Example:

$$\varphi(z) = z - \frac{z^3}{3}$$

$$\varphi'(z) = 1 - z^2 \qquad \varphi'(z) = 0 \quad \text{at} \quad z = \pm 1$$

$$\varphi''(z) = -2z \qquad \varphi''(1) = -2 \qquad \varphi''(-1) = 2.$$

Critical points at $z_0 = \pm 1$ of order $m = 2$.

Valleys $\cos(\theta m + \psi) < 0$

$$\varphi''(z) = a e^{i\psi}$$

- For $z_0 = 1 \rightarrow \varphi''(1) = -2 \rightarrow \psi = \pi$
- For $z_0 = -1 \rightarrow \varphi''(-1) = 2 \rightarrow \psi = 0.$

Steepest descent: $\theta m + \psi = (2k+1)\pi \quad , \; k = 0, 1.$

$$\theta = -\frac{\psi}{m} + \frac{(2k+1)\pi}{m}$$

- For $z_0 = 1$

$$\theta = -\frac{\pi}{2} + \frac{(2k+1)\pi}{2} \qquad \theta_0 = 0$$

$$\theta_1 = \pi$$

- For $z_0 = -1$

$$\theta_0 = \frac{\pi}{2} \quad , \quad \theta_1 = \frac{3\pi}{2}$$

$$\varphi(x,y) = u(x,y) + i v(x,y).$$

$$v(x,y) = \mathrm{Im}\left(z - \frac{z^3}{3}\right)$$

$$z = x + iy \qquad \varphi(x,y) = \left(x + iy - \frac{(x+iy)^3}{3}\right)$$

$$= x + iy - \frac{(x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3)}{3}$$

$$= \left(x - \frac{x^3}{3} - xy^2 \right) + i \left(y - x^2 y + \frac{y^3}{3} \right)$$

$$V(x,y) = \operatorname{Im}\left(z - \frac{z^3}{3} \right) = y\left(1 - x^2 + \frac{y^2}{3} \right) = V(1,0) = 0$$

$\uparrow$ For $z_0 = 1$.

In a neighbourhood of $z_0 = 1$, the curves of steepest ascent / descent are

a) $y = 0$ b) $1 - x^2 + \frac{y^2}{3}$

As $U(x,y) = x\left(1 - \frac{x^2}{3} + y^2 \right) \Rightarrow y = 0$ steepest descent.

$$U(x,y)\Big|_{y=0} = x\left(1 - \frac{x^2}{3} \right)$$

$$U(x,y)\Big|_{1-x^2+\frac{y^2}{3}} = x\left(1 - \frac{x^2}{3} + 3x^2 - 3 \right) = x\left(-2 + \frac{8}{3}x^2 \right)$$

Analogously, for $z_0 = -1$., $V(x,y) = y\left(1 - x^2 + \frac{y^2}{3} \right) = V(-1,0) = 0$

$$U(x,y)\Big|_{y=0} = x\left(1 - \frac{x^2}{3} \right)$$

$$U(x,y)\Big|_{1-x^2+\frac{y^2}{3}} = x\left(-2 + \frac{8}{3}x^2 \right)$$

Example:

$$\varphi(z) = \cosh(z) - \frac{z^2}{2}.$$

$\varphi'(z) = \operatorname{senh}(z) - z$ $\qquad \varphi'(z) = 0$ iff $z = 0.$

$\varphi''(z) = \cosh(z) - 1$ $\qquad \varphi''(0) = 0.$

$\varphi'''(z) = \sinh(z)$ $\qquad \varphi'''(0) = 0$

$\varphi^{(4)}(z) = \cosh(z)$ $\qquad \varphi^{(4)}(0) = 1 \neq 0.$

$\therefore$ $\varphi(z)$ has a critical point at $z_0 = 0$ of order $m = 4$

$$\varphi^{(4)}(z) = \cosh(z) = \cosh(z)\, e^{i\varphi} \qquad \varphi = 0.$$

$m\theta + \varphi = (2k+1)\pi$ $\qquad$ steepest descent.

