

Representations.

Let G be a topological group (metrizable). Let $(V, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Let $GL(V)$ be the set of linear invertible matrices $T: V \rightarrow V$.

A representation of G in V is a group homomorphism $\eta: G \rightarrow GL(V)$ such that $(x, v) \mapsto \eta(x)v$ is continuous. The representation η is called unitary in V i.e.,

$$\langle \eta(x)v, \eta(x)w \rangle = \langle v, w \rangle ; \forall v, w \in V, x \in G.$$

A closed subspace $W \subseteq V$, it's called invariant for η , if $\eta(x)W \subseteq W$ $\forall x \in G$. The representation η is called irreducible, if there are no non-trivial closed self-subspaces i.e., the only invariant subspaces are 0 and V .

Example: The identity map $P: U(n) \rightarrow GL(\mathbb{C}^n) = GL_n(\mathbb{C})$ is a unitary representation.

Lemma: P is irreducible.

Proof: By definition, $U(n)$ consists of the linear operators of \mathbb{C}^n such that they are unitary with respect to the internal product $\langle v, w \rangle = v^t \bar{w}$.

Let $V \subseteq \mathbb{C}^n$ be a subspace which is not self-subspace, also the zero, and neither all the space. Let $W = V^\perp$ the orthogonal complement i.e.,

$$W = \{w \in \mathbb{C}^n / \langle w, v \rangle = 0, \forall v \in V\}$$

Thus, $\mathbb{C}^n = V \oplus W$, let e_1, \dots, e_l be a orthonormal base of V and e_{l+1}, \dots, e_n other one from W . So the operator T given by

$$T(e_1) = e_{l+1} ; T(e_{l+1}) = e_1 .$$

$$T(e_j) = e_j ; j \neq 1, l+1 .$$

is unitary. Thus $T \in U(n)$, but T do not let stable to V .

Let $T: H \rightarrow H$, T is unitary if and only if, for each orthonormal base (e_j) , the formula $T e_j$ is orthonormal.

Exponential map

A serie $A \in \text{Mat}_n(\mathbb{C})$ of the form $\sum_{v=0}^{\infty}$ converges by the definition if the succession of partial sums

$$S_k = \sum_{v=1}^k A_v$$

converges.

Proposition: For each $A \in \text{Mat}_n(\mathbb{C})$, the series

$$\exp(A) = \sum_{v=0}^{\infty} \frac{A^v}{v!}$$

Converges and define an element in $\text{GL}_n(\mathbb{C})$. If $A, B \in \text{Mat}_n(\mathbb{C})$ and $AB = BA$, then $\exp(A+B) = \exp(A)\exp(B)$. In particular

$$\exp(-A) = \exp(A^{-1})$$

Proof: Let's remember that in $\text{Mat}_n(\mathbb{C})$

$$\|A\|_1 = \sum_{i,j} |a_{ij}|, \text{ if } A = (a_{ij})$$

Lemma: For $A, B \in \text{Mat}_n(\mathbb{C})$

$$\|AB\|_1 \leq \|A\|_1 \|B\|_1$$

In particular for $j \in \mathbb{N}$, $\|A^j\|_1 \leq \|A\|_1^j$

Proof: Let $A = (a_{ij})$, $B = (b_{ij})$

$$\begin{aligned} \|AB\|_1 &= \sum_{i,j=1}^n \left| \sum_{k=1}^n a_{ik} b_{kj} \right| \leq \sum_{i,j,k=1}^n |a_{ik} b_{kj}| \\ &\leq \sum_{i,j,k,l=1}^n |a_{ik}| |b_{lj}| = \|A\|_1 \|B\|_1 \end{aligned}$$

Lemma: Let $(A_v)_{v \geq 0}$ be a succession of matrices in $\text{Mat}_n(\mathbb{C})$. Let's suppose that

$$\sum_{v=0}^{\infty} \|A_v\|_1 < \infty$$

So the series $\sum_{v=0}^{\infty} A_v$ converges.

Proof: Let $b_k = \sum_{v=0}^k A_v$, we have to prove that the succession (B_k) converges. It is sufficient to show that it is a succession of Cauchy.

Thus, given $\epsilon > 0$; $\exists K_0$, such that for $m > k > K_0$

$$\epsilon > |b_m - b_k| = \sum_{v=k+1}^m \|A_v\|_1 \geq \left\| \sum_{v=k+1}^m A_v \right\|_1 = \|B_m - B_k\|_1.$$

Then (B_k) is Cauchy in $\text{Mat}_n(\mathbb{C})$, thus converges.

