11. Cl(3)

$$\mathbb{R}^3 \ni x = x^i e_i$$

such that

$$e_i^2 = 1 \qquad e_i \wedge e_j = -e_j \wedge e_i \;;\quad i,j = 1,2,3.$$

$$e_i e_j + e_j e_i = 2\delta_{ij}$$

$$Cl(3) = \Lambda^0(\mathbb{R}^3) + \Lambda^1(\mathbb{R}^3) + \Lambda^2(\mathbb{R}^3) + \Lambda^3(\mathbb{R}^3)$$

scalars vectors bivectors trivectors

Λ^0	1	$e_{ij} = e_i \circ e_j$
Λ^1	e_1, e_2, e_3	$\dim 2^3 = 8.$
Λ^2	$e_{12}, e_{23}, e_{31}.$	
Λ^3	e_{123}	

Using:

$$* e^H = (e^K \circ e^K)\, e^K$$

$$* 1 = (e_{123} \circ e_{123})\, e_{123} \longrightarrow e_{123} \circ e_{123} = e_{123123}$$

$$= -e_{123} \qquad\qquad = e_{112323}$$

$$= e_{2323}$$

$$= -e_{2233} = -1.$$

$$\therefore\; e_{123} = i \quad \text{Complex unit.}$$

$$* e_i = (e_{jK} \circ e_{jK})\, e_{jK} \;;\quad i,j,K \text{ cyclic.}$$

$$= -e_{jK} \longrightarrow e_{jK} \circ e_{jK} = e_{jKjK} = -e_{jjKK} = -1.$$

$$* e_{jK} = (e_i \circ e_i)\, e_i = e_i.$$

$$* e_{123} = (1 \circ 1)\, 1 = 1.$$

$$e_{jK} = e_j \circ e_K = \cancel{e_j \cdot e_K}^{\,0} + e_j \wedge e_K$$

$$e_{jK} = e_j \wedge e_K = (*e_j, e_K)\, e_{123}$$

$$= -(e_{Ki}, e_K)\, e_{123}$$

$$= (e_{iK}, e_K)\, e_{123}$$

$$= i e_i$$

$$e_i \wedge e_j = i\epsilon_{ij}^{k} e_k$$

$$e_i \circ e_j = e_i \cdot e_j + i\epsilon_{ij}^{k} e_k$$

$$e_i e_j = \delta_{ij} + i\epsilon_{ij}^{k} e_k$$ Pauli algebra.

Also note

$$u \times v = *(u \wedge v).$$

Even subalgebra:

$$Cl^{+}(3) := \Lambda^0 + \Lambda^2 := \{ r + isv \mid r, s \in \mathbb{R},\ v \in \mathbb{R}^3 \}$$

$$Cl^{+}(3) \cong \mathbb{H} = \{ (1, u, v, w) / u^2 = v^2 = w^2 = -1,\ uvw = -1 \}$$

$$(u, v, w) \longleftrightarrow (e_{12}, e_{23}, e_{13})$$

Rotations in $\mathbb{R}^3$

III. $Cl(3) \longrightarrow \mathbb{R}^{1,3} = \{ x / x = x^0 e_0 + x^i e_i \}$ $x^i \in \mathbb{R}$, $i = 1,2,3$
$\mu = 0,1,2,3.$

Such that

$$x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$
$$= \eta^{\mu\nu} x_\mu x_\nu$$ Minkowski metric.

diag(+,−,−,−).

$$\left.\begin{aligned} e_0^2 = 1 = -e_1^2 = -e_2^2 = -e_3^2 \\ e_i \wedge e_j = -e_j \wedge e_i \end{aligned}\right\} \ e_i e_j + e_j e_i = 2\eta_{ij}$$

Dirac algebra

$$Cl(1,3) = \Lambda^0 + \Lambda^1 + \Lambda^2 + \Lambda^3 + \Lambda^4$$

$2^4 = 16.$

Λ^0	scalars	$1.$ $1.$
Λ^1	vectors	$e_0, e_1, e_2, e_3.$
Λ^2	bivectors	$e_{01}, e_{02}, e_{03}, e_{12}, e_{23}, e_{31}.$ $\sigma_1 \ \sigma_2 \ \sigma_3 \ *\sigma_3 \ *\sigma_1 \ *\sigma_2$
Λ^3	trivectors	$e_{012}, e_{023}, e_{031}, e_{123}.$
Λ^4	4-vectors	$e_{0123}.$ $-\sigma_{123}.$

$$e_{0123} \circ e_{0123} = e_{01230123} = -e_{00123123}$$

$$= -e_{123123} = -e_{112323}$$

$$= -e_{2323} = -1$$

$e_{0123} = i$ imaginary unit.

