Computational Geometry

- Algorithms for geometric problems.

- Relevant for Games, Computer Graphics, Robotics and GIS.

- We focus on “combinatorial computational geometry”, that is, the objects under study are basic geometrical objects, such as points, lines segments, polygons and polyhedra.

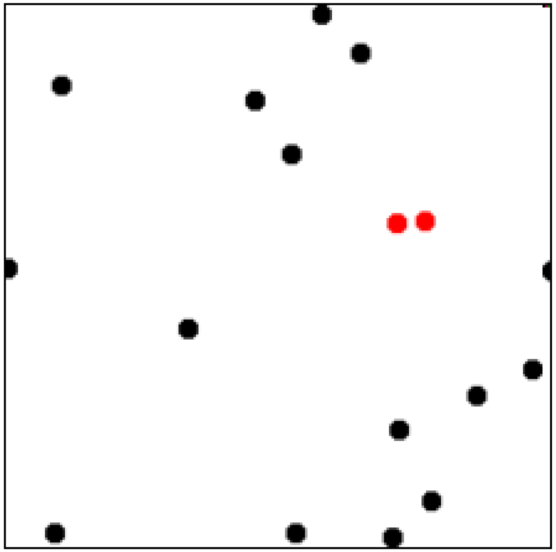
Computational Geometry

Readings:

- S. Skiena, M. Revilla, Programming Challenges, Chapter 13
- S. Skiena, The Algorithm Design Manual, Chapter 17
- T. Cormen et al., Introduction to Algorithms, Chapter 33
- David Goldberg. "What Every Computer Scientist Should Know About Floating-Point Arithmetic". ACM Computing Surveys, 23 (1): 5–48, 1991 (link)

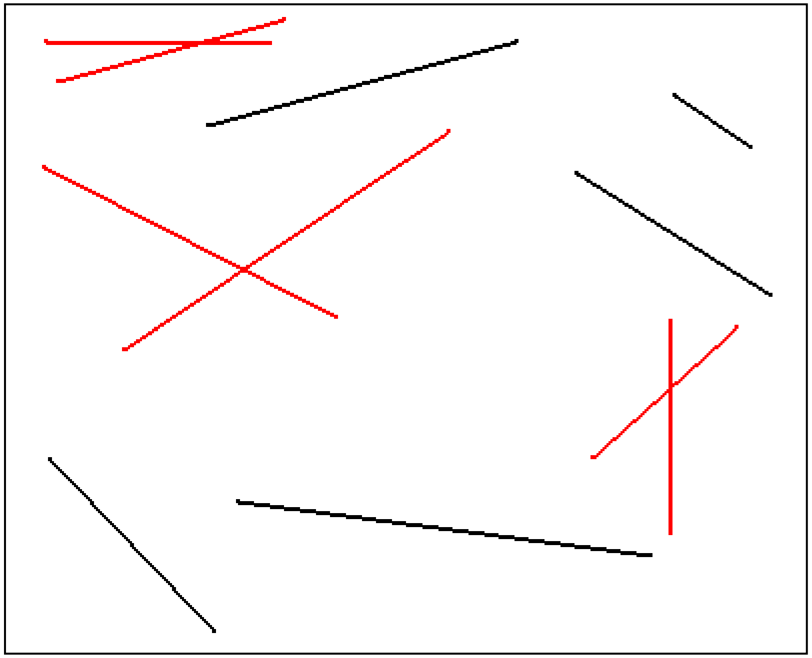
Computational Geometry

- Closest pair problem



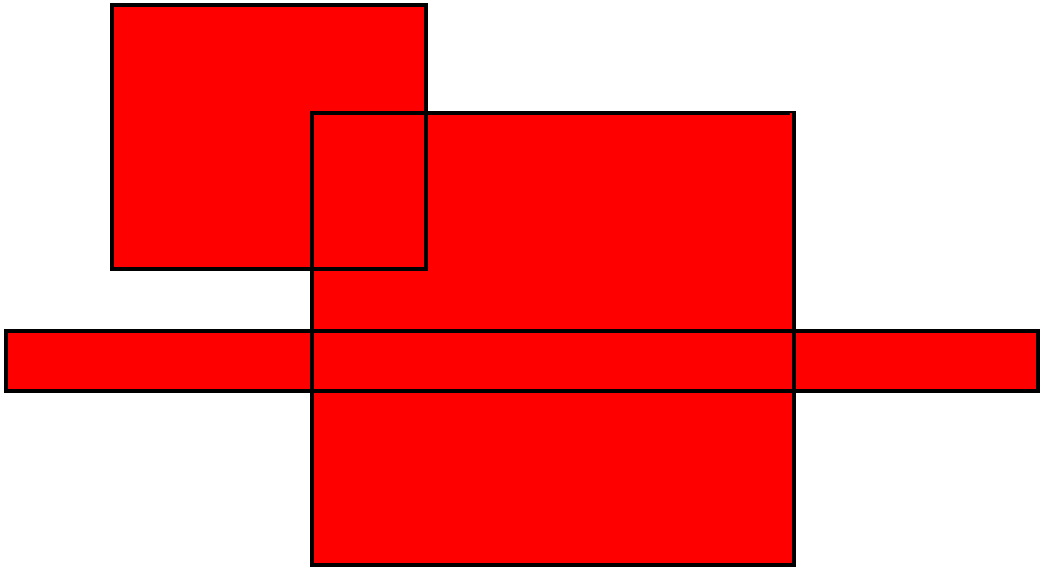
Computational Geometry

- Line intersection problem



Computational Geometry

- Area of the union of rectangles problem

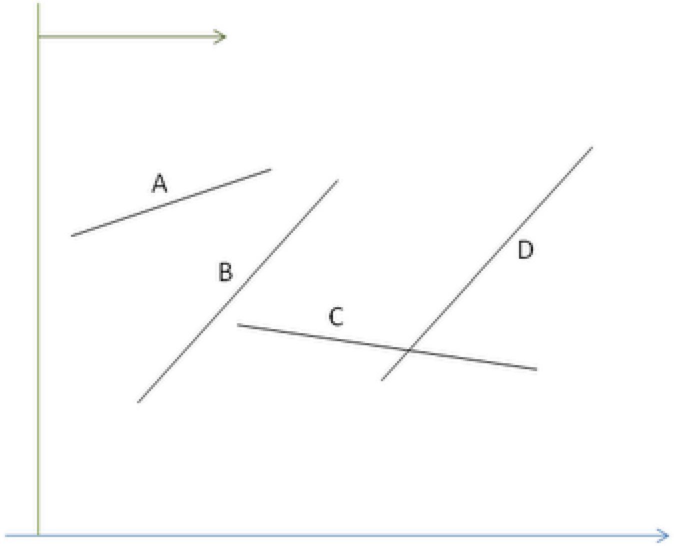


Computational Geometry

- Special strategies for computational geometry:
  - Line Sweep
    - A line sweeps the plane, stopping at some points.
    - Perform geometric operations at each stop.
    - Assumes that the objects are sorted in one dimension.
    - The solution is obtained once the line passed all objects.
    - For 3D, it is called plane sweep.

Computational Geometry

- Special strategies for computational geometry:
  - Line Sweep



Computational Geometry

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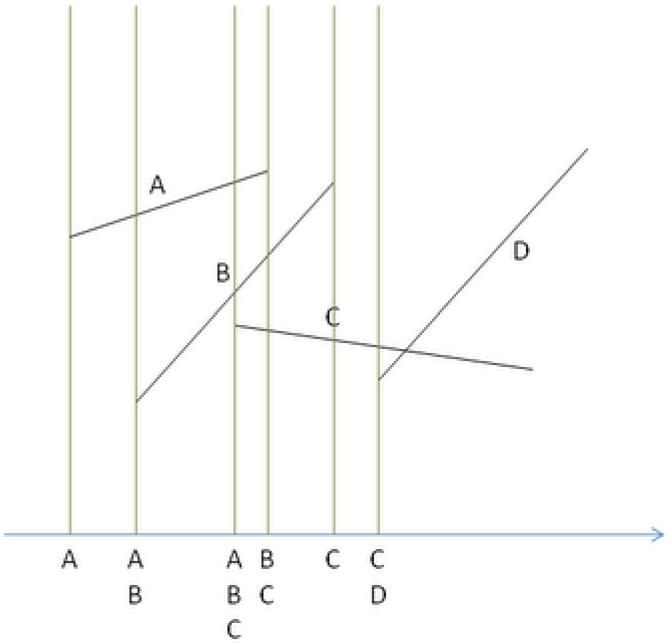

Computational Geometry

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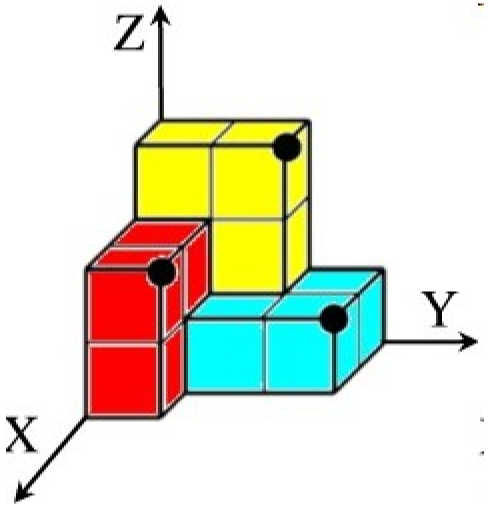

