Example: Let X be differentiable vector field $X = x^2 dx$ on $\mathbb{R}$. We want to find the one parameter group associated to X

Answer: First introduce $y = y(x)$ such that $X = \partial y$, then

$$X(y) = x^2 \frac{\partial y}{\partial x} = 1$$

$$dy = \frac{1}{x^2} dx$$

$$y(x) = -\frac{1}{x} + C$$

$$X = \frac{d}{dt} = \frac{dx}{dt} \partial_x = x^2 \partial_x$$

$$\frac{dx}{dt} = x^2$$

$$-\frac{1}{x} = t + C$$

$$x(t) = \frac{-1}{t+C}$$

$$\dot{\sigma}(t) = \frac{d\sigma(t)}{dt} = X_{\sigma(t)} = x^2(t).$$

$$= \left(\frac{-1}{t+C}\right)^2$$

$$\sigma(t) = \frac{-1}{t+C}$$

Take C in $t=0$

$$x(0) = \frac{-1}{C} \quad \text{or} \quad C = \frac{-1}{x}$$

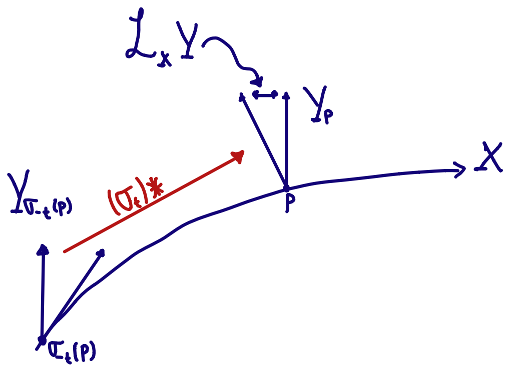
$$\sigma(t) = \frac{1}{x^{-1} - t} = \frac{x}{1 - tx}$$

- $\left.\frac{d\sigma_t(x)}{dt}\right|_{t=0} = \left.\frac{-x(-x)}{(1-tx)^2}\right|_{t=0} = x^2 = X_{\sigma(t)}$
- $(\sigma_t \circ \sigma_s)(x) = \sigma_t(\sigma_s(x)) = \sigma_t\left(\frac{x}{1-xs}\right) = \frac{x/(1-xs)}{1 - t(x/(1-xs))}$

$$= \frac{x}{1-(t+s)x} = \sigma_{t+s}(x).$$

Definition: Let $X \in \mathfrak{X}(M)$ generates a local one-parameter group of transformations σ_t on M. If $Y \in \mathfrak{X}(M)$, we define its Lie derivative along X to be

$$\mathcal{L}_X Y := \lim_{t\to 0} \frac{Y - (\sigma_t)_* Y}{t}$$



Tangent map of σ_t is used to drag the vector field Y forgard along the integral curves from a point $\sigma_{-t}(p)$ to p and the result is compared with the original value Y_p of the vector field Y.

(X_p tangent vector field associated to the flow of σ_t)

$$(\mathcal{L}_X Y)_p(f) = \lim_{t\to 0} \frac{1}{t}\left[Y_p(f) - ((\sigma_t)_* Y)_p(f)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[Y_p(f) - Y_{\sigma_{-t}(p)}(f\circ\sigma_t)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[Y_p f - (Yf)\circ\sigma_{-t}(p) - Y_{\sigma_{-t}(p)}(f\circ\sigma_t - f)\right]$$

but, $X_p(f) = \dfrac{d(f\circ\sigma_t)}{dt}$

then,
$$(\mathcal{L}_X Y)_p(f) = X(Y_p(f)) - Y(X_p(f))$$
$$= [X, Y]_p(f).$$

The Lie derivative may be extend to all tensor fields.

Take any diffeomorphism $\varphi: M \longrightarrow N$, and define an induced map $\tilde{\varphi}: \mathcal{T}^r_s(M) \longrightarrow \mathcal{T}^r_s(M)$, such that

1. For vector fields, set
$$\tilde{\varphi} = \varphi_*$$

II. For scalar fields $f: M \to \mathbb{R}$

$$\tilde{\varphi} f := f \circ \varphi^{-1}$$

III. For covectorfields set

$$\tilde{\varphi} = (\varphi^{-1})^{*}$$ Inverse of pushforward.