$k = 0, 1, 2, 3.$

$$\theta = -\frac{\varphi}{m} + \frac{2k+1}{m}\pi$$

$$\theta_0 = \frac{\pi}{4}, \quad \theta_1 = \frac{3\pi}{4}, \quad \theta_2 = \frac{5\pi}{4}, \quad \theta_3 = \frac{7\pi}{4}$$

$$\cosh(x+iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$$

$$\varphi(x,y) = \cosh(x+iy) - \frac{(x-iy)^2}{2}$$

$$= \underbrace{\left(\cosh(x)\cos(y) - \frac{x^2}{2} + \frac{y^2}{2}\right)}_{=:\,U(x,y)} + i\underbrace{\left(\sinh(x)\sin(y) - xy\right)}_{=:\,V(x,y)}$$

$$V(x,y) = \operatorname{Im}(\varphi(z)) = \sinh(x)\sin(y) - xy = V(0,0) = 0$$

$$U(x,y) = \operatorname{Re}(\varphi(z)) = \cosh(x)\cos(y) - \frac{x^2}{2} + \frac{y^2}{2} = U(0,0) = 1.$$

Example:

$$I(\lambda) = \int_0^1 e^{i\lambda z} \log(z)\, dz.$$

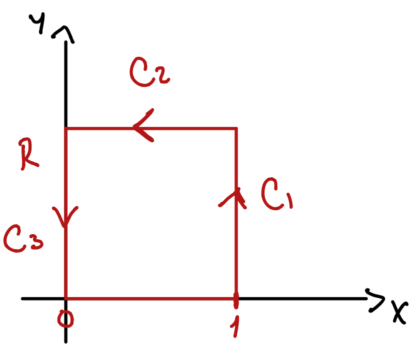
$$\Rightarrow \varphi(z) = iz = i(x+iy) = -y + ix.$$

$$\varphi'(z) = i \neq 0 \qquad \text{No saddle-points.}$$

steepest descent / ascent. paths are given by

$v = \operatorname{Im}(\varphi(z)) = x = \text{constant}.$

$u = \operatorname{Re}(\varphi(z)) = -y$ For $y>0$ the curves $x=\text{constant}$ are steepest descent paths



we deform the path as no continuous steepest descent path passes through the two end points.

$$\Rightarrow \int_0^1 e^{i\lambda z} \log(z)\, dz = -\int_{C_1+C_2+C_3} e^{i\lambda z} \log(z)\, dz.$$

- for C_2: $z = x + iR \Rightarrow \int_{C_2} = -\int_0^1 \log(x+iR)\, e^{i\lambda(x+iR)}\, dx$

and
$$\left| \int_{C_2} \right| \leq e^{-R\lambda} \int_0^1 |\log(x+iR)|\, dx$$

$$\Rightarrow \left| \int_{C_2} \right| \text{ goes to zero as } R \to \infty$$

• for C_1:

$$z = 1 + iy \qquad dz = i\,dy$$

$$\int_{C_1} = i\int_0^R dy\, \log(1+iy)\, e^{i\lambda(1+iy)}$$

$$= i e^{i\lambda} \int_0^R \log(1+iy)\, e^{-\lambda y}\, dy$$

• for C_3:

$$z = iy \qquad dz = i\,dy$$

$$\int_{C_3} = -i\int_0^R dy\, \log(iy)\, e^{i\lambda i y} = -i\int_0^R \log(iy)\, e^{-\lambda y}\, dy$$

$$I(\lambda) \overset{R\to\infty}{=\!=\!=} \underbrace{i\int_0^\infty \log(iy)\, e^{-\lambda y}\, dy}_{=:I_1} \underbrace{- i e^{i\lambda}\int_0^\infty \log(1+iy)\, e^{-\lambda y}\, dy}_{=:I_2}$$