Computational Geometry

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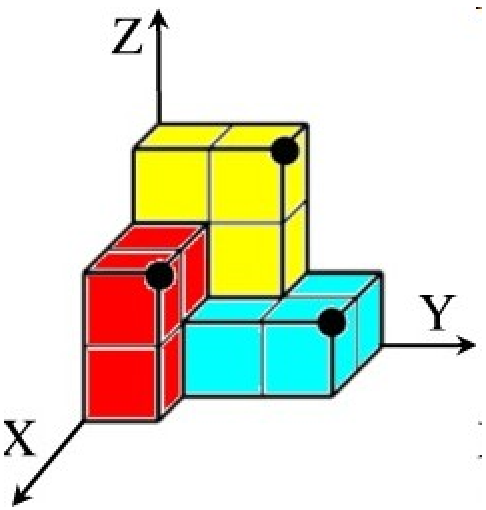

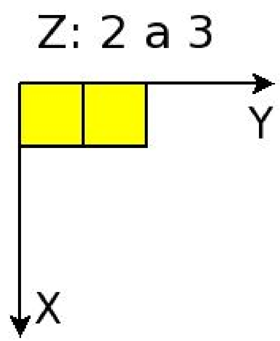

Computational Geometry

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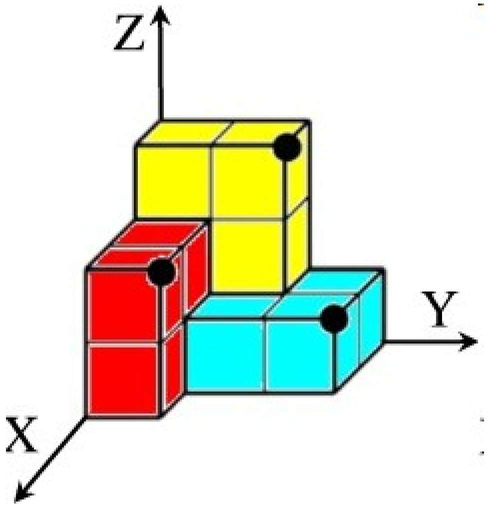

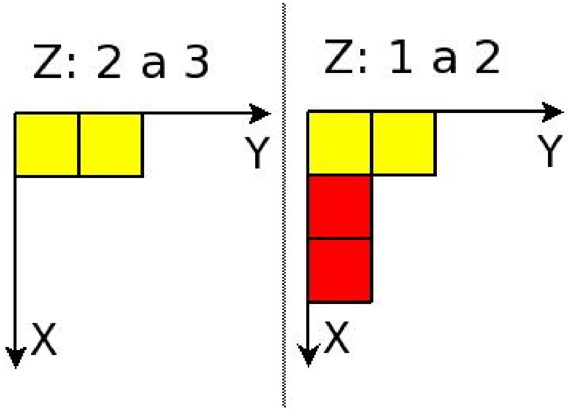

Computational Geometry

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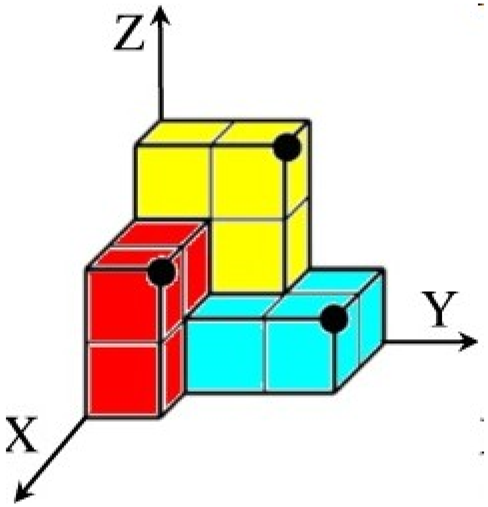
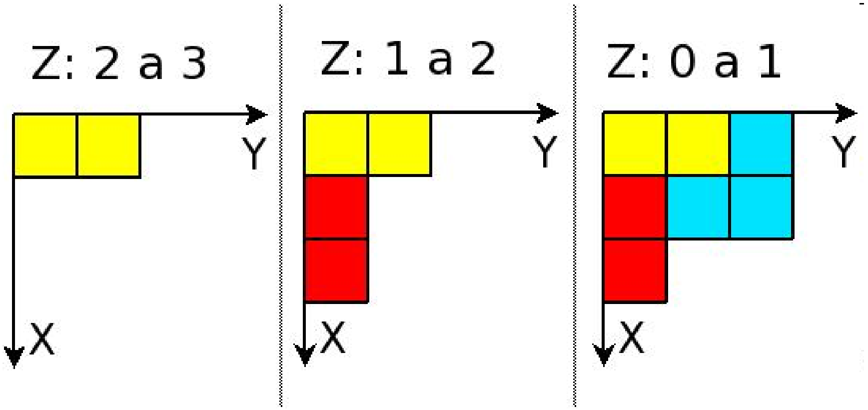
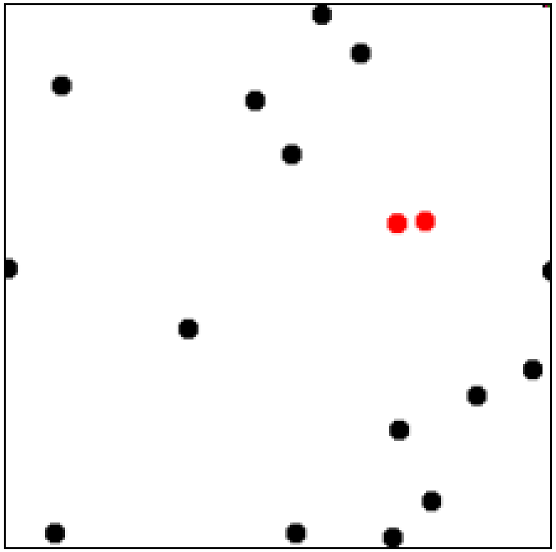




Computational Geometry

- Closest pair problem (given *n* points in 2D)



Computational Geometry

- Closest pair problem
- Brute force: $O(n^2)$
  - Compute all the distances between the *n* points and pick the pair with the smallest distance.

Computational Geometry

- Closest pair problem

- Divide-and-Conquer: O(n log n)

Step 1: Sort points along the x-coordinate.

Step 2: Divide: Split the points into 2 equal subsets, and solve recursively, the left and right subsets (δ_1 and δ_2).

Step 3: Merge: Find the minimum distance between points in the left and in the right subset (δ_c). The minimum distance is $\min(\delta_1, \delta_2, \delta_c)$. How to do it efficiently?

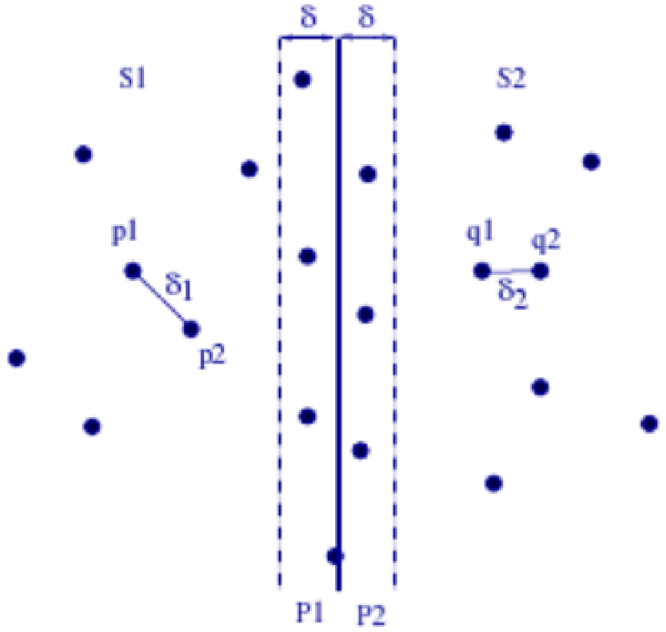
Computational Geometry

- Closest pair problem
- Divide-and-Conquer: O(n log n)

Step 3:

How to compute the minimum distance between the points in P1 and P2?

