Feynman rules in energy-momentum space.

Fourier transform of quantities in coordinates space.

$$\Rightarrow \mathcal{K}(p_1 t_1, p_0 t_0) = \int \exp\left(\frac{-i}{\hbar} p_1 \cdot x_1\right) K(x_1 t_1; x_0 t_0) \exp\left(\frac{i}{\hbar} p_0 \cdot x_0\right) dx_0 dx_1$$

Amplitude for observaying a particle with conditions $p_1 t_1$ if one has observed a particle with conditions $p_0 t_0$.

For the free propagator

$$\mathcal{K}(p_1 t_1, p_0 t_0) = \Theta(t_1 - t_0) \left(\frac{2\alpha}{i}\right)^{3/2} \int \exp\left[\frac{i}{\hbar}(p_0 x_0 - p_1 x_1)\right] \exp\left[\frac{im(x_1 - x_0)}{2\hbar(t_1 - t_0)}\right] dx_0 dx_1$$

$$\alpha = \frac{m}{2\hbar(t_1 - t_0)}$$

Change variables

$$x := x_0 - x_1 \quad \text{and} \quad X := x_0 + x_1$$

$$p := p_0 - p_1 \quad \text{and} \quad P := p_0 + p_1$$

such that

$$\begin{aligned} P \cdot x + p \cdot X &= (p_0 + p_1) \cdot (x_0 - x_1) + (p_0 - p_1) \cdot (x_0 + x_1) \\ &= p_0 x_0 \cancel{- p_0 x_1} \cancel{+ p_1 x_0} - p_1 x_1 + p_0 x_0 \cancel{+ p_0 x_1} \cancel{- p_1 x_0} - p_1 x_1 \\ &= 2(p_0 x_0 - p_1 x_1) \end{aligned}$$

and $dx_0 x_1 = \frac{1}{2} dX dx$.

$$\Rightarrow \mathcal{K}(p_1 t_1, p_0 t_0) = \Theta(t_1 - t_0) \left(\frac{2\alpha}{i}\right)^{3/2} \underbrace{\int \exp\left[\frac{i}{2\hbar}(pX)\right] dX}_{①} \underbrace{\int \exp\left[\frac{i}{2\hbar}(Px)\right] e^{i\alpha x^2} dx}_{②}$$

① $\int \exp\left[\frac{i}{2\hbar}(pX)\right] dX = 8(2\pi\hbar)^3 \delta p = 8(2\pi\hbar)^3 \delta(p_0 - p_1)$

momentum conservation

② $\displaystyle\int \exp\left[\frac{i}{2\hbar}(P\cdot x)\right]e^{i\alpha x^2}dx = \int \exp\left[\frac{i}{2\hbar}(P\cdot x + i\alpha x^2)\right]dx$

$$\int_{-\infty}^{\infty} \exp(-ax^2+bx+c)dx = \exp\left(\frac{b^2}{4a}+c\right)\left(\frac{\pi}{a}\right)^{1/2}$$

$$= \exp\left[\frac{-P^2/4\hbar^2}{4(-i\alpha)}\right]\left(\frac{\pi}{-i\alpha}\right)^{3/2}$$

$$\Longrightarrow \mathcal{K}(p_1t_1, p_0t_0) = (2\pi\hbar)^3\theta(t_1-t_0)\delta(p_0-p_1)\exp\left(\frac{-ip_0^2(t_1-t_0)}{2m\hbar}\right)$$

Inverse

$$K_0(X_1t_1; X_0t_0) = \frac{1}{(2\pi\hbar)^6}\int \exp\left[\frac{i}{\hbar}(p_1\cdot x_1)\right]\mathcal{K}_0(p_1t_1;p_0t_0)\exp\left[\frac{-i}{\hbar}(p_0\cdot X_0)\right]dp_1dp_0$$

$$= \theta(t_1-t_0)\frac{1}{(2\pi\hbar)^3}\int \exp\left[\frac{i}{\hbar}\left((q\cdot(X_1-X_0)) - \frac{q^2}{2m}(t_1-t_0)\right)\right]dq$$

Now, consider Fourier on t-dependence

$$K_0(p_1\varepsilon_1; p_0\varepsilon_0) = \exp\left(\frac{i}{\hbar}\varepsilon_1t_1\right)\mathcal{K}_0(p_1t_1;p_0t_0)\exp\left(\frac{-i}{\hbar}\varepsilon_0t_0\right)dt_0t_1$$

$$= (2\pi\hbar)^3\delta(p_0-p_1)\int\theta(\tau)\exp\left(\frac{-ip_0^2\tau}{2m\hbar}\right)\exp\left[\frac{i}{\hbar}(\varepsilon_1t_1-\varepsilon_0t_0)\right]dt_0dt_1$$

$\tau := t_1 - t_0$ change $t_1 \longmapsto \tau$

$$K_0 = (2\pi\hbar)^3\delta(p_0-p_1)\underbrace{\int\exp\left(\frac{-i}{\hbar}(\varepsilon_0-\varepsilon_1)t_0\right)dt_0}_{①}\underbrace{\int\theta(\tau)\exp\left[\frac{i}{\hbar}\left(\varepsilon_1-\frac{p_0^2}{2m}\right)\tau\right]d\tau}_{②}$$

① $\displaystyle\int \exp\left(\frac{-i}{\hbar}(\varepsilon_0-\varepsilon_1)t_0\right)dt_0 = 2\pi\hbar\,\delta(\varepsilon_0-\varepsilon_1)$

② $\displaystyle\int\theta(\tau)\exp\left[\frac{i}{\hbar}\left(\varepsilon_1-\frac{p_1^2}{2m}\right)\tau\right]d\tau = \int_0^{\infty}\exp\left[\frac{i}{\hbar}\left(\varepsilon_1-\frac{p_1^2}{2m}\right)\tau\right]d\tau.$

define $\omega := \frac{1}{\hbar}\left(E_1 - \frac{p_1^2}{2m}\right)$

$$\Rightarrow \int_0^\infty e^{i\omega\tau}\, d\tau$$

does not converge!