Homework: Prove that $\text{Mat}_n(\mathbb{C})$, all Cauchy's succession converges to the norm $\|\cdot\|_1$ or $\|\cdot\|_2$.

To prove the proposition, less to prove that

$$\sum_{v=0}^{\infty} \frac{\|A^v\|_1}{v!} < \infty$$

thus

$$\sum_{v=0}^{\infty} \frac{\|A^v\|_1}{v!} \leq \sum_{v=0}^{\infty} \frac{\|A\|_1^v}{v!} < \infty$$

Since the exponential in \mathbb{R} converges. For the rest of the proposition, let $A, B \in \text{Mat}_n(\mathbb{C})$, $AB = BA$.

$$\begin{aligned} \exp(A+B) &= \sum_{v=0}^{\infty} \frac{(A+B)^v}{v!} = \sum_{v=0}^{\infty} \frac{1}{v!} \sum_{k=0}^v \binom{v}{k} A^k B^{v-k} \\ &= \sum_{v=0}^{\infty} \sum_{k=0}^v \frac{1}{k!(v-k)!} A^k B^{v-k} = \sum_{v=0}^{\infty} \sum_{k=0}^v \frac{1}{v!k!} A^v B^k = \exp(A)\exp(B). \end{aligned}$$

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Proposition: For each $A \in \text{Mat}_n(\mathbb{C})$

$$\det(\exp(A)) = \exp(\text{tr}A)$$

Proof: let $S \in \text{GL}_n(\mathbb{C})$, then

$$\det(\exp(SAS^{-1})) = \det(S \exp(A) S^{-1}) \det(\exp(A))$$

and

$$\exp(\text{tr}(SAS^{-1})) = \exp(\text{tr}A)$$

Thus, both sides of the proposition are invariants under the conjugation.

By Jordan's theorem, all square matrix is conjugated to an upper triangular matrix.

Let's suppose

$$A = \begin{pmatrix} a & & * \\ & \ddots & \\ & & a_n \end{pmatrix}$$

Geometrical multiplicity
Algebraic multiplicity

for $v \geq 0$

$$A^v = \begin{pmatrix} a^v & & * \\ & \ddots & \\ & & a_n^v \end{pmatrix}$$

then

$$\exp(A) = \begin{pmatrix} e^a & & * \\ & \ddots & \\ & & e^{a_n} \end{pmatrix}$$

$$\det(\exp(A)) = e^{a_1} \cdot \dots \cdot e^{a_n} = e^{a_1 + \dots + a_n} = \exp(\text{tr}(A))$$

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Let $G \subseteq GL_n(\mathbb{C})$ be a closed subgroup. The Lie's algebra of G is given by

$$\text{Lie}(G) = \{ \underline{X} \in \text{Mat}_n(\mathbb{C}) / \exp(t \cdot \underline{X}) \in G, \forall t \in \mathbb{R} \}$$

Example:

i) The linear special group $SL_n(\mathbb{C})$, which consist of the matrices $A \in \text{Mat}_n(\mathbb{C})$, such that $\det(A) = 1$, his Lie algebra

$$\text{Lie}(SL_n(\mathbb{C})) = sl_n(\mathbb{C}) = \{ \underline{X} \in \text{Mat}_n(\mathbb{C}) / \text{tr}(\underline{X}) = 0 \}$$

ii) $U(n)$ Lie's algebra.

$$\text{Lie}(U(n)) = u(n) = \{ \underline{X} \in \text{Mat}_n(\mathbb{C}) / \underline{X}^* = -\underline{X} \}$$

where $\underline{X}^* = \overline{\underline{X}}^t$.

Proposition: (Docarmo, Differential Geometry)

Let G be a closed subgroup of $GL_n(\mathbb{C})$. So $\text{Lie}(G)$ is a real vectorial sub space of $\text{Mat}_n(\mathbb{C})$

If $\underline{X}, \underline{Y} \in \text{Lie}(G)$, then $[\underline{X}, \underline{Y}] \stackrel{\text{def}}{=} \underline{X}\underline{Y} - \underline{Y}\underline{X}$

it is called a Lie's parenthesis of \underline{X} and \underline{Y} .

Let $\pi: G \rightarrow GL(V)$ be a representation of finite dimension. Then for each $X \in \text{Lie}(G)$ the map $t \mapsto \pi(\exp(t \cdot X))$, $t \in \mathbb{R}$, is infinitely differentiable.