$\delta = \min(\delta_1, \delta_2)$



Computational Geometry

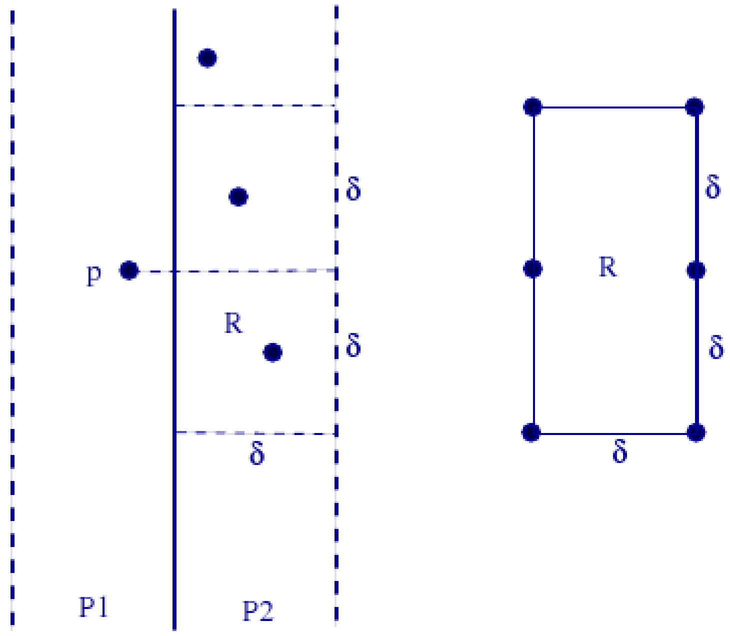
- Closest pair problem

- Divide-and-Conquer: O(n log n)

  Step 3:

For each point p, check neighbors at a distance at most δ of p

There can be at most 6 neighbors at a distance δ of p.



Computational Geometry

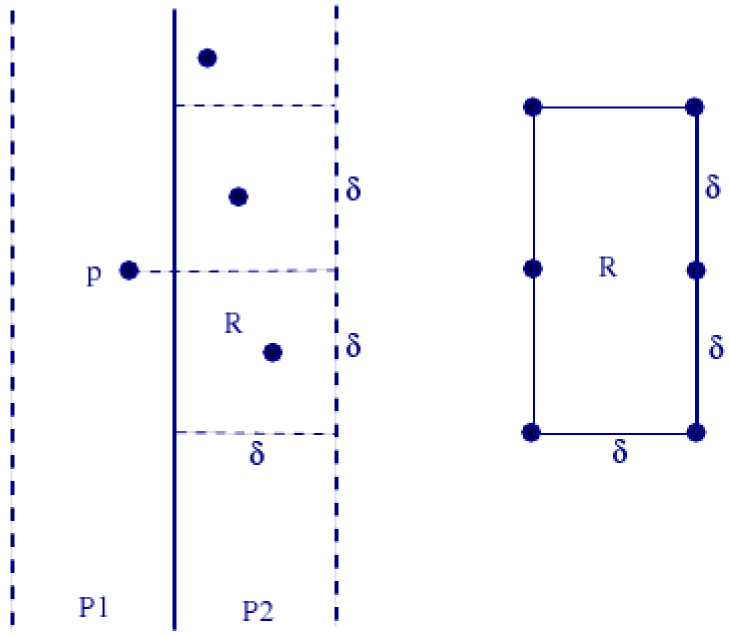
- Closest pair problem

- Divide-and-Conquer: O(n log n)

Step 3 in O(n):

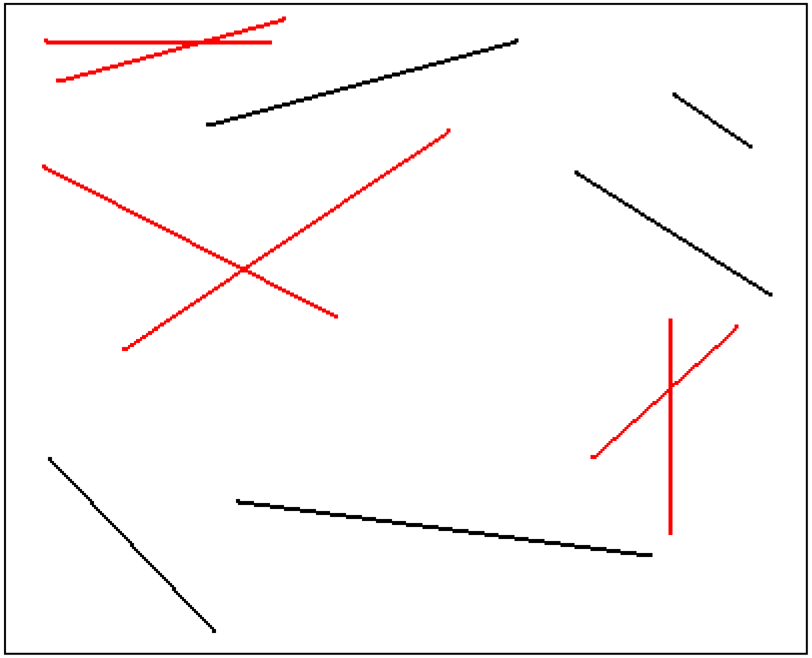
- Project points in P1 and P2 on the vertical line
- For each point, compute the minimum distance from the 6 neighbors

(All points must be sorted in y)



Computational Geometry

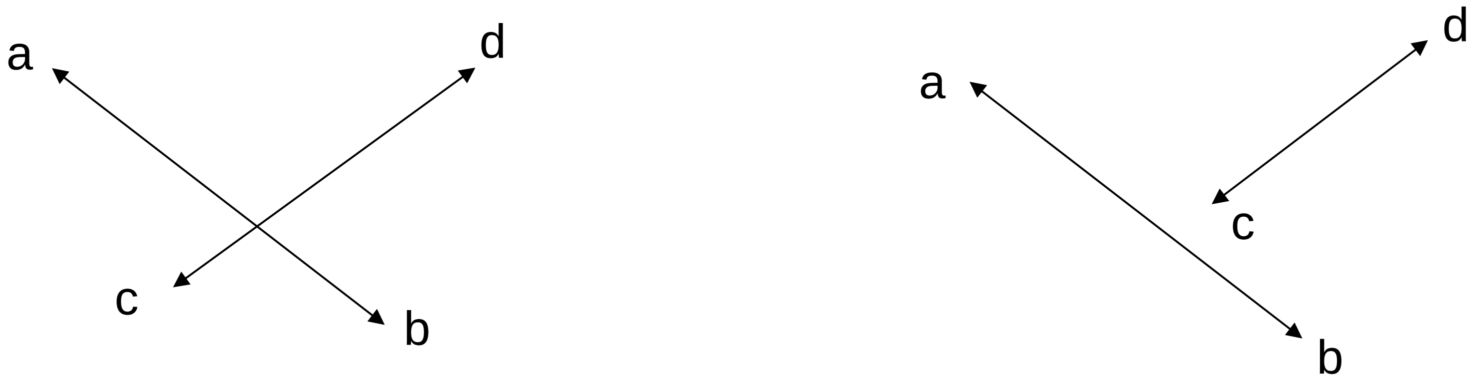
- Line intersection problem (given n lines and k intersections)



Computational Geometry

- Line intersection problem

- Two segments *ab* and *cd* intersect if and only if:
  - the endpoints *a* and *b* are on opposite sides of line *cd*, and
  - the endpoints *c* and *d* are on opposite sides of line *ab*.



Computational Geometry

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Computational Geometry

- Line intersection problem

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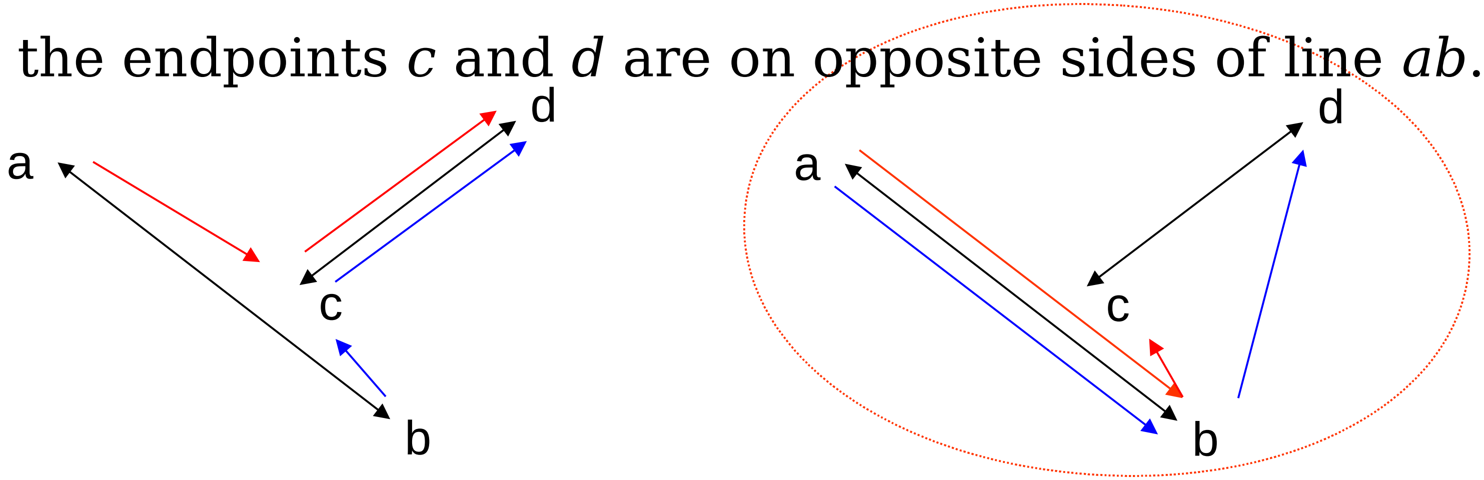
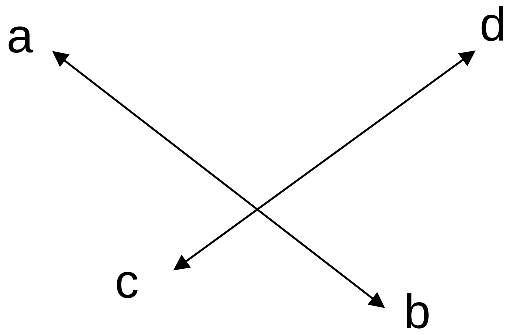


