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CSC373S: Algorithm Design, Analysis & Complexity

Lecture 29

Monday March 20, 2017
based on notes by Denis Pankratov

NP Complete Problems *(cont.)*

Definition: $G = (V, E)$ is an undirected graph
$S \subseteq V$ is a vertex cover if every edge $e \in E$ is incident on at least 1 vertex from S
If $|S| = k$, then S is called a k-cover

Example:

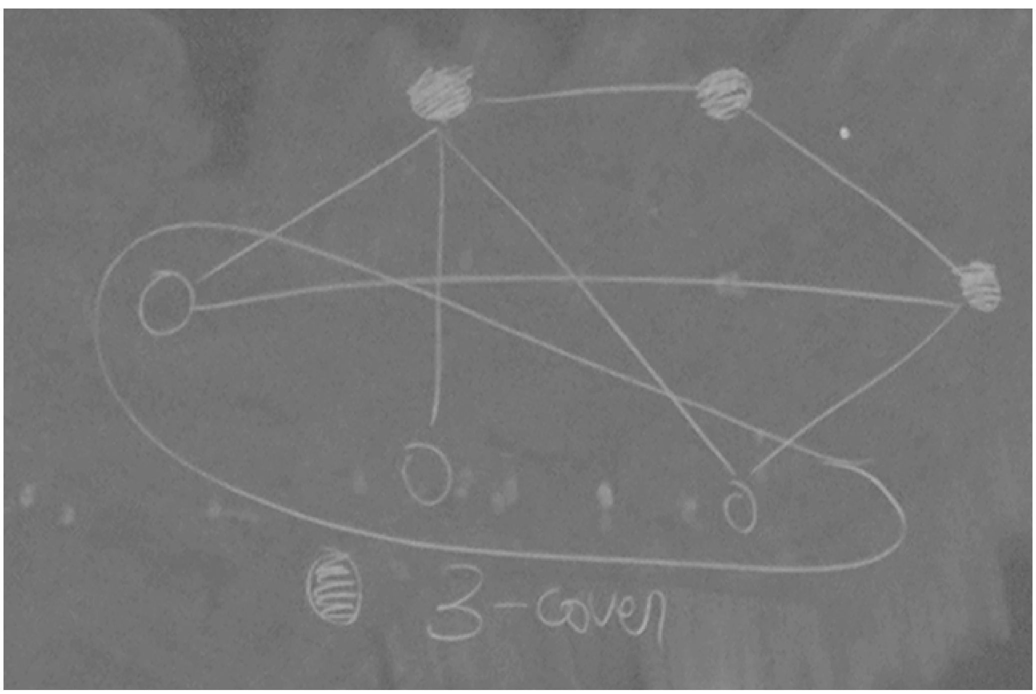

Figure 1: 3-cover

Vertex Cover Decision Problem
Input: $G = (V, E)$ undirected graph
$k \in \mathbb{Z}$
Output: 1 if G has a k-cover
0 otherwise

$L_{VC} = \{< G, k > | G \text{ has } k\text{-cover}\}$

Claim: $L_{VC} \in NPC$
Proof:

1. $L_{VC} \in NP$
   Certificate set $S \subseteq V$. The verifier checks that $|S| = k$ & that every edge of G is incident on a node from S

   Clearly, the certificate size is *polynomial* in the size of the input & the verifier runs in polynomial time
   (*Note:* Even though this might sound trivial, you need to specify this to get full marks!)

