Representations.

Let G be a topological group (metrizable). Let $(V, \langle\cdot\rangle)$ be a Hilbert space. Let $GL(V)$ be the set of linear invertible matrices $T: V \to V$.

A representation of G in V is a groups homomorphism $\eta: G \longrightarrow GL(V)$ such that $(x,v) \longmapsto \eta(x)v$ is continuous. The representation η is called unitary in V i.e.,

$$\langle \eta(x)v, \eta(x)w \rangle = \langle v, w \rangle \ ; \ \forall v, w \in V, x \in G.$$

A closed subspace $W \subseteq V$, it's called invariant for η, if $\eta(x)W \subseteq W$ $\forall x \in G$. The representation η is called irreducible, if there are no invariable closed self-subspaces i.e., the only invariant subspaces are 0 and V.

Example: The identity map $\rho: U(n) \longrightarrow GL(\mathbb{C}^n) = GL_n(\mathbb{C})$ is a unitary representation.

Lemma: ρ is irreducible.

Proof: By definition, $U(n)$ consists of the linear operators of $\mathbb{C}^n$ such that they are unitary with respect to the internal product $\langle v, w \rangle = v^t \overline{w}$.

Let $V \subseteq \mathbb{C}^n$ be a subspace which is not self-subspace., also the zero, and neither all the space. Let $W = V^{\perp}$ the orthogonal complement i.e.,

$$W = \}w \in \mathbb{C}^n / \langle w, v \rangle = 0, \forall v \in V\{$$

Thus, $\mathbb{C}^n = V \oplus W$, let $e_1, \ldots, e_l$ be a orthonormal base of V and $e_{l+1}, \ldots, e_n$ other one from W. So the operator T given by

$$T(e_1) = e_{l+1} \quad ; \quad T(e_{l+1}) = e_1 .$$

$$T(e_j) = e_j \quad ; \quad j \neq 1, l+1.$$

is unitary. Thus $T \in U(n)$, but T do not let stable to V.

Let $T: H \longrightarrow H$, T is unitary if and only if, for each orthonormal base (e_j), the formula Te_j is orthonormal.

Exponential map

A serie $A \in Mat_n(\mathbb{C})$ of the form $\sum_{\nu=0}^{\infty}$ converges by the definition if the succession of partial sums

$$S_k = \sum_{\nu=1}^{\infty} A_\nu$$

Converges.

Proposition: For each $A \in Mat_n(\mathbb{C})$, the series

$$\exp(A) = \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!}$$

Converges and define an element in $GL_n(\mathbb{C})$. If $A, B \in Mat_n(\mathbb{C})$ and $AB = BA$, then $\exp(A+B) = \exp(A)\exp(B)$. In particular

$$\exp(-A) = \exp(A^{-1})$$

Proof: Let's remember that in $Mat_n(\mathbb{C})$

$$\|A\|_1 = \sum_{i,j}^{n} |a_{ij}| \quad , \text{if } A = (a_{ij})$$

Lemma: For $A, B \in Mat_n(\mathbb{C})$

$$\|AB\|_1 \leq \|A\|_1 \|B\|_1$$

in particular for $j \in \mathbb{N}$, $\|A^j\|_1 \leq \|A\|_1^j$

Proof: Let $A = (a_{ij})$, $B = (b_{ij})$

$$\|AB\|_1 = \sum_{i,j=1}^{n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \leq \sum_{i,j,k=1}^{n} |a_{ik} b_{kj}|$$

$$\leq \sum_{i,j,k,l=1}^{n} |a_{ik}| |b_{lj}| = \|A\|_1 \|B\|_1$$

Lemma: Let $(A_\nu)_{\nu \geq 0}$ be a succession of matrices in $Mat_n(\mathbb{C})$. Let's suppose that

$$\sum_{\nu=0}^{\infty} \|A_\nu\|_1 < \infty$$

So the series $\sum_{\nu=0}^{\infty} A_\nu$ converges.

Proof: Let $b_k = \sum_{\nu=0}^{k} A_\nu$, we have to prove that the succession (B_k) converges. It is sufficient to show that it is a succession of Cauchy.

Thus, given $\varepsilon > 0$; $\exists k_0$, such that for $m \geq k \geq k_0$

$$\varepsilon > |b_m - b_k| = \sum_{\nu=k+1}^{m} \|A_\nu\|_1 \geq \left\| \sum_{\nu=k+1}^{m} A_\nu \right\|_1 = \|B_m - B_k\|_1$$

Then (B_k) is Cauchy in $Mat_n(\mathbb{C})$, thus converges.