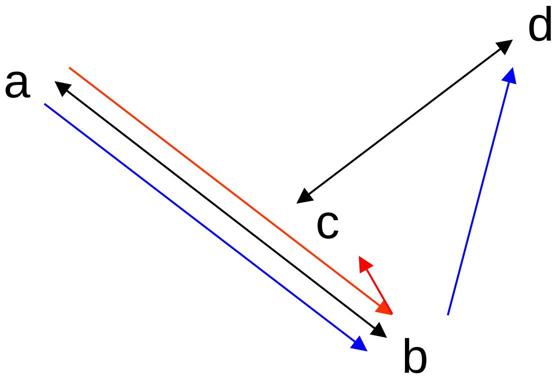
Computational Geometry

- Line intersection problem

- CW(a,b,c) tests whether a,b,c are in counterclockwise order.
- Two segments ab and cd intersect if and only if:

$$\mathrm{CW}(a,c,d) \neq \mathrm{CW}(b,c,d) \text{ and } \mathrm{CW}(a,b,c) \neq \mathrm{CW}(a,b,d)$$

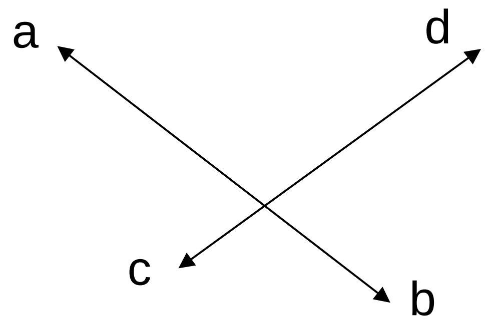




Computational Geometry

- Line intersection problem
- CW(a,b,c) tests whether a,b,c are in counterclockwise order.
- CW(a,b,c) is True if det(M) >0, False otherwise.

$$M = \begin{vmatrix} 1 & x_a & y_a \\ 1 & x_b & y_b \\ 1 & x_c & y_c \end{vmatrix}$$

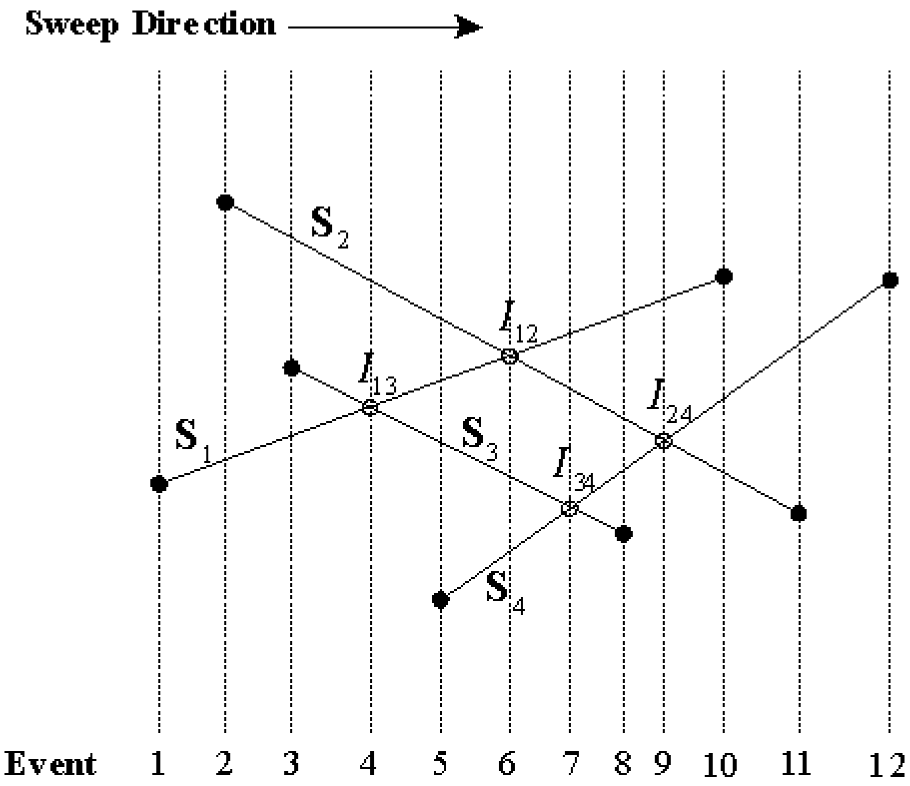


Computational Geometry

- Line intersection problem
- Brute force: $O(n^2)$
  - Detect intersection between every pair of lines.

Computational Geometry

- Line intersection problem



Bentley-Ottmann algorithm by sweep-line

Computational Geometry

- Line intersection problem

Sweep line: O((n+k) log n)

Step 1: Sort 2n points in x-coordinate and let $Q = \{p_1,\ldots,p_{2n}\}$

Step 2: From left to right, for each point *p in Q*:

Step 2.1: If *p* is left-endpoint of *l*, add intersection between *l* and its neighboring lines to Q.

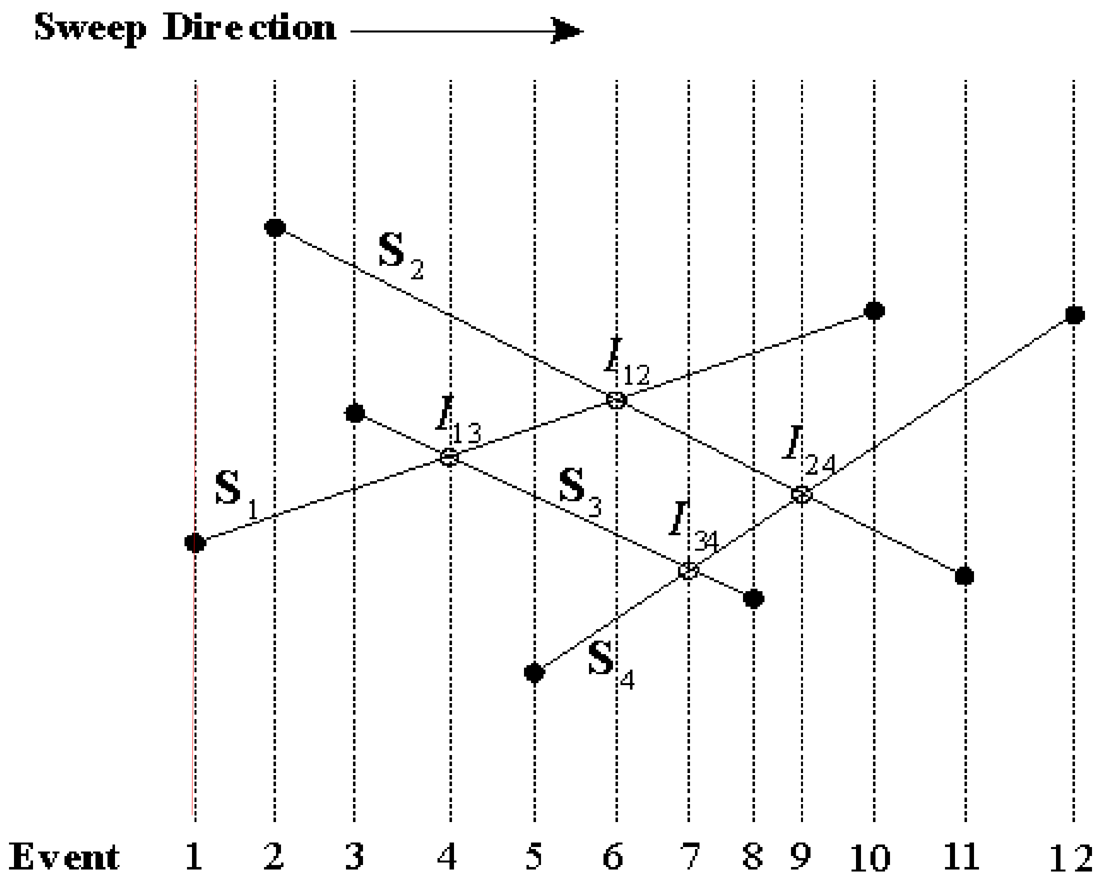
Step 2.2: If *p* is right-endpoint of *l*, add intersection between its neighboring lines to *Q* (to the right of *p*).

