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CSC373S: Algorithm Design, Analysis & Complexity

Lecture 19

Friday February 17, 2017
based on notes by Denis Pankratov

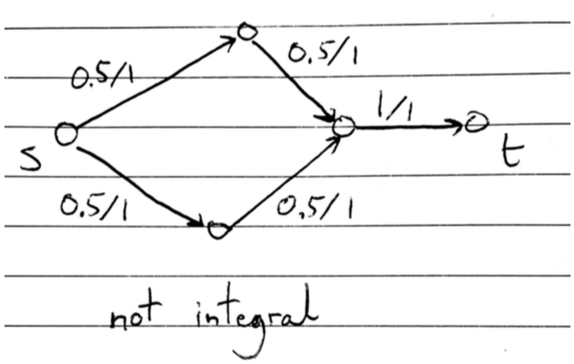
Network Flows *(continued)*

Runtime:

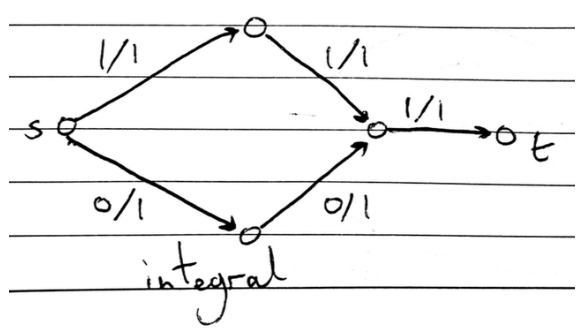
1. Edmonds-Karp $\mathcal{O}(|V||E|^2)$
2. Dinic's Algo $\mathcal{O}(|V|^2|E|)$
3. Push-relabel $\mathcal{O}(|V|^3)$
4. Based on nearly linear laplacian solvers

Lemma: If in a given flow network all capacities are integral, then there exists a max-flow that is integral (i.e. all values in flow are integers).

Example:



(a) *Not integral*



(b) *Integral*

Figure 1: *Visualization of Example*

Proof: FF on a flow network w/ integral capacities terminates & produces integral flow that is max (by correctness of FF).

Multi-Source Multi-Sink Max-Flow

Input: $G = (V, E)$ directed, no back edges
$c : E \Rightarrow \mathbb{R}_{>0}$ capacities
$s_1, .., s_k$ - sources (no incoming edges)
$t_1, .., t_l$ - sinks (no outgoing edges)

Output: multi-source multi-sink max-flow

Solution: reduce it to regular max flow.

Construct a flow network:

$$G' = (V', E'), s, t, c'$$

$$V' = V \cup \{s, t\}$$

Where s is a super source and t is a super sink
Add edges $(s, s_i) \in E$ w/ $c'(s, s_i) = \infty$
Add edges $(t_i, t) \in E$ w/ $c'(t_i, t) = \infty$
Add all other edges E w/ their capacities

