Quantization of Gauge fields through path integral

$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = -i[D_\mu, D_\nu] \quad ; \quad D_\mu = \partial_\mu + igA_\mu(x)$$

$$A_\mu(x) = A^a_\mu(x) T_a \quad ; \quad [T_a, T_b] = i f_{abc} T_c \, ; \; \mathrm{Tr}\, T_a T_b = \frac{1}{2}\delta_{cb}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] = F^a_{\mu\nu} T_a$$

Consider

$$D^\mu F_{\mu\nu} = 0 = \partial^\mu F_{\mu\nu} + ig[A_\mu, F_{\mu\nu}]$$

$$\partial^\mu F^a_{\mu\nu} + igf^{abc} A^{b\mu} F^c_{\mu\nu} = 0$$

I. Free theory: $g \to 0$ (Pertorvative)

II. Photon: $(\partial_\alpha \partial^\alpha \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu = 0$

but:

$$(K^2 \eta^{\mu\nu} - K^\mu K^\nu)^{-1} \longrightarrow \text{No exists} \longrightarrow$$ there is no a covariant propagator

Path Integral:

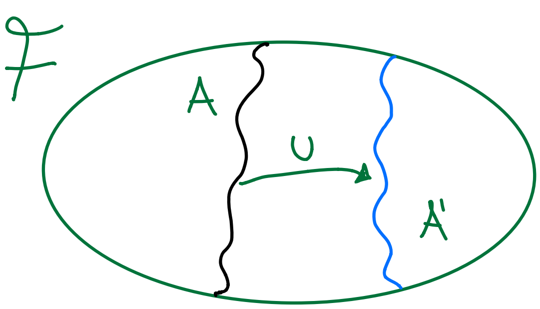
$$W[J^\mu_a] \propto \int DA^\mu_a \, e^{i\int dx [\mathcal{L}_{YM} + J^\mu_a A^a_\mu]}$$

$$\prod_a DA^\mu_a$$

III. No gauge invarian.

IV. Above all $A^\mu_a(x)$

Including: $\{A_\mu\} \ni A_\mu, A'_\mu = U(x) A_\mu U(x) - ig^{-1} U(x) \partial_\mu U^\dagger(x)$.



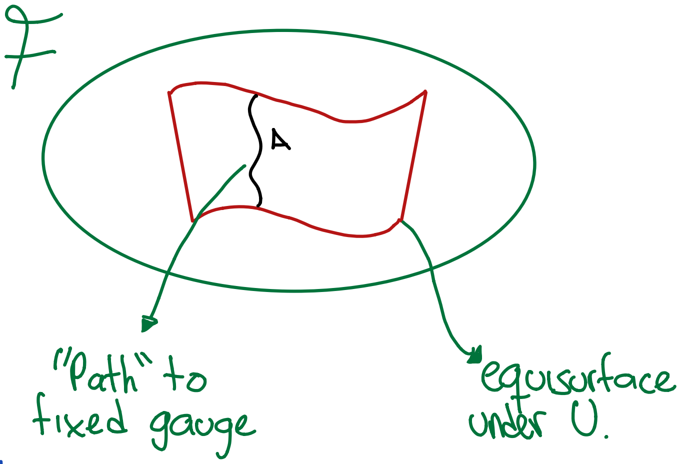
but:

$$S_{YM}[A] = S_{YM}[A']$$

then,

$$\int DA^\mu_a \, e^{iS_{YM}} \longrightarrow \infty !!$$

Answer: fix the gauge.



$=[A_\mu]$

Fix the gauge $\equiv$ Fix the class of equivalence.

