A POSTERIORI ERROR ANALYSIS – $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ FORMULATION FOR THE EDDY CURRENT PROBLEM

1. Introduction. In electrical power engineering applications, the displacement currents in a metallic conductor are negligible compared to the conduction current. In these contexts, the displacement currents can be omitted from Maxwell's equations, leading to a magneto-quasistatic submodel, commonly referred to in the literature as the eddy current problem. Mathematically, this submodel provides a satisfactory approximation to the complete Maxwell system solution at low frequencies.

2. Problem Setting. The eddy current model is obtained by dropping the displacement currents from Maxwell equations [2, Chapter 10]. The aforementioned problem consist in to determine the electromagnetic fields induced in a three-dimensional conducting domain Ω_{C} by a given source current density $\boldsymbol{J}_{\mathrm{D}}$. The eddy current model equations restricted to a bounded domain $\Omega \subset \mathbb{R}^3$ where $\Omega_{\mathrm{D}} := \Omega \backslash \overline{\Omega}_{\mathrm{C}}$ is the insulator domain (which includes the dielectric materials) is given by

Find $\boldsymbol{E} \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{C}})$ *and* $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}; \Omega)$ *such that:*

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{H} &= \sigma \boldsymbol{E} && \text{in } \Omega_{\mathrm{C}}, \\ i\omega\mu \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} &= \mathbf{0} && \text{in } \Omega_{\mathrm{C}}, \\ \mathbf{curl}\, \boldsymbol{H} &= \boldsymbol{J}_{\mathrm{D}} && \text{in } \Omega_{\mathrm{D}}, \\ \operatorname{div}(\mu \boldsymbol{H}) &= 0 && \text{in } \Omega, \\ \boldsymbol{H} \times \boldsymbol{n} &= \boldsymbol{f}_{\mathrm{D}} && \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{2.1}$$

where the unknowns $\boldsymbol{E}$ and $\boldsymbol{H}$ are the magnetic and electric field, respectively. The magnetic permeability μ and the electric conductivity σ are scalar functions satisfying:

$$\begin{aligned} &0 < \mu_{\min} \leqslant \mu \leqslant \mu_{\max} && \text{in } \Omega, \\ &0 < \sigma_{\min} \leqslant \sigma \leqslant \sigma_{\max} && \text{in } \Omega_{\mathrm{C}}, \qquad \sigma = 0 \text{ in } \Omega_{\mathrm{D}}. \end{aligned}$$

The data of the problem are the tangential trace of the magnetic field $\boldsymbol{f}_{\mathrm{D}}$ and the source current density $\boldsymbol{J}_{\mathrm{D}} \in \mathrm{L}^2(\Omega)^3$, for which we suppose

$$\operatorname{supp} \boldsymbol{J}_{\mathrm{D}} \subset \Omega_{\mathrm{D}} \qquad \text{and} \qquad \operatorname{div} \boldsymbol{J}_{\mathrm{D}} = 0 \qquad \text{in } \Omega_{\mathrm{D}}.$$

Next, we will introduce the $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ potential formulation, which is a classical formulation for the eddy current problem (2.1) in terms of three potentials functions: $\boldsymbol{A}$, V and ψ. This formulation was analyzed in [1] (see also [4]). To this purpose, let Ω_{A} be an open set such that

$$\overline{\Omega}_{\mathrm{C}} \cap \operatorname{supp} \boldsymbol{J}_{\mathrm{D}} \subset \Omega_{\mathrm{A}} \qquad \text{and} \qquad \overline{\Omega}_{\mathrm{A}} \subset \Omega.$$

We denote by Ω_{A}^j, $j = 1, \ldots, m_{\mathrm{A}}$, the connected components of Ω_{A}. We assume that each Ω_{A}^j is a convex polyhedron and $\overline{\Omega_{\mathrm{A}}^j}$ are mutually disjoint. Besides, we respectively denote by $\Gamma_{\boldsymbol{A}}$ and Γ_{C} the boundaries of Ω_{A} and Ω_{C} with $\boldsymbol{n}_{\mathrm{A}}$ and $\boldsymbol{n}_{\mathrm{C}}$ their outward unit normal vectors, respectively.

