$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2$$

$$V(r) = \frac{1}{2}K - K\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

Timelike Geodesics $(K=1)$

$$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$$0 = \frac{dV(r)}{dr} = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} = \frac{1}{r^4}\left(r^2M - rL^2 + 3ML\right)$$

$$R_\pm = \frac{-(-L^2) \pm \sqrt{(-L^2)^2 - 4M(3ML^2)}}{2M}$$

$$= \frac{L^2 \pm \sqrt{L^4 - 12M^2L^2}}{2M}$$

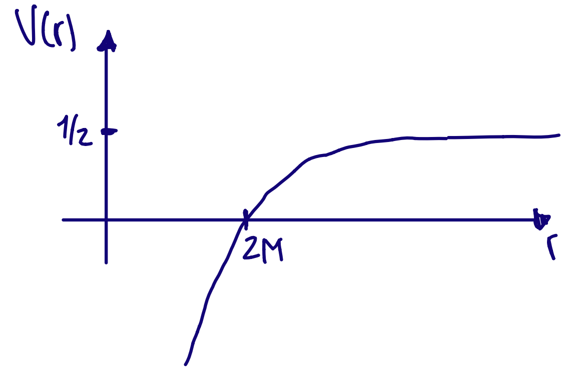
$$= \frac{L^2 \pm L\sqrt{L^2 - 12M^2}}{2M} \quad ; \quad L^2 \geq 12M^2$$

a. $L^2 = 12M^2$:

$$0 = V = \frac{1}{r^3}\left(\frac{r^3}{2} - Mr^2 + \frac{L^2 r}{2} - ML^2\right)$$

$$V(2M) = \frac{1}{(2M)^3}\left(4M^3 - 4M^3 + 12M^3 - M(12M^2)\right) = 0$$

There are no extrema of V.



A particle heading towards the center of atraction will fall directly to the $r=2M$ surface and will continue in to the $r=0$ singularity

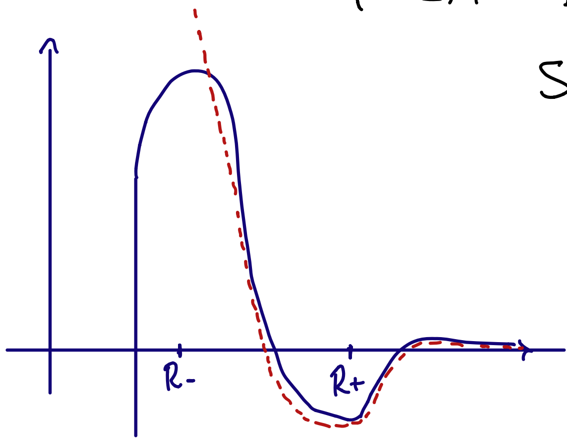
b. $L^2 > 12M^2$: The extremum R_+ is a minimum and R_- is a maximum of $V(r)$

$$\frac{\partial^2 V(r)}{\partial r^2} = -\frac{4}{r^5}\left(\cancel{\quad}\right)\Big|_{R_\pm} + \frac{1}{r^4}(2Mr - L^2)\Big|_{R_\pm}$$

$$2MR_+ - L^2$$

$$2M\left(\frac{L^2 + L\sqrt{\quad}}{2M}\right) - L^2 = L\sqrt{\quad} > 0$$

$$2MR_- - L^2 = 2M\left(\frac{L^2 - L\sqrt{\quad}}{2M}\right) - L^2 < 0$$



Stable orbits only at $r = R_+$

Null geodesics

$$K = 0 \Rightarrow V(r) = \frac{L^2}{2r^3}(r - 2M)$$

$$0 = \frac{\partial V}{\partial r} = \frac{L^2}{2}\left[-\frac{3}{r^4}(r-2M) + \frac{1}{r^3}\right] = \frac{L^2}{2r^4}(-2r + 6M)$$