Step 2.3: If *p* is an intersection between l_1 and l_2, add intersection between l_1 and l_2 and their neighboring lines to *Q*.

- Q is a priority queue that keeps the line with the minimum x-coordinate
- Use a balanced binary tree to maintain the order of the segments at each step in the y-coordinate

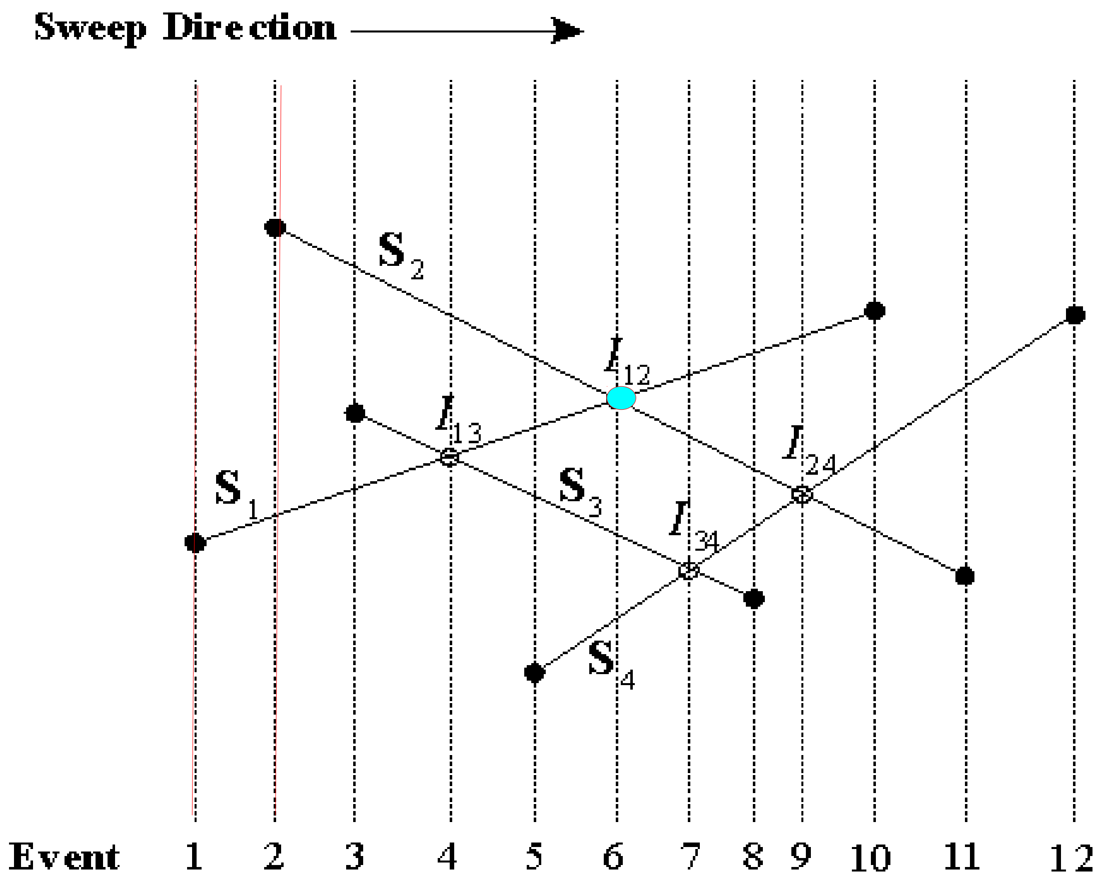
Computational Geometry

- Line intersection problem



Computational Geometry

- Line intersection problem

