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CSC373S: Algorithm Design, Analysis & Complexity

Lecture 18

Wednesday February 15, 2017
based on notes by Denis Pankratov

Network Flows *(continued)*

Last time: Proof of correctness of FF Max-Flow Min-Cut Theorem
$\Rightarrow$ algorithm to compute min-cut

1. compute max flor f
2. compute $S = \{v | v \text{ is reachable from } s \text{ in } G_f\}$ (BFS)
   Runtime: $\mathcal{O}(|V| + |E|) = O(|E|)$ if no redundant nodes

Example: More *Complex* Now

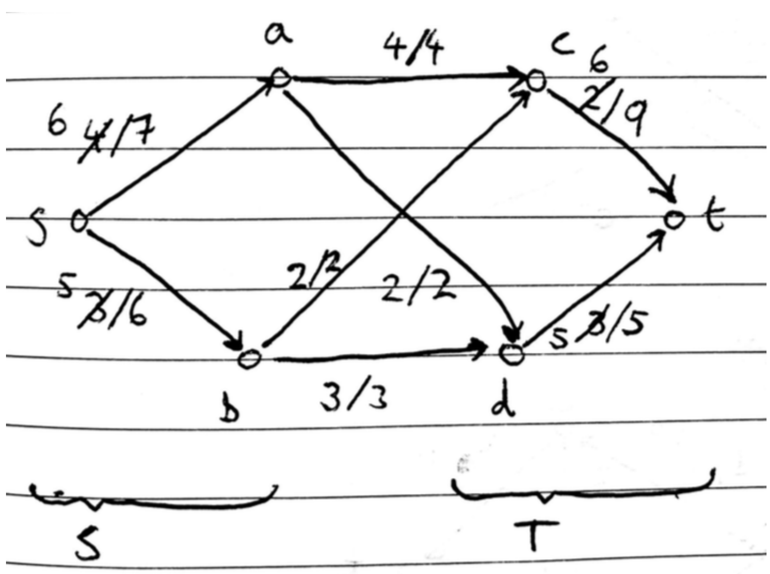


Figure 1: *After* $P_1..P_4$; S vertices to the left, T to the right

P	$C_f(P)$
$P_1 = (s, b, d, t)$	3
$P_2 = (s, b, c, t)$	2
$P_3 = (s, a, c, t)$	4
$P_4 = (s, a, d, t)$	2

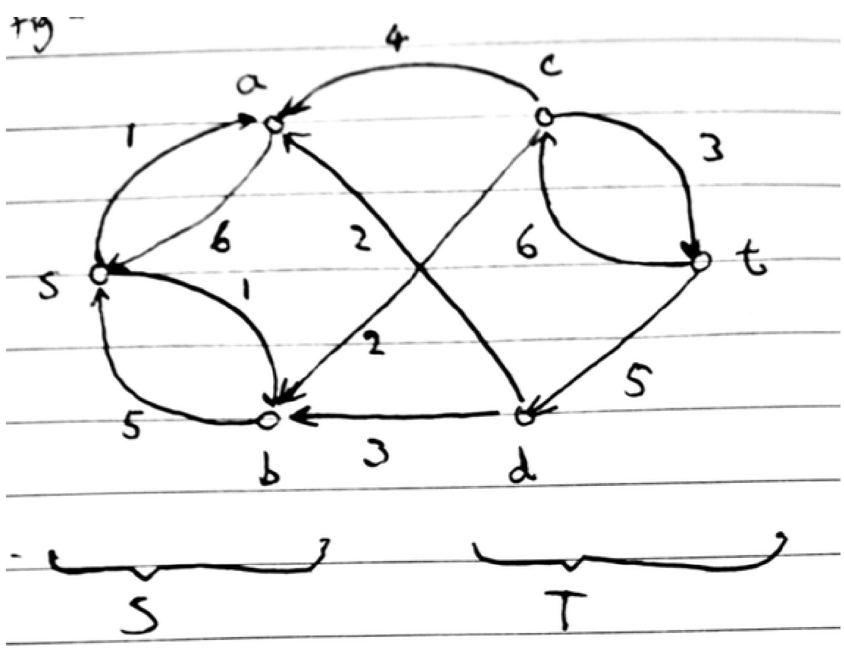


Figure 2: *Residual Graph*

Overall, we have that:

