Watson's Lemma

Let $f(t)$ be a complex valued function of a real variable t, such that:

I. f is continuous on $(0,\infty)$

II. As $t \to 0^+$, $f(t) \sim \sum_{k=0}^{\infty} a_k t^{\rho_k - 1}$ with $0 < Re(\rho_0) < \dots < \lim_{k\to\infty} Re(\rho_k) = \infty$

III. For some fixed $c > 0$, $f(t) = \mathcal{O}(e^{ct} t^{\rho_N - 1})$ as $t \longrightarrow \infty$

then we have

$$I(x) := \int_0^\infty e^{-xt} f(t)\,dt \sim \sum_{k=0}^{\infty} a_k \frac{\Gamma(\rho_k)}{x^{\rho_k}}$$

as $x \longrightarrow \infty$

Proof: The conditions (I – III) guarantee that $I(x)$ converges for $x > 0$, and the conditions (II – III) imply.

$$\left| f(t) - \sum_{k=0}^{N-1} a_k - t^{\rho_k - 1} \right| \leq K_n e^{ct} \left| t^{\rho_N - 1} \right| \quad \text{for } t > 0.$$

then

$$e^{-xt} f(t) - \sum_{k=0}^{N-1} e^{-xt} a_k t^{\rho_k - 1} \leq K_n e^{-(x-c)t}\, t^{\rho_N - 1}$$

$$\left| \int_0^\infty e^{-xt} f(t) - \underbrace{\sum_{k=0}^{N-1} e^{-xt} a_k t^{\rho_k - 1} dt}_{=: I_1} \right| \leq K_n \underbrace{\int_0 e^{-(x-c)t} \left| t^{\rho_N - 1} \right| dt}_{=: I_2.}$$

with a change of variables in I_1, $u = xt$ we have

$$I_1 = \frac{1}{x^{\rho_k}} \int_0^\infty e^{-u} u^{\rho_k - 1} du = \frac{\Gamma(\rho_k)}{x^{\rho_k}}$$

and for I_2 with the substitution

$$u := (x-c)t \quad \text{and} \quad dt = \frac{du}{|x-c|}$$

$$I_2 = \frac{1}{|x-c|^{\rho_N}} \int_0^\infty e^{-u} |t^{\rho_N - 1}| dt - \frac{1}{|x-c|^{\rho_N}} \Gamma(Re(\rho_N))$$

Finally,

$$\left| I(x) - \sum_{k=0}^{N-1} a_k \frac{\Gamma(\rho_k)}{x^{\rho_k}} \right| \leq K_N \frac{\Gamma(Re(\rho_N))}{|x-c|^{\rho_N}}$$

Therefore

$$I(x) := \int_0^\infty e^{-xt} f(t) dt = \sum_{k=0}^{N-1} a_k \frac{\Gamma(\rho_k)}{x^{\rho_k}} + \mathcal{O}\left(\frac{1}{x^{\rho_N}}\right)$$

Laplace transform.

Example:

- $$I(x) = \int_0^5 \frac{e^{-xt}}{1+t^2} dt, \quad \text{for large } x.$$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \dots \text{ around } t=0$$

$$= \sum_{k=0}^{\infty} (-1)^k (t)^{2k}$$

And by watson's:

i) Substitute this expansion into the integral.

ii) Interchange integral and summation

iii) Extend from 5 to ∞.

So $a_k = (-1)^k$ and $\rho_k = 2k+1$, then

$$I(x) = \sum_{k=0}^{N-1} (-1)^k \frac{\Gamma(2k+1)}{x^{2k+1}} + \mathcal{O}\left(\frac{1}{x^{2N+1}}\right) \text{ as } x \to \infty$$

$$I(x) = \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \frac{6!}{x^7} + \dots \quad \text{as } x \to \infty$$