$$I(x)=\int_a^b e^{-x\phi(t)}f(t)\,dt \sim e^{-x\phi(a)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)C_n}{x^{(n+\beta)/\alpha}} \quad \text{as } x\longrightarrow\infty$$

Theorem: If and g are C^n differentiable at z, then.

$$\frac{d^n}{dz^n}f(z)g(z)=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(z)\,g^{(n-k)}(z)$$

Proof: For $n=1 \rightsquigarrow$ standard Leibniz's rule.

$$\frac{d}{dz}f(z)g(z)=f'(z)g(z)+f(z)g'(z)$$

Assume that it is valid for n and try for $n+1$.

$$\frac{d^{n+1}}{dz^{n+1}}f(z)g(z)=\frac{d}{dz}\left(\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(z)\,g^{(n-z)}(z)\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}\left[f^{(k+1)}(z)g^{(n-k)}(z)\,f^{(k)}(z)\,g^{(n-k+1)}(z)\right]$$

$$=f(z)g^{(n+1)}(z)+\sum_{k=1}\binom{n}{k}f^{(k)}g^{(n-k+1)}(z)$$

$$+\underbrace{\sum_{k=0}^{n-1}\binom{n}{k}f^{(k+1)}(z)\,g^{(n-k)}(z)}+f^{(n+1)}(z)g(z)$$

Change $j=k+1$

$$\sum_{j=1}^{n}\binom{n}{j-1}f^{(j)}(z)\,g^{(n-j+1)}(z)$$

then

$$\frac{d^{n+1}}{dz^{n+1}}f(z)g(z)=f(z)g^{(n+1)}(z)+\sum_{k=1}^{n}\left[\binom{n}{k}\binom{n}{k-1}\right]f^{(k)}(z)\,g^{(n-k+1)}(z)+f^{(n+1)}(z)g(z)$$

$$\binom{n}{k}\binom{n}{k-1}=\frac{n!}{(n-k)!\,k!}+\frac{n!}{(n-k+1)!\,(k-1)!}=\frac{n!}{(k-1)!\,(n-k)!}\left[\frac{1}{k}+\frac{1}{n-k+1}\right]$$

$$=\frac{n!}{(k-1)!\,(n-k)!}\;\frac{(n+1)}{k(n-k+1)}=\frac{(n+1)!}{k!\,(n-k+1)!}=\binom{n+1}{k}$$

then

$$\frac{d^{n+1}}{dz^{n+1}} f(z)g(z) = f(z)g^{(n+1)}(z) + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)}(z) g^{(n+1-k)}(z) + f^{(n+1)}(z) g(z)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(z)\, g^{(n+1-k)}(z)$$

now, consider

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad , \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

forefore

$$f(z)g(z) = \sum_{n=0}^{\infty} C_n (z-z_0)^n$$

Where

$$C_n = \frac{1}{n!} \frac{d^n}{dz^n} \bigg|_{z=z_0}$$

$$f(z)g(z) = \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} f^{(k)}(z_0)\, g^{(n-k)}(z_0) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \frac{g^{(n-k)}(z_0)}{(n-k)!}$$

$$= \sum_{k=0}^{n} a_n b_{n-k} .$$

now

$$g(y) = \frac{f(t)}{\phi'(t)} \longrightarrow g(y)\phi'(t) = f(t)$$

using another notation

$$\frac{f(z)}{g(z)} = h(z) \longrightarrow f(z) = h(z)g(z)$$

therefore

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = \left(\sum_{k=0}^{\infty} c_k (z-z_0)^k\right) \left(\sum_{j=0}^{\infty} b_j (z-z_0)^j\right)$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{n} c_k b_{n-k} (z-z_0)^{k+(n-k)}$$

$$a_n = \sum_{k=0}^{n} c_k b_{n-k}$$

for $n=0$: $\quad a_0 = b_0 c_0 \longrightarrow c_0 = \dfrac{a_0}{b_0}$

for $n \geq 1$:

$$a_n = \sum_{k=0}^{n} c_k b_{n-k} = c_n b_0 + \sum_{k=0}^{n-1} c_k b_{n-k}.$$

$$c_n = \frac{a_n - \sum_{k=0}^{n-1} c_k b_{n-k}}{b_0}$$

Recurrence relation.

$$c_1 = \frac{a_1 - c_0 b_1}{b_0} = \frac{a_1 - \frac{a_0}{b_0} b_1}{b_0} = \frac{a_1 b_0 - a_0 b_1}{b_0^2}$$

$$c_2 = \frac{a_2 - c_0 b_2 - c_1 b_1}{b_0} = \frac{a_2 - \frac{a_0}{b_0} b_2 - \left(\frac{a_1 b_0 - a_0 b_1}{b_0^2}\right) b_1}{b_0}$$

to finish the proof of Ederlyi's theorem. Check that the remaining part (c,b) is legligible

$$\varepsilon := \inf_{c \leq t \leq b} \phi(t) - \phi(a)$$

Assume x_0 such that $I(x_0)$ is absolutely convergent.

