We are interested in a particular integral representative of Weyl ordering

$$Q_\hbar^{\omega}(f(p,q)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} du\,dv\,\hat{f}(u,v)\,e^{-\frac{i}{\hbar}(u\hat{p}+v\hat{q})} \;;\; u,v\in\mathbb{R}^n,\ f\in A_c.$$

where

$$\hat{f}(u,v) := \mathcal{F}^{-1}[f(p,q)] = \int_{\mathbb{R}^{2n}} dq\,dp\,f(p,q)\,e^{i(up+vq)}$$

is the Fourier inverse of $f(p,q)$.

Homework: Check that $Q_\hbar^{\omega}$ obeys the properties of Weyl's ordering.

Example:

$$Q_\hbar^{\omega}(pq) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du\,dv\,dq\,dp\;pq\,e^{i(up+vq)}\,e^{-i(u\hat{p}+v\hat{q})}$$

Using Backer–Campbell–Hausdorf (BCH) formula. Check.

$$\exp(X)\exp(Y) = \exp\left(X+Y+\frac{1}{2}[X,Y]+\dots\right)$$

and defining $X=-iv\hat{q}$, $Y=-iu\hat{p}$.

$$\exp(-iv\hat{q},-iu\hat{p}) = \exp\left(\frac{1}{2}(-iu)(-iv)[\hat{q},\hat{p}]\right)\exp(-iv\hat{q})\exp(-iu\hat{p}).$$

$$= \exp\left(i\frac{uv}{2}\right)\exp(-iv\hat{q})\exp(-iu\hat{p})\,\varphi(q)$$

$$[\exp(-iu\hat{p})\varphi](q) = \varphi(q+u)$$

$$[\exp(-iv\hat{q})\varphi](q) = \exp(-ivq)\,\varphi(q).$$

$$Q_\hbar^{\omega}(pq) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du\,dv\,dp\,dq\,(pq)\,e^{i(up+vq)}\,e^{\frac{iuv}{2}}\,e^{-iv\hat{q}}\,e^{-iu\hat{p}}$$

$$= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du\,dv\,dp\,dq\;pq\,e^{\frac{iuv}{2}}\,e^{iv(q-\hat{q})}\,e^{iu(p-\hat{p})}$$

Change $q \longmapsto q' := q+\frac{u}{2}$, $dq' = dq$.

$$\Rightarrow Q_\hbar^\omega(pq)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du\,dv\,dp\,dq'\; p\left(q'-\frac{u}{2}\right)\cancel{e^{\frac{iuv}{2}}}\, e^{iv(q'-\cancel{\frac{u}{2}}-\hat{q})}\, e^{iu(p-\hat{p})}$$

$$=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du\,dv\,dp\,dq'\; p\left(q'-\frac{u}{2}\right) e^{iv(q'-\hat{q})}\, e^{iu(p-\hat{p})}.$$

$$=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du\,dv\,dp\,dq'\; pq'\, e^{iv(q'-\hat{q})}\, e^{iu(p-\hat{p})}.$$

$$-\frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} du\,dv\,dp\,dq'\; p\frac{u}{2}\, e^{iv(q'-\hat{q})}\, e^{iu(p-\hat{p})}.$$

$$=\int_{\mathbb{R}^2} dp\,dq'\; pq'\,\delta(q'-\hat{q})\,\delta(p-\hat{p})-\frac{1}{2\pi}\int_{\mathbb{R}^3} du\,dp\,dq'\; p\frac{u}{2}\, e^{iu(p-\hat{p})}\,\delta(q'-\hat{q})$$

$$=\hat{p}\hat{q}-\frac{1}{2\pi}\int_{\mathbb{R}^3} du\,dp\;\; p\frac{u}{2}\, e^{iu(p-\hat{p})}$$

$$p\frac{u}{2}\, e^{iu(p-\hat{p})}=\frac{p}{2i}\frac{\partial}{\partial p} e^{iu(p-\hat{p})}$$

$$\Rightarrow \frac{1}{2\pi}\int du\,dp\;\frac{p}{2i}\frac{\partial}{\partial p} e^{iu(p-\hat{p})}$$

$$=\frac{1}{2\pi}\int dp\,\frac{p}{2i}\frac{\partial}{\partial p}\int du\, e^{iu(p-\hat{p})}$$

$$=\int dp\,\frac{p}{2i}\frac{\partial}{\partial p}\delta(p-\hat{p})=-\int dp\,\frac{\partial}{\partial p}\left(\frac{p}{2i}\right)\delta(p-\hat{p}).$$

$$=-\frac{1}{2i} \qquad \therefore\; Q_\hbar^\omega(p,q)=\hat{p}\hat{q}+\frac{1}{2i}$$

$$qp \longmapsto \frac{\hat{p}\hat{q}+\hat{q}\hat{p}}{2}$$

$$[\hat{q},\hat{p}] \longmapsto \hat{q}\hat{p}-\hat{p}\hat{q}$$

$$\hat{q}\hat{p} \longmapsto \hat{p}\hat{q} + [\hat{q},\hat{p}]$$

$$qp \longmapsto \frac{\hat{p}\hat{q} + \hat{p}\hat{q} + [\hat{q},\hat{p}]}{2}$$

$$= \hat{p}\hat{q} + \left[\frac{-i\hbar}{2}\right] = \hat{p}\hat{q} + \frac{1}{2i}$$

