

$$H = \frac{1}{2} (p^2 + q^2)$$

$$\begin{aligned} W(p, q) &= \frac{2(-1)^n}{\pi \hbar} e^{-2H/\hbar} L_n\left(\frac{4H}{\hbar}\right) \\ &= \frac{2(-1)^n}{\pi \hbar} e^{-(p^2+q^2)/\hbar} L_n\left(\frac{2(p^2+q^2)}{\hbar}\right) \end{aligned}$$

$$A_c \ni w \longleftrightarrow \psi \in \mathcal{H}.$$

$$\Psi_n(x) = A_n e^{-x^2/2\hbar} H_n\left(\frac{x}{\sqrt{\hbar}}\right) ; \text{ take } \hbar=1.$$

$$\begin{aligned} \Rightarrow W(q, p) &= \int dz \Psi_n\left(q + \frac{z}{2}\right) \overline{\Psi_n\left(q - \frac{z}{2}\right)} e^{-ipz} \\ &= \int dz |A_n|^2 e^{-(q+z/\hbar)^2/2} H_n\left(q + \frac{z}{2}\right) \overline{e^{-(q-z/\hbar)^2/2} H_n\left(q - \frac{z}{2}\right)} e^{-ipz} \\ &= |A_n|^2 \int dz e^{-q^2 - z^2/4} H_n\left(q + \frac{z}{2}\right) H_n\left(q - \frac{z}{2}\right) e^{-ipz} \end{aligned}$$

Parity of Hermite $H_n(-x) = (-1)^n H_n(x)$.

$$H_n\left(q - \frac{z}{2}\right) = H_n\left(-\left(\frac{z}{2} - q\right)\right) = (-1)^n H_n\left(\frac{z}{2} - q\right)$$

$$W(q, p) = |A_n|^2 (-1)^n e^{-q^2} \int dz e^{-z^2/4} H_n\left(\frac{z}{2} + q\right) H_n\left(\frac{z}{2} - q\right) e^{-ipz}$$

changing $\frac{z}{2} =: q$; $dz = 2dq$.

$$W(q, p) = 2 |A_n|^2 (-1)^n e^{-q^2} \int dy e^{-y^2} H_n(y+q) H_n(y-q) e^{-ipy}$$

Identify:

$$L_n(2(a^2+b^2)) = e^{b^2} \int dx e^{-x^2} H_n(x-a) H_n(x+a) e^{-2ibx}$$

$$W(q, p) = 2 |A_n|^2 (-1)^n e^{-(p^2+q^2)} L_n(2(a^2+b^2))$$

② Bopp shifts:

$$f(p, q) * g(p, q) = f\left(p - \frac{i\hbar}{2} \partial q, q + \frac{i\hbar}{2} \partial p\right) g(p, q)$$

$$\begin{aligned}
 H * W &= H\left(p - \frac{i\hbar}{2}\partial_q, q + \frac{i\hbar}{2}\partial_p\right) W(p, q) \\
 &= \frac{1}{2} \left[\left(p - \frac{i\hbar}{2}\partial_q\right)^2 + \left(q + \frac{i\hbar}{2}\partial_p\right)^2 \right] W(p, q) \\
 &= \frac{1}{2} \left[\left(p^2 - i\hbar p \partial_q - \frac{\hbar^2}{4} \partial_q^2\right) + \left(q^2 + i\hbar q \partial_p - \frac{\hbar^2}{4} \partial_p^2\right) \right] W(p, q) \\
 &= \frac{1}{2} (p^2 + q^2) W + \frac{i\hbar}{2} (q \partial_p - p \partial_q) W - \frac{\hbar^2}{8} (\partial_p^2 + \partial_q^2) W
 \end{aligned}$$

We want $H * W = EW$

$$\text{Re:} \left[\frac{1}{2} (p^2 + q^2) - \frac{\hbar^2}{8} (\partial_p^2 + \partial_q^2) \right] W = EW$$

$$\text{Im:} (q \partial_p - p \partial_q) W = 0.$$

③ Star-exponential:

Theorem: The energy spectrum may be obtained by the support of the Fourier transform of

$$\text{Exp}_* \left(\frac{-itH}{\hbar} \right)$$

Proof: First, note the identity resolution.

$$1 = \int |q\rangle \langle q| dq$$

corresponds to

$$\sum_n W_n (2\pi\hbar) = 1.$$

Homework: Write the theorem in terms of Wigner.