The magnetic vector potential $\boldsymbol{A} : \Omega_{\mathrm{A}} \to \mathrm{C}$ is given by $\boldsymbol{A}|_{\Omega_{\mathrm{A}}^j} := \boldsymbol{A}_j$, $j = 1, \ldots, m_A$, where each $\boldsymbol{A}_j : \Omega_{\mathrm{A}}^j \to \mathrm{C}$ is characterized by

$$\mu \boldsymbol{H} = \mathbf{curl}\, \boldsymbol{A}_j \text{ in } \Omega_{\mathrm{A}}^j, \qquad \operatorname{div} \boldsymbol{A}_j = 0 \text{ in } \Omega_{\mathrm{A}}^j, \qquad \boldsymbol{A}_j \cdot \boldsymbol{n}_{\mathrm{A}} = 0 \text{ on } \partial\Omega_{\mathrm{A}}^j. \tag{2.2}$$

On the other hand, an electric scalar potential $V : \Omega_{\mathrm{C}} \to \mathrm{C}$ is introduced to satisfy

$$V \in \mathrm{H}^1(\Omega_{\mathrm{C}}), \qquad \boldsymbol{E} = -i\omega \boldsymbol{A} - i\omega\, \mathbf{grad}\, V \qquad \text{in } \Omega_{\mathrm{C}}.$$

Finally, a magnetic scalar potential $\psi : \Omega_{\psi} \to \mathrm{C}$ is defined on $\Omega_{\psi} := \Omega \backslash \overline{\Omega}_{\mathrm{A}}$ such that

$$\boldsymbol{H} = \omega\, \mathbf{grad}\, \psi \text{ in } \Omega_{\psi}.$$

Consequently, Problem (2.1) can be rewritten in terms of the three potential functions $\boldsymbol{A}$, V and ψ (see [1, Secction 3]) as follows:

$$
\begin{aligned}
\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)+i\omega\sigma\boldsymbol{A}+i\omega\sigma\,\mathbf{grad}\,V &= \mathbf{0} && \text{in } \Omega_{\mathrm{C}},\\
\operatorname{div}\left(-i\omega\sigma\boldsymbol{A}-i\omega\sigma\,\mathbf{grad}\,V\right) &= 0 && \text{in } \Omega_{\mathrm{C}},\\
\mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right) &= \boldsymbol{J}_{\mathrm{D}} && \text{in } \Omega_{\mathrm{A}}\backslash\overline{\Omega}_{\mathrm{C}},\\
\left.\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\right|_{\Omega_{\mathrm{C}}}\times\boldsymbol{n}_{\mathrm{C}}-\left.\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\right)\right|_{\Omega_{\mathrm{A}}\backslash\overline{\Omega}_{\mathrm{C}}}\times\boldsymbol{n}_{\mathrm{C}} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{C}},\\
\operatorname{div}\left(\mu\,\mathbf{grad}\,\psi\right) &= 0 && \text{in } \Omega_{\psi},\\
\operatorname{div}\boldsymbol{A} &= 0 && \text{in } \Omega_{\mathrm{A}},\\
\boldsymbol{A}\cdot\boldsymbol{n}_{\mathrm{A}} &= 0 && \text{on } \Gamma_{\boldsymbol{A}},\\
\mathbf{grad}\,\psi\times\boldsymbol{n} &= \boldsymbol{f}_{\mathrm{D}} && \text{on } \Gamma,\\
\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\cdot\boldsymbol{n}_{\mathrm{A}}-\omega\,\mathbf{grad}\,\psi\cdot\boldsymbol{n}_{\mathrm{A}} &= 0 && \text{on } \Gamma_{\boldsymbol{A}},\\
\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}\times\boldsymbol{n}_{\mathrm{A}}-\omega\,\mathbf{grad}\,\psi\times\boldsymbol{n}_{\mathrm{A}} &= \mathbf{0} && \text{on } \Gamma_{\boldsymbol{A}},\\
\left(i\omega\sigma\boldsymbol{A}+i\omega\sigma\,\mathbf{grad}\,V\right)\cdot\boldsymbol{n}_{\mathrm{C}} &= 0 && \text{on } \Gamma_{\mathrm{C}}.
\end{aligned}
\tag{2.3}
$$