Define the weyl transform

$$Q^{\omega}_{\hbar} : L^1(\mathbb{R}^2) \longrightarrow \mathcal{L}(\mathcal{H})$$

$$Q^{\omega}_{\hbar}(f) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{f}(u,v) e^{-i(u\hat{p} - v\hat{q})} du dv$$

$\tilde{f} :=$ Fourier inverse of f.

which must be understood in the weak sense. For $\varphi_1, \varphi_2 \in \mathcal{H}$.

$$(Q^{\omega}_{\hbar}(f)\varphi_1, \varphi_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} du dv \, \tilde{f}(u,v) (e^{-i(u\hat{p}+v\hat{q})} \varphi_1, \varphi_2)$$

sense of convergence in the norm of Hilbert space. Let's evaluate $(\hat{A}\varphi)(z)$

$$(\hat{A}\varphi)(z) = (Q^{\omega}_{\hbar}(A)\varphi)(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} du dv dp dq \, A(p,q) e^{i(vq+up+\frac{uv}{2})} e^{-ivz} \varphi(z-u).$$

Integral in v brings. $\delta(q + \frac{u}{2} - z)$

$$\Rightarrow (\hat{A}\varphi)(z) = \frac{1}{2\pi} \int_{\mathbb{R}^3} du dp dq \, A(p,q) \, \delta(q + \frac{u}{2} - z) e^{iup} \varphi(z-u)$$

Integrating in q.

$$(\hat{A}\varphi)(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} du dp \, A\left(p, z - \frac{u}{2}\right) e^{iup} \varphi(z-u)$$

change $u \longmapsto x = z - u$; $dx = -du$.

$$(\hat{A}\varphi)(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} dx dp \, A\left(p, \frac{z+x}{2}\right) e^{i(z-x)p} \varphi(x)$$

$$(\hat{A}\psi)(z) = \int_{\mathbb{R}} dx \left(\frac{1}{2\pi} \int_{\mathbb{R}} dp\, A\left(p, \frac{z+x}{2}\right) e^{i(z-x)p} \right) \psi(x)$$

$=: K_{\hat{A}}(z,x)$ integral Kernel.

$$= \int_{\mathbb{R}} dx\, K_{\hat{A}}(z,x)\psi(x)$$

We may use this Kernel to obtain an expression for A (The symbol of A).

$$K_{\hat{A}}(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} dp\, A\left(p, \frac{x+y}{2}\right) e^{i(x-y)p}.$$

Change

$$X = \frac{x+y}{2}; \qquad z := x-y$$

$$x = X + \frac{z}{2}, \qquad y = X - \frac{z}{2}$$

$$K_{\hat{A}}\left(X + \frac{z}{2}, X - \frac{z}{2}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} dp\, A(p, X) e^{ipz}$$

Inverting Fourier

$$\int K_{\hat{A}}\left(X + \frac{z}{2}, X - \frac{z}{2}\right) e^{-iPz} = \frac{1}{2\pi} \int_{\mathbb{R}^2} dp dz\, A(p, X) e^{ipz} e^{-iPz}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} dp dz\, A(p,X)\, e^{-i(P-p)z} = \int_{\mathbb{R}} dp A(p,X)\, \delta(p-P)$$

$$= A(P, X) = \mathrm{symb}[\hat{A}]$$

We can go back and forth between operators (Kernels) and symbols on phase space!

<u>Wigner function.</u>

Consider the projector operator $\hat{P}_\psi$ that projects on the state ψ

$$(\hat{P}_\psi \phi)(z) = \psi(z) \int_{\mathbb{R}} dy\, (\psi,\phi)(y) = \psi(z) \int_{\mathbb{R}} \overline{\psi(y)}\, \phi(y)\, dy$$

its integral Kernel is given by

$$K_{\hat{P}_\psi}(x,y) = \psi(x)\overline{\psi(y)}$$

its symbol will be called the wigner function.

$$W[\psi](pq) := \mathrm{sym}[\hat{P}_\psi](p,q)$$

$$W[\psi](pq) = \int dz\, \psi\left(q+\frac{z}{2}\right)\overline{\psi\left(q-\frac{z}{2}\right)}\, e^{-ipz}$$

in particular

$$\frac{1}{2\pi}\int_{\mathbb{R}} dp\, W[\psi](pq) = \psi(q)\overline{\psi(q)} = |\psi(q)|^2$$

$$\frac{1}{2\pi}\int_{\mathbb{R}} dp\, W[\psi](pq) = |g(p)|^2$$

where $g(p) := \frac{1}{2\pi}\tilde{\psi}(p) \longleftarrow$ Fourier transform of ψ.

we may generalize wigner of two states ψ, Φ that is the weyl-wigner symbol of the kernel $\psi(x)\overline{\Phi(y)}$.

Theorem: for an arbitrary operator $\hat{A}$, we can obtain expectation values by the following formula.

$$(\psi, \hat{A}\psi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} W[\psi](p,q)\, A(p,q)\, dp\, dq.$$