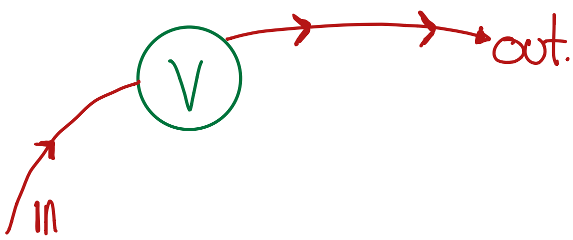
change $\tau \longmapsto i\tau$ Wick rotation.

$\omega \longmapsto \omega + i\varepsilon \qquad \varepsilon > 0\ ;\ \varepsilon \ll 1$

$$\Rightarrow \int_0^\infty e^{i(\omega + i\varepsilon)\tau}\, d\tau.$$

Scattering matrix (S-matrix)

Relation of the initial state and the final state of a system undergoing a scattering process



We want the S matrix unitary!

Initial condition: $\varphi_{in}(x_i t_i)$ plane wave

Assumption: $V \longrightarrow 0$ for large $t < 0$ and $t_1 < t$.

In first born approximation

$$\psi^{(+)}(x_f t_f) = \int K_0(x_f t_f ; x_i t_i)\, \varphi_{in}(x_i t_i)\, dx_i$$

$$- \frac{i}{\hbar} \int K_0(x_f t_f ; x t)\, V(x t)\, K_0(x t ; x_i t_i)\, \varphi_{in}(x_i t_i)\, dx_i\, dx\, dt.$$

(+) means $\psi^{(+)}(x_f t_f)$ corresponds to a wave that was free at $t = -\infty$, then $K_0(xt; x't')$ retarded propagator vanishing for $t' < t$.

we want to find ψ_{out} and

$$S = \int \overline{\psi_{out}(x_f t_f)}\, \psi^{(1)}(x_f t_f)\, dx_f$$

$$= \int \overline{\psi_{out}(x_f t_f)}\, K_0(x_f t_f ; x_i t_i)\, \psi_{in}(x_i t_i)\, dx_i\, dx_f.$$

$$- \frac{i}{\hbar} \int \overline{\psi_{out}(x_f t_f)}\, K_0(x_f t_f ; x t)\, V(x t)\, K_0(x t, x_i t_i)\, \psi_{in}(x_i t_i)\, dx_f\, dx_i\, dx\, dt.$$

$$\psi_{in}(x t) = \frac{1}{\sqrt{\tau}} \exp\left[\frac{i}{\hbar}(p_i \cdot x - E_i t)\right]$$

$$\psi_{out}(x t) = \frac{1}{\tau} \exp\left[\frac{i}{\hbar}(p_f \cdot x - E_f t)\right]$$

$E = \frac{p^2}{2m}$; τ volume of a normalization box.

$$\tau = (2\pi)^3$$

The first integral becomes

$$I_1 := \int \overline{\psi_{out}(x_f t_f)}\, K_0(x_f t_f ; x_i t_i)\, \psi_{in}(x_i t_i)\, dx_i\, dx_f.$$

Also a free plane wave

$$I_1 = \int \frac{1}{\sqrt{2}} \exp\left[\frac{-i}{\hbar}(p_f \cdot x - E_f t)\right] \frac{1}{\sqrt{2}} \exp\left[\frac{i}{\hbar}(p_i \cdot x - E_i t)\right] dx$$

$$= \frac{1}{(2\pi)^3} \int \exp\left[\frac{-i}{\hbar}(p_f - p_i) x - (\cancelto{0}{(E_f - E_i)} t)\right] dx$$

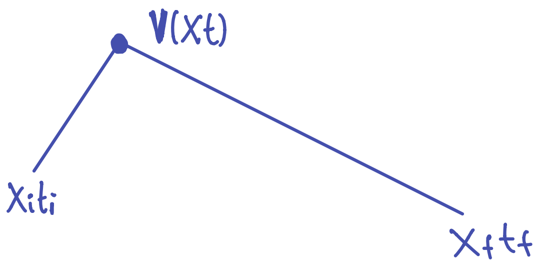
$$= \delta(K_f - K_i)$$

Where $K_f := \frac{p_f}{\hbar}$ and $K_i := \frac{p_i}{\hbar}$

$$S_{fi} = \delta(K_f - K_i) - \frac{i}{\hbar}\int \underbrace{\Psi_{out}(x_f t_f) K_0(x_f t_f; xt) K_0(xt; x_i t_i) \Psi_{in}(x_i t_i)}_{=: A_{fi}} dx_f dx_i dx dt.$$

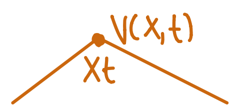
- Scattering matrix.
- No interaction $\longrightarrow$ momentum conservation.
- Interaction (up to first order) $\longrightarrow$ Amplitude that an "out" state results from a particular "in" state.

Feynman rules



a) 

$K_0(X_2 t_2, X_1 t_1)$ Free propagator.

b) 

$-\frac{i}{\hbar} V(x,t)$ and interaction over x and f.

c) finally, we multiply by $\overline{\Psi_{out}}$ and Ψ_{in} at the end of the diagram and integrate over $dx_i dx_f$.