Renormalization

Process to remove ultraviolet divergences:

I. Regularization: Asolating the divergences through an analytical process.

II. Renormalization: Extraction of divergences.

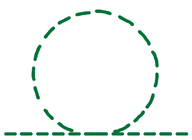
$$\varphi_R(x) = \frac{1}{Z^{1/2}} \varphi_B(x)$$

Should result in a finite theory.

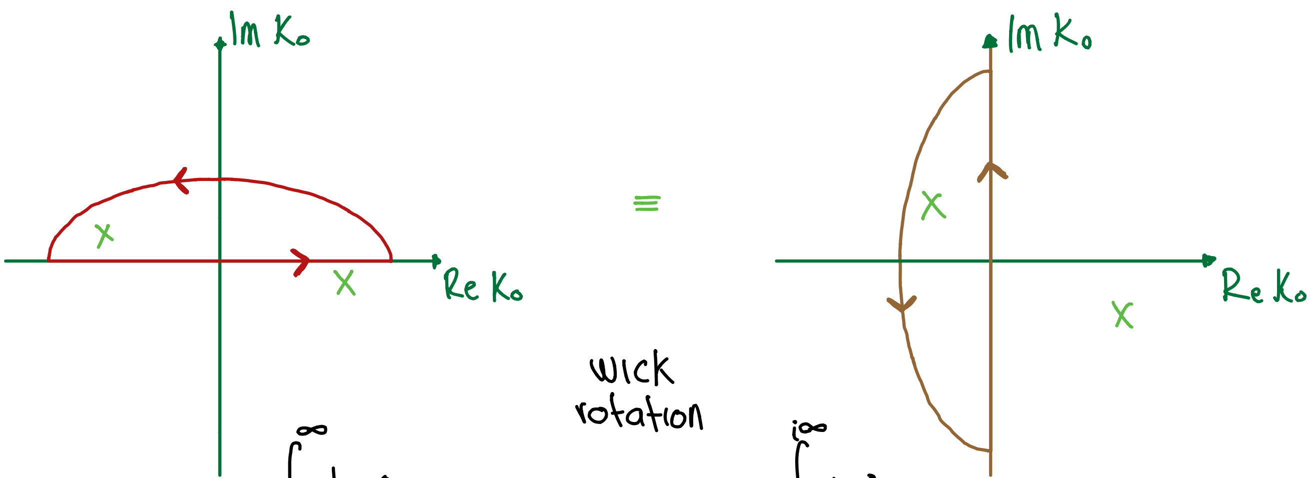
Dimensional regularization

Method: Loops integrals in d-dimensions $\longrightarrow d \to 4$; for $\lambda\varphi^4$.

Autoenergy:



$$-i\lambda_B \Delta_F(0) = (-i\lambda_B) \int \frac{d^4K}{(2\pi)^4} \frac{i}{K^2 - m_B^2 + i\epsilon}$$



$$\int_{-\infty}^{\infty} dK^0 \xrightarrow{\text{wick rotation}} \int_{-i\infty}^{i\infty} dK^0$$

Introduce "Euclidean" variables.

$$K^0 = i K_E^0$$

$$\vec{K} = \vec{K}_E$$

$$\longrightarrow K_E^2 := (K_E^0)^2 + |\vec{K}_E|^2$$

Therefore,

$$-i\lambda\Delta_F(0) = (-i\lambda_B)\int \frac{d^4K_E}{(2\pi)^4}\,\frac{1}{K_E^2+m_B^2} \quad 4D$$

Consider:

$$I(\omega, m_B) = \int \frac{d^{2\omega}K_E}{(2\pi)^{2\omega}}\,(K_E^2+m_B^2)^{-1} \rightarrow 2\omega D$$

Integration by spherically coordinates

$$d^{2\omega}K_E = \Omega_{2\omega}\,K_E^{2\omega-1}\,dK_E$$

$$\Omega_{2\omega} = \frac{2\omega\pi^{\omega}}{\Gamma(\omega+1)} \longrightarrow I(\omega,m_B) = \frac{2\omega\pi^{\omega}}{(2\pi)^{2\omega}\Gamma(\omega+1)}\int_0^{\infty} dK_E\,\frac{K_E^{2\omega-1}}{K_E^2+m_B^2}$$

$$K_E^2 = m_B^2 X \; ; \quad dK_E = \frac{m_B^2}{2K_E}\,dx = \frac{m_B^2}{2m_B\sqrt{X}}\,dx = \frac{m_B}{2\sqrt{X}}\,dx.$$