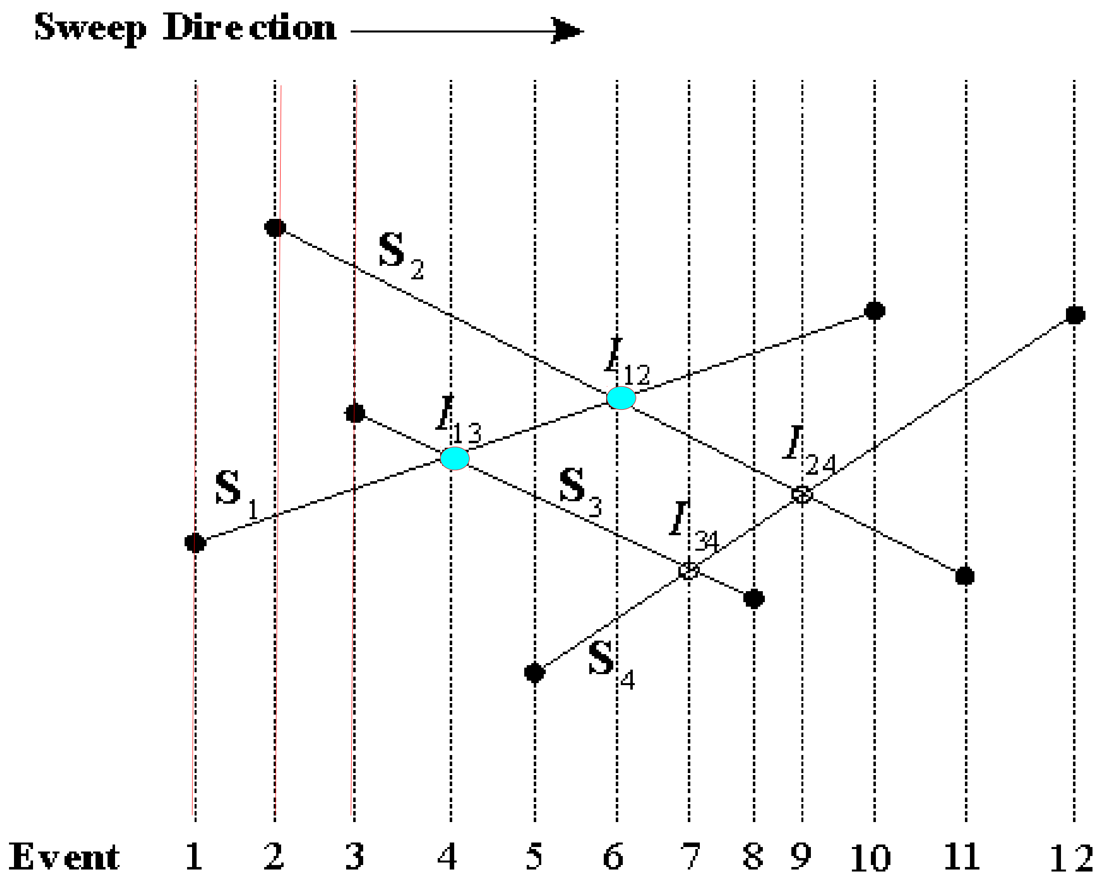


- Check intersection of S1 and S2

- Add intersection I12

Computational Geometry

- Line intersection problem

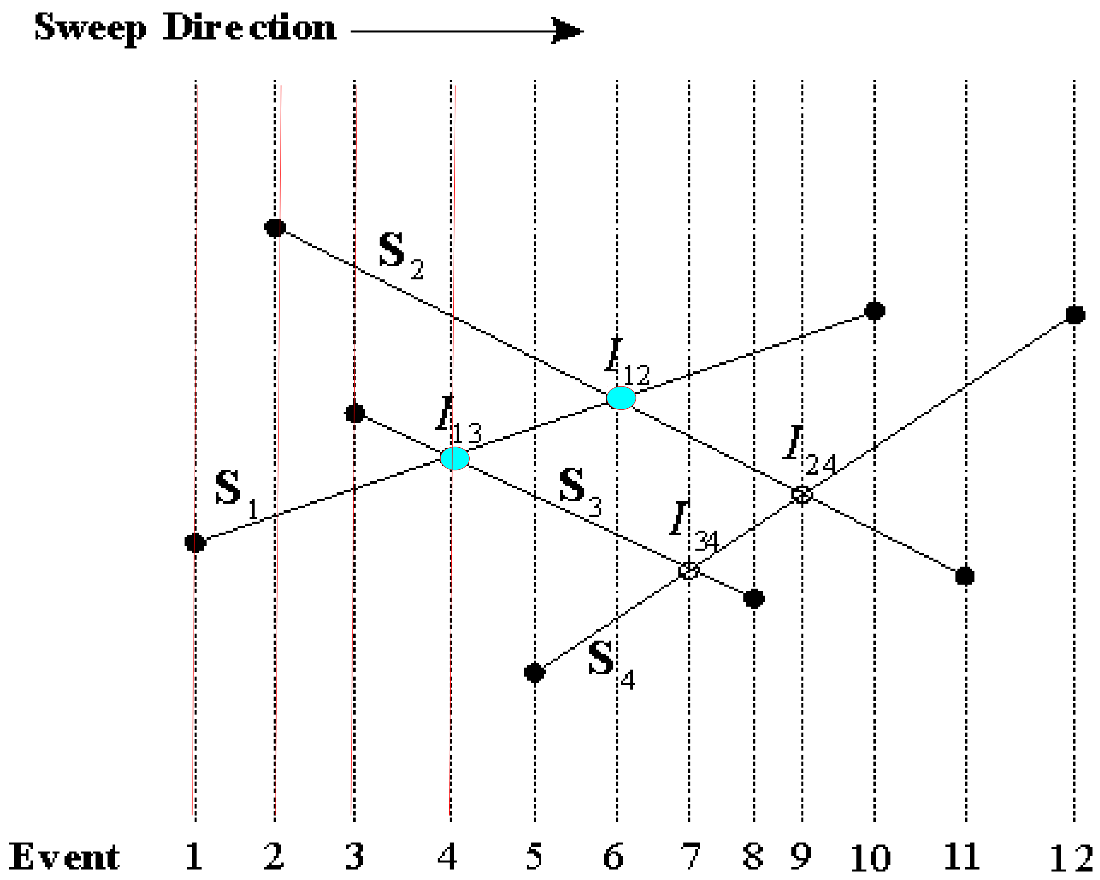


- Check intersection of S3 and S1 and of S3 and S2

- Add intersection I13

Computational Geometry

- Line intersection problem

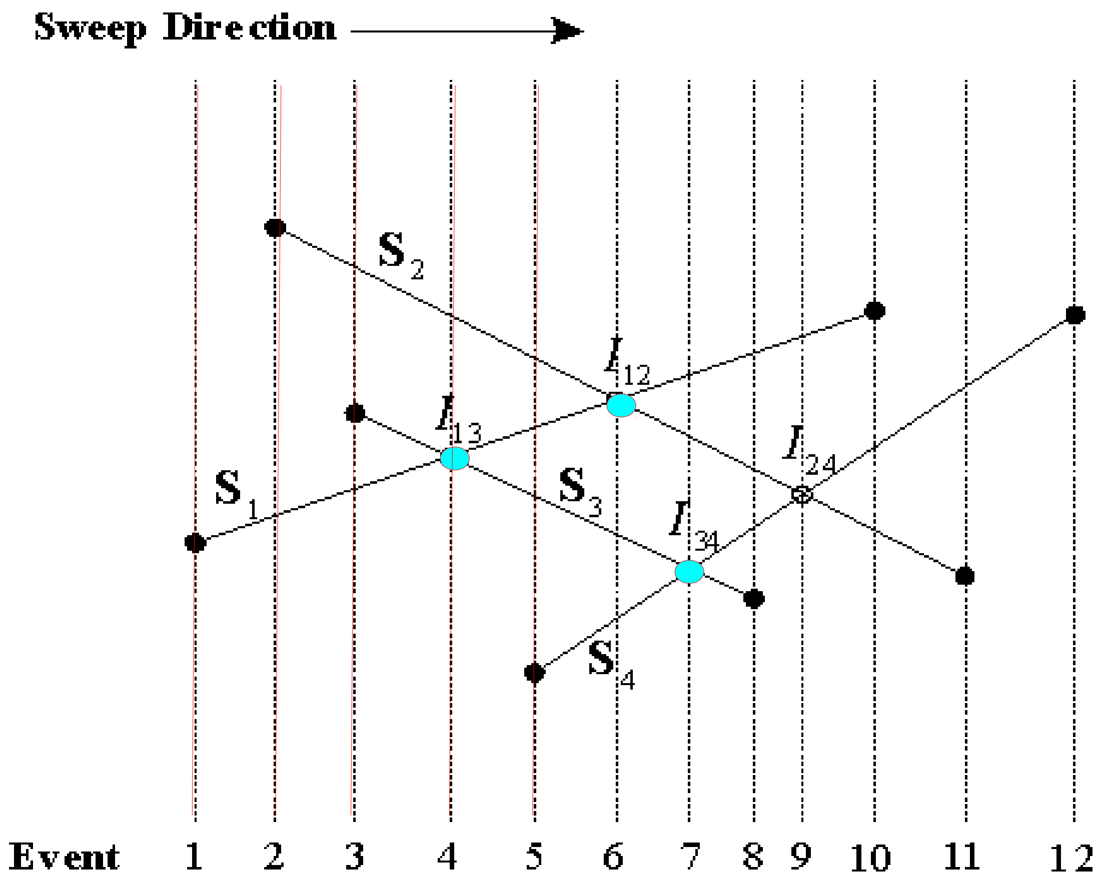


- Swap position of S1 and S3

- Check intersection of S1 and S2

Computational Geometry

