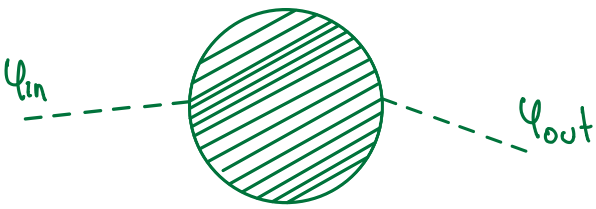
Interacting fields



$S = S[\varphi_0]$: Path integral with fixed boundary conditions, $\varphi_0(x)$

$$\varphi_0 \longrightarrow \begin{cases} \varphi_{in} & t \to -\infty \\ \varphi_{out} & t \to \infty \end{cases}$$

In vacuum ($\varphi_0 \to 0$):

$$W[J] \propto \int D\varphi \, e^{iS[\varphi, J]}$$

where,

$$S[\varphi, J] = \int dx \, (\mathcal{L} + J\varphi) = S_0[\varphi, J] + S_{int}[\varphi]$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

clearly, if $\mathcal{L}_{int} \to 0$, then $W[J] \to W_0[J]$.

therefore,

$$e^{iS[\varphi,J]} = e^{iS_{int}[\varphi]} e^{iS_0[\varphi, J]}$$

Using,

$$\frac{\delta}{\delta J(x)} e^{iS_0[\varphi, J]} = i\varphi(x) e^{iS_0[\varphi, J]}$$

$$\longrightarrow \quad \varphi(x) \longrightarrow -i \frac{\delta}{\delta J(x)}$$

therefore

$$e^{iS[\varphi,J]} = e^{i\int dx \, \mathcal{L}_{int}\left(-i \frac{\delta}{\delta J(x)}\right)} e^{iS_0[\varphi, J]}$$

$$W[J] \propto e^{i\int dx \, \mathcal{L}_{int}\left(-i \frac{\delta}{\delta J(x)}\right)} W_0[J]$$

$W[0] = 1$

with no-null boundary conditions:

$$\underset{\langle \varphi_{out} | \varphi_{in} \rangle^J}{\overset{\textstyle W[J, \varphi_0]}{\shortparallel}} \quad : \int D\varphi \quad \text{valued over } \varphi \to \varphi_0 \text{ in the boundary.}$$

How are related $W[J, \varphi_0]$ and $W[J]$?

$$W[J, \varphi_0] \propto \int D\varphi \, e^{iS[\varphi, J]}$$

Is evidently $W[J,\varphi_0] \propto e^{i\int dx\, \mathcal{L}_{int}\left(-i\frac{\delta}{\delta J(x)}\right)} W_0[J,\varphi_0]$

Consider in $W_0[J,\varphi_0]$ the change of variable.

$$W[J,\varphi_0] \propto \int D\varphi\, e^{iS[\varphi,J]} : \qquad \varphi(x) \longrightarrow \varphi(x) + \varphi_0(x)$$

$\varphi(x) \to 0$ in $t \to \pm\infty$

and $(\Box + m^2)\varphi_0(x) = 0.$

The action

$$S[\varphi,J] = \int dx \left[-\frac{1}{2}\varphi(\Box + m^2)\varphi + J\varphi\right]$$

$$\varphi \to \varphi + \varphi_0 \longrightarrow \int dx \left[-\frac{1}{2}\varphi(\Box + m^2)\varphi + J\varphi + J\varphi_0\right]$$

$$= S_0[\varphi,J] + \int dx\, J(x)\varphi_0(x)$$

therefore,

$$W_0[J,\varphi_0] = e^{\int dx\, J(x)\varphi_0(x)}\, W_0[J]$$

Using,

$$\frac{\delta W_0[J]}{\delta J(x)} = -\int dy\, \Delta_f(x-y)\, J(y)\cdot W_0[J]$$

and as,

$$(\Box_x + m^2)\Delta_f(x-y) = -i\delta(x-y),$$

then,

$$(\Box_x + m^2)\frac{\delta}{\delta J(x)} W_0 = -\int dy\, (\Box_x + m^2)\Delta_f(x-y)\, J(y)\, W_0$$

$$= iJ(x)\, W_0.$$

$$W_0[J,\varphi_0] = e^{\int dx\, \varphi_0(x)(\Box_x + m^2)\frac{\delta}{\delta J(x)}} \cdot W_0[J]$$

therefore,

$$W_0[J,\varphi_0] = e^{\int dx\, \varphi_0(x)(\Box_x + m^2)\frac{\delta}{\delta J(x)}} \cdot W[J]$$

$$\propto e^{\int dx\, \varphi_0(x)(\Box_x + m^2)\frac{\delta}{\delta J(x)}}\, e^{i\int dy\, \mathcal{L}_{int}\left(-i\frac{\delta}{\delta J(y)}\right)}\, W_0[J]$$

then, we found that,

$$S[\varphi_0] = \exp\left(\int dx\, \varphi_0(x) K_x \frac{\delta}{\delta J(x)} \right) W[J] \Big|_{J=0.}$$

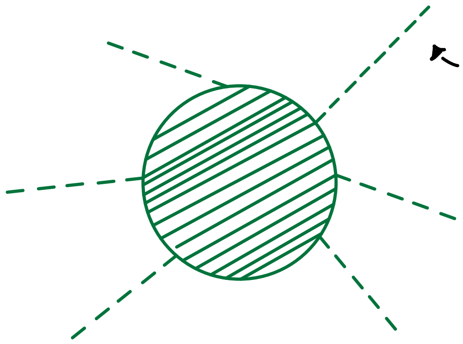
where $K_x = (\square_x + m^2)$

expanding in series:

$$S[\varphi_0] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \left(\prod_{i=1}^{n} dx_i \right) \varphi_0(x_1) \cdots \varphi_0(x_n) K_{x_1} \cdots K_{x_n} \frac{\delta^n W[J]}{\delta J(x_n) \cdots \delta J(x_1)} \Big|_{J=0}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \left(\prod_{i=1}^{n} dx_i \right) \varphi_0(x_1) \cdots \varphi_0(x_n) K_{x_1} \cdots K_{x_n} G^{(n)}(x_1, \ldots, x_n)$$

