$$f * g = fg + \sum_{n=1}^{\infty} \hbar^n P_n(f,g)$$

where

$$P(f,g) = f(p,q)\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^j}\frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j}\frac{\overrightarrow{\partial}}{\partial q^j}\right)\right]^n g(p,q)$$

$$\{f,g\}_m := f*g - g*f$$

$$\{f,g\}_m = \{f,g\} + \mathcal{O}(\hbar^3)$$

Schrödinger-like equation

In standard quantum mechanics: $\hat{H}|\psi\rangle = \hat{E}|\psi\rangle$

Multiply from the right by $\langle\psi| \in \mathcal{H}^*$

$$\hat{H}\hat{\rho} = E\hat{\rho}$$

Define $\quad H(p,q) := (Q_\hbar^\omega)^{-1}[\hat{H}] \quad$ classic Hamiltonian

and $\quad W(p,q) := (Q_\hbar^\omega)^{-1}[\hat{\rho}] \quad$ wigner function.

$$\Rightarrow (Q_\hbar^\omega[H]\, Q_\hbar^\omega[W])(p,q) = E(Q_\hbar^\omega[EW(p,q)]$$

$$H(p,q) * W(p,q) = EW(p,q) \qquad *\text{-genvalue equation.}$$

therefore, $W(p,q)$ characterizes the quantum dynamics of the system through the $*$-product.

Also, Liouville's theorem $\partial_t f = \{f,H\}$ gives the dynamical evolution of an arbitrary observable.

This may be deformed into

$$\partial_t f = \{f,H\}_m \qquad \text{Moyal's equation.}$$

Bopp shifts:

We know

$$e^{\alpha\frac{\partial}{\partial\alpha}} f(x) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\alpha^n\left(\frac{\partial}{\partial x}\right)^n\right) f(x)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\alpha^n\left(\frac{\partial f(x)}{\partial x}\right)^n = f(x+\alpha)$$

Taylor arround $x=\alpha$.

$$f(p,q) * g(p,q) = f(p,q) \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] g(q,p)$$

$$= f(p,q)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n\left(\frac{\overleftarrow{\partial}}{\partial q}\right)^n\left(\frac{\overrightarrow{\partial}}{\partial p}\right)^n\right]\left[\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{i\hbar}{2}\right)^m(-1)^m\left(\frac{\overleftarrow{\partial}}{\partial p}\right)^m\left(\frac{\overrightarrow{\partial}}{\partial q}\right)^m\right] g(p,q)$$

$$= f\left(p - \frac{i\hbar}{2}\frac{\partial}{\partial q},\ q + \frac{i\hbar}{2}\frac{\partial}{\partial p}\right) g(p,q)$$

*-exponentials and time evolution

In ordinary quantum mechanics we have the Heisenberg representation. Let $\Psi(t)$ follows the schrödinger equation

$$-i\hbar\frac{d}{dt}\Psi(t) = \hat{U}(t,t_0)\Psi(t_0)$$

where $\hat{U}(t,t_0)$ is an unitary operator

$$-i\hbar\frac{d}{dt}U(t,t_0) = HU(t,t_0)$$

$$\frac{dU(t,t_0)}{U(t,t_0)} = \frac{i}{\hbar}\hat{H}dt$$

$$\ln(U(t,t_0)) = \frac{i}{\hbar}\hat{H}t$$

$$U(t,t_0) = e^{i\hat{H}t/\hbar}.$$

unitary operator for time evolution