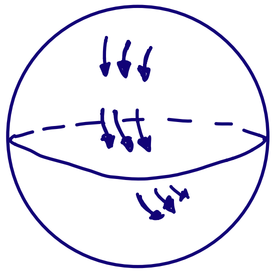
Vector fields



Given a function f and a vector field in $\mathbb{R}$, v, we can form the directional derivative of f in the v direction, and we will denote by $v(t)$.

Let $x^1, \dots, x^n$ in $\mathbb{R}^n$, and ∂_μ the partial derivative $\frac{\partial}{\partial x^\mu}$

So, if v has components $(v^1, \dots, v^n)$, then

$$v(f) = \vec{v} \cdot \nabla f = \sum_{\mu=1}^{n} v^\mu \partial_\mu f = v^\mu \partial_\mu f.$$

As $v(f) = v^\mu \partial_\mu f$

$$v = v^\mu \partial_\mu \quad \text{is an operator.}$$

Let's define vector fields in a manifold M.

The set of smooth functions (in reals) in a manifold M is denoted by $C^\infty(M)$, which are an algebra over the real numbers i.e., is closed under the sum and point multiplication.

$$f + g = g + f$$

$$f + (g + h) = (f + g) + h$$

$$f(gh) = (fg)h$$

$$(f + g)h = fh + gh$$

$$1f = f$$

$$\alpha(\beta f) = (\alpha\beta)f$$

$$\alpha(f + g) = \alpha f + \alpha g$$

$$(\alpha + \beta)f = \alpha f + \beta f$$

with $f, g, h \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$.

Is a commutative algebra $fg = gf$.

A vector field $v \in M$, is defined as a function from $C^\infty(M)$ to $C^\infty(M)$, which satisfies the following properties.

$$v(f + g) = v(f) + v(g)$$

$$v(\alpha f) = \alpha v(f)$$

$$v(fg) = v(f)g + f v(g).$$

$$\forall\, f,g \in C^\infty(M),\ \alpha \in \mathbb{R}.$$

Let $\mathrm{Vect}(M)$ the set of all the vector fields of M.

Given $v, w \in \mathrm{Vect}(M)$, let's define $v + w$.

$$(v+w)(f) = v(f) + w(f)$$

and given $v \in \mathrm{Vect}(M)$ and $g \in C^\infty(M)$, let's define gv by

$$(gv)(f) = g\,v(f)$$

Homework: Show that $v + w$ and $gw \in \mathrm{Vect}(M)$.

Prove that the following relations are valid for all $v, w \in \mathrm{Vect}(M)$ and $f, g \in C^\infty(M)$

$$\left.\begin{aligned} f(v+w) &= fv + fw \\ (f+g)v &= fv + gv \\ (fg)v &= f(gv) \\ 1v &= v \end{aligned}\right\}$$

$\mathrm{Vect}(M)$ form a modulus over $C^\infty(M)$.

$v = v^\mu \partial_\mu$, the vector fields $\{\partial_\mu\}$ generate $\mathrm{Vect}(\mathbb{R}^n)$.

Homework: Prove that if $v^\mu \partial_\mu = 0$, i.e., $v^\mu \partial_\mu f = 0$ $\forall f \in C^\infty(M)$, then $v^\mu = 0$, $\forall\, \mu = 1, \dots, n$.

Idea:

$$v^\mu \partial_\mu x^1 = v^1 \frac{\partial x^1}{\partial x^1} + \dots + v^n \frac{\partial x^n}{\partial x^1} = 0$$

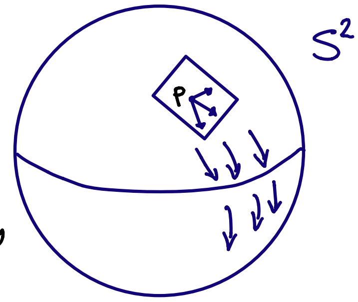
$$v^1 = 0$$

It follows for the other cases.

Tangent vectors

A vector field is though in M as a rule of asgne a row for each point of M (this class of vector is known as tangent vector).

To have the precise definition of tangent vector in $p \in M$ this vector must allow us the directional derivative in the point p.



For example, given a vector field $v \in M$, we can take the directional derivative of $f \in C^\infty(M)$, $v(f)$ and evaluate it in p.

$$V_p : C^\infty(M) \longrightarrow \mathbb{R}$$

$$V_p(f) = v(f)(p).$$

And think V_p as a tangent vector of p.

$$V_p(f+g) = V_p(f) + V_p(g)$$

$$V_p(\alpha f) = \alpha V_p(f)$$

$$V_p(fg) = V_p(f)g(p) + f(p)V_p(g)$$

Let T_pM the tangent space in p, denote the set of tangent vectors in $p \in M$.

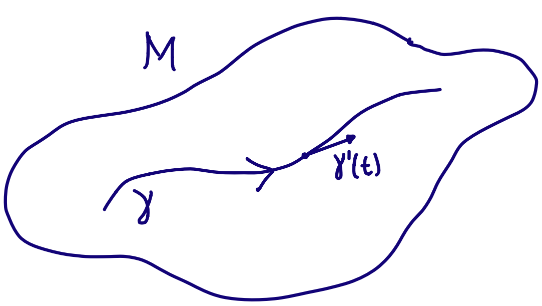
Why tangent vectors are given by rows?

First, we can sum tangent vectors $v, w \in T_pM$.

$$(v+w)(f) = v(f) + w(f)$$

$$(\alpha v)(f) = \alpha v(f)$$

This makes tangent space a vector space



The curve is $\gamma : \mathbb{R} \to M$, smooth, i.e., $\forall f \in C^\infty(M)$.

$f(\gamma(t))$ is smooth in t. Given a curve $\gamma : \mathbb{R} \to M$ and any $t \in \mathbb{R}$, the tangent vector $\gamma'(t)$, must be a vector in $T_{\gamma(t)}M$.

Thus, we define $\gamma'(t)$, as the function from $C^\infty(M)$ to $\mathbb{R}$, that takes any $f \in C^\infty(M)$ to the derivative

$$\frac{d}{dt} f(\gamma(t)) = \gamma'(t)[f]$$

↳ $\gamma'(t)$ derives functions in the direction of γ.