Anomaly

Definition: The existence of a symmetry at classical level that doesn't stay at quantum level.

Noether theorem: Symmetry $\longrightarrow \partial^\mu j_\mu = 0$

$\rightarrow$ Anomaly : $\langle \partial^\mu j_\mu \rangle = a(x)$

Specific case: Chiral theories

$$\begin{array}{ccc} G_L & \times & G_R \\ \downarrow & & \downarrow \\ \psi_L & & \psi_R \end{array}$$

then,

$$W \propto \int DA^\mu_L D\psi D\bar{\psi} D\eta^* e^{iS_{eH}}$$

↳ No gauge invariant in general. (BRS)

↳ Possible violation of Slavnov-Taylor.

↳ S would depend on ξ !!

↳ Non-physical theory.

Unquantificable

Example: Axial Electrodynamics.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + iqV_\mu + igA_\mu\gamma_5)\psi - \frac{1}{4}F^2 - \frac{1}{4}G^2$$

$$\left.\begin{array}{l} F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \\ G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{array}\right\} : \quad U_V(1) \times U(1)_A$$

Gauge:

$$\psi \longrightarrow \psi' = e^{-iq\Lambda(x)} e^{-ig\alpha(x)\gamma_5}\psi$$

$$V_\mu \longrightarrow V'_\mu = V_\mu + \partial_\mu \Lambda(x)$$

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x)$$

$$\psi_L \longrightarrow e^{-i\theta_L(x)}\psi_L$$

$$\psi_R \longrightarrow e^{-i\theta_R(x)}\psi_R$$

$$\theta_{L,R} = q\Lambda \mp g\alpha$$

then

$$D\psi \longrightarrow \left[\operatorname{Det} e^{-iq\Lambda - ig\alpha\gamma_5}\right]^{-1} D\psi,$$

but

$$D\bar{\psi} \longrightarrow \bar{\psi}\left[\operatorname{Det} e^{iq\Lambda - ig\alpha\gamma_5}\right]^{-1},$$

therefore,

$$D\psi D\bar{\psi} \longrightarrow \operatorname{Det} e^{i2g\alpha\gamma_5}\, D\psi D\bar{\psi}$$

In a formal way,

$$\operatorname{Det} e^{2ig\alpha(x)\gamma_5} = e^{\operatorname{Tr}(2ig\alpha(x)\gamma_5)}$$

$\operatorname{Tr} \longrightarrow$ Sum over Dirac indices and space-time.

to evaluate the Jacobian:

(Fujikawa, Suzuki. Path integral & Quantum anomalies. Oxford.)

$$\psi(x) = \sum_n a_n \varphi_n(x)$$
$$\bar{\psi}(x) = \sum_n \bar{\varphi}_n(x)\, \bar{b}_n$$

$a_n, b_n \longrightarrow$ Grassmann.

where

$$i\not{D}\varphi_n(x) = \lambda_n \varphi_n \quad : \text{Energy eigenfunctions.}$$

$$\sum_n \varphi_n(x)\varphi_n^{+}(y) = \delta(x-y)\mathbb{1}$$

The basis formally diagonalise the action:

$$\int dx\, \bar{\psi}(i\not{D} - m)\psi = \lim_{N\to\infty} \sum_{n=1}^{N} (\lambda_n - m)\bar{b}_n a_n$$

$$\to \quad D\psi D\bar{\psi} = \lim_{N\to\infty} \prod_{n=1}^{N} da_n d\bar{b}_n$$

Under the chiral transformation:

$$\psi \longrightarrow \psi' := \sum a'_n \varphi_n(x)$$

$$e^{-ig\alpha\gamma_5}\psi = \psi - ig\alpha\gamma_5\psi$$
$$= \sum a_n\varphi_n(x) - ig\alpha\gamma_5 \sum a_n \varphi_n(x)$$

then,

$$a_n \longrightarrow a'_n = a_n + \sum_m i\int dx\, \varphi_n^{+}\alpha(x)\gamma_5\varphi_m a_m$$

Similarly,

$$b'_n = \bar{b}_n + \sum_m i\,\bar{b}_m \int dx\, \varphi_m^{+}\alpha(x)\gamma_5\varphi_n$$

finally,

$$\prod_{n=1}^{N} da_n' \, d\bar{b}_n' = \det\left[\delta_{nm} - ig\int dx\, \varphi_n^\dagger \alpha \gamma_5 \varphi_m\right]^{-1} \prod_{n=1}^{N} da_n$$

$$\times \det\left[\delta_{nm} - ig\int dx\, \varphi_n^\dagger \alpha \gamma_5 \varphi_m\right]^{-1} \prod_{n=1}^{N} d\bar{b}_n$$

