$$W[\psi](p,q) = \mathrm{sym}[\hat{P}_\psi](p,q) = \int \psi\left(q+\frac{z}{2}\right)\overline{\psi\left(q-\frac{z}{2}\right)}\, e^{-ipz/\hbar}\, dz$$

$$\langle \psi, \hat{A}\psi\rangle = \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}^2} W[\psi](p,q)A(p,q)\,dp\,dq}_{=:I} \quad ; \; |\psi\rangle \in \mathcal{H},\ \langle\psi| \in \mathcal{H}^*$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}} \psi\left(q+\frac{z}{2}\right)\overline{\psi\left(q-\frac{z}{2}\right)}\, e^{-ipz}\, dz\right]\left[\int_{\mathbb{R}} K_{\hat{A}}\left(q+\frac{z'}{2}, q-\frac{z'}{2}\right) e^{-iz'p}\, dz'\right] dp\,dq$$

Integrating in p brings $\delta(z+z')$, and thus integrating in z'.

$$I = \int_{\mathbb{R}^2} dq\,dz\, \psi\left(q+\frac{z}{2}\right)\overline{\psi\left(q-\frac{z}{2}\right)}\, K_{\hat{A}}\left(q-\frac{z}{2}, q+\frac{z}{2}\right)$$

Change $y' := q - \frac{z}{2}$, $\quad y := q + \frac{z}{2} \iff z = y - y'$, $\quad q = \frac{y'+y}{2}$.

$$dz\,dq = dz \wedge dq = (dy - dy') \wedge \frac{1}{2}(dy' + dy)$$

$$= \frac{1}{2}\left[dy \wedge dy' - dy' \wedge dy\right] = dy \wedge dy' = dy\,dy'$$

$$I = \int_{\mathbb{R}} dy\,dy'\, \overline{\psi(y')}\, K_{\hat{A}}(y',y)\psi(y) = \int_{\mathbb{R}} dy'\, \overline{\psi(y')}\left[\int_{\mathbb{R}} K_{\hat{A}}(y',y)\psi(y)\,dy\right]$$

$$= \int_{\mathbb{R}} dy'\, \overline{\psi(y')}\,(\hat{A}\psi)(y') = (\psi, \hat{A}\psi)$$

Note: Symbols may depend on $\hbar$! $\longleftarrow$ This encloses the quantum behaviour at the phase space level.

Example:

$$\mathrm{symb}\left[\hat{p}\hat{q} + \frac{\hbar}{2i}\right] = \mathrm{symb}[\hat{p}\hat{q}] + \mathrm{symb}\left[\frac{\hbar}{2i}\right]$$

$$= \mathrm{symb}[\hat{p},\hat{q}] + \frac{\hbar}{2i} = pq$$

$$\Longrightarrow \mathrm{sym}[\hat{p},\hat{q}] = pq - \frac{\hbar}{2i}$$

This may define a semiclassical observable if for any $A \in A_c$ (such that $\hat{A} \in \hat{A}_a = Q^{w}_{\hbar}[A]$). It allows an asymptotic expansion.

$$A(p,q) \sim \hbar \sum_{n=0}^{\infty} A_n(p,q)\hbar^n$$

$a \in \mathbb{R}$. $a_n(p,q)$ do not depend on $\hbar$.

$A_0(p,q)$ is the classical observable corresponding to the quantum observable $\hat{A}$.

