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CSC373S: Algorithm Design, Analysis & Complexity

Lecture 17

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based on notes by Denis Pankratov

Network Flows

Input: $G = (V, E)$ directed graphs
$c : E \to \mathbb{R}_{>0}$ capacities
$s, t \in V$

- no back edges
- edges into S
- no edges out of t

Output: $f-$ max-flow
Residual Graph: Assume $G, s, t, c-$ flow graph.
$f-$ some feasible flow on G.

Residual capacities:

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v), & (u, v) \in E \\ f(v, u), & (v, u) \in E \end{cases}$$

Meaning: **Case 1:** $(u, v) \in E$
We can potentially add at most $c(u, v) - f(u, v)$ to the flow

Case 2: $(v, u) \in E$
We can *undo* at most $f(v, u)$ flow

Residual graph $G_f = (V, E_f)$

$$E_f = \{(u, v) | c_f(u; v) > 0\}$$

Augmenting Path: A path P from s to t in G_f

Capacity of P:

$$C_f(P) = \min\{c_f(e) | e \in P\}$$

Augmenting the flow f by pushing more flow along P

$$f \uparrow f_p(u,v) = \begin{cases} f(u,v), & (u,v),(v,u) \notin P, (u,v) \in E \\ f(u,v) + c_f(P) & (u,v) \in P \\ f(u,v) - c_f(P) & (v,u) \in P \end{cases}$$

