$$H=\frac{1}{2}(p^2+q^2)$$

$$\omega(p,q)=\frac{2(-1)^n}{\pi\hbar}e^{-2H/\hbar}L_n\left(\frac{4H}{\hbar}\right)$$

$$=\frac{2(-1)^n}{\pi\hbar}e^{-(p^2+q^2)/\hbar}L_n\left(\frac{2(p^2+q^2)}{\hbar}\right)$$

$$A_c \ni W \longleftrightarrow \psi\in\mathcal{H}.$$

$$\psi_n(x)=A_n e^{-x^2/2\hbar}H_n\left(\frac{x}{\sqrt{\hbar}}\right) \quad ; \quad \text{take } \hbar=1.$$

$$\Rightarrow W(q,p)=\int dz\,\psi_n\left(q+\frac{z}{2}\right)\overline{\psi_n\left(q-\frac{z}{2}\right)}e^{-ipz}$$

$$=\int dz|A_n|^2e^{-(q+z/2)^2/2}H_n\left(q+\frac{z}{2}\right)e^{-(q-z/2)^2/2}H_n\left(q-\frac{z}{2}\right)e^{-ipz}$$

$$=|A_n|^2\int dz\,e^{-q^2-z^2/4}H_n\left(q+\frac{z}{2}\right)H_n\left(q-\frac{z}{2}\right)e^{-ipz}$$

Parity of Hermite $H_n(-x)=(-1)^nH_n(x)$.

$$H_n\left(q-\frac{z}{2}\right)=H_n\left(-\left(\frac{z}{2}-q\right)\right)=(-1)^nH_n\left(\frac{z}{2}-q\right)$$

$$W(q,p)=|A_n|^2(-1)^ne^{-q^2}\int dz\,e^{-z^2/4}H_n\left(\frac{z}{2}+q\right)H_n\left(\frac{z}{2}-q\right)e^{-ipz}$$

changing $\frac{z}{2}=:y$; $dz=2dy$.

$$W(q,p)=2|A_n|^2(-1)^ne^{-q^2}\int dy\,e^{-y^2}H_n(y+q)H_n(y-q)e^{-2ipy}$$

Identify:

$$L_n(2(a^2+b^2))=e^{b^2}\int dx\,e^{-x^2}H_n(x-a)H_n(x+a)e^{-2ibx}$$

$$W(q,p)=2|A_n|^2(-1)^ne^{-(p^2+q^2)}L_n(2(a^2+b^2))$$

② Bopp shifts:

$$f(p,q)*g(p,q)=f\left(p-\frac{i\hbar}{2}\partial q,\ q+\frac{i\hbar}{2}\partial p\right)g(p,q)$$

$$H*W = H\left(p - \frac{i\hbar}{2}\partial q,\ q + \frac{i\hbar}{2}\partial p\right) W(p,q)$$

$$= \frac{1}{2}\left[\left(p - \frac{i\hbar}{2}\partial q\right)^2 + \left(q + \frac{i\hbar}{2}\partial p\right)^2\right] W(p,q)$$

$$= \frac{1}{2}\left[\left(p^2 - i\hbar p\partial q - \frac{\hbar^2}{4}\partial q^2\right) + \left(q^2 + i\hbar q\partial p - \frac{\hbar^2}{4}\partial p^2\right)\right] W(p,q)$$

$$= \frac{1}{2}(p^2+q^2)W + \frac{i\hbar}{2}(q\partial p - p\partial q)W - \frac{\hbar^2}{8}(\partial^2 p + \partial^2 q)W$$

We want $H*W = EW$

$$\text{Re}: \left[\frac{1}{2}(p^2+q^2) - \frac{\hbar^2}{8}(\partial^2 p + \partial^2 q)\right] W = EW$$

$$\text{Im}: (q\partial p - p\partial q)W = 0.$$

③ Star-exponential:

Theorem: The energy spectrum may be obtained by the support of the fourier transform of

$$\text{Exp}_*\left(\frac{-itH}{\hbar}\right)$$

Proof: First, note the identity resolution.

$$1 = \int |q\rangle\langle q|\, dq$$

corresponds to

$$\sum_n W_n (2\pi\hbar) = 1.$$

Homework: Write the theorem in terms of wigner.

