Homework: Isotropic coordinates

Slices $t =$ constant

$\Sigma :=$ Euclidean 3-Space

We want $ds^2 = -A(r)\,dt^2 + B(r)\,d\sigma^2$

$$d\sigma^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2$$

$r \longmapsto \rho = \rho(r)$, and assume

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + [\lambda(\rho)]^2 (d\rho^2 + \rho^2 d\Omega^2)$$

such that

$$\lambda^2 \rho^2 = r^2$$

and

$$\lambda^2 d\rho^2 = \frac{1}{1 - \frac{2m}{r}}\,dr^2$$

If and only if

$$\frac{d\rho^2}{\rho^2} = \frac{dr^2}{r^2\left(1 - \frac{2m}{r}\right)} = \frac{dr^2}{r^2 - 2mr}$$

If and only if

$$\frac{d\rho}{\rho} = \pm \frac{dr}{\sqrt{r^2 - 2mr}}$$

show that

$$ds^2 = -\frac{\left(1 - \frac{1}{2}\frac{m}{\rho}\right)^2}{\left(1 + \frac{1}{2}\frac{m}{\rho}\right)^2}dt^2 + \left(1 + \frac{1}{2}\frac{m}{\rho}\right)^4 [d\rho^2 + \rho^2 d\Omega^2]$$

Also, show that the isotropic form of Schwarzschild admits the Killing vector fields

$$\frac{\partial}{\partial t}, \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

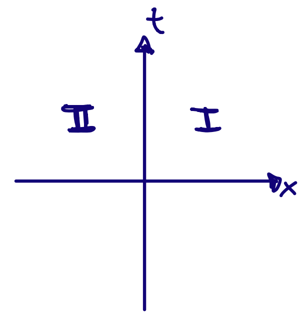
and find all the commotators!

Kruskal extension of Schwarzschild

I. Consider a 2D metric

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2 \qquad \begin{array}{l} x \in \mathbb{R} \\ t \in \mathbb{R}^+ \end{array}$$

As $t \longrightarrow 0^+$ seems to contain a singularity



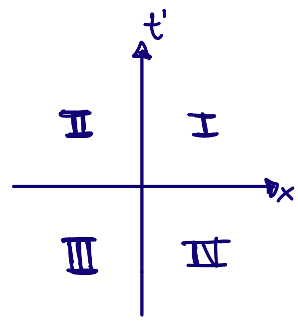
$$t \longmapsto t' = \frac{1}{t}$$

$$dt' = -\frac{1}{t^2}\,dt$$

$$dt'^2 = \frac{1}{t^4}dt^2$$

$$ds^2 = -dt'^2 + dx^2$$

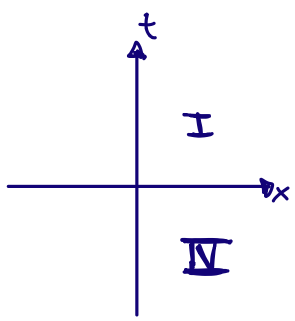
flat 2D-metric



$t' > 0$ portion of Minkowski spacetime.

Then, the apparent singularity $t=0$ represents $t' \longrightarrow \infty$ in Minkowski, and thus it is not singular at all.

II. Rindler spacetime



$$g_{\mu\nu} = \begin{pmatrix} -x^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ds^2 = -x^2dt^2 + dx^2 \qquad t \in \mathbb{R},\ x \in \mathbb{R}^+$$

$\hookrightarrow$ "Singularity" at $x=0$

since $\det g\big|_{x=0} = 0 \longrightarrow g^{\mu\nu} \longrightarrow \infty$ at $x \longrightarrow 0$

For all geodesics

$$0 = g_{ab}K^aK^b = -x^2\dot{t}^2 + \dot{x}^2$$

dot means derivative with respect to an affine parameter.

