Rigidity, Mapping Class Groups and Stochastic Topology.

Noé Bárcenas.
Joint work with O. Arizmendi
and R.Chávez Cáliz.

Centro de Ciencias Matemáticas.
UNAM.

V Escuela de Análisis Topológico de Datos.

Overview

We propose the use of random topology to provide a probabilistic insight to the specific phenomena: **rigidity**.
For this, we will first motivate the study of rigidity in the curve complex associated to a surface, after that we can give the stochastic approach.

What is rigidity?

Rigidity phenomena called mathematicians attention because it **uses the structure of the objects to describe morphisms between them.**

Rigidity in mapping class group

We know surfaces.

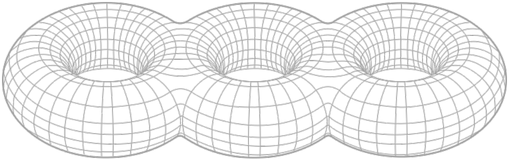

Figure: Surface of genus 3

We can associate a group to the surface, $Mod(S)$, the mapping class group. Rigidity: Group homomorphisms

$$Mod(S) \to Mod(S^{'})$$

are induced by manipulation of the given surfaces.

Why the mapping class group ?

Let S be a 2- dimensional smooth manifold without boundary. S can be furnished with a Riemannian Metric, even with a complex structure. Is it unique? NO.
There is a **whole space $T(S)$ of complex structures on S.** (Homeomorphic to an euclidean ball of dimension $6g - 6$ if the surface has genus g).

Mapping Class Group!.

Definition

The **mapping class group**, $Mod(S)$ is the (discrete!) group of (orientation preserving) homeomorphisms of the surface, modulo the subgroup of homeomorphisms which are homotopic to the identity.

$$Mod(S) = Homeo^{+}(S) \diagup Homeo_0(S).$$

(Versions for a surface with a finite number of boundary components).

Curve complex

How to understand $Mod(S)$**?** Consider a closed surface. A curve α inside it is the image of a continuous map. It is essential if no component of $S - \alpha$ is a disk. We will consider now the (discrete) collection of isotopy classes of essential curves.

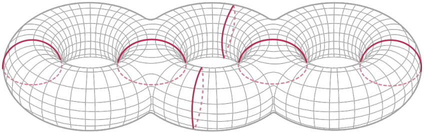

Figure: Some essential curves

The Curve complex!

And we will put an edge connecting them if they admit a realization without intersection.

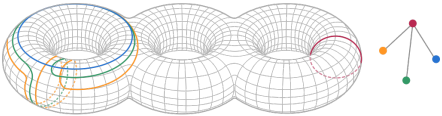

Figure: Construction of the curve complex

One gets a simplicial (flag) complex, denoted by $\mathcal{C}(S)$, the curve complex.

Rigidity of the Curve complex

The mapping class group acts on the Curve complex.

Theorem (Luo)

Let S be a closed surface of genus at least 2. The (simplicial) automorphisms of the curve complex $\mathcal{C}$ are in bijective correspondence with isotopy classes of homeomorphisms of the surface.

$$Aut(\mathcal{C}(S)) \cong Mod^*(S).$$

Here, such classes are responsible for the rigidity phenomenon.
The automorphisms of the curve complex are all geometric.The group of simplicial automorphism of the curve complex is the extended Mapping Class group. Ivanov's Rigidity meta-conjecture.
Version of Margulis Superrigidity Theorem.

Recall the curve complex, $\mathcal{C}(S_g)$ is a flag complex of a **graph**. We have the following theorem

Theorem

Let S_g be a genus g closed surface, where g is t least 2. The curve graph has the following properties:

- *It is connected.*
- *Every vertex has infinite degree.*
- *It has clique number equal to $3g - 3$*
- *It has infinite diameter.*

Rigidity in graphs

Definition

Let Γ be a simplicial graph and let $H < \Gamma$ be a vertex-induced subgraph. A function $f : y \to \Gamma$ is **locally injective** if $f|_{star(v)}$[1] is injective for all $v \in V(y)$.

Definition

$H < \Gamma$ is **rigid** if every locally injective function defined in H can be extended to an automorphism of Γ.

[1] The *star*(v) is the vertex-induced subgraph with vertices $\{v\} \cup N(v)$ (v plus its neighborhood).

A vertex $v \in V$ in a graph it's called to be uniquely determined by $A \subset V(G)$, denoted $v =< A >$, if v is the unique neighbor of every element of B, i.e.

$$\{v\} = \bigcap_{w \in B} lnk(w)$$

Definition

The first rigid expansion of $Y \subset \Gamma$ is the vertex-induced subgraph whose vertices are

$$V(Y) \cup \{v \in V(\Gamma) : \exists A \subset V(Y) \text{ where } v =< A >\}$$

Why Stochastic topology?

- We would like to answer the rather vague question: **How common is rigidity?**
- Doing rigid expansion by hand its hard
- There is a whole translation of the mapping class group rigidity phenomenon in terms of random graph theory.

Slogan: The Curve complex is very similar to a Random Graph in the sense of Erdös-Rényi with very specific parameters, obtained in the limit (The Rado Graph).
Deterministic counterpart: Behring-Gaster's result.

Aknowledgements

This is Ricardo Chávez-Cáliz MSc. Thesis, in process at UNAM.