$$|f| = 6 + 5 = 11$$

$$c(S, T) = 4 + 2 + 2 + 3 = 11$$

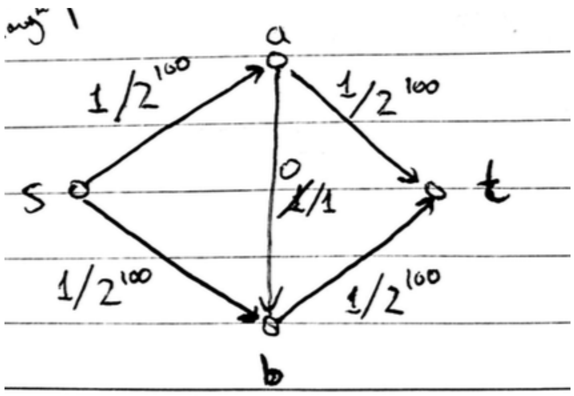
<u>Runtime of FF method:</u>
(**worst-case** over inputs & choice of augmenting paths)

Case 1: Capacities are arbitrary positive real numbers.

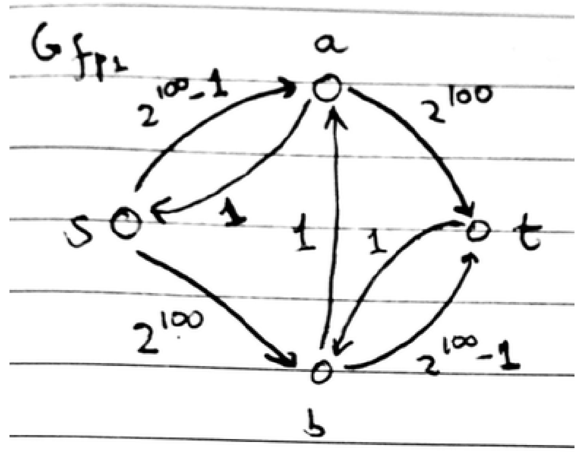
If some capacities are irrational, FF might not terminate.
Moreover, such examples FF might converge to a flow that is not maximum,

Case 2: Capacities are $\mathbb{N}$

Let $f*$ be a max flow, FF can be made ro run $\mathcal{O}(|f*||E|)$ each time we augment w/ path P, value of the flow increases by $c(P) \geq 1$.
$\Rightarrow$ number of augmentations $\leq |f*|$
Each augmentation path can be found by DFS/BFS in time $\mathcal{O}(|E|)$ assuming no redundant nodes (<u>not</u> polynomial time).



(a) *After 2 augmentation paths*, P_1, P_2



(b) *Residual Graph*

Figure 3: *Example*

P	$C_f(P)$
$P_1 = (s, a, b, t)$	1
$P_2 = (s, b, a, t)$	1

Need to perform these steps 2^{100} times to get max-flow!

Can we make FF run in polynomial time?
Yes! *Edmonds-Karp Algorithm*: run FF, use shortest unweighted path for augmentation. Can be done by BFS!

Claim: EK runs in $\mathcal{O}(|V||E|^2)$ times

Definition: Let G, s, t, c a flow network
$f-$ a flow
$d_f(u, v) =$ length of shortest unweighted path in G_f

Lemma 1: Let f be a flow, f' be an augmented flow after 1 step of EK. Then, $\forall v, d_f(s, v) \leq d_{f'}(s, v)$.

Proof (by contradiction): Assume $\exists v, d_f(s, v) > d_{f'}(s, v)$.
Pick such v with smallest value of $d_{f'}(s, v)$.

Let u be the node immediately preceeding v on shortest path in $G_{f'}$.

