Poincaré group.

Let $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, with scalar product

$$xy = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 = g_{\mu\nu} x^\mu y^\nu$$

this space is called the Minkowski space. The linear transforms in this space

$$x \longrightarrow x' = \Lambda x + y$$
$$x'^\mu = \Lambda^\mu_\nu x^\nu + y^\mu, \qquad \mu, \nu = 0, 1, 2, 3.$$

$\longrightarrow$ Translations.

which them satisfy.

$$g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}$$

$\longrightarrow$ Lorentz transformation.

$$x' \cdot y' = g_{\mu\nu} x'^\mu y'^\nu$$
$$= g_{\mu\nu} \Lambda^\mu_\rho x^\rho \Lambda^\nu_\sigma y^\sigma$$
$$= g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho y^\sigma$$

Called inhomogeneous lorentz transformations and they are denoted by (Λ, y) and the homogeneous part $(\Lambda, 0)$ leaves invariant the scalar product.

$$x'y' = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\nu_\sigma x^\rho x^\sigma = xy.$$

this inhomogeneous transformations form a group, called Poincaré group **P**.

$$(\Lambda_1, y_1)(\Lambda_2, y_2) = (\Lambda_1, \Lambda_2, y_1 + \Lambda_1 y_2)$$

$$x \longrightarrow x' = \Lambda_2 x + y_2$$
$$x' \longrightarrow x'' = \Lambda_1 (\Lambda_2 x + y_2) + y_1$$
$$= \Lambda_1 \Lambda_2 x + \Lambda_1 y_2 + y_1$$
$$= (\Lambda_1 \Lambda_2, \Lambda_1 y_2 + y_1)$$

$$P = L \times T_4$$

L - Homogeneous part.

T_4 - Translations

In the natural topology of the matrices, L has fourth disconected parts.

$$(g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = g_{\rho\sigma})$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x \cdot y = g_{\mu\nu} x^{\mu} y^{\nu} = \sum_{\mu,\nu=0}^{3} g_{\mu,\nu} x^{\mu} y^{\nu}$$

$$g_{\mu\nu} \Lambda^{\mu}_{0} \Lambda^{\nu}_{0} = 1$$

$$\rightarrow (\Lambda^{0}_{0})^2 = 1, \quad \sum_{k=1}^{3} (\Lambda^{k}_{0})^2 \geq 1.$$

$$\rightarrow \Lambda^{0}_{0} \geq 1 \quad \text{or} \quad \Lambda^{0}_{0} \leq -1.$$

the fourth parts of L are

- $L^{\uparrow}_{+} = \det \Lambda = +1, \quad \text{sign } \Lambda^{0}_{0} = +1.$

  This part contains to the identity matrix I and it is a group. It is called eigen lorentz group.

- $L^{\uparrow}_{-} : \det \Lambda = -1, \quad \text{sgn } \Lambda^{0}_{0} = +1$

  Contains to the element

  $$I_s X = (X^{0}_{1} - X^{1}, -X^{2}, -X^{3})$$

- $L^{\downarrow}_{-} : \det \Lambda = -1, \quad \text{sign } \Lambda^{0}_{0} = -1$

  $$I_t X = (-X^{0}, X^{1}, X^{2}, X^{3})$$

- $L^{\downarrow}_{+} : \det \Lambda = +1, \quad \text{sgn } \Lambda^{0}_{0} = -1$

  $$I_s I_t$$

$$L^{\uparrow}_{+} \simeq SO(3,1) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$$

The group SL(2,C)

Consist in the complex matrices 2x2.

