Integral spectrum

Let's consider a partition of (a,b).

$$a < \lambda_0 < m \leq \lambda_1 < \ldots < \lambda_{n-1} \leq M < \lambda_n < b.$$

with norm of the partition $\Delta = \max|\lambda_{i+1} - \lambda_i| < \varepsilon$. We choose any $\mu_i \in [\lambda_i, \lambda_{i+1}]$. Sum the inequalities

$$\lambda_1 (E_{\lambda_2} - E_{\lambda_1}) \leq A(E_{\lambda_2} - E_{\lambda_1}) \leq \lambda_2 (E_{\lambda_2} - E_{\lambda_1})$$

$$\sum_{k=0}^{n-1} \lambda_k (E_{\lambda_{k+1}} - E_{\lambda_k}) \leq A\left(\sum_{k=0}^{n-1} (E_{\lambda_{k+1}} - E_{\lambda_k})\right) = A \leq \sum_{k=0}^{n-1} \lambda_{k+1} (E_{\lambda_{k+1}} - E_{\lambda_k})$$

Let's consider the operator $\sum_{k=0}^{n-1} \mu_k (E_{\lambda_{k+1}} - E_{\lambda_k})$. As $-\varepsilon \leq \lambda_k - \mu_k$ and $\lambda_{k+1} - \mu_k \leq \varepsilon$, we have

$$-\varepsilon I \leq \sum_{k=0}^{n-1} (\lambda_k - \mu_k)(E_{\lambda_{k+1}} - E_{\lambda_k}) \leq A - \sum_{k=0}^{n-1} \mu_k (E_{\lambda_{k+1}} - E_{\lambda_k})$$
$$\leq \sum_{k=0}^{n-1} (\lambda_{k+1} - \mu_k)(E_{\lambda_{k+1}} - E_{\lambda_k}) \leq \varepsilon I.$$

Due to the property, $-\varepsilon I \leq T \leq \varepsilon I$, then $\|T\| \leq \varepsilon$, we have

$$\left\| A - \sum_{k=0}^{n-1} \mu_k (E_{\lambda_{k+1}} - E_{\lambda_k}) \right\| \leq \varepsilon$$

In the limit $\varepsilon \to 0$, and calling to this limit the integral, we can define the notion of spectrum integral

$$A = \int_a^b \lambda \, dE_\lambda = \int_m^M \lambda \, dE_\lambda = \int_{-\infty}^{\infty} \lambda \, dE_\lambda.$$

As $E_\lambda \equiv I$ if $\lambda \geq M$, and $E_\lambda = 0$ if $\lambda = m$; i.e., $dE_\lambda = 0$ out of $[m, M]$

$$A = \sum_{i=1}^{n} \lambda_i P_{\lambda_i}$$

Theorem: (Hilbert) For all self-adjoint bounded operator A in a Hilbert space $\mathcal{H}$ such that $mI \leq A \leq MI$, exists a spectral family $\{E_\lambda\}$, for $\lambda \in \mathbb{R}$ of orthogonal projections such that:

I. $E_\lambda = 0$ if $\lambda < m$ and $E_\lambda = I$, if $\lambda > M$

II. $E_{\lambda+0} = E_\lambda$ Continuity by the right.

III. $E_{\lambda_1} \leq E_{\lambda_2}$ for $\lambda_1 < \lambda_2$

which means that $\{E_\lambda\}$ is a spectral family.

IV.
$$A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda,$$
with the integral converging in the operators norm.

V. E_λ are strong limits of polynomies in A, therefore commute with any operator B that commutes with A.

VI.
$$\|Ax\|^2 = \int \lambda^2 \, d\langle E_\lambda x, x\rangle$$

VII A family $\{E_\lambda\}$ that satisfy (I) - (IV) is unique.

Proof: We have already proved (I) - (V). lets prove (VII).

Note that $\langle E_\lambda x, x\rangle$ is a function of λ, and
$$\int \lambda^2 \, d\langle E_\lambda x, x\rangle$$
may be extend as a Riemann integral.

Going back to the definition of $\int \lambda \, dE_\lambda$
$$\Delta E_{\lambda_i} = E_{\lambda_{i+1}} - E_{\lambda_i}$$
are orthogonal projectors $\Delta E_{\lambda_i} \perp \Delta E_{\lambda_j}$, $i \neq j$. Then
$$\|\Delta E_{\lambda_i} x\|^2 = \langle \Delta E_{\lambda_i} x, x\rangle = \langle E_{\lambda_i} x, x\rangle = \langle E_{\lambda_{i+1}} x, x\rangle - \langle E_{\lambda_i} x, x\rangle$$
As the sum $\sum_i \mu_i \Delta E_{\lambda_i}$ converges to the operator A.

It follows that $\sum_i \mu_i \Delta E_{\lambda_i} x$ converges to Ax, for any x

Then for the Riemann sum
$$\begin{aligned}\|\sum_i \mu_i \Delta E_{\lambda_i} x\|^2 &= \langle \sum_i \mu_i \Delta E_{\lambda_i} x, \sum_j \mu_j \Delta E_{\lambda_j} x\rangle \\ &= \sum_i \mu_i^2 \langle \Delta E_{\lambda_i} x, \Delta E_{\lambda_i} x\rangle \\ &= \|Ax\|^2.\end{aligned}$$

$\blacksquare$

Proposition: Let φ be a continuous function in $[m,M]$. Let's define the integral

$$\int_m^M \varphi(\lambda)\, dE_\lambda$$

as the limit of the sum

$$\sum_i \varphi(\mu_i)\, \Delta E_{\lambda i}$$

when the norm of the partition goes to zero.

The integral converges in the norm of the operator to $\varphi(A)$ that for definition has

$$\varphi(A) = \int_m^M \varphi(\lambda)\, dE_\lambda$$

and

$$\| \varphi(A) x \|^2 = \int_m^M \varphi^2(\lambda)\, d\langle E_\lambda x, x \rangle.$$