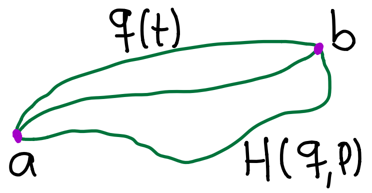
Path integral



$$\langle b|a\rangle = \lim_{N\to\infty}\int \prod_{i=1}^{N} \frac{dp_i}{2\pi}\, dq_i\, e^{i\left(p_i \frac{\delta q_i}{\delta t} - H\right)\delta t}$$

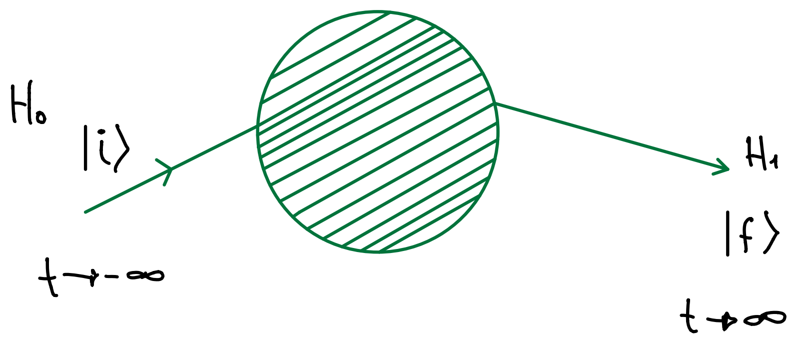
$$= \int Dp(t)\, Dq(t)\, e^{i\int_a^b dt(p\dot{q} - H)}$$

then, $\langle b|a\rangle \propto \int Dq(t)\, e^{iS[q]}$

$$S[q] = \int dt\, \mathcal{L}(q,\dot{q})$$

Dispersion Problem:

$H = H_0 + H_1$



$A_{fi} = ?$

S matrix:

$$S = \mathcal{T} \exp\left(-i\int_{-\infty}^{\infty} dt\, H_I(t)\right)$$

$$S_{fi} = \langle f|S|i\rangle$$

Then,

$$\langle f|e^{-iH\tau}|i\rangle = \int dq_f\, dq_i \langle f|q_f\rangle\langle q_f|e^{-iH\tau}|q_i\rangle\langle q_i|i\rangle$$

$$= \int dq_f\, dq_i\, \varphi_f^*(q_f)\langle q_f|e^{-iH\tau}|q_i\rangle \varphi_i^*(q_i)$$

In particular: $Z = \langle 0|e^{-iH\tau}|0\rangle$ dispersion of amplitude of $|0\rangle \to |0\rangle$.

Path integral: $Z \equiv \int Dq(t)\, e^{iS[q]}$

Quantization of the scalar field:

$$q \to \varphi : \; L \longrightarrow \int d^3x \, \mathcal{L} \to S[\varphi] = \int d^4x \, \mathcal{L}(\varphi)$$

$$\varphi \to \hat{\varphi} : \; \langle q_a, t_a | q_b, t_b \rangle \longrightarrow \langle \varphi_b(\bar{x}, t'') | \varphi_a(\bar{x}', t') \rangle$$

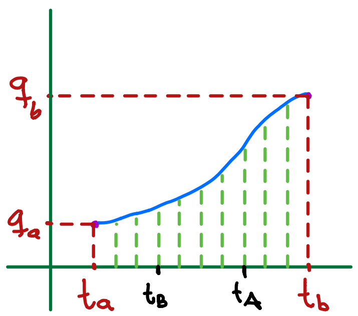
$$\lim_{\substack{t' \to -\infty \\ t'' \to \infty}} : \int D\pi \, D\varphi \, e^{i\int d^4x \mathcal{L}(\varphi)}$$

therefore, $\quad Z \propto \int D\varphi \, e^{iS[\varphi]}$

$$\langle 0 | T(\hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n)) | 0 \rangle = ?$$