Thus,

$$\text{Exp}_*\left(\frac{-itH}{\hbar}\right) = \text{Exp}_*\left(\frac{-itH}{\hbar}\right) * 1 = \text{Exp}_*\left(\frac{-itH}{\hbar}\right) * \sum_n W_n$$

$$= \sum_n e^{-itE_n/\hbar} W_n$$ *-genvalue equation.

$$F(E_m) = \int_0^\infty \text{Exp}_*\left(\frac{-itH}{\hbar}\right) * 1\, e^{itE_m/\hbar}\, dt$$

$$= \int_0^\infty \sum_n e^{-itE_n/\hbar} e^{itE_m/\hbar} W_n dt$$

$$= \int_0^\infty \sum_n e^{-it/\hbar (E_m - E_n)} W_n dt$$

$$= \sum_n \left[\int_0^\infty e^{-it/\hbar (E_m - E_n)} dt \right] W_n$$

$$= \sum_n \delta(E_m - E_n) W_n \propto W_n$$

$$\Rightarrow Exp_*\left(\frac{-itH}{\hbar}\right) = \int W_m e^{-itE_m/\hbar} dE_m$$

For the harmonic oscillator,

$$H = \frac{1}{2}(p^2 + q^2)$$

$$\phi(t, H) := Exp_*\left(\frac{-itH}{\hbar}\right) = \sum_{n=0}^\infty \left(\frac{-it}{\hbar}\right) \frac{1}{n!} (H*)^n$$

first try

$$H * f(H) = Hf(H) + \frac{i\hbar}{2} H \left(q\frac{\partial f}{\partial p} - p\frac{\partial f}{\partial q} \right) + \left(\frac{i\hbar}{2}\right)^2 \frac{1}{2!} H (\overleftarrow{\partial}_q^2 \overrightarrow{\partial}_p^2 - 2\overleftarrow{\partial}_{qp} \overrightarrow{\partial}_{pq} + \overleftarrow{\partial}_p^2 \overrightarrow{\partial}_q^2) f(H)$$

$$\cancel{+ \dots} \to 0$$

$$\frac{\partial f}{\partial p} = \frac{\partial H}{\partial p} \frac{\partial f}{\partial H} = p \frac{df}{dH} \Rightarrow q\frac{\partial f}{\partial p} - p\frac{\partial f}{\partial q} = q\left(p\frac{df}{dH}\right) - p\left(q\frac{df}{dH}\right) = 0$$

$$\frac{\partial f}{\partial q} = q \frac{df}{dH}$$

$$\partial_p^2 f = \partial_p(\partial_p f) = \partial_p\left(p\frac{df}{dH}\right) = p^2 \frac{d^2 f}{dH^2} + \frac{df}{dH}$$

$$\partial_q^2 f = q^2 \frac{d^2 f}{dH^2} + \frac{df}{dH}$$

$$H * H(f) = Hf - \frac{\hbar^2}{8}\left[(p^2+q^2)\frac{d^2f}{dH^2} + 2\frac{df}{dH}\right] = Hf - \frac{\hbar^2}{4}\frac{df}{dH} - \frac{\hbar^2}{4}H\frac{d^2f}{dH^2}$$

we may define f(H) as

$$(H*)^n =: f_n(H)$$

$$\Rightarrow H * f_n(H) = f_{n+1}(H) = Hf_n(H) - \frac{\hbar^2}{4}f_n'(H) - \frac{\hbar^2}{4}Hf_n''(H)$$

Recurrence relation, also we have.

$$\phi(t,H) = \sum_{n=0}^{\infty}\left(-\frac{it}{\hbar}\right)^n \frac{1}{n!} f_n(H)$$

$\Rightarrow \phi(t,H)$ also follows the recurrence relation.

$$\Rightarrow i\hbar\frac{d\phi}{dt} = H*\phi$$

$$i\hbar\frac{d\phi}{dt} = H\phi - \frac{\hbar^2}{4}\frac{\partial\phi}{\partial H} - \frac{\hbar^2}{4}H\frac{\partial^2\phi}{\partial H^2} \qquad (\#)$$

Try

$$\phi(t,H) = g_1(t)\exp\left[\frac{2H}{i\hbar}g_2(t)\right]$$

- $$\frac{\partial\phi}{\partial t} = g_1'\exp\left[\frac{2H}{i\hbar}g_2(t)\right] + g_1\frac{2H}{i\hbar}g_2'\exp\left[\frac{2H}{i\hbar}g_2(t)\right]$$

$$= \left(g_1' + \frac{2H}{i\hbar}g_1g_2'\right)\exp\left[\frac{2H}{i\hbar}g_2(t)\right] = \left(\frac{g_1'}{g_1} + \frac{2H}{i\hbar}g_2'\right)\phi$$

$$\Rightarrow i\hbar\frac{\partial\phi}{\partial t}\left(i\hbar\frac{g_1'}{g_1} + 2Hg_2'\right)\phi$$

- $$\frac{\partial\phi}{\partial H} = \frac{2g_1g_2}{i\hbar}\exp\left[\frac{2H}{i\hbar}g_2(t)\right] = \frac{2g_2}{i\hbar}\phi$$

$$\Rightarrow -\frac{\hbar^2}{4}\frac{\partial\phi}{\partial H} = \frac{i\hbar}{2}g_2\phi$$