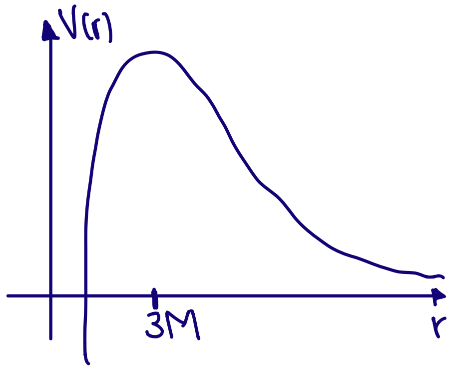
$$\Leftrightarrow -3(r-2M) + r = 0.$$

$$-2r + 6M = 0$$

Independent of L $\boxed{r = 3M}$

$$\frac{L^2}{2}\left[\left(-\frac{4}{r^5}\right)(-2r+6M) + \frac{1}{r^4}(-2)\right]$$

$$\frac{L^2}{2r^5}(8r - 24M - 2r)$$



Unstable photon orbits exist at $r=3M$.

$$\frac{\partial^2 V}{\partial r^2} = \frac{3L^2}{r^5}(r - 4M)\Big|_{r=3M} < 0$$

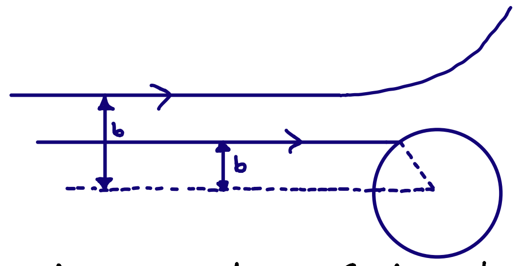
Gravity affects the propagation of light rays in the strong regime

Minimum energy

$$\frac{1}{2}E^2 = V(R=3M) = \frac{L^2}{2(3M)^3}(3M - 2M) = \frac{L^2 M}{2(27M^3)}$$

then

$$\frac{L^2}{E^2} = 27M^2$$



In flat spacetime $\frac{L}{E} =: b$ impact parameter of the light ray.

Impact parameter := distance of closest approach to $r=0$.

Define b_a as non-flat "apparent" impact parameter and let b_c be the critical impact parameter.

$$b_c = 3^{3/2} M$$

It corresponds to minimal energy.

Light bending effect.

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} = \frac{1}{2}E^2$$

$$\dot{r} = \sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}}$$

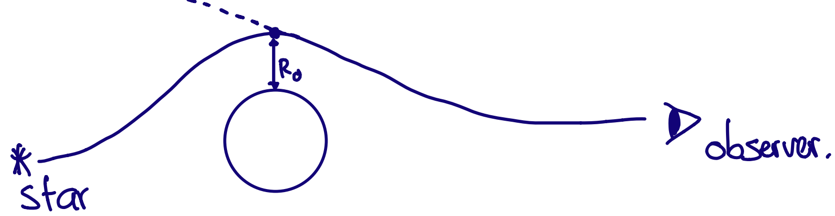
$$\dot{\phi} = \frac{L}{r^2} \longrightarrow L = \dot{\phi} r^2$$

$$\frac{d\phi}{dr} = \frac{d\phi}{d\tau}\frac{d\tau}{dr} = \frac{L}{r^2}\left[E^2 - \frac{L^2}{r^3}(r-2M)\right]^{-1/2}$$

$$\frac{dr}{d\tau}\frac{d\phi}{dr} = \frac{d\phi}{d\tau}$$

we wish to find $\Delta\phi = \phi_{+\infty} - \phi_{-\infty}$

R_0 := Turning point.



