$$d : C^\infty(M) \longrightarrow \Omega^1(M)$$

$$C^\infty(M) \ni \phi^* f = f \circ g : M \longrightarrow \mathbb{R}$$

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu$$

$$f \in C^\infty(N), \quad f : N \longrightarrow \mathbb{R}$$

$$\phi^* df = d(\phi^* f)$$

Proof:

$$(\phi^* df)_p (v)_p = (df)_q (\phi_* v) = (\phi_* v)(f)(q) = v(\phi^* f)(p) = d(\phi^* f)_p (v).$$

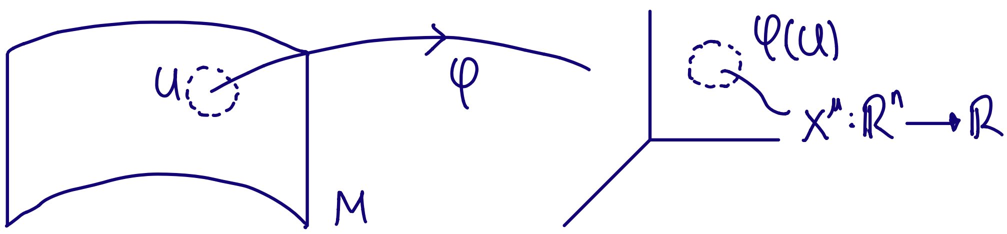
Homework: Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, $\phi(t) = \sin t$ and dx a 1-form in $\mathbb{R}$, prove that

$$\phi^* dx = \cos t\, dt$$

$$\mathbb{R} \xrightarrow{\phi} \mathbb{R} \xrightarrow{dx} \mathbb{R}$$

Change of coordinates

Let M be an n-dim manifold, a chart is diffeomorphism φ and from an open $U \subseteq M$, in $\mathbb{R}^n$. This allow us to make calculus in U, as if they would calculus in $\mathbb{R}^n$.



We can use φ, to take the pullback of the coordinate functions x^μ from $\mathbb{R}^n$ to U instead of calling this coordinate functions $\varphi^* x^*$, usually is called x^μ

Note: It is not confusing as long as we know it is a chart $\varphi : U \longrightarrow \mathbb{R}^n$.

The functions x^μ in U, are called local coordinates of U, thus $f : U \longrightarrow \mathbb{R}$, $f(x^1, \dots, x^n)$.

In a similar way, the vector fields ∂_μ, are the basis of $\mathbb{R}^n$. We may use the pushforward for φ^{-1} to take them to U. $\varphi^{-1}{}_* \partial_\mu$ will be labeled ∂_μ, a vector field u in U.

$$U = u^\mu \partial_\mu$$

In the same way, the 1-forms dx^μ in $\mathbb{R}^n$ we pass them to U, using $\varphi^* dx^\mu$ we denote them as dx^μ

$$\omega = \omega_\mu dx^\mu.$$

If we take coordinate functions $x^1, \dots, x^n \in \mathbb{R}^n$, gives a basis

$$\left\{ \frac{\partial}{\partial x^\mu} \right\}$$

for $\mathrm{Vect}(\mathbb{R}^n)$

$$v = v^\mu \partial_\mu, \quad \text{with} \quad v^\mu \in C^\infty(\mathbb{R}^n)$$

Let's suppose other coordinates in $\mathbb{R}^n$ $x^{1'}, x^{2'}, \dots, x^{n'}$; such that

$$\left\{ \frac{\partial}{\partial x'^\nu} \right\}$$

Is another basis for $\mathrm{Vect}(\mathbb{R}^n)$

$$v = v'^\mu \partial'_\mu$$

We need to know how v'^μ is related with v^μ. So, as $\{\partial_\mu\}$ and $\{\partial'_\nu\}$ are basis

$$\partial_\mu = T^\nu{}_\mu \partial'_\nu$$

To find out who is $T^\nu{}_\mu$, apply

$$\partial_\mu x'^\lambda = T^\mu_\nu \partial'_\nu x'^\lambda = T^\nu_\mu \frac{\partial x'^\lambda}{\partial x'^\nu} = T^\nu_\mu \delta^\lambda_\nu = T^\lambda_\mu$$

Then,

$$T^\lambda_\mu = \frac{\partial x'^\lambda}{\partial x^\mu}$$

Therefore

$$\partial_\mu = \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu \quad = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}$$

$$v = v'^\mu \partial'_\mu = v^\mu \partial_\mu \longrightarrow v'^\nu \partial'_\nu = v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu$$

then,

$$v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} v^\mu$$

Homework: Prove that

$$dx'^{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}} dx^{\mu}$$

and, if ω is a 1-form in $\mathbb{R}^n$

$$\omega = \omega_\mu dx^\mu = \omega'_\mu dx'^\mu$$

$$\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu$$

P-forms

Let V be a vector space, we want multiply 2 vectors in V in some way and we want them to

$$\vec{v} \times \vec{\omega} = -\vec{\omega} \times \vec{v}$$

We will call to this new product, generalized cross product, exterior product or wedge product $(\wedge)$

The exterior algebra over V, is denoted by $\wedge V$ and it is the generated algebra by V with the relations.

$$v \wedge w = -w \wedge v, \qquad \forall\, v, w \in V$$

This means that we start with vectors in V, and one element 1. And we make an algebra, taking all the linear convinations from the products in the form

$$v_1 \wedge v_2 \wedge \cdots \wedge v_p \;;\; v_i \in V$$

and they satisfy

$$v_1 \wedge v_2 = -v_2 \wedge v_1 \;, \quad \forall\, v_1, v_2 \in V$$

Example: Let V a 3-dim vector space. Then $\wedge V$, is the linear combination of exterior products of elements of V.

Suppose V, has a basis dx, dy, dz.

$$1 \in \wedge V,\ dx, dy, dz \in \wedge V$$

and its linear combinations. If $v, w \in V$

$$v = v_x dx + v_y dy + v_z dz$$

$$w = w_x dx + w_y dy + w_z dz$$

$$v \wedge w = (v_x dx + v_y dy + v_z dz) \wedge (w_x dx + w_y dy + w_z dz)$$
$$= (v_x w_y - v_y w_x) dx \wedge dy + (v_y w_z - v_y w_z) dy \wedge dz$$
$$+ (v_z w_x - v_x w_z) dz \wedge dx.$$

Homework: Let $u = u_x dx + u_y dy + u_z dz.$

$$u \wedge v \wedge w = \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} dx \wedge dy \wedge dz$$

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

Let $a, b, c, d \in V \longrightarrow$ 3-dim

$$a \wedge b \wedge c \wedge d = 0.$$

In general, in any vector space V. We denote $\Lambda^p V$ as the subspace of ΛV that exists in the linear combinations.

$$v_1 \wedge \dots \wedge v_p$$

$\Lambda^0 V$ is $\mathbb{R}$.

Homework: Let V an n-dim vector space. Proof that $\Lambda^p V$ is empty if $p > n$ and $0 \leq p \leq n$,

$$\dim \Lambda^p V = \frac{n!}{p!(n-p)!}$$

A vector space V is said the direct sum of subspaces $V_1, \dots, V_n$ if $v \in V$ is expressed in a unique way as

$$v = v_1 + v_2 + \dots + v_n, \quad \text{with} \quad v_i \in V_i$$

$$V = \bigoplus_{i=1}^{n} V_i$$

Then

$$\Lambda V = \bigoplus \Lambda V$$

and

$$\dim(\Lambda V) = 2^n.$$