α - Infinitesimal

$$\det\left[\mathbb{1} - i\int \alpha \cdots\right]^{-2} = \det\left[\exp \cdot \ln\left(\mathbb{1} - i\int \alpha\right)^{-2}\right]$$

$$= \exp\left[\mathrm{tr}\left(-2\ln\left(\mathbb{1} - i\int \alpha\right)\right)\right]$$

$$J := \exp \cdot \mathrm{tr}\left(2ig\int dx\, \varphi_n^\dagger \alpha(x) \gamma_5 \varphi_m\right)$$

$$J = \exp\left[2ig\int dx\, \alpha(x) \lim_{N\to\infty} \sum_{n=1}^{N} \varphi_n^\dagger(x) \gamma_5 \varphi_n(x)\right]$$

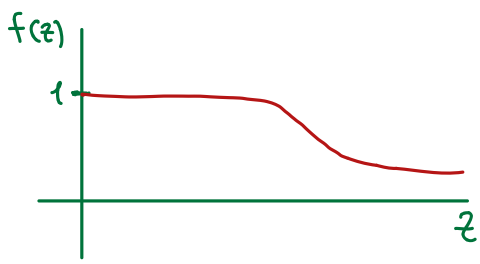
non-null $\qquad$ measure of the anomaly $=: a$

Evaluate $a(x)$:

$$a(x) = \lim_{N\to\infty} \lim_{M\to\infty} \sum_{n}^{N} \varphi_n^\dagger(x) \gamma_5 \underbrace{f\left(\frac{\lambda_n^2}{M^2}\right)}_{\text{Regulator}} \varphi_n(x)$$

$f(\lambda) \to$ smooth function.
$\quad \hookrightarrow$ Keeps gauge invariance.

$f(0) = 1$; $f(z\to\infty) = 0$.



Using: $\quad i\not{D}\varphi_n = \lambda_n \varphi_n$

$$a(x) = \lim_{M\to\infty} \mathrm{tr}\left(\gamma_5 f\left(\frac{-\not{D}^2}{M^2}\right)\right)$$

$$\{\varphi_n\} \longrightarrow \{e^{ikx}\}$$

$$a(x) = \lim_{M\to\infty} \mathrm{tr}\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x} \gamma_5 f\left(-\frac{\not{D}^2}{M^2}\right) e^{ik\cdot x}$$

$$D_\mu = \partial_\mu + iqV_\mu + igA_\mu\gamma_5 \rightarrow \not{D}^2 = \gamma^\mu D_\mu \gamma^\nu D_\nu = \gamma^\mu\gamma^\nu \tilde{D}_\mu D_\nu$$

$$\tilde{D}_\mu = D_\mu - 2igA_\mu\gamma_5 \qquad = \not{D}^2 + \frac{i}{4}[\gamma^\mu,\gamma^\nu](qF_{\mu\nu} + gG_{\mu\nu}\gamma_5) + 2ig\not{A}\gamma_5\not{D}$$

Moreover,
$$D_\mu \chi(x) e^{ikx} = [(ik_\mu + D_\mu)\chi]e^{ikx}$$

$\rightarrow k_\mu \rightarrow Mk_\mu$:

$$a(x) = \lim_{M\to\infty} \mathrm{tr}\int \frac{d^4k}{(2\pi)^4} M^4\gamma_5 f\Big[-\left(ik_\mu + \frac{D_\mu}{M}\right)^2$$
$$-\frac{i}{4M^2}[\gamma^\mu,\gamma^\nu](qF_{\mu\nu} + gG_{\mu\nu}\gamma_5) - \frac{2ig}{M}\not{A}\gamma_5\left(i\not{k} + \frac{\not{D}}{M}\right)\Big]$$

expanding f at $\mathcal{O}^4(1/M^4)$ and using $\mathrm{tr}\,\gamma_5 = 0 = \mathrm{tr}\,\gamma_5[\gamma^\mu,\gamma^\nu]$
$= \mathrm{tr}\,\gamma_5\gamma_\mu$