b_a must be greater than b_c

Turning point R_0 at $V(R_0) = \frac{E^2}{2}$

$$K = 0 \to \frac{1}{2}E^2 = V.$$

$$b_a = \frac{L}{E} \qquad \frac{L^2}{2R_0^3}(R_0 - 2M) = \frac{E^2}{2}$$

$$b_c = \frac{L}{E_{min}} \qquad \frac{L^2}{E^2}(R_0 - 2M) = R_0^3$$

$$\boxed{R_0^3 - b^2(R_0 - 2M) = 0}$$

$$R_0 = \frac{2b}{\sqrt{3}} \cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{3^{3/2}M}{b}\right)\right]$$

$$\frac{d\phi}{dr} = \frac{L}{r^2}\frac{1}{\sqrt{E^2 - \frac{L^2}{r^2} + \frac{2ML^2}{r^3}}}$$

$$= \frac{1}{r^2}\frac{1}{\sqrt{\frac{1}{b^2} - \frac{1}{r^2} + \frac{2M}{r^3}}}$$

$$= \frac{1}{\sqrt{r^4 b^{-2} - r^4 + 2Mr}}$$

$$\Delta\phi = \int_{-\infty}^{\infty} \frac{dr}{\sqrt{r^4 b^{-2} - r^4 + 2Mr}}$$

$$= 2\int_{R_0}^{\infty} \frac{dr}{\sqrt{r^4 b^{-2} - r^4 + 2Mr}}$$

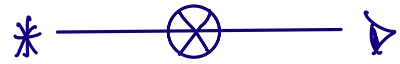
change:

$$u := \frac{1}{r} \qquad du = -\frac{1}{r^2}\,dr$$

$$r \to R_0 \qquad u \to \frac{1}{R_0}$$

$$r \to \infty \qquad u \to 0$$

$$\Delta\phi = 2\int_0^{1/R_0} \frac{du}{(b^{-2}-u^2+2Mu^3)^{1/2}}$$



- For the case of flat spacetime $M=0$ and then $R_0=b$.

$$\Delta\phi\Big|_{M=0} = 2\int_0^{1/R_0} \frac{du}{(b^{-2}-u^2)^{1/2}}$$

$$= 2\arcsin\left(\frac{u}{b^{-1}}\right)\Bigg|_0^{1/R_0}$$

$$= 2\arcsin\left(\frac{b}{R_0}\right)$$

$$= 2\arcsin 1 = 2\frac{\pi}{2} = \pi$$

- For $M \neq 0$ will be a deflection of light $\Rightarrow \Delta\phi \neq \pi$

$$\Delta\phi = 2\int_0^{1/R_0} \frac{du}{\left(\frac{1}{b^2}-u^2+2Mu^3\right)^{1/2}}$$

$$= 2\int_0^{1/R_0} \frac{du}{\left[\frac{1}{\left(\frac{R_0^3}{R_0-2M}\right)}-u^2+2Mu^3\right]^{1/2}}$$

$$b^2 = \frac{R_0^3}{R_0-2M} \quad \rightarrow \quad \boxed{R_0^3 = (R_0-2M)b^2}$$

$$\Delta\phi = \int_0^{1/R_0} \frac{du}{\sqrt{R_0^{-2}-2MR_0^{-3}-u^2+2Mu^3}}$$

Weierstrass elliptic.

Let us work at fixed R_0 and to first order in M.

$$\frac{\partial(\Delta\phi)}{\partial M}\Bigg|_{M=0} = 2\int_0^{1/R_0} \frac{R^{-3}-u^3}{(R_0^{-2}-2MR_0^{-3}-u^2+2Mu^3)^{3/2}}\,du\Bigg|_{M=0}$$

$$= 2\int_0^{1/b} \frac{b^{-3}-u^3}{(b^{-2}-u^2)^{3/2}}\,du = 4b^{-1}$$

finally

$$\delta := \Delta\phi - \pi \approx M\left(\frac{\partial\Delta\phi}{\partial M}\bigg|_{M=0}\right)$$

$$= \frac{4M}{b} \longleftarrow$$ Taylor expansioned at first order of M.

$\delta\phi \rightsquigarrow$ light bending of rays!