According to the properties of the potential functions previously mentioned, we can notice that $\boldsymbol{A}$, V and ψ respectively belong to the spaces:

$$
\boldsymbol{\mathcal{X}}:=\mathrm{H}_0(\mathrm{div};\Omega_{\mathrm{A}})\cap\mathrm{H}(\mathbf{curl};\Omega_{\mathrm{A}}),\quad \mathcal{M}:=\prod_{j=1}^{m_{\mathrm{C}}}\mathrm{H}^1(\Omega_{\mathrm{C}}^j)/\mathbb{C},\quad \mathrm{H}^1(\Omega_{\psi})/\mathbb{C},
$$

endowed respectively with the norms

$$
\|\boldsymbol{Z}\|_{\boldsymbol{\mathcal{X}}}^2:=\|\boldsymbol{Z}\|_{0,\Omega_{\mathrm{A}}}^2+\|\operatorname{div}\boldsymbol{Z}\|_{0,\Omega_{\mathrm{A}}}^2+\|\mathbf{curl}\,\boldsymbol{Z}\|_{0,\Omega_{\mathrm{A}}}^2,\quad |U|_{1,\Omega_{\mathrm{C}}}:=\|\mathbf{grad}\,U\|_{0,\Omega_{\mathrm{C}}},\quad |\varphi|_{1,\Omega_{\mathrm{C}}}:=\|\mathbf{grad}\,\varphi\|_{0,\Omega_{\mathrm{C}}},
$$

for any $\boldsymbol{Z}\in\boldsymbol{\mathcal{X}}$, $U\in\mathcal{M}$ and $\varphi\in\mathrm{H}^1(\Omega_{\psi})/\mathbb{C}$.

Once the previous considerations have been given, we want to consider a variational formulation for Problem (2.3). To this end, we need to introduce the bilinear form $\mathscr{A}$ defined on $(\boldsymbol{\mathcal{X}}\times\mathcal{M}\times\mathrm{H}^1(\Omega_{\psi})/\mathbb{C})\times(\boldsymbol{\mathcal{X}}\times\mathcal{M}\times\mathrm{H}^1(\Omega_{\psi})/\mathbb{C})$ by

$$
\begin{aligned}
\mathscr{A}((\boldsymbol{A},V,\psi),(\boldsymbol{Z},U,\varphi)) :=& \int_{\Omega_{\mathrm{A}}}\frac{1}{\mu}\left(\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\bar{\boldsymbol{Z}}+\operatorname{div}\boldsymbol{A}\,\operatorname{div}\bar{\boldsymbol{Z}}\right)+\omega^2\int_{\Omega_{\psi}}\mu\,\mathbf{grad}\,\psi\cdot\mathbf{grad}\,\bar{\varphi}\\
&+i\omega\int_{\Omega_{\mathrm{C}}}\sigma\left(\boldsymbol{A}+\mathbf{grad}\,V\right)\cdot\left(\bar{\boldsymbol{Z}}+\mathbf{grad}\,\bar{U}\right)-\omega\left\langle\mathbf{grad}\,\psi\times\boldsymbol{n}_{\mathrm{A}},\pi_\tau(\bar{\boldsymbol{Z}})\right\rangle_{\Gamma_{\boldsymbol{A}}}+\omega\left\langle\mathbf{grad}\,\bar{\varphi}\times\boldsymbol{n}_{\mathrm{A}},\pi_\tau(\boldsymbol{A})\right\rangle_{\Gamma_{\boldsymbol{A}}}.
\end{aligned}
$$

Let us notice that for any $\boldsymbol{w}\in\mathrm{H}(\mathbf{curl};\Omega_{\psi})$, its tangential trace on $\Gamma_{\boldsymbol{A}}$ also belongs to $\mathrm{H}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_{\boldsymbol{A}}};\Gamma_{\boldsymbol{A}})$ and, consequently, $\langle\boldsymbol{w}\times\boldsymbol{n}_{\mathrm{A}},\pi_\tau(\boldsymbol{v})\rangle_{\Gamma_{\boldsymbol{A}}}$ is also well defined.

