Problem Set 2: Answer Key

Prof. Thomas Hintermaier TA: Ricardo Duque Gabriel

Question 1

To show that the steady state level of consumption under the optimal solution is smaller than the golden rule level of consumption, we proceed in two steps: first we show that the steady state level of capital under the optimal solution is lower than the golden rule level of capital, and in the second step we show that this implies that the steady state level of consumption under the optimal solution is smaller than the golden rule level of consumption.

We derive the optimal level of capital using the Euler equation evaluated at the steady state ($c_{t+1} = c_t = c^*$):

$$\beta \frac{u'(c^*)}{u'(c^*)}(F'(k^*) + 1 - \delta) = 1 \tag{1}$$

$$F'(k^*) = \frac{1}{\beta} - 1 + \delta = \delta + \theta \tag{2}$$

The golden rule level of capital is characterized by:

$$F'(k^{\#}) = \delta \tag{3}$$

As long as $\theta > 0$, it holds that $F'(k^*) > F'(k^{\#})$. Using the assumption that $F''(k) < 0$, we have that $k^{\#} > k^*$.

Imposing a steady state on the resource constraint implies the function:

$$c = F(k) - \delta k \tag{4}$$

Note that this describes steady-state consumption as a function of steady-state capital.

This is a concave function, its second derivative is $F''(k)$. Therefore, the value of consumption at the golden rule is the unique maximum. Since k^* is different (in particular, smaller) from the golden rule level $k^{\#}$, consumption for this constant level of k is smaller than the maximum.

Question 2

2. a) The Lagrangian for this economy is:

$$\mathcal{L}_t = \sum_{\substack{\bar{s}=2n-1 \\ n=1,2,3,...}}^{\infty} \beta^{\bar{s}} u(c_{t+\bar{s}}) + \lambda_{t+\bar{s}}(F(k_{t+\bar{s}}) - c_{t+\bar{s}} - k_{t+\bar{s}+1} + \underbrace{(1-\delta)k_{t+\bar{s}}}_{\text{depreciation}}) \tag{5}$$

$$+ \sum_{\substack{\bar{s}=2n-2 \\ n=1,2,3,...}}^{\infty} \beta^{\bar{s}} u(c_{t+\bar{s}}) + \lambda_{t+\bar{s}}(F(k_{t+\bar{s}}) - c_{t+\bar{s}} - k_{t+\bar{s}+1} + \underbrace{k_{t+\bar{s}}}_{\text{no depreciation}}) \tag{6}$$

2. b) The derivative of the Lagrangian with respect to k_{t+318} is:

$$\frac{\partial \mathcal{L}_t}{\partial k_{t+318}} = \lambda_{t+318}\left(F'(k_{t+318}) + 1\right) - \lambda_{t+317} = 0 \tag{7}$$

Question 3

3. a) From the Euler equation evaluated at the steady state (check Question 1) we get:

$$F'(k^*) = \delta + \theta \tag{8}$$

Using the functional form of the production function $F(k) = \bar{A}k^{\alpha}$:

$$\alpha \bar{A} k^{*\alpha-1} = \delta + \theta$$

This implies:

$$k^* = \left(\frac{\delta + \theta}{\alpha \bar{A}}\right)^{\frac{1}{\alpha-1}} \tag{9}$$

Evaluated at $\alpha = 0.5$, $\bar{A} = 1$, $\beta = 0.95$ and $\delta = 0.1$, the steady state level of capital is $k^* = 10.731$. Using the resource constraint, the steady state level of consumption is:

$$c^* = F(k^*) - \delta k^* = \bar{A}k^{*\alpha} - \delta k^* = 2.203$$

3. b) Given the utility function $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, it follows:

$u'(c) = c^{-\sigma}$,
$u''(c) = -\sigma c^{-\sigma-1}$

Using the parameter value $\sigma = 2$ and the result for the steady state $c^* = 2.203$, we can evaluate these derivatives at the steady state: $u'(c^*) = 0.206$ and $u''(c^*) = -0.187$.

For the given production function $F(k) = \bar{A}k^{\alpha}$, we have:

$F'(k) = \alpha \bar{A} k^{\alpha-1}$,
$F''(k) = \alpha(\alpha - 1)\bar{A}k^{\alpha-2}$

Using the parameter values $\alpha = 0.5$, $\bar{A} = 1$ and evaluating the derivative at the steady state level $k^* = 10.731$ we get $F''(k^*) = -0.007$.