- Line intersection problem

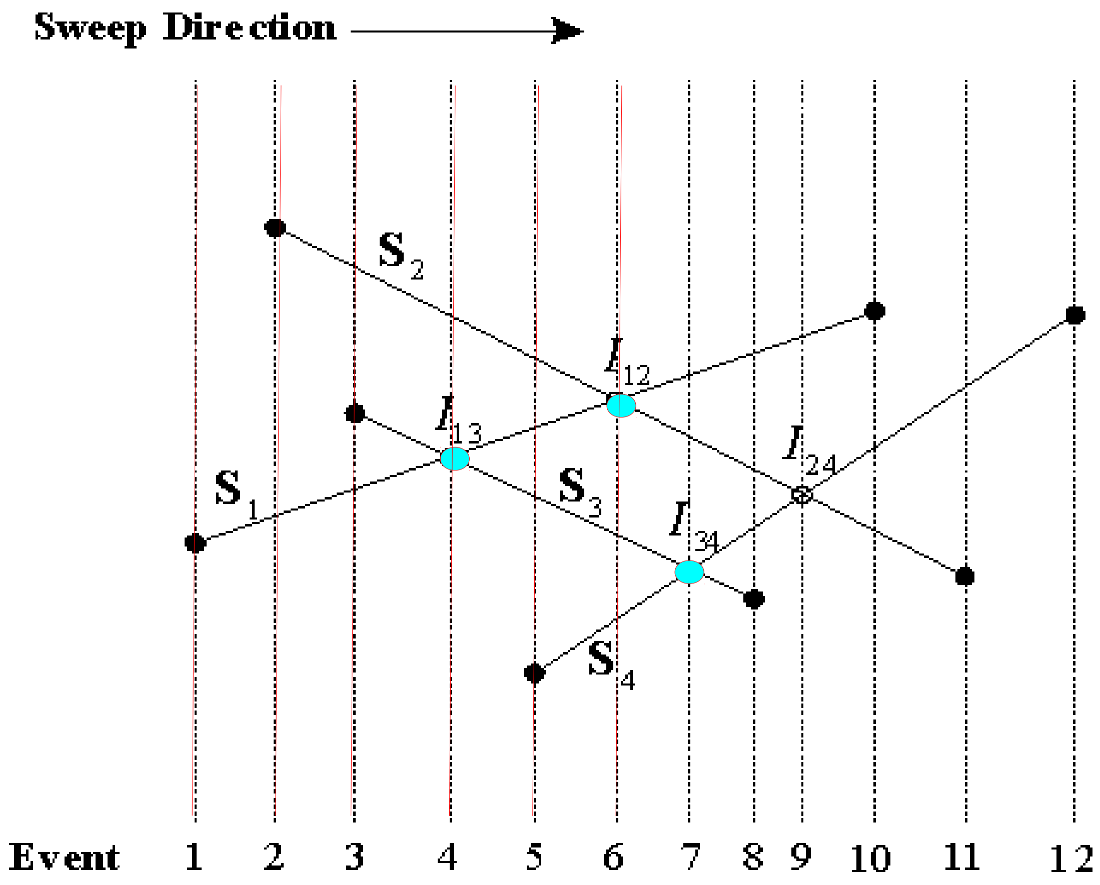


- Check intersection of S4 and S3

- Add intersection I34

Computational Geometry

- Line intersection problem

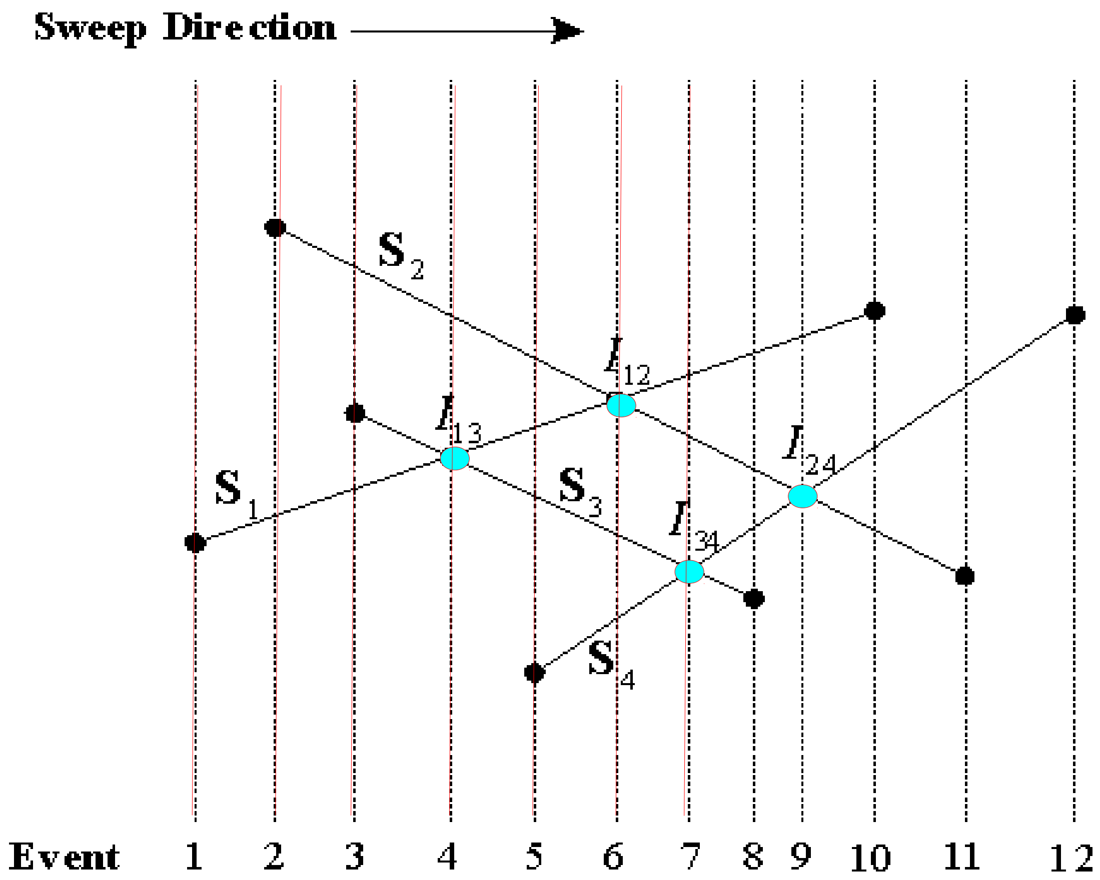


- Swap position of S1 and S2

- Check intersection of S2 and S3

Computational Geometry

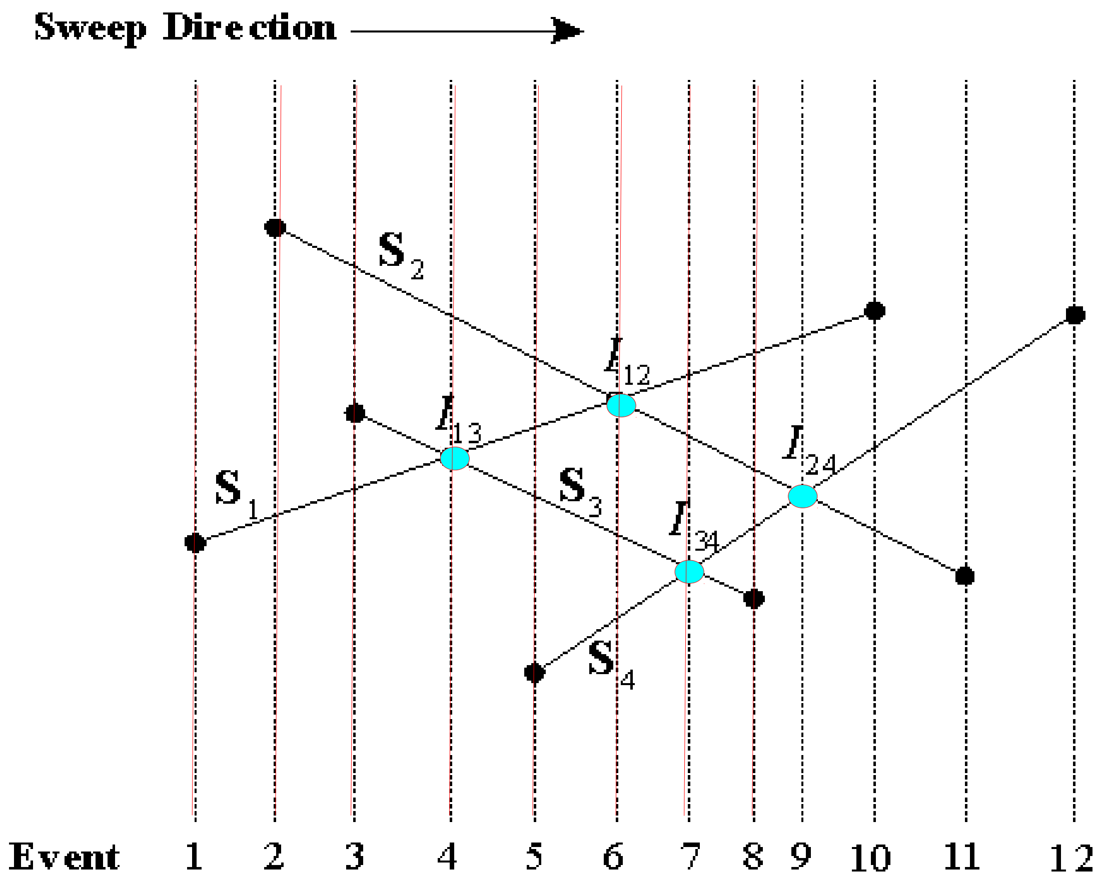
- Line intersection problem



- Swap position of S3 and S4
- Check intersection of S2 and S4
- Add intersection I24