Therefore, we have the following variational formulation for Problem (2.3) (see [1, Section 4]):

Problem 1. *Find* $(\boldsymbol{A},V,\psi)\in\boldsymbol{\mathcal{X}}\times\mathcal{M}\times\mathrm{H}^1(\Omega_{\psi})/\mathbb{C}$ *such that:*

$$
\begin{aligned}
&\omega\,\mathbf{grad}\,\psi\times\boldsymbol{n}=\boldsymbol{f}_{\mathrm{D}} \qquad \text{in } \mathrm{H}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma};\Gamma),\\
&\mathscr{A}((\boldsymbol{A},V,\psi),(\boldsymbol{Z},U,\varphi))=\int_{\Omega_{\mathrm{A}}}\boldsymbol{J}_{\mathrm{D}}\cdot\bar{\boldsymbol{Z}}, \qquad \forall(\boldsymbol{Z},U,\varphi)\in\boldsymbol{\mathcal{X}}\times\mathcal{M}\times\mathrm{H}^1_{\Gamma}(\Omega_{\psi}).
\end{aligned}
$$

The following result regarding the existence and uniqueness of solution of Problem 1 was proved in Theorem 4.1 of [1].

THEOREM 2.1. *The bilinear form $\mathscr{A}$ is elliptic on $\boldsymbol{\mathcal{X}} \times \mathcal{M} \times \mathrm{H}^1(\Omega_\psi)/\mathbb{C}$, i.e., there exists $\alpha > 0$ such that*

$$|\mathscr{A}((\boldsymbol{Z}, U, \varphi), (\boldsymbol{Z}, U, \varphi))| \geqslant \alpha \left(|\boldsymbol{Z}|^2_{1,\Omega_\mathrm{A}} + |U|^2_{1,\Omega_\mathrm{C}} + |\varphi|^2_{1,\psi} \right),$$

for any $(\boldsymbol{Z}, U, \varphi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times \mathrm{H}^1(\Omega_\psi)/\mathbb{C}$. Therefore, Problem 1 has a unique solution $(\boldsymbol{A}, V, \psi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times \mathrm{H}^1(\Omega_\psi)/\mathbb{C}$.

Now, in what follows to obtain a discrete formulation of this problem, we further assume that all the domains are Lipschitz polyhedra. Let $\{\mathcal{T}_h\}_h$ be a family of tetrahedral meshes of $\overline{\Omega}$ such that, for each mesh, all the elements $K \in \mathcal{T}_h$ are completely included in one of the three subdomains $\overline{\Omega}_\mathrm{A}$, $\overline{\Omega}_\mathrm{C}$ or $\overline{\Omega}_\psi$. We define $\mathcal{T}_h^B := \left\{K \in \mathcal{T}_h : K \subset \overline{\Omega}_B\right\}$ for $B \in \{\mathrm{C}, \mathrm{A}, \psi\}$, moreover let $\mathcal{T}_h := \mathcal{T}_h^\mathrm{C} \cup \mathcal{T}_h^{\mathrm{A}\backslash\mathrm{C}} \cup \mathcal{T}_h^\psi$ where $\mathcal{T}_h^{\mathrm{A}\backslash\mathrm{C}} := \left\{K \in \mathcal{T}_h : K \subset \overline{\Omega}_\mathrm{A} \backslash \Omega_\mathrm{C}\right\}$. The discrete problem will be obtained by considering the following finite element subspaces:

$$\begin{aligned}
\boldsymbol{\mathcal{X}}_h &:= \left\{\boldsymbol{Z}_h \in \boldsymbol{\mathcal{X}} : \ \boldsymbol{Z}_h|_K \in \mathbb{P}_1^3(K) \ \forall K \in \mathcal{T}_h^\mathrm{A}\right\},\\
\mathcal{M}_h &:= \left\{U_h \in \mathcal{M} : \ U_h|_K \in \mathbb{P}_1(K) \ \forall K \in \mathcal{T}_h^\mathrm{C}\right\},\\
\mathcal{Q}_h &:= \left\{\varphi_h \in \mathrm{H}^1(\Omega_\psi) : \ \varphi_h|_K \in \mathbb{P}_1(K) \ \forall K \in \mathcal{T}_h^\psi\right\},\\
\mathcal{Q}_{\Gamma,h} &:= \left\{\varphi_h \in \mathcal{Q}_h : \ \varphi_h|_\Gamma = 0\right\}.
\end{aligned}$$

