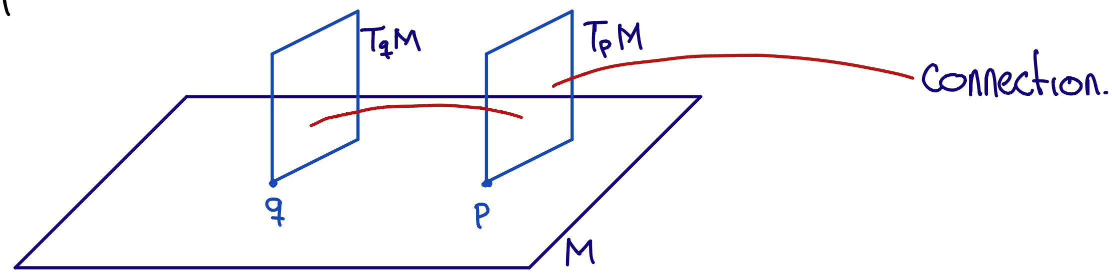
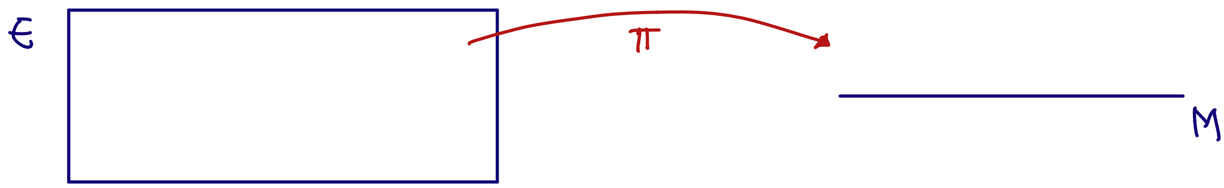
Fiber Bundles

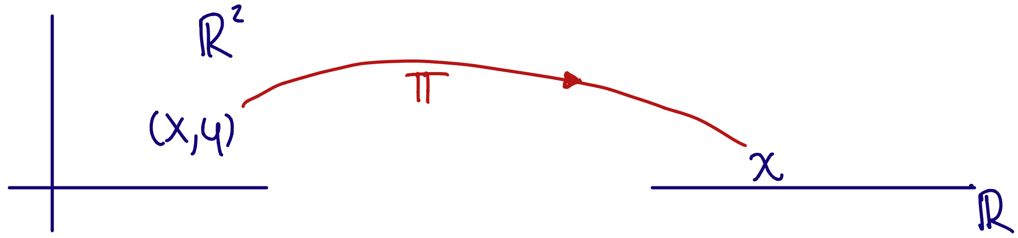
A vector field v in M asigns to each $p \in M$, a vector, i.e., $V_p \in T_pM$



A bundle, is an structure composed of a manifold E, a manifold M and a map $\pi : E \longrightarrow M$.



Let $M = \mathbb{R}$, $E = \mathbb{R}^2$ and $\pi : E \longrightarrow M$, the projection in the x-axis.



The manifold E, is called Total space, M is the base space and π the projection map.

For each $p \in M$, $E_p = \{q \in E : \pi(q) = p\} \longrightarrow$ Fibers.

Total space E,

$$E = \bigcup_{p \in M} E_p$$

The tangent bundle of a manifold is an example.

The total space TM is the sum of the tangent spaces to M.

$$TM = \bigcup_{p \in M} T_pM,$$

and the projection $\pi : TM \longrightarrow M$ maps each tangent vector $v \in T_pM$, $p \in M$.

