Homework: Proof that

$$\frac{\partial \mathcal{L}_{EH}}{\delta g_{\mu\nu,\sigma\rho}} = (-g)^{1/2}\left[\frac{1}{2}\left(g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}\right) - g^{\mu\nu}g^{\sigma\rho}\right]$$

We will follow a shortcut:

Define $\mathfrak{g}^{\mu\nu} := (-g)^{1/2} g^{\mu\nu}$

$$S = \int_\Omega \mathfrak{g}^{\mu\nu} R_{\mu\nu}$$

$$\delta S = \int_\Omega \left(\delta \mathfrak{g}^{\mu\nu} R_{\mu\nu} + \mathfrak{g}^{\mu\nu}\delta R_{\mu\nu}\right)$$

1. First integral:

$$\int_\Omega \delta \mathfrak{g}^{\mu\nu} R_{\mu\nu} = \int_\Omega R_{\mu\nu}\, \delta\left((-g)^{1/2} g^{\mu\nu}\right)$$

$$= \int_\Omega R_{\mu\nu}\left(\delta(-g)^{1/2} g^{\mu\nu} + (-g)^{1/2}\delta g^{\mu\nu}\right)$$

$\delta(-g)^{1/2}$ Jacobi's formulation.

$$\frac{d}{dt}(\det M) = \mathrm{tr}\left(\mathrm{adj}\, M \frac{dM}{dt}\right)$$

then

$$\delta(-g)^{1/2} = \frac{1}{2}(-g)^{1/2} g^{\alpha\beta}\,\delta g_{\alpha\beta}$$

$$\frac{1}{2}\frac{-1}{(-g)^{1/2}}\delta g = -\frac{1}{2}\frac{1}{(-g)^{1/2}}\left[(g) g^{\alpha\beta}\,\delta g_{\alpha\beta}\right]$$

$$0 = \delta(\delta^\mu_\eta) = \delta(g^{\mu\nu} g_{\nu\eta})$$

$$= \delta g^{\mu\nu} g_{\nu\eta} + g^{\mu\nu}\delta g_{\nu\eta} = 0$$

then

$$\delta g^{\mu\nu} g_{\nu\eta} = -g^{\mu\nu}\delta g_{\nu\eta}$$

$$g^{\mu\rho} = \delta g^{\mu\nu} g_{\nu\eta}\, g^{\eta\rho} = -g^{\mu\nu} g^{\eta\rho}\delta_{\nu\eta}$$

$$\int_\Omega R_{\mu\nu}\left[\left(\frac{1}{2}(-g)^{1/2} g^{\alpha\beta}\delta g_{\alpha\beta}\right)g^{\mu\nu} + (-g)^{1/2}\left(-g^{\mu\sigma}g^{\rho\nu}\delta g_{\sigma\rho}\right)\right]$$

$$= \int_\Omega (-g)^{1/2} R_{\mu\nu}\left[\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right]\delta g_{\alpha\beta}$$

$$= \int_\Omega (-g)^{1/2}\left[\frac{1}{2}Rg^{\alpha\beta} - R^{\alpha\beta}\right]\delta g_{\alpha\beta}$$

$$= -\int_\Omega (-g)^{1/2}\left[R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right]\delta g_{\alpha\beta}$$

$$= -\int_\Omega (-g)^{1/2} G^{\alpha\beta}\delta g_{\alpha\beta}$$

2. Second Integral:

$$\int_\Omega g^{\mu\nu}\delta R_{\mu\nu}$$

First consider a coordinate system such that

$$\Gamma^{\mu}_{\rho\sigma} \overset{*}{=} 0$$

$$R^{\sigma}_{\mu\rho\nu} \overset{*}{=} \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho}$$

A variation in Γ

$$\Gamma^{\sigma}_{\mu\nu} \longmapsto \Gamma^{\sigma}_{\mu\nu} + \delta\Gamma^{\sigma}_{\mu\nu}$$

induces a variation in the Riemann tensor

$$R^{\sigma}_{\mu\rho\nu} \longmapsto R^{\sigma}_{\mu\rho\nu} + \delta R^{\sigma}_{\mu\rho\nu}$$

where

$$\delta R^{\sigma}_{\mu\rho\nu} \overset{*}{=} \partial_\rho(\delta\Gamma^{\sigma}_{\mu\nu}) - \partial_\nu(\delta\Gamma^{\sigma}_{\mu\rho})$$
$$= \nabla_\rho(\delta\Gamma^{\sigma}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\sigma}_{\mu\rho})$$

As this is the tensor equation, then it must hold for every coordinate system.

$$\delta R^{\sigma}_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^{\sigma}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\sigma}_{\mu\rho})$$ Palatini equation

then

$$\delta R_{\mu\nu} = \nabla_\sigma(\delta\Gamma^\sigma_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\sigma_{\mu\sigma})$$

$$\int_\Omega g^{\mu\nu}\delta R_{\mu\nu}$$

$$= \int_\Omega g^{\mu\nu}\left[\nabla_\sigma(\delta\Gamma^\sigma_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\sigma_{\mu\sigma})\right]$$

$$= \int_\Omega \left[\nabla_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\sigma})\right]$$

$$= \int_\Omega \nabla_\sigma\left[g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\nu_{\mu\nu}\right]$$

$$= \int_{\partial\Omega}\left(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\nu_{\mu\nu}\right)$$

