Volume form element

Let $p \in M$ and $\xi : M \to \mathbb{R}^n$ be a coordinate system such that $\xi = (X^1, \dots, X^n)$.

Consider a positive orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM, then

$$X^i = \sum_j a^{ij} e_j \in T_pM.$$

$$g_{ik}(p) = \langle X_i, X_k \rangle (p) = \sum_{j,l} a_{ij} (a_{kl})^t \langle e_j, e_l \rangle$$

$$= \sum_{j,l} a_{ij} (a_{kl})^t \delta_{jl} = \sum_j a_{ij} a_{jk}$$

$$\mathrm{Vol}(X_1^{(p)}, \dots, X_n^{(p)}) = (\det a_{ij}) \,\mathrm{vol}(e_{1(p)}, \dots, e_{n(p)})$$

$$= \det a_{ij} = \sqrt{\det(g_{ij})} =: \sqrt{g}$$

Let $\eta = (y^1, \dots, y^n)$ be a different coordinate system

$$\eta : M \to \mathbb{R}^n$$

then

$$Y_i(p) := \left.\frac{\partial}{\partial y^i}\right|_p \quad \text{and} \quad h_{ij}(p) = \langle Y_i, Y_j \rangle (p).$$

Also

$$\sqrt{\det(g_{ij}(p))} = \mathrm{vol}(X_{1(p)}, \dots, X_{n(p)}) = J^{-1} \mathrm{Vol}(Y_{1(p)}, \dots, Y_{n(p)})$$

$$J_{ij} := \partial y_i^{-1} \circ dX_k = \frac{\partial X_i}{\partial y^j}$$

$J :=$ Jacobian of the transformation

$$J = \det(J_{ij})$$

$$= J^{-1} \sqrt{\det(h_{ij})}$$

derivative of a determinant.

$$dx^1 \wedge \cdots \wedge dx^n = \left(\frac{\partial X^1}{\partial y^{j_1}} dy^{j_1}\right) \wedge \left(\frac{\partial X^2}{\partial y^{j_2}} dy^{j_2}\right) \wedge \cdots \wedge \left(\frac{\partial X^n}{\partial y^{j_n}} dy^{j_n}\right)$$

$$= J \, dy^1 \wedge \cdots \wedge dy^n$$

therefore $\sqrt{-g}\, dx^1 \wedge \cdots \wedge dx^n$ is invariant.

$$= J^{-1} \sqrt{-h}\, J d^n y$$

$$\sqrt{-g}\, d^n x = \sqrt{-h}\, d^n y$$

Induced metric (Will be relevant for ADM)

Let $M^m \subseteq N^n$ $(m \le n)$ be a submanifold with metric g_N.

If $f: M \longrightarrow N$ is the embedding, then the pullback map f^* induces the natural metric $g_M = f^* g_N$ on M such that

$$g_{M\,ij}(x) = g_{N\,ab}(f(x)) \frac{\partial f^a}{\partial x^i} \frac{\partial f^b}{\partial x^j} \qquad 1 \le i,j \le m$$

$$1 \le a,b \le n$$

$$f: M \longrightarrow N$$

$$f^*: T^*N \longrightarrow T^*M$$

where f^a denotes the coordinate of $f(x)$ $(f \circ \xi^{-1}: \xi(U) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R})$.

Example: let (θ, φ) be polar coordinates of S^2 and define f by the embedding $S^2 \subset \mathbb{R}^3$

$$f(\theta,\varphi) \longmapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$\delta_{ab} = g_{ab}$ in $\mathbb{R}^3$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \delta_{ab} \frac{\partial f^a}{\partial x^\mu} \frac{\partial f^b}{\partial x^\nu} dx^\mu dx^\nu$$

$$= \sum_a \frac{\partial f^a}{\partial x^\mu} \frac{\partial f^a}{\partial x^\nu} dx^\mu dx^\nu$$

$$\frac{\partial f^1}{\partial \theta} = \cos\theta\cos\varphi \qquad \frac{\partial f^1}{\partial \varphi} = -\sin\theta\sin\varphi$$

$$\frac{\partial f^2}{\partial \theta} = \cos\theta\sin\varphi \qquad \frac{\partial f^2}{\partial \varphi} = \sin\theta\cos\varphi$$

$$\frac{\partial f^3}{\partial \theta} = -\sin\theta \qquad \frac{\partial f^3}{\partial \varphi} = 0$$

