Watson's Lemma

$$I(x) = \int_0^\infty e^{-xt} f(t)\,dt \sim \sum_{k=0}^\infty \frac{a_k \Gamma(\rho_k)}{x^{\rho_k}}$$

$$f(x) \sim \sum_{k=0}^\infty a_k t^{\rho_k - 1}$$

Example:

$$I(x) := \int_0^{\pi/4} e^{-xt} \sqrt{1+\cos(t)}\,dt$$

$$f(t) = \sqrt{1+\cos(t)} = \sqrt{2}\cos\left(\frac{t}{2}\right)$$

$$f(t) = \sqrt{2} \sum_{k=0}^\infty \frac{(-1)^k}{2k!} \left(\frac{t}{2}\right)^{2k}$$

$$t^{2k} = t^{\rho_k - 1}$$

$$\rho_k = 2k+1$$

$$a_k = \frac{\sqrt{2}(-1)^k}{(2k)!} \frac{1}{2^k}$$

$$I(x) \sim \sum_{k=0}^\infty \frac{\sqrt{2}(-1)^k}{\cancel{(2k)!}\, 2^{2k}} \frac{\cancel{\Gamma(2k+1)}}{x^{2k+1}}$$

$$I(x) \sim \sqrt{2} \sum_{k=0}^\infty \frac{(-1)^k}{2^{2k}} \frac{1}{x^{2k+1}}$$

$$I(x) \sim \sqrt{2}\left(\frac{1}{x} - \frac{1}{4x^3} + \frac{1}{16x^5}\right) + \mathcal{O}\left(\frac{1}{x^7}\right)$$

Modified Bessel function

Example:

$$K_0(x) = \int_1^\infty (s^2-1)^{1/2} e^{-sx}\,ds$$

Change $s = t+1$

$$K_0(x) = e^{-x} \int_0^\infty (t^2+2t)^{-1/2} e^{-xt}\,dt$$

$$f(t) = (t^2+2t)^{-1/2} = (2t)^{-1/2}\left(1+\frac{t}{2}\right)^{-1/2}$$

Binomial expansion $|t|<2$

$$f(t) = (2t)^{-1/2}\sum_{k=0}^{\infty}\binom{k-1/2}{-1/2}\frac{t^k}{2^k} = (2t)^{-1/2}\sum_{k=0}^{\infty}\frac{(-1)^k(k-1/2)!}{k!\,(-1/2)!}\left(\frac{t}{2}\right)^k$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(k+1/2)}{k!\,\Gamma(1/2)}\,\frac{1}{2^{k+1/2}}\;t^{k-1/2}$$

$$a_k = \frac{(-1)^k\,\Gamma(k+1/2)}{k!\,\sqrt{\pi}}\;\frac{1}{2^{k+1/2}}$$

$$t^{k-1/2} = t^{\rho_k-1}\quad,\qquad \rho_k = k+\frac{1}{2}$$

$$K_0(x) \sim \sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(k+1/2)}{\sqrt{\pi}\,k!\,2^{k+1/2}}\;\frac{\Gamma(k+1/2)}{x^{k+1/2}}$$

$$K_0(x) = \frac{[\Gamma(1/2)]^2}{\sqrt{\pi}\,\sqrt{2}}\,\frac{1}{\sqrt{x}} - \frac{[\Gamma(3/2)]^{3/2}}{\sqrt{\pi}\,(2)^{3/2}}\,\frac{1}{x^{3/2}} + O\left(\frac{1}{x^{5/2}}\right)$$

as $x\to\infty$

Laplace type integrals again

$$I(x) := \int_a^b e^{-x\phi(t)}\,f(t)\,dt$$

(a,b) real interval, x large positive parameter, ϕ and f continuous.

Assume, for simplicity, ϕ has a unique minimum in $[a,b]$ which occurs at $t=a$.

Theorem (Ederlyi): For

$$I(x) = \int_a^b e^{-x\phi(t)}\,f(t)\,dt$$

assume:

I) $\phi(t) > \phi(a)$ for all $t\in(a,b)$, and for every $\delta>0$ the infimum of $\phi(t)-\phi(a)$ in $[a+\delta,b]$ is possitive.

II) $\phi'(t)$ and $f(t)$ are continuous in a neighbourhood of $t=a$ (except possible $t=a$).

III) Assume ϕ and t may be Taylor expanded as

$$\phi(t)=\phi(a)+\sum_{k=0}^{\infty}a_k(t-a)^{k+\alpha}$$

$$\phi'(t)=\sum_{k=0}^{\infty}a_k(k+\alpha)(t-a)^{k+\alpha-1}$$

$$f(t)=\sum_{k=0}^{\infty}b_k(t-a)^{k+\beta-1}$$

where $\alpha>0$, $Re(\beta)>0$, and $a_0\neq 0\neq b_0$.