$$I(\omega,m_B) = \frac{2\omega\pi^{\omega}}{(2\pi)^{2\omega}\Gamma(\omega+1)}\int_0^{\infty} dx\,\frac{m_B}{2\sqrt{X}}\,\frac{(m_B X^{1/2})^{2\omega-1}}{m_B^2(X^2+1)}$$

$$= \frac{\omega\, m_B^{2\omega-2}}{2^{2\omega}\pi^{\omega}\Gamma(\omega+1)}\int_0^{\infty} dx\,\frac{X^{\omega-1}}{X^2+1}.$$

$$I(\omega,m_B) = \frac{m_B^{2\omega-2}}{(4\pi)^{2\omega}}\,\Gamma(1-\omega) \longrightarrow \text{Poles in } \omega = 1,2,\dots$$

Regularization scale:

$$m_B^{2\omega-2} = m_B^2\,(M^2)^{\omega-2}\left(\frac{m_B^2}{M^2}\right)^{\omega-2}$$

then,

$$I(\omega,m_B) = m_B^2\,(M^2)^{\omega-2}\,A^{\omega-2}\,\Gamma(1-\omega)$$

$$A = \frac{m_B^2}{4\pi M^2} \; ; \qquad A^{\omega-2}\Gamma(1-\omega) = \frac{1}{\omega-2} - (1+\Gamma'(1)) + \ln(A) + \mathcal{O}(\omega-2)$$

$$\Gamma'(1) = -\gamma_E = -0.577$$

Euler-Mascheroni constant.

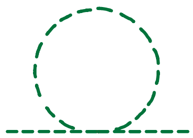
$$-i\lambda_B\Delta_F(0)\longrightarrow(-i\lambda_B)\frac{m_B^2}{16\pi^2}\cdot(M^2)^{\omega-2}\cdot\left[\frac{1}{\omega-2}-(1+\Gamma'(1))+\ln(A)+\theta(\omega-2)\right]$$

$$[S]=0\rightarrow[\mathcal{L}]=2\omega=[(\partial\varphi)^2]\rightarrow[\varphi]=\omega-1$$

$$\rightarrow[\lambda\varphi^4]=2\omega\rightarrow[\lambda]=2\omega-4(\omega-1)$$

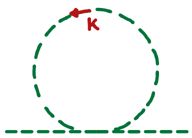
$$=4-2\omega$$

M appear to correct the dimensions of the loop factor.



$$\sim[m^2]\longrightarrow\Gamma^{(2)}.$$

Loop Integrals



$$\rightarrow I(\omega,m)=i2m\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2-m^2+i\epsilon)^{-1}=\frac{m^{2\omega-2}}{(4\pi)^\omega}\Gamma(1-\omega)$$

$\downarrow d/dm$

$$\frac{d}{dm}I(\omega,m)=i2m\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2-m^2+i\epsilon)^{-2}=(2\omega-2)\frac{m^{2\omega-3}}{(4\pi)^\omega}\Gamma(1-\omega)$$

$$\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2-m^2+i\epsilon)^{-2}=\frac{i\,m^{2\omega-4}}{(4\pi)^\omega}(1-\omega)\Gamma(1-\omega)$$

$$=\frac{i\,m^{2\omega-4}}{(4\pi)^\omega}\Gamma(2-\omega)$$

Iterating:

$$\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2-m^2+i\epsilon)^{-n}=i(-1)^n\frac{m^{2\omega-2n}}{(4\pi)^\omega}\frac{\Gamma(n-\omega)}{\Gamma(n)}$$

Expression valid $\forall\, n$. Is regular in $\omega=2$ for $n>2$ as we hoped.

Poles in $n, n+1, \dots$

with $k'=k+p$ we can show that:

$$\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}(k^2+2pk-m^2+i\epsilon)^{-n}=i(-1)^n\frac{\Gamma(n-\omega)}{\Gamma(n)}\frac{(m^2+p^2)^{\omega-n}}{(4\pi)^\omega}$$

