Example:

$$I(x) = \int_0^x t^{-1/2} e^{-t}\, dt \quad \text{as} \quad x \longrightarrow \infty$$

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt, \quad Re(z) > 0.$$

$$I(x) = \underbrace{\int_0^\infty t^{-1/2} e^{-t}\, dt}_{=:\Gamma(1/2)} - \underbrace{\int_x^\infty t^{-1/2} e^{-t}\, dt}_{=:I_1}$$

$$I_1(x) = -\int_x^\infty t^{-1/2} e^{-t}\, dt = \int_x^\infty t^{-1/2} \frac{d}{dt}(e^{-t})\, dt$$

$$= t^{-1/2} e^{-t}\Big|_x^\infty - \int_x^\infty \frac{d}{dt}(t^{-1/2}) e^{-t}\, dt$$

$$= t^{-1/2} e^{-t}\Big|_x^\infty + \frac{1}{2}\int_x^\infty t^{-3/2} e^{-t}\, dt$$

Leading order.

$$I(x) = \sqrt{\pi} - \frac{e^{-x}}{x^{1/2}} \quad \text{as} \quad x \longrightarrow \infty$$

Repeated integration by parts gives the asymptotic expansion

$$I(x) = \int_0^x t^{-1/2} e^{-t}\, dt \sim \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}}\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^n (1)(3)(5)\cdots(2n-1)}{(2x)^n}\right]$$

as $x \longrightarrow \infty$

Example:

Munroe effect.

$$F(x) = \int_a^b f(t) e^{ixt}\, dt, \quad f(t) \in C^N$$

$$F(x) = \int_a^b f(t)\left(-\frac{i}{x}\right) \frac{d}{dt}\left(e^{ixt}\right) dt$$

$$= \left(-\frac{i}{x}\right) f(t) e^{ixt} \Big|_a^b - \left(-\frac{i}{x}\right) \int_a^b \frac{d}{dt} f(t)\, e^{ixt}\, dt.$$

$$= \frac{i}{x}\left(f(a) e^{iax} - f(b) e^{ibx}\right) + \frac{i}{x} \int_a^b \frac{d}{dt} f(t)\, e^{ixt}\, dt.$$

Integrating by parts successively.

$$F(x) = \sum_{n=0}^{\infty} \left(\frac{i}{x}\right)^{n+1} \left[e^{iax} f^{(n)}(a) - e^{ibx} f^{(n)}(b) \right] + \underbrace{\left(\frac{i}{x}\right)^{N} \int_a^b f^{(N)}(t)\, e^{ixt}\, dt}_{\mathcal{O}\left(\frac{1}{x^N}\right)}$$

Example:

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt \quad , \quad x \longrightarrow 0'$$

(Stielges transform of e^{-t})

$$I(x) = -\int_0^\infty \frac{1}{1+xt} \frac{d}{dt}\left(e^{-t}\right) dt.$$

$$= -\frac{1}{1+xt} e^{-t} \Big|_0^\infty + \int_0^\infty \frac{d}{dt}\left(\frac{1}{1+xt}\right) e^{-t}\, dt.$$

$$= 1 - \int_0^\infty \frac{x}{(1+xt)^2} e^{-t}\, dt$$

$$= 1 + \int_0^\infty \frac{x}{(1+xt)^2} \frac{d}{dt}\left(e^{-t}\right) dt$$

$$= 1 + x\left[\frac{e^{-t}}{(1+xt)^2} \Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(\frac{1}{(1+xt)^2}\right) e^{-t}\, dt \right]$$

$$= 1 - x - x(-2x)\int_0^\infty \frac{1}{(1+xt)^3} e^{-t}\,dt$$

$$= 1 - x + 2x^2 - \dots + (-1)^{n-1}(n-1)!\,x^{n-1} + (-1)^n n!\,x^n \int_0^\infty \frac{e^{-t}}{(1+xt)^{n+1}}\,dt$$

then

$$I(x) \sim \sum_{n=0}^{\infty} (-1)^n n!\,x^n \quad \text{as} \quad x \longrightarrow 0^+$$

Example: Laplace Integrals

$$I(x) := \int_a^b f(t) e^{x\phi(t)}\,dt$$

In the literature they put "$-x$", but they are talking about minimums.

