$$F = B + E \wedge dt$$

$$\Rightarrow dF = 0.$$

$$\begin{aligned} dF = d(B + E\wedge dt) &= dB + d(E\wedge dt) \\ &= dB + dE \wedge dt + (-1) E \wedge \cancelto{0}{d(dt)} \\ &= dB + dE\wedge dt \end{aligned}$$

If ω is a differential form,

$$\omega = \omega_I dx^I = \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$d\omega = \partial_\mu \omega_I dx^\mu \wedge dx^I \qquad I - \text{multi-index}, \quad \mu = 0, 1, 2, 3.$$

we can write $d\omega$ as the sum of the spacelike part

$$d_s \omega = \partial_i \omega_I dx^i \wedge dx^I, \qquad i = 1, 2, 3.$$

and the timelike part

$$dt \wedge \partial_t \omega = \partial_0 \omega_I dx^0 \wedge dx^I.$$

Thus,

$$\begin{aligned} dF &= dB + dE \wedge dt \\ &= d_s B + dt \wedge \partial_t B + (d_s E + \cancelto{0}{dt \wedge \partial_t E}) \wedge dt \\ &= d_s B + (\partial_t B + d_s E) \wedge dt = 0 \end{aligned}$$

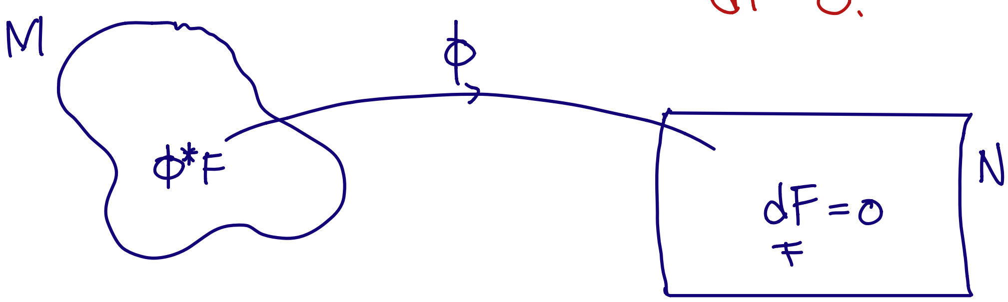
Then

$$d_s B = 0 \longrightarrow \nabla \cdot \vec{B} = 0$$

$$\partial_t B + d_s E = 0 \longrightarrow \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0.$$

$$\Updownarrow$$

$$dF = 0.$$



$$0 = \phi^*(dF) = d(\phi^* F).$$

Metric

In the euclidean space, $\mathbb{R}^3$, we measure distances and angles using the vector inner-product.

$$v \cdot w = v^1 w^1 + v^2 w^2 + v^3 w^3$$

and the norm

$$\|v\|^2 = v \cdot v.$$

In the Minkowski space, we measure distances using a generalization of the inner-product.

$$v \cdot w = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

if $x \in \mathbb{R}^4$

I. $x \cdot x > 0$, spacelike

II. $x \cdot x < 0$, timelike

III. $x \cdot x = 0$, null.

The notion of metric generalizes these concepts:

A semi-Riemannian metric or simplely metric, in a vector space V, is a map

$$g : V \times V \longrightarrow \mathbb{R}$$

is bilinear, i.e., linear in each entrance

$$g(cv + v', w) = c g(v, w) + g(v', w)$$

$$g(v, cw + w') = c g(v, w) + g(v, w')$$

is symmetric

$$g(v, w) = g(w, v)$$

and non-degenerate, if

$$g(v, w) = 0 \quad , \quad \forall w \in V, \quad \text{then} \quad v = 0$$

- v is spacelike if $g(v, v) > 0$.
- v is timelike if $g(v, v) < 0$.
- v is null if $g(v, v) = 0$. orthogonal to each other.

If $g(v,w)=0$ we say that v and w are orthogonal

Let a metric in V, we can find always an orthonormal base of V, i.e., $\{e_\mu\}$ such that $g(e_\mu,e_\nu)=0$ if $\mu\neq\nu$ and ± 1 if $\mu=\nu$.

The number ± 1 is independent of the orthonormal base.

If the number is $+1$ is p and q (-1), we say that it has signature (p,q).

For example the Minkowski space has signature $(3,1)$

$$\eta(v,w)=-v^0w^0+v^1w^1+v^2w^2+v^3w^3.$$