IV. Demand linearity

$$\tilde{\varphi}(T \otimes S) = \tilde{\varphi}(T) \otimes \tilde{\varphi}(S)$$

for any tensor fields T and S

These properties imply $\tilde{\varphi}\langle \omega, X \rangle = \langle \tilde{\varphi}\omega, \tilde{\varphi}X \rangle$ for an arbitrary covector ω and vector X fields

$$\begin{aligned}\langle \tilde{\varphi}\omega, \tilde{\varphi}X \rangle (p) &= \langle (\varphi^{-1})^{*} \omega_{\varphi^{-1}(p)}, \varphi_{*} X_{\varphi^{-1}(p)} \rangle \\ &= \langle \omega_{\varphi^{-1}(p)}, X_{\varphi^{-1}(p)} \rangle \\ &= \langle \omega, X \rangle (\varphi^{-1}(p))\end{aligned}$$

Homework: Show that for an arbitrary vector field X and $f \in \mathcal{F}(M)$ is fulfilled

$$\tilde{\varphi} X (\tilde{\varphi}(f)) = \tilde{\varphi}(X(f)).$$

For an arbitrary tensor, thus we have

$$\tilde{\varphi} T(\omega^1, \dots, \omega^r, X_1, \dots, X_s) = \tilde{\varphi} T(\tilde{\varphi}\omega^1, \dots, \tilde{\varphi}\omega^r, \tilde{\varphi}X_1, \dots, \tilde{\varphi}X_s)$$

Definition: The Lie derivative of a smooth tensor field T with respect to vector field X. $\mathcal{L}_X T$ is defined as.

$$\mathcal{L}_X T := \lim_{t \to 0} \frac{1}{t} (T - \tilde{\sigma}_t T)$$

- When T is a scalar field

$$\begin{aligned}(\mathcal{L}_X f)_p &= \lim_{t \to 0} \frac{1}{t} (f - \tilde{\sigma}_t f) \\ &= \lim_{t \to 0} \frac{1}{t} (f - f \circ \sigma_t^{-1}) \\ &= \frac{d f(\sigma_t(p))}{dt} \bigg|_{t=0} = X_p(f)\end{aligned}$$

Or in local coordinates

$$\mathcal{L}_X f = X^i \partial_{x_i} f.$$

- In the basis

$$\mathcal{L}_{\partial x_i} \partial x_j = [\partial x_i, \partial x_j] = \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_j \partial x_i} = 0$$

- $\mathcal{L}_X Y = [X, Y] = -[Y, X] = -\mathcal{L}_Y X$
- $\mathcal{L}_X Y = \mathcal{L}_X (Y^i \partial_{x^i})$

$$= [Y^i{}_{,j} X^j - Y^j X^i{}_{,j}] \partial_{x_i}$$

- $\mathcal{L}_X \langle \omega, Y \rangle = X \langle \omega, Y \rangle = \langle \mathcal{L}_X \omega, Y \rangle + \langle \omega, \mathcal{L}_X Y \rangle$

Proof:

$$\begin{aligned}\mathcal{L}_X \langle \omega, Y \rangle &= X \langle \omega, Y \rangle \\ &= \frac{d}{dt} \tilde{\sigma}_t \langle \omega, Y \rangle \Big|_{t=0} \\ &= \frac{d}{dt} \langle \tilde{\sigma}_t \omega, \tilde{\sigma}_t Y \rangle \Big|_{t=0} \\ &= \left\langle \frac{d}{dt} \tilde{\sigma}_t \omega, \tilde{\sigma}_t Y \right\rangle_{t=0} + \left\langle \tilde{\sigma}_t \omega, \frac{d}{dt} \tilde{\sigma}_t Y \right\rangle_{t=0} \\ &= \langle \mathcal{L}_X \omega, Y \rangle + \langle \omega, \mathcal{L}_X Y \rangle\end{aligned}$$

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- $(\mathcal{L}_X \omega)_j = \langle \mathcal{L}_X \omega, \partial_{x_j} \rangle = \mathcal{L}_X \langle \omega, \partial_{x_j} \rangle - \langle \omega, \mathcal{L}_X \partial_{x_j} \rangle$

but,

$$\begin{aligned}\mathcal{L}_X (\partial_{x_j}) &= -\mathcal{L}_{\partial x_j} X = -[\partial_{x_j}, X^i \partial_{x_i}] \\ &= -[\partial_{x_j}(X^i \partial_{x_i}) - X^i \partial_{x_i}(\partial_{x_j})] \\ &= -(\partial_{x_j} X^i) \partial_{x_i} - \cancel{X^i \partial^2_{x_i x_j}} + \cancel{X^i \partial^2_{x_j x_i}} \\ &= -X^i{}_{,j} \partial_{x_i}\end{aligned}$$

for $\omega = \omega^k dx^k$

$$(\mathcal{L}_X \omega)_j = \mathcal{L}_X \omega_j + \langle \omega, X^i{}_{,j} \partial x_i \rangle$$

j-th component

$$= X^i \frac{\partial \omega_j}{\partial x^i} + \omega_i X^i_{,j}$$

$$\mathcal{L}_X \omega = \left(X^i \frac{\partial \omega_j}{\partial x^i} + \omega_i X^i_{,j} \right) dx^j$$

