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CSC373S: Algorithm Design, Analysis & Complexity

Lecture 20

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based on notes by Denis Pankratov

Linear Programming (LP)

Definition: $f : \mathbb{R}^n \to \mathbb{R}$ is linear if $\exists a_1, .., a_n \in \mathbb{R}$ so that $f(x_1, .., x_n) = a_2x_2 + \cdots + a_nx_n$

Linear constraints; assume f is linear:

1. $f(x_1, .., x_n) = b$ equality constraint
2. $$\text{inequality constraint} \begin{cases} f(x_1, .., x_n) \leq b \\ f(x_1, .., x_n) \geq b \end{cases}$$

LP Problem:
Maximize/minimize a given linear function subject to given linear constraints

Example

x_1 units of product 1 per day
x_2 units of product 2 per day
$1 -per unit revenue for product 1
$6 -per unit revenue for product 2
200 units - demand for product 1
300 units - demand for product 2
400 units - overall capacity of the factory
Maximize revenue $\max x_1 + 6x + 2$ is the objective function
Subject to:

$$\text{constraints} \begin{cases} x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{cases}$$

$\max c^T x$ is subject to $Ax \leq b, x \geq 0$

Definition: Let $(\bar{x}_1, \bar{x}_2, .., \bar{x}_n \in \mathbb{R}^n$
If it satisfies all constraints, it is called a feasible solution.
Set of all feasible solutions is called a feasible region.

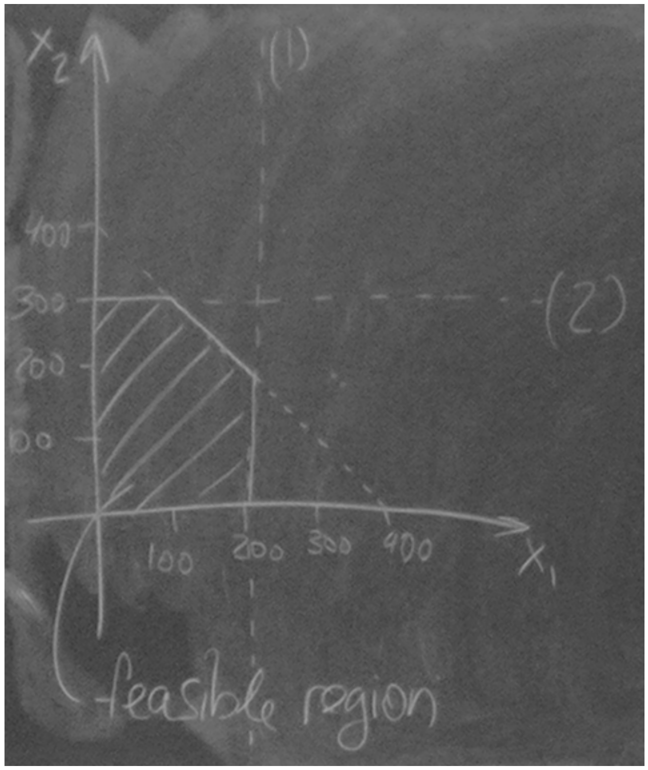

Figure 1: *Visualization*

In general, a single inequality constraint defines a *half-space*.
Feasible region = intersection of all such *half-spaces*.

Use matrix notation.

Example:
$C = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$
Feasible region (*see Figure 1, above*) $F \subseteq \mathbb{R}^n$

$$F = \{\bar{x} \in \mathbb{R}^n | A\bar{x} \leq b\}$$

This is called a *polyhedron*.
If F is bounded, it is called a polytope.

Fact: A polytrope $F = \{\bar{x} \in \mathbb{R}^n | A\bar{x} \leq b\}$ is convex, meaning $\forall \bar{x}, \bar{y} \in F, \forall \alpha \in [0,1], \alpha\bar{x} + (1-\alpha)\bar{y} \in F$
Example: objective $x_1 + 6x_2$.
Fix value $\upsilon \in \mathbb{R}$
$x_1 + 6x_2 = \upsilon$ - line with slope $-1/6$
$x_2 = -1/6 x_1 + 1/6 \upsilon$

$c^T x = v$

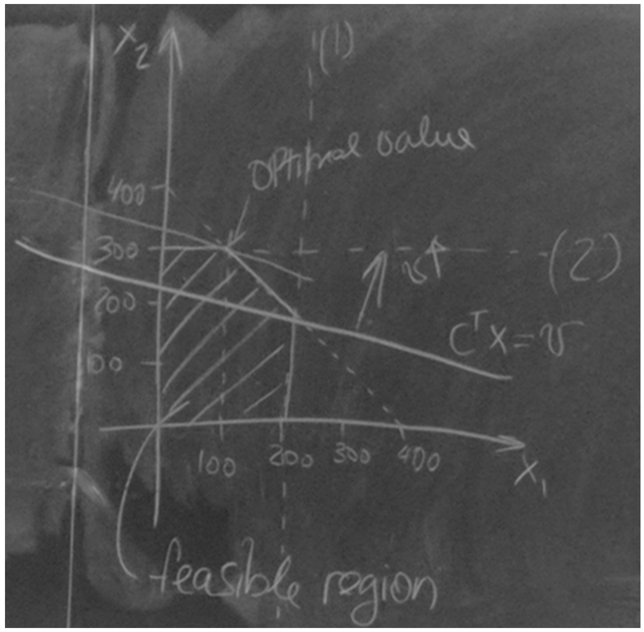

Figure 2: Updated *Figure 1*

Generally, $c^T x = v$ defines a hyperplane.
v is achievable if and only if that hyperplane intersects the feasible region.

Definition: $\bar{x} \in \mathbb{R}^n$ is optimal if it is feasible & optimizes the objective function.
Theorem: Every linear program satisfies exactly one of the following:

1. LP is infeasible (e.g. $x_1 \geq 3, x_1 \leq 2$)
2. LP is unbounded (e.g. $\max x_1 + x_2$ so that $x_1, x_2 \geq 0$)
3. There is a vertex of the feasible region that gives an optimal solution

Complexity of LP

LP is solvable in *polytime* (interior-point method, ellipsoid).
Simplex - **worst-case exponential time** but works extremely well in practice.

Powerful Tool

- Given problem, try to express it as an LP
- Use off-the-shelf solvers

Examples

Shortests s to t path. **Input:** $G = (V, E)$ diagraph
$w : E \to \mathbb{R}_{\geq 0}$
$s, t \in V$

Output: Length of shortest s to t path

Reduction: Introduce variables d_v for $v \in V$
number of variables $= |V|$ number of constraints $= |E| + 1$ $\max d_t$ so that $\forall (u, v) d_v \leq d_u + w(u, v)$
$d_s = 0$

Consider $d(s, v) = \min_{(u,v) \in E} \{d(s, u) + w(u, v)\}$

$\Rightarrow d(s, v)$ is the largest real number so that $\forall (u, v) \in E, d(s, v) \leq d(s, u) + w(u, v)$

Examples: Max-Flow
Input: $G = (V, E)), s, t, c$ - flow network
Output: Max-flow

Reduction: Introduce variables $f_{u,v}$ for $(u, v) \in E$
number of variables $= |E|$
number of constraints $= 2|E| + |V| - 2$
$\max \sum_{(s,u) \in E} f_{s,u}$ so that $\forall (u, v) \in E, f_{u,v} \leq C(u, v)$
$\forall v \in V$
$\{s, t\}, \sum_{(u,v) \in E} f_{u,v} = \sum_{(v,u) \in E} f_{v,u}$
$\forall (u, v) \in E, f_{u,v} \geq 0$