Renormalization Group

$\tilde{\Gamma}^{(n)} = \tilde{\Gamma}^{(n)}(M) \longrightarrow$ dispersion amplitudes and other direct observables should not do so.

How does $\tilde{\Gamma}^{(n)}$ change when M changes?

$\Gamma^{(n)}(p_1, \dots, p_n; \hat{g}, \xi, m, M) \longleftarrow$ n implicit Lorentz index

renormalization parameters

$$\frac{\delta}{\delta A_\mu} ; \frac{\delta}{\delta \psi} ; \frac{\delta}{\delta \bar{\psi}}$$

$n = n_A + n_\psi$

$\downarrow \varphi_B = Z_\varphi^{1/2} \varphi.$

then,

$$\Gamma^{(n)}(p_1, \dots, p_n; \hat{g}, \xi, m, M) = Z_A^{n_A/2} Z_\psi^{n_\psi/2} \tilde{\Gamma}_B^{(n)}(p_1, \dots, p_n; g_B, \xi_B, m_B)$$

$f(M)$

So, $M \frac{\partial}{\partial M} \tilde{\Gamma}^{(n)}$:

$$\frac{\partial}{\partial \ln(M)} \tilde{\Gamma}^{(n)} + \frac{\partial \hat{g}}{\partial \ln(M)} \cdot \frac{\partial \tilde{\Gamma}^{(n)}}{\partial \hat{g}} + \frac{\partial \xi}{\partial \ln(M)} \cdot \frac{\partial \tilde{\Gamma}^{(n)}}{\partial \xi} + \frac{\partial m}{\partial \ln(M)} \cdot \frac{\partial \tilde{\Gamma}^{(n)}}{\partial m} =$$

$$= \frac{n_A}{2} Z_\psi^{n_\psi/2} Z_A^{n_A/2} Z_A^{-1} \frac{\partial Z_A}{\partial \ln(M)} \cdot \tilde{\Gamma}_B^{(n)} + \frac{n_\psi}{2} Z_\psi^{n_\psi/2} Z_A^{n_A/2} Z_\psi^{-1} \frac{\partial Z_\psi}{\partial \ln(M)} \cdot \tilde{\Gamma}_B^{(n)}$$

Defining,

$$\beta_{\hat{g}} := \frac{\partial \hat{g}}{\partial \ln(M)} \qquad \gamma_m = -m^{-1} \frac{\partial m}{\partial \ln(M)}$$

$$\beta_{\xi} := \frac{\partial \xi}{\partial \ln(M)} \qquad \gamma_A = Z_A^{-1} \frac{\partial Z_A}{\partial \ln(M)}$$

$$\gamma_\psi = Z_\psi^{-1} \frac{\partial Z_\psi}{\partial \ln(M)}$$

write,

$$\left(\frac{\partial}{\partial \ln(M)} + \beta_{\hat{g}} \frac{\partial}{\partial \hat{g}} + \beta_\xi \frac{\partial}{\partial \xi} + \gamma_m \frac{\partial}{\partial \ln(m)} - n_A \gamma_A - n_\psi \gamma_\psi \right) \tilde{\Gamma}^{(n)}(p_1, \dots, p_n; \hat{g}, \xi, m, M) = 0$$

Renormalization Group equations.

MS ($\overline{MS}$): The coefficients of the equation does not depend on m/M.

To find the coefficients: $g_B = g Z_2 Z_\varphi^{-1} Z_A^{-1/2}$, etc.