PROBLEM 2. *Find $(\boldsymbol{A}_h, V_h, \psi_h) \in \boldsymbol{\mathcal{X}}_h \times \mathcal{M}_h \times \mathcal{Q}_{\Gamma,h}$ such that:*

$$\mathscr{A}((\boldsymbol{A}_h, V_h, \psi_h), (\boldsymbol{Z}_h, U_h, \varphi_h)) = \int_{\Omega_\mathrm{A}} \boldsymbol{J}_\mathrm{D} \cdot \bar{\boldsymbol{Z}}_h, \qquad \forall (\boldsymbol{Z}_h, U_h, \varphi_h) \in \boldsymbol{\mathcal{X}}_h \times \mathcal{M}_h \times \mathcal{Q}_{\Gamma,h}.$$

If the solution of continuous problem is smooth enough, the standard finite element error analysis techniques yield the following result (see [1, Theorem 5.1]). From now on, in order to avoid writing unnecessary constants, the inequality $a \leqslant Cb$, where $C > 0$ regardless of the mesh size, will be written simply as $a \lesssim b$.

THEOREM 2.2. *Let $(\boldsymbol{A}, V, \psi)$ and $(\boldsymbol{A}_h, V_h, \psi_h)$ be the solutions to Problems 1 and 2, respectively. If $\boldsymbol{A} \in \mathrm{H}^{1+s}(\Omega_\mathrm{A})^3$, $V \in \mathrm{H}^{1+s}(\Omega_\mathrm{C})$ and $\psi \in \mathrm{H}^{1+s}(\Omega_\psi)$ with $s > 0$, then*

$$|\boldsymbol{A} - \boldsymbol{A}_h|_{1,\Omega_\mathrm{A}} + |V - V_h|_{1,\Omega_\mathrm{C}} + |\psi - \psi_h|_{1,\Omega_\psi} \lesssim \ h^{\min\{1,s\}} \left(\|\boldsymbol{A}\|_{1+s,\Omega_\mathrm{A}} + \|V\|_{1+s,\Omega_\mathrm{C}} + \|\psi\|_{1+s,\Omega_\psi} \right).$$

Finally, we want to emphasize about the convenience for choosing the domain Ω_A so that its connected components be convex polyhedron. In fact, it can be proved in that case the magnetic vector potential $\boldsymbol{A}$ satisfying (2.2) belongs to $\mathrm{H}^{1+s}(\Omega_\mathrm{A})^3$ for some $s > 0$ and thus the regularity condition about $\boldsymbol{A}$ in Theorem 2.2 has meaning. Moreover, $\boldsymbol{\mathcal{X}} = \left\{\boldsymbol{Z} \in \mathrm{H}^1(\Omega_\mathrm{A})^3 : \ \boldsymbol{Z} \cdot \boldsymbol{n}_\mathrm{A} = 0\right\}$ if and only if each Ω_A^j is a convex polyhedron (see [3, Theorem I.3.9]). Consequently, under the aforementioned considerations about the domain Ω_A, it make sense approximate the vector potential $\boldsymbol{A}$ by using finite element functions from $\boldsymbol{\mathcal{X}}_h$. More details about this issue can be found in [1, Section 5].

3. A posteriori error analysis. In this section we derive a reliable and efficient residual based error estimator for the $\boldsymbol{A}, V - \boldsymbol{A} - \psi$ formulation (see Problem 1). To this purpose, we begin by introducing further notations and definitions. Let $\mathcal{F}_h$ be the set of faces induced by the mesh $\mathcal{T}_h$, we write

$$\mathcal{F}_h := \mathcal{F}_{h,\mathrm{C}}^{\mathrm{int}} \cup \mathcal{F}_{h,\mathrm{A}\backslash\mathrm{C}}^{\mathrm{int}} \cup \mathcal{F}_{h,\psi}^{\mathrm{int}} \cup \mathcal{F}_h^{\Gamma_\mathrm{C}} \cup \mathcal{F}_h^{\Gamma_{\boldsymbol{A}}} \cup \mathcal{F}_h^{\Gamma},$$