- $$\frac{\partial^2\phi}{\partial H^2} = \frac{4g_2^2}{\hbar^2}\phi$$

$$\Rightarrow -\frac{\hbar^2}{4}\frac{\partial^2\phi}{\partial H^2} = g_2^2\phi$$

from (#), we have

$$i\hbar \frac{g_1'}{g_1} + 2Hg_2' = H + \frac{i\hbar}{2} g_2 + 2g_2^2 H$$

$$H(2g_2' - 1 - 2g_2^2) + i\hbar\left(\frac{g_1'}{g_1} - \frac{1}{2} g_2\right) = 0$$

$$2g_2' - 1 - 2g_2^2 = 0 \longrightarrow g_2' = \frac{1+g_2^2}{2}$$

$$\frac{g_1'}{g_1} - \frac{1}{2} g_2 = 0$$

$$\frac{dg_2}{1+g_2^2} = \frac{1}{2} dt$$

$$\arctan(g_2) = \frac{t}{2}$$

$$g_2 = \tan\left(\frac{t}{2}\right)$$

$$\frac{g_1'}{g_1} = \frac{1}{2} \tan\left(\frac{t}{2}\right)$$

$$\frac{dg_1}{g_1} = \frac{1}{2} \tan\left(\frac{t}{2}\right) dt$$

$$\ln(g_1) = \ln\left(\sec\left(\frac{t}{2}\right)\right)$$

$$g_1 = \sec\left(\frac{t}{2}\right)$$

$$\mathrm{Exp}_*\left(\frac{-itH}{\hbar}\right) = \sec\left(\frac{t}{2}\right) \exp\left(\frac{2H}{i\hbar} \tan\left(\frac{t}{2}\right)\right)$$

In order to obtain the Wigner function expand $\phi(t, H)$

$$\mathrm{Exp}_*\left(\frac{-itH}{\hbar}\right) = \int W_m e^{itE_m/\hbar} dE_m$$

$$\Longleftrightarrow \qquad W_m = \int \overline{\mathrm{Exp}_*\left(\frac{-itH}{\hbar}\right)} e^{-itE_m/\hbar} dt$$

Define $\quad \tau := \frac{t}{2}, \quad R := \frac{p^2+q^2}{\hbar}, \quad \alpha := \frac{2E_m}{\hbar}$

$$W_m = 2\int \frac{e^{iR\tan(\tau)}}{\cos(\tau)} e^{-i\alpha\tau} d\tau$$

Define $\quad z := e^{i\tau}$

$$\cos(\tau) = \frac{e^{i\tau} + e^{-i\tau}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$\sin(\tau) = \frac{e^{i\tau} - e^{-i\tau}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\tan(\tau) = \frac{z^2-1}{z^2+1}\,\frac{1}{i}$$

$$e^{-i\alpha\tau} = (e^{i\tau})^{-\alpha} = z^{-\alpha}, \quad dz = ie^{i\tau}d\tau, \quad d\tau = \frac{dz}{zi}$$

$$W_m = 2\int \frac{e^{\frac{iR}{i}\left(\frac{z^2-1}{z^2+1}\right)} z^{-\alpha}}{\frac{z^2+1}{2z}} \frac{dz}{iz} = \frac{4}{i}\int \frac{e^{R\left(\frac{z^2-1}{z^2+1}\right)} z^{-\alpha}}{z^2+1}\, dz$$

$$\frac{z^2-1}{z^2+1} = 1 + \frac{2z^2}{z^2+1}$$

$$e^{R\left(\frac{z^2-1}{z^2+1}\right)} = e^{R\left(1+\frac{2z^2}{z^2+1}\right)} = e^{-R}\, e^{2R\left(\frac{z^2}{z^2+1}\right)}$$

Define

$$\frac{-\lambda}{1-\lambda} := \frac{z^2}{z^2+1}$$

$$-\lambda = \frac{z^2}{z^2+1}(1-\lambda)$$

$$\left(\frac{z^2}{z^2+1} - 1\right)\lambda = \frac{z^2}{z^2+1}$$

$$\left(\frac{\cancel{z^2} - \cancel{z^2} - 1}{z^2+1}\right)\lambda = \frac{z^2}{z^2+1}$$

$$\frac{-\lambda}{z^2+1} = \frac{z^2}{z^2+1}$$

$$-\lambda = z^2$$

$$z = (-\lambda)^{1/2}$$

$$z^{-\alpha} = (-\lambda)^{-\alpha/2}$$

$$dz = \frac{-1}{2(-\lambda)^{1/2}} d\lambda$$

$$W_m = -\frac{2}{i} e^{-R} \int \frac{e^{2R\lambda/(\lambda-1)}}{} \frac{1}{(-\lambda)^{(\alpha+1)/2}} d\lambda$$

Using Laguerre integral representation

$$\frac{1}{2\pi i} \int \frac{e^{R\left(\frac{\lambda}{1-\lambda}\right)}}{1-\lambda} \frac{d\lambda}{\lambda^{l+1}} = L_l(2R)$$

where $l = \frac{\alpha+1}{2}$

$$= -\frac{2}{i} e^{-R} (2\pi i) L_{\frac{\alpha+1}{2}}(2R)$$

Therefore

$$\propto \; e^{-R} L_l(2R)$$