Example:

IV. In $\mathcal{H}$, let's fix $y \in \mathcal{H}$ and let's define $f(x) = \langle x, y \rangle$, this functional we can identify with

$$\|f(x)\| \leq \|x\| \, \|y\|$$

we will denote by $E^{\#}$ the functional's space on E.

Theorem: Let $f \in E^{\#} / \{0\}$, then:

1. $\dim E / \operatorname{Ker} f = 1$.

2. If $f, g \in E^{\#} / \{0\}$ and $\operatorname{Ker} f = \operatorname{Ker} g$, there is $\lambda \neq 0$ such that $\lambda f = g$.

Proof:

1. Let $x' \in E$, $f(x') \neq 0$. Let's take $x_0 = \frac{x'}{f(x')}$, we note that $f(x_0) = 1$. For all $x \in E$.

$$y = x - f(x) x_0 \in \operatorname{Ker} f$$

then $x = f(x) x_0 + y$, $y \in \operatorname{Ker} f$, this decomposition is unique i.e.,

$$x = \lambda x_0 + z, \quad z \in \operatorname{Ker} f,$$

then $\lambda = f(x)$, $z = y$. Thus

$$E / \operatorname{Ker} f = \{\lambda x_0 + \operatorname{Ker} f : \lambda \in \mathbb{R} \ (\text{or } \mathbb{C})\}$$

and finally $\dim E / \operatorname{Ker} f = 1$.

2. Let $x = f(x) x_0 + y$, let's apply g, then

$$g(x) = f(x) g(x_0) \longrightarrow g = g(x_0) f.$$

■

Bounded linear functionals

Let $X = (E, \|\cdot\|)$ be a normed space. We will call to $f \in X^{\#}$ a bounded linear functional if there exists $c > 0$, such that

$$|f(x)| \leq C \|x\|$$

i.e., f is bounded in a bounded set.

We say that f is a continuous functional, if $x^n \longrightarrow x$, implies that $f(x_n) \longrightarrow f(x)$.

Let X^* the set of the bounded functionals in X. Is a vectorial space with respect to the sum of functionals. Let's define a norm in X^*, let $f \in X^*$

$$\|f\|^* = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|$$

In particular is fulfilled

$$|f(x)| \leq \|f\|^* \|x\| = \|f\| \|x\|$$

Proposition: f is a bounded functional if and only if f is a continuous function.

Proof: ($\rightarrow$) If f is bounded

$$|f(x_n) - f(x_0)| = f|(x_n - x_0)| \leq \|f\| \|x_n - x_0\|$$

If $x_n \rightarrow x_0$, then $|f(x_n) - f(x_0)| \longrightarrow 0$.

($\leftarrow$) Si $x_n \longrightarrow 0$, luego $f(x_n) \longrightarrow 0$. If f is not bounded, for each $n \in \mathbb{N}$, there is x_n with $\|x_n\| = 1$ and $|f(x_n)| > n$. But in this case

$$\left| f\left(\frac{x_n}{n}\right) \right| > 1, \quad \text{where} \quad \frac{x_n}{n} \longrightarrow 0.$$

This is a contradiction. ■

In fact if a linear functional f is continuous in $x=0$, then is continuous in all X.

We see also that if f is continuous, $\operatorname{Ker} f$ is closed, the contrary claim, if $f \in X^\#$ and $\operatorname{Ker} f$ is closed, then is continuous.

Because of the definition of norm

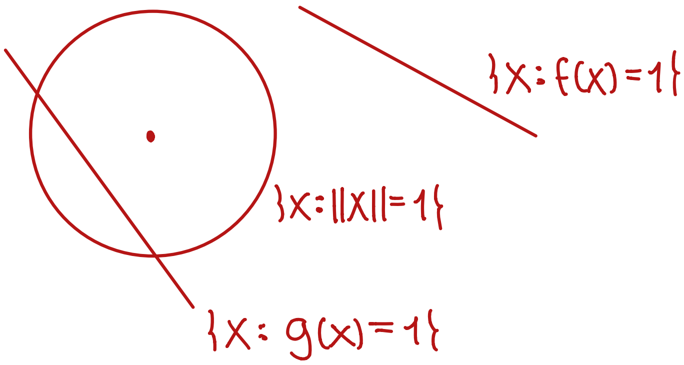
$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup\{|f(x)| : \|x\| = 1\}$$