Thus,

$$\begin{aligned}
 \text{Exp}_* \left(\frac{-itH}{\hbar} \right) &= \text{Exp}_* \left(\frac{-itH}{\hbar} \right) * 1 = \text{Exp}_* \left(\frac{-itH}{\hbar} \right) * \sum_n W_n \\
 &= \sum_n e^{-itE_n/\hbar} W_n \quad \text{* - eigenvalue equation.}
 \end{aligned}$$

$$F(E_m) = \int_0^\infty \text{Exp}_* \left(\frac{-itH}{\hbar} \right) * 1 e^{itm/\hbar} dt$$

$$\begin{aligned}
&= \int_0^\infty \sum_n e^{-itE_n/\hbar} e^{itE_m/\hbar} W_n dt \\
&= \int_0^\infty \sum_n e^{-it/\hbar(E_m - E_n)} W_n dt \\
&= \sum_n \left[\int_0^\infty e^{-it/\hbar(E_m - E_n)} dt \right] W_n \\
&= \sum_n \delta(E_m - E_n) W_n \propto W_n
\end{aligned}$$

$$\Rightarrow \text{Exp}_* \left(-\frac{itH}{\hbar} \right) = \int W_m e^{-itE_m/\hbar} dE_m$$

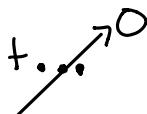
for the harmonic oscillator.

$$H = \frac{1}{2} (p^2 + q^2)$$

$$\phi(t, H) := \text{Exp}_* \left(-\frac{itH}{\hbar} \right) = \sum_{n=0}^{\infty} \left(-\frac{it}{\hbar} \right) \frac{1}{n!} (H*)^n$$

first try

$$H*f(H) = Hf(H) + \frac{it}{2} H \left(q \frac{\partial f}{\partial p} - p \frac{\partial f}{\partial q} \right) + \left(\frac{it}{2} \right)^2 \frac{1}{2!} H \left(\frac{\partial^2 f}{\partial p^2} - 2 \frac{\partial^2 f}{\partial p \partial q} + \frac{\partial^2 f}{\partial q^2} \right) f(H)$$



$$\frac{\partial f}{\partial p} = \frac{\partial H}{\partial p} \frac{\partial f}{\partial H} \quad \frac{\partial f}{\partial q} = p \frac{\partial f}{\partial H} \rightarrow q \frac{\partial f}{\partial p} - p \frac{\partial f}{\partial q} = q \left(p \frac{\partial f}{\partial H} \right) - p \left(q \frac{\partial f}{\partial H} \right) = 0$$

$$\frac{\partial f}{\partial q} = q \frac{\partial f}{\partial H}$$

$$\frac{\partial^2 f}{\partial p^2} = \frac{\partial}{\partial p} \left(p \frac{\partial f}{\partial H} \right) = p^2 \frac{d^2 f}{dH^2} + \frac{df}{dH}$$

$$\frac{\partial^2 f}{\partial q^2} = q^2 \frac{d^2 f}{dH^2} + \frac{df}{dH}$$

$$H * H(f) = Hf - \frac{\hbar^2}{8} \left[(P^2 + Q^2) \frac{d^2 f}{dH^2} + 2 \frac{df}{dH} \right] = Hf - \frac{\hbar^2}{4} \frac{df}{dH} - \frac{\hbar^2}{4} H \frac{d^2 f}{dH^2}$$