Homework: Prove that $Mat_n(\mathbb{C})$, all Cauchy's succession converges to the norm $\|\cdot\|_1$ or $\|\cdot\|_2$.

To prove the proposition, less to prove that

$$\sum_{\nu=0}^{\infty} \frac{\|A^\nu\|_1}{\nu!} < \infty$$

thus

$$\sum_{\nu=0}^{\infty} \frac{\|A^\nu\|_1}{\nu!} \leq \sum_{\nu=0}^{\infty} \frac{\|A\|_1^\nu}{\nu!} < \infty$$

Since the exponential in $\mathbb{R}$ converges. For the rest of the proposition, let $A, B \in Mat_n(\mathbb{C})$, $AB = BA$.

$$\exp(A+B) = \sum_{\nu=0}^{\infty} \frac{(A+B)^\nu}{\nu!} = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \sum_{k=0}^{\nu} \binom{\nu}{k} A^k B^{\nu-k}$$

$$= \sum_{\nu=0}^{\infty} \sum_{k=0}^{\nu} \frac{1}{k!(\nu-k)!} A^k B^{\nu-k} = \sum_{\nu=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\nu! k!} A^\nu B^k = \exp(A)\exp(B).$$

$\blacksquare$

Proposition: For each $A \in Mat_n(\mathbb{C})$

$$\det(\exp(A)) = \exp(\operatorname{tr} A)$$

Proof: Let $S \in GL_n(\mathbb{C})$, then

$$\det(\exp(SAS^{-1})) = \det(S\exp(A)S^{-1}) \det(\exp(A))$$

and

$$\exp(\operatorname{tr}(SAS^{-1})) = \exp(\operatorname{tr} A)$$

Thus, both sides of the proposition are invariants under the conjugation.

By Jordan's theorem, all squar matrix is conjugated to an upper triangular matrix.

Let's suppose

Geometrical multiplicity
Algebraic multiplicity

$$A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix}$$

for $\nu \geq 0$

$$A^{\nu} = \begin{pmatrix} a_1^{\nu} & & * \\ & \ddots & \\ & & a_n^{\nu} \end{pmatrix}$$

then

$$\exp(A) = \begin{pmatrix} e^{a_1} & & * \\ & \ddots & \\ & & e^{a_n} \end{pmatrix}$$

$$\det(\exp(A)) = e^{a_1} \cdot \ldots \cdot e^{a_n} = e^{a_1 + \ldots + a_n} = \exp(\mathrm{tr}\, A)$$

■

Let $G \subseteq GL_n(\mathbb{C})$ be a closed subgroup. The Lie's algebra of G is given by

$$\mathrm{Lie}(G) = \{ \underline{X} \in \mathrm{Mat}_n(\mathbb{C}) / \exp(t \cdot \underline{X}) \in G, \forall\, t \in \mathbb{R} \}$$

Example:

i) The linear special group $SL_n(\mathbb{C})$, which consist of the matrices $A \in \mathrm{Mat}_n(\mathbb{C})$, such that $\det(A) = 1$, his Lie algebra

$$\mathrm{Lie}(SL_n(\mathbb{C})) = \mathfrak{sl}_n(\mathbb{C}) = \{ \underline{X} \in \mathrm{Mat}_n(\mathbb{C}) / \mathrm{tr}(\underline{X}) = 0 \}$$

ii) $U(n)$ Lie's algebra.

$$\mathrm{Lie}(U(n)) = \mathfrak{u}(n) = \{ \underline{X} \in \mathrm{Mat}_n(\mathbb{C}) / \underline{X}^* = -\underline{X} \}$$

where $\underline{X}^* = \overline{\underline{X}}^t$.

Proposition: (Docarmo, Diferential Geometry)

Let G be a closed subgroup of $GL_n(\mathbb{C})$. So $\mathrm{Lie}(G)$ is a real vectorial sub space of $\mathrm{Mat}_n(\mathbb{C})$

If $\underline{X}, \underline{Y} \in \mathrm{Lie}(G)$, then $[\underline{X}, \underline{Y}] \overset{\text{def}}{=} \underline{X}\underline{Y} - \underline{Y}\underline{X}$

It is called a Lie's parenthesis of $\underline{X}$ and $\underline{Y}$.