Extending to a general tensor T of type (r,s)

$$(\mathcal{L}_X T)^{ij\ldots}{}_{kl\ldots} = \frac{\partial}{\partial x^m}\left(T^{ij\ldots}{}_{kl\ldots}\right) X^m - T^{mj\ldots}{}_{kl\ldots} X^i_{,m}$$
$$- T^{im\ldots}{}_{kl\ldots} X^j_{,m} - \ldots + T^{ij\ldots}{}_{ml\ldots} X^m_{,k}$$
$$+ T^{ij\ldots}{}_{km\ldots} X^m_{,l} + \ldots$$

- For the components of a vector field Y

$$(\mathcal{L}_X Y)^i = [X, Y]^i = X^j \partial_{x_j} Y^i - Y^j \partial_{x_j} X^i$$
$$= X^j \nabla_j Y^i - Y^j \nabla_j X^i$$

- For the components of a covector field ω

$$(\mathcal{L}_X \omega)_i = X^j \partial_{x_j} \omega_i + \omega_j \partial_{x_i} X^j$$
$$= X^j \nabla_j \omega_i + \omega_j \nabla_i X^j$$

- For an arbitrary tensor

$$(\mathcal{L}_X T)^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = X^m \nabla_m T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$$
$$- \sum_{i=1}^{k} T^{a_1 \ldots m \ldots a_k}{}_{b_1 \ldots b_l} \nabla_m X^{a_i}$$
$$+ \sum_{i=1}^{l} T^{a_1 \ldots a_k}{}_{b_1 \ldots m \ldots b_l} \nabla_{b_i} X^m$$

- $$(\mathcal{L}_X g)_{ab} = \cancelto{0}{X^m \nabla_m g_{ab}} + g_{mb} \nabla_a X^m + g_{am} \nabla_b X^m$$
$$= \nabla_a (g_{mb} X^m) + \nabla_b (g_{am} X^m)$$
$$= \nabla_a X_b + \nabla_b X_a$$

Definition: Let M be a pseudo-Riemannian manifold with metric g, and $X \in \mathfrak{X}(M)$. If a displacement ϵX (ϵ infinitesimal) generates an isometry, then X is called a Killing vector field.

If $f: x^\mu \longrightarrow x^\mu + \epsilon X^\mu$ is an isometry

$$\frac{\partial}{\partial x^\mu}(x^\sigma + \epsilon X^\sigma)\frac{\partial}{\partial x^\nu}(x^\lambda + \epsilon X^\lambda)\, g_{\sigma\lambda}(x + \epsilon X) = g_{\mu\nu}(x)$$

$$(\delta^\sigma_\mu + \epsilon\, \partial_\mu X^\sigma)(\delta^\lambda_\nu + \epsilon\, \partial_\nu X^\lambda)[g_{\sigma\lambda}(x) + \epsilon X^\alpha \partial_\alpha g_{\sigma\lambda}] = g_{\mu\nu}(x).$$

$$g_{\mu\nu}(x) + \epsilon\underbrace{[\partial_\mu X^\sigma g_{\sigma\nu} + \partial_\nu X^\lambda g_{\mu\lambda} + X^\alpha \partial_\alpha g_{\mu\nu}]}_{=0} + O(\epsilon^2) = g_{\mu\nu}(x)$$

$$\therefore \mathcal{L}_X g_{\mu\nu} = 0 \qquad \text{Killing equation.}$$

Let $\phi_\epsilon: M \to M$ generates the Killing vector X, then $\mathcal{L}_X g = 0$ shows that the local geometry does not change as we move along ϕ_ϵ, therefore X represents the direction of the symmetry of a manifold.

Homework: Show that if ∇ is the Levi-Civita connection, then the Killing equation may be written as

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = \partial_\mu X_\nu + \partial_\nu X_\mu - 2\Gamma^\lambda{}_{\mu\nu} X_\lambda = 0.$$