We want $x \longrightarrow \infty$, f and ϕ continuous functions.

$$I(x) = \frac{1}{x}\int_a^b \frac{f(t)}{\phi'(t)} \frac{d}{dt}\left(e^{x\phi t}\right) dt$$

$$= \frac{1}{x}\frac{f(t)}{\phi'(t)} e^{x\phi(t)}\Big|_a^b - \frac{1}{x}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right) e^{x\phi(t)}\,dt.$$

Integrating by parts again will introduce $\frac{1}{x^2}$ terms... and so on!

leading terms

$$I(x) \sim \frac{1}{x}\frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{1}{x}\frac{f(a)}{\phi'(a)} \qquad \text{as} \quad x \longrightarrow \infty$$

Failure of integration by parts:

For laplace type integrals, we expect to get an expansion of the form.

$$I(x) \sim e^{x\phi(b)} \sum_{n=1}^{\infty} A_n x^{-n} \qquad \text{for} \quad x \longrightarrow \infty$$

If not, integration by parts is not a good technique!

Example: $\displaystyle\int_0^\infty e^{-xt^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}$ (Gaussian) $x \longrightarrow \infty$

$$\left(\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right)^2 = \left(\int_{-\infty}^{\infty} e^{-ay_1^2}\,dy_1\right)\left(\int_{-\infty}^{\infty} e^{-ay_2^2}\,dy_2\right)$$

$$= \iint_0^\infty e^{-a(y_1^2+y_2^2)}\,dy_1\,dy_2 = \int_0^{2\pi}\int_0^\infty e^{-ar^2}\,dr\,(r\,d\theta)$$

$$= 2\pi \int_0^\infty e^{-ar^2}\,r\,dr$$

$s := ar^2$

$ds = 2ar\,dr \longrightarrow r\,dr = \dfrac{ds}{2a}$

$$= \frac{\pi}{a}\int_0^\infty e^{-s}\,ds = \frac{\pi}{a}\left(-e^{-s}\right)\Big|_0^\infty = \frac{\pi}{a}$$

then
$$\int_0^\infty e^{-ay^2}\,dy = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$I(x) = \int_0^\infty e^{-xt^2}\,dt = \underbrace{\frac{e^{-xt^2}}{-2xt}\Bigg|_0^\infty}_{\text{Divergent!}} - \frac{1}{x}\int_0^\infty \frac{d}{dt}\left(\frac{1}{-2t}\right)e^{-xt^2}\,dt$$

when $f(t) = 1$, $\phi(t) = -t^2$.

Laplace method

We want to study Laplace Integrals.

$$I(x) = \int_a^b f(t)\,e^{x\phi(t)}\,dt$$

where $f(t)$, $\phi(t)$ are real continuous functions.

Theorem: (Laplace) Assume $\phi(t)$ attains its maximum on $t = c \in (a,b)$ and that $f(c) \neq 0$. Thus, the major contribution to the asymptotic expansion of $I(x)$ $(x \longrightarrow \infty)$ comes from a neighbourhood of the point $t = c$.

We will consider here ϕ has only a maximum and thus

$$\phi'(c)=0 \quad , \quad \phi''(c)<0$$

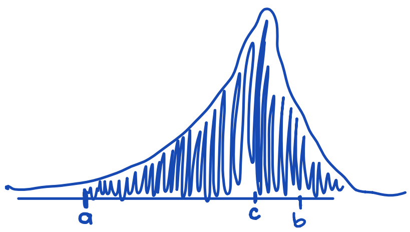
we may taylor expand $\phi(t)$ and $f(t)$ around $t=c$.

