Watson's Lemma

$$I(x) = \int_0^\infty e^{-xt} f(t)\,dt \sim \sum_{k=0}^{\infty} \frac{a_k \Gamma(\rho_k)}{x^{\rho_k}}$$

$$f(x) \sim \sum_{k=0}^{\infty} a_k t^{\rho_k - 1}$$

Example:

$$I(x) := \int_0^{\pi/4} e^{-xt} \sqrt{1+\cos t}\, dt$$

$$f(t) = \sqrt{1+\cos t} = \sqrt{2} \cos\left(\frac{t}{2}\right)$$

$$f(t) = \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k!} \left(\frac{t}{2}\right)^{2k} \qquad t^{2k} = t^{\rho_k - 1}$$

$$a_k = \frac{\sqrt{2}(-1)^k}{(2k)!} \frac{1}{2^k} \qquad \rho_k = 2k+1$$

$$I(x) \sim \sum_{k=0}^{\infty} \frac{\sqrt{2}(-1)^k}{\cancel{(2k)!}\, 2^{2k}} \frac{\cancel{\Gamma(2k+1)}}{x^{2k+1}}$$

$$I(x) \sim \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}} \frac{1}{x^{2k+1}}$$

$$I(x) \sim \sqrt{2}\left(\frac{1}{x} - \frac{1}{4x^3} + \frac{1}{16x^5}\right) + O\left(\frac{1}{x^7}\right)$$

Modified Bessel function

Example: $$K_0(x) = \int_1^\infty (s^2-1)^{1/2} e^{-sx}\, ds$$

Change $\quad s = t + 1$

$$K_0(x) = e^{-x}\int_0^\infty (t^2+2t)^{-1/2} e^{-xt}\,dt$$

$$f(t) = (t^2+2t)^{-1/2} = (2t)^{-1/2}\left(1+\frac{t}{2}\right)^{-1/2}$$

Binomial expansion $|t|<2$

$$f(t) = (2t)^{-1/2}\sum_{k=0}^{\infty}\binom{k-1/2}{-1/2}\frac{t^k}{2^k} = (2t)^{-1/2}\sum_{k=0}^{\infty}\frac{(-1)^k(k-1/2)!}{k!\,(-1/2)!}\left(\frac{t}{2}\right)^k$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(k+1/2)}{k!\,\Gamma(1/2)}\,\frac{1}{2^{k+1/2}}\;t^{k-1/2}$$

$$a_k = \frac{(-1)^k\,\Gamma(k+1/2)}{k!\,\sqrt{\pi}}\,\frac{1}{2^{k+1/2}}$$

$$t^{k-1/2} = t^{\beta_k-1}$$

$$\beta_k = k+\frac{1}{2}$$

$$K_0(x) \sim \sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(k+1/2)}{\sqrt{\pi}\;k!\;2^{k+1/2}}\;\frac{\Gamma(k+1/2)}{x^{k+1/2}}$$

$$K_0(x) = \frac{[\Gamma(1/2)]^2}{\sqrt{\pi}\,\sqrt{2}}\,\frac{1}{\sqrt{x}} - \frac{[\Gamma(3/2)]^2}{\sqrt{\pi}\;2^{3/2}}\,\frac{1}{x^{3/2}} + O\left(\frac{1}{x^{5/2}}\right)$$

as $x\to\infty$

Laplace type integrals again

$$I(x) := \int_a^b e^{-x\phi(t)} f(t)\, dt.$$

(a,b) real interval x large positive parameter ϕ and f continuous.

Assume, for simplicity, ϕ has a unique minimum in $[a,b]$ wich occurs at $t=a$

Theorem: (Erdélyi)

For
$$I(x) = \int_a^b e^{-x\phi(t)} f(t)\, dt$$

assume:

1) $\phi(t) > \phi(a)$ for all $t \in (a,b)$, and for every $\delta > 0$ the infimum of $\phi(t) - \phi(a)$ in $[a+\delta, b)$ is positive.

11) $\phi'(t)$ and $f(t)$ are continuous in a neighbourhood of $t=a$ (except possible at $t=a$)

111) Assume ϕ and f may be Taylor expanded as

$$\phi(t) = \phi(a) + \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha} \rightarrow \phi'(t) = \sum_{k=0}^{\infty} a_k (k+\alpha)(t-a)^{k+\alpha-1}$$

$$f(t) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$$

Where $\alpha > 0$, $\mathrm{Re}(\beta) > 0$ and $a_0 \neq 0 \neq b_0$

1v) The integral $I(x)$ converges (absolutly) for $x \to \infty$, then

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{c_n}{x^{(n+\beta)/\alpha}} \quad \text{as } x \to \infty$$

Where $C_n = C_n(a_n, b_n)$

Proof:

By (I) and (II), there exists a number $c \in (a,b)$ such that $\phi(t)$, $\phi'(t)$ and $f(t)$ are continuous in $(a,c]$.

$$\phi'(c) = \lim_{c \to a^+} \frac{\phi(c) - \phi(a)}{c-a} > 0$$

Define $y := \phi(t) - \phi(a)$ and $Y := \phi(c) - \phi(a)$

Hot trick:

$$e^{x\phi(a)} \underbrace{\int_a^c e^{-x\phi(t)} f(t)\,dt}_{\text{Looks like } I(x)} = \int_a^c e^{-x(\phi(t)-\phi(a))} f(t)\,dt$$

$$= \underbrace{\int_0^Y e^{-xy} g(y)\,dy}_{\text{Watson-like}}$$

$$\boxed{\begin{aligned} g(y)dy &= f(t)dt \\ g(y)\frac{dy}{dt} &= f(t)dt \\ g(y)\frac{dy}{dt} &= f(t) \\ \text{or} \quad g(y) &= \frac{f(t)}{\phi'(t)} \end{aligned}}$$