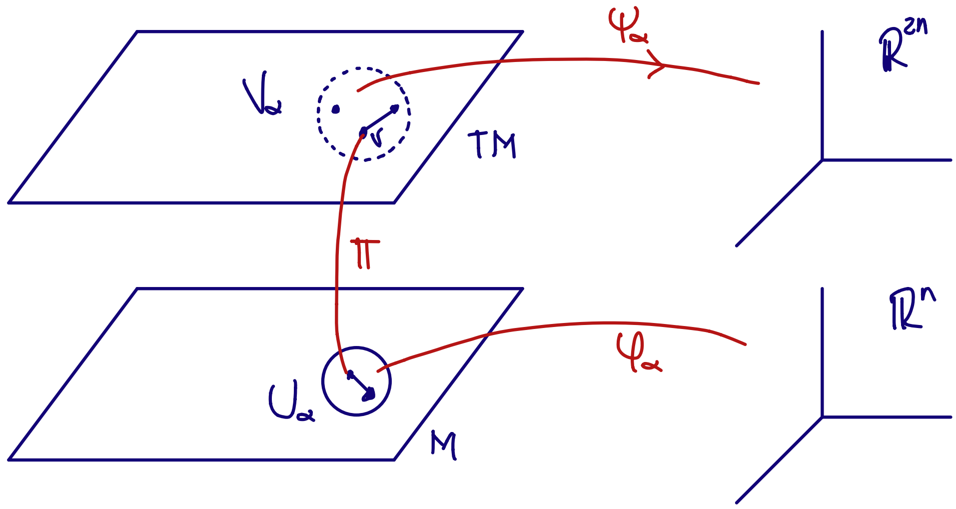
We must give to T_pM the structure of manifold.

M is a manifold of dimension n, locally looks like $\mathbb{R}^n$.

Specifically a point TM, is to give $p \in M$ and $v \in T_pM$ i.e., TM locally looks like $\mathbb{R}^n \times \mathbb{R}^n$.

Let's define a chart in TM, using the chart of M, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ of M. Let V_α be a subset of TM given by

$$V_\alpha = \{ v \in TM, \pi(v) \in U_\alpha \}$$



$$\psi_\alpha(v) = \{ \varphi_\alpha(\pi(v)), (\varphi_\alpha)_* v \}$$

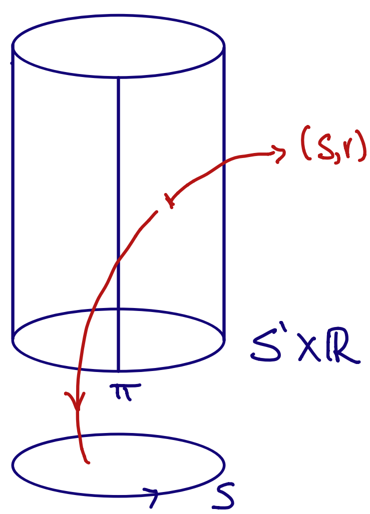
Given two manifolds M and F the trivial boundle over M, with fibers in F, is simply the cartesian product $E = M \times F = (p,f)$ and projection.

$$\pi(p,f) = p \quad ; \forall \ (p,f) \in M \times F$$

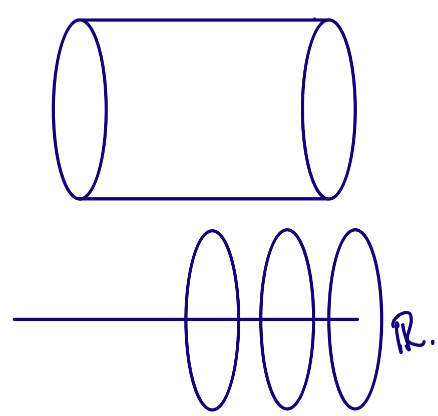
The trivial bundle $E = M \times F$, the bundle over p

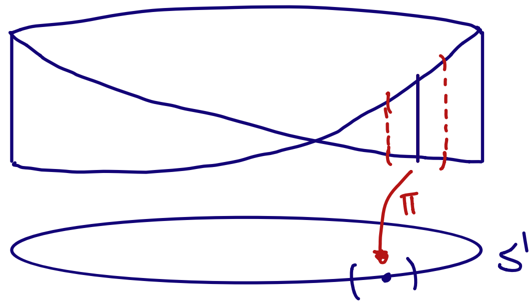
$$E_p = \{p\} \times F$$

The cylinder



$R \times S^1$





Given two fiber bundles $\pi: E \to M$, $\pi': E' \to M'$, a morphism $\psi: E \to E'$, with a map $\phi: M \to M'$ such that ψ map each fiber E_p in the fibers $E'_{\phi(p)}$.