$\sum$ n-particles process :



each exterior line: $\varphi_0(x) \cdot K_x$

In the momentum representation:

$$G^{(n)}(x_1, \ldots, x_n) = \int \left(\prod_{i=1}^{n} \frac{dp_i^4}{(2\pi)^4} \right) e^{i \sum p_i x_i}\, \tilde{G}^{(n)}(p_1, \ldots, p_n) (2\pi)^4 \delta\left(\sum p_i\right)$$

then,

$$K_{x_i} G^{(n)}(x_1, \ldots, x_n) \longrightarrow \times (m^2 - p_i^2)\, \tilde{G}(p_1, \ldots, p_n)$$

therefore,

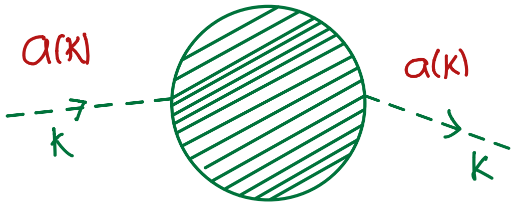
$$S[\varphi_0] = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int \left(\prod_{i=1}^{n} \frac{dp_i}{(2\pi)^4} \right) (2\pi)^4 \delta\left(\sum p_i\right) (m^2 - p_1^2) \cdots (m^2 - p_n^2)$$
$$\times\, \tilde{G}^{(n)}(p_1, \ldots, p_n) \prod_{i=1}^{n} \left[\int dx_i\, \varphi_0(x_i)\, e^{i p_i x_i} \right]$$

Due to,

$$\varphi_0(x) = \int \tilde{d}^3 k \left[a(k) e^{-ikx} + a^*(k) e^{+ikx} \right]$$

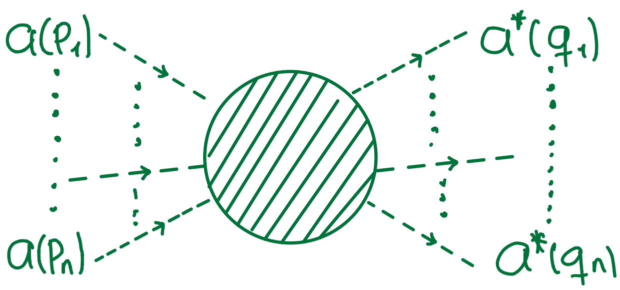
then, $$\int dx\, e^{ipx}\varphi_0(x) = \int d^3K\, \frac{2\pi}{2E_K}\left[a_K\,\delta^4(p-K) + a^*(K)\,\delta^4(p+K)\right]$$

$E_K=\sqrt{K^2+m^2}$ $\longrightarrow$ $\tilde{G}^{(n)}$ must be evaluated in mass layer for all outer lines.



Just one factor will contribute in each specific process.

$n \longrightarrow m$ particles.



From $\int \tilde{d}^3K \longrightarrow \rho(K) = \frac{1}{(2\pi)^3\, 2K_0}$

step covariant function

: Normalization factor.

$$S_{fi} = \left[\rho(p_1)\cdots\rho(q_m)\right]^{-1} \left.\frac{\delta S[\varphi_0]}{\delta a(p_1)\cdots\delta a(p_n)\,\delta a^*(q_1)\cdots\delta a^*(q_m)}\right|_{\substack{a=0\\a^*=0.}}$$

$$= (2\pi)^4\,\delta(p_1+\cdots+p_n-q_1-\cdots-q_m)\,\mathcal{M}_{fi}$$

Invariant amplitude

$$S=\sum_{n=0}^{\infty}\frac{(i)^n}{n!}\int \tilde{d}^3K_1\cdots\tilde{d}^3K_n\,(2\pi)^4\,\delta\left(\sum K\right)(m^2-K_1^2)\cdots(m^2-K_n^2)\,\tilde{G}^{(n)}(K_1,\dots,K_n)$$

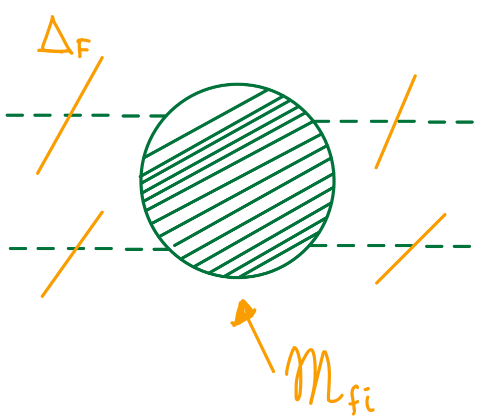
$$\times\left[a(K_1)+a^*(-K_1)\right]\cdots\left[a(K_n)+a^*(-K_n)\right]$$

finally

$$\mathcal{M}_{fi} = (-i)^{n+m}\,(p_1^2-m^2)\cdots(q_m^2-m^2)\,\tilde{G}^{(n)}(p_1,\dots,p_n,-q_1,\dots,-q_m)$$

$\Pi\,\tilde{\Delta}_F(p_i)^{-1}$: Cuts the outer legs

$\downarrow$



$$\mathcal{M}_{fi} = \tilde{\Gamma}^{(n)}(p_1,\dots,p_n,-q_1,\dots,-q_m)$$