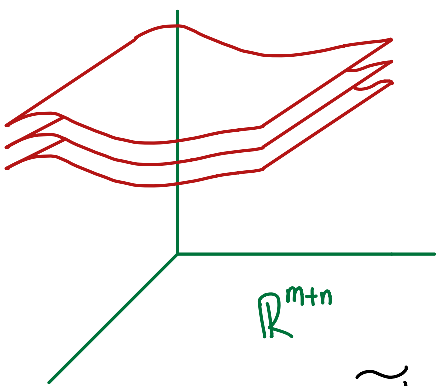
then, the integration space $\mathcal{F}/U$

Analogy: $\mathbb{R}^{n+m}$

(Coleman, 1973)

$F_i(x_1,\dots,x_{n+m})=0$

$i=1,\dots,m.$



$S=S(x_{m+1},\dots,x_{m+n})$

clearly,
$$W=\int dx_1\cdots dx_{m+n}\,e^{iS}\longrightarrow\infty$$

the finite part:

$$\widetilde{W}=\int dx_{m+1}\dots dx_{m+n}\,e^{iS}$$
$$=\int dx_{m+1}\cdots dx_{m+n}\int dF_1\cdots dF_m\,e^{iS}\prod_{i=1}^{m}\delta(F_i)$$

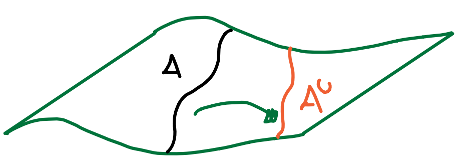
$$\widetilde{W}=\int dx_1\cdots dx_{n+m}\,e^{iS}\det\left(\frac{\partial F_i}{\partial x_k}\right)\prod_{i=1}^{m}\delta[F_i(x)]$$

Jacobian$\Big|_{\text{null surfaces.}}$

Faddeev-Popov Procedure (PLB 25 (1967) 29)

Analize J=0

$$S[A]=\int d^4x\,\mathcal{L}_{YM}(A)\longrightarrow\text{gauge invariant.}$$



$$S[A]=S[A^U]$$

$U=e^{-ig\Lambda(x)}$

$\Lambda(x)=T_a\Lambda_a(x)$

Gauge condition: $F_a(A)=0$

Spetially,

$$F(A)=\partial_\mu A^\mu_a-f_a(x)=0$$

$$Z \propto \int DA^a_\mu \, e^{iS[A]} \; ;$$

Definition:

$$\Delta[A] = \int DU \, \delta[F_a(A^U)]$$

$$\int DU \equiv \int D\Lambda_a(x) \qquad \prod_a \delta(F_a)$$

$$Z \propto \int DA^a_\mu \, \Delta^{-1}[A] \int DU \, \delta[F_a(A^U)] \, e^{iS[A]}$$

$$\int DA^{U_a}_\mu \equiv \int DA^a_\mu$$

$$Z \propto \int DU \underbrace{\int DA^a_\mu \, \Delta^{-1}[A] \, \delta[F_a(A)] \, e^{iS[A]}}_{=: \bar{Z}}$$

valued in a finite-fixed norm

$$\Delta[A] = \int DU \delta[F_a(A)]$$

$$\int DU := \int D\Lambda_a = \int DF_a \det\left(\frac{\delta \Lambda_b(x)}{\delta F_a(x')}\right)$$

$$\Delta[A] = \det\left(\frac{\delta \Lambda_b(x)}{\delta F_a(x')}\right)\Bigg|_{F_a=0} \qquad \Delta^{-1}[A] = \det\left(\frac{\delta F_a(x')}{\delta \Lambda_b(x)}\right)\Bigg|_{F=0}$$

We will say that for A, being an anti-hermitian matrix

$$\int d\theta_1^* \cdots d\theta_n^* d\theta_1 \cdots d\theta_n \exp(-\theta^\dagger A \theta) = \det A.$$

θ_i - Grassmann ; $\quad \theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$

Path integral:

$$\int D\eta^* D\eta \exp\left(i \int dx dy \, \eta^*_a(x) B_{ab}(y) \eta_b(y)\right) \propto \det(-B)$$

$$B_{ab}(x,y) = \frac{-\delta F_a(x)}{\delta \Lambda_b(y)}.$$

then,

$$\det\left(\frac{\delta F_a(x)}{\delta \Lambda_b(y)}\right) \propto \int D\eta^* D\eta \, e^{-i \int dx dy \, \eta^*_a(x) \frac{\delta F_a(x)}{\delta \Lambda_b(y)} \eta_b(y)}$$

η_a : Ghost.