Let $\Pi: G \longrightarrow GL(V)$ be a representation of finite dimension. Then for each $\underline{X} \in Lie(G)$ the map $t \longrightarrow \Pi(\exp(t \cdot \underline{X}))$, $t \in \mathbb{R}$. is infinitely differential.

Let

$$\Pi(X) = \frac{d}{dt}\Big|_{t=0} \Pi(\exp(t \cdot \underline{X})) \in End(V)$$

End:= endomorphism

The map $\underline{X} \longrightarrow \Pi(\underline{X})$ satisfy

$$\Pi([\underline{X}, \underline{Y}]) = [\Pi(\underline{X}), \Pi(\underline{Y})]$$

A closed subgroup G of $GL_n(\mathbb{C})$ is called path-connected, if all point $x, y \in G$, may be join by a continuous curve i.e.,

$$\gamma : [0,1] \longrightarrow G \quad , \gamma(0) = X, \quad \gamma(1) = Y$$

Example: $\mathbb{R}^{\times} = GL_1(\mathbb{R})$ is not path-connected.

A representation $\Pi : Lie(G) \longrightarrow End(V)$ is called $*$-representation if $\underline{X} \in Lie(G)$, then

$$\Pi(\underline{X})^{*} = \Pi(-\underline{X})$$

Multiplicidad geométrica y algebraica de un valor propio.

Definición: (multiplicidad geométrica)

Sea λ un valor propio de A, entonces la multiplicidad geométrica de λ, $(MG(\lambda))$ es la dimensión del espacio propio asociado a λ.

$$MG(\lambda) = \dim E_\lambda = Nul(A - \lambda I)$$

$$Av = \lambda v$$

$$(A - \lambda I)v = 0$$

$$MG(\lambda) = \dim E_\lambda = \dim \ker(A - \lambda I) = Nul(A - \lambda I)$$

Definición (Multiplicidad algebraica)

Sea $A_{n\times n}$, λ un valor propio de A, llamamos multiplicad algebraica de λ ($MA(\lambda)$) a la multiplicidad que λ tiene como raíz de $P(\lambda)$.

Si $P(\lambda) = (\lambda - a_1)^{k_1}(\lambda - a_2)^{k_2}(\lambda - a_3)^{k_3}\cdots(\lambda - a_n)^{k_n}$

Para $a_1, a_2, a_3, \dots, a_n$ raíces del polinomio. y $k_1, k_2, k_3, \dots, k_n$ las multiplicidades que tiene a_i, $i \in 1, \dots, n$.

$$MA(a_1) = k_1, \quad MA(a_2) = k_2$$

ejemplos:

1. (Multiplicidad algebraica

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P(\lambda) = |A - I\lambda|$$

$$|A - \lambda I| = \begin{pmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = (3-\lambda)(3-\lambda)(-\lambda)^2 = 0$$

$$= (9 - 6\lambda + \lambda^2)\lambda^2 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)(\lambda - 3) = 0$$

$$\left.\begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 0 \end{matrix}\right\} \text{Multiplicidad } 2.$$

$$MA(3) = 2$$

$$MA(0) = 2.$$

2.(Multiplicidad geométrica)

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

v.p $= 2$ $\quad MA(2) = 3$

$$MG(2) = 3 - rg \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 3 - 2 = 1$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$(3-\lambda)^3 + 2 - (3-\lambda) - 6 = 0$

$\lambda_1 = 5 \qquad MA(A) = 1$

$\lambda_2 = 2 \qquad MA(2) = 2$

$$MG(2) = 3 - rg \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 3 - 1 = 2$$

$$MG(5) = 3 - rg \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = 3 - 2 = 1$$

Forma canónica de Jordan

$$A = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 1 & -1 \\ -1 & 1 & -2 \end{bmatrix}$$

$$\begin{aligned}
\text{Det } |A - \lambda I| &= (-2-\lambda)^2 (1-\lambda) + 1 + (-2-\lambda) + 2(-2-\lambda) \\
&= (-2-\lambda)^2 (1-\lambda) + (-2-\lambda - 4 - \lambda) + 1 \\
&= (4 + 4\lambda + \lambda^2)(1-\lambda) - 6 - 3\lambda \\
&= 4 + \cancel{4\lambda} + \lambda^2 - \cancel{4\lambda} - 4\lambda^2 - \lambda^3 - 6 - 3\lambda + 1 \\
&= -\lambda^3 - 3\lambda^2 - 3\lambda - 1.
\end{aligned}$$