Canonical Quantization II

$$(\Box + m^2)\phi = 0 \longrightarrow \phi_k = e^{\pm ikx} \qquad k^2 = m^2$$

$$\omega_k = \sqrt{\vec{k}^2 + m^2}$$

Source:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

then

$$(\Box + m^2)\phi(x) = j(x)$$

which solution is

$$\phi(x) = \phi_0(x) + i\int d^4y\, D(x-y)\, j(y)$$

propagator

$$(\Box + m^2)D(x+y) = -i\,\delta^4(x-y)$$

Fourier expansion:

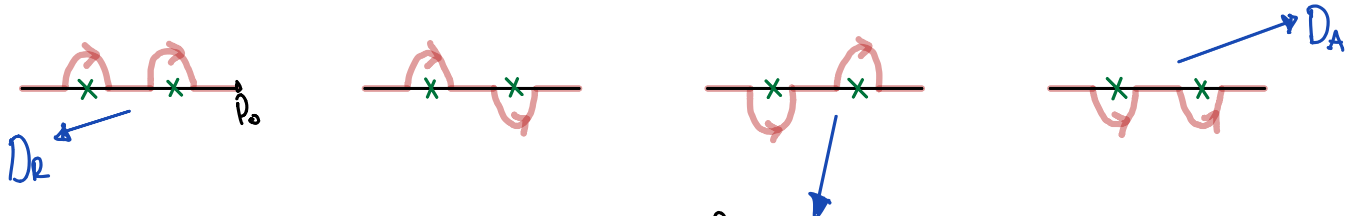
$$D(z) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \tilde{D}(p)$$

then

$$(-p^2 + m^2)\tilde{D}(z) = -i \longrightarrow \tilde{D}(p) = \frac{i}{p^2 - m^2}$$

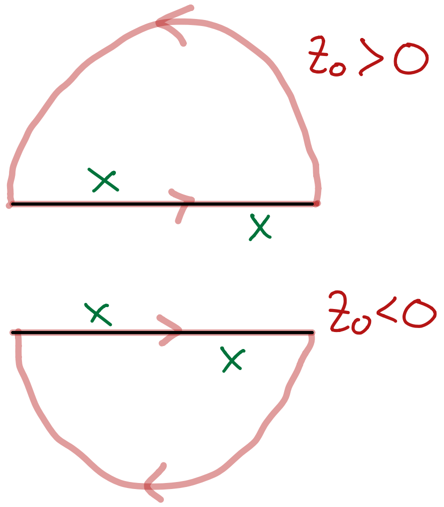
$$\longrightarrow D(z) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ipz}$$

$D(z)$ has poles at $p^2 = m^2$ —— $(p^0)^2 = E^2 = \vec{p}^2 + m^2$



$$\Delta_F(z) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ipz}$$

Feynman propagator



$$-\oint \frac{dp^0}{2\pi}\,\frac{e^{-ip^0 z_0}}{[p^0-(E-i\epsilon)][p^0+(E-i\epsilon)]}$$

$$\underset{\epsilon\to 0}{=}\frac{1}{2E}\left[\Theta(z_0)e^{-iEz_0}+\Theta(-z_0)e^{iEz_0}\right]$$

$$\Delta_F(z)=\int \tilde{d}^3p\left[\Theta(z_0)e^{-ip\cdot z}+\Theta(-z_0)e^{ipz}\right]$$

Consider

$$\int \tilde{d}^3p\; e^{-ip(x-y)}=\int \tilde{d}^3p\,\tilde{d}^3q\; e^{-ipx}e^{iqy}(2\pi)^3 2E\,\delta^3(\vec{p}-\vec{q})$$

$$=\int \tilde{d}^3p\,\tilde{d}^3q\; e^{-ipx}e^{iqy}\;\langle 0|[a_p,a_q^\dagger]|0\rangle$$

$$\langle 0|[a_p,a_q^\dagger]|0\rangle = \langle 0|a_p a_q^\dagger|0\rangle$$

Then.

$$\int \tilde{d}^3p\; e^{-ip(x-y)}=\langle 0|\phi(x)\phi(y)|0\rangle$$

Therefore,

$$\Delta_F(x-y)=\Theta(x_0-y_0)\langle 0|\phi(x)\phi(y)|0\rangle+\Theta(y_0-x_0)\langle 0|\phi(x)\phi(y)|0\rangle$$

