Fermi Gas

$$\ln(\Xi(T,V,\mu)) = \sum_j \ln\{1 + \exp(-\beta(E_j - \mu))\}$$

$$\langle n_j \rangle = \frac{1}{\exp(\beta(E_j - \mu)) + 1}$$ Fermi-Dirac Distribution

$$\Phi(T,V,\mu) = -\frac{1}{\beta}\ln(\Xi(T,V,\mu)) = -pV$$

then

$$p(T,\mu) = k_B T \lim_{V\to\infty} \frac{1}{V}\ln(\Xi(T,V,\mu))$$

$$U = \sum_j E_j \langle n_j \rangle = \sum_j \frac{E_j}{\exp(\beta(E_j - \mu)) + 1}$$

$$N = \sum_j \langle n_j \rangle = \sum_j \frac{1}{\exp(\beta(E_j - \mu)) + 1}$$

$$E_j = E_{j,\sigma} = \frac{\hbar^2 k^2}{2m}$$ Free fermions

In the thermodynamic limit

$$\ln(\Xi) = \gamma \frac{V}{(2\pi)^3}\int d^3\vec{k}\, \ln\left\{1 + \exp\left(-\beta\left(\frac{\hbar^2 \vec{k}^2}{2m} - \mu\right)\right)\right\}$$

$$\gamma = 2S + 1$$ Spin multiplicity

$$\langle n_j \rangle = \left(\exp\left(\frac{\beta\hbar^2 k^2}{2m} - \beta\mu\right) + 1\right)^{-1}$$

$$N = \gamma \frac{V}{(2\pi)^3}\int d^3\vec{k}\left(\exp\left(\frac{\beta\hbar^2 k^2}{2m} - \beta\mu\right) + 1\right)^{-1}$$

$$U = \gamma \frac{V}{(2\pi)^3}\int d^3\vec{k}\,\frac{\hbar^2 k^2}{2m}\left(\exp\left(\frac{\beta\hbar^2 k^2}{2m} - \beta\mu\right) + 1\right)^{-1}$$

$$\ln(\Xi) = \gamma V \int_0^\infty D(E) \ln\left\{ 1 + \exp(-\beta(E-\mu)) \right\} dE$$

$$N = \gamma V \int_0^\infty D(E) f(E)\, dE$$

$$f(E) := \frac{1}{\exp(\beta(E_j-\mu)) + 1}$$

$$U = \gamma V \int_0^\infty E\, D(E) f(E)\, dE$$

$$D(E) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} = C E^{1/2}$$

states Density

Integrating by parts

$$\ln(\Xi) = \gamma V C \int_0^\infty E^{1/2} \ln\left\{ 1 + \exp(-\beta(E-\mu)) \right\} dE$$

$$= \frac{2}{3} E^{3/2} \ln\left\{ 1 + \exp(-\beta(E-\mu)) \right\} \Big|_0^\infty - \int_0^\infty E^{3/2} \frac{-\beta \exp(-\beta(E-\mu)\, dE}{1-\exp(-\beta(E-\mu))}$$

therefore,

$$\ln(\Xi) = \frac{2}{3} \beta \gamma V C \int_0^\infty E^{3/2} f(E)\, dE$$

$$= \frac{2}{3} \frac{\gamma V}{K_B T} \int_0^\infty E\, D(E) f(E)\, dE$$

$$= \frac{2}{3} \frac{V}{K_B T} \langle E \rangle = \frac{2}{3} \frac{V}{K_B T} u = \frac{2}{3} \frac{U}{K_B T}$$

As $\phi = -K_B T \ln(\Xi) = -pV \rightarrow U = \frac{3}{2} pV$