CSC236 Assignment 1

1.)

Let N denote the set of all natural numbers.

$f_1 = 1$, $f_2 = 1$. $\forall n \in N$, $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$.

P(n): $\forall n \in N$, $n \geq 1$, $\gcd(f_n, f_{n+1}) = 1$.

The base case for P(n) is when n = 1.

P(1): $f_1 = 1$, $f_{1+1} = f_2 = 1$. Therefore $\gcd(f_1, f_2) = \gcd(1, 1) = 1$. Therefore P(1) holds.

P(k): Suppose $\exists k \in N$, $k \geq 1