Homework Number

Your Name

Student Number

Course Number

Course Name

October 17, 2022

UNIVERSIDADE DE MACAU
UNIVERSITY OF MACAU

Problem 1

Smoothness

A differential function f is said to be L-smooth if its gradietns are Lipschitz continuous, that is

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$$

let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function. If f is L-smooth then prove the following inequality:

- (15 pt) Prove $\langle \nabla^2 f(x)v, v\rangle \leq L\|v\|_2^2, \quad \forall x, v \in \mathbb{R}^d$
- (15 pt) Prove $f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|_2^2$

Solution.

- $\langle \nabla^2 f(x)v, v\rangle \leq L\|v\|_2^2, \quad \forall x, v \in \mathbb{R}^d$

  Proof.

  Before proving the original inequality, we firstly prove the hessian matrix of the L-smooth differential function is negative semi-definite.

  $\because ||\nabla f(x) - \nabla f(y)|| \leq L||x - y||$
  $\therefore ||x - y|| \cdot ||\nabla f(x) - \nabla f(y)|| \leq L||x - y||^2$
  $\because \langle (x - y), \nabla f(x) - \nabla f(y)\rangle \leq ||x - y|| \cdot ||\nabla f(x) - \nabla f(y)||$(Cauchy-Schwartz Inequality)
  Rearranging the above terms, we would have: $\langle Lx - \nabla f(x) + Ly - \nabla f(y), (x - y)\rangle \geq 0$

  ......

- $f(y) \leq f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|_2^2$

  Proof.

  The original inequality can be written as:

  $$f(y) - f(x) - \nabla f(x)^T(y - x) \leq \frac{L}{2}\|y - x\|_2^2$$

  Observing the left hand side of the inequality, let $z(t) = x + t(y - x)$ and $g(t) = f(z(t))$. Then by the Newton-Leibniz formula:
  ......

Problem 2

Gradient descent rate with line search in strongly convex function

Suppose the function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex and twice differentiable, i.e $\nabla^2 f(x) \succeq lI$ with constant $l > 0$. Also, its gradient is Lipschitz continuous with constant $L > 0$, i.e. we have that $\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|$ for any x, y.

......

Solution.

......