Lie algebra

$$\forall\ \vec{\alpha}, \vec{\beta} \in \mathbb{R}^N\ \exists\ \vec{\gamma} \in \mathbb{R}^N / D(\vec{\alpha}) \cdot D(\vec{\beta}) = D(\vec{\gamma})$$

$$e^{i\vec{\alpha}\vec{X}} \cdot e^{i\vec{\beta}\vec{X}} = e^{i\vec{\gamma}\vec{X}}$$

In general $\quad \vec{\gamma} \neq \vec{\alpha} + \vec{\beta}$

$$i\vec{\gamma}\vec{X} = \ln(1 + e^{i\vec{\alpha}\vec{X}} \cdot e^{i\vec{\beta}\vec{X}} - 1)$$

Use: $\quad \ln(1+x) = x - \frac{1}{2}x^2 + \ldots$

After algebra:

$$i\vec{\delta}\vec{X} \approx i\vec{\alpha}X + i\vec{\beta}X - (\vec{\alpha}\vec{X})(\vec{\beta}X)$$
$$- \frac{1}{2}(\vec{\alpha}\vec{X})^2 - \frac{1}{2}(\vec{\beta}\vec{X}) + \frac{1}{2}(\vec{\alpha}\vec{X} + \vec{\beta}\vec{X})^2$$

Observation: If $[X_a, X_b] = 0$, then $\vec{\delta} = \vec{\alpha} + \vec{\beta}$; in general.

G is abelian.

$$i\vec{\gamma}\vec{X} = i\vec{\alpha}\vec{X} + i\vec{\beta}\vec{X} - \frac{1}{2}[\vec{\alpha}\vec{X}, \vec{\beta}\vec{X}]$$

$$[\vec{\alpha}\vec{X}, \vec{\beta}\vec{X}] = -2i(\vec{\delta} - \vec{\alpha} - \vec{\beta}) \cdot \vec{X} = i\vec{\gamma}\vec{X}$$

$\vec{\gamma} \in \mathbb{R}^N$ closes the algebra.

then,

$$\alpha_a \beta_b [X_a, X_b] = i\gamma_c X_c.$$
$$\gamma_c = \alpha_a \beta_b f_{abc}.$$

$$\boxed{[X_a, X_b] = i f_{abc} X_c}$$ Lie algebra.

structure constants $\quad f_{abc} = -f_{bac}.$

Unitary representations

$$D^{-1}(\vec{\alpha}) = D(\alpha)^\dagger$$

$$e^{-i\vec{\alpha}\vec{X}} = e^{-i\alpha X^\dagger}$$

then,

$$X^\dagger = X \longrightarrow [X_a, X_b]^\dagger$$
$$= -i f^*_{abc} X_c$$
$$= i f_{bac} X_c$$
$$= -i f_{abc}.$$

Structure Constants: $f^*_{abc} = f_{abc}.$

Jacobi Identity.

$$[X_a,[X_b, X_c]] + \text{Cyclic permutations} = 0.$$
$$[f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde}] = 0.$$

Adjoint representations.

$$[T_a]_{bc} \equiv -i f_{abc}.$$

then,

$$[T_a, T_b]_{cd} = -f_{ace} f_{bed} + f_{bce} f_{aed}$$
$$= f_{cae} f_{bed} + f_{bce} f_{aed}.$$
$$= -f_{abe} f_{ced}. \qquad T_a \in M(\mathbb{C}).$$
$$= i f_{abe}(i f_{ced})$$
$$[T_a, T_b] = i f_{abc} T_c. \checkmark \longrightarrow \dim T_a = N$$
$$= \dim \mathfrak{a}$$

Any abelian Lie algebra of dim=n are

$$\overset{\text{iso}}{\approx} \bigoplus_n \text{algebra 1-D.}$$

Group: All the irreducible representations of an abelian group are 1-D.

Inner product: $\mathrm{Tr}(T_a T_b) = K^a \delta_{ab}$

Theorem: $K^a < 0$: It lacks irreducible representations of finite dimension, not trivials.

No compact groups.

Example:
- Poincaré Group
- Lorentz Group.

$K^a < 0$: Compact algebras.

$$Tr[T_a, T_b] = \lambda \delta_{ab}, \quad \lambda > 0.$$

In this basis.

$$f_{abc} = -i\, Tr([T_a, T_b] T_c)\, \lambda^{-1}$$

$$\begin{aligned} Tr([T_a, T_b] T_c) &= Tr(T_a T_b T_c - T_b T_a T_c) \\ &= Tr(T_b T_c T_a - T_c T_b T_a) \\ &= Tr([T_b, T_c] T_a) \end{aligned}$$

finally,

$$\begin{aligned} f_{abc} &= f_{bca} = f_{cab} \\ &= -f_{bac} = -f_{acb} = -f_{cba} \end{aligned}$$

Then f_{abc} is totally antisymetric.

$$[T_a]_{bc} = i f_{abc}$$

$$\longrightarrow T_a^{\dagger} = T_a \quad \text{unitary Representation.}$$

$$Tr(T_a) = 0.$$

Example: $SU(2)$

$$\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Definition: An invariant subalgebra (SubAlg) is a subset of generators that map themselves under commutation, with any element of the algebra.

Given, $X \in$ Subalg : $\forall Y \in$ alg, $[X, Y] \in$ Subalg.

Exponential map:
$$h = e^{iX} \quad ; \quad g = e^{iY}$$
then,
$$g^{-1} h g = e^{iX'} \in H.$$
where $e^{iX'} \in H_{inv}$.

Definition: Simple algebra: It has no invariant subalgebras, no trivials.

Generates a simple group.

Theorem: An adjoint representation of a simple algebra is an irreducible representation.

Definition: An abelian invariant subalgebra has just one generator that commutes with all the generators of the group, $U(1)$.

Abelian invariant subalgebras correspond to $K^a = 0$.

In general.
$$Tr(X_a, X_b) = K^a \delta_{ab}$$
$$Tr X^2 = 0.$$