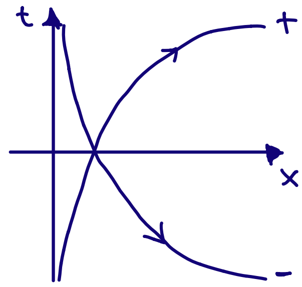
$$x^2\dot{t}^2 = \dot{x}^2$$

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{x^2}\left(\frac{dx}{d\tau}\right)^2$$

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2}$$

$$\frac{dt}{dx} = \pm \frac{1}{x}$$

$$t = \pm \ln x + \text{Constant}$$



"+" outgoing geodesics

"−" ingoing geodesics

Define null coordinates

$$u := t - \ln x$$

$$v := t + \ln x$$

$$e^{v-u} = e^{(t+\ln x)-(t-\ln x)} = e^{2\ln x} = x^2$$

$$du = dt - \frac{1}{x} dx$$

$$dv = dt + \frac{1}{x} dx$$

$$du\,dv = dt^2 - \frac{1}{x^2} dx^2$$

$$ds^2 = -x^2 dt^2 + dx^2 = -e^{v-u} du\,dv$$

$$g_{\mu\nu} = \begin{pmatrix} 0 & -e^{v-u} \\ -e^{v-u} & 0 \end{pmatrix}$$

Note that $u, v \in \mathbb{R}$, then corresponds to $x > 0$ Rindles spacetime

Choose now

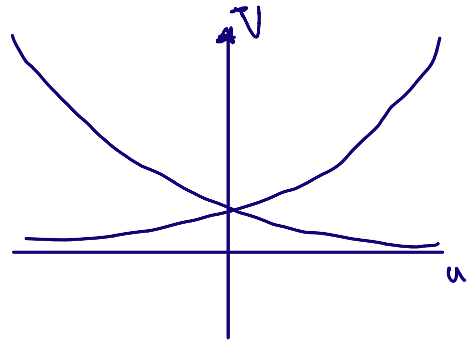
$$U = U(u) := -e^{-u}$$

$$V = V(v) := e^{v}$$

$$g(U,V) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\det g = -1.$$

$$dU = e^{-u}\,du$$
$$dV = e^{v}\,dv$$
$$ds^2 = -dU\,dV$$



Original Rindler $\longrightarrow$ $U<0,\ V>0$

as ds^2 contains no longer any singularity at $U=0=V$.

Then, we may extend the spacetime by allowing $U\in\mathbb{R}$, $V\in\mathbb{R}$

One last coordinate transformation

$$T := \frac{U+V}{2} \qquad X := \frac{V-U}{2}$$

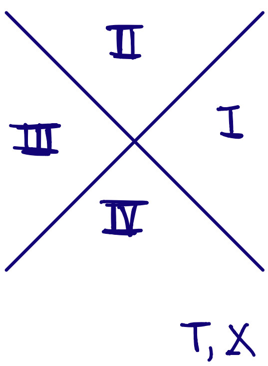
$$dT = \frac{dU+dV}{2} \qquad dX = \frac{dV-dU}{2}$$

$$-dT^2 + dX^2 = -dU\,dV = ds^2$$

Minkowski

$T, X \in \mathbb{R}$

Going backwards: $(x,t) \longmapsto (X,T)$

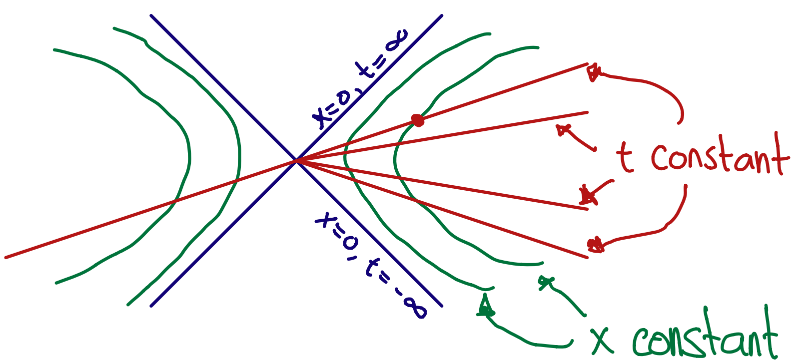


$$x = (X^2 - T^2)^{1/2}$$

$$t = \tanh^{-1}\left(\frac{T}{X}\right) = \frac{1}{2}\ln\left(\frac{T+X}{T-X}\right)$$

$$X + T = V \qquad X = x\cosh t$$

$$X - T = -U \qquad T = x\sinh t$$



Rindler spacetime $X>|T|$ Region I $x>0$, $T\in\mathbb{R}$.

Null lines $X=\pm T$ are mislabeled by $x=0$, $t=\pm\infty$.