Back to weyl & wigner: Some properties.

$$1)\ A(x,p) = \int dy\, e^{-ipy/\hbar} \left\langle x+\frac{y}{2}, \hat{A}\left(x-\frac{y}{2}\right)\right\rangle$$

Proof:

$$K_{\hat{A}}(q,q') = \langle q, Q^{w}_{\hbar}[A]q'\rangle$$

$$= \int dp\, A\left(p, \frac{q+q'}{2}\right) e^{ip(q-q')/\hbar}$$

Define

$$\mathrm{Tr}[\hat{A}] = \mathrm{Tr}[Q^{w}_{\hbar}[A]] := \int dq \langle q, \hat{A}q\rangle = \int dq \int dp\, A(p,q)$$

using $K_{\hat{A}}(q,q)$

Now, taking $q = x + \frac{u\hbar}{2}$, $q' = x - \frac{u\hbar}{2}$ in the kernel.

$$\left\langle x+\frac{u\hbar}{2}, \hat{A}\left(x-\frac{u\hbar}{2}\right)\right\rangle = \int dp\, A(p,x)\, e^{iup/\hbar}$$

by Fourier

$$A(p,x) = \frac{1}{2\pi}\int du \left(\int dp\, A(p,x)\, e^{iup/\hbar}\right) e^{-iup/\hbar}$$

$$= \frac{1}{2\pi}\int du \left\langle x+\frac{u\hbar}{2}, \hat{A}\left(x-\frac{u\hbar}{2}\right)\right\rangle e^{-iup/\hbar}$$

$\blacksquare$

2) $A(p,q) = \frac{1}{2\pi}\int du\, e^{ixu/\hbar}\left\langle p+\frac{u\hbar}{2}, \hat{A}\left(p,\frac{u\hbar}{2}\right)\right\rangle$

Proof: By completeness relation

$$\mathbb{1} = \int dp\,|p\rangle\langle p|, \text{ and } \langle p|p'\rangle = \delta(p-p')$$

$$A(p,q) = \int du\, e^{-iup/\hbar}\left\langle x+\frac{up}{\hbar}\middle| p'\right\rangle\left\langle p'\middle|A\middle|p''\right\rangle\left\langle p''\middle| x-\frac{y}{\hbar}\right\rangle dp'dp''$$

$$\langle x|p\rangle = \frac{1}{\hbar^{1/2}}\exp\left(\frac{ixp}{\hbar}\right)$$

$$A(p,q) = \int du\, e^{-iup/\hbar}\left[\frac{1}{\hbar^{1/2}}\exp\left(\frac{ip(x+u\hbar/2)}{\hbar}\right)\right]\langle p'|Ap''\rangle$$

$$\times\left[\frac{1}{\hbar^{1/2}}\exp\left(\frac{ip(x+u\hbar/2)}{\hbar}\right)\right]dp'dp''$$

$$= \frac{1}{\hbar}\int du\,dp'dp''\, e^{-iup/\hbar}e^{-ix(p''-p')/\hbar}e^{\frac{iu}{2}(p'+p'')}\langle p'|Ap''\rangle$$

Integrating in u brings $\hbar\,\delta\left(p-\left(\frac{p'+p''}{2}\right)\right)$

change $y = p'-p''$, $z = p'+p'' \longrightarrow p' = \frac{y+z}{2}$, $p'' = \frac{z-y}{2}$

$$dp'dp'' \longmapsto \frac{1}{2}dydz.$$

$$A(p,q) = \frac{1}{2}\int dydz\,\delta\left(p-\frac{z}{2}\right)e^{ixy/\hbar}\left\langle\frac{y+z}{2}\middle|\hat{A}\left(\frac{z-y}{2}\right)\right\rangle$$

$$\delta[f(x)] = \frac{1}{|f'(x)|}\delta(x).$$

$$= \int dy\, e^{ixy/\hbar}\left\langle p+\frac{y}{2}\middle|A\left(p-\frac{y}{2}\right)\right\rangle$$