For I_1:

$$\log(iy) = \log(i) + \log(y) = \log(y) + \log e^{i\pi/2}$$

$$= \log(y) + i\frac{\pi}{2} \qquad \omega = \lambda y.$$

$$I_1 = i\int_0^\infty e^{-\lambda y}\left(\log(y) + i\frac{\pi}{2}\right) dy = \frac{-\pi}{2\lambda} + \frac{i}{\lambda}\int_0^\infty e^{-\omega} \log\left(\frac{\omega}{\lambda}\right) d\omega$$

$$= -\frac{\pi}{2\lambda} + \frac{i}{\lambda}\int_0^\infty e^{-\omega}\log(\omega)\,d\omega - \frac{i}{\lambda}\log(\lambda)\int_0^\infty e^{-\omega}\,d\omega$$

$$= -\frac{\pi}{2\lambda} - \frac{i}{\lambda}\log(\lambda).$$

$$\int_0^\infty e^{-\omega}\log\omega\, d\omega = -\gamma \qquad \text{Euler-Mascheroni constant.}$$

- for I_2: Use watson's Lemma.

$$I_2 = i e^{i\lambda} \int_0^\infty \left(\sum_{n=1}^\infty \frac{(-1)^{n-1} (iy)^n}{n} \right) e^{-\lambda y} \, dy.$$

$$\sim e^{i\lambda} \sum_{n=1}^\infty \frac{(-i)^{n+1} \Gamma(n+1)}{n \lambda^{n+1}}$$

$$\therefore I(\lambda) \sim -\frac{i \log(\lambda)}{\lambda} - \frac{2\pi}{\lambda} - \frac{i\gamma}{\lambda} - e^{i\lambda} \sum_{n=1}^\infty \frac{(-i)^{n+1} \Gamma(n+1)}{n \lambda^{n+1}}$$

Example.

$$I(\lambda) = \int_{-\infty}^\infty \frac{e^{i\lambda(z + z^3/3)}}{2z^2+1} \, dz$$

$$\varphi(z) = i\left(z + \frac{z^3}{3} \right)$$

$\varphi'(z) = i(1+z^2)$ $\qquad$ $\varphi'(z) = 0$ iff $z_0 = \pm i$

Saddle points at $z_0 = \pm i$

$\varphi''(z) = 2iz$ $\qquad$ $\varphi''(i) = -2$, $\varphi''(-i) = 2$

order $M=2$.

Directions of steepest descent:

$z_0 = i$ $\qquad$ $\varphi''(i) = -2 \to \psi = \pi$

$$2\theta + \pi = (2k+1)\pi \quad , k=0,1.$$

$$\theta = -\frac{\pi}{2} + \frac{2k+1\pi}{2}$$

$$\theta_0 = 0 \quad , \quad \theta_1 = \pi$$

$$z_0 = -i \quad \to \quad \varphi''(-i) = 2 \qquad \psi = 0.$$

$$\theta = (2k+1)\frac{\pi}{2} \quad , \quad k = 0, 1.$$

$$\theta_0 = \frac{\pi}{2} \quad , \quad \theta_1 = \frac{3\pi}{2}$$

Now, set $z = Re^{i\phi}$ $\to$ $z + \frac{z^3}{3} = Re^{i\phi} + \frac{R^3 e^{3i\theta}}{3}$

then, for large R.

$$e^{i\lambda(z + z^3/3)} \sim O(e^{\frac{-\lambda R^3}{3}\sin(3\phi)})$$ that decays as long as $\sin(3\phi) > 0$.

$$\sin(3\phi) > 0 \quad \to \quad 0 < 3\phi < \pi \quad \text{or} \quad 0 < \phi < \frac{\pi}{3}$$