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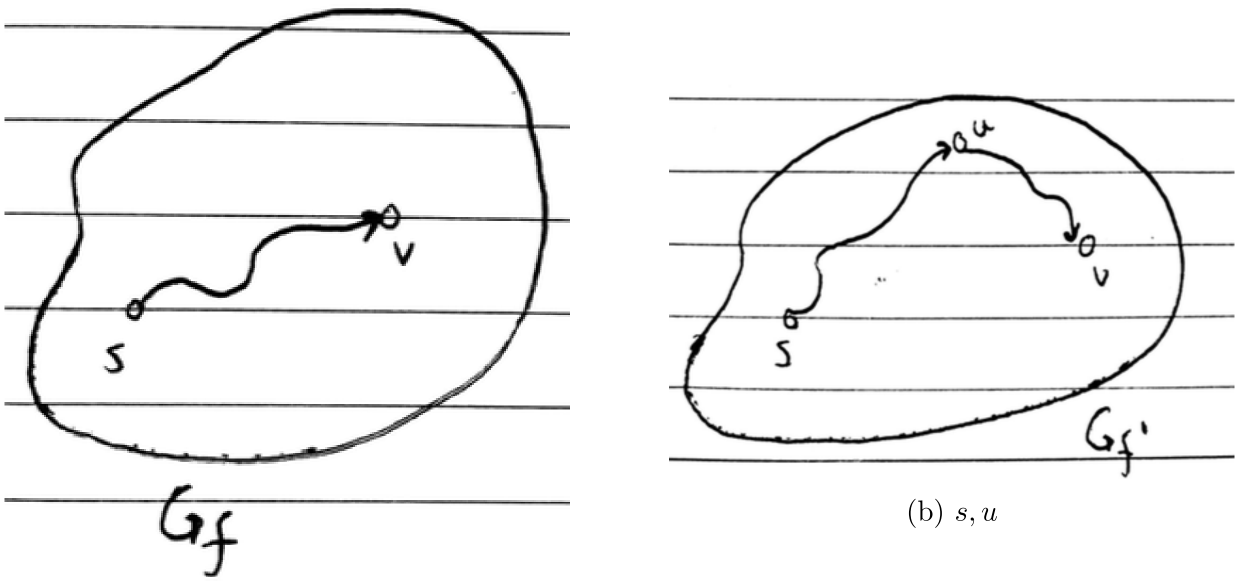

(b) s, u

(a) s, v

Figure 4: *Visualization*

1. by choice of $v, d_f(s,u) \leq d_{f'}(s,u) = d_{f'}(s,v) - 1$

2. $(u,v) \notin E_f$
   if $(u,v) \in E_f, d_f(s,v) \leq d_f(s,u) + 1 \leq d_{f'}(s,u) + 1 = d_{f'}(s,v)$

$(u,v) \notin E_f, (u,v) \in E_{f'}$
$\Rightarrow (v,u)$ was on the augmenting path.

By the way EK chooses paths:
(Keeping in mind that $d_f(s,u) \leq d_{f'}(s,u)$)

1. $d_f(s,u) = d_f(s,v) + 1$

2. $d_{f'}(s,u) = d_{f'}(s,v) - 1$

With both (1) and (2):

$$\Rightarrow d_{f'}(s,v) \geq d_f(s,v) + 2G_f$$
$$d_{f'}(s,v) > d_f(s,v)$$

<u>Definition:</u> An edge (u,v) is *critical* for augmenting path P if $C_f(P) = c_f(u,v)$

Lemma 2: An edge e can become critical $\leq |V|/2$ times throughout the entire run of EK.

<u>Aside:</u> Whenever we have an augmenting path, some edge becomes critical
$\Rightarrow$ number of augmentations $\leq$ number of times some edge becomes critical
$\leq \mathcal{O}(|V||E|)$
$\Rightarrow$ overall runtime $\mathcal{O}(|V||E|^2)$

<u>Proof:</u> Let f be a flow when (u, v) becomes critical.

Then,

$$d_f(s, v) = d_f(s, u) + 1$$

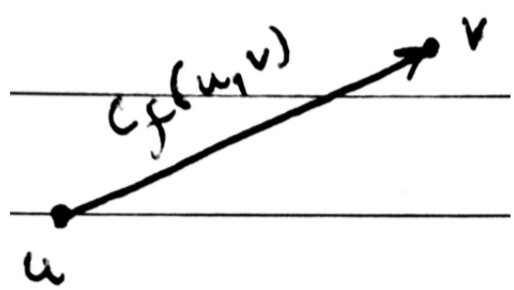

Figure 5: Needs to be *undone* to become critical again

Before (u, v) becomes critical again, some flow on u to v needs to be undone.

Let f' be the flow immediately before flow on (u, v) gets undone.

$$\begin{aligned} d_{f'}(s, u) &= d_{f'}(s, v) + 1 \\ &\geq d_f(s, v) + 1 \text{ (by previous lemma)} \\ &\geq d_f(s, u) + 2 \end{aligned}$$

$\Rightarrow$ shortest path from s to u in G_f increases by at least 2 each time (u, v) becomes critical. This can start at 1 and end at $\leq |V| \Rightarrow {}^{|V|}/_2$ times (u, v) can be critical