Even subalgebra:

$$Cl^+(3) = \Lambda^0 + \Lambda^2 + \Lambda^4$$

$$\exp(B\theta) \in Cl^+(1,3)$$

$$B \in \Lambda^2(\mathbb{R}^{1,3})$$ spinor

$$\dim Cl^+(1,3) = 8 = \dim Cl(3).$$

$$Cl^+(1,3) = Cl(3)$$

$$\sigma_k := e_0 e_k$$ basis for $\mathbb{R}^3$.

$$\sigma_1\sigma_2\sigma_3 = \sigma_{123} = e_{01}e_{02}e_{03} = e_{010203}$$

$$= -e_{012003} = -e_{0123}.$$

$$* e_{01} = i e_{01} = e_{0123} e_{01}$$

$$= e_{012301} = -e_{001231}$$

$$= -e_{1231} = e_{23}$$

$$B = a_1 e_{01} + a_2 e_{02} + a_3 e_{03} + b_3 e_{12} + b_1 e_{23} + b_2 e_{31}.$$

$$= a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 + b_3 * \sigma_3 + b_1 * \sigma_1 + b_2 * \sigma_2.$$

$$= \vec{A} + i\vec{C}$$

$$B \in \Lambda^2(\mathbb{R}^{1,3})$$

$$\vec{A}, \vec{C} \in \Lambda^1(\mathbb{R}^3)$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = (d \wedge A)_{ij}$$

$$F = d \wedge A$$

$$F = \vec{E} + i\vec{B}$$

From the vector space $T_p^*(M)$ of differentials (1-forms) we may get differential forms by exterior product

p-forms will be elements of $\Lambda^p(T_p^* M)$

$$\sum a_H dx^{h_1} \wedge \cdots \wedge dx^{h_p}, \quad a_H \in \mathcal{F}(M).$$

If ω and η are p- and q-forms respectively, we can see

$$\omega \wedge \eta = \sum a_H b_K dx^H \wedge dx^K.$$

Example:

$$\omega = X_1 dx_1 + X_2 dx_2 + X_3 dx_3$$

$$\eta = A_1 dx_2 dx_3 + A_2 dx_3 dx_1 + A_3 dx_1 dx_2$$

$$\begin{aligned} \omega \wedge \eta = & X_1 A_1 dx_1 \wedge dx_2 \wedge dx_3 \\ & + X_2 A_2 dx_2 \wedge dx_3 \wedge dx_1 \\ & + X_3 A_3 dx_3 \wedge dx_1 \wedge dx_2 \\ & = (X_1 A_1 + X_2 A_2 + X_3 A_3) dx_1 \wedge dx_2 \wedge dx_3 . \end{aligned}$$

Exterior derivative.

Let $d : \Lambda^p(T_x^* M) \longrightarrow \Lambda^{p+1}(T_x^* M)$, such that

I. $d(\omega + \eta) = d\omega + d\eta$ distributive.

II. $d(a\omega) = a d\omega, \quad a \in \mathbb{R}$ $\mathbb{R}$-linearity.

III. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$ Leibniz

IV. $d(d\omega) = 0$ $d^2 = 0$

V. For each function $f \in \mathcal{F}(M)$

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i$$

example: Let $\omega = a_H dx^H$ be a p-form, and let $d\omega = \frac{\partial a_H}{\partial x^i} dx^i \wedge dx^H$.

III. $\omega = a_H dx^H, \ \eta = b_K dx^K$

$$d(\omega \wedge \eta) = d(a_H b_K \, dx^H \wedge dx^K)$$

$$= \frac{\partial(a_H b_K)}{\partial x^i} dx^i \wedge dx^H \wedge dx^K$$

$$= \left(\frac{\partial a_H}{\partial x^i} b_K + a_H \frac{\partial b_K}{\partial x^i} \right) dx^i \wedge dx^H \wedge dx^K.$$

$$= \left(\frac{\partial a_H}{\partial x^i} dx^i \wedge dx^H \right) \wedge (b_K dx^K) + (-1)^p (a_H dx^H) \wedge \left(\frac{\partial b_K}{\partial x^i} dx^i \wedge dx^K \right)$$

$$= d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$$

IV. $d(d\omega) = d\left(\frac{\partial a_H}{\partial x^i} dx^i \wedge dx^H \right)$

$$= \frac{\partial^2 a_H}{\partial x^i} dx^i \wedge dx^H = \frac{1}{2} \sum \underbrace{\left(\frac{\partial^2 a_H}{\partial x^j \partial x^i} - \frac{\partial^2 a_H}{\partial x^i \partial x^j} \right)}_{=0} dx^j \wedge dx^i \wedge dx^H$$

$d^2 = 0$ stands for the equality of mixed partial derivatives.

V. $f \in \mathcal{F}(M)$

$$df = \frac{\partial f}{\partial x^i} dx^i$$