IV) The integral $I(x)$ converges (absolutly) for $x\to\infty$, then.

$$I(x)\sim e^{-x\phi(a)}\sum_{n=0}^{\infty}\Gamma\left(\frac{n+\beta}{\alpha}\right)\frac{C_n}{x^{(n+\beta)/\alpha}}, \text{ as } x\to\infty$$

where $C_n=C_n(a_n,b_n)$

Proof: By (I) and (II), there exists a number $c\in(a,b)$, such that $\phi(t),\phi'(t)$ and $f(t)$ are continuous in $[a,b]$.

$$\phi'(c)=\lim_{c\to a^+}\frac{\phi(c)-\phi(a)}{c-a}>0$$

Define $y:=\phi(t)-\phi(a)$ and $Y:=\phi(c)-\phi(a)$

Hot trick:

$$e^{x\phi(a)}\underbrace{\int_a^c e^{-x\phi(t)}f(t)\,dt}_{\text{looks like } I(x).}=\int_a^c e^{-x(\phi(t)-\phi(a))}f(t)\,dt$$

$$=\underbrace{\int_a^c e^{-xy}\,g(y)\,dy}_{\text{Watson-like.}}$$

$$\boxed{\begin{aligned} g(y)\,dy&=f(t)\,dt\\ g(y)\frac{dy}{dt}&=f(t)\\ \text{or}\quad g(y)&=\frac{f(t)}{\phi'(t)}\end{aligned}}$$

Also, $y=\phi(t)-\phi(a)=\sum_{k=0}^{\infty}a_k(t-a)^{k+\alpha}$ as $t\to a^+$.

Use reversion series:

If $y = a_1 x + a_2 x^2 + a_3 x^3 + \dots$, $a_0 = 0$ Non-constant term.

we may invert x in terms of a series expansion in y

$$x = A_1 y + A_2 y^2 + A_3 y^3 + \dots \qquad A_0 = 0 \text{ Non-constant term}$$

$$y = a_1(A_1 y + A_2 y^2 + A_3 y^3 + \dots) + a_2(A_1 y + A_2 y^2 + A_3 y^3 + \dots)^2$$
$$+ a_3(A_1 y + A_2 y^2 + A_3 y^3 + \dots)^3 + \dots$$
$$y = a_1 A_1 y + (a_1 A_2 + a_2 A_1^2) y^2 + (a_1 A_3 + a_2 A_1 A_2 + a_3 A_1^3) y^3 + \dots$$

then

$$a_1 A_1 = 1 \longrightarrow A_1 = \frac{1}{a_1}$$

$$a_1 A_2 + a_2 A_1^2 = 0 \longrightarrow A_2 = \frac{-a_2 A_1^2}{a_1} = -\frac{a_2}{a_1^3}$$

$$A_3 = a_1^{-5}(2a_2^2 a_1 - a_3)$$

In our case

$$y = \sum_{k=0}^{\infty} a_k (t-a)^{k+\alpha} , \quad t \longrightarrow a^+$$

we obtain

$$t - a = \sum_{k=1}^{\infty} d_k y^{k/\alpha} , \quad \text{as } y \longrightarrow 0^+$$

$$= d_1 y^{1/\alpha} + d_2 y^{2/\alpha} + d_3 y^{3/\alpha} + \dots$$

$$y = a_0 (t-a)^{\alpha} + a_1 (t-a)^{1+\alpha} + a_2 (t-a)^{2+\alpha} + \dots$$

$$y = a_0 (d_1 y^{1/\alpha} + d_2 y^{2/\alpha} + d_3 y^{3/\alpha} + \dots)^{\alpha}$$
$$+ a_1 (d_1 y^{1/\alpha} + d_2 y^{2/\alpha} + d_3 y^{3/\alpha} + \dots)^{1+\alpha}$$
$$+ a_2 (d_1 y^{1/\alpha} + d_2 y^{2/\alpha} + d_3 y^{3/\alpha} + \dots)^{2+\alpha} + \dots$$
$$= a_0 d_1^{\alpha} y + (a_0 d_2^{\alpha} + \frac{1}{\alpha} a_1 d_1^2 d_2^{\alpha-1}) y^2 + \dots$$

$$d_1 = \frac{1}{a_0^{1/\alpha}} , \quad d_2 = \frac{-a_1}{\alpha\, a_0^{1+2/\alpha}}$$

then

$$g(y)\frac{dy}{dt} = f(t)$$

$$g(y)\frac{d}{dt}\left(\sum_{j=0}^{\infty} a_j (t-a)^{j+\alpha}\right) = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$$

$$g(y)\sum_{j=0}^{\infty} a_j (j+\alpha)(t-a)^{j+\alpha-1} = \sum_{k=0}^{\infty} b_k (t-a)^{k+\beta-1}$$

equating same powers

$$j+\alpha-1 = k+\beta-1$$

$$j = k+\beta-\alpha$$

$$g(y)\, a_{k+\beta-\alpha}(k+\beta) = b_k$$

finally,

$$g(y) \sim \sum_{k=0}^{\infty} c_k\, y^{(k+\beta)/\alpha - 1}$$

$$I(x) \sim e^{-x\phi(a)} \sum_{n=0}^{\infty} \frac{c_n \Gamma\left(\frac{n+\beta}{\alpha}\right)}{x^{(n+\beta)/\alpha}}$$

Until here we have made the half proof.