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad a_{ik} \in \mathbb{C} \quad , \quad i,k = 1,2.$$

This group is conected $[\pi : SL(2,\mathbb{C}) \longrightarrow SU(2)]$. We may parametrize it by

$$a = a_0 e + \sum_{k=1}^{3} a_k \sigma_k, \quad a_0, a_k \in \mathbb{C}, \quad e \text{ is the identity matrix } I_{2x2}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli's matrices

any point of the Minkowski space may be represent as a matrix.

$$x \longmapsto x = x^0 e + \sum_{k=1}^{3} x^k \sigma_k$$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = X \longmapsto \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = a$$

$$\|x\|^2 = g_{\mu\nu} x^\mu x^\nu \qquad \det(a) = x^{0^2} - x^{3^2} - x^{1^2} - x^{2^2}$$

Using $a \in SL(2,\mathbb{C})$, we may define a linear map of $SL(2,\sigma)$ in itself.

$$x' = a x a^\dagger$$

Preserves the inner product.

$$(x')^2 = \det(x') = \det(axa^\dagger) = \det(x) = (x)^2$$

Let's write this correspondence as.

$$x' = a x a^\dagger \longrightarrow x' = \Lambda(a) x$$

$$\Lambda_o^o = |a_0|^2 + \sum_{k=1}^{3} |a_k|^2$$

$$\Lambda_o^k = a_0 \overline{a_k} + \overline{a_0} a_k + i \varepsilon^{klm} a_l \overline{a_m}$$

$$\Lambda_k^l = \delta_k^l \left(|a_0|^2 - \sum_{t=1}^{3} |a_t|^2 \right) a_k \overline{a_l} + \overline{a_k} a_l + i \varepsilon^{klm} (\overline{a_0} a_m - a_0 \overline{a_m})$$

$$\varepsilon^{klm} \begin{cases} 1 \,, & klm \text{ is permutation} \\ 0 & \text{is odd}. \\ -1 & \text{there are representations.} \end{cases}$$

moreover.

$$\Lambda^\nu_\nu (a) = Tr(\Lambda(a)) = Tr(a)^2 = 4|a_0|^2$$

Since $SL(2,\mathbb{C})$ is connected, the map on L gives that the image of $SL(2,\mathbb{C})$ is $L_+^\uparrow$.

We can invert

$$a_0 e + \sum_{k=1}^{3} a_k \overline{\sigma_k} = D^{-1} \left\{ \Lambda^\nu_\nu e + \sum_{k=1}^{3} (\Lambda_o^k + \Lambda_k^o + i \varepsilon^{ok\varphi\tau} \Lambda^\tau_\varphi) \sigma_k \right\}$$

$$D^2 = 4 - Tr(\Lambda\Lambda) + (Tr(N^2) + i \varepsilon^{\mu\nu\varsigma\tau} \Lambda^\varsigma_\mu \Lambda^\tau_l$$

In other words

$$SL(2,\mathbb{C}) \ni \pm e \longrightarrow I_{4\times4} \in L_+^\uparrow$$

$$L_+^\uparrow \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$$

Subgroups of $SL(2,\mathbb{C})$

Given the equivalence between $L_+^\uparrow$ and $SL(2,\mathbb{C})$ is easier to study $SL(2,\mathbb{C}) \times T_4$

$SU(2) \subseteq SL(2,\mathbb{C})$: which is formed by

$$u^\dagger = u^{-1}$$

and its convenient to parametrize $SU(2)$ by

$$u = u_0 e + i \sum_{k=1}^{3} u_k \sigma_k, \qquad u_0, u_k \in \mathbb{R}$$

$SU(1,1)$, $v \in SL(2,\mathbb{C})$: formed by

$$v^\dagger \sigma_3 = \sigma v^{-1}$$

its parametrized by

$$v = v_0 e + v_1 \sigma_1 + v_2 \sigma_2 + i v_3 \sigma_3$$

with

$$v_0^2 - v_1^2 - v_2^2 + v_3^2 = 1, \quad v_0, v_k \in \mathbb{R}$$

$SL(2,\mathbb{R}) \subseteq SL(2,\mathbb{C})$: formed by

$$a^\dagger \sigma_2 = \sigma_2 a^{-1}$$

and its parametrized by

$$a = a_0 e + a_1 \sigma_1 + i a_2 \sigma_2 + a_3 \sigma_3,$$

$$a_0, a_k \in \mathbb{R}.$$