Completion

Definition: A succession $(X_n) \in \mathbb{X}$, is called of Cauchy, if and only if, for each $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that $\|X_n - X_m\| < \varepsilon$, for all $n, m \in \mathbb{N}$, and $m, n > N$ i.e.,

$$\|X_n - X_m\| \longrightarrow 0 \quad \text{if} \quad n > m \longrightarrow \infty .$$

Properties:

I) $\lim\limits_{n \to \infty} \|X_n\|$ exists, since the triangle inequality shows that

$$\big| \|X_n\| - \|X_m\| \big| \leq \|X_n - X_m\|$$

therefore, the succession of real numbers $(\|X_n\|)$ is a Cauchy's succession on $\mathbb{R}$, then converges.

II) (X_n) is a bounded succession, since $(\|X_n\|)$ converges, thus is bounded i.e., there is $M \in \mathbb{R}$, such that $\|X_n\| \leq M, \forall n \in \mathbb{N}$.

Definition: A normed space $\mathbb{X}$ is called complete if and only if all Cauchy's succession (X_n) in $\mathbb{X}$ converges to an element $x \in \mathbb{X}$. The spaces $(\mathbb{R}^n, \|\cdot\|)$ and $(\mathbb{C}^n, \|\cdot\|)$ are complete with the euclidean norm. However, $(\mathbb{Q}, \|\cdot\|)$ is not complete.

Definition: A complete normed space $(\mathbb{X}, \|\cdot\|)$ is called Banach space

The steps to follow for proving if a space is complete:

For an arbitrary Cauchy's succession:

1. Find an element x, which we hope that it be the limit of X_n
2. Verify that $x \in \mathbb{X}$
3. Prove that $\|X_n - X\| \longrightarrow 0$ if $n \longrightarrow \infty$

Examples:

1. The space $C[a, b]$ is complete.

Let (X_n) a Cauchy's succession in $C[a,b]$, equiped with the supremum norm. So $\forall \varepsilon > 0$, there is $N(\varepsilon) \in \mathbb{N}$, such that

$$\sup_{t \in [a,b]} |X_n(t) - X_m(t)| < \varepsilon , \quad \forall\, n, m \in N(\varepsilon).$$

therefore, for each $t \in [a, b]$, there is a limit.

$$\lim_{n\to\infty} X_n(t) = X(t).$$

moreover, for $n > N(\varepsilon)$ fixed, taking m tending to infinity and computing the supremum we have

$$\sup_{t\in[a,b]} |X_n(t) - X(t)| < \varepsilon$$

Using the fact that if a succession of a continuous function (X_n) converges uniformly to $X(t)$, the limit is a continuous function.

Thus $C[a,b]$ equiped with the norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|$$

is a complete normed space.

11. The space l_p, $1 \leq p < \infty$, is a complete normed space.

$$(x_n) \in l_p, \quad \|x_n\|_p = \left(\sum_{i=1}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

Let $(X_m^n)_m$ be a Cauchy's succession with the norm $\|\cdot\|_p$. For all $\varepsilon > 0$ there are $n, k > N(\varepsilon)$ such that

$$\sum_{i=1}^{\infty} |X_m^n - X_m^k|^p < \varepsilon^p \qquad (*)$$

then if $n, k > N(\varepsilon)$, $|X_m^n - X_m^k| < k$, that means that each succession $(X_m^n)_n$ is a Cauchy's succession on $\mathbb{R}$ (or $\mathbb{C}$) and by completion of $\mathbb{R}$ (or $\mathbb{C}$) there is X_m, such that

$$\lim_{n\to\infty} X_m^n = X_m$$

Let $x = (X_m)_n$, then for all $M \in \mathbb{N}$.

$$\sum_{m=1}^{M} |X_m|^p = \lim_{n\to\infty} \sum_{m=1}^{M} |X_m^n|^p \leq \sup \|X^n\|_p^p < c < \infty$$

for some $c > 0$, since the cauchy's successions are bounded. Finally, we get that for any $M \in \mathbb{N}$, from

$$\sum_{m=1}^{M} |X_m^n - X_m^k|^p < \varepsilon^p \text{ ; with } n, k \text{ large enough.}$$

taking $K \to \infty$

$$\sum_{m=1}^{M} |x_m^n - x_m|^p < \varepsilon^p$$

and now $M \longrightarrow \infty$

$$\sum_{m=1}^{\infty} |x_m^n - x_m|^p < \varepsilon^p$$

thus $\|x^n - x\|_p < \varepsilon$ i.e., $x^n \longrightarrow x$.