$$ds^2 = \frac{\partial f^1}{\partial \theta}\frac{\partial f^1}{\partial \theta} d\theta\, d\theta + 2\frac{\partial f^1}{\partial \theta}\frac{\partial f^1}{\partial \varphi} d\theta\, d\varphi + \frac{\partial f^1}{\partial \varphi}\frac{\partial f^1}{\partial \varphi} d\varphi\, d\varphi$$

$$+ \frac{\partial f^2}{\partial \theta}\frac{\partial f^2}{\partial \theta} d\theta\, d\theta + 2\frac{\partial f^2}{\partial \theta}\frac{\partial f^2}{\partial \varphi} d\theta\, d\varphi + \frac{\partial f^2}{\partial \varphi}\frac{\partial f^2}{\partial \varphi} d\varphi\, d\varphi$$

$$+ \frac{\partial f^3}{\partial \theta}\frac{\partial f^3}{\partial \theta} d\theta\, d\theta + \cancelto{0}{2\frac{\partial f^3}{\partial \theta}\frac{\partial f^3}{\partial \varphi} d\theta\, d\varphi} + \cancelto{0}{\frac{\partial f^3}{\partial \varphi}\frac{\partial f^3}{\partial \varphi} d\varphi\, d\varphi}$$

$= \cos^2\theta\cos^2\varphi\, d\theta d\theta \cancel{- 2\cos\theta\cos\varphi\sin\theta\sin\varphi\, d\theta d\varphi}$

$+ \sin^2\theta\sin^2\varphi\, d\varphi d\varphi + \cos^2\theta\sin^2\varphi\, d\theta d\theta$

$\cancel{+ 2\cos\theta\sin\varphi\sin\theta\cos\varphi\, d\theta d\varphi} + \sin^2\theta\cos^2\varphi\, d\varphi d\varphi$

$+ \sin^2\theta\, d\theta d\theta$

$= \cos^2\theta\cos^2\varphi\, d\theta d\theta + \sin^2\theta\sin^2\varphi\, d\varphi d\varphi$

$+ \cos^2\theta\sin^2\varphi\, d\theta d\theta + \sin^2\theta\cos^2\varphi\, d\varphi d\varphi + \sin^2\theta\, d\theta d\theta$

$= \cos^2\theta\,\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{\to 1}\, d\theta d\theta + \sin^2\theta\, d\theta d\theta$

$+ \sin^2\theta\,\underbrace{(\sin^2\varphi + \cos^2\varphi)}_{\to 1}\, d\varphi d\varphi$

$= \underbrace{(\cos^2\theta + \sin^2\theta)}_{\to 1}\, d\theta d\theta + \sin^2\theta\, d\varphi d\varphi$

$= d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi = g_{\mu\nu}\, d\theta \otimes d\varphi$

Homework: Let $f: T^2 \longrightarrow \mathbb{R}^3$ be an embedding of the two-forms T^2 into $\mathbb{R}^3$ defined by

$$f(\theta,\varphi) \longmapsto ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta)$$

where $R > r$. Show that the induced metric is

$$= r^2\, d\theta \otimes d\theta + (R + r\cos\theta)^2\, d\varphi d\varphi$$

Connections

Let V and W be vector fields on a pseudo-Riemannian manifold M. We want to define a new vector rate of change of W in the V_p-direction

In $\mathbb{R}^n$: $$\frac{\partial W^\mu}{\partial x^\nu} := \lim_{\Delta x \to 0} \frac{W^\mu(\ldots, x^\nu + \Delta x^\nu, \ldots) - W^\mu(\ldots, x^\nu, \ldots)}{\Delta x^\nu}$$

It works in $\mathbb{R}^n$ but there is no natural way to parallel transport a vector on a manifold!

Axiomatise it!

Definition: An affine connection ∇ on a smooth manifold M is a function

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$(X, Y) \longrightarrow \nabla_X Y$$

such that

I. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ — $\mathcal{F}(M)$ linear in first entry

II. $\nabla_X (aY + bZ) = a\nabla_X Y + b\nabla_X Z$ — $\mathbb{R}$-linear in second entry

III. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$ — Not a tensor!

$\therefore \nabla_X Z$ is called the covariant derivative of Z with respect to X for the connection ∇.

Theorem: On a pseudo-Riemann manifold M there is a unique connection ∇ such that besides (I) to (III) it follows

IV. $[V,W] = \nabla_V W - \nabla_W V$ — Torsion-free condition.

V. $X\langle V,W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$ — Isometric Condition

$\forall X,V,W \in \mathfrak{X}(M)$.

Notes:

- $\langle X,Y\rangle =: g(X,Y)$
- Torsion tensor $\quad \mathcal{T}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$

∇ + (I) to (V) $\longrightarrow$ Levi-Civita connection on M.