We ask that the variation of Γ's in the boundary vanishes!

$$0 = \delta S = \int_M (-g)^{1/2} G^{\alpha\beta}\delta g_{\alpha\beta} + \cancel{\int_{\partial M}(g^{\mu\nu} - g^{\mu\sigma})\delta\Gamma}^{\to 0}$$

$$\Longleftrightarrow \quad \boxed{G^{\alpha\beta} = 0}$$

Palatini Lagrangian

$$\mathcal{L}_P = (-g)^{1/2} R$$

$$R = g^{\mu\nu}\left[\Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}\right]$$

Consider g and Γ as independ fields

$$\mathcal{L}_P = \mathcal{L}_P(g, \Gamma, \partial\Gamma)$$

Field equations:

1.

$$\frac{\partial \mathcal{L}_P}{\partial g^{\alpha\beta}} - \frac{\partial}{\partial x^\gamma}\frac{\partial \mathcal{L}_P}{\partial g^{\alpha\beta}_{\ ,\gamma}} = 0 \qquad \Longrightarrow \qquad G^{\alpha\beta}$$

11.

$$\frac{\partial \mathfrak{L}_P}{\partial \Gamma^\gamma_{\alpha\beta}} - \frac{\partial}{\partial x^\delta}\left(\frac{\partial \mathfrak{L}_P}{\partial \Gamma^\gamma_{\alpha\beta,\delta}}\right) = 0$$

$$\frac{\partial \mathfrak{L}_P}{\partial \Gamma^\gamma_{\alpha\beta}} = \mathfrak{g}^{\mu\nu}\left[\frac{\partial \Gamma^\rho_{\mu\nu}}{\partial \Gamma^\gamma_{\alpha\beta}}\Gamma^\sigma_{\rho\sigma} + \Gamma^\rho_{\mu\nu}\frac{\partial \Gamma^\sigma_{\rho\sigma}}{\partial \Gamma^\gamma_{\alpha\beta}} - \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial \Gamma^\gamma_{\alpha\beta}}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\sigma}\frac{\partial \Gamma^\sigma_{\rho\nu}}{\partial \Gamma^\gamma_{\alpha\beta}}\right]$$

$$= \mathfrak{g}^{\mu\nu}\left[\delta^{\alpha\beta\rho}_{\mu\nu\gamma}\Gamma^\sigma_{\rho\sigma} + \Gamma^\rho_{\mu\nu}\delta^{\alpha\beta\sigma}_{\rho\sigma\gamma} - \delta^{\alpha\beta\rho}_{\mu\sigma\gamma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\sigma}\,\partial\Gamma^{\alpha\beta\sigma}_{\rho\sigma\gamma}\right]$$

$$= \mathfrak{g}^{\alpha\beta}\Gamma^\sigma_{\gamma\sigma} + \mathfrak{g}^{\mu\nu}\Gamma^\alpha_{\mu\nu}\delta^\beta_\gamma - \mathfrak{g}^{\alpha\nu}\Gamma^\beta_{\gamma\nu} - \mathfrak{g}^{\mu\beta}\Gamma^\alpha_{\mu\gamma}$$

$$\frac{\partial \mathfrak{L}_P}{\partial \Gamma^\gamma_{\alpha\beta,\delta}} = \mathfrak{g}^{\mu\nu}\left(\delta^{\alpha\beta\delta\sigma}_{\mu\nu\sigma\gamma} - \delta^{\alpha\beta\delta\sigma}_{\mu\sigma\nu\gamma}\right)$$

$$= \mathfrak{g}^{\alpha\beta}\delta^\delta_\gamma - \mathfrak{g}^{\alpha\delta}\delta^\beta_\gamma$$

$$\partial_\delta\left(\frac{\partial \mathfrak{L}_P}{\partial \Gamma^\gamma_{\alpha\beta,\delta}}\right) = \mathfrak{g}^{\alpha\beta}{}_{,\gamma} - \mathfrak{g}^{\alpha\delta}{}_{,\delta}\,\delta^\beta_\gamma$$

Field equations for Γ:

$$\mathfrak{g}^{\alpha\beta}{}_{,\gamma} - \mathfrak{g}^{\alpha\mu}{}_{,\mu}\,\delta^\beta_\gamma = \mathfrak{g}^{\alpha\beta}\Gamma^\mu_{\gamma\mu} + \mathfrak{g}^{\mu\nu}\Gamma^\alpha_{\mu\nu}\delta^\beta_\gamma - \mathfrak{g}^{\alpha\mu}\Gamma^\beta_{\gamma\mu} - \mathfrak{g}^{\mu\beta}\Gamma^\alpha_{\mu\gamma}$$

$$\Longleftrightarrow$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma}\right)$$