# Normal distribution (Gaussian)

$$W_N(N_1) = \frac{N!}{N_1!\,N_2!}\, p^{N_1} q^{N_2} \longrightarrow \begin{cases} W_N(0) = q^N \xrightarrow[N\gg 1]{} 0 \\ W_N(N) = p^N \xrightarrow[N\gg 1]{} 0 \end{cases} \Big\}\ \text{maximum.}$$

Let $\tilde{N}_1 := N_{1_{max}} = rN$ with $0 < r < 1$.

$$f(N_1) = \ln(W_1(N_1)) = \ln(N!) - \ln(N_1!) - \ln(N-N_1)! + N_1 \ln(p) + (N-N_1)\ln(q)$$

if $N \to \infty$ and $N_1$ and $N-N_1$ are of order $N$ near of the maximum.

then,

$$f(N_1) \approx N\ln(N) - N - N_1\ln(N_1) + N_1 - (N-N_1)\ln(N-N_1) + (N-N_1) + N_1\ln(p) + (N-N_1)\ln(q).$$

$$\frac{\partial f}{\partial N_1} \approx -\ln(N_1) - 1 + 1 + \ln(N-N_1) + 1 - 1 + \ln(p) - \ln(q).$$

$$= -\ln(N_1) + \ln(N-N_1) + \ln(p) - \ln(q)$$

$$\longrightarrow -\ln(\tilde{N}_1) + \ln(N-\tilde{N}_1) + \ln(p) - \ln(q) = 0 \longrightarrow \ln\left(\frac{N-\tilde{N}_1}{\tilde{N}_1}\right) = \ln\left(\frac{q}{p}\right)$$

$$\longrightarrow \frac{N}{\tilde{N}_1} - 1 = \frac{q}{p} \longrightarrow \frac{N}{\tilde{N}_1} = \frac{q}{p} + 1 \longrightarrow Np = (q+p)\tilde{N}_1$$

$$\longrightarrow \tilde{N}_1 = Np, \text{ but } \langle N_1\rangle = Np \longrightarrow \tilde{N}_1 = \langle N_1\rangle$$

Now, $$\frac{\partial^2 f}{\partial N_1^2} = -\frac{1}{N_1} - \frac{1}{N-N_1} \longrightarrow \left.\frac{\partial^2 f}{\partial N_1^2}\right|_{N_1=\tilde{N}_1} = -\frac{1}{Np} - \frac{1}{N-Np}$$

$$-\frac{1}{Np} - \frac{1}{Nq} = \frac{-N(p+q)}{N^2pq} = -\frac{1}{Npq}$$

but $\langle(\Delta N_1)^2\rangle = Npq$.

$$\longrightarrow \left.\left(\frac{\partial^2 f}{\partial N_1^2}\right)\right|_{N_1=\tilde{N}_1} = -\frac{1}{\langle(\Delta N_1)^2\rangle} < 0$$

Expanding by taylor

$$f(N_1) = \ln(W_N(N_1)) = \ln(W_N(\tilde{N}_1)) - \frac{1}{2Npq}(N_1 - \tilde{N}_1)^2 + \dots$$

$$\exp[f(N_1)] = W_N(\tilde{N}_1)\exp\left[\frac{-(N_1-\tilde{N}_1)^2}{2Npq} + \dots\right]$$

for $N \gg 1$.

$$P_G(N_1) = P_0 \exp\left[\frac{-(N_1 - \langle N_1\rangle)^2}{2(\Delta N_1^*)^2}\right] \quad \text{where} \quad \tilde{N}_1 = \langle N_1 \rangle$$

$$Npq = \langle(\Delta N_1)^2\rangle = (\Delta N_1^*)^2$$

As $\sum_{N_1} P_G(N_1) = 1 \longrightarrow \int_{-\infty}^{\infty} P_0 \exp\left[-\frac{x^2}{2(\Delta N_1^*)^2}\right] dx = 1$

$$\longrightarrow P_0 \left[2\pi(\Delta N_1^*)^2\right] = 1 \longrightarrow P_0 = \frac{1}{[2\pi(\Delta N_1^*)^2]^{1/2}}$$

therefore

$$P_G(N_1) = [2\pi(\Delta N_1^*)^2]^{-1/2} \exp\left[\frac{-(N_1 - \langle N_1\rangle)^2}{2(\Delta N_1^*)^2}\right]$$

Normal distribution

I. $\langle N_1 \rangle_G = \int_{-\infty}^{\infty} N_1 P_G(N_1)\, dN_1 = \langle N_1 \rangle = Np.$

II. $\langle(\Delta N_1)^2\rangle_G = \int_{-\infty}^{\infty} (N_1 - \langle N_1\rangle_G)^2 P_G(N_1)\, dN_1 = (\Delta N_1^*)^2 = Npq.$

$$\frac{\partial^3 f}{\partial N_1^3} \approx \frac{1}{N_1^2} - \frac{1}{(N-N_1)^2} \longrightarrow \left.\frac{\partial^3 f}{\partial N_1^3}\right|_{N_1=\tilde{N}_1} = \frac{1}{N^2p^2} - \frac{1}{(N-Np)^2}$$

$$= \frac{(N-Np)^2 - N^2p^2}{N^4p^2q^2} = \frac{N^2(q^2-p^2)}{N^4p^2q^2} = \frac{q^2-p^2}{N^2p^2q^2} = \frac{(q-p)(q+p)}{N^2p^2q^2}$$

$$= \frac{q-p}{N^2p^2q^2}$$

$$\frac{1}{2Npq}|N_1-\tilde{N}_1|^2 >> \frac{|q-p|}{6N^2p^2q^2}|N_1-\tilde{N}_1|^3$$

$$\text{if } |N_1-\tilde{N}_1| \approx \frac{3Npq}{|q-p|} \longrightarrow P_G \sim P_0 \exp\left[-\frac{1}{2Npq}\frac{9N^2p^2q^2}{|q-p|^2}\right]$$

$$P_G \longrightarrow 0 \quad \text{if} \quad N \longrightarrow \infty$$