Ex:

$$I(x) = \int_0^x t^{-1/2} e^{-t} \, dt \quad \text{as} \quad x \to \infty$$

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \, dt, \quad Re(z) > 0.$$

$$I(x) = \underbrace{\int_0^\infty t^{-1/2} e^{-t} \, dt}_{=: \Gamma(1/2)} - \underbrace{\int_x^\infty t^{-1/2} e^{-t} \, dt}_{=: I_1}$$

$$I_1(x) = -\int_x^\infty t^{-1/2} e^{-t} \, dt = \int_x^\infty t^{-1/2} \frac{d}{dt}(e^{-t}) \, dt$$

$$= t^{-1/2} e^{-t} \Big|_x^\infty - \int_x^\infty \frac{d}{dt}(t^{-1/2}) e^{-t} \, dt$$

$$= t^{-1/2} e^{-t} \Big|_x^\infty + \frac{1}{2} \int_x^\infty t^{-3/2} e^{-t} \, dt$$

Leading order

$$I(x) = \sqrt{\pi} - \frac{e^{-x}}{x^{1/2}} \quad \text{as} \quad x \to \infty$$

Repeated integration by parts gives the asymptotic expansion.

$$I(x) = \int_0^x t^{-1/2} e^{-t} \, dt \sim \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} \left[ 1 - \sum_{n=1}^\infty \frac{(-1)^n \, (1)(3)(5)\cdots(2n-1)}{(2x)^n} \right]$$

as $x \to \infty$

Ex: Munroe effect

$$F(x) = \int_a^b f(t)\, e^{ixt}\, dt \qquad ; \quad f(t) \in C^N$$

$$F(x) = \int_a^b f(t) \left(\frac{-i}{x}\right) \frac{d}{dt}\left(e^{ixt}\right) dt$$

$$= \left(\frac{-i}{x}\right) f(t)\, e^{ixt} \Big|_a^b - \left(\frac{-i}{x}\right) \int_a^b \frac{df(t)}{dt}\, e^{ixt}\, dt$$

$$= \frac{i}{x}\left(f(a)\, e^{iax} - f(b)\, e^{ibx}\right) + \frac{i}{x} \int_a^b \frac{df(t)}{dt}\, e^{ixt}\, dt.$$

Integrating by parts successively

$$F(x) = \sum_{n=0}^{\infty} \left(\frac{i}{x}\right)^{n+1} \left[e^{iax} f^{(n)}(a) - e^{ibx} f^{(n)}(b)\right] + \underbrace{\left(\frac{i}{x}\right)^N \int_a^b f^{(N)}(t)\, e^{ixt}\, dt}_{O\left(\frac{1}{x^N}\right)}$$

Ex

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt \qquad x \to 0^+$$

(Stielges transform of $e^{-t}$)

$$I(x) = -\int_0^\infty \frac{1}{1+xt} \frac{d}{dt}\left(e^{-t}\right) dt$$

$$= -\frac{1}{1+xt}\, e^{-t} \Big|_0^\infty + \int_0^\infty \frac{d}{dt}\left(\frac{1}{1+xt}\right) e^{-t}\, dt$$

$$= 1 - \int_0^\infty \frac{x}{(1+xt)^2}\, e^{-t}\, dt$$

$$= 1 + \int_0^\infty \frac{x}{(1+xt)^2} \frac{d}{dt}(e^{-t})\, dt$$

$$= 1 + x\left[ \frac{e^{-t}}{(1+xt)^2} \Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(\frac{1}{(1+xt^2)}\right) e^{-t}\, dt \right]$$

$$= 1 - x - x(-2x)\int_0^\infty \frac{1}{(1+xt)^3} e^{-t}\, dt$$

$$= 1 - x + 2x^2 - \dots + (-1)^{n-1}(n-1)!\, x^{n-1}$$

$$+ (-1)^n n!\, x^n \int_0^\infty \frac{e^{-t}}{(1+xt)^{n+1}}\, dt.$$

then

$$I(x) \sim \sum_{n=0}^\infty (-1)^n n!\, x^n \quad \text{as} \quad x \to 0^+$$

Ex: Laplace Integrals.

