Definition: A Riemannian metric on a differentiable manifold M is a correspondence which associates to each point an inner product $\langle \cdot, \cdot \rangle_p$ on the tangent space T_pM which varies differentiably in the following sense:

If $\xi : U \subset M \longrightarrow \mathbb{R}^n$ is a coordinate system around p with

$$\xi(x^1, \dots, x^n) = q \in \xi(U)$$

and

$$\frac{\partial(q)}{\partial x^i} = dx^q(0, \dots, 1, \dots, 0)$$

then

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x^j} \right\rangle_p = g_{ij}(x^1, \dots, x^n)$$

is a differentiable function on $U \subseteq M$.

Definition: A Riemannian manifold is a differentiable manifold M furnished with a metric tensor g which satisfies the following:

at each $p \in M$.

I. $g_p(u, v) = g_p(v, u)$ symmetric.

II. $g_p(u, u) \geq 0$ (equality only holds for $u = 0$).

$\downarrow$

g_p is a positive-definite bilinear form.

Note:

a. Positive-definiteness may be weakened to non-degeneracy

$g_p(u, v) = 0, \quad \forall\, v \in T_pM.$

$\rightarrow u = 0$

Semi-Riemannian
Pseudo-Riemannian.

b. Pseudo-Riemannian metrics appear in the context of relativistic theories.

c. Whenever we have a complex vector space

$(v, w) = \overline{(w, v)} \longrightarrow$ Hermitian inner product replaces symmetric property.

d. Every inner product is a metric space. Take for example

$$g(u,v)=\langle u-v,u-v\rangle$$

e. If we have a complete inner product space, then is a Hilbert space.

Before: $\langle\cdot,\cdot\rangle: T_p^*M \times T_pM \longrightarrow \mathbb{R}$

Now: $g_p: T_pM \times T_pM \longrightarrow \mathbb{R}$

In order to get g_p consider

$$g_p(u,\cdot): T_pM \longrightarrow \mathbb{R}$$
$$v \longmapsto g_p(u,v) \in \mathbb{R}.$$

$g_p(u,\cdot)$ may be identified with a one-form $\omega_u \in T_p^*M$.

$$g_p(u,\cdot)=\omega_u$$
$$g_p(u,v)=\langle\omega_u,v\rangle \in \mathbb{R}.$$

Analogously $\omega \in T_p^*M$ induces $V_\omega \in T_pM$ by

$$\langle\omega,u\rangle_p = g_p(V_\omega,u)$$

$\therefore$ g_p gives rise to an isomorphism between T_pM and T_p^*M.

let $\xi=(x^1,\ldots,x^n)$ be a coordinate system. Since g is a tensor of type $(0,2)$, it may be expanded in terms of

$dx^i \otimes dx^j$ as $g_p = g_{ij}(p)\,dx^i dx^j$

The metric tensor components are given by

$$g_{ij}(p)=g_p\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x^j}\right)=g_{ji}(p).$$

then, for vector fields

$$v=\sum_i v^i\partial_i \quad , \quad \omega=\sum_i \omega^j\partial_j$$

$$g_p(v,w)=\langle\omega_v,\omega\rangle=\langle v_i\,dx^i,\omega^j\partial_j\rangle$$
$$=v_i\,\omega^j\langle dx^i,\partial_j\rangle$$
$$=v_i\,\omega^j\,\delta^i_j=v_i\,\omega^i=g_{ij}v^j\omega^i.$$

g_{ij} may be regarded as a matrix and we have g_{ij} is invertible

Notation: $(g_{ij})^{-1} =: g^{ij}$

$$g_{ij} g^{jk} = \delta_i^k$$

$v = v^i \partial_i$

$\omega_j = g_{ji} v^i$

$\omega_v = \omega_j dx^j$

$= g_{ji} v^i dx^j$

let $g := \det g_{ij}$, then $\det g^{ij} = g^{-1}$

$$\begin{array}{ccc} T_p{}^*M & \underset{g^{ij}}{\overset{g_{ij}}{\rightleftarrows}} & T_pM \\ \omega_a = g_{ab} u^a & & u^b = g^{ba}\omega_a \end{array}$$

let $g_p(v, \omega)$ be a pseudo-Riemannian metric, analoge in $\mathbb{R}^n$:

let $p, q \in \mathbb{Z}$ such that $p+q=n$, and consider the inner product.

$$\langle v, w \rangle = -\sum_{i=0}^{p} v_i w_i + \sum_{j=p+1}^{n} v_j w_j .$$

then $\mathbb{R}^{p,q}$ is the pseudo-Euclidean space.

For $n \geq 2$: $\mathbb{R}^{1,q} :=$ minkowski space $(g \longmapsto n)$

For $n = 4$: $\mathbb{R}^{1,3}$ is the simplest relativistic space

Let M^n be a pseudo-Riemannian manifold and let $p+q=n$.

If $p=0 \longrightarrow M^{0,q} = M^q$ Riemann manifold.

If $p=1$ and $n \geq 2 \longrightarrow M^{1,2} :=$ Lorentzian manifold.

EJTP manifold

Definition: A tangent vector to M at p is:

I. Spacelike $\langle v, v \rangle > 0$

II. Null $\langle v, v \rangle = 0$ and $v \neq 0$

III. Timelike $\langle v, v \rangle < 0$

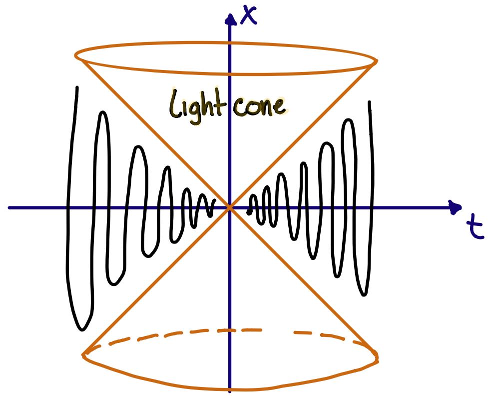
Trichotomy law for pseudo-Riemannian manifold.

Physics notes:

a. Null vector are also called lightlike

b. Set of null vectors in $T_pM \longrightarrow$ Null cone at $p \in M$

C. The category for a tangent vector defines its causal character



Differential forms

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I. Exterior algebra:

Let K denote a field, then $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and let $V :=$ n-dim oriented vector space over $\mathbb{R}$

$$\alpha \in V, \quad \alpha := \alpha^i e_i$$

for each $p \leq n$ we shall construct vector spaces $\Lambda^p V$ over $\mathbb{R}$

Let $\wedge$ define an exterior product as a bilinear, associative and skew-symmetric product on V.

$$\wedge : V \times V \longrightarrow \Lambda^2 V$$

$$(\alpha, \beta) \longmapsto \alpha \wedge \beta$$

such that,

$$(a_1\alpha_1 + a_2\alpha_2) \wedge \beta = a_1 \alpha_1 \wedge \beta + a_2 \alpha_2 \beta$$

$$\alpha \wedge (b_1\beta_1 + b_2\beta_2) = b_1 \alpha \wedge \beta_1 + b_2 \alpha \wedge \beta_2$$

$$\alpha \wedge \beta = -\beta \wedge \alpha \quad (\Longleftrightarrow \alpha \wedge \alpha = 0)$$