Steepest descent paths satisfy

$$V(x,y) = \text{Im}(\varphi(z)) = x\left(1 + \frac{x^2}{3} - y^2\right)$$

$$U(x,y) = \text{Re}(\varphi(z)) = y\left(\frac{y^2}{3} - 1 - x^2\right)$$

Steepest descent are either $x = 0$ or $1 + \frac{x^2}{3} - y^2 = 0$

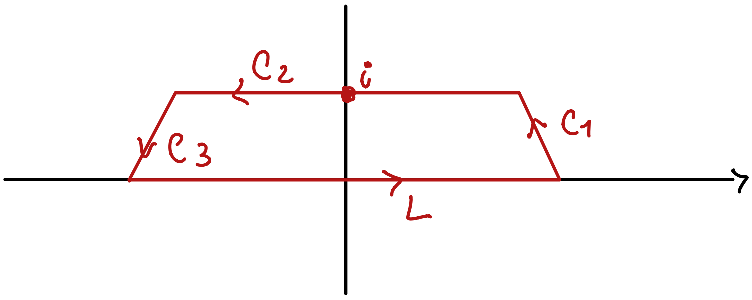
$$U_{x=0} = y\left(\frac{y^2}{3} - 1\right)$$

$$U_{1+\frac{x^2}{3}-y^2=0} = y\left(-\frac{8y^2}{3} + 2\right)$$

for $z_0 = i$ $(x_0 = 0, y_0 = 1)$ steepest descent $1 + \frac{x^2}{3} - y^2 = 0$

for $z_0 = -i$ $(x_0 = 0, y_0 = -1)$ steepest descent $x = 0$.

We deform the original contour to the upper-half plane on the steepest descent path through $t=i$



$$\oint_{L+C_1+C_2+C_3} \frac{e^{i\lambda\left(z+\frac{z^3}{3}\right)}}{2z^2+1}\,dz = 2\pi i \,\mathrm{Res}\left(z=\frac{i}{\sqrt{2}}\right)$$

$$2\pi i\,\mathrm{Res}\left(z=\frac{i}{\sqrt{2}}\right) = 2\pi i\left(\frac{e^{i\lambda\left(\frac{i}{\sqrt{2}}+\left(\frac{i}{\sqrt{2}}\right)^3\frac{1}{3}\right)}}{4\left(\frac{i}{\sqrt{2}}\right)}\right) = \frac{\sqrt{2}}{2}\pi e^{i\lambda\left(\frac{i}{\sqrt{2}}-\frac{i}{6\sqrt{2}}\right)}$$

$$= \frac{\sqrt{2}}{2}\pi e^{-5\lambda/6\sqrt{2}}$$

$$\int_L = \cancel{\int_{C_1}}^{\to 0} - \int_{C_2} - \cancel{\int_{C_3}}^{\to 0} + \frac{\sqrt{2}}{2}\pi e^{-5\lambda/6\sqrt{2}} \quad \text{as } R\to\infty$$

for C_2:

$$z = i+t \qquad , t\in\mathbb{R}$$

$$z+\frac{z^3}{3} = (i+t)+\frac{(i+t)^3}{3}$$

$$= i+t+\frac{i^3+3i^2t+3it^2+t^3}{3}$$

$$= \left(\cancel{t}-\cancel{t}+\frac{t^3}{3}\right)+i\left(1-\frac{1}{3}+t^2\right)$$

$$= \frac{t^3}{3}+i\left(\frac{2}{3}+t^2\right)$$

$$i\lambda\left(z+\frac{z^3}{3}\right) = \frac{it^3\lambda}{3} - \lambda\left(\frac{2}{3}+t^2\right)$$

This term oscillates.

Also, as $t \to 0$ $\quad \frac{1}{1+2z^2} \longrightarrow \frac{1}{-2+1} = -1$

and (up to leading order) the oscillates term goes to 1.

$$\to \int_{-\infty}^{\infty} \frac{e^{i\lambda(z+z^3/3)}}{2z^2+1}\,dz \sim \int_{-\infty}^{\infty} \frac{e^{-2\lambda/3}\,e^{-\lambda t^2}}{(-1)}\,dt + \frac{\sqrt{2}}{2}\pi e^{-5\lambda/\sqrt{2}\,6}$$

$$\sim -\frac{\sqrt{\pi}}{\sqrt{\lambda}} e^{-2\lambda/3} + \frac{\sqrt{2}}{2}\pi e^{-5\lambda/\sqrt{2}\,6}. \qquad \text{as } \lambda \to \infty$$