Let's assume $x \geq x_0$

$$x(\phi(t) - \phi(a)) = (x - x_0)(\phi(t) - \phi(a)) + x_0(\phi(t) - \phi(a))$$

$$\geq (x - x_0)\varepsilon + x_0(\phi(t) - \phi(a))$$

$$I(x) = \int_a^b e^{-x\phi(t)} f(t)\,dt \qquad \int_a^b \longrightarrow \int_a^c + \int_c^b$$

then

$$\int_c^b e^{-x(\phi(t) - \phi(a))} f(t)\,dt \leq \int_c^b e^{-(x-x_0)\varepsilon} f(t)\,dt + \int_c^b e^{-x(\phi(t) - \phi(a))} f t\,dt$$

as the parts containing x_0 are convergent.

$$\left| e^{x\phi(a)} \int_c^b e^{-x\phi(t)} f(t)\,dt \right| \leq K e^{-\varepsilon x}$$

where K is given by

$$K = e^{x_0(\varepsilon+\phi(a))} \int_c^b e^{-x_0\phi(t)} |f(t)|$$

is an appropriate constant.

$\therefore$ Most of the contribution to the asymptotic behaviour of $I(x)$ comes from the interval $(a,c]$ ∎

Theorem (Perron's formula): The coefficients above are explicitly given by

$$C_n = \frac{1}{\alpha n!}\left[\frac{d^n}{dx^n}\left\{ G(t) \left(\frac{t-a}{\phi(t)-\phi(a)}\right)^{\frac{n+\beta}{\alpha}}\right\}\right]_{t=a}$$

where

$$G(t) \sim \sum_{k=0}^{\infty} b_k (t-a)^k$$

and

$$\phi(t) \sim \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$$

Proof: Let's define l as the label of the first non-vanishing coefficient in this last expression (appart from a_0) and define.

$$\phi_l(t) := \frac{\phi(t)-\phi(a)-a_0(t-a)^{\alpha}}{(t-a)^{\alpha+l}} \sim \sum_{k=0}^{\infty} a_{k+l}(t-a)^k \quad \text{as } t \to a^+$$

$$I(x) = \int_a^b e^{-x\phi(t)} f(t)\,dt = e^{-x\phi(a)} \int_a^b e^{-a_0 x (t-a)^{\alpha}} \underbrace{e^{-x(t-a)^{\alpha+l}\phi_l(t)} f(t)}_{:=h(x,t)}\,dt$$

$$= e^{-x\phi(a)} \int_a^b e^{-a_0 x(t-a)^{\alpha}} h(x,t)\,dt$$

change $z := (t-a)x^{1/\alpha}$, $dz = x^{1/\alpha}dt$

$$I(x) = \frac{e^{-x\phi(a)}}{x^{1/\alpha}} \int_0^{1/\alpha(b-a)} e^{-a_0 z^\alpha} h(x, x^{-1/\alpha}z + a)\,dz$$

set $s := x^{-1/\alpha} z$

$$h(x, s+a) = e^{-x(t-a)^{\alpha+l}\phi_l(t)} f(t) = \exp(-z^\alpha s^l \phi_l(s+a)) f(s+a)$$

$$t - a = s$$

$$t = s + a.$$

$$(t-a)^{\alpha+l} = s^{\alpha+l} = s^\alpha s^l = s^l(x^{-1}z^\alpha)$$

using Taylor-series.

$$\exp(-z^\alpha s^l \phi_l(s-a)) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k}{d\omega^k}\left[e^{-z^\alpha \omega^l \phi_l(\omega+a)}\right]_{\omega=0} s^k, \text{ as } s \to 0^+$$

and considering

$$f(x) = \sum_{k=0}^{\infty} b_k (x-a)^{k+\beta-1}$$

then

$$h(x, s+a) \sim \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k}\left[e^{-z^\alpha \omega^l \phi_l(\omega+a)}\right]_{\omega=0} s^{k+\beta-1}$$

$$I(x) \sim \frac{e^{-x\phi(a)}}{x^{1/\alpha}} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{b_{n-k}}{k!}\left[\frac{d^k}{d\omega^k}\int_0^{1/\alpha(b-a)} e^{-(a_0+\omega^l\phi_l(\omega+a))z^\alpha} s^{k+\beta-1}\right]_{\omega=0} dz$$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{b_{n-k}}{k!}\left[\frac{d^k}{d\omega^k}\int_0^{1/\alpha(b-a)} e^{-(a_0+\omega^l\phi_l(\omega+a))z^\alpha} s^{k+\beta-1}\right]_{\omega=0} \frac{dz}{x^{(n+\beta)/\alpha}}$$

