Volume of a hypersphere

$$V_n(R) = \int \cdots \int dx_1 \cdots dx_n$$

$$0 \leq x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2$$

Como $V_n(R) \sim R^n \longrightarrow V_n(R) = A_n R^n$

$$\longrightarrow \delta V_n(R) = n A_n R^{n-1} \delta R = S_n(R)\, \delta R$$

area of the hypersphere of radius R.

$$\Omega_n(R; \delta R) = S_n(R)\, \delta R = C_n R^{n-1} \delta R$$

with $C_n := n A_n$

volumen of the hyperspheric shell of radius R and width.

Let

$$\left[\int_{-\infty}^{\infty} \exp(-ax^2)\, dx \right]^n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-ax_1^2 - \cdots - ax_n^2)\, dx_1 \cdots dx_n = \left(\frac{\pi}{a} \right)^{n/2}.$$

In the other hand,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-ax_1^2 - \cdots - ax_n^2)\, dx_1 \cdots dx_n = \int_0^{\infty} \exp(-aR^2)\, n A_n R^{n-1}\, dR$$

Let $x = aR^2$, then $dx = 2aR\,dR$

$$\int_0^{\infty} \exp(-x)\, n A_n \left(\frac{x}{a} \right)^{\frac{n-1}{2}} \frac{\sqrt{a}}{2a\sqrt{x}}\, dx$$

$$= \frac{n A_n}{2a^{n/2}} \int_0^{\infty} x^{\frac{n}{2}-1} \exp(-x)\, dx = \frac{n A_n}{2a^{n/2}} \Gamma\left(\frac{n}{2} \right)$$

therefore,

$$\left(\frac{\pi}{a} \right)^{n/2} = \frac{n A_n}{2a^{n/2}} \Gamma\left(\frac{n}{2} \right) \longrightarrow n A_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2} \right)}$$

$$\longrightarrow C_n = n A_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\longrightarrow \Omega_n(R;\delta R) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} R^{n-1}\delta R$$

if $n = 3N \longrightarrow C_{3N} = \dfrac{2\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}\right)} = \dfrac{2\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}+1-1\right)}$

$$= \frac{2\pi^{3N/2}}{\sqrt{2\pi\left(\frac{3N}{2}-1\right)}\left(\frac{3}{2}N-1\right)^{(3/2N-1)} e^{-(3/2N-1)}\left\{1+\frac{1}{12(3/2N-1)}+\dots\right\}}$$

$\approx N^{-3/2N} b^N \quad ; \quad b :=$ constant.

$$\Gamma(x+1) = \sqrt{2\pi x}\, x^x e^{-x}\left\{1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{138}{51840x^3}+\dots\right\}$$