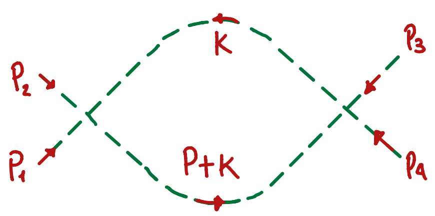
Deriving with respect to p_μ, p_ν

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu (k^2+2pk-m^2+i\epsilon)^{-n} = i(-1)^n \frac{\Gamma(n-\omega)}{\Gamma(n)} \frac{(m^2+p^2)^{\omega-n}}{(4\pi)^\omega}(-p_\mu)$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} k_\mu k_\nu (k^2+2pk-m^2+i\epsilon)^{-n}$$

$$= i(-1)^n \frac{\Gamma(n-\omega)}{\Gamma(n)} \frac{(m^2+p^2)^{\omega-n}}{(4\pi)^\omega}\left[p_\mu p_\nu (n-\omega-1) - \frac{1}{2}\eta_{\mu\nu}(m^2+p^2)\right]$$

Vertex : $\tilde{\Gamma}^{(4)}$



$$\longrightarrow \int \frac{d^4k}{(2\pi)^4}(k^2-m_B^2+i\epsilon)^{-1}\left[(p+k)^2-m_B^2+i\epsilon\right]^{-1} \equiv J.$$

$P = P_1+P_2+P_3+P_4$

Feynman Integral.

$a = (p+k)^2 - m_B^2$

$b = k^2 - m_B^2$

$$\frac{1}{ab} = \int_0^1 dx\,(ax+b(1-x))^{-1}$$

$$\longrightarrow J = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\int_0^1 dx\left[(p+k)^2x - m_B^2x + (k^2-m_B^2)(1-x)+i\epsilon\right]^{-2}$$

$$k^2x + p^2x + 2pkx - m_B^2x + k^2 - k^2x - m_B^2 + m_B^2x + p^2x^2 - p^2x^2$$

$$= (k+px)^2 + p^2x(1-x) - m^2$$

$$k' = k+px \quad ; \quad -\mu^2(x) = p^2x(1-x) - m^2$$

then

$$J = \int_0^1 dx \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\left[k'^2 - \mu^2(x) + i\epsilon\right]^{-2},$$

finally

$$J(\omega, m_B, s) = \frac{i\,\Gamma(2-\omega)}{(4\pi)^\omega}\int_0^1 dx\left[m_B^2 - p^2x(1-x)\right]^{\omega-2}.$$

Adimensionalizing the integral with M: $\quad A = \dfrac{m_B^2}{4\pi M^2}$

$$J = i\frac{1}{(4\pi)^2}(M^2)^{\omega-2}\int_0^1 dx\, A^{\omega-2}\left[1-\frac{p^2x(1-x)}{m_B^2}\right]^{\omega-2}\Gamma(2-\omega)$$

$$= i\frac{1}{16\pi^2}(M^2)^{\omega-2}\int_0^1 dx\left[\frac{1}{2-\omega}+\Gamma'(1)-\ln(A)-\ln\left(1-\frac{p^2}{m_B^2}x(1-x)\right)\right.$$
$$\left.+\mathcal{O}(\omega-2)\right]$$

Pole + finite part

$$\lambda_B^2 J = \frac{i}{16\pi^2}\lambda_B^2(M^2)^{\omega-2}\left[\frac{1}{2-\omega}+\Gamma'(1)-F(s,m_B,M)\right]+\mathcal{O}(\omega-2)$$

$$F(s,m_B,M)=\int_0^1 dx\,\ln\left[\frac{m_B^2-sx(1-x)}{4\pi M^2}\right]$$

$s=p^2=(p_1+p_2)^2$; if $s<4m_B^2 \to F$ is easy to integrate.

Remark: J has infrared divergences: $m_B^2\to 0$ and $s\to 0$.

$\to$ Does not contribute to observables in QED

$\to$ Must be regularizated.

Feynman Integral

$$\frac{1}{ab}=\int_0^1 dx\,(ax+b(1-x))^{-2}$$

Is particular case from

$$\frac{1}{a^\alpha b^\beta}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx\,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}}\quad ;\ (\alpha,\beta>0)$$

Also the utility.

$$\frac{1}{a_1\cdots a_n}=\Gamma(n)\int_0^1 dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{n-2}}dx_{n-1}\left[a_1(1-x_1)+a_2(x_1-x_2)+\cdots+a_n x_{n-1}\right]^{-n}$$

$$\frac{1}{a^{\alpha_1}\cdots a^{\alpha_n}}=\frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}\int_0^1 dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{n-2}}dx_{n-1}\frac{(1-x_1)^{\alpha_1-1}(x_1-x_2)^{\alpha_2-1}\cdots(x_{n-1})^{\alpha_n-1}}{[a_1(1-x_1)+a_2(x_1-x_2)+\cdots+a_nx_{n-1}]^{(\alpha_1+\cdots+\alpha_n)}}$$