where $\mathcal{F}_{h,\mathrm{C}}^{\mathrm{int}}$, $\mathcal{F}_{h,\mathrm{A}\backslash\mathrm{C}}^{\mathrm{int}}$, $\mathcal{F}_{h,\psi}^{\mathrm{int}}$ are the tetrahedral faces lying in the interior of Ω_C, $\Omega_\mathrm{A} \backslash \overline{\Omega_\mathrm{C}}$ and Ω_ψ, respectively, and $\mathcal{F}_h^{\Gamma_\mathrm{C}}$, $\mathcal{F}_h^{\Gamma_{\boldsymbol{A}}}$, $\mathcal{F}_h^{\Gamma}$ are the set of faces lying in Γ_C, $\Gamma_{\boldsymbol{A}}$, Γ, respectively. Additionally, $\mathcal{F}_{h,\mathrm{A}}^{\mathrm{int}}$ denotes the set of interior faces lying in Ω_A.

Furthermore, for any $K \in \mathcal{T}_h^\mathrm{C}$ we define by $\mathcal{F}_{h,K}^{\mathrm{C},\mathrm{int}} := \mathcal{F}_{h,\mathrm{C}}^{\mathrm{int}} \cap \partial K$. We will use similar notations to define $\mathcal{F}_{h,K}^{\mathrm{A},\mathrm{int}}$, $\mathcal{F}_{h,K}^{\mathrm{A}\backslash\mathrm{C},\mathrm{int}}$ and $\mathcal{F}_{h,K}^{\psi,\mathrm{int}}$.

3.1. Local error indicators and global error estimator. Let $(\boldsymbol{A}, V, \psi)$ and $(\boldsymbol{A}_h, V_h, \psi_h)$ be the solutions to Problems 1 and Problem 2, respectively. Hence, the error on $\boldsymbol{A}$, V and ψ take the form:

$$\boldsymbol{e}_{\boldsymbol{A}} := \boldsymbol{A} - \boldsymbol{A}_h, \quad e_{\mathrm{V}} := V - V_h, \quad e_{\psi} := \psi - \psi_h.$$

We propose the following local error indicators:

- For any $K \in \mathcal{T}_h^{\mathrm{C}}$:

$$\begin{aligned}\eta_{K,\mathrm{C}}^2 := h_K^2 \Bigg[& \left\| - \mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}_h\right) + \mathbf{grad}\left(\frac{1}{\mu}\,\mathrm{div}\,\boldsymbol{A}_h\right) - i\omega\sigma\left(\boldsymbol{A}_h + \mathbf{grad}\,V_h\right)\right\|_{0,K}^2 \\ & + \left\|\mathrm{div}\left(i\omega\sigma\left(\boldsymbol{A}_h + \mathbf{grad}\,V_h\right)\right)\right\|_{0,K}^2 \Bigg] \\ & + \sum_{F \in \mathcal{F}_{h,K}^{\mathrm{C,int}}} h_F \Bigg[\left\| [\![i\omega\sigma(\boldsymbol{A}_h + \mathbf{grad}\,V_h) \cdot \boldsymbol{n}_F]\!] \right\|_{0,F}^2 \\ & \qquad + \left\| \left[\!\!\left[\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}_h \times \boldsymbol{n}_F \right]\!\!\right] \right\|_{0,F}^2 + \left\| \left[\!\!\left[\frac{1}{\mu}\,\mathrm{div}\,\boldsymbol{A}_h \right]\!\!\right] \right\|_{0,F}^2 \Bigg] \\ & + \sum_{F \in \mathcal{F}_{h,K}^{\Gamma_{\mathrm{C}}}} h_F \left\| i\omega\sigma(\boldsymbol{A}_h + \mathbf{grad}\,V_h) \cdot \boldsymbol{n}_{\mathrm{C}} \right\|_{0,F}^2 .\end{aligned}$$

- For any $K \in \mathcal{T}_h^{\mathrm{A}\backslash\mathrm{C}}$:

$$\begin{aligned}\eta_{K,\mathrm{A}\backslash\mathrm{C}}^2 := h_K^2 & \left\| \boldsymbol{J}_{\mathrm{D}} - \mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}_h\right) + \mathbf{grad}\left(\frac{1}{\mu}\,\mathrm{div}\,\boldsymbol{A}_h\right)\right\|_{0,K}^2 \\ & + \sum_{F \in \mathcal{F}_{h,K}^{\mathrm{A}\backslash\mathrm{C,int}}} h_F \left(\left\| \left[\!\!\left[\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}_h \times \boldsymbol{n}_F \right]\!\!\right] \right\|_{0,F}^2 + \left\| \left[\!\!\left[\frac{1}{\mu}\,\mathrm{div}\,\boldsymbol{A}_h \right]\!\!\right] \right\|_{0,F}^2 \right) \\ & + \sum_{F \in \mathcal{F}_{h,K}^{\Gamma_{\boldsymbol{A}}}} h_F \left\| \left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}_h - \omega\,\mathbf{grad}\,\psi_h \right) \times \ \boldsymbol{n}_{\mathrm{A}} \right\|_{0,F}^2 .\end{aligned}$$

