Time evolution of classical Hamiltonian systems.

Definition: The flow generated by a Hamiltonian field X_H associated to $\hat{H}$ is called the Hamiltonian flow Φ_t^H, and for pure states it generates time evolution.

$$\xi(t) = \Phi_t^H(\xi(0)).$$

Also a trajectory for a pure state $\xi \in M$ is calculated by

$$\dot{\xi} = X_H\Big|_{\xi} = P(dH)\Big|_{\xi} = -\frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j} + \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j}\Big|_{\xi}$$

$$= \dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}_i\frac{\partial}{\partial p_i}\Big|_{\xi} = \frac{d}{dt}\Big|_{\xi}.$$

where we used Hamilton's equations.

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

For mixed states we may obtain equations of motion from the probability conservation law.

$$0 = \frac{d\rho}{dt} = \frac{d\rho}{dt} + \{\rho, H\} \qquad \text{Liouville equation.}$$

$$L(\rho, H): \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$$

The Hamilton flow $\Phi_t^H = e^{t\mathcal{L}_{X_H}}$ induces for every t an automorphism U_t^H of the algebra $\hat{A}_c$.

$$U_t^H \hat{A} := (\Phi_t^H)^* A\,\bullet\,; \qquad \hat{A} \in \hat{A}_c,\ A \in A_c.$$

and such that its action may be interpreted as the time evolution of $\hat{A}$.

$$\hat{A}(t) = U_t^H \hat{A}(0) = e^{t\mathcal{L}_{X_H}} A(0)\,\bullet$$

$$= e^{tX_H} A(0)\,\bullet$$

Thus, differentiation with respect to t:

$$\frac{d\hat{A}(t)}{dt} = X_H(e^{tX_H}A(0)\,\bullet) = X_H\hat{A}(t) = \{\hat{A}, \hat{H}\}$$

Heisenberg-like evolution.

In the Schrödinger-like representation we have $d\mathcal{S}/dt=0$.

$$\frac{d\langle\hat{A}\rangle_{\rho(t)}}{dt}=\frac{d}{dt}\int_M A(\xi)\cdot\rho(\xi)d\xi.$$

$$=\int_M \frac{d}{dt}A(\xi)\cdot\rho(\xi)d\xi+\cancel{\int_M A(\xi)\cdot\frac{d}{dt}\rho(\xi)d\xi}^{\to 0}.$$

$$=\int_M \frac{d}{dt}\hat{A}(\xi)\,\rho(\xi)d\xi=\int_M \}A(\xi),\hat{H}(\xi)\{\,\rho(\xi)d\xi.$$

$=\langle\}\hat{A}(t),\hat{H}(t)\{\rangle$ Ehrenfest theorem.

Homework: $\langle\hat{A}(0)\rangle_{\rho(t)}=\langle\hat{A}(t)\rangle_{\rho(0)}$.

Quantization of classical theory

Consider a classical Hamiltonian system.

$$(M,\mathcal{P},\hat{\mathcal{H}})$$

$M:=$ Classical phase space (with local coordinates (q,p)'s)

$\mathcal{P}:=$ Poisson tensor (in local coordinates $\mathcal{P}=\partial q^i\wedge\partial p_i$

$\hat{\mathcal{H}}:=$ Hamiltonian Classical operator.

Also $A_c:=(C^\infty(M),\cdot)$ algebra of classical observables.

Let h denote a Hilbert space and let $B(h)$ be the set of bounded linear operators over h. Define $\circ$ as the composition between operators.

$\Rightarrow A_q:=(B(h),\circ)$ is the algebra of quantum observables.

$\hookrightarrow$ associative but non-commotative in general

Definition: A quantization rule is given by an invertible map $Q_\hbar: A_c\longrightarrow A_q$ which depends on a real parameter $\hbar>0$ such that for any pair of classical observables $f_1,f_2\in A_c$ then $Q_\hbar$ follows.

i) $\lim\limits_{\hbar\to 0}\frac{1}{2}Q_\hbar^{-1}\left(Q_\hbar(f_1)\circ Q_\hbar(f_2)+Q_\hbar(f_2)\circ Q_\hbar(f_1)\right)=f_1\cdot f_2.$

ii) $\lim\limits_{\hbar\to 0}Q_\hbar^{-1}(\}Q_\hbar(f_1),Q_\hbar(f_2)\{_\hbar)=\}f_1,f_2\{$

where $\{\cdot,\cdot\}_\hbar := \frac{1}{i\hbar}[\cdot,\cdot]$; being $[\cdot,\cdot]$ the standard commutator.

(I) & (II) guarantee that the quantization rule $Q_\hbar$ has a classical limit.

Notation: For $f \in A_c$, we take $\hat{f} := Q_\hbar(f) \in A_Q$, Also

$$[\hat{f}_1,\hat{f}_2] = Q(i\hbar\{f_1,f_2\}) =: i\hbar\,\widehat{\{f_1,f_2\}}$$

$$[\hat{f}_1,\hat{f}_2] = i\hbar\,\widehat{\{f_1,f_2\}}$$ Bohr correspondence

Idea: Fix $Q_\hbar \rightsquigarrow$ Determines QM

However, due to the non-commutative of the composition of operators, we must impose an ordering rule of factors

Example: Weyl-ordering $\longrightarrow$ Produces symmetric operators.

$$1 \longmapsto Q_\hbar^w(1) =: \mathbb{1}$$

$$q \longmapsto Q_\hbar^w(q) =: \hat{q}$$

$$p \longmapsto Q_\hbar^w(p) =: \hat{p}$$

$$p^n q^m \longmapsto Q_\hbar^w(p^n q^m) =: \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\hat{q}^k \circ \hat{p}^n \circ \hat{q}^{m-k}$$

Example:

$$pq \longmapsto Q_\hbar^w(pq) = \frac{1}{2}\sum_{k=0}^{1}\binom{1}{k}\hat{q}^k \circ \hat{p} \circ \hat{q}^{1-k}$$

$$= \frac{1}{2}(\hat{p}\circ\hat{q} + \hat{q}\circ\hat{p})$$

$$pq^2 \longmapsto \frac{1}{2^2}(\hat{p}\circ\hat{q}\circ\hat{q} + \hat{q}\circ\hat{p}\circ\hat{q} + \hat{q}\circ\hat{q}\circ\hat{p})$$

Example: Normal ordering

$$1 \longmapsto Q_\hbar^N(1) =: \hat{\mathbb{1}}$$

$$p \longmapsto Q_\hbar^N(p) =: \hat{p}$$

$$q \longmapsto Q_\hbar^N(q) =: \hat{q}$$

$$p^n q^m \longmapsto Q_\hbar^N(p^n q^m) =: \hat{p}^n \circ \hat{q}^m$$

$$pq \longmapsto Q_\hbar^N(pq) =: \hat{p}\circ\hat{q} \neq \hat{q}\circ\hat{p}.$$