3. c) In period $t = 17$ the vector of deviations of consumption and capital from their respective steady state values is:

$$\begin{bmatrix} c_{17} - c^* \\ k_{17} - k^* \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{10}$$

We can calculate the deviations of c_{19} and k_{19} from their respective steady state values by using the given system:

$$\begin{bmatrix} c_{18} - c^* \\ k_{18} - k^* \end{bmatrix} = A \begin{bmatrix} c_{17} - c^* \\ k_{17} - k^* \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} c_{19} - c^* \\ k_{19} - k^* \end{bmatrix} = A \begin{bmatrix} c_{18} - c^* \\ k_{18} - k^* \end{bmatrix} \tag{12}$$

This two transformations can be combined as:

$$\begin{bmatrix} c_{19} - c^* \\ k_{19} - k^* \end{bmatrix} = AA \begin{bmatrix} c_{17} - c^* \\ k_{17} - k^* \end{bmatrix} = A^2 \begin{bmatrix} c_{17} - c^* \\ k_{17} - k^* \end{bmatrix} = \begin{bmatrix} 1.0073 & -0.0078 \\ -1 & 1.0526 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.9903 \\ 0.1716 \end{bmatrix} \tag{13}$$

Question 4

4. a) For this problem we have the following Euler and capital accumulation equations:

$$0.95 \frac{c_{t+1}^{-2}}{c_t^{-2}} (0.5 k_{t+1}^{-0.5} + 0.9) = 1 \tag{14}$$

$$k_{t+1} = k_t^{0.5} + 0.9 k_t - c_t \tag{15}$$

In the Euler equation we use (15) to replace k_{t+1}:

$$c_{t+1} = c_t \left(0.95(0.5(k_t^{0.5} + 0.9k_t - c_t)^{-0.5} + 0.9) \right)^{0.5} \tag{16}$$

Equations (16) and (15) describe the evolution of next period's consumption and capital in terms of today's consumption and capital.

4. b) Starting from $k_1 = 1$ and $c_1 = 1$, and using (16) and (15) we get:

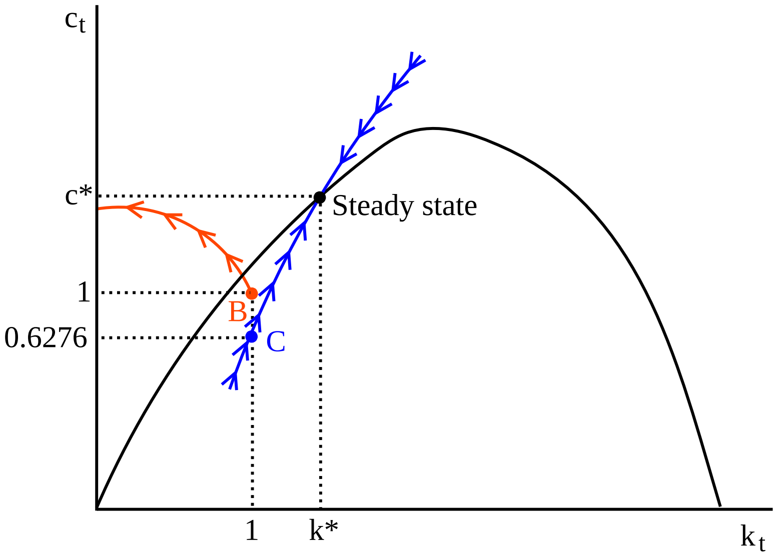
$k_1 = 1, c_1 = 1$
$k_2 = 0.9, c_2 = 1.16$
$k_3 = 0.59, c_3 = 1.41$
$k_4 = -0.11$

The path for consumption and capital already would lead to a negative capital stock in period 4 for these starting values. Therefore, it makes no sense to pursue this path further.

4. c) Starting from $k_1 = 1$ and $c_1 = 0.627582223701021$, and using (16) and (15) we get:

$k_1 = 1, c_1 = 0.628$
$k_2 = 1.27, c_2 = 0.709$
$k_3 = 1.56, c_3 = 0.788$

Using these starting values, we converge to the steady state. The following figure illustrates the two different paths: the path under (b), starting from the initial pair B of capital and consumption, and the path under (c), starting from the initial pair C of capital and consumption.



4. d) MATLAB code can be found in the respective folder on ecampus.