Notice that,

$$[Q,\phi(x)]=-\phi(x)\;;\quad [Q,\phi^\dagger(x)]=-\phi^\dagger(x).$$

if

$$Q|q\rangle=q|q\rangle \longrightarrow Q\,\phi|q\rangle=(q-1)\,\phi|q\rangle \rightarrow \phi|q\rangle\propto|q-1\rangle$$

$$Q\,\phi^\dagger|q\rangle=(q+1)\,\phi^\dagger|q\rangle\rightarrow \phi^\dagger|q\rangle\propto|q+1\rangle$$

Definition: ("Ordered timelike product")

$$T(\phi(x)\chi(y))=\Theta(x_0-y_0)\phi(x)\phi(y)+\Theta(x_0-y_0)\phi(y)\phi(x)$$

therefore,

$$\Delta_F(x-y)=\langle 0|T(\phi(x)\phi(y))|0\rangle$$

$$\Delta_F(x-y)=\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle$$ complex case

Dirac field

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi \rightarrow \pi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = i\psi^\dagger \longrightarrow H = \bar\psi(-i\vec\gamma\cdot\vec\nabla + m)\bar\psi$$

solution:

$$\psi(x) = \sum_s \int \widetilde{d^3p}\,[a(p,s)u(p,s)e^{-ipx} + b^\dagger(p,s)v(p,s)e^{ipx}]$$

then

$$H = \sum_s \int \widetilde{d^3p}\,\mathcal{E}[a^\dagger(p,s)a(p,s) - b(p,s)b^\dagger(p,s)]$$

observation:

$$[b(p,s), b^\dagger(q,s)] = (2\pi)^3\, 2\mathcal{E}_p\, \delta^3(\vec p - \vec q)$$

Under normal ordering $\rightarrow$ negative energy!

We may say that:

$$\{\psi_\alpha(\vec x, t), \psi^\dagger_\beta(\vec x', t)\} = \delta^3(\vec x - \vec x')\,\delta_{\alpha\beta}$$

other null

$$a(p,s) = u^\dagger(p,s)\int d^3x\, e^{ipx}\psi(x)$$

$$b^\dagger(p,s) = v^\dagger(p,s)\int d^3x\, e^{-ipx}\psi(x)$$

$$\longrightarrow \{a(p,s), a^\dagger(q,r)\} = (2\pi)^3 2\mathcal{E}_p\,\delta^3(p-q)\,\delta_{rs} = \{b(p,s), b^\dagger(q,r)\}$$

other null

Fock space:

$$b(p,s)|0\rangle \equiv 0$$

$$a(p,s)|0\rangle \equiv 0$$

$$|1_{bs}\rangle = b^\dagger(p,s)|0\rangle \longrightarrow b^\dagger(p,s)|1_{bs}\rangle = b^\dagger(p,s)b(p,s)|0\rangle$$

$$= |b^\dagger(p,s), b^\dagger(p,s)|0\rangle = 0$$

It can't be generated states with two identical particles. $\longrightarrow$ Exclusion Principle

$$\{a^\dagger(p,s), a^\dagger(q,r)\} = 0 \longrightarrow a^\dagger(p,s)\,a^\dagger(q,r)|0\rangle = -a^\dagger(q,r)\,a^\dagger(p,s)|0\rangle$$

Fermi's statistic

$$:Q: = \int d^3x :\psi^\dagger\psi: = \int d^3p \sum_s [a^\dagger(p,s)a(p,s) - b^\dagger(\vec{p},s)b(p,s)]$$

$a^\dagger(p,s) \longrightarrow$ generate particles : $E>0, \vec{p}, s, q=+1$

$b^\dagger(p,s) \longrightarrow$ generate particles: $E<0, \vec{p}, s, q=-1.$

Propagator:

$$(i\partial\!\!\!/ - m)\psi(x) = f(x)$$

Green function: $(i\partial\!\!\!/_x - m)S(x-y) = i\delta^4(x-y)$

then

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\cancel{p\!\!\!/ - m}} e^{-ip(x-y)}$$

$$= \frac{i(p\!\!\!/ - m)}{p^2 - m^2}$$

$\Lambda_+ = \frac{p\!\!\!/ + m}{2}$: energy projector (particles)

