The Poincare group

Transformations that preserves

$$ds^2 = \eta_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

Non-homogeneous group $\quad x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

10-parameters

Lorentz group:

(non-compact)

$$L = O(1,3) = \{ \Lambda \in GL(4,\mathbb{R}) / \Lambda^T \eta \Lambda = \eta \}$$

$$L\,\text{alg}: \mathfrak{o}(1,3) : \{ a \in M_{4\times 4}(\mathbb{R}) / a^T = -\eta a \eta$$

Proper: $L_+ = SO(1,3) = \{ \Lambda \in O(1,3) / \det \Lambda = +1 \}$

Improper: $L_- = \{ \Lambda \in O(1,3) / \det \Lambda = -1 \} \rightarrow$ Is not a subgroup

$$T, P \in L.$$

Orthochronous transformations

Are defined by

$$L^\uparrow = \{ \Lambda \in O(1,3) / \Lambda^0{}_0 \geq +1$$

non-orthochronous:

$$L^\downarrow = \{ \Lambda / \Lambda^0{}_0 \leq -1 \}$$

then the orthochronous Lorentz group or restricted Lorentz group is.

$$L_+ \cap L^\uparrow$$

The connected part of the poincaré group is

$$J^{\mu\nu},\ P^\alpha \longrightarrow \text{hermitian.}$$

Poincaré algebra:

- $i[J^{\mu\nu}, J^{\alpha\beta}] = \eta^{\nu\alpha} J^{\mu\beta} - \eta^{\mu\alpha} J^{\nu\beta} - \eta^{\beta\mu} J^{\alpha\nu} + \eta^{\beta\nu} J^{\alpha\mu}$
- $i[P^\mu, J^{\alpha\beta}] = \eta^{\mu\alpha} P^\beta - \eta^{\mu\beta} P^\alpha$
- $[P^\mu, P^\nu] = 0$

$\quad \hookrightarrow$ Generates a subgroup.

Angular momentum.

$$\vec{J} \equiv (J^{23}, J^{31}, J^{12}) \rightarrow SO(3)$$

$$J^{i} = \frac{1}{2}\epsilon^{ijk} J^{jk}$$

Boost's:

$$\vec{K} = (J^{01}, J^{02}, J^{03}) \rightarrow K^{i} = J^{0i}$$

$$\Lambda = \exp(-i\vec{\theta}\vec{J} + i\vec{n}\vec{K})$$

algebra:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad [J_i, P_j] = i\epsilon_{ijk} P_k$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \qquad [K_i, P_i] = -iH\delta_{ij}$$

$$[K_i, K_j] = -i\epsilon_{jkj} J_k \qquad [K_i, H] = -iP_j$$

$$[J_i, H] = [P_i, H] = [H, H] = 0 \qquad SO(3) \overset{\text{iso}}{\equiv} \frac{SU(2)}{Z_2}$$

Representation in 4-dim (3+1): x^{μ} ; $(J^{\mu\nu})^{\rho}{}_{\sigma} = i(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\mu}_{\sigma})$

Tensorial representations:

$$4 \otimes 4 \rightarrow T^{\mu\nu} \text{ : reducible.}$$

$$4 \otimes 4 = 6 \oplus 9 \oplus 1$$

Anti-symmetric

$$A^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$$

Symmetric s/Trace.

$$S^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) - T\eta^{\mu\nu}$$

Trace

$$T = \eta_{\mu\nu} T^{\mu\nu}$$

Casimir:

Poincare: $O(1,3) \otimes$ Trans

is not semisimple

by Wigner-Inonu contraction

$p^{\mu} p_{\mu} \rightarrow m = 2$

Pauli-Lubanski

$$W^{\mu} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_{\sigma}$$

$SO(1,4)$: Semisimple : $\ell = 2$

If $p^2 > 0$: static $p^{\mu} = (m, 0)$

then,

$$W^{\mu} = -\left(\frac{m}{2}\right) \epsilon^{\mu\nu\rho 0} J_{\nu\rho} = \frac{m}{2} \epsilon^{0\mu\nu\rho} J_{\nu\rho}$$

$$W^{0} = 0 \quad ; \quad W^{i} = \frac{m}{2} \epsilon^{ijk} J^{k} = m J^{i}$$

$$\therefore -W^{\mu} W_{\mu} = m^2 s(s+1) \rightarrow \text{spin}$$

Spin: Quantum-Relativistic intrinsic property.

$\rightarrow$ Irreducible representation : (m, s)

If $p^2 = 0$: the states has just one grade of freedom.