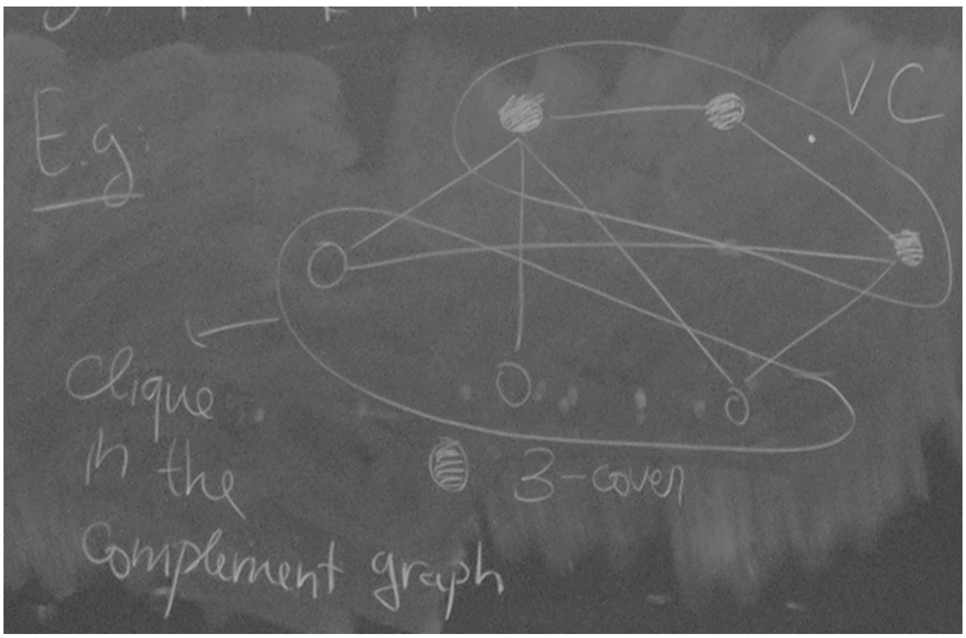

Figure 2: *Visualization*

2.
   There is no edge between remaining nodes. Complement graph replaces each non-edge with an edge.
   $L_{CLIQUE} \leq_p L_{VC}$
   Given $G = (V, E)$ & k. We need to construct $G' = (V', E')$ & k' so that G has a k-clique $\Leftrightarrow G'$ has a k'-cover, $V' = V, E' = \binom{V}{2} \setminus E$ - the complement graph
   $k' = n - k$ where $n = |V|$
   Clearly, this constructions runs in polynomial time.

   $\Rightarrow$ G has a k-clique S NTS: $V \setminus S$ is a vertex cover in G'.
   Let $e = \{u, v\} \in E' \Rightarrow \{u, v\} \notin E$
   $\Rightarrow$ either $u \notin S$ or $v \notin S$

$\Rightarrow$ either $u \in V \setminus S$ or $v \in V \setminus S$

$\Rightarrow V \setminus S$ is an $(n-k)$ vertex cover in G'

G has a k-clique $\Leftarrow G'$ has a $(n-k)$-cover
$\Leftarrow$ Let $T \subseteq V$ be an $(n-k)$ vertex cover in G'
NTS: $V \setminus T$ is a k-clique in G
Let $u, v \in V \setminus T$ suppose $\{u, v\} \notin E \Rightarrow \{u, v\} \in E'$
$\Rightarrow \{u, v\}$ is not covered by T - contradiction!

$\Rightarrow \{u, v\} \in E$ ☑

Hamiltonian Cycle

Definition: $G = (V, E)$ is an undirected graph
A Hamiltonian cycle is a simple cycle that visits every vertex.

Example:
Fact: $L_{HAM-CYCLE} = \{< G > | G \text{ has a Hamiltonian cycle}\}$ is NP-complete
Proof: CLRS ☑

Definition: $G = (V, E)$ is an undirected graph
A path from u to v $(u \neq v)$ is Hamiltonian if it is simple & visits every node of G.

Example:

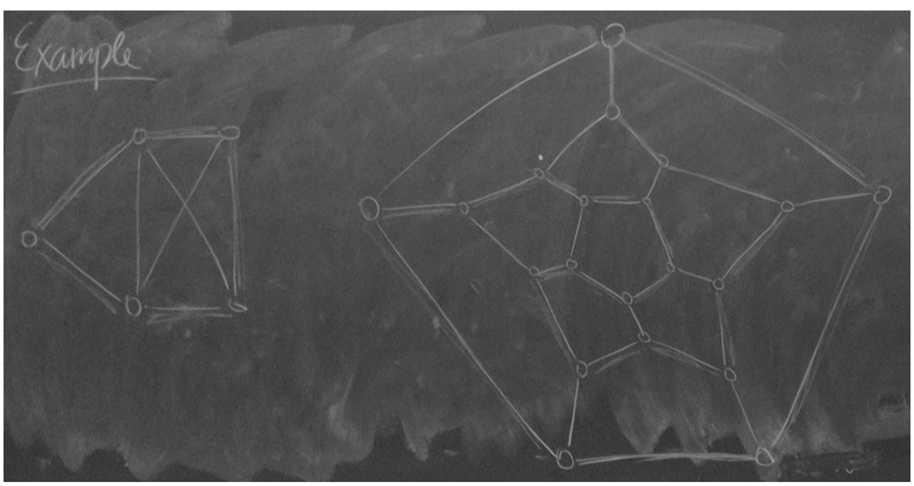

Figure 3: *Visualization*

$L_{HAM-PATH} = \{< G > | G \text{ has a Ham path}\}$
Claim: $L_{HAM-PATH} \in NPC$
Proof:

