

Heisenberg's representations.

$$U(t, t_0) = e^{itH/\hbar}$$

$$\langle f \rangle_\psi = \langle \psi, \widehat{f(t_0)} \psi \rangle \quad \text{as } \psi(t) = U(t, t_0) \psi(t_0).$$

$$\begin{aligned} &= \frac{\langle U(t, t_0) \psi(t_0), f U(t, t_0) \psi(t_0) \rangle}{\langle U(t, t_0) \psi(t_0), U(t, t_0) \psi(t_0) \rangle} \\ &= \frac{\langle \psi(t_0), U^{-1}(t, t_0) f U(t, t_0) \psi(t_0) \rangle}{\langle \psi(t_0), U^{-1}(t, t_0) U(t, t_0) \psi(t_0) \rangle} \\ &= \frac{\langle \psi(t_0), f(t) \psi(t_0) \rangle}{\langle \psi(t_0), \psi(t_0) \rangle} \end{aligned}$$

where

$$f(t) := U^{-1}(t, t_0) f(t_0) U(t, t_0)$$

Also, taking the conjugate to Schrödinger

$$-i\hbar \frac{\partial}{\partial t} U^\dagger(t, t_0) = U^\dagger(t, t_0) H$$

for an arbitrary $f \in \hat{A}_Q$

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} f(t) &= i\hbar \left[\left(\frac{\partial}{\partial t} U^{-1}(t, t_0) \right) f(t_0) U(t, t_0) + U^{-1}(t, t_0) \frac{\partial}{\partial t} U(t, t_0) \right] \\ &= i\hbar \left[-\frac{1}{i\hbar} U^\dagger(t, t_0) H f(t_0) U(t, t_0) + U^{-1}(t, t_0) f(t_0) \frac{1}{i\hbar} H U(t, t_0) \right] \\ &= U^\dagger(t, t_0) \left[-H f(t_0) + f(t_0) H \right] U(t, t_0) \\ &= U^\dagger(t, t_0) [H, f(t_0)] U(t, t_0) \\ &= [H, f](t) \quad \text{equivalent to Liouville's.} \end{aligned}$$

Now, in our case we want to reproduce something similar.

Definition: The $*$ -exponential is defined as

$$\text{Exp}_*(a) := \sum_{n=0}^{\infty} \frac{1}{n!} (a^*)^n$$

Where $(a*)^n := a*a*\dots*a$ (n-times)

In particular, we may propose an $*$ -unitary evolution operator.

$$U_*(q, p, t) := \text{Exp}_* \left(\frac{itH}{\hbar} \right) = 1 + \left(\frac{it}{\hbar} \right) H + \frac{1}{2!} \left(\frac{it}{\hbar} \right)^2 H * H + \dots$$

for an arbitrary Hamiltonian $H \in A_c$.

As in quantum mechanics, one may show that a solution to Moyal's equation.

$$d_t f = \{f, H\}_M$$

is given by

$$W(q, p, t) = U_*^{-1}(q, p, t) * W(q, p, 0) * U_*(q, p, t).$$

Homework: Verify the solution to the Moyal's equation.

Example: Harmonic oscillator.

$$H = \frac{1}{2} (p^2 + q^2)$$

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$$H * W = \epsilon W$$

$$H * W = H(p, q) \exp \left[\left(\frac{i\hbar}{2} \right) \left(\overleftrightarrow{\partial}_q \overleftrightarrow{\partial}_p - \overleftrightarrow{\partial}_p \overleftrightarrow{\partial}_q \right) \right] W(p, q)$$

$$= H(p, q) \left[1 + \frac{i\hbar}{2} \overleftrightarrow{p} + \left(\frac{i\hbar}{2} \right)^2 \frac{1}{2!} \overleftrightarrow{p}^2 + \dots \right] W(p, q)$$

$$H \overleftrightarrow{p} W = \frac{\partial H}{\partial q} \frac{\partial W}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial W}{\partial q} = q \frac{\partial W}{\partial p} - p \frac{\partial W}{\partial q}$$

$$\begin{aligned} H \overleftrightarrow{p}^2 W &= H \left(\overleftrightarrow{\partial}_q^2 \overleftrightarrow{\partial}_p^2 - 2 \overleftrightarrow{\partial}_q \overleftrightarrow{\partial}_p \overleftrightarrow{\partial}_p \overleftrightarrow{\partial}_q + \overleftrightarrow{\partial}_p^2 \overleftrightarrow{\partial}_q^2 \right) W \\ &= \frac{\partial^2 W}{\partial p^2} + \frac{\partial^2 W}{\partial q^2} \end{aligned}$$

$$H * W = (HW)(p, q) + \frac{i\hbar}{2} \left(q \frac{\partial W}{\partial p} - p \frac{\partial W}{\partial q} \right) - \frac{\hbar^2}{8} \left(\frac{\partial^2 W}{\partial p^2} + \frac{\partial^2 W}{\partial q^2} \right) = \epsilon W$$

$$\operatorname{Re}(H * w) = HW - \frac{\hbar^2}{8} \left(\frac{\partial^2 w}{\partial p^2} + \frac{\partial^2 w}{\partial q^2} \right) = \epsilon W$$

$$\operatorname{Im}(H * w) = q \frac{\partial w}{\partial p} - p \frac{\partial w}{\partial q} = 0$$