$f(k^2)$

but,
$$\mathrm{tr}\,\gamma_5[\gamma^\mu,\gamma^\nu][\gamma^\alpha,\gamma^\beta] = -16\,\varepsilon^{\mu\nu\alpha\beta}$$

we can show the result for $A_\mu \rightarrow 0$:

$$a(x) = -\frac{q^2}{32}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\cdot\int\frac{d^4k}{(2\pi)^4}f''(k^2)$$

$$d^4k \rightarrow i\,d^4k_E \qquad : i\int d^4k_E f''(-k_E^2) = i\pi^2\int_0^\infty dx\, f'(-x) = i\pi^2$$
$$= i\pi^2 k_E^2\, dk_E^2$$

therefore,
$$a(x) = \frac{q^2}{32\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

$$\rightarrow D\psi D\bar{\psi} \rightarrow \exp\left(2i\int dx\, g\alpha(x)\frac{q^2}{16\pi^2}\tilde{F}^{\mu\nu}F_{\mu\nu}\right)D\psi D\bar{\psi}$$

In the absence of A_μ:

$$\mathcal{L} \longrightarrow \mathcal{L} + g(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\gamma_5\psi$$

j_5^μ

then

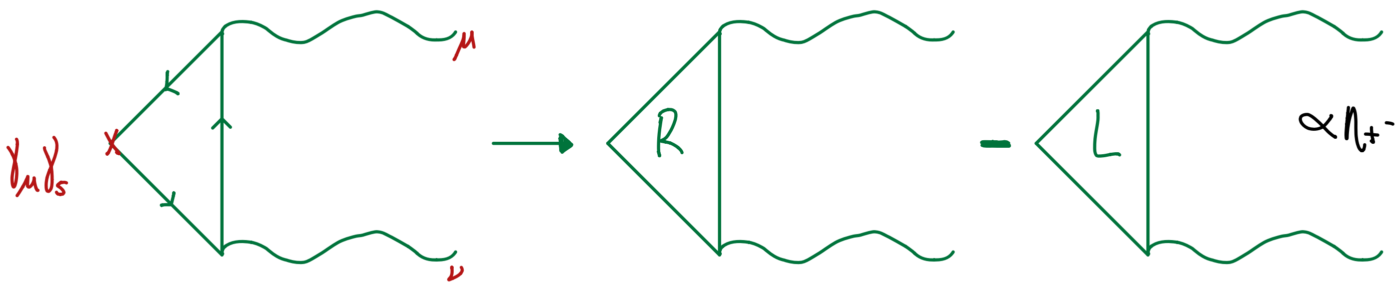
$$W \to \int D\Psi D\bar{\Psi} \exp i \left[\int dx \left(\mathcal{L} - \alpha\, \partial_\mu j_5^\mu(x) + g\alpha \frac{q^2}{8\pi^2} \tilde{F}\cdot F \right) \right]$$

the invariance of W, imply:

$$\partial_\mu j_5^\mu(x) = \frac{q^2}{8\pi^2} \tilde{F}^{\mu\nu} F_{\mu\nu}$$

Adler, Bell, Jacking anomaly.

from perturbative analysis:



$\propto n_+ - n_-$

then,

$$J = \exp[-2i\alpha(x) q^2 \underbrace{(n_+ - n_-)}_{=: n}]$$

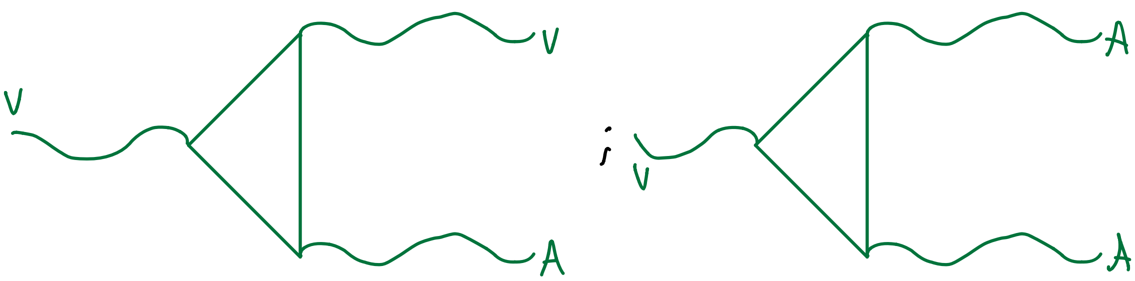
$=: n \to$ Pontryagin index

Non-abelian theories:

Vertices involve the factor

$$A_r^{abc} := \mathrm{tr}(T^a \{T^b, T^c\})_r \quad \to \text{zero to achieve } a(x) = 0$$

in the same way, we will have mixed anomalies.



$[G_1]^2 \cdot G_2$

SM:

$[SU(2)]^3$; $[SU(2)]^2 U(1) \to T_r Y = 0$

$[U(1)]^3 \longrightarrow T_r Y^3 = 0$

because $T_r T^a = 0$ for G non-abelians.

... OK!!