The Fibonacci Sequence and Generalizations

Linear Algebra - MAS3105

Ricardo Frumento

Dr. Brian W. Curtin
Mathematics & Statistics

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Contents

1 Abstract

This project will work on a general form for all elements of the Fibonacci Sequence using the concepts of eigenvalue, eigenvector and diagonalizaton. MatLab functions will be used to understand how to proceed on a computer. A generalization for Lucas sequence is discussed using two examples, one that the same method works and one that does not.

2 Motivation

A sequence is a list of numbers written in a specific order:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots \quad [6]$$

The study of such collections has proven to be useful because there is a famous sequence that shows up constantly in nature. The Fibonacci sequence also known as the Fibonacci numbers is:

$$F(0) = 0, \quad F(1) = 1,$$

$$F(k) = F(k-1) + F(k-2), \quad k > 1 \quad [2]$$

The applications can range from computer algorithms like the Fibonacci heap to biological systems such as the pattern on broccoli's surface. There are records about the sequence in Indian mathematics as early as 400 BC [2] but the first time it appeared on occidental mathematics was to study the population growth of theoretical rabbits.

2.1 The Rabbit Problem

The question was how many rabbits could be born on ideal circumstances from a pair of one male and one female rabbits, assuming a rabbit can mate at the age of one month and always produces a pair of one male and one female rabbits after one month and there are no deaths in the community [1]. This is clearly wrong on the biological perspective but for mathematics purposes it works perfectly. With this initial conditions it is possible to start the sequence. Consider f_k to be the number of rabbit pairs and k to be the number of months passed:

$$f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5$$

With this it is possible to write a general law for the population growth as:

$$f_k = f_{k-1} + f_{k-2}$$

To find how many rabbits will exist after one year it is only a matter of continuing the computations. The result is 144 but it is clear that this is not applicable when the element being found has a higher index.
One way to find an easier solution for higher indexes is to use linear algebra concepts. The first step is to rewrite the problem as system of linear equations:

$$f_{k+1} = f_{k+1}$$

$$f_{k+2} = f_k + f_{k+1}$$

Let $\mathbf{u}_k = \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix}$. The system can be written as a matrix equation of the form $\mathbf{u}_{k+1} = A\mathbf{u}_k$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. It is important to note that $\mathbf{u}_k$ can be written in terms of $\mathbf{u}_0$ as

$$\mathbf{u}_k = A\mathbf{u}_{k-1} = AA\mathbf{u}_{k-2} = \cdots = A^k\mathbf{u}_0$$

With this any number in the sequence can be found, now it is only a matter of finding the best way to compute A^k. Here is where eigenvalues and eigenvectors can be useful.

2.2 Eigenvectors and Eigenvalues

An eigenvector of an $n \times n$ matrix A is a nonzero vector such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$.[4] A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda I) = 0 \quad [4]$$

Using the matrix found:

$$\det(\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$$

$$\det(\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}) = 0$$

$$-\lambda + \lambda^2 - 1 = 0$$

This give two eigenvalues

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

To find the eigenvectors there must be a nontrivial solution to

$$(A - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\mathbf{v} = 0$$

$$\left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \end{array}\right] \; R_2 + \tfrac{1}{\lambda}R_1 \;\rightarrow \left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \; -\tfrac{R_1}{\lambda} \;\rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

It is important to note that

$$\lambda_1\lambda_2 = -1$$

Then the two eigenvectors found are

$$v_1 = \begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix}$$

One way to check this is to compute $A\mathbf{v} = \lambda\mathbf{v}$. But the question was an easy way to find A^k and this did not solve it. One more concept is required, diagonalizable matrices. This is important because it simplifies the computation.
Let D be a diagonal matrix $n \times n$ and $\{d_1, d_2, \ldots, d_n\}$ be the elements of the diagonal, then D^k is a diagonal matrix and the elements of the diagonal are $\{{d_1}^k, {d_2}^k, \ldots, {d_n}^k\}$.

2.3 Diagonalizable Matrices

The information about the eigenvalues and eigenvectors of a matrix A can be shown by using a factorization of the form $A = PDP^{-1}$, where D is a diagonal matrix with the eigenvalues as diagonal entries and P is an invertible matrix formed by the eigenvectors. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors [4]. With the vectors found it is easy to see that they are linearly independent then

$$P = \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$$

Matrix P is invertible and the process of finding the inverse is show in Appendix A, and the MatLab function inv() was used to find the inverse, this is shown in Appendix C. Using this form, finding A^k is simple

$$A^2 = AA = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDDP^{-1} = PD^2P^{-1}$$

This process can be expanded to A^k and then

$$A^k = (PDP^{-1}) \dots (PDP^{-1}) = PD^kP^{-1}$$

As proposed, using diagonalization, eigenvalues and eigenvectors is easier to find an arbitrary element of the Fibonacci sequence.