Also, $y = \phi(t) - \phi(a) = \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha}$ as $t \to a^+$

Use series reversion:

If $y = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ $\quad a_0 = 0$ Non-constant term

We may invert x in terms of a series expansion in y

$X = A_1 y + A_2 y^2 + A_3 y^3 + \dots$

$A_0 = 0$ Non-constant term

$y = a_1 (A_1 y + A_2 y^2 + A_3 y^3 + \dots) + a_2 (A_1 y + A_2 y^2 + A_3 y^3 + \dots)^2$
$+ a_3 (A_1 y + A_2 y^2 + A_3 y^3 + \dots)^3 + \dots$

$y = a_1 A_1 y + (a_1 A_2 + a_2 A_1^2) y^2 + (a_1 A_3 + a_2 A_1 A_2 + a_3 A_1^3) y^3 + \dots$

then

$a_1 A_1 = 1 \qquad A_1 = \frac{1}{a_1}$

$a_1 A_2 + a_2 A_1^2 = 0 \qquad A_2 = \frac{-a_2 A_1^2}{a_1} = \frac{-a_2}{a_1^3}$

$A_3 = a_1^{-5} (2a_2^2 a_1 - a_3)$

In our case

$$y = \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha} \quad , \quad t \to a^+$$

we obtain

$$t - a = \sum_{k=1}^{\infty} d_k y^{k/\alpha} \quad \text{as} \quad y \to 0^+$$

$$= d_1 y^{1/\alpha} + d_2 y^{2/\alpha} + d_3 y^{3/\alpha} + \dots$$

$y = a_0 (t-a)^{\alpha} + a_1 (t-a)^{1+\alpha} + a_2 (t-a)^{2+\alpha} + \dots$
$y = a_0 (d_1 y^{1/\alpha} + d_2 y^{2/\alpha} + d_3 y^{3/\alpha} + \dots)^{\alpha}$
$+ a_1 (d_1 y^{1/\alpha} + d_2 y^{2/\alpha} + d_3 y^{3/\alpha} + \dots)^{\alpha+1}$

$$+a_2\left(d_1 y^{1/\alpha}+d_2 y^{2/\alpha}+d_3 y^{3/\alpha}+\dots\right)^{\alpha+2}+\dots$$

$$=a_0 d_1^{\alpha} y+\left(a_0 d_2^{\alpha}+\frac{1}{\alpha}a_1 d_1^2 d_2^{\alpha-1}\right)y^2+\dots$$

$$d_1=\frac{1}{a_0^{1/\alpha}} \qquad d_2=\frac{-a_1}{\alpha\, a_0^{1+2/\alpha}}$$

then

$$g(y)\frac{dy}{dt}=f(t)$$

$$g(y)\frac{d}{dt}\left(\sum_{j=0}^{\infty} a_j (t-a)^{j+\alpha}\right)=\sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$$

$$g(y)\sum_{j=0}^{\infty} a_j (j+\alpha)(t-a)^{j+\alpha-1}=\sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$$

Equating same powers.

$$j+\alpha-1=k+\beta-1$$
$$j=k+\beta-\alpha$$

Multiplication and division of series.

$$g(y)\, a_{k+\beta-\alpha}(k+\beta)=b_k$$

Finally,

$$g(y)\sim\sum_{k=0}^{\infty} c_k y^{(k+\beta)/\alpha-1}$$

$$I(x)\sim e^{-x\phi(a)}\sum_{n=0}^{\infty}\frac{c_n\Gamma\left(\frac{n+\beta}{\alpha}\right)}{x^{(n+\beta)/\alpha}}$$

Until here we made the half proof.

Adding, Multiplying, and Dividing Series

let

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

Now, lets sum

$$\begin{array}{rl} & 1 + x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3!} + \dots \\ + & \quad x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \\ \hline \end{array}$$

$$e^x + \ln(x+1) = 1 + 2x + \frac{x^3}{2} - \frac{5}{24}x^4 + \dots$$

General Idea:

If
$$f(x) = a_0 + a_1x + a_2x^2 + \dots$$
$$g(x) = b_0 + b_1x + b_2x^2 + \dots$$

then

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots$$

Multiplication:

$$\begin{array}{rl} f(x)g(x) = & a_0 + a_1x + a_2x^2 + \dots \\ \times & b_0 + b_1x + b_2x^2 + b_3x^3 \\ \hline = & a_0b_0 + a_1b_0x + a_2b_0x^2 + a_3b_0x^3 + \dots \\ & + a_0b_1x + a_1b_1x^2 + a_2b_1x^3 + \dots \end{array}$$

$$+ a_0 b_2 x^2 + a_1 b_2 x^3 + \dots$$

$$\vdots \qquad \dots$$

$$= \sum_{n=0}^{\infty} C_n x^n, \text{ where } \quad C_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_j b_{n-j}$$

Division: if $b_0 \neq 0$

$$\frac{f(x)}{g(x)} = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = \sum_{n=0}^{\infty} C_n x^n$$

Where

$$a_n = \sum_{j=0}^{n} b_j C_{n-j}$$

So that

$$C_n = \frac{1}{b_0} \left(a_n - \sum_{j=1}^{n} b_j C_{n-j} \right)$$