Algorithm:

```
def Ford-Fulkerson(G,s,t,c):
  init flow f to all 0 flow
  while some P - augmenting path - in G_f:
    f = f_uparrow_f_p
  return f
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Lemma: $|f \uparrow f_p| = |f| + c_P(f)$

Example:

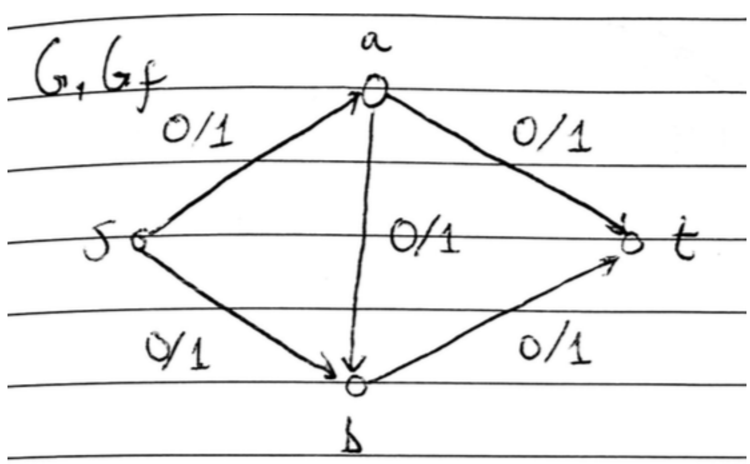
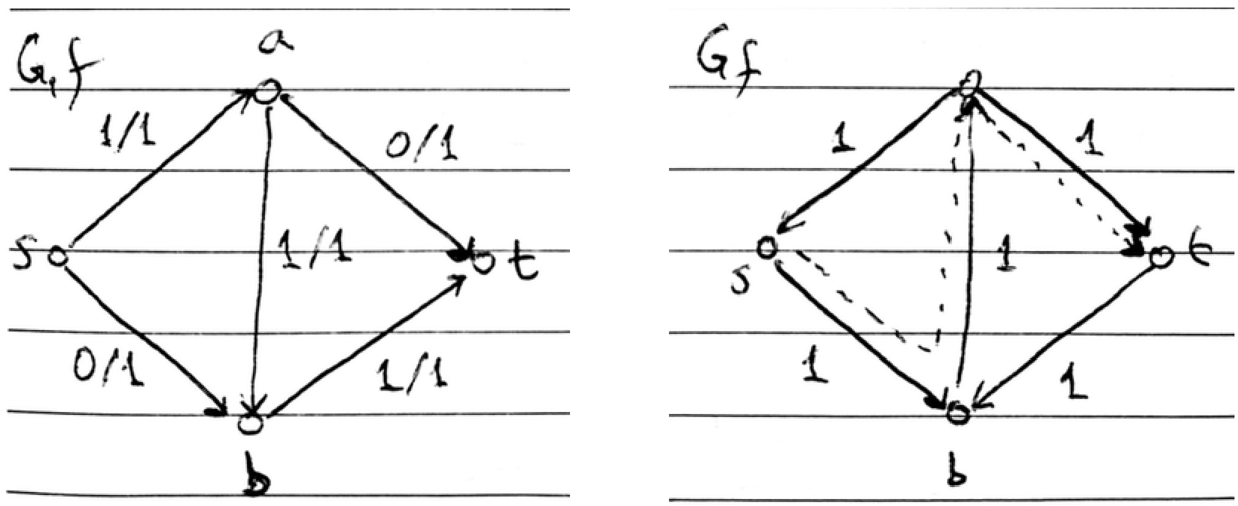


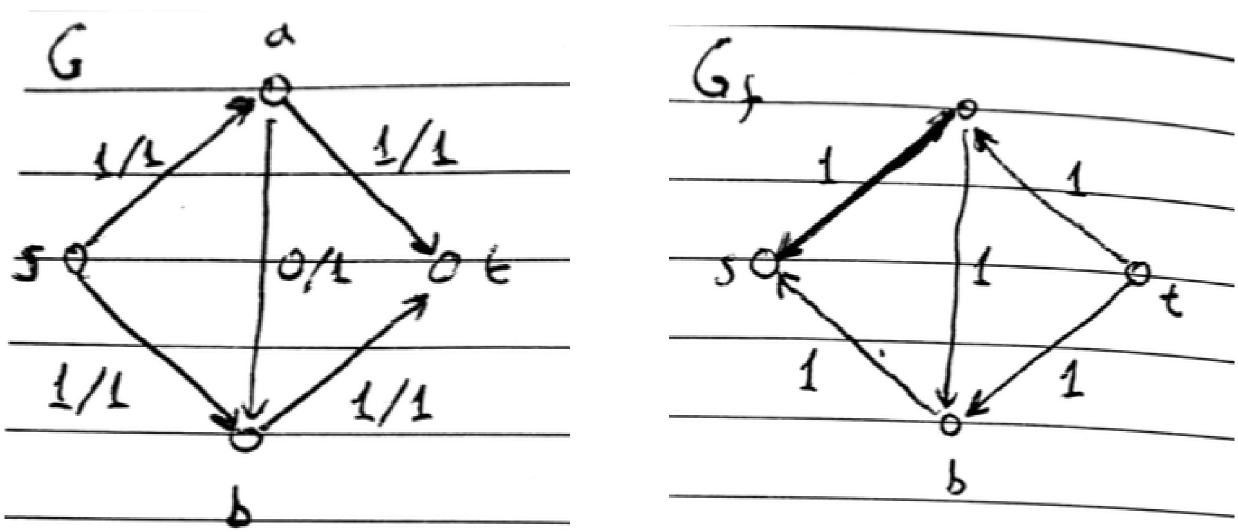
Figure 1: Initial graph



(a) Choose $P_1 = (s, a, b, t)$

(b) $P_2 = (s, b, a, t)$

Figure 2: Steps 1-2



(a) *Intermediate state*

(b) *End*

Figure 3: Steps 3-4

No agumenting path (caption last pic) $\Rightarrow$ FF terminates w/ f so that $|f| = 2$

<u>Lemma:</u> $|E_f| \leq 2|E|$

Assuming FF terminates, we will show it finds the max flow.

<u>Definition:</u> Let G, s, t be given then an (s,t)-cut, whic is a partition (S, T) of V so that:

- $S \cap T = \varnothing, S \cup T = V$
- $s \in S, t \in T$

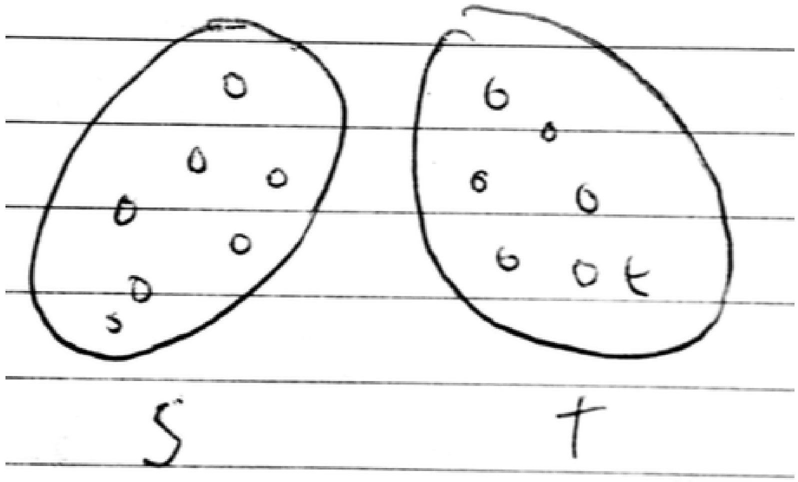


Figure 4: Visualization of S, T

Notation: $U \subseteq V$

$$out(U) = \{(u, v) \in E | u \in U, v \notin U\}$$
$$in(U) = \{(v, u) \in E | u \in U, v \notin U\}$$

Definition: Capacity of cut (S, T) is $c(S, T)\ sum_{e \in out(S)} c(e)$