$W^2 = 0$

$\downarrow$

$W_{\mu} = h P_{\mu}$

helicity $\rightarrow h = \hat{p} \cdot \vec{J}$

pseudo scalar

Spinors:

$$[J_i, J_k] = i \epsilon_{ijk} J_k$$
$$[J_i, K_j] = i \epsilon_{ijk} K_k$$
$$[K_i, K_j] = -i \epsilon_{ijk} J_k$$

we define:

$$\vec{J}^{\pm} = \frac{\vec{J} \pm i \vec{K}}{2}$$

then,

$$[J_i^+, J_j^+] = i \epsilon_{ijk} J_k^+$$
$$[J_i^-, J_j^-] = i \epsilon_{ijk} J_k^-$$
$$[J_i^+, J_i^-] = 0$$

$$SO(1,3)^{\mathbb{C}} = SU(2) \otimes SU(2)$$

rep: (j_-, j_+)

then,

$\dim: (2j_- + 1)(2j_+ + 1)$

$\vec{J} = \vec{J}^+ + \vec{J}^-$; spin: $|j_+ - j_-| \leq s \leq j_+ + j_-$: Integer steps.

irreducible representations.

Irreducible representations

$(0,0)$: $\dim = 1$ $\quad \vec{J}^\pm = 0 \longrightarrow \vec{J}, \vec{K} = 0$

	$(1/2, 0)$	$(0, 1/2)$	: $\dim = 2$; spin $= 1/2$
	ψ_L	ψ_R	Fermions (bi-comp)
Weyl Representation	left-handed	Right-handed	
$\vec{J}^+$	0	$\vec{\sigma}/2$	$\vec{J} = \vec{\sigma}/2$
$\vec{J}^-$	$\vec{\sigma}/2$	0	$\vec{K} = -i(\vec{J}^+ - \vec{J}^-)$
			$= \pm i\vec{\sigma}/2 \neq \vec{K}^+$

$P = \vec{J} \longmapsto \vec{J}$

$L \longleftrightarrow R$

then,

$$\psi_{L,R} \xrightarrow{\Lambda_{L,R}} \exp\{(-i\vec{\theta} \pm \vec{\eta}) \cdot \vec{\sigma}/2\} \psi_{L,R}$$

from the transformation properties:

$$\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R$$

$$(\sigma^2 \sigma^i \sigma^2 = -\sigma^{i*}$$

Therefore

$$\sigma^2 \psi_L^* \xrightarrow{\Lambda} \sigma^2 (\Lambda_L \psi_L)^* = (\sigma^2 \Lambda_L^* \sigma^2)\sigma^2 \psi_L^*$$

$$= \Lambda_R (\sigma^2 \psi_L^*)$$

then,

$$\sigma^2 \psi_L^* \in \overline{(0, 1/2)}$$

Definition: Charge conjugacy: $\psi_L^c \equiv i\sigma^2 \psi_L^*$

similarly : $\psi_R^c \equiv -i\sigma^2 \psi_R^* \in \overline{(1/2, 0)}$

Dirac representation: $(1/2, 0) \oplus (0, 1/2)$

$\psi = \psi_L + \psi_R$: $\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow$ 4-components.

Rep$(1/2, 1/2)$: dim 4 ; $1 \geq j \geq |1/2 - 1/2| = 0 \longrightarrow j = 0, 1.$

$$(1/2, 0) \oplus (0, 1/2) \longrightarrow (\psi_{L\alpha}, \xi_{R\beta}) \quad ; \quad \alpha, \beta = 1, 2.$$

Definition: Covariant vectors:

$$\psi_R \equiv i\sigma^2 \psi_L^*$$

$$\xi_L \equiv -i\sigma^2 \xi_R^*$$

$$\boxed{\begin{array}{l} \xi_R^\dagger \sigma^\mu \psi_R \\ \xi_L^\dagger \bar{\sigma}^\mu \psi_L \end{array}} \quad \partial A_\mu$$

Complex Spin 1.

Transformation: $\Lambda^\mu{}_\nu$ – real.

this

$$\text{rep} \equiv \text{rep}(x^\mu)$$

$$\left.\begin{array}{l} (1/2, 0) \otimes (1/2, 0) = (0, 0) \oplus (1, 0) \\ (0, 1/2) \otimes (0, 1/2) = (0, 0) \oplus (0, 1) \end{array}\right\} \begin{array}{l} \text{Tensorial representation} \\ T^{\mu\nu} \text{ selfdual and} \\ \text{anti-selfdual. } \mathbb{C}. \end{array}$$