Feynman propagator:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ - m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

$$S_F(x-y) = \langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle$$

Electromagnetic field

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

I. $\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F_{\mu 0} \longrightarrow \Pi_0 = 0$ primary constrain

$\Pi_i = i$

$\longrightarrow [A_\mu(\vec{x},t), \Pi_\nu(\vec{x}\,',t)] \neq i\eta_{\mu\nu}\delta^3(\vec{x} - \vec{x}\,')$!!

II. Equation of motion: $\partial^\nu F_{\nu\mu} = 0 \longrightarrow \Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = 0$

$\Box A_\mu = 0$, in the Lorentz norm: $\partial_\nu A^\nu = 0$ ...

$$(\Box \eta_{\mu\nu} - \partial_\nu \partial_\mu) A^\nu = 0$$

the Green function:

$$\tilde{G}_{\mu\nu} \propto [-K^2 \eta_{\mu\nu} + K_\mu K_\nu]^{-1}$$

Does not exist!

"forced solution": fix the norm

- A_μ has two degrees of physical degrees.

Lorentz: $\partial_\mu A^\mu = 0 \longrightarrow A_\mu = \epsilon_\mu e^{\pm iKx}$; $K^2 = 0$; $K \cdot \epsilon = 0$

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda \; ; \; \Box \Lambda = 0 : \; \epsilon_0 = 0 \longrightarrow \vec{K} \cdot \vec{\epsilon} = 0$$

then

$$A_\mu(x) = \sum_\lambda \int \tilde{d}^3 p \, [a(p,\lambda) \epsilon_\mu^\lambda(p) e^{-ipx} + a^\dagger(p,\lambda) \epsilon_\mu^{\lambda *} e^{ipx}]$$

$$\epsilon_{\mu(p)}^{\lambda *} \epsilon_{(p)}^{\lambda' \mu} = -\delta^{\lambda\lambda'} \quad ; \quad \lambda = 0, \dots, 3.$$

but:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 \longrightarrow \Pi_0 = -\partial_\mu A^\mu$$

The norm should not be imposed at the operator level. (off-shell)

Gupte - Blenler:

$$\partial_\mu A^\mu |0\rangle = 0$$

Radiation norm: $\quad \phi = 0 \,, \quad \nabla \vec{A} = 0 = \Pi_0$

then

$$[A_i(\vec{x},t), \Pi_j(\vec{x}',t)] = -i \delta_{tr}^{ij} (\vec{x} - \vec{x}')$$

transverse Delta Dirac: $\displaystyle \int \frac{d^3 K}{(2\pi)^3} e^{i\vec{K}(\vec{x}-\vec{x}')} \left(\delta^{ij} - \frac{K^i K^j}{K^2} \right)$

then $[\nabla \cdot \vec{A}_i E_j] = 0$

Lorentz propagator:

$$\Box A_\mu = J_\mu(x) \longrightarrow \Box_x D_{\mu\nu}(x-y) = i\delta(x-y)\,1_{\mu\nu}$$

then,

$$\tilde{D}_F^{\mu\nu}(q) = \frac{-i}{q^2+i\epsilon}\,1_{\mu\nu}$$

therefore,

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4}\,\tilde{D}_F^{\mu\nu}(q)\,e^{-iq(x-y)}$$

$$= \langle 0|T(A^\mu(x)A^\nu(y)|0\rangle$$

R_ξ - gauge:

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{1}{2\xi}(\partial A)^2$$

$$\Pi_0 = -\frac{1}{\xi}(\partial\cdot A)$$

$$\rightarrow \tilde{D}_F^{\mu\nu}(\eta) = \frac{i}{q^2+i\epsilon}\left(-\eta^{\mu\nu} + (1-\xi)\frac{K^\mu K^\nu}{K^2}\right)$$

$\xi \longrightarrow 1$: Feynman gauge

$\xi \longrightarrow 0$: Landau gauge

Indefinition of $\mathcal{L}$

$\xi \rightarrow \infty$: no gauge, no propagator