Linear operator

Let X, Y be Banach spaces, and let $T: X \longrightarrow Y$ a linear map, usually called operator, defined on X. T is said that is bounded if there exists a c, such that

$$\|Tx\|_Y \leq C \|X\|_X \quad \text{for all } x \in X.$$

If T is bounded, we define

$$\|T\| = \sup_{X \neq 0} \frac{\|Tx\|}{Tx}$$

Plainly $\|Tx\| < C$, with $\|X\| = 1$, then $\|T\| \leq C$.

The family of bounded linear operators is a linear space with respect to the sum and multiplication by scalar. We can see that $\|T\|$ defines a norm.

We will write as $\mathcal{L}(X, Y)$ the set of bound operators with the norm previously defined.

From the definition of norm, $\|Tx\| \leq \|T\| \cdot \|x\|$.

Theorem: Let X be a normed space and Y a complete normed space. Then $\mathcal{L}(X, Y)$ is from Banach.

Proof: Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy's succession on $\mathcal{L}(X, Y)$ for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$

$$\|A_n - A_m\| \leqslant \varepsilon$$

this implies that for all $x \in X$, and $m, n \geq N$

$$\begin{aligned} \|A_n(x) - A_m(x)\|_Y &= \|(A_n - A_m)(x)\|_Y \\ &\leq \|A_n - A_m\| \, \|X\| \\ &\leq \varepsilon \|X\| \end{aligned}$$

therefore, for all $x \in X$, the sequence $\{A_n(x)\}_{n=1}^{\infty}$ is Cauchy in Y. As Y is Banach this has a limit. Let $A(x) \in Y$ and therefore let's define for all $x \in X$.

$$A(x) = \lim_{n \to \infty} A_n(x).$$

A is a linear bounded operator, then

$$\|A(x)\|_Y \leq \sup_{n\in\mathbb{N}} \|A_n(x)\| \leq \|x\|_X \sup_{n\in\mathbb{N}} \|A_n\|$$

So,

$$\|A\| \leq \sup_{n\in\mathbb{N}} \|A_n\| \supset A \in \mathcal{L}(X,Y)$$

Now, we have to prove that $A_n \longrightarrow A$.

As $\{A_n\}_{n=1}^{\infty}$ is Cauchy, for all $\varepsilon>0$, exists $N\in\mathbb{N}$, such that for $n,m \geq N$.

$$\|A_m - A_n\| < \varepsilon.$$

this means that for such m,n and x, such that $\|x\|\leq 1$, we have that

$$\|A_m x - A_n x\| < \varepsilon.$$

Taking $m \longrightarrow \infty$, we have that $\|Ax - A_n x\| \leq \varepsilon$, for all x, with $\|x\|\leq 1$. Then $\|A-A_n\|\leq\varepsilon$ for any $n>N$, implying that

$$\lim_{n\to\infty} \|A-A_n\| = 0.$$

$\blacksquare$

Observations:

I. A is a bounded operator if and only if A is continuous

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

II. The set $\operatorname{Ker} A = \{x : Ax = 0\}$ is a closed subspace.

III. The theorem implies that for any normed space X, the dual space X^* is complete,

$$\mathcal{L}(X,\mathbb{R}) = X^*$$

Examples:

1. In $C([0,1])$, define

$$Af = \int_0^1 K(t,\tau) f(\tau)\, d\tau$$

with K a continuous function in two variables.

Prove that Af is bounded.

Ans:

$$\|Af\|_{C[0,1]} \leq \max_t |f| \max_t \int_0^1 |K(t,\tau)| d\tau$$

$$\sup_{f\neq 0} \frac{\|Af\|_{C[0,1]}}{\max\limits_{\substack{t \\ f\neq 0}} |f|} \leq \max_t \int_0^1 |K(t,\tau)| d\tau$$

$$\|A\| \leq \max_t \int_0^1 |K(t,\tau)| d\tau$$

II. The translation space in ℓ_2, defined by

$$Tx = (0, a_1, a_2, \dots, a_n, \dots)$$

for $a_n \in \ell_2$, $\|Tx\| = \|x\|$, then $\|T\| = 1$.

III. let $(\alpha_{ij})_{i,j=1}^{\infty}$ an infinity matrix and let

$$K^2 = \sum_{i,j=1}^{\infty} |\alpha_{ij}|^2 < \infty$$

then the operator A defined in ℓ_2 by

$$A((\alpha_i)_{i=1}^{\infty}) = (B_i)$$

with

$$B_i = \sum_{i=1}^{\infty} \alpha_{ij} \alpha_i \quad ; \text{for } i \in \mathbb{N}$$

is a linear bounded operator.

$$\|A(\alpha_i)\|_{\ell_2} \leq K \|(\alpha_i)\|_{\ell_2}.$$