- For any $K \in \mathcal{T}_h^{\psi}$:

$$\begin{aligned}\eta_{K,\psi}^2 := h_K^2 \left\| \mathrm{div}\left(\omega^2\mu\,\mathbf{grad}\,\psi_h\right)\right\|_{0,K} & + \sum_{F \in \mathcal{F}_{h,K}^{\psi,\mathrm{int}}} h_F \left\| [\![\omega^2\mu\,\mathbf{grad}\,\psi_h \cdot \boldsymbol{n}_F]\!] \right\|_{0,F}^2 . \\ + \sum_{F \in \mathcal{F}_{h,K}^{\Gamma_{\boldsymbol{A}}}} h_F & \left\| \left(\omega^2\mu\,\mathbf{grad}\,\psi_h - \ \omega \ \ \mathbf{curl}\,\boldsymbol{A}_h\right) \cdot \boldsymbol{n}_{\mathrm{A}} \right\|_{0,F}^2 .\end{aligned}$$

- For any $F \in \mathcal{F}_h^{\Gamma_{\mathrm{C}}}$:

$$\eta_{F,\mathrm{C}}^2 := h_F \left(\left\| \left[\!\!\left[\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{A}_h \times \boldsymbol{n}_F \right]\!\!\right] \right\|_{0,F}^2 + \left\| \left[\!\!\left[\frac{1}{\mu}\,\mathrm{div}\,\boldsymbol{A}_h \right]\!\!\right] \right\|_{0,F}^2 \right).$$

Therefore, the (global) error estimator is defined as follows

$$\eta^2 := \sum_{K \in \mathcal{T}_h^{\mathrm{C}}} \eta_{K,\mathrm{C}}^2 + \sum_{K \in \mathcal{T}_h^{\mathrm{A}\backslash\mathrm{C}}} \eta_{K,\mathrm{A}\backslash\mathrm{C}}^2 + \sum_{K \in \mathcal{T}_h^{\psi}} \eta_{K,\psi}^2 + \sum_{F \in \mathcal{F}_h^{\Gamma_{\mathrm{C}}}} \eta_{F,\mathrm{C}}^2. \tag{3.1}$$

THEOREM 3.1. *The following estimate holds true*

$$|\boldsymbol{e}_{\boldsymbol{A}}|_{1,\Omega_{\mathrm{A}}} + |e_{\mathrm{V}}|_{1,\Omega_{\mathrm{C}}} + |e_{\psi}|_{1,\Omega_{\psi}} \lesssim \eta\,.$$

THEOREM 3.2. *The efficiency condition holds true, i.e.,*

$$\eta \lesssim |\boldsymbol{e}_{\boldsymbol{A}}|_{1,\Omega_{\mathrm{A}}} + |e_{\mathrm{V}}|_{1,\Omega_{\mathrm{C}}} + |e_{\psi}|_{1,\Omega_{\psi}}\,.$$

4. Numerical results. In this section, we will report some numerical results obtained using piecewise linear polynomials of degree one on tetrahedral meshes. These meshes have been generated with the TetGen mesh generator. We employ a MATLAB code specifically designed to illustrate the properties of the a posteriori estimator. Initially, we apply this code to solve a test problem with a known analytical solution. The known analytical solution contributes to the validation of the computer code and the verification of the error estimates proven in the preceding theorems (Theorem 2.2 and Theorem 3.2). Finally, we extend the method to a problem with a more realistic geometric configuration.