Computational Geometry

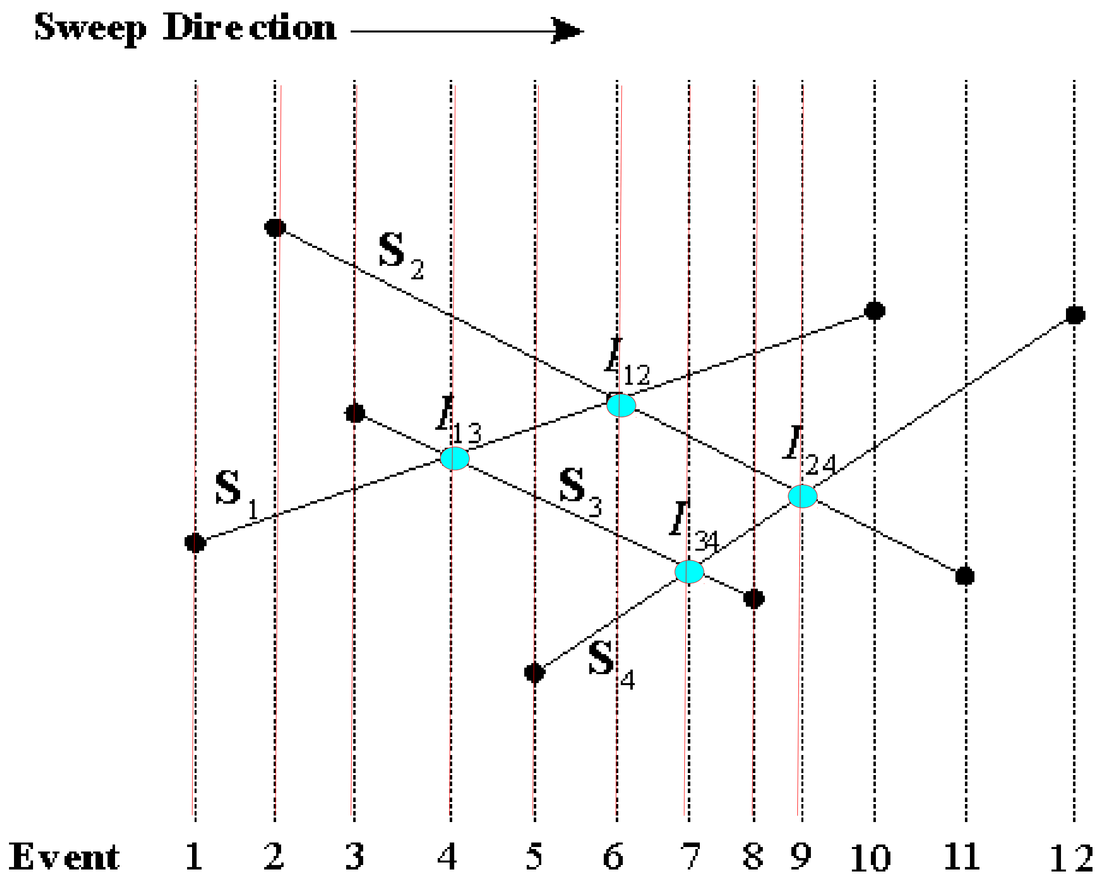
- Line intersection problem



- Remove S3

Computational Geometry

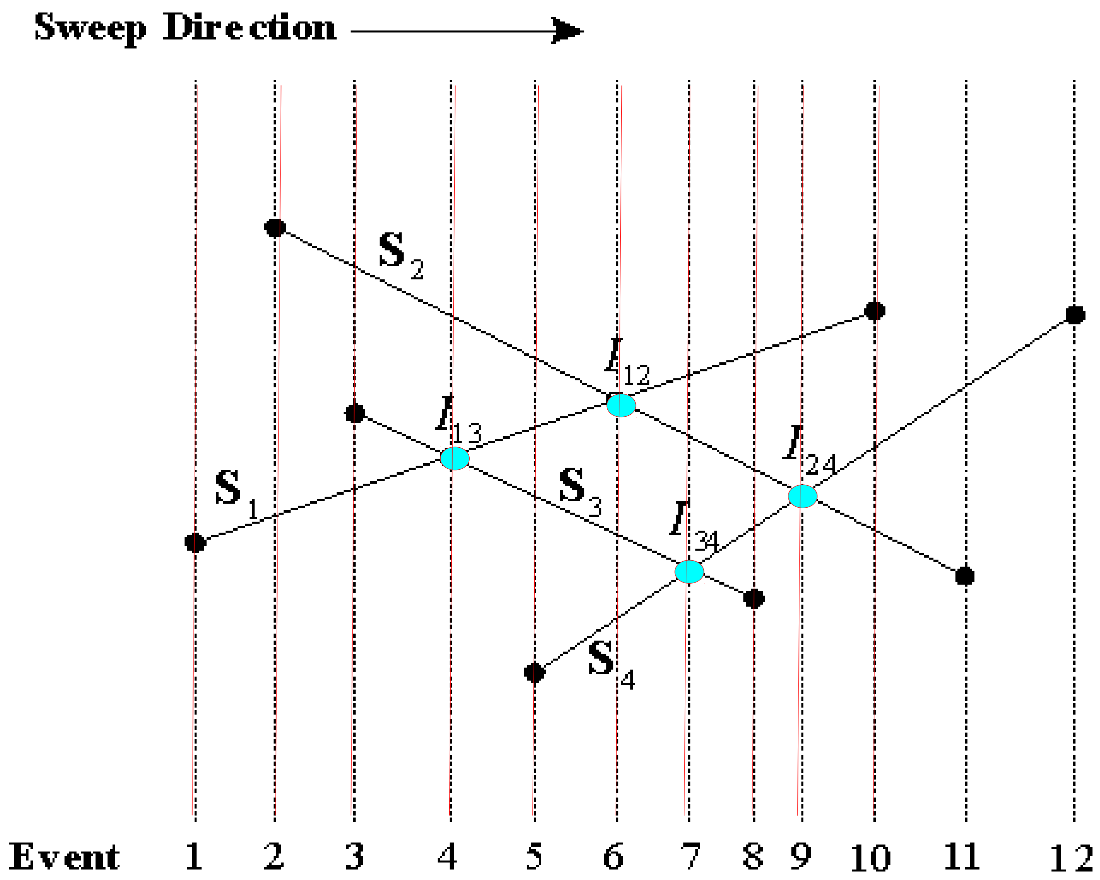
- Line intersection problem



- Swap position of S2 and S4
- Check intersection of S1 and S4

Computational Geometry

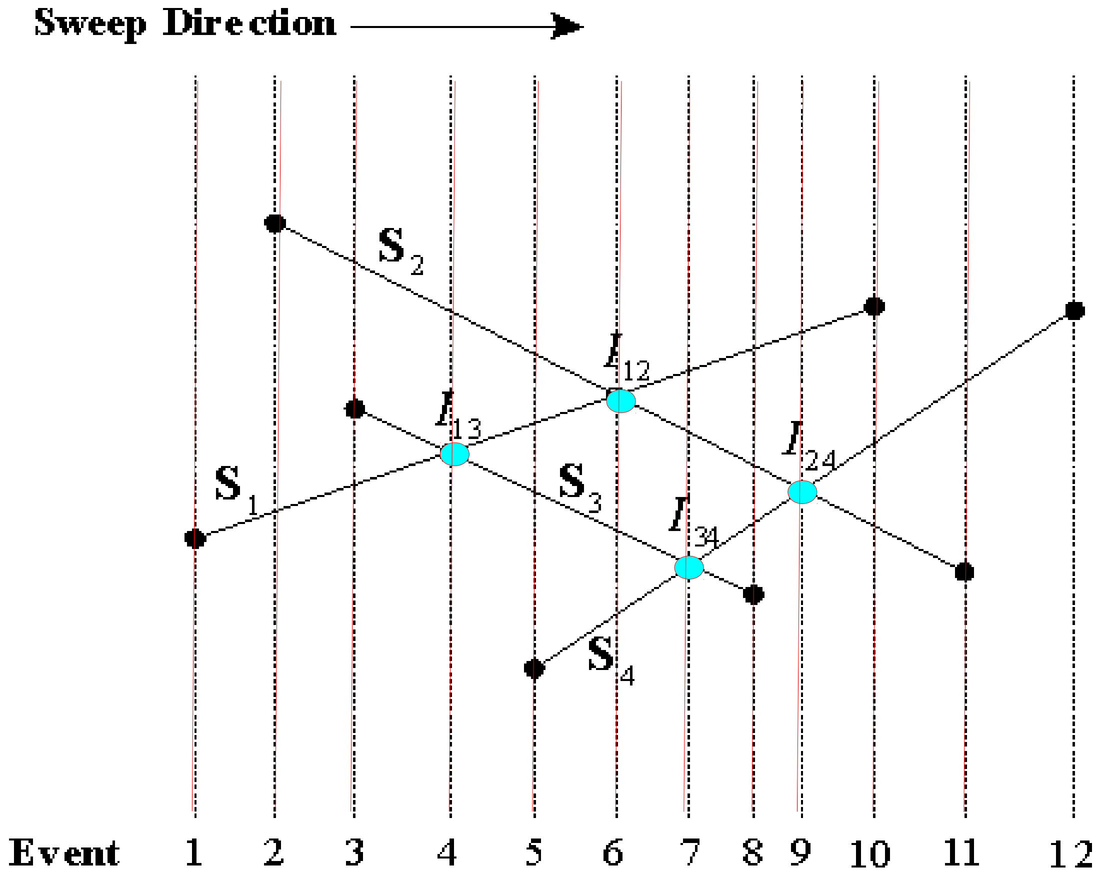
- Line intersection problem



-Remove S1

Computational Geometry

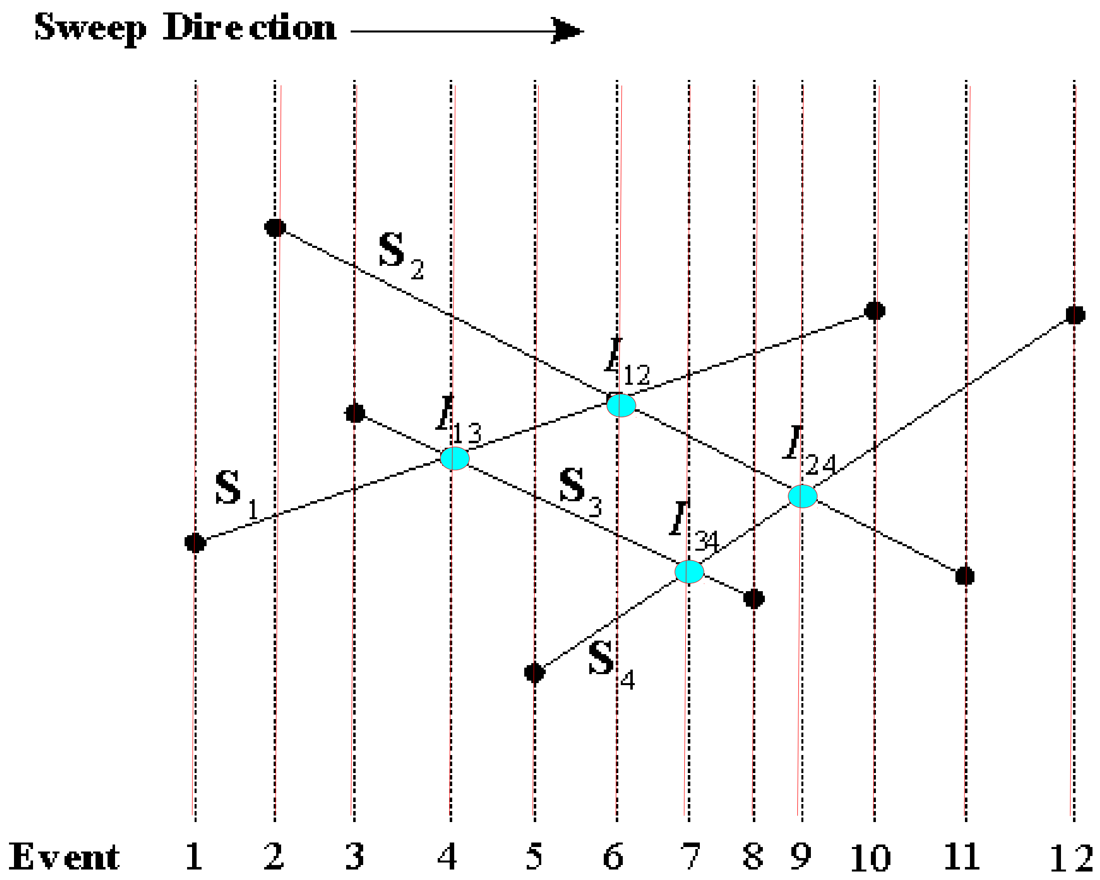
- Line intersection problem



-Remove S2

Computational Geometry

- Line intersection problem



-Remove S3