To a loop (MS, $\overline{MS}$)

$$\beta_{\hat{g}} = -b\hat{g}^3$$

$$b = \frac{1}{16\pi^2}\left(\frac{11}{3}C_1 - \frac{4}{3}\sum_r C_r\right)$$

$C_r\delta_{ab} = Tr(T_a T_a)_r$

$C_1 = C_r = adj.$

$$\beta_\xi = \left[\left(\frac{13}{16} - \frac{\xi^2}{2}\right)C_1 - \frac{4}{3}\sum_r C_r\right]\frac{\hat{g}^2 \xi}{8\pi^2}$$

$$\gamma_m = \frac{3C_3^r \hat{g}^2}{8\pi^2} := b_m \hat{g}_2$$

$C_3^r \delta_{ij} = (T_a T_b)^r_{ij}$

$C_3^r = \frac{d_{C_1}}{d_r} C_r$

$$\gamma_A = \left[\left(\frac{13}{6} - \frac{\xi}{2}\right)C_1 - \frac{4}{3}\sum_r C_r\right]\frac{\hat{g}^2}{16\pi^2}$$

$$\gamma_\varphi = \frac{\xi C_3^r}{16\pi^2}\hat{g}^2$$

Rescaling momenta & RGE

$\lambda\varphi^4$:

$$\Gamma[\varphi] = \sum \frac{i^n}{n!}\int \prod^n d^4x_i \Gamma^{(n)}(x_i)\varphi(x_1)\cdots\varphi(x_n)$$

as $[\Gamma] = 0 = -4n + [\Gamma^{(n)}] + n$

$$[\Gamma^{(n)}(x_i)] = 4n - n$$

So, $\tilde{\Gamma}^{(n)}(P_\alpha)\,2\pi\delta^4(\sum P) = \int \prod^n d^4x_i\, e^{i\sum P_j x_j}\Gamma^{(n)}(x_i)$

therefore, $[\tilde{\Gamma}^{(n)}(P_\alpha)] - 4 = -4n + [\Gamma^{(n)}(x_i)]$

$$= -4n + 4n - n$$

finally

$$d_r := [\tilde{\Gamma}^{(n)}(P_\alpha)] = 4 - n$$

For Gauge field theories:

$$d_\Gamma = 4 - n_A - \frac{3}{2} n_\psi$$

In 2ω: $d_\Gamma = 2\omega + n_A(1-\omega) + n_\psi(1/2 - \omega)$

$\Gamma^{(n)}$ is a homogeneous function of degree d_Γ in p, m and M under $p_i \to s p_i$:

$$\tilde{\Gamma}^{(n)}(sp_i; \hat{g}, \xi, m, M) = s^{d_\Gamma} \tilde{\Gamma}^{(n)}(p_i; \hat{g}, \xi, s^{-1}m, s^{-1}M)$$

then,

$$\left(s\frac{\partial}{\partial s} + m\frac{\partial}{\partial m} + M\frac{\partial}{\partial M} \right) \tilde{\Gamma}^{(n)}(sp_i; \hat{g}, \xi, m, M) = d_\Gamma \tilde{\Gamma}^{(n)}(sp_i; \hat{g}, \xi, m, M)$$

RGE:

$$\left(M\frac{\partial}{\partial M} + \beta_{\hat{g}} \frac{\partial}{\partial \hat{g}} + \beta_\xi \frac{\partial}{\partial \xi} - \gamma_m m \frac{\partial}{\partial m} - n_A \gamma_A - n_\psi \gamma_\psi \right) \tilde{\Gamma}^{(n)}(sp_i; \hat{g}, \xi, m, M)$$

finally

$$\left(-s\frac{\partial}{\partial s} + \beta_{\hat{g}} \frac{\partial}{\partial \hat{g}} + \beta_\xi \frac{\partial}{\partial \xi} - (1+\gamma_m) m \frac{\partial}{\partial m} - n_A \gamma_A - n_\psi \gamma_\psi + d_\Gamma \right) \tilde{\Gamma}^{(n)}(sp_i; \hat{g}, \xi, m, M)$$

Equation that valuates the rescaling in p.