$$\Rightarrow W = W(z) \quad ; \quad z := \frac{2(p^2 + q^2)}{\hbar} = \left(\frac{4}{\hbar} p \right) \frac{dw}{dz}$$

$$\frac{\partial w}{\partial p} = \frac{\partial z}{\partial p} \frac{dw}{dz} = \frac{4p}{\hbar} \frac{dw}{dz} \quad \frac{\partial w}{\partial q} = \frac{\partial z}{\partial q} \frac{dw}{dz} = \left(\frac{4}{\hbar} q \right) \frac{dw}{dz}$$

$$\frac{d^2 w}{dp^2} = \frac{\partial}{\partial p} \left(\frac{\partial w}{\partial p} \right) = \frac{\partial}{\partial p} \left(\frac{4p}{\hbar} \frac{dw}{dz} \right) = \frac{4}{\hbar} \frac{dw}{dz} + \left(\frac{4q}{\hbar} \right)^2 \frac{d^2 w}{dz^2}$$

Real:

$$\frac{\hbar z}{4} W(z) - \frac{\hbar^2}{8} \left(\frac{4}{\hbar} \frac{dw}{dz} + \frac{16p^2}{\hbar^2} \frac{d^2 w}{dz^2} + \frac{4}{\hbar} \frac{dw}{dz} + \frac{16q^2}{\hbar^2} \frac{d^2 w}{dz^2} \right) = \epsilon W(z).$$

$$2(p^2 + q^2) \frac{d^2 w}{dz^2} + \hbar \frac{dw}{dz} - \frac{\hbar z}{4} w + \epsilon w = 0.$$

$$\hbar z \frac{d^2 w}{dz^2} + \hbar \frac{dw}{dz} + \left(\epsilon - \frac{\hbar z}{4} \right) w = 0.$$

$$\left[z \frac{d^2}{dz^2} + \frac{dw}{dz} + \left(\frac{\epsilon}{\hbar} - \frac{z}{4} \right) \right] w = 0 \quad (\#)$$

$$w(z) = e^{-z/2} L(z).$$

$$\frac{dw}{dz} = -\frac{1}{2} e^{-z/2} L + e^{-z/2} \frac{dL}{dz} = \left(-\frac{1}{2} L + \frac{dL}{dz} \right) e^{-z/2}$$

$$\begin{aligned} \frac{d^2 w}{dz^2} &= \left(-\frac{1}{2} \frac{dL}{dz} + \frac{d^2 L}{dz^2} \right) e^{-z/2} + \left(-\frac{1}{2} L + \frac{dL}{dz} \right) \left(-\frac{1}{2} e^{-z/2} \right) \\ &= \left(\frac{d^2 L}{dz^2} - \frac{dL}{dz} + \frac{1}{4} L \right) e^{-z/2} \end{aligned}$$

from (#), then

$$\left\{ z \left(\frac{d^2 L}{dz^2} - \frac{dL}{dz} + \frac{1}{4} L \right) + \left(-\frac{1}{2} L + \frac{dL}{dz} \right) + \left(\frac{\epsilon}{\hbar} - \frac{z}{4} \right) L \right\} e^{-z/2} = 0$$

$$z \frac{d^2 L}{dz^2} + (1-z) \frac{dL}{dz} + \left(\frac{\epsilon}{\hbar} - \frac{z}{4} + \frac{z}{4} - \frac{1}{2} \right) L = 0$$

$$=: n \in \mathbb{Z}^+$$

$$\therefore L = L_n(z) = L_n\left(\frac{4H}{\hbar}\right) \quad \text{Laguerre equation.}$$

$$n = \frac{E}{\hbar} - \frac{1}{2}$$

$$\Rightarrow E_n = \hbar(\omega)\left(n + \frac{1}{2}\right)$$

$$W(p, q) = \alpha_n e^{-2H/\hbar} L_n\left(\frac{4H}{\hbar}\right)$$

$$W_n(p, q) = \alpha_n e^{-(p^2+q^2)/\hbar} L_n\left(\frac{2(p^2+q^2)}{\hbar}\right)$$

Change $Q := \operatorname{arccot}\left(\frac{p}{q}\right)$

$$H := \frac{p^2 + q^2}{2}$$

such that

$$dp \wedge dq = dH \wedge dQ$$

$$\begin{aligned} \Rightarrow 1 &= \int dH dQ \alpha_n e^{-2H/\hbar} L_n\left(\frac{4H}{\hbar}\right) \\ &= \alpha_n \int_0^\pi dQ \int_0^\infty dH e^{-2H/\hbar} L_n\left(\frac{4H}{\hbar}\right) = \alpha_n \int \frac{dx}{\left(\frac{\hbar}{4}\right)} e^{-x/2} L_n(x) \end{aligned}$$

$$x = \frac{4H}{\hbar}$$

Gradshteyn 7.414.6

$$\int_0^\infty dx e^{-ax} L_n(x) = (a-1)^n a^{-n-1}.$$

$$= \alpha_n \pi \left(\frac{\hbar}{4}\right) \left[\left(\frac{1}{2} - 1\right)^n \left(\frac{1}{2}\right)^{-n-1}\right] = \frac{\alpha_n \pi \hbar (-1)^n}{2}$$