We may define $f(H)$ as

$$(H*)^n =: f_n(H)$$

$$\Rightarrow H * f_n(H) = f_{n+1}(H) = H f_n(H) - \frac{\hbar^2}{4} f_n'(H) - \frac{\hbar^2}{4} H f_n''(H)$$

Recurrence relation, also we have.

$$\phi(t, H) = \sum_{n=0}^{\infty} \left(-\frac{it}{\hbar}\right)^n \frac{1}{n!} f_n(H)$$

$\Rightarrow \phi(t, H)$ also follows the recurrence relation.

$$\Rightarrow i\hbar \frac{d\phi}{dt} = H * \phi$$

$$i\hbar \frac{d\phi}{dt} = H\phi - \frac{\hbar^2}{4} \frac{\partial\phi}{\partial H} - \frac{\hbar^2}{4} H \frac{\partial^2\phi}{\partial H^2} \quad (\#)$$

Try

$$\phi(t, H) = g_1(t) \exp\left[\frac{2H}{i\hbar} g_2(t)\right]$$

$$\bullet \frac{\partial\phi}{\partial t} = g'_1 \exp\left[\frac{2H}{i\hbar} g_2(t)\right] + g_1 \frac{2H}{i\hbar} g'_2 \exp\left[\frac{2H}{i\hbar} g_2(t)\right]$$

$$= \left(g'_1 + \frac{2H}{i\hbar} g_1 g'_2\right) \exp\left[\frac{2H}{i\hbar} g_2(t)\right] = \left(\frac{g'_1}{g_1} + \frac{2H}{i\hbar} g'_2\right) \phi$$

$$\Rightarrow i\hbar \frac{\partial\phi}{\partial t} \left(i\hbar \frac{g'_1}{g_1} + 2H g'_2 \right) \phi$$

$$\bullet \frac{\partial\phi}{\partial H} = \frac{2g_1 g_2}{i\hbar} \exp\left[\frac{2H}{i\hbar} g_2(t)\right] = \frac{2g_2}{i\hbar} \phi$$

$$\Rightarrow -\frac{\hbar^2}{4} \frac{\partial\phi}{\partial H} = \frac{i\hbar}{2} g_2 \phi$$

$$\bullet \frac{\partial^2\phi}{\partial H^2} = \frac{4g_2^2}{\hbar^2} \phi$$

$$\Rightarrow -\frac{\hbar^2}{4} \frac{\partial^2\phi}{\partial H^2} = g_2^2 \phi$$

from (#), we have

$$i\hbar \frac{g'_1}{g_1} + 2Hg'_2 = H + \frac{i\hbar}{2} g_2 + 2g_2^2 H$$

$$H(2g'_2 - 1 - 2g_2^2) + i\hbar \left(\frac{g'_1}{g_1} - \frac{1}{2} g_2 \right) = 0$$

$$2g'_2 - 1 - 2g_2^2 = 0 \rightarrow g'_2 = \frac{1 + g_2^2}{2}$$

$$\boxed{\frac{g'_1}{g_1} - \frac{1}{2} g_2 = 0}$$



$$\frac{dg_2}{1+g_2^2} = \frac{1}{2} dt$$

$$\arctan(g_2) = \frac{t}{2}$$

$$\frac{g'_1}{g_1} = \frac{1}{2} \tan\left(\frac{t}{2}\right)$$

$$\frac{dg_1}{g_1} = \frac{1}{2} \tan\left(\frac{t}{2}\right) dt$$

$$g_2 = \tan\left(\frac{t}{2}\right)$$

$$\ln(g_1) = \ln|\sec\left(\frac{t}{2}\right)|$$

$$g_1 = \sec\left(\frac{t}{2}\right)$$

$$\text{Exp}\left(-\frac{i\hbar H}{\hbar}\right) = \sec\left(\frac{t}{2}\right) \exp\left(\frac{2H}{i\hbar} \tan\left(\frac{t}{2}\right)\right)$$

