$$\langle q_f t_f | q_i t_i \rangle = \int dq_1 \ldots dq_n \langle q_f t_f | q_n t_n \rangle \langle q_n t_n | q_{n-1} t_{n-1} \rangle \ldots \langle q_1 t_1 | q_i t_i \rangle$$

$$\langle q_{j+1} t_{j+1} | q_j t_j \rangle = \delta(q_j - q_{j+1}) - \frac{i\tau}{\hbar} \langle q_{j+1} | (\hat{H} | q_j \rangle)$$

$$\hat{H} | q_j \rangle, \quad \hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(q).$$

① $$\langle q_{j+1} | \left(\frac{\hat{p}^2}{2m} | q_j \rangle \right) = \int dp' dp \, \langle q_{j+1} | p' \rangle \langle p' | \left(\frac{\hat{p}^2}{2m} | p \rangle \langle p | q_j \rangle \right)$$

$$\langle q | p \rangle = \frac{1}{(2\pi\hbar)^{1/2}} \exp\left(\frac{ipq}{\hbar} \right)$$

$$\Rightarrow \langle q_{j+1} | \left(\frac{\hat{p}^2}{2m} | q_j \rangle \right) = \int \frac{dp\, dp'}{2\pi\hbar} e^{i/\hbar (p' q_{j+1} - p q_j)} \frac{p^2}{2m} \langle p' | p \rangle \qquad =: \delta(p' - p)$$

$$= \int \frac{dp}{2\pi\hbar} e^{i/\hbar (q_{j+1} - q_j)} \frac{p^2}{2m}$$

② $$\langle q_{j+1} | (V(\hat{q}) | q_j \rangle) = V\left(\frac{q_{j+1} - q_j}{2} \right) \langle q_{j+1} | q_j \rangle = V(\bar{q}_j) \delta(q_{j+1} - q_j) = \int \frac{dp}{2\pi\hbar} e^{i/\hbar (q_j - q_{j+1})} V(\bar{q}_j)$$

Must be evaluated at a point $q \in [q_j, q_{j+1}]$

Irrelevated average as we want the limit $q_{j+1} - q_j \to 0$

$$\therefore \langle q_{j+1} | (\hat{H} | q_j \rangle) = \int \frac{dp}{2\pi\hbar} e^{i/\hbar\, p (q_{j+1} - q_j)} \left(\frac{p^2}{2m} + V(\bar{q}_j) \right)$$

$$= \int \frac{dp}{2\pi\hbar} e^{i/\hbar\, p (q_{j+1} - q_j)} H(p, \bar{q}_j)$$

finally

$$\langle q_{j+1} t_{j+1} | q_j t_j \rangle = \int \frac{dp_j}{2\pi\hbar} \exp\left(\frac{i}{\hbar} \left[p_j (q_{j+1} - q_j) - \tau H(p_j, \bar{q}_j) \right] \right)$$

If we consider the full interval $[t_i, t_f]$

$$\langle q_f t_f | q_i t_i \rangle = \lim_{n\to\infty} \int \prod_{j=0}^{n} dq_j \prod_{j=0}^{n} \frac{dp_j}{2\pi\hbar} \exp\left[\frac{i}{\hbar} \sum_{j=0}^{n} [p_j(q_{j+1}-q_j) - \tau H(p_j, \bar{q}_j)] \right]$$

$$q_0 := q_i \quad , \quad q_{n+1} := q_f \qquad \dot{q} = \frac{dq}{dt}$$

Notation:

$$\mathcal{D}q := \prod_{j=0}^{\infty} dq_j \qquad \mathcal{D}p := \prod_{j=0}^{\infty} dp_j$$

$$\langle q_f t_f | q_i t_i \rangle = \int \frac{\mathcal{D}p\,\mathcal{D}q}{2\pi\hbar} \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} dt\, [p\dot{q} - H(p,q)] \right]$$

$$= \int \frac{\mathcal{D}p\,\mathcal{D}q}{2\pi\hbar} \exp\left[\frac{i}{\hbar} \int dt\, \mathcal{L} \right]$$

$$= \int \frac{\mathcal{D}p\,\mathcal{D}q}{2\pi\hbar} \exp\left(\frac{i}{\hbar} S \right) \quad , \quad S := \text{classical action.}$$

S matrix → Perturbative

↳ Scattering.