$$\phi(t)=\phi(c)+\cancel{\frac{1}{1!}\,\phi'(t)\Big|_{t=c}(t-c)}^{\to 0}+\frac{1}{2!}\,\phi''(t)\Big|_{t=c}(t-c)^2+\ldots$$

by definition.

$$I(x)=\int_a^b f(c)\,e^{x(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2)}\,dt$$

$$=f(c)e^{x\phi(c)}\int_a^b e^{\frac{x}{2}\phi''(t-c)^2}\,dt$$



$$\int_a^b \longmapsto \int_{c-\varepsilon}^{c+\varepsilon} \longmapsto \int_{-\infty}^{\infty}$$

$$=f(c)e^{x\phi(c)}\int_{-\infty}^{\infty} e^{-as^2}\,ds$$

$s:=t-c$

$ds=dt$

$a:=-\frac{1}{2}\phi''(c)$

$$=f(c)e^{x\phi(c)}\frac{\sqrt{2\pi}}{\sqrt{|\phi''(c)|x}}$$

Example:

$$I(\lambda)=\int_{-1}^{1}\frac{\sin(t)}{t}\,e^{-\lambda\cosh(t)}\,dt \quad , \quad \lambda\longrightarrow 0$$

$$f(t)=\frac{\sin(t)}{t} \quad , \quad \phi(t)=-\cosh(t)$$

$\phi'(t)=-\sinh(t)$ $\qquad$ $\phi''(t)=-\cosh(t)$

$\phi'(0)=0$ $\qquad$ $\phi''(0)=-1<0$

$$I(\lambda)\sim(1)\,e^{-\lambda}\frac{\sqrt{2\pi}}{\sqrt{\lambda}}=\frac{\sqrt{2\pi}}{\sqrt{\lambda}}\left(1-\lambda+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\ldots\right)$$

Higher order asymptotics for Laplace Integrals

Take higher order in $f(t)$ Taylor expanded.

$$I(x) = \int_a^b f(t) e^{x\phi(t)} dt \simeq \int_a^b \left(f(c) + f'(c)(t-c) + f''(c)\frac{(t-c)^2}{2!} + \dots \right) e^{x\left(\phi(c) + \phi''(c)\frac{(t-c)^2}{2}\right)} dt$$

$$t \to s = t - c$$

then

$$I(x) \sim e^{x\phi(c)} f(c) \sqrt{\frac{2\pi}{x|\phi''(c)|}} + \cancel{e^{x\phi(c)} \int_a^b f'(c) s e^{\frac{x}{2}\phi''(c)s^2} ds} \to 0$$

$$+ e^{x\phi(c)} \int_a^b \frac{1}{2} f''(c) s^2 e^{\frac{x}{2}\phi''(c)s^2} ds + \dots$$

therefore

$$\int s^2 e^{-as} ds$$

$$u := -\frac{x}{2}\phi''(c)s^2$$

$$du = -x\phi''(c) s\, ds$$

$$s = \sqrt{\frac{-2u}{\phi''(c)x}} = \sqrt{\frac{2u}{x|\phi''(c)|}}$$

$$= \int_0^\infty \frac{\sqrt{2u}\; e^{-u}\, du}{\sqrt{x|\phi''(c)|}\;\; x|\phi''(c)|}$$

$$= \frac{2^{1/2}}{(x|\phi''(c)|)^{3/2}} \int_0^\infty u^{1/2} e^{-u} du = \frac{2^{1/2}}{(x|\phi''(c)|)^{3/2}} \Gamma\left(\frac{3}{2}\right) = \sqrt{\frac{\pi}{2}} \frac{1}{(x|\phi''(c)|)^{3/2}}$$

$$\therefore \quad I(x) = e^{x\phi(c)} \left[\frac{f(c)\sqrt{2\pi}}{(x|\phi''(c)|)^{1/2}} + \sqrt{\frac{\pi}{2}} \frac{1}{(x|\phi''(c)|)^{3/2}} + \dots \right]$$