Consider

$$\langle q_b | t_b | Q(t_A) Q(t_B) | q_a, t_a \rangle \qquad t_A > t_B$$



$$= \int \prod_i dq_i \langle q_b, t_b | q_{N-1}, t_{N-1} \rangle \cdots \langle q_{A+1}, t_{A+1} | Q_{H_A} | q_A, t_A \rangle$$

$$\times \langle q_{B+1}, t_{B+1} | Q(t_B) | q_B, t_B \rangle \cdots \langle q_1, t_1 | q_a, t_a \rangle$$

$$= \int \prod_i dq_i \; \underbrace{q_A \, q_B}_{\text{permutables}} \langle q_b, t_b | q_{N-1}, t_{N-1} \rangle \cdots \langle q_1, t_1 | q_a, t_a \rangle$$

$$= \langle q_b, t_b | Q(t_B) Q(t_A) | q_a, t_a \rangle \qquad t_B > t_A$$

therefore,

$$\langle q_b, t_b | T[Q(t_A) \, | \, Q(t_B)] | q_a, t_a \rangle = \int D_p D_q \, q(t_A) q(t_B) e^{iS(q,p)}$$

In general,

$$\langle b | T(Q(t_n) \cdots Q(t_1)) | a \rangle = \int D_p D_q \, q(t_1) \cdots q(t_n) e^{iS(q,p)}$$

$$\downarrow$$

$$\langle 0 | \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) | 0 \rangle = \int D\pi \, D\varphi \, \varphi(x_1) \cdots \varphi(x_n) \, e^{iS[\varphi]}$$

with fixed boundary conditions: $\varphi(t \to \pm\infty)$

then, $\int D\pi$ can generally be "incorporated" into a normalization factor.

Definition:

$$W[J] = \int D\pi \, D\varphi \, e^{i\int d^4x \left(\mathcal{L} + J(x)\varphi(x)\right)}$$

$$\equiv {}_{out}\langle 0 | 0 \rangle^{J}_{in}$$

by definition, we require:

1. $W[J=0] = 1$ (Normalisation), is enough with consider in general

$$W[J] \equiv \frac{1}{Z}\int D\varphi \, e^{iS[\varphi,J]} \; ; \quad Z \equiv \int D\varphi \, e^{iS[\varphi]} = \left[\int D\pi\right]^{-1}$$

II.

$$\left.\frac{\delta W}{\delta J(x)}\right|_{J=0} = \int D\pi \, D\varphi \, \frac{\delta}{\delta J(x)} \left. e^{iS[\varphi,J]}\right|_{J=0}$$

$$= \int D\pi \, D\varphi \, (i)\left(\frac{\delta S[\varphi,J]}{\delta J(x)}\right) \left. e^{iS[\varphi,J]}\right|_{J=0}$$

but,
$$\frac{\delta S}{\delta J(x)} = \varphi(x) \longrightarrow \left.\frac{\delta W}{\delta J(x)}\right|_{J=0} = i\int D\pi \, D\varphi \; \varphi(x) e^{iS[\varphi]}$$

Iterate:
$$\left.\frac{\delta^n W[J]}{\delta J(x_n)\cdots\delta J(x_1)}\right|_{J=0} = (i)^n \langle 0 | T(\varphi(x_1)\cdots\varphi(x_n)) | 0 \rangle$$

then,
$$G^{(n)}(x_1,\ldots,x_n) = (-i)^n \left.\frac{\delta^n W[J]}{\delta J(x_n)\cdots\delta J(x_1)}\right|_{J=0}$$

$W[J]$: Generating function:

$$W[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1\cdots d^4x_n \left.\frac{\delta^n W[J]}{\delta J(x_n)\cdots\delta J(x_1)}\right|_{J=0} J(x_1)\cdots J(x_n)$$

$$\langle 0 | 0 \rangle^J = W[J] = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^4x_1\cdots d^4x_n (i)^n G^{(n)}(x_1,\ldots,x_n) J(x_1)\cdots J(x_n)$$