Fixing the gauge

$$W[J^a_\mu] \propto \int DA_\mu \det\left|\frac{\delta F_a(x)}{\delta \Lambda_b(y).}\right| \delta[F_a]\, e^{i\int dx\,(\mathcal{L}_{YM} + J^a_\mu A^\mu_a)}$$

$F_a = \partial_\mu A^\mu_a - f_a(x)$: Unitary norm or R_ξ gauge

$$W[J] \propto \Phi_\xi \cdot \int DA_\mu \cdots$$

$$\Phi_\xi = \int Df_a\, e^{-i/2\xi \int dx\, f_a^2(x)}.$$

but,

$$\int Df_a\, e^{-i/2\xi \int dx\, f_a^2} \cdot \delta[F_a] = e^{-i/2\xi \int dx\, (\partial_\mu A^\mu_a)^2}$$

therefore

$$W[J^a_\mu] \propto \int DA_\mu \det\left|\frac{\delta F_a(x)}{\delta \Lambda_b(y).}\right| e^{i\int dx\,(\mathcal{L}_{YM} + J^a_\mu A^\mu_a - 1/2\xi(\partial_\mu A^\mu_a)^2)}$$

Term that fix the gauge

Consider the infinitesimal transformation

$$\delta A^\mu_a(x) \simeq \partial^\mu \Lambda_a(x) + g f_{abc} \Lambda_b(x) A^\mu_c(x)$$

then

$$\frac{\delta A^\mu_a(x)}{\delta \Lambda_b(y)} = \delta_{ab}\, \partial^\mu_x\, \delta^4(x-y) + g f_{abc}\, \delta^4(x-y)\, A^\mu_c(x)$$

Taking

$$F_a = \partial_\mu A^\mu_a - f_a(x)$$

Independent from Λ_a

Then,

$$\frac{\delta F_a(x)}{\delta \Lambda_b(y)} = \partial_{x\mu} \{ [\delta_{ab} \partial^\mu_x + g f_{abc} A^\mu_c(x)]\, \delta(x-y) \}$$

Therefore,

$$\det\left|\frac{\delta F_a(x)}{\delta \Lambda_b(y).}\right| \propto \int D\eta^* D\eta\, e^{-i\int dx\,dy\, \eta^*_a(x) \frac{\delta F_a(x)}{\delta \Lambda_b(y)} \eta_b(y)}$$

$$= \int D\eta^* D\eta\, e^{i\int dx dy\, \eta_a^*(x)\left[\partial_{x\mu}\right]\left[\partial_{ab}\partial_x^\mu + g f_{abc} A_c^\mu(x)\right]\delta(x-y)\}\eta_b(y)}$$

$$= \int D\eta^* D\eta\, e^{i\int dx\, \partial_\mu \eta_a^*(x)\left[\partial^\mu \eta_a(x) + g f_{abc}\eta_b(x) A_c^\mu\right]}$$

$\mathcal{L}_{FP} = \partial_\mu \eta_a^* D_{ab}^\mu \eta_b(x) =:$ Faddeev-Popov Lagrangian.

where

$$D_{ab}^\mu = (\delta_{ab}\partial^\mu + g f_{abc} A_c^\mu),$$

finally

$$W[J_\mu^a] \propto \int DA_\mu D\eta^* D\eta\, e^{i\int dx\left(\mathcal{L}_{YM} - 1/2\xi\,(\partial_\mu A_a^\mu)^2 + \mathcal{L}_{FP} + J_\mu^a A_a^\mu\right)}$$

Generating function in R_ξ-Gauge

Ghost: Grassmann fields that in $g\to 0$, obey $\Box\eta = 0$.

its couplings are of gauge: $\partial_\mu \eta^\dagger D^\mu \eta$

QED: $f_{abc} = 0 \longrightarrow$ Ghost uncoupling $\rightarrow$ does not have contribution to $W[J]$