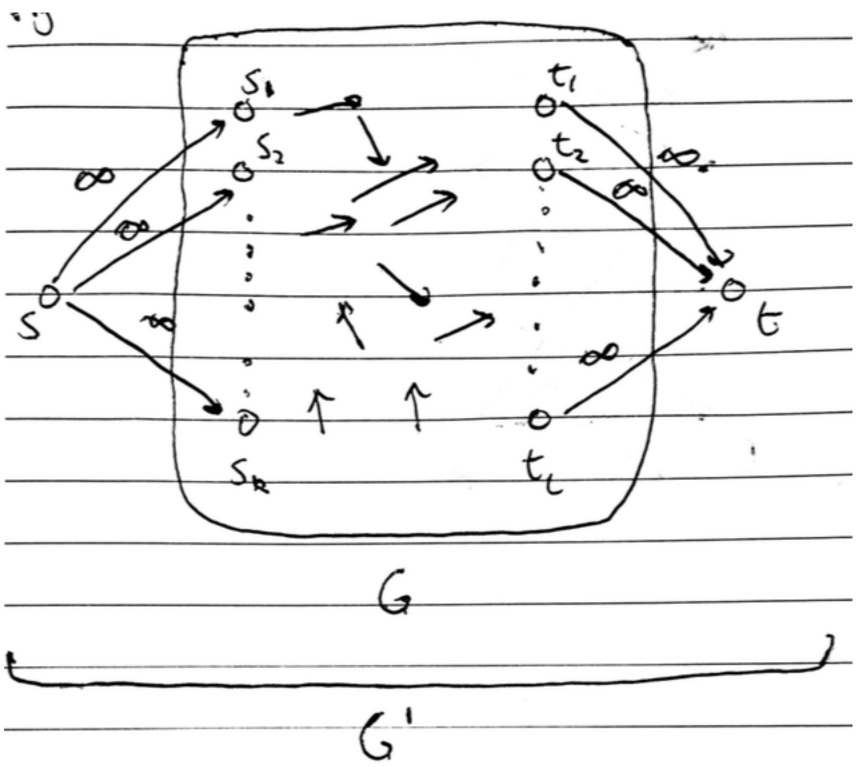


Figure 2: *Visualization*

Exercise: Show that max-flow in G' corresponds to max-flow in G & *vice versa*.

Maximum Bipartite Matching

Definition: Let $G = (V, E)$ be an undirected graph. G is bipartite if $\exists L, R \subseteq V$ so that $L \cap R = \varnothing, L \cup R = V$ and $\forall e \in E, |e \cap L| = |e \cap R| = 1$.

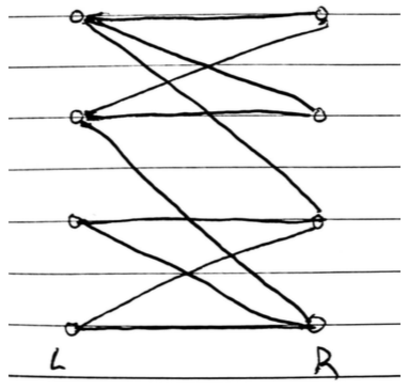


Figure 3: *Bipartite Graph*

Definition: $M \subseteq E$ is a matching if $\forall e_1, e_2 \in M, e_1 \cap e_2 = \varnothing$

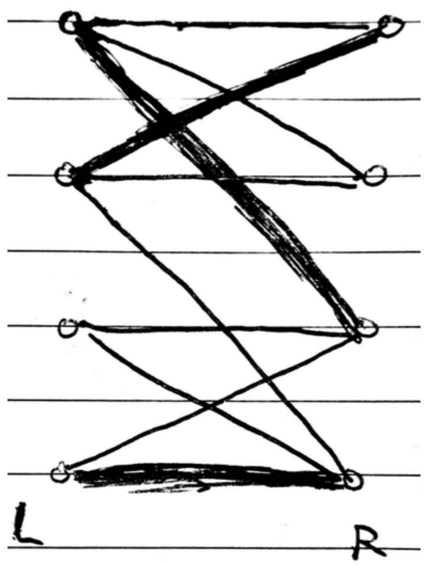


Figure 4: *M in Bipartite Graph* (M is shown thicker)

Input: $G = (L \dot{\cup} R, E)$ - bipartite ($\dot{\cup} \equiv \sqcup \equiv$ disjoint union)

Output: $M \subseteq M$ - matching of max size

Solution: Reduce this problem to network flows.

