Homework 1. Markov chains

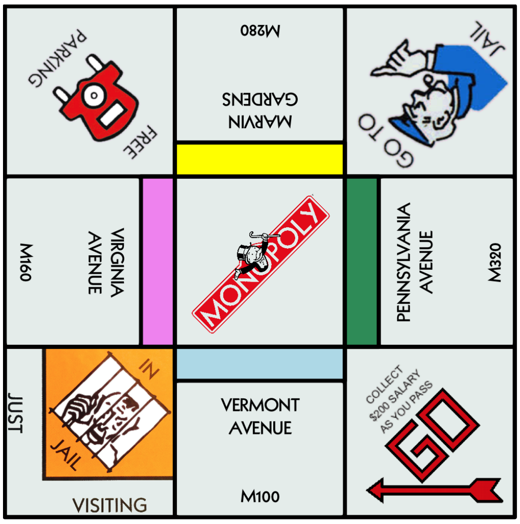


Figure 1: Simplified version of the "Monopoly" board game. In this version, the board is reduced to 8 distinct positions.

Consider the simplified version of the classic game "Monopoly" depicted in Fig. 1. In this homework, you will describe the motion of a single player using a Markov chain. To that purpose, consider that the player rolls a single die in each play. Moreover, the jail does not prevent the player from continuing to play. In other words, when landing on the "GO TO JAIL" cell, the player jumps back to the "JAIL" cell, but resumes play immediately.

Exercise 1.

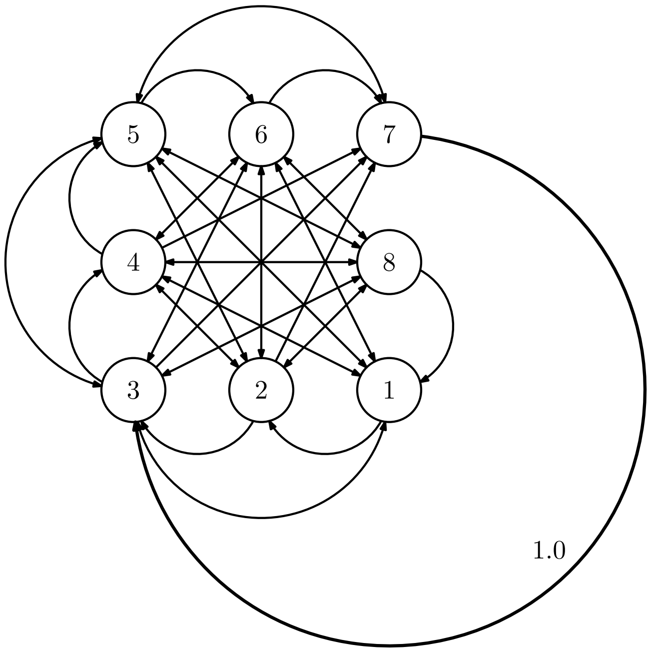
(a) Write down the transition diagram representing the motion of the player.

(b) Write down the Markov chain model corresponding to the transition diagram in (a).

(c) Suppose that the player departs from the "COLLECT $200" cell at time step $t = 0$. Compute the probability of the player being in each cell at time step $t = 3$.

Solution:

(a) The transition diagram becomes:



where all unmarked transitions correspond to a probability $p = \frac{1}{6}$.

(b) The Markov chain is specified as a pair $(\mathcal{X}, \boldsymbol{P})$, where

$$\mathcal{X} = \{1, 2, 3, 4, 5, 6, 7, 8\},$$

with the correspondence:

1: "COLLECT \$200"

2: "VERMONT AVENUE"

3: "JAIL"

4: "VIRGINIA AVENUE"

5: "FREE PARKING"

6: "MARVIN GARDENS"

7: "GO TO JAIL"

8: "PENNSYLVANIA AVENUE"

The corresponding transition probabilities are:

$$
\boldsymbol{P} = \begin{bmatrix}
0.000 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.000 \\
0.000 & 0.000 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 \\
0.167 & 0.167 & 0.000 & 0.000 & 0.167 & 0.167 & 0.167 & 0.167 \\
0.167 & 0.167 & 0.167 & 0.000 & 0.000 & 0.167 & 0.167 & 0.167 \\
0.167 & 0.167 & 0.167 & 0.167 & 0.000 & 0.000 & 0.167 & 0.167 \\
0.000 & 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.000 & 0.000
\end{bmatrix}.
$$

(c) Computing $\boldsymbol{P}^3$ yields

$$
\boldsymbol{P}^3 = \begin{bmatrix}
0.111 & 0.088 & 0.227 & 0.116 & 0.111 & 0.125 & 0.120 & 0.102 \\
0.106 & 0.088 & 0.204 & 0.120 & 0.116 & 0.130 & 0.130 & 0.106 \\
0.106 & 0.079 & 0.199 & 0.120 & 0.116 & 0.130 & 0.134 & 0.116 \\
0.116 & 0.079 & 0.190 & 0.116 & 0.116 & 0.130 & 0.134 & 0.120 \\
0.120 & 0.088 & 0.190 & 0.106 & 0.111 & 0.130 & 0.134 & 0.120 \\
0.120 & 0.093 & 0.199 & 0.106 & 0.102 & 0.125 & 0.134 & 0.120 \\
0.111 & 0.139 & 0.278 & 0.083 & 0.083 & 0.111 & 0.111 & 0.083 \\
0.102 & 0.102 & 0.269 & 0.102 & 0.097 & 0.116 & 0.116 & 0.097
\end{bmatrix},
$$

and the distribution over states for $t = 3$ comes

$$
\boldsymbol{\mu}_3 = \begin{bmatrix} 0.111 & 0.088 & 0.227 & 0.116 & 0.111 & 0.125 & 0.120 & 0.102 \end{bmatrix}.
$$