1. $L_{HAM-PATH} \in NP$ - exercise

2. $L_{HAM-CYCLE} \leq_p L_{HAM-PATH}$
   Given $G = (V, E)$, we need to construct $G' = (V', E')$ so that G has a Ham cycle $\Leftrightarrow$ G' has a Ham path

Pick arbitrary $v \in V$ & duplicate it v'
neighbors(v) = neighbors(v') & no edge between v & v'

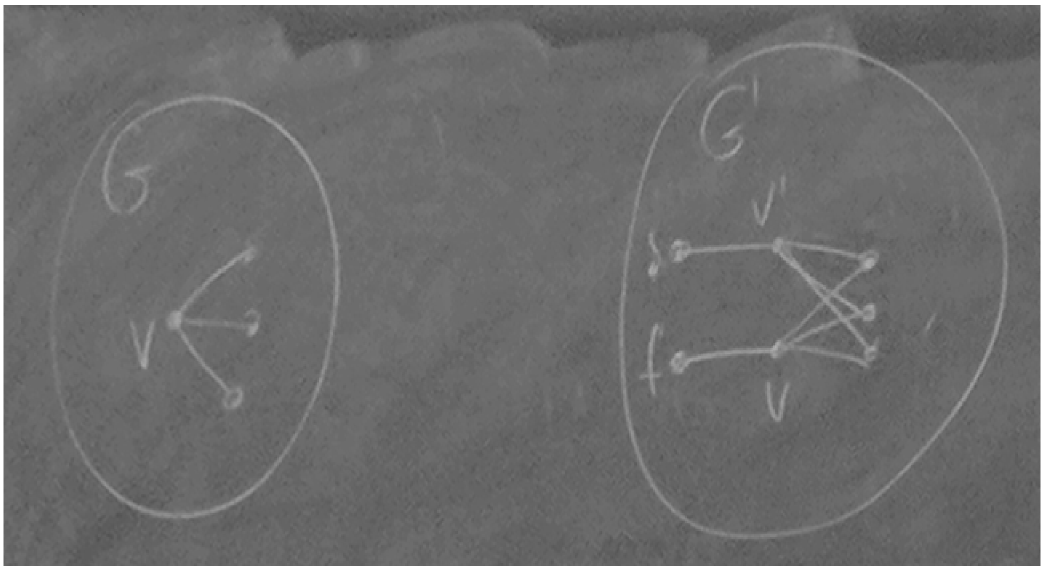

Figure 4: *Visualization*

Introduce two new nodes f & s
f connected only to v & s connected only to v'
This construction can be done in polytime

<u>Claim:</u> G has a Ham cycle $\Leftrightarrow$ G' has a Ham Path
<u>Proof:</u> $\Rightarrow$ G has a Ham Cycle $< v, u_1, .., u_{n-1}, v \Rightarrow < f, v, u_1, .., u_{n-1}, v', s >$
This is a Ham path in G' $\Leftarrow$ Let $< G' > \in L_{HAM-PATH}$
Note: Every Ham-Path in G' starts at f & finishes at S (or vice versa) $\leftarrow$ Exercise

Let $< f, v, u, .., u_{n-1}, v', s >$ be a Ham path in G' $\Rightarrow < v, u_1, .., u_{n-1}, v >$ is a Ham cycle in G. ☑

$L_{TSP} = \{< G, c, k > | G$ has a stour of cost $\leq k$, G is a complete directed graph$\}$

<u>Claim:</u> $L_{HAM-CYCLE} \leq_p L_{TSP}$
<u>Proof:</u> Given $G = (V, E)$ is undirected
Let G' = complete directed graph on V

$$c(u, v) = \begin{cases} 0, & \text{if } \{u, v\} \in E \\ 1, & \text{otherwise} \end{cases}$$

$k = 0$
Claim: $< G > \in L_{HAM-CYCLE} \Leftrightarrow < G', c, k > \in L_{TSP}$ (exercise)