The equation may be solved by using "running parameters" that are solutions to:

$$s\frac{\partial \bar{g}(s)}{\partial s} = \beta_{\hat{g}}(\bar{g}(s)) \quad ; \quad \bar{g}(1) = \hat{g}(M) =: \hat{g}$$

$$s\frac{\partial \bar{\xi}(s)}{\partial s} = \beta_\xi(\bar{g}(s), \bar{\xi}(s)) \quad ; \quad \bar{\xi}(1) = \xi(M) =: \xi$$

$$s\frac{\partial \bar{m}(s)}{\partial s} = -[1 + \gamma_m(\bar{g}(s), \bar{\xi}(s)] \bar{m}(s) \; ; \quad \bar{m}(1) = m(M) = m$$

From its definition

$$\beta_{\hat{g}} = M\frac{\partial \hat{g}}{\partial M} \left(= s\frac{\partial \hat{g}}{\partial s} \right) \longrightarrow \bar{g}(s) = \hat{g}(sM).$$

Similarly from:

$$\beta_\xi = M\frac{\partial \xi}{\partial M} \longrightarrow \bar{\xi}(s) = \xi(sM)$$

scaled Parameters

and

$$m\gamma_m = -M\frac{\partial m}{\partial M} \longrightarrow \bar{m}(s) = s^{-1}m(sM)$$

with these definitions:

$$\left(-s\frac{\partial}{\partial s} + s\frac{\partial \bar{g}}{\partial s}\cdot\frac{\partial}{\partial g} + s\frac{\partial \bar{\xi}}{\partial s}\cdot\frac{\partial}{\partial \xi} - s\frac{\partial \bar{m}}{\partial s}\cdot\frac{\partial}{\partial m} - n_A\gamma_A - n_\psi\gamma_\psi + d_r\right)\tilde{\Gamma}^{(n)}(sp_i;\hat{g},\xi,m,M) = 0$$

which solution is

$$\tilde{\Gamma}^{(n)}(sp_i;\hat{g},\xi,m,M) =$$

$$= s^{d_r}\exp\left(-\int_1^s \frac{ds'}{s'}\left[n_A\gamma_A(\bar{g}(s),\bar{\xi}(s)) + n_\psi\gamma_\psi(\bar{g}(s),\bar{\xi}(s))\right]\right)\tilde{\Gamma}^{(n)}(p_i;\bar{g}(s),\bar{\xi}(s),\bar{m}(s),M)$$

$\tilde{\Gamma}^{(n)}(sp)$ is governed by $\bar{g},\bar{\xi},\bar{m}$

Note:

I.

$$\beta_\xi = \left[\left(\frac{13}{16} - \frac{\xi}{2}\right)C_1 - \frac{4}{3}\sum_r C_r - \frac{4}{3}\sum_r C_r\right]\frac{g\xi}{8\pi^2}$$

In general $\bar{\xi}(s)$ changes with s.

Landau gauge: $\xi = 0 \to \bar{\xi}(1) = 0 \to \bar{\xi}(s) = 0$

II.

$$s\frac{\partial \bar{m}(s)}{\partial s} = -(1+\gamma_m)\bar{m}(s) \longrightarrow \frac{\partial \bar{m}}{\partial s} \sim s^{-1}$$

$\bar{m}(s)$ decrease faster than $\bar{g}(s)$ $(g(\ln(s)))$.

the behavior to great momenta may generalised using $\bar{m}(s) \to 0$

Asymptotic freedom

Consider

$$\beta_g(\bar{g}) = -b\bar{g}^3 = s\frac{\partial \bar{g}}{\partial s}.$$

So

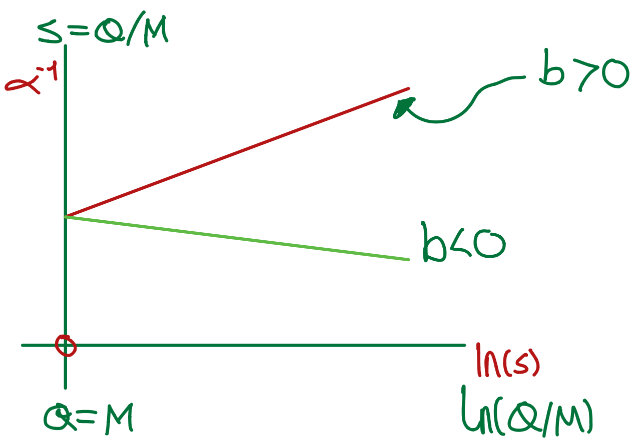
$$-\bar{g}^{-3}\frac{\partial \bar{g}}{\partial \ln(s)} = b \longrightarrow \frac{\partial \bar{g}^{-2}}{\partial \ln(s)} = 2b$$

