Proposition: For all $\varepsilon > 0$, there is only a finite number of linearly independent vectors, with eigenvalues λ_i, such that $|\lambda_i| \geq \varepsilon$. This is, exist a finite number of eigenvalues $\lambda_i \in \sigma_p$, such that $|\lambda_i| \geq \varepsilon$ and each λ_i, has finite multiplicity i.e.,

$$\dim \operatorname{Ker} \lambda < \infty$$

Proof: If that were not the case, $\{x_i\}_{i=1}^{\infty}$ vectors λ_i and $Tx_i = \lambda_i x_i$, $|\lambda_i| \geq \varepsilon$. Consider $E_k = \operatorname{span}\{x_i\}_{i=1}^{k} \subsetneq E_{k+1}$. By the above theorem, let

$$y_k \in E_k, \text{ with } \|y_k\| = 1 \text{ and } d(y_k, E_{k-1}) \geq \frac{1}{2}$$

we will show that $\{T(y_k/\lambda_k)$ not contains a Cauchy subsequence, which implies that T is not compact, since y_k/λ_k is bounded (By $|\lambda_k| \geq \varepsilon$).

Let $y_k = \sum_{i=1}^{k} a_i x_i$, then

$$T(y_k/\lambda_k) = a_k x_k + \sum_{i=1}^{k-1}\left(\frac{a_i \lambda_i}{\lambda_k}\right) x_i = y_k + z_k,$$

with $z_k \in E_{k-1}$.

Then for any $k > n$,

$$\left\|T\left(\frac{y_k}{\lambda_k}\right) - T\left(\frac{y_n}{\lambda_n}\right)\right\| = \|y_k - \underbrace{(y_n - z_k + z_n)}_{\in E_{k-1}}\| \geq \frac{1}{2}$$

and there is not exists a Cauchy subsequence of $T\left(\frac{y_k}{\lambda_k}\right)$

■

The structure of σ_p (Point spectrum) is the following:

It is at most a sequence converging to zero and each λ_i has finite multiplicity.

Let's see that if T is compact, then $\sigma(T) = \{\sigma_p(T), 0\}$.

Proposition: Let T be a compact operator, $\lambda \neq 0$. Then $\Delta_\lambda = \overline{\Delta_\lambda}$ i.e., Δ_λ is a closed subspace for $\lambda \neq 0$.

Remember that $\Delta_\lambda := \operatorname{Im}(T - \lambda I)$

Proof: Let $T_\lambda x = y$, $E_y = \{z : T_\lambda z = y\}$. We see that $E_y = x + E_0$, where $T_\lambda x = y$ and $E_0 = \operatorname{Ker} T_\lambda$. let's proof the following statement

Proposition: Let $\alpha(y) = \inf\{\|z\| : z \in E_y\}$. Then exists a constant C independent of y, such that

$$\alpha(y) \leq C\|y\|.$$

Proof: Let's assume that is false i.e., exists $\tilde{y}_n$ such that

$$\frac{\|\tilde{y}_n\|}{\alpha(\tilde{y}_n)} \longrightarrow 0.$$

As $T_\lambda x = y$ if and only if $T_\lambda(\gamma x) = \gamma y$, for γ an scalar, the function $\alpha(y)$ is homogeneous i.e.,

$$\alpha(\gamma y) = \gamma\alpha(y) \quad , \gamma \geq 0.$$

Taking $y_n = \dfrac{\tilde{y}_n}{\alpha(\tilde{y}_n)}$, we obtain $\alpha(y_n) = 1$ and $y_n \longrightarrow 0$.

Let x_n such that $T_\lambda x_n = y_n \longrightarrow 0$ and $\|x_n\| \leq 2$ (since $\alpha(y_n) = 1$ and we can choose x_n near to 1).

For the compacity of T, exists a subsequence x_{n_k} such that $Tx_{n_k} \longrightarrow w$ and $(T - \lambda I)x_{n_k} = T_\lambda x_{n_k} = y_{n_k} \longrightarrow 0$.

Thus $\lambda x_{n_k} \longrightarrow w$ which it means that $x_{n_k} \longrightarrow w/\lambda := x_0$

Thus

$$T_\lambda x_0 = 0 \quad \text{and} \quad x_0 \in E_0.$$

Then $x_{n_k} - x_0 \in E_{y_{n_k}}$, and therefore $\alpha(y_{n_k}) \longrightarrow 0$, which contradicts that $\alpha(y_n) = 1$

$\blacksquare$

Going back to the prove, let $y_n \in \Delta_\lambda$, and $y_n \longrightarrow y$. As $\{y_n\}$ is bounded, by the above statement, exists x_n, with $\|x_n\| < C$, such that

$$y_n = T_\lambda x_n \longrightarrow y.$$

Then, exists x_{n_k} such that $Tx_{n_k} \longrightarrow z$, which implies that

$$x_{n_k} \longrightarrow (z - y)/\lambda = x_0, \text{ and } Tx_0 = \lambda x_0 = y$$

Thus $y \in \Delta_\lambda$.

$\blacksquare$

Corollary: let T be a compact operator and T^* its dual (is compact).

$$\Delta_\lambda^* = \operatorname{Im} T_\lambda^* \subseteq X^*$$

Proposition:

I. $\Delta_\lambda = X$ implies $\operatorname{Ker} T_\lambda = 0$

(or equivalently if $\ker T_\lambda \neq 0$, then $\Delta_\lambda \neq X$)

II. In a similar way

$$\Delta_\lambda^* = X^*, \text{ then } \operatorname{Ker} T_\lambda^* = 0.$$

Proof: If this were not the case, it would exist $x_0 \in \operatorname{Ker} T_\lambda$, such that $x_0 \neq 0$. As $\Delta_\lambda = \operatorname{Im} T_\lambda = X$, exists x_1 such that $T_\lambda x_1 = x_0$. In a similar way, for each k, exists x_k, we have

$$T_\lambda x_k = x_{k-1} \quad , \quad k = 1, 2, \ldots$$

for each x_k, we have $T_\lambda^k x_k = x_0 \neq 0$, but $T_\lambda^{k+1} x_k = T x_0 = 0$.

Then if

$$N_k = \{ x : T_\lambda^k x = 0 \} = \operatorname{Ker} T_\lambda^k$$

we have

$$N_{k+1} \supsetneq N_k$$

By the previous lemma, exists $y_k \in N_k$, $\|y_k\| = 1$ and

$$d(y_k, N_{k-1}) \geq \frac{1}{2}$$

Let's proof that $\{Ty_k\}$ does not have a Cauchy subsequence contradicting the compacity of T.

Let $k > n$

$$\|Ty_k - Ty_n\| = \|T_\lambda y_k - \lambda y_k - T_\lambda y_n - \lambda y_n\|$$
$$= \|\lambda y_k - (\lambda(y_n - T\lambda y_k - T_\lambda y_n)\|$$
$$\geq |\lambda| \frac{1}{2}$$

and there is not a Cauchy subsequence of $\{Ty_k\}$. ∎