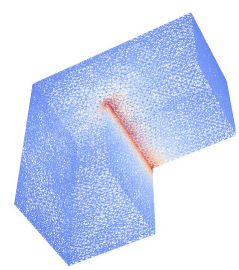
4.1. A test with known analytical solution. We approximate the solution of the following source problem: find $(\boldsymbol{A}, V, \psi) \in \boldsymbol{\mathcal{X}} \times \mathcal{M} \times \mathrm{H}^1_\Gamma(\Omega_\psi)$ solution to the weak formulation of the following equations

$$\begin{aligned}
\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{A}) + i\boldsymbol{A} + i\,\mathbf{grad}\,V &= \boldsymbol{f}_1 && \text{in } \Omega_{\mathrm{C}},\\
\mathrm{div}\,(-i\boldsymbol{A} - i\,\mathbf{grad}\,V) &= f_2 && \text{in } \Omega_{\mathrm{C}},\\
\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{A}) &= \boldsymbol{f}_3 && \text{in } \Omega_{\mathrm{A}}\backslash\overline{\Omega}_{\mathrm{C}},\\
\mathrm{div}\,(\mathbf{grad}\,\psi) &= f_4 && \text{in } \Omega_\psi,\\
(\mathbf{curl}\,\boldsymbol{A} - \mathbf{grad}\,\psi)\cdot\boldsymbol{n}_{\mathrm{A}} = \boldsymbol{g}\cdot\boldsymbol{n}_{\mathrm{A}} & && \text{on } \Gamma_{\boldsymbol{A}},\\
(\mathbf{curl}\,\boldsymbol{A} - \mathbf{grad}\,\psi)\times\boldsymbol{n}_{\mathrm{A}} = \boldsymbol{g}\times\boldsymbol{n}_{\mathrm{A}} & && \text{on } \Gamma_{\boldsymbol{A}},\\
(i\boldsymbol{A} + i\,\mathbf{grad}\,V)\cdot\boldsymbol{n}_{\mathrm{C}} = \boldsymbol{f}_5\cdot\boldsymbol{n}_{\mathrm{C}} & && \text{on } \Gamma_{\mathrm{C}},
\end{aligned}$$

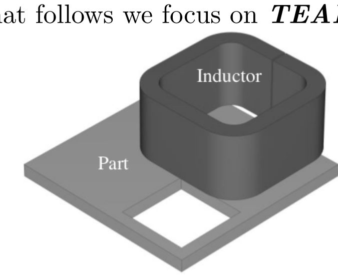
where $\Omega := (0,4)^3$, $\Omega_{\mathrm{C}} := L$, $\Omega_{\mathrm{A}} := (1,3)^3$, $\Omega_\psi := \Omega\backslash\overline{\Omega_{\mathrm{A}}}$ and without loss of generality we assume that the angular frequency ω is equal to one, the magnetic permeability μ is equal to one within domain Ω, and the conductivity σ is equal to one within domain Ω_{C}.

The data $\boldsymbol{f}_1,\ f_2,\ \boldsymbol{f}_3,\ f_4,\ \boldsymbol{g}$ and $\boldsymbol{f}_5$ has been chosen so that the analytical solution be

$$\begin{aligned}
\boldsymbol{A}(x,y,z) &= 0,\\
V(r,\theta) &= r^{2/3}\sin\left(\frac{2}{3}\theta\right),\\
\psi(x,y,z) &= 0.
\end{aligned}$$



4.2. An application problem in a more realistic geometry. In this case, we present a numerical example with a some more complex geometry. We have computed the eddy currents induced by a nonconvex conductor domain. In what follows we focus on ***TEAM Workshop Problem 7***.



REFERENCES

[1] Acevedo, Ramiro and Rodríguez, Rodolfo, *Analysis of the A, $V-A-\psi$ potential formulation for the eddy current problem in a bounded domain*, Electron. Trans. Numer. Anal., 26 (2007), pp. 270–284.

[2] A. Bossavit, *Computational Electromagnetism*, Academic Press Inc., San Diego, CA, 1998.

[3] V. Girault and P. A. Raviart, *Finite Element Methods for Navier Stokes Equations*, Springer, New York, 1986.

[4] P. J. Leonard and D. Rodger, *Finite element scheme for transient 3D eddy currents*, IEEE Trans. Magn., 24 (1988), pp. 90-93.

[5] Verfürth, Rüdiger, *A Posteriori Error Estimation Techniques for Finite Element Methods*, Oxford University Press., (2013).