Let E be a Banach space and let E_1 be a subspace of E. E_1 is a Banach space if and only if E_1 is closed. (Exercise).

Homework: Prove that l_∞, with the supremum norm is a Banach space.

$$(x_n) \in l_\infty, \quad \sup |x_n| < \infty$$

Definition: Let E, F normed spaces. We say that a linear map $T: E \longrightarrow F$ is a isometry if and only if

$$\|Tx\| = \|x\|_F ; \quad \forall x \in E.$$

Theorem (Completion): let E be normed vectorial space. There exists a complete normed space $\hat{E}$. A linear operator $T: E \longrightarrow \hat{E}$ such that:

a) $\|Tx\| = \|x\| \ \forall x \in E$ (Isometry)

b) $\operatorname{Im} T (:= TE)$ is dense in $\hat{E}$ ($\overline{TE} = \hat{E}$). We will call to $\hat{E}$ the completion of E.

Proof: Let $\mathcal{E}$ be the space of all cauchy's successions on E.

$$\bar{x} = (x_i \in E)_{i=1}^{\infty}$$

Let's introduce a seminorm in $\mathcal{E}$.

$$p(\bar{x}) = \lim_{i \to \infty} \|x_i\|$$

where $\bar{x} = (x_i)$ is from Cauchy. This limit exist, by the properties of Cauchy's succession.

Let's define $N = \{ \bar{x} : p(\bar{x}) = 0 \}$ i.e., is the space of the successions that converge to zero. So p defines a norm

In the quotient space $\mathcal{E}/N=\hat{\mathcal{E}}$, by the same formula

$$P(\bar{x})=\lim_{i\to\infty}\|x_i\|$$

for any representative $\bar{x}$, from the equivalence class

$$[\bar{x}]=\bar{x}+N\in\mathcal{E}/N$$

The operator $T:E\longrightarrow\hat{E}$ is defined as $Tx=[x]$, where $\bar{x}$ is the constant succession $\bar{x}=(x,x_1,\ldots)$ which is from Cauchy.

To prove the theorem we have to show two things

1. TE is dense in E.

II. $\hat{E}$ is complete.

I. Let $\bar{x}=(x_n)\in\mathcal{E}$. for all $\varepsilon>0$, there exists $N\in\mathbb{N}$, such that $\|x_n-x_m\|<\varepsilon$ for $m>n>N$. Let's define $\bar{x}_n\in\mathcal{E}$, by $\bar{x}_n=(x_n,\ldots,x_n,\ldots)$ a constant succession i.e.,

$$Tx_n=[\bar{x}_n]$$

So, the distance from $[\bar{x}]$ to $[\bar{x}_n]$ is

$$p([\bar{x}]-[\bar{x}_n])=p(\bar{x}-\bar{x}_n)\leq\varepsilon$$

Then for each $[\bar{x}]\in\hat{E}$. we may estimate with elements of TE, also is clear that for a Cauchy's succession $\bar{x}=(x_n)$, the succession Tx_n, of constant successions converges to $[\bar{x}]$.

II. Let's consider a Cauchy's succession in $\hat{\mathcal{E}}$. $[\bar{x}^{(n)}]$, we have that

$$p([\bar{x}^{(n)}]-[\bar{x}^{(m)}])=p(\bar{x}^{(n)}-\bar{x}^{(m)})\longrightarrow 0. \quad \text{if } n,m\longrightarrow\infty$$

Taking $\varepsilon_n>0$ converging to zero, and $x_n\in E$, such that

$$p(T^n-Tx_n)<\varepsilon_n$$

we can do it since in I we proved that $\overline{TE}=\hat{E}$.

Then for a succession $\bar{x}^{(0)}=(x_n)$ from cauchy in E, actually

$$\|x_n - x_m\| = p(Tx_n - Tx_m)$$

$$\leq p(Tx_n - \bar{x}^{(n)}) + p(\bar{x}^{(n)} - \bar{x}^{(m)}) + p(Tx_m - \bar{x}^{(m)}) \longrightarrow 0.$$

if $n, m \longrightarrow \infty$

Then $\bar{x}^{(0)} = (x_n)$ belongs to $\mathcal{E}$, and

$$[\bar{x}^{(0)}] = \lim_{n \to \infty} [\bar{x}^{(n)}],$$

actually

$$p([\bar{x}^{(0)}] - [\bar{x}^{(n)}]) = p(\bar{x}^{(0)} - \bar{x}^{(n)})$$

$$\leq p(\bar{x}^{(n)} - Tx_n) + p(Tx_n - \bar{x}^{(0)}) \longrightarrow 0$$

if $n \longrightarrow \infty$.

$\blacksquare$