$$= \sup\left\{ \frac{1}{\|x\|} : f(x) = 1 \right\}$$

$$= \frac{1}{\inf\{\|x\| : f(x) = 1\}}$$

The quantity $\rho_f = \inf\{\|x\| : f(x) = 1\}$ is the distance from the hiperplane $\{x : f(x) = 1\}$ to zero. Thus the norm $\|f\|$ is $1/\rho_f$.

This means that the functional f has norm less than 1, but the functional g, greater than 1.



Bounded linear functions in a Hilbert space.

Theorem (Riesz's representation): for each $\varphi \in \mathcal{H}^*$, there is an unique $y \in \mathcal{H}$, such that $\varphi(x)=\langle x,y\rangle$, $x \in \mathcal{H}$ i.e., any continuous linear functional in a Hilbert space is represented for some element and of the same space and satisfy $\varphi(x)=\langle x,y\rangle$, also this correspondence is a isometry

$$\|\varphi\|^* = \|y\|$$

Proof: For each $y \in \mathcal{H}$, we define the functional $\varphi_y(x)=\langle x,y\rangle$. Is clear that φ_y is a linear functional, of the Cauchy-Schwarz

$$|\varphi_y(x)| \leq \|x\|\,\|y\|$$

which implies that

$$\sup_{x\neq 0} \frac{|\varphi_y(x)|}{\|x\|} = \|\varphi_y\| \leq \|y\|$$

taking $x=y$, we obtain

$$|\varphi_y(y)| = \|y\|^2 \leq \|\varphi\|\,\|y\|$$

and thus

$$\|\varphi_y\| \leq \|y\|.$$

Now, let's proof that any functional $\varphi \in \mathcal{H}^*$, has the form

$$\varphi(x)=\langle x,y\rangle.$$

In the case that $\varphi=0$, is the trivial case.

Let $L=\operatorname{Ker}\varphi$, $\varphi \neq 0$. As φ is continuous, L is closed and $\dim H/L = 1$, let's take $L^\perp$ ($\dim L^\perp = 1$).

Then, $L^{\perp} = \operatorname{span}\{\hat{y}\}$, for $\hat{y} \in \mathcal{H} \setminus \{0\}$.

The vector $\hat{y}$ defines a linear functional given by

$$\varphi_{\hat{y}}(x) = \langle x, \hat{y} \rangle.$$

then

$$\operatorname{Ker} \varphi_y = \{\hat{y}\}^{\perp} = (L^{\perp})^{\perp} = L$$

thus $\operatorname{Ker} \hat{y} = \operatorname{Ker} y$.

By the second part of the proposition $\varphi = \lambda \hat{y}$, we have

$$\varphi(x) = \lambda \varphi_y(x) = \lambda \langle x, y \rangle = \langle x, y \rangle$$

moreover,

$$\| \varphi \| = \| y \|$$

$\mathfrak{X}^*$, is the dual space to $\mathfrak{X}$, and has a dual norm

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}$$

■

Proposition: For any normed space $\mathfrak{X}$, the dual space $\mathfrak{X}^*$, always is complete i.e., is of Banach.

Theorem (Hahn-Banach): Let E be a subspace of $\mathfrak{X}$, and let let $f_0 \in E^*$. Then there exists an extension $f \in \mathfrak{X}^*$ such that $f|_E = f_0$ (i.e., $f(x) = f_0(x)$, for $x \in E$) and $\|f\|_{\mathfrak{X}^*} = \|f_0\|_{E^*}$ i.e.,

$$\sup_{\substack{x \neq 0 \\ x \in \mathfrak{X}}} \frac{|f(x)|}{\|x\|} = \sup_{x \in E/\{0\}} \frac{|f(x)|}{\|x\|}$$