CSC236 Assignment 2

1.)

P(t): The root of rooted tree t has the largest label of all the nodes in t.

The goal is to prove for all $t \in T$, P(t).

Base Case: Let t be a single node.

Then the label on the root of t is clearly the largest label of all the nodes in t since t is a single node, and so P(t).

Induction Step: Let $t_1, t_2 \in T$. Suppose $P(t_1)$ and $P(t_2)$, i.e., the roots of t_1 and t_2 have the largest label of t_1 and t_2 respectively. [IH]

WTP: Let t be a rooted tree with a new root r connected to the roots of t_1 and t_2. The goal is to prove P(t).

By IH, the roots of t_1 and t_2 have the largest label of t_1 and t_2 respectively. By definition of T, if a node w with label a_w is a child of a node v with label a_v, $a_v > a_w$. Since the roots of t_1 and t_2 are both childs of the new root r, in order for t to be in T, the new root r must have a larger label than the roots of t_1 and t_2. Therefore, the root of t has the largest label of all the nodes in t and P(t).

2.)

$P((x, y))$: There exists $k \in N$, such that $(x, y) = (2^{k+1} + 1, 2^k + 1)$.

The goal is to prove for all $(x, y) \in M$, $P((x, y))$.

Base Case: Let $(x, y) = (3, 2)$.

Then $P((x, y))$ since $(2^{0+1} + 1, 2^0 + 1) = (2 + 1, 1 + 1) = (3, 2)$, so $k = 0$.

Induction Step: Let $(x, y) \in M$. By defintion, M includes $(3x - 2y, x)$.

Suppose $P((x, y))$, i.e., there exists some $k \in N$, such that $(x, y) = (2^{k+1} + 1, 2^k + 1)$. [IH]

WTP: $P((3x - 2y, x))$, i.e., there exists some $m \in N$, such that $(3x - 2y, x) = (2^{m+1} + 1, 2^m + 1)$.

$(x, y) = (2^{k+1} + 1, 2^k + 1)$ # By IH

So $x = 2^{k+1} + 1$ and $y = 2^k + 1$

$(3x - 2y, x) = (3(2^{k+1} + 1) - 2(2^k + 1), 2^{k+1} + 1)$ # $x = 2^{k+1} + 1$ and $y = 2^k + 1$

$(3x - 2y, x) = (3 \cdot 2^{k+1} + 3 - 2 \cdot 2^k - 2, 2^{k+1} + 1)$

$(3x - 2y, x) = (3 \cdot 2^{k+1} + 3 - 2^{k+1} - 2, 2^{k+1} + 1)$ # $2 \cdot 2^k = 2^{k+1}$

$(3x - 2y, x) = (2 \cdot 2^{k+1} + 1, 2^{k+1} + 1)$ # $3 - 2 = 1$ and $3 \cdot 2^{k+1} - 2^{k+1} = 2 \cdot 2^{k+1}$

$(3x - 2y, x) = (2^{k+2} + 1, 2^{k+1} + 1)$ # $2 \cdot 2^{k+1} = 2^{k+2}$

$(3x - 2y, x) = (2^{(k+1)+1} + 1, 2^{(k+1)} + 1)$

$(3x - 2y, x) = (2^{m+1} + 1, 2^m + 1)$ # $m = k + 1$

Therefore, there exists some $m \in N$, such that $(3x - 2y, x) = (2^{m+1} + 1, 2^m + 1)$ so $P((3x - 2y, x))$.

3.)

$P(f)$: There is a formula f' such that $f' \in H$ and $f \Leftrightarrow f'$.

The goal is to prove for all $f \in G$, $P(f)$.

Base Case: Let $f = x$.

Let $f' = f$. Then $c_{not}(f') = 0$ and $c_{and}(f') = 0$ so $c_{not}(f') = c_{and}(f')$ so $f' \in H$ and $f \Leftrightarrow f'$, therefore $P(f)$.

Induction Step: Let $f_1, f_2 \in G$. By definition, G includes $\neg f_1$ and $(f_1 \wedge f_2)$.

Suppose $P(f_1)$ and $P(f_2)$, i.e., there are formulas $f_1', f_2' \in H$ such that $f_1 \Leftrightarrow f_1'$ and $f_2 \Leftrightarrow f_2'$. [IH]

WTP:

(A) $P(\neg f_1)$, i.e., There is a formula f' such that $f' \in H$ and $\neg f_1 \Leftrightarrow f'$.

(B) $P((f_1 \wedge f_2))$, i.e., There is a formula f' such that $f' \in H$ and $(f_1 \wedge f_2) \Leftrightarrow f'$.

Case (A):

Let $f = \neg f_1$.

$f_1 \Leftrightarrow f_1'$ # By IH, $f_1' \in H$ such that $f_1 \Leftrightarrow f_1'$

$\neg f_1 \Leftrightarrow \neg f_1'$

$\neg f_1 \Leftrightarrow \neg(f_1' \wedge f_1')$ # Absorption, Negation laws, $\neg a \Leftrightarrow \neg(a \wedge a)$

Let $f' = \neg(f_1' \wedge f_1')$. Since $c_{not}(f') = c_{and}(f')$, $f' \in H$ and since $f \Leftrightarrow f'$, $P(\neg f_1)$.

Case (B):

Let $f = (f_1 \wedge f_2)$.

$f_1 \Leftrightarrow f_1'$ and $f_2 \Leftrightarrow f_2'$ # By IH, $f_1', f_2' \in H$ such that $f_1 \Leftrightarrow f_1'$ and $f_2 \Leftrightarrow f_2'$

$(f_1 \wedge f_2) \Leftrightarrow (f_1' \wedge f_2')$

$(f_1 \wedge f_2) \Leftrightarrow (f_1' \wedge f_2') \wedge (f_1' \vee f_2')$ # Associative, Absorption laws, $(f_1' \wedge f_2') \wedge (f_1' \vee f_2') \Leftrightarrow (f_1' \wedge f_2')$

$(f_1 \wedge f_2) \Leftrightarrow (f_1' \wedge f_2') \wedge \neg(\neg f_1' \wedge \neg f_2')$ # DeMorgan's laws, $(f_1' \vee f_2') \Leftrightarrow \neg(\neg f_1' \wedge \neg f_2')$

Let $f' = (f_1' \wedge f_2') \wedge \neg(\neg f_1' \wedge \neg f_2')$. Since $c_{not}(f') = c_{and}(f')$, $f' \in H$ and since $f \Leftrightarrow f'$, $P((f_1 \wedge f_2))$.