In order to obtain the Wigner function expand $\phi(t, H)$

$$\text{Exp}\left(-\frac{i\hbar H}{\hbar}\right) = \int W_m e^{itE_m/\hbar} dE_m$$

$$\Leftrightarrow W_m = \int \overline{\text{Exp}\left(-\frac{i\hbar H}{\hbar}\right)} e^{-itE_m/\hbar} dt$$

$$\text{Define } \tau := \frac{t}{2}, \quad R := \frac{P^2 + q^2}{\hbar}, \quad \alpha := \frac{2E_m}{\hbar}$$

$$W_m = 2 \int \frac{e^{iR\tan(\tau)}}{\cos(\tau)} e^{-i\alpha\tau} d\tau$$

$$\text{Define } z := e^{i\tau}$$

$$\cos(\tau) = \frac{e^{i\tau} + e^{-i\tau}}{2} = z + \frac{1}{z} = \frac{z^2 + 1}{2z}$$

$$\sin(\tau) = \frac{e^{i\tau} - e^{-i\tau}}{2i} = z - \frac{1}{z} = \frac{z^2 - 1}{2iz}$$

$$\tan(\tau) = \frac{z^2 - 1}{z^2 + 1} \cdot \frac{1}{i}$$

$$e^{-i\omega\tau} = (e^{i\tau})^{-\alpha} = \bar{z}^{-\alpha}, \quad dz = ie^{i\tau}d\tau, \quad d\tau = \frac{dz}{zi}$$

$$W_m = 2 \int \frac{e^{\frac{iR}{i}(\frac{z^2-1}{z^2+1})} z^{-\alpha}}{\frac{z^2+1}{2z}} \frac{dz}{iz} = \frac{4}{i} \int \frac{e^{\frac{R(z^2-1)}{z^2+1}} z^{-\alpha}}{z^2+1} dz$$

$$\frac{z^2 - 1}{z^2 + 1} = 1 + \frac{2z^2}{z^2 + 1}$$

$$e^{\frac{R(\frac{z^2-1}{z^2+1})}{i}} = e^{R(1 + \frac{2z^2}{z^2+1})} = e^{-R} e^{2R(\frac{z^2}{z^2+1})}$$

Define

$$\frac{-\lambda}{1-\lambda} := \frac{z^2}{z^2+1}$$

$$-\lambda = \frac{z^2}{z^2+1} (1-\lambda)$$

$$\left(\frac{z^2}{z^2+1} - 1 \right) \lambda = \frac{z^2}{z^2+1}$$

$$\left(\frac{z^2 - z^2 - 1}{z^2+1} \right) \lambda = \frac{z^2}{z^2+1}$$

$$\frac{-\lambda}{z^2+1} = \frac{z^2}{z^2+1}$$

$$-\lambda = z^2$$

$$z = (-\lambda)^{1/2}$$

$$\bar{z}^{-\alpha} = (-\lambda)^{-\alpha/2}$$

$$dz = \frac{-1}{2(-\lambda)^{\alpha/2}} d\lambda$$

$$W_m = -\frac{2}{i} e^{-R} \int_{-\infty}^R e^{2R\lambda/(x-1)} \frac{1}{(-\lambda)^{(\alpha+1)/2}} d\lambda$$

Using Laguerre integral representation

$$\frac{1}{2\pi i} \int \frac{e^{R(\frac{\lambda}{1-x})}}{1-\lambda} \frac{d\lambda}{\lambda^{x+1}} = L_x(2R)$$

$$\text{where } l = \frac{\alpha+1}{2}$$

$$= -\frac{2}{i} e^{-R}(2\pi i) L_{\frac{\alpha+1}{2}}(2R)$$

Therefore

$$\propto e^{-R} L_x(2R)$$