Usually:

$$\alpha(s) = \frac{\bar{g}^2(s)}{4\pi} \longrightarrow \frac{\partial s^{-1}}{\partial \ln(s)} = 8\pi b \longleftarrow \text{constant}$$

$$\alpha(1) = \frac{\hat{g}^2(M)}{4\pi} =: \alpha(M)$$

$$: \quad \alpha^{-1}(s) = \alpha^{-1} + 8\pi b \ln(s)$$



$\alpha^{-1}(s) \to \infty$
$s \to \infty$

$\downarrow$

$\bar{g}^2(s) \to 0$ — asymtotic freedom

$\downarrow$

$\bar{g}^2 \to \infty$
$s \to 0$

big energy = small distances

Theory strongly coupled to small moments.

QCD: (SU(3))

$$b = \frac{1}{16\pi^2}\left(11 - \frac{2}{3}N_f\right)$$

$$N_f = 3 \to b = \frac{9}{16\pi^2} > 0$$

Quarks are not asymtotic states.

QCD is strongly coupled to strong distances.

QED:

U(1)
$C_r = q^2$

$$b = \frac{1}{16\pi^2}\left(-\frac{4}{3} N_G \left(1 + 0 + \frac{4}{9} + \frac{1}{9}\right)\right) = -\frac{56}{27}\frac{N_G}{16\pi^2} < 0$$

$e(1)\,;\, u(2/3)\,;\, d(-1/3)\,;\, \nu(0)$

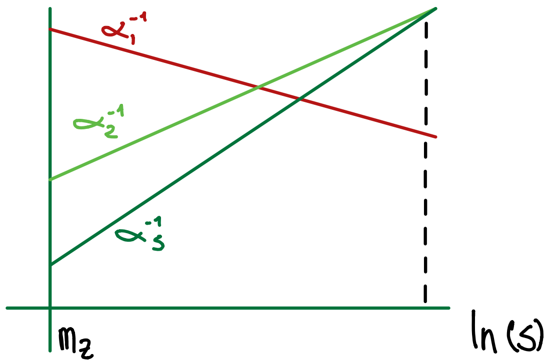
QED is not asymfotically free.

$\bar{e}(s)$ grow up with s.

SM: by convension:

$M = m_Z$

$SU(3) \times SU(2) \times U(1)_Y$



$Q_U \approx 10^{16}$ GeV.

Suggest $\alpha_1 \approx \alpha_2 \approx \alpha_3$ to high energy

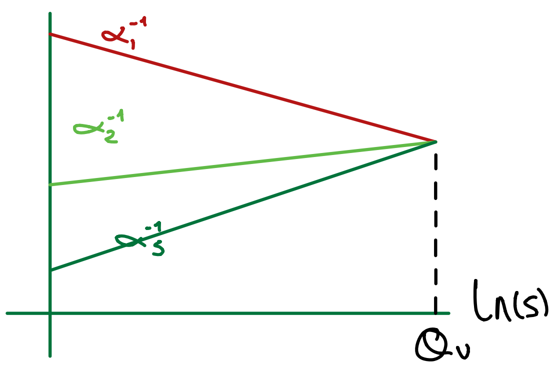
Just one α suggest only one class of interaction.

GUT:

$G \supset SU(3) \times SU(2) \times U(1)$

$G \simeq SU(5)\,,\; SO(10)\,,\; E_8 \,\ldots$

+ SUSY



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