In literature they put "$-x$", but they are talking about minimums.

$$I(x) := \int_a^b f(t)\, e^{x\phi(t)}\, dt$$

we want $x \to \infty$, $f$ and $\phi$ continuous functions.

$$I(x) = \frac{1}{x}\int_a^b \frac{f(t)}{\phi'(t)} \frac{d}{dt}\left(e^{x\phi(t)}\right) dt$$

$$= \frac{1}{x}\frac{f(t)}{\phi'(t)} e^{x\phi(t)} \Big|_a^b \underbrace{- \frac{1}{x}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right) e^{x\phi(t)}\, dt}$$

Integrating by parts again will introduce $\frac{1}{x^2}$ terms ... and so on!

Leading term

$$I(x) \sim \frac{1}{x}\frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{1}{x}\frac{f(a)}{\phi'(a)} \quad \text{as} \quad x \to \infty$$

## Failure of integration by parts.

For Laplace type integrals, we expect to get an expansion of the form

$$I(x) \sim e^{x\phi(b)} \sum_{n=1}^{\infty} A_n x^{-n} \quad \text{for} \quad x \to \infty$$

If not, integration by parts is not a good technique!

Ex:

$$\int_0^{\infty} e^{-xt^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{x}} \qquad \text{(Gaussian)} \qquad x \to \infty$$

$$\left(\int_{-\infty}^{\infty} e^{-ay^2} dy\right)^2 = \left(\int_{-\infty}^{\infty} e^{-ay_1^2} dy_1\right)\left(\int_{-\infty}^{\infty} e^{-ay_2^2} dy_2\right)$$

$$= \iint_0^{\infty} e^{-a(y_1^2+y_2^2)} dy_1 dy_2 = \int_0^{2\pi}\int_0^{\infty} e^{-ar^2} dr\,(r\,d\theta)$$

$$= 2\pi \int_0^{\infty} e^{-ar^2} r\,dr$$

$s := ar^2$

$ds = 2ar\,dr \quad dr = \frac{ds}{2a}$

$$= \frac{\pi}{a}\int_0^{\infty} e^{-s} ds = \frac{\pi}{a}\left(-e^{-s}\right)_0^{\infty} = \frac{\pi}{a}$$

then $\displaystyle\int_0^{\infty} e^{-ay^2} dy = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$$I(x) = \int_0^\infty e^{-xt^2} dt = \frac{e^{-xt^2}}{-2xt}\Bigg|_0^\infty - \frac{1}{x}\int_0^\infty \frac{d}{dt}\left(\frac{1}{-2t}\right) e^{-xt^2} dt$$

Divergent!

when $f(t) = 1$, $\phi(t) = -t^2$

## 1) Laplace Method.

We want to study Laplace Integrals

$$I(x) = \int_a^b f(t)\, e^{x\phi t} dt$$

$f(t)$ and $\phi(t)$ real and continuous functions

**Thm:** (Laplace)

Assume $\phi(t)$ attains its maximum on $t = c \in (a,b)$ and that $f(c) \neq 0$. Thus, the major contribution to the asymptotic expansion of $I(x)$ $(x \to \infty)$ comes from a neighbourhood of the point $t = c$.

We will consider here $\phi$ has only a maximum an thus

$$\phi'(c) = 0 \quad \text{and} \quad \phi''(c) < 0$$

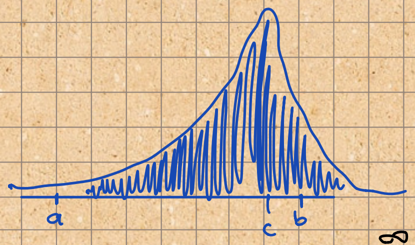
We may Taylor expand $\phi(t)$ and $f(t)$ around $t = c$.