Flow across cut (S, T). Given some flow f:

$$f(S, T) = \sum_{e \in out(S)} f(e) - \sum_{e \in in(S)} f(e)$$

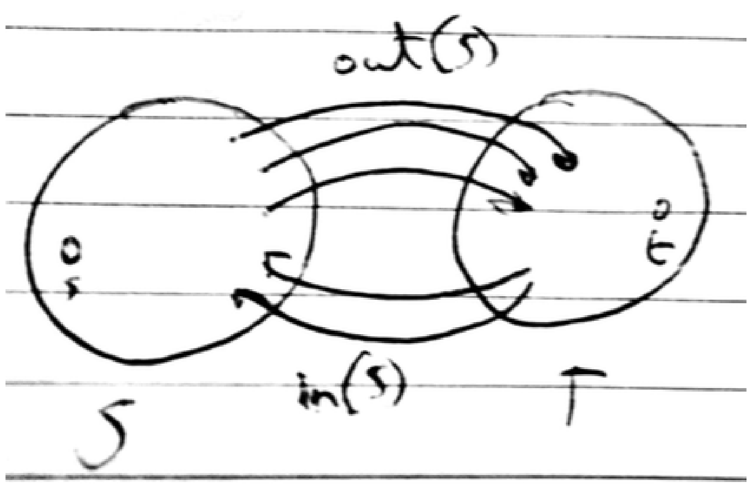


Figure 5: Visualization of S, T with $in(S), out(S)$

Lemma: for any flow f and for any cut $(S, T), f(S, T) \leq c(S, T)$

Proof:

$$\begin{aligned} f(S,T) &= \sum_{e \in out(S)} f(e) - \sum_{e \in in(S)} f(e) \\ &\leq \sum_{e \in out(s)} f(e) \\ &\leq \sum_{e \in out(s)} c(e) = c(S,T) \end{aligned}$$

Lemma: for any cut $(S, T), f(S, T) = |f|$

Proof:

$$\begin{aligned} |f| &= f^{out}(s) \\ &= f^{out}(s) - f^{in}(s)(\text{the latter is equal to } 0) \\ &= \sum_{v \in S} f^{out}(v) - f^{in}(v)(*) \end{aligned}$$

Case 1: $(u, v) \in E, u \in S, v \in S$
$f(u, v)$ appears exactly twice in (*), once positively for $f^{out}(u)$, once negatively for $f^{in}(v)$
$\Rightarrow f(u, v)$ disappears from (*)

Case 2: $u \in S, v \in T$
$f(u, v)$ appears exactly once positively in $f^{out}(u)$

Case 3: $u \in T, v \in S$
$f(u, v)$ appears exactly once positively in $f^{in}(v)$

Case 4: $u \in T, v \in T, f(u, v)$ does not appear in (*)

see (*) $= \sum_{e \in out(S)} f(e)$(Case 2) $- \sum_{e \in in(S)} f(e)$(Case 3) $= f(S, T)$