$$x^{(-1/\alpha)(n+\beta-1)} = x^{-\frac{n+\beta}{\alpha}} x^{1/\alpha}$$

Change

$$\rho := (a_0 + \omega^l \phi_l(\omega+a)) z^\alpha$$

then
$$dz = \frac{d\rho\ \rho^{1/\alpha - 1}}{\alpha[a_0 + \omega^l \phi_l(\omega+a)]^{1/\alpha}}$$

$$z^{n+\beta-1} = \frac{\rho^{(n+\beta-1)/\alpha}}{(a_0+\omega^l\phi_l(\omega+a))^{(n+\beta-1)/\alpha}}$$

$$I(x) = e^{-x\phi(a)} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{b_{n-k}}{k!}\frac{d^k}{d\omega^k}\left[\left(\int_0^\infty e^{-\rho}\rho^{(n+\beta-1)/\alpha}\,d\rho\right)\left(\frac{1}{\alpha[a_0+\omega^l\phi_l(\omega+a)]^{\frac{(n+\beta)}{\alpha}}}\right)\right]_{\omega=0}\frac{1}{x^{\frac{(n+\beta)}{\alpha}}}$$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\frac{d^k}{d\omega^k}\left[\frac{1}{\alpha[a_0+\omega^l\phi_l(\omega+a)]^{\frac{(n+\beta)}{\alpha}}}\right]_{\omega=0}\frac{1}{x^{\frac{(n+\beta)}{\alpha}}}$$

Using definition of ϕ_l

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\frac{d^k}{d\omega^k}\left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right]^{\frac{n+\beta}{\alpha}}\Bigg|_{\omega=0}\frac{1}{x^{(n+\beta)/\alpha}}$$

Before, we knew.

$$I(x) \sim e^{-x\phi(a)}\sum_{n=0}^{\infty}\Gamma\left(\frac{n+\beta}{\alpha}\right)\frac{C_n}{x^{(n+\beta)/\alpha}}$$

$$\therefore\quad C_n = \frac{1}{\alpha}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\frac{d^k}{dt^k}\left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right]^{\frac{n+\beta}{\alpha}}\Bigg|_{t=a}$$

$$= \frac{1}{\alpha\, n!}\frac{d^n}{dt^n}\left\{f(t)\left[\frac{(t-a)^\alpha}{\phi(t)-\phi(a)}\right]^{\frac{n+\beta}{\alpha}}\Bigg|_{t=a}\right\}$$

$\blacksquare$

$$x^{\left(\frac{-1}{\alpha}\right)(n+\beta-1)} = x^{-\frac{n+\beta}{\alpha}} \, x^{1/\alpha}$$

Change

$$\rho := (a_0 + \omega^l \phi_l(\omega - a)) \, z^\alpha$$

then

$$dz = \frac{d\rho \quad \rho^{\frac{1}{\alpha}-1}}{\alpha[a_0+\omega^l\phi_l(\omega+a)]^{1/\alpha}}$$

$$z^{n+\beta-1} = \frac{\rho^{(n+\beta-1)/\alpha}}{[a_0+\omega^l\phi_l(\omega+a)]^{(n+\beta-1)/\alpha}}$$

$$I(x) = e^{-x\phi(a)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k}\left[\left(\int_0^\infty e^{-\rho} \rho^{\frac{n+\beta-1}{\alpha}} \, d\rho\right)\left(\frac{1}{\alpha[a_0+\omega^l\phi_l(\omega+a)]^{n+\beta/\alpha}}\right)\right]_{\omega=0} *$$

$$* \; \frac{1}{x^{(n+\beta)/\alpha}}$$

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k}\left[\frac{1}{[a_0+\omega^l \phi_l(\omega+a)]^{(n+\beta)/\alpha}}\right]_{\omega=0} \frac{1}{x^{(n+\beta)/\alpha}}$$

Using definition of ϕ_1

$$= e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{d\omega^k}\left[\frac{(t-a)^{\alpha}}{\phi(t)-\phi(a)}\right]^{\frac{n+\beta}{\alpha}}\Bigg|_{\omega=0} \frac{1}{x^{(n+\beta)/\alpha}}$$

Before we knew

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{c_n}{x^{(n+\beta)/\alpha}}$$

$\therefore$

$$c_n = \frac{1}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \frac{d^k}{dt^k}\left[\frac{(t-a)^{\alpha}}{\phi(t)-\phi(a)}\right]^{(n+\beta)/\alpha}\Bigg|_{t=a}$$

$$= \frac{1}{\alpha n!} \frac{d^n}{dt^n}\left\{ f(t)\left[\frac{(t-a)^{\alpha}}{\phi(t)-\phi(a)}\right]^{(n+\beta)/\alpha}\Bigg|_{t=a} \right\}$$