle x, x\rangle$ then $0 \leq \langle Bx, x\rangle \leq \|x\|^2$, which implies B is bounded.

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Proposition: Let A such that

$$mI \leq A \leq MI.$$

for some $m, M \in \mathbb{R}$, let P be a polynomial such that $P(z) \geq 0$ $\forall z \in [m, M]$, then $P(A) \geq 0$.

Lemma: Si $A \geq 0$, $B \geq 0$ and $AB = BA$, $AB \geq 0$.

Lemma: Let $A \geq 0$. Then exists an operator X, such that $X^2 = A$ and $X \geq 0$. We will write $\sqrt{A}$ by X. Moreover, for all B such that $AB = BA$, is fullfiled $\sqrt{A}B = B\sqrt{A}$.

This lemma implies the previous.

$$\langle ABx, x\rangle = \langle \sqrt{A}\sqrt{A}Bx, x\rangle = \langle B\sqrt{A}x, \sqrt{A}x\rangle = 0.$$

Proof: We want to find $X \geq 0$, such that $X^2 = A$. We can assume that $0 \leq A \leq I$. Let $B = I - A$, $Y = I - X$. Then $A = I - B$, $X = I - Y$, $0 \leq B \leq I$ and the equation to be solved is

$$(I - Y)^2 = I - 2Y + Y^2 = I - B,$$

i.e.,

$$Y = \frac{B + Y^2}{2}$$

We will solve by approximating the solution by a sequence Y_n, defined by

$$Y_{n+1} = \frac{1}{2}(B + Y_n^2), \qquad Y_0 = 0.$$

By induction, we can see that:

I. $Y_n \geq 0$ and Y_n is a polynomial with non-negative coefficients for all $n \in \mathbb{N}$.

II. $Y_n \leq I$, since by supposition $Y_{n-1} \leq I$, which implies that $Y_n \leq I$, because $B \leq I$ and $Y_{n-1}^2 \leq I$. The latter is obtained from $0 \leq Y_{n-1} \leq I$, $\|Y_{n-1}\| \leq 1$.

$$\langle Y_{n-1}^2 X, X \rangle = \langle Y_{n-1} X, Y_{n-1} X \rangle \leq \|X\|^2.$$

III. $Y_{n+1} - Y_n \geq 0$, and they are polynomials with non-negative coefficients.

$$Y_{n+1} - Y_n = \frac{1}{2}(Y_n^2 - Y_{n-1}^2) = \frac{1}{2}(Y_n + Y_{n-1})(Y_n - Y_{n-1})$$

where we have used $Y_n Y_{n-1} = Y_{n-1} Y_n$, by (I) are polynomials in B.

Moreover $Y_{n+1} + Y_n$ is a polynomial in B. Then $Y_1 - Y_0 = B/2$.

Then by induction we assume that $Y_n - Y_{n-1}$ is a polynomial in B, we derive the same conclusion for $Y_{n+1} - Y_n$.

Thus, by induction, $Y_{n+1} - Y_n$ is a polynomial in B, and as a result

$$Y_{n+1} - Y_n \geq 0, \qquad \forall\, n = 0, 1, \ldots$$

By the theorem (Converge of operators), $Y_n \to Y_\infty$ for some operator Y_∞ (in a strong way). Clearly

$$Y_\infty = \frac{1}{2}(B + Y_\infty^2)$$

It means that $X = I - Y_\infty$ is $\sqrt{A}$. Also $0 \leq Y_\infty \leq I$ implies that $X \geqslant 0$.

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Proof: If $m < M$, then $P(z) \geqslant 0$ for $z \in (m, M)$, implies that $P(z)$ may be written as

$$P(z) = c \prod_{\alpha_i \leq m} (z - \alpha_i) \prod_{\beta_i \leq M} (\beta_i - z) \prod \left((z - \gamma_i)^2 - \delta_i^2\right)^2, \quad \gamma_i, \delta_i \in \mathbb{R}.$$

for some $c > 0$. Since α_i are the real roots to the left of m, β_i are the real roots to the right of M, and the other term counts the complex roots.

thus

$$A - \alpha_i I \geqslant 0 \quad , \quad \beta_i I - A \geqslant 0$$

and

$$(A - \gamma_i I)^2 + \delta_i^2 I \geqslant 0$$

As all of this operators are commutative, by the previous lemma

$$P(A) \geqslant 0.$$

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