Fruitful insights from Octavio Arizmendi, but also a group of experts on the curve complex and mapping class groups (around Jesús Hernández) based in Morelia.

Rigidity calculations

With this model in mind we have the following calculations to study rigidity phenomena.

- What is the probability that a vertex v is uniquely determined by a set of size k? (Event E_1)
- What is the probability that a set of size k generate a rigid expansion? (Event E_3)

Uniquely determined vertex

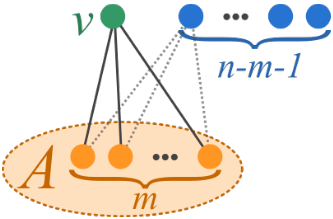

Figure: Probability of uniquely determined vertices

If $< A >= v$ with $|A| = m$ there's a edge between v and every vertex in A, and none of the $n - m - 1$ remaining vertices is also connected to every vertex in A, i.e.

$$\mathbb{P}(E_1(m)) = p^m(1 - p^m)^{n-m-1}$$

For the second question we have that if A_k does not generate a rigid expansion is because none of the possible subsets of A_k determined uniquely a vertex outside of A_k.
We have that the probability that none of the vertices outside of A_k is uniquely determined by $A_m \subset A_k$ is $\rho_{m,k} = (1 - \mathbb{P}(E_1(m))^{n-k}$.

$$\mathbb{P}(A_k \text{ generates a rigid expansion}) = 1 - \Pi_{m=1}^{k} (\rho_{k,m})^{\binom{k}{m}}$$

Calculations

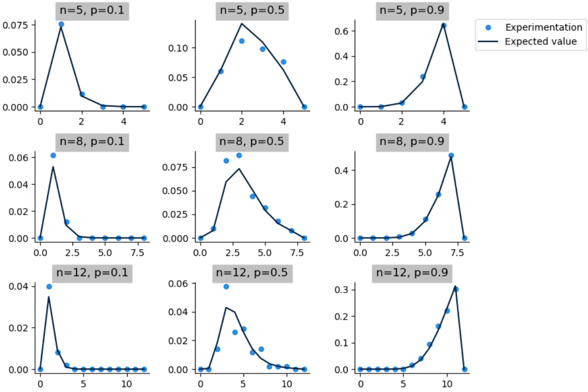

Figure: Probability of uniquely determined vertex, varying k in A_k

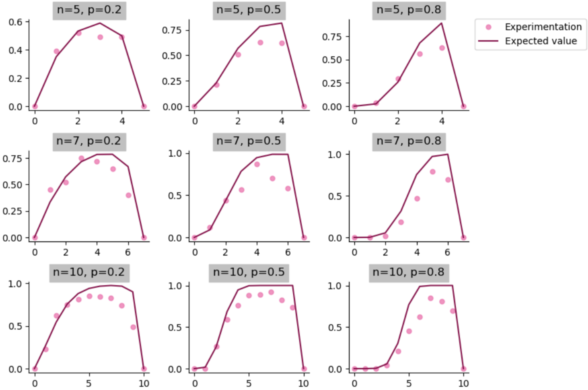

Figure: Probability of having a rigid expansion, varying k in A_k

Connectedness

Theorem

Let $\omega(n)$ be a function that tends to infinity arbitrarily slow as n tends to infinity

- *If $p \geq \frac{log(n)+\omega(n)}{n}$ then*

$$\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{G}(n,p) \textit{ is connected}) = 1$$

- *If $p \leq \frac{log(n)-\omega(n)}{n}$ then*

$$\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{G}(n,p) \textit{ is disconnected}) = 1$$

Vertices have infinite degree

Theorem

Let $\epsilon > 0$ be fixed, $\epsilon n^{-3/2} \leq p = p(n) \leq 1 - \epsilon n^{-3/2}$, let $k = k(n)$ be a natural number and set $\lambda_k = \lambda_k(n) = n \cdot b(k; n-1, p)$. Then the following assertions hold.

- *If* $\lim \lambda_k(n) = 0$, *then* $\lim P(X_k = 0) = 1$.
- *If* $\lim \lambda_k(n) = \infty$, *then* $\lim P(X_k > t) = 1$ *for every fixed* t.
- *If* $0 < \lim \lambda_k(n) < \lim \lambda_k(n) < \infty$, *then* X_k *has asymptotically Poisson distribution with mean* λ_k*:*

$$P(X_k = r) \sim e^{\lambda_k} \cdot \lambda_k^r / r!$$

for every fixed r.

Clique Number and asymptotic of genera.

Theorem

Let $r = r(n) = O(n^{1/3})$ *and let* $p = p(n)$, $0 < p < 1$, *be such that*

$$\binom{n}{r} p^{\binom{r}{2}} \to \infty \text{ and } \binom{n}{r+1} p^{\binom{r+1}{2}} \to 0$$

Then a.e G_p *has clique number* r

Infinite diameter

Theorem

Let c be a positive constant, $d = d(n) \geq 2$ a natural number, and define $p = p(n, c, d), 0 < p < 1$, by

$$p = \frac{\left(n \cdot log(n^2/c)\right)^{1/d}}{n}$$

Suppose that $pn/(log\ n)^3 \to \infty$. Then in $G(n, p)$ we have

$$\lim_{n\to\infty} \mathbb{P}(diam\ G = d) = e^{-c/2} \text{ and } \lim_{n\to\infty} \mathbb{P}(diam\ G = d+1) = 1-e^{-c/2}$$