Let

$$\pi(X) = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp(t \cdot X)) \in \text{End}(V)$$

$\text{End} := \text{endomorphism}$

The map $X \mapsto \pi(X)$ satisfy

$$\pi([X, Y]) = [\pi(X), \pi(Y)]$$

A closed subgroup G of $GL_n(\mathbb{C})$ is called path-connected, if all point $x, y \in G$, may be join by a continuous curve i.e.,

$$g: [0, 1] \longrightarrow G, \quad g(0) = x, \quad g(1) = y$$

Example:

$\mathbb{R}^x = GL_1(\mathbb{R})$ is not path-connected.

A representation $\pi: \text{Lie}(G) \rightarrow \text{End}(V)$ is called *-representation if $X \in \text{Lie}(G)$, then

$$\pi(X)^* = \pi(-X)$$

Multiplicidad geométrica y algebraica de un valor propio.

Definición: (multiplicidad geométrica)

Sea λ un valor propio de A , entonces la multiplicidad geométrica de λ , $(MG(\lambda))$ es la dimensión del espacio propio asociado a λ .

$$MG(\lambda) = \dim E_\lambda = \text{Nul}(A - \lambda I)$$

$$Av = \lambda v$$

$$(A - \lambda I)v = 0$$

$$MG(\lambda) = \dim E_\lambda = \dim \text{Ker}(A - \lambda I) = \text{Nul}(A - \lambda I)$$

Definición (Multiplicidad algebraica)

Sea λ un valor propio de A , llamamos multiplicidad algebraica de λ a la multiplicidad que tiene como raíz en $P(\lambda)$.

$$\text{Si } P(\lambda) = (\lambda - \alpha_1)^{k_1} (\lambda - \alpha_2)^{k_2} (\lambda - \alpha_3)^{k_3} \dots (\lambda - \alpha_n)^{k_n}$$

para $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ raíces del polinomio y $k_1, k_2, k_3, \dots, k_n$ las multiplicidades que tienen $\alpha_i, i \in 1, \dots, n$.

$$MA(\alpha_1) = k_1, MA(\alpha_2) = k_2$$

Ejemplos:

1. (Multiplicidad algebraica)

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad P(\lambda) = |A - I\lambda|$$

$$|A - \lambda I| = \begin{pmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 2 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = (3-\lambda)(3-\lambda)(-\lambda)^2 = 0$$

$$= (9 - 6\lambda + \lambda^2)\lambda^2 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)(\lambda - 3) = 0$$

$$\left. \begin{array}{l} \lambda_1 = 3 \\ \lambda_2 = 0 \end{array} \right\} \text{Multiplicidad 2.}$$

$$MA(3) = 2$$

$$MA(0) = 2$$

2. (Multiplicidad geométrica)

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\text{v. p} = 2 \quad \text{MA}(2) = 3$$

$$\text{MG}(2) = 3 - \text{rg} \begin{pmatrix} 0 & | & 2 & 3 \\ 0 & | & 0 & 2 \\ 0 & | & 0 & 0 \end{pmatrix} = 3 - 2 = 1$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 4 & 1 & 3 \end{pmatrix}$$

$$(3-\lambda)^3 + 2 - (3-\lambda) - 6 = 0$$

$$\lambda_1 = 5 \quad \text{MA}(5) = 1$$

$$\lambda_2 = 2 \quad \text{MA}(2) = 2$$

$$\text{MG}(2) = 3 - \text{rg} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 3 - 1 = 2$$

$$\text{MG}(5) = 3 - \text{rg} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = 3 - 2 = 1$$

Forma canónica de Jordan

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}$$

$$\begin{aligned} \text{Def } |A - \lambda I| &= (-2-\lambda)^2(1-\lambda) + 1 \\ &\quad + (-2-\lambda) + 2(-2-\lambda) \\ &= (-2-\lambda)^2(1-\lambda) + (-2-\lambda) \\ &\quad - 4(-\lambda) + 1 \\ &= (4+4\lambda+\lambda^2)(1-\lambda) - 6 - 3\lambda \\ &= 4 + 4\cancel{\lambda} + \lambda^2 - \cancel{4\lambda} - 4\lambda^2 - \lambda^3 - 6 \\ &\quad - 3\lambda + 1 \\ &= -\lambda^3 - 3\lambda^2 - 3\lambda - 1. \end{aligned}$$