Rigidity, Mapping Class Groups and Stochastic Topology.

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What is rigidity?

Surfaces.
We will associate a group to a a surface, $Mod(S)$, the mapping class group. Rigidity: Group homomorphisms

$$Mod(S) \to Mod(S^{'})$$

are induced by manipulation of the given surfaces.

Mapping class group ?

Let S be a 2- dimensional smooth manifold without boundary. S can be furnished with a Riemannian Metric, even with a complex structure. Is it unique? NO.
There is a whole space $T(S)$ of complex structures on S. (Homeomorphic to an euclidean Ball of dimension $6g - 6$ if the surface has genus g).

Example of a non-rigid phenomenon. Continuation.

Homeomorphisms of the surface act on this space:

$$S \xrightarrow{\varphi} S',$$

But homeomorphisms that are homotopic to the identity fix the complex structure we started with. (These, the ones non-homotopic to *id* are responsible for this non-rigid aspect of the theory of surfaces: the complex structure is not unique, and there is a whole Moduli).

Mapping Class group!

Consider a closed surface. A curve α inside it is the image of a continuous map. It is essential if no component of $S - \alpha$ is a disk. We will consider now the (discrete) collection of isotopy classes of essential curves.
And we will put an edge connecting them if they admit a realization without intersection.
One gets a simplicial (flag) complex, denoted by $\mathcal{C}(S)$, the curve complex.

Rigidity of the Curve complex

A homeomorphism $\varphi : S \to S$, acts on the vertices by permuting the isotopy class of the curves. Homeomorphisms which are isotopic to the identity act trivially. For one second, allow for non-orientation-preserving homeomorphisms as well.

Theorem (Luo)

Let S be a closed surface of genus at least 2. The (simplicial) automorphisms of the curve complex $\mathcal{C}$ are in bijective correspondence with isotopy classes of homeomorphisms of the surface.

$$Aut(\mathcal{C}(S)) \cong Mod^*(S).$$

Here, such classes are responsible for the rigidity phenomenon. The automorphisms of the curve complex are all geometric.The group of simplicial automorphism of the curve complex is the extended Mapping Class group.
Ivanov's Rigidity meta-conjecture.
Version of Margulis superrigidity Theorem.

Definition

The mapping class group, Mod(S) is the (discrete!) group of (orientation preserving) homeomorphisms of the surface, modulo the subgroup of homeomorphisms which are homotopic to the identity.

$$Mod(S) = Homeo^{+}(S) \diagup Homeo_0(S).$$

(Versions for a surface with a finite number of boundary components).

Example

- Let D^2 denote the 2-dimensional disk. $Mod(D^2)$ is the trivial group. Any orientation preserving homeo has a fixed point. It is homotopic to the identity via a radial homotopy.
- Let A be the annulus. $Mod(A)$ is the group of integers.
- Let T be the torus. $H_1(T) \cong \mathbb{Z}^2$,

$$Mod(T) \to Aut^+(H_1(T)) = Aut^+(\mathbb{Z}^2) \cong Sl_2(\mathbb{Z})$$

Recall the curve complex, $\mathcal{C}(S)$. Its vertices are isotopy classes of essential curves. There is an edge if two such curves have a disjoint realization.

Theorem

Let S_g be a genus g closed surface, where g is t least 2. The curve complex has the following properties:

- *It is connected.*
- *Every vertex has infinite degree.*
- *It has clique number equal to $3g-3$*
- *It has infinite diameter.*

Explanation

Aim of this Effort: give Probabilistic proofs of properties of the curve complex, based on the theory of random graphs.
The easy ones: the four fundamental properties listed above. (Nice didactical introduction to fundamental ideas of random graph theory). Reason to be delivered here in the TDA-Stochastic topology School. This gives probabilistic proofs of results about mapping class groups.

The more involved one, but also adressed here: Luo's proof of rigidity of the curve complex.

Slogan: The Curve complex is very similar to a Random Graph in the sense of Erdös-Rényi with very specific parameters, obtained in the limit (The Rado Graph).
Deterministic counterpart: Behring- Gaster's result.

Aknowledgements

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The four theorems

Connectedness

Vertices have infinite degree

Clique Number and assymptotics of genera.

Infinite diameter

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