

We are interested in a particular integral representative of Weyl ordering

$$Q_n^\omega(f(p,q)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} du dv \hat{f}(u,v) e^{-\frac{i}{\hbar}(u\hat{p}+v\hat{q})}; u, v \in \mathbb{R}^n, f \in A_c.$$

where

$$\hat{f}(u,v) := \mathcal{F}^{-1}[f(p,q)] = \int_{\mathbb{R}^n} dq dp f(p,q) e^{i(u p + v q)}$$

is the Fourier inverse of $f(p,q)$.

Homework: Check that Q_n^ω obeys the properties of Weyl's ordering.

Example:

$$Q_n^\omega(pq) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dq dp pq e^{i(u p + v q)} e^{-i(u\hat{p} + v\hat{q})}$$

Using Backer-Campbell-Hausdorff (BCH) formula.

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X,Y] + \dots)$$

↳ Check.

and defining $X = -iv\hat{q}$, $Y = -iu\hat{p}$.

$$\exp(-iv\hat{q}, -iu\hat{p}) = \exp\left(\frac{1}{2}(-iu)(-iv)[\hat{q}, \hat{p}]\right) \exp(-iv\hat{q}) \exp(-iu\hat{p}).$$

$$= \exp\left(\frac{iuv}{2}\right) \exp(-iv\hat{q}) \exp(-iu\hat{p}) \psi(q)$$

$$[\exp(-iu\hat{p})\psi](q) = \psi(q + u)$$

$$[\exp(-iv\hat{q})\psi](q) = \exp(-ivq)\psi(q).$$

$$Q_n^\omega(pq) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq (pq) e^{i(u p + v q)} e^{\frac{iuv}{2}} e^{-iv\hat{q}} e^{-iu\hat{p}}$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq (pq) e^{\frac{iuv}{2}} e^{i(v(q-\hat{q}))} e^{i(u(p-\hat{p}))}$$

Change $q \mapsto q' = q + \frac{u}{2}$, $dq' = dq$.

$$\begin{aligned}
\Rightarrow Q_h^\omega(p, q) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq' P\left(q' - \frac{u}{2}\right) e^{\frac{iuv}{2}} e^{iv(q' - \frac{u}{2} - \hat{q})} e^{iu(p - \hat{p})} \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq' P\left(q' - \frac{u}{2}\right) e^{iv(q' - \hat{q})} e^{iu(p - \hat{p})} \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq' pq' e^{iv(q' - \hat{q})} e^{iu(p - \hat{p})} \\
&\quad - \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq' \frac{pu}{2} e^{iv(q' - \hat{q})} e^{iu(p - \hat{p})} \\
&= \int_{\mathbb{R}^2} dp dq' pq' \delta(q' - \hat{q}) \delta(p - \hat{p}) - \frac{1}{2\pi} \int_{\mathbb{R}^3} du dp dq' p \frac{u}{2} e^{iu(p - \hat{p})} \delta(q' - \hat{q}) \\
&= \hat{p} \hat{q} - \frac{1}{2\pi} \int_{\mathbb{R}^3} du dp p \frac{u}{2} e^{iu(p - \hat{p})} \\
&\quad \hat{p} \frac{u}{2} e^{iu(p - \hat{p})} = \frac{p}{2i} \frac{\partial}{\partial p} e^{iu(p - \hat{p})} \\
\Rightarrow \frac{1}{2\pi} \int du dp \frac{p}{2i} \frac{\partial}{\partial p} e^{iu(p - \hat{p})} &= \frac{1}{2\pi} \int dp \frac{p}{2i} \frac{\partial}{\partial p} \int du e^{iu(p - \hat{p})} \\
&= \int dp \frac{p}{2i} \frac{\partial}{\partial p} \delta(p - \hat{p}) = - \int dp \frac{\partial}{\partial p} \left(\frac{p}{2i}\right) \delta(p - \hat{p}) \\
&= -\frac{1}{2i} \quad \therefore Q_h^\omega(p, q) = \hat{p} \hat{q} + \frac{1}{2i}
\end{aligned}$$

$$qp \mapsto \frac{\hat{p}\hat{q} + \hat{q}\hat{p}}{2}$$

$$[\hat{q}, \hat{p}] \mapsto \hat{q}\hat{p} - \hat{p}\hat{q}$$

$$\begin{aligned}
 \hat{q}\hat{p} &\longmapsto \hat{p}\hat{q} + [\hat{q}, \hat{p}] \\
 q\hat{p} &\longmapsto \frac{\hat{p}\hat{q} + \hat{p}\hat{q} + [\hat{q}, \hat{p}]}{2} \\
 &= \hat{p}\hat{q} + \left[-\frac{i\hbar}{2} \right] = \hat{p}\hat{q} + \frac{i}{2\hbar}
 \end{aligned}$$