3 Description and Solution

3.1 Solving the Fibonacci Sequence

The first 13 Fibonacci numbers are

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
0	1	1	2	3	5	8	13	21	34	55	89	144

Using the method described above, any number can be computed quickly

$$\mathbf{u}_5 = A^5\mathbf{u}_5 = (PDP^{-1})^5\mathbf{u}_0 = PD^5P^{-1}\mathbf{u}_0 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

The first element of the vector is the fifth Fibonacci number. It is possible to further the symbolic computation so the result will be directly the element

$$\mathbf{u}_k = A^k\mathbf{u}_0 = PD^kP\mathbf{u}_0 =$$

$$= \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} (\frac{1}{\sqrt{5}}) \begin{bmatrix} 1 & \lambda_1 \\ -1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2\lambda_1^k & -\lambda_1\lambda_2^k \\ \lambda_1^k & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 \\ -1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2\lambda_1^k + \lambda_1\lambda_2^k & -\lambda_2\lambda_1^{k+1} + \lambda_1\lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k & \lambda_1^{k+a} - \lambda_2^{k+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2\lambda_1^{k+1} + \lambda_1\lambda_2^{k+1} \\ \lambda_1^{k+1} - \lambda_2^{k+1} \end{bmatrix}$$

The first entry of $\mathbf{u}_k$ is the k^{th} element of the Sequence then

$$f_k = \frac{1}{\sqrt{5}}(\lambda_1\lambda_2^{k+1} - \lambda_2\lambda_1^{k+1}) = \frac{1}{\sqrt{5}}(\lambda_1^k - \lambda_2^k)$$

This relation can find any Fibonacci number.

3.2 The Golden Ratio

There is another particular characteristic involved with the Fibonacci numbers. It is called the golden ratio and in mathematics this ratio is obtained when two numbers satisfy the condition

$$\frac{a+b}{a} = \frac{a}{b} = \phi, \quad a > b > 0[3]$$

$$\phi = \frac{1+\sqrt{5}}{2}$$

Using the expression for the k^{th} Fibonacci number it is possible to find an aproximation to the golden ratio

$$\frac{f_{k+1}}{f_k} = \frac{\frac{1}{\sqrt{5}}(\lambda_1^{k+1} - \lambda_2^{k+1})}{\frac{1}{\sqrt{5}}(\lambda_1^k - \lambda_2^k)}$$

When k becomes large enough, the second term goes to zero and the expression becomes

$$\frac{\frac{1}{\sqrt{5}}(\lambda_1^{k+1})}{\frac{1}{\sqrt{5}}(\lambda_1^k)} = \frac{1+\sqrt{5}}{2} = \phi$$

3.3 Generalizing Solutions

To further the generalization it is possible to study sequences of the form

$$f_{k+2} = af_{k+1} + bf_k, \quad a, b \in \Re$$

This is called a Lucas sequence. Following are two examples and their first ten elements

$$f_{k+2} = 3f_{k+1} - 2f_k$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
0	1	3	7	15	31	63	127	255	511

$$f_{k+2} = 2f_{k+1} - f_k$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
0	1	2	3	4	5	6	7	8	9

For the first sequence it is easy to see that the elements are the powers of two deducted one. Applying the same work used for the Fibonacci the k^{th} element is

$$f_k = 2^k - 1$$

The computation of this result is on Appendix B.
For the second sequence it is even easier to find the pattern and the k^{th} element is

$$f_k = k$$

But a problem shows up when applying the same previous work. The two eigenvalues are the same and this generates only one linearly independent eigenvector, breaking the condition that a matrix is invertible if and only if it is formed by linearly independent eigenvector. The computations are shown on Appendix B.

4 Discussion

The Fibonacci numbers are Nature's numbering system [5]. In this project it was explored a solution for a general form for the elements using linear algebra, the computations were not difficult but were prone to error, on the process of calculating the f_k there are three matrix multiplication. The use of computers in this project helped to avoid that. For example, this project used the MatLab functions eig() to discover the eigenvectors of a matrix, and inv() to find the inverse of a matrix. Using MatLab helped with checking if the computations made by hand were accurate and also decreased the time to finish finding general forms for the elements of the three sequences. The problem this project found is when the sequence results in a system of equations with a matrix that presents two equal eigenvalues the method collapses. There is no way of creating an invertible matrix from linearly dependent vectors. One way to overcome this problem might be the use of general forms of eigenvectors but this was not tested here.

5 Conclusion

Eigenvalues, eigenvectors and diagonalization of matrices are powerful tools in understanding the behaviour of the Fibonacci sequence. The MatLab methods shown in Appendix C make the computations much easier. It were necessary less than 10 lines of code to find f_{10} from scratch. That efficiency is matched by the computations done by hand after the k^{th} was found but larger indexes the computer methods are going to be the only alternative. When generalizing the work for Lucas sequences the MatLab routines will probably be easier because the matrices might have more complicated elements and the computations by hand are prone to error.

References

[1] Fibonacci numbers and nature. http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/, Sep 2016. Accessed on 2020-10-14.