Construct flow-network $G' = (L \dot{\cup} R \cup \{s, t\}, E'), s, t, c'$

1. Add edges $(s,l) \in E', \forall l \in L$
   $c'(s,l) = 1$

2. Add edges $(r,t) \in E', \forall r \in R$
   $c'(r,t) = 1$

3. $\forall e \in E, e = \{l,r\}, l \in L, r \in R$
   Add edge $(l,r) \in E', c'(l,r) = 1$

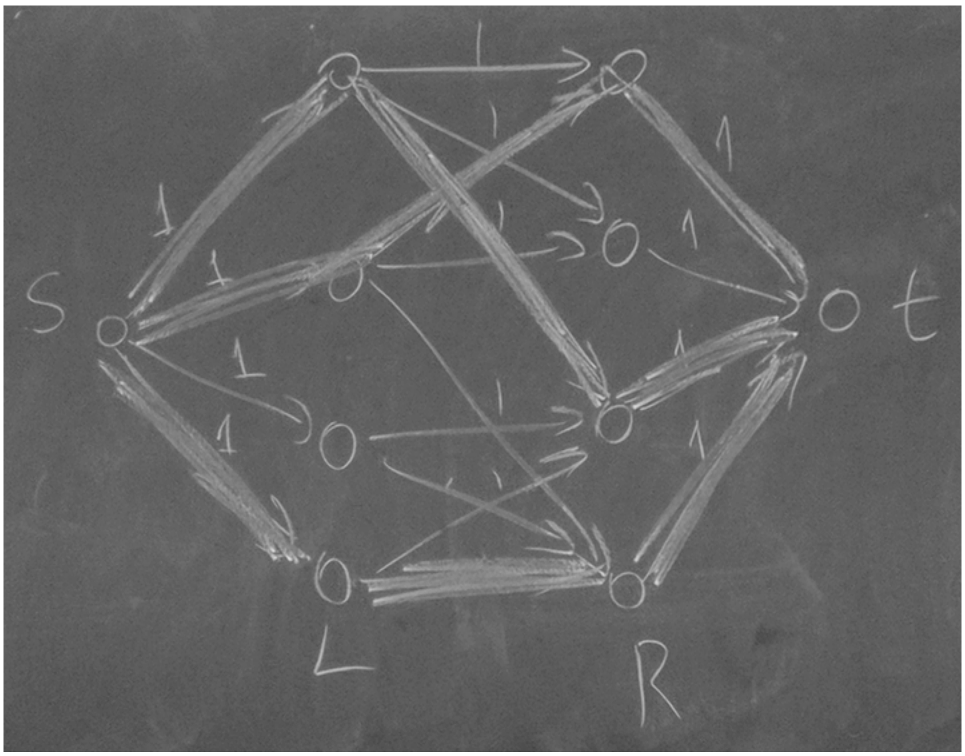


Figure 5: *Visualization*

<u>Lemma 1:</u> If M is a matching in G $\exists$ flow f in G' so that $|M| = |f|$

<u>Proof:</u> Let $M = \{\{l_1, r_1\}, \{l_2, r_2\}, .., \{l_k, r_k\}\}, l_i \in L, r_i \in R$

<u>Define f:</u>

$$f(s, l_i) = 1, \forall i \in [k]$$

$$f(r_i, t) = 1, \forall i \in [k]$$

$$f(l_i, r_i) = 1, \forall i \in [k]$$

$$f(e) = 0 \text{ elsewhere}$$

Easy to see f is a valid flow,

$$|f| = \sum_{i=1}^{k} f(s, l_i) = k = |M|$$

Corollary: Max flow in $G' \geq$ Max-Matching in G

Lemma 2: Let f be an integral flow in $G', \exists M-$ a matching in G so that $|M| = |f|$

Proof: Let $M = \{\{l, r\} | f(l, r) = 1\}$

1. $|M| = |f|$
2. M is a valid matching at most one unit of flow enters $l \in L$
   $\Rightarrow$ at most 1 unit of flow leaves $l \in L$
   $\Rightarrow \forall e_1, c_2 \in M, |e_1 \cap e_2 \cap L| = 0$

By a symmetric argument, no edges of M overlap on R as well.

Corollary 2: Max flow in $G' \leq$ Max-matching in G (uses the lemma about integral max-flow)

Runtime: max-matching $\leq |V|$
$\Rightarrow$ max number of augmentations $\leq |V|$
$\Rightarrow$ overall runtime $\mathcal{O}(|V||E|)$ if no isolated vertices