Define the Weyl transform

$$\begin{aligned}
 Q_h^\omega : L^1(\mathbb{R}^2) &\longrightarrow \mathcal{L}(\mathcal{H}) \\
 Q_h^\omega(f) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(u, v) e^{-i(u\hat{p} - v\hat{q})} du dv
 \end{aligned}$$

\tilde{f} := Fourier inverse of f .

which must be understood in the weak sense. For $\psi_1, \psi_2 \in \mathcal{H}$.

$$(Q_h^\omega(f)\psi_1, \psi_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} du dv \tilde{f}(u, v) (e^{-i(u\hat{p} + v\hat{q})} \psi_1, \psi_2)$$

Sense of convergence in the norm of Hilbert space. Let's evaluate $(\hat{A}\psi)(z)$

$$(\hat{A}\psi)(z) = (Q_h^\omega(A)\psi)(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq A(p, q) e^{i(vq + up + \frac{uq}{2})} e^{-ivz} \psi(z-u).$$

Integral in v brings $\delta(q + \frac{u}{2} - z)$

$$\Rightarrow (\hat{A}\psi)(z) = \frac{1}{2\pi} \int_{\mathbb{R}^3} du dp dq A(p, q) \delta(q + \frac{u}{2} - z) e^{iup} \psi(z-u)$$

Integrating in q .

$$(\hat{A}\psi)(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} du dp A(p, z - \frac{u}{2}) e^{iup} \psi(z-u)$$

change $u \mapsto x = z-u$; $dx = -du$.

$$(\hat{A}\psi)(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} dx dp A(p, \frac{z+x}{2}) e^{i(z-x)p} \psi(x)$$

$$(\hat{A}\psi)(z) = \int_{\mathbb{R}} dx \left(\frac{1}{2\pi} \int_{\mathbb{R}} dp A\left(p, \frac{z+x}{2}\right) e^{i(z-x)p} \right) \psi(x)$$

$$=: K_{\hat{A}}(z, x) \quad \text{Integral Kernel.}$$

$$= \int_{\mathbb{R}} dx K_{\hat{A}}(z, x) \psi(x)$$

We may use this Kernel to obtain an expression for A (The symbol of \hat{A}).

$$K_{\hat{A}}(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} dp A\left(p, \frac{x+y}{2}\right) e^{i(x-y)p}$$

Change

$$X = \frac{x+y}{2}; \quad z := x-y$$

$$x = X + \frac{z}{2}, \quad y = X - \frac{z}{2}$$

$$K_{\hat{A}}\left(X + \frac{z}{2}, X - \frac{z}{2}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} dp A(p, X) e^{ipz}$$

Inverting Fourier

$$\int K_{\hat{A}}\left(X + \frac{z}{2}, X - \frac{z}{2}\right) e^{-ipz} = \frac{1}{2\pi} \int_{\mathbb{R}^2} dp dz A(p, X) e^{ipz} e^{-ipz}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} dp dz A(p, X) e^{-i(p-p)z} = \int_{\mathbb{R}} dp A(p, X) \delta(p - P)$$

$$= A(P, X) = \text{Symbol}[\hat{A}]$$

We can go back and forth between operators (Kernels) and symbols on phase space!

Wigner function.

Consider the projector operator \hat{P}_{ψ} that projects on the state ψ

$$(\hat{P}_{\psi}\phi)(z) = \psi(z) \int_{\mathbb{R}} dy (\psi, \phi)(y) = \psi(z) \int_{\mathbb{R}} \overline{\psi(y)} \phi(y) dy$$

its integral Kernel is given by

$$K_{\hat{P}_\Psi}(x, y) = \Psi(x) \overline{\Psi(y)}$$

Its symbol will be called the **wigner function**.

$$\omega[\Psi](p, q) := \text{sym}[\hat{P}_\Psi](p, q)$$

$$\omega[\Psi](p, q) = \int dz \Psi\left(q + \frac{z}{2}\right) \overline{\Psi\left(q - \frac{z}{2}\right)} e^{-ipz}$$

In particular

$$\frac{1}{2\pi} \int_{\mathbb{R}} dp \omega[\Psi](p, q) = \Psi(q) \overline{\Psi(q)} = |\Psi(q)|^2$$

$$\frac{1}{2\pi} \int_{\mathbb{R}} dp \omega[\Psi](p, q) = |g(p)|^2$$

where $g(p) := \frac{1}{2\pi} \tilde{\Psi}(p)$ \longleftarrow Fourier transform of Ψ .

We may generalize wigner of two states Ψ, Φ that is the weyl-wigner symbol of the kernel $\Psi(x) \overline{\Phi(y)}$.

Theorem: for an arbitrary operator \hat{A} , we can obtain expectation values by the following formula.

$$(\Psi, \hat{A}_\Psi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega[\Psi](p, q) A(p, q) dp dq.$$