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[3] Golden ratio. https://en.wikipedia.org/wiki/Golden_ratio, Sep 2020. Accessed on 2020-10-16.

[4] J. J. M. David C. Lay, Steven R. Lay. *Linear Algebra and Its Applications*. Pearson Education, Inc., 2021.

[5] S. Sinha. The fibonacci numbers and its amazing applications. 6:7–14, 01 2019.

[6] J. Stewart. *Essential Calculus Early Transcendentals*. Brooks/Cole, Cengage Learning, 2007.

A Finding the Inverse

$$P = \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} -\lambda_2 & -\lambda_1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \ -\tfrac{R_1}{\lambda_2} \quad =$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{\lambda_1}{\lambda_2} & -\frac{1}{\lambda_2} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \ R_2 - R_1 \quad =$$

$$= \left[\begin{array}{cc|cc} 1 & \frac{\lambda_1}{\lambda_2} & -\frac{1}{\lambda_2} & 0 \\ 0 & 1 - \frac{\lambda_1}{\lambda_2} & \frac{1}{\lambda_2} & 1 \end{array}\right] \ R_1 - \tfrac{\lambda_1}{\lambda_2} R_2 \quad =$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{\lambda_2} - \frac{\lambda_1}{\lambda_2(\lambda_2 - \lambda_1)} & -\frac{\lambda_1}{(\lambda_2 - \lambda_1)} \\ 0 & 1 & \frac{1}{(\lambda_2 - \lambda_1)} & \frac{\lambda_2}{(\lambda_2 - \lambda_1)} \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{(\lambda_2 - \lambda_1)} & -\frac{\lambda_1}{(\lambda_2 - \lambda_1)} \\ \frac{1}{(\lambda_2 - \lambda_1)} & \frac{\lambda_2}{(\lambda_2 - \lambda_1)} \end{bmatrix}$$

$$\lambda_2 - \lambda_1 = -\sqrt{5}$$

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \lambda_1 \\ -1 & -\lambda_2 \end{bmatrix}$$

B Applying the Method

For $f_{k+2} = 3f_{k+1} - 2f_k$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{u}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue

$$\det(A - \lambda) = 0, \quad \lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det(\begin{bmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}) = 0$$

$$\lambda^2 - 3\lambda + 2 = 0, \quad \lambda_1 = 1, \lambda_2 = 2$$

Eigenvector

$$A\mathbf{v} = \lambda\mathbf{v} \to (A - \lambda)\mathbf{v} = 0, \quad A - \lambda = \begin{bmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -\lambda & 1 & 0 \\ -2 & 3-\lambda & 0 \end{array}\right]$$

$$\lambda_1 = 1$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \; R_2 - 2R_1 \to \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2$$

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right] \; R_2 - R_1 \to \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, P^{-1} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$f_k = A^k u_0 = PD^kP^{-1}u_0 = \begin{bmatrix} 2^k - 1 \\ 2^{k+1} - 1 \end{bmatrix}$$

For $f_{k+2} = 2f_{k+1} - f_k$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{u}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue

$$\det(A - \lambda) = 0, \quad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 - 2\lambda + 1 = 0, \quad \lambda_1 = \lambda_2 = 1$$

The equal eigenvalues do not create linearly independent eigenvectors so the method fails.

C MatLab Computations

Initial given values

```
A = [0 1; 1 1]
```

```
A = 2x2
     0     1
     1     1
```

```
u0 = [0; 1]
```

```
u0 = 2x1
     0
     1
```

```
lam1 =(1+sqrt(5))/2;
lam2 = (1-sqrt(5))/2;
D = [lam1 0; 0 lam2]
```

```
D = 2x2
    1.6180         0
         0   -0.6180
```

```
P = [-lam2 -lam1; 1 1]
```

```
P = 2x2
    0.6180   -1.6180
    1.0000    1.0000
```

Finding $\mathbf{u}_5$

```
u5 = (A^5)*u0
```

```
u5 = 2x1
     5
     8
```

Finding the eigenvalues

```
eigeValue = eig(A)
```

```
eigeValue = 2x1
   -0.6180
    1.6180
```

Checking the given eigenvalues

```
lam2*[-lam1 1]'
```

```
ans = 2x1
    1.0000
   -0.6180
```

```
A*[-lam1 1]'
```

```
ans = 2x1
    1.0000
   -0.6180
```

```
lam1*[-lam2 1]'
```

```
ans = 2x1
    1.0000
    1.6180
```

```
A*[-lam2 1]'
```

```
ans = 2x1
    1.0000
    1.6180
```

Computing A^{10} and $\mathbf{u}_{10}$

```
Pinv = inv(P)
```

```
Pinv = 2x2
    0.4472    0.7236
   -0.4472    0.2764
```

```
D10 = D^10
```

```
D10 = 2x2
  122.9919         0
         0    0.0081
```

```
A10 = P*D10*Pinv
```

```
A10 = 2x2
    34    55
    55    89
```

```
u10 = A10*u0
```

```
u10 = 2x1
    55
    89
```