$$\phi(t) = \phi(c) + \underbrace{\cancel{\frac{1}{1!}\phi'(t)\Big|_{t=c}(t-c)}}_{\text{by def.}}^{\to 0} + \frac{1}{2!}\phi''(t)\Big|_{t=c}(t-c)^2 + \ldots$$

$$I(x) = \int_a^b f(c)\, e^{x\left(\phi(c) + \frac{1}{2}\phi''(c)(t-c)^2\right)} dt$$

$$= f(c)\, e^{x\phi(c)} \int_a^b e^{\frac{x}{2}\phi''(t-c)^2}\, dt$$

$$\int_a^b \longmapsto \int_{c-\varepsilon}^{c+\varepsilon} \longmapsto \int_{-\infty}^{\infty}$$




$$= f(c)\, e^{x\phi(c)} \int_{-\infty}^{\infty} e^{-as^2}\, ds$$

$s := t - c$

$ds = dt$

$a := -\frac{1}{2}\phi''(c)$

$$= f(c)\, e^{x\phi(c)} \frac{\sqrt{2\pi}}{\sqrt{|\phi''(c)|x}}$$

Ex:

$$I(\lambda) = \int_{-1}^{1} \frac{\sin(t)}{t} e^{-\lambda \cosh t}\, dt \quad , \lambda \to 0$$

$f(t) = \frac{\sin(t)}{t}$ $\qquad \phi(t) = -\cosh(t)$

$\phi'(t) = -\sinh(t)$ $\quad , \quad \phi''(t) = -\cosh(t)$

$\phi(0) = -1$ $\qquad \phi''(0) = 0$ $\qquad \phi'' = -1 < 0$

$$I(\lambda) \sim (1)\, e^{-\lambda} \frac{\sqrt{2\pi}}{\sqrt{\lambda}} = \frac{\sqrt{2\pi}}{\sqrt{\lambda}}\left(1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \dots\right)$$

Higher order asymptotics for Laplace integrals.

Take higher order in $f(t)$ Taylor expanded

$$I(x)=\int_a^b f(t)e^{x\phi(t)}dt \simeq \int_a^b \left(f(c)+f'(c)(t-c)+f''(\phi)\frac{(t-c)^2}{2!}+\dots\right)e^{x(\phi(c)+\frac{1}{2}\phi''(c)(t-c)^2)}dt$$

$t \to s = t-c.$

then

$$I(x)\sim e^{x\phi(c)} f(c)\sqrt{\frac{2\pi}{x|\phi''(c)|}} + \cancel{e^{x\phi(c)}\int_a^b f'(c)\, s\, e^{\frac{x}{2}\phi''(c)s^2} ds} \to 0$$

$$+\, e^{x\phi(c)}\int_a^b \frac{1}{2} f''(c)s^2 e^{\frac{x}{2}\phi''(c)s^2} + \dots$$

therefore.

$$\int s^2 e^{-as^2} ds$$

$u := -\frac{x}{2}\phi''(c)s^2$

$du = -x\,\phi''(c)\,s\,ds$

$$=\int_0^\infty \frac{\sqrt{2u}}{\sqrt{x|\phi''(c)|}}\, e^{-u}\, \frac{du}{x|\phi''(c)|}$$

$s=\sqrt{\frac{-2u}{x\phi''(c)}}=\sqrt{\frac{2u}{x|\phi''(c)|}}$

$$=\frac{2^{1/2}}{(x|\phi''(c)|)^{3/2}}\int_0^\infty u^{1/2}e^{-u}du = \frac{2^{1/2}}{(x|\phi''(c)|)^{3/2}}\,\Gamma(3/2)$$

$$=\sqrt{\frac{\pi}{2}}\,\frac{1}{(x|\phi''(c)|)^{3/2}}$$

$$\therefore\quad I(x)=e^{x\phi(c)}\left[\frac{f(c)\sqrt{2\pi}}{(x|\phi''(c)|)^{1/2}}+\sqrt{\frac{\pi}{2}}\,\frac{1}{(x|\phi''(c)|)^{3/2}}+\dots\right]$$