Free bosons in normal region ($\mu < 0$)

$$\frac{1}{V}\ln(\Xi(\beta,V,z)) = -\frac{1}{V}\ln(1-z) - \frac{1}{V}\sum_{j\neq 0}\ln(1-z\exp(-\beta\varepsilon_j))$$

in the thermodynamic limit

$$\frac{1}{V}\ln(\Xi(\beta,V,z)) \longrightarrow -\gamma C\int_0^\infty \varepsilon^{1/2}\ln(1-z\exp(-\beta\varepsilon))\,d\varepsilon$$

Remember that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$$= \gamma C\int_0^\infty \varepsilon^{1/2}\left(z\exp(-\beta\varepsilon) + \frac{1}{2}z^2\exp(-2\beta\varepsilon) + \ldots\right)d\varepsilon$$

$$= \gamma C\sum_{n=1}^\infty\int_0^\infty \varepsilon^{1/2}\frac{z^n}{n}e^{-n\beta\varepsilon}\,d\varepsilon = \gamma C\sum_{n=1}^\infty\frac{z^n}{n}\frac{\Gamma(1/2+1)}{(n\beta)^{1/2+1}}$$

Remember that

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}},$$

and that

$$\Gamma\left(\frac{1}{2}+m\right) = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m}\sqrt{\pi}$$

$$= \gamma C(k_BT)^{3/2}\sum_{n=1}^\infty\frac{z^n}{n^{5/2}}\frac{\sqrt{\pi}}{2} = \frac{\gamma\sqrt{\pi}}{8\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_BT)^{3/2}\sum_{n=1}^\infty\frac{z}{n^{5/2}}$$

$$= \gamma\frac{1}{8\pi^{3/2}}\left(\frac{2(2\pi)^2 m k_BT}{h^2}\right)^{3/2}\sum_{n=1}^\infty\frac{z}{n^{5/2}}$$

$$= \gamma\left(\frac{2\pi m k_BT}{h^2}\right)^{3/2}\sum_{n=1}^\infty\frac{z}{n^{5/2}} = \frac{\gamma}{\lambda^3}\sum_{n=1}^\infty\frac{z^n}{n^{5/2}}$$

$$\lambda := \frac{h}{(2\pi m k_BT)^{1/2}}$$ wave thermic length

Then, we define

$$g_\alpha(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$$

$$g_\alpha(1) = \zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

Riemann zeta function

then

$$\frac{1}{V} \ln(\Xi(\beta, V, z) = \frac{\gamma}{\lambda^3} g_{5/2}(z)$$

$$N = z \frac{\partial}{\partial z} \ln(\Xi(\beta, V, z)) = z V \frac{\gamma}{\lambda^3} \frac{\partial}{\partial z} g_{5/2}(z) = \frac{z V \gamma}{\lambda^3} \sum_{n=1}^{\infty} \frac{z^{n-1}}{n^{5/2-1}}$$

$$= \frac{V\gamma}{\lambda^3} \sum_{n=1}^{\infty} \frac{z^n}{n^{3-2}} = \frac{\gamma V}{\lambda^3} g_{3/2}(z)$$

$$U = -\frac{\partial}{\partial \beta} \ln(\Xi(\beta, V, z)) = \frac{3\gamma V}{2\beta\lambda^3} g_{5/2}(z)$$

$$C_V = C_V(T, v) = \frac{1}{N} \left(\frac{\partial U}{\partial T}\right)_{V,N} = -\frac{K_B \beta^2}{N} \left(\frac{\partial U}{\partial \beta}\right)_{V,N}$$

$$\left(\frac{\partial U}{\partial \beta}\right)_N = \frac{\partial(U, N)}{\partial(\beta, N)} = \frac{\partial(U, N)}{\partial(\beta, z)} \frac{\partial(\beta, z)}{\partial(\beta, N)}$$

$$= \left(\frac{\partial U}{\partial \beta}\right)_z - \left(\frac{\partial U}{\partial z}\right)_\beta \frac{\left(\frac{\partial N}{\partial \beta}\right)_z}{\left(\frac{\partial N}{\partial z}\right)_\beta}$$

$$\left(\frac{\partial U}{\partial \beta}\right)_z = -\frac{15\gamma V}{4\beta^2\lambda^3} g_{5/2}$$

Here is β!

$$\left(\frac{\partial U}{\partial z}\right)_\beta = \frac{3}{2} \frac{\gamma V}{\beta \lambda^3 z} g_{3/2}(z)$$

$$\left(\frac{\partial N}{\partial \beta}\right)_z = -\frac{3}{2} \frac{\gamma V}{\beta \lambda^3} g_{3/2}(z)$$

$$\left(\frac{\partial N}{\partial Z}\right)_\beta = \frac{\gamma V}{\lambda^3 Z}\, g_{1/2}(Z)$$

then

$$C_v = \frac{3}{2} K_B \left\{ \frac{5}{2}\frac{g_{5/2}(Z)}{g_{3/2}(Z)} - \frac{3}{2}\frac{g_{3/2}(Z)}{g_{1/2}(Z)} \right\}$$

In the classical limit $g(z) \approx z$

Remember, $g_\alpha(Z) = Z + \frac{Z^2}{2^\alpha} + \frac{Z^3}{3^\alpha} + \dots \approx Z$

In the Bose-Einstein transition $Z=1$ and $T=T_0$.

$$g_{1/2}(1) \to \infty$$
$$g_{3/2}(1) = \zeta\left(\frac{3}{2}\right) = 2.612$$
$$g_{5/2}(1) = \zeta\left(\frac{5}{2}\right) = 1.342$$

then, C_v is finite. Now for the entropy $S(T,V,Z)$.

$$S = -\left(\frac{\partial \phi}{\partial T}\right)_{V,\mu} = V\left(\frac{\partial P}{\partial T}\right)_\mu$$

As $\frac{1}{V}\ln(\Xi(\beta,V,Z)) = \frac{\gamma}{\lambda^3} g_{5/2}(Z)$

$$P = -\left(\frac{\partial \phi}{\partial V}\right) = \frac{1}{\beta}\frac{\partial}{\partial V}\ln(\Xi) = \frac{\gamma}{\lambda^3}\frac{1}{\beta} g_{5/2}(Z) = p(T,\mu)$$

finally,

$$S = \frac{K_B T \gamma V}{\lambda^3}\left\{ \frac{5}{2} g_{5/2}(Z) - g_{3/2}(Z)\ln(Z) \right\}$$

$$= K_B N \left\{ \frac{5}{2}\frac{g_{5/2}(Z)}{g_{3/2}(Z)} - \ln(Z) \right\}.$$