Corollary: for any flow f for any cut $(S, T), |f| \leq c(S, T)$. In particular, max-flow $\leq$ min-cut.

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0.1 Max-Flow Min-Cut Theorem

The following are equivalent:

1. f is a max-flow
2. no augmenting path in G_f
3. there exist a cut (S, T) so that $|f| = c(S, T)$

<u>Proof:</u>

(1) $\Rightarrow$ (2): By contrapositive:

$$[\neg(2) \Rightarrow \neg(1)]$$

Suppose P is an augmenting path in G_f
$\Rightarrow |f \uparrow f_p| = |f| + c_f(P) > |f|$
$\Rightarrow f$ is not max

(2) $\Rightarrow$ (3): Let $S = \{v \in V |$ there is a path from s to v in $G_f\}$
$T := V\ S$
Note: $t \in T$ because no augmenting path exists

Let $(u, v) \in out(S), v \not{i}nS$, then $f(u, v) = c(u, v)$.
Otherwise, $c_f(u, v) > 0$
$\Rightarrow v \in S$

Let $(v, u) \in in(S)$, then $f(v, u) = 0$.
Otherwise, $\Rightarrow c_f(u, v) > 0$ and $v \in S$

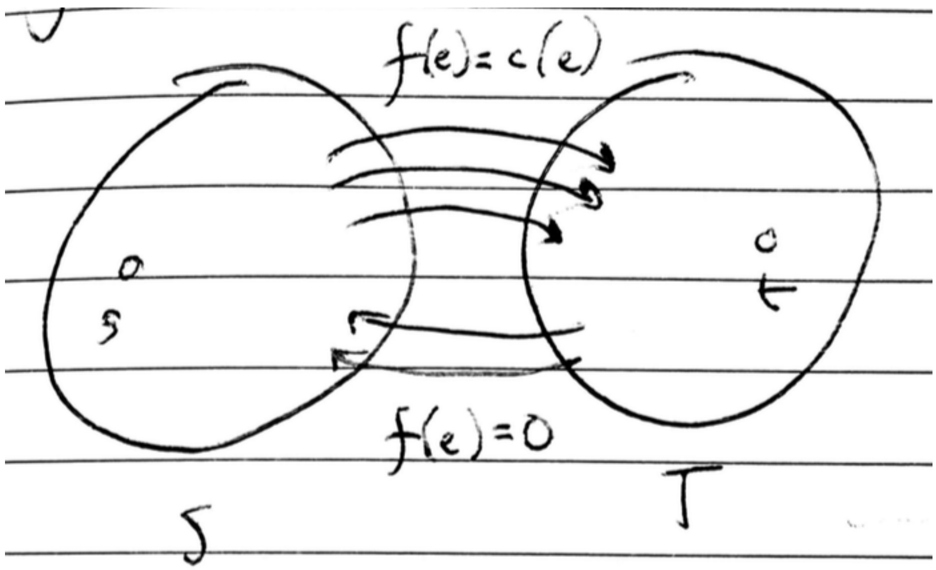

Figure 6: Visualization of S, T with $f(e) = c(e), f(e) = 0$

$$
\begin{aligned}
f(S,T) &= \sum_{e \in out(S)} f(e) - \sum_{e \in in(S)} f(e) \\
&= \sum_{e \in out(S)} c(e) - O \\
&= c(S,T)
\end{aligned}
$$

(3) $\Rightarrow$ (1): It was already done in previous lemmas.