3) $\mathrm{Tr}[\hat{A}\hat{B}] = \frac{1}{\hbar}\int dq\,dp\, A(q,p)B(q,p)$

Proof: $A(q,p)=\int dy\, e^{-ipy/\hbar}\left\langle q+\frac{y}{2}\right|\hat{A}\left|q-\frac{y}{2}\right\rangle$

$$B(q,p)=\int dy'\, e^{-ipy'/\hbar}\left\langle q+\frac{y'}{2}\right|\hat{B}\left|q-\frac{y'}{2}\right\rangle$$

$$\int dp\,dq\, A(q,p)B(q,p)=\int dp\,dq\,dy\,dy'\, e^{-ipy/\hbar}\left\langle q+\frac{y}{2}\right|\hat{A}\left|q-\frac{y}{2}\right\rangle$$
$$\times\, e^{-ipy'/\hbar}\left\langle q+\frac{y'}{2}\right|\hat{B}\left|q-\frac{y'}{2}\right\rangle$$

Integrating in p brings $\hbar\delta(y-y')$, thus integrating in y':

$$I=\int dq\,dy\,\langle dq\,dy\left\langle q+\frac{u}{2}\right|\hat{A}\left|q-\frac{u}{2}\right\rangle\left\langle q-\frac{u}{2}\right|\hat{B}\left|q+\frac{y}{2}\right\rangle$$

Change

$$u:=q-\frac{y}{2},\quad v:=q+\frac{y}{2}$$

$$du\,dv=dq\,dy.$$

$$I=\hbar\int du\,dv\,\langle v|\hat{A}|u\rangle\langle u|\hat{B}|u\rangle$$
$$A|u\rangle\langle v|$$
$$=\hbar\int dv\,\langle v|\hat{A}\hat{B}|v\rangle=\hbar[\hat{A},\hat{B}].$$

$\blacksquare$

4) $W[\mathbb{1}](p,q)=1.$

Proof:

$$W[\mathbb{1}](p,q)=\int e^{-ipy/\hbar}\left\langle x+\frac{y}{2}\right|\mathbb{1}\left|x-\frac{y}{2}\right\rangle dy.$$
$$=\int dy\, e^{-ipy/\hbar}\,\delta\!\left(\left(x+\frac{y}{2}\right)-\left(x-\frac{y}{2}\right)\right)$$
$$=\int dy\, e^{-ipy/\hbar}\,\delta(y)$$
$$=1.$$

$\blacksquare$

5) Wigner function is normalized.

$$\int dq\,dp\, W(q,p) = 1.$$

Proof:

$$\int dq \int W[\psi](p,q)\,dp = \int dq\, |\psi[q]|^2 = 1.$$

$\blacksquare$

6) Let W_a be the wigner function associated to the operator

$$\hat{\rho}_a := |\psi_a\rangle\langle\psi_a|$$

Proof:

$$Tr[\hat{\rho}_a \hat{\rho}_b] = \frac{1}{\hbar}\int dp\,dq\, W_a(p,q)\, W_b(p,q)$$

$\blacksquare$

7) As $Tr[\hat{\rho}_a \hat{\rho}_b] = \hbar |\langle\psi_a|\psi_b\rangle|^2$

Proof: Thus, for orthogonal states

$$\langle\psi_a|\psi_b\rangle = 0.$$

$$\Longrightarrow \int dp\,dq\, W_a(p,q)\, W_b(p,q) = 0$$

$\therefore$ Wigner may be negative at some points in phase space

$$\langle\psi|A\psi\rangle = \int dp\,dq\, W(p,q)\, A(p,q).$$

8) Wigner function is real.

Proof:

$$W(q,p) = \int dz\, e^{-ipz/\hbar}\, \psi\left(q + \frac{z}{2}\right) \overline{\psi\left(q - \frac{z}{2}\right)}$$

$$W^*(q,p) = \int dz\, e^{ipz/\hbar}\, \overline{\psi\left(q + \frac{z}{2}\right)}\, \psi\left(q - \frac{z}{2}\right)$$

change $z \longmapsto y = -z$

$$W^*(q,p) = \int dy\, e^{-ipy/\hbar}\, \overline{\psi\left(q - \frac{z}{2}\right)}\, \psi\left(q + \frac{z}{2}\right) = W(p,q).$$

$\blacksquare$