In order that perturbation thecniques are applicable, the integral of the potential $V(x)$ must be small compared to $\hbar$.

$$\exp\left[\frac{-i}{\hbar} \int_{t_i}^{t_f} V(x,t)\,dt \right] = 1 - \frac{i}{\hbar} \int_{t_i}^{t_f} V(x,t)\,dt - \frac{1}{2!\hbar^2} \left[\int_{t_i}^{t_f} V(x,t)\,dt \right]^2 + \ldots$$

we will also expand $K(q_f t_f, q_i t_i)$ as $K = K_0 + K_1 + \ldots$

$K_0 :=$ free propogator

$$K_0 = N \int \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} \frac{1}{2} m \dot{x}^2\, dt \right]$$

$$N := \int \frac{\mathcal{D}p}{2\pi\hbar} \longrightarrow \text{momentum Volume}$$

$$K_0 = \lim_{n \to \infty} \left(\frac{m}{i\hbar\tau} \right)^{(n+1)/2} \int_{-\infty}^{\infty} \prod_{j=1}^{n} dx_j \exp\left[\frac{im}{2\hbar\tau} \sum_{j=0}^{n} (x_{j+1} - x_j)^2 \right]$$

Discrete form

Theorem: Show

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\frac{\pi}{\alpha} \right)^{1/2} \quad , \alpha > 0$$

Gaussian

Proof:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx dy$$

Polar coordinates

$$= \int_0^{2\pi} \int_0^{\infty} e^{-\alpha r^2} r dr d\theta = 2\pi \int_0^{\infty} e^{-\alpha r^2} r dr$$

$$r^2 = u \quad , \quad r dr = \frac{du}{2}$$

$$I = 2\pi \int_0^{\infty} e^{-\alpha u} \frac{du}{2} = \pi \left[-\frac{1}{\alpha} e^{-\alpha u} \right]_0^{\infty} = \frac{\pi}{\alpha}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$\blacksquare$

Theorem: Show

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \exp\left(\frac{b^2}{4a} + c \right) \left(\frac{\pi}{a} \right)^{1/2} \quad ; \quad a \neq 0.$$

Proof: Define $g(x) := -ax^2 + bx + c$.

Let $\bar{x}$ be the value of x that gives a minimum for $g(x)$

$$0 = g'(x) = -2ax + b \qquad \bar{x} = \frac{b}{2a}.$$

such that

$$g(\bar{x}) = -a\left(\frac{b}{2a}\right)^2 + b\left(\frac{b}{2a}\right) + c$$

$$= \frac{-b^2}{4a} + \frac{b^2}{2a} + c$$

$$= \frac{b^2}{4a} + c$$

we may write $g(x)$ as

$$g(x) = g(\bar{x}) - a(x-\bar{x})^2$$

$$= \left(\frac{b^2}{4a} + c\right) - a(x^2 - 2x\bar{x} + \bar{x}^2)$$

$$= \left(\frac{b^2}{4a} + c\right) - ax^2 - 2ax\left(\frac{b}{2a}\right) + \frac{b^2}{4a}$$

$$= -ax^2 + bx + c$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{g(x)} dx = g(\bar{x}) \int_{-\infty}^{\infty} e^{-a(x-\bar{x})^2} dx$$

Changing $u := x - \bar{x}$, $du = dx$

$$I = e^{g(\bar{x})} \int_{-\infty}^{\infty} e^{-au^2} du = \exp\left(\frac{b^2}{4a} + c\right)\left(\frac{\pi}{a}\right)^{1/2}$$

■

Theorem: Show that

$$\int_{-\infty}^{\infty} \exp\left[i\lambda\left[(x-a)^2 + (x_2-x_1)^2 + \dots + (b-x_n)^2\right]\right] dx_1 \cdots dx_n = \left(\frac{i^n \pi^n}{(n+1)\lambda}\right)^{1/2} \exp\left[\frac{i\lambda}{n+1}(b-a)^2\right]$$

Proof: By induction:

a) For $n=1$: $\int_{-\infty}^{\infty} \exp\left[i\lambda\left[(x-a)^2 + (b-x)^2\right]\right] dx$

$$= \int_{-\infty}^{\infty} dx \exp\left[i\lambda\left(x^2 - 2ax + a^2 + b^2 - 2bx + x^2\right)\right]$$

$$= \int_{-\infty}^{\infty} dx \exp\left[\underbrace{2i\lambda}_{-a} x^2 - \underbrace{2i\lambda(a+b)}_{+b} x + \underbrace{i\lambda(a^2+b^2)}_{+c}\right]$$

$$= \exp\left[\frac{-4\lambda^2(a+b)^2}{4(-2i\lambda)} + i\lambda(a^2+b^2)\right]\left(\frac{\pi}{-2i\lambda}\right)^{1/2}$$

$$= \exp\left[\frac{i\lambda(a-b)^2}{2}\right]\left(\frac{i\pi}{2\lambda}\right)^{1/2}$$

b) Assume that it is true for n and prove for n+1.

$$I_{n+1} := \int_{-\infty}^{\infty} \exp\left[i\lambda\left((x_1-a)^2 + \ldots + (b-x_{n+1})^2\right)\right] dx_1 \ldots dx_{n+1}$$

$$= \left(\frac{i^n \pi^n}{(n+1)\lambda^n}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[\frac{i\lambda}{n+1}(x_{n+1}-a)^2\right] \exp\left[i\lambda(b-x_{n+1})^2\right] dx_{n+1}$$

$$= \left(\frac{i^n \pi^n}{(n+1)\lambda^n}\right)^{1/2} \int_{-\infty}^{\infty} \underbrace{\exp\left[i\lambda\left[\frac{(x_{n+1}-a)^2}{n+1} + (b-x_{n+1})^2\right]\right]}_{=: J} dx_{n+1}$$

Defining $x_{n+1} - a =: y$

$$\int_{-\infty}^{\infty} \exp\left[i\lambda\left[\frac{y^2}{n+1} + (b-a-y)^2\right]\right] dx_{n+1}$$

$$\frac{y^2}{n+1} + (b-a-y)^2 = \frac{1}{n+1} y^2 + (b-a)^2 - 2(b-a)y + y^2$$

$$= \frac{n+2}{n+1} y^2 - 2(b-a)y + (b-a)^2$$

$$= \left(\frac{n+2}{n+1}\right)\left(y - \frac{n+1}{n+2}(b-a)\right)^2 + \frac{1}{n+2}(b-a)^2$$

changing $z := y - \frac{n+1}{n+2}(b-a)$

$$I_{n+1} = \left(\frac{i^n \pi^n}{(n+1)\lambda^n}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[i\lambda\left(\frac{n+2}{n+1}\right) z^2 + \frac{i\lambda}{n+2}(b-a)^2\right] dz.$$

$$= \exp\left[\frac{i\lambda}{n+2}(b-a)^2\right]\left(\frac{i^n \pi^n}{(n+1)\lambda^n}\right)^{1/2}\left[\frac{\pi(n+1)}{-i\lambda(n+2)}\right]^{1/2}$$

$$= \left(\frac{i^{n+1}\pi^{n+1}}{(n+2)\lambda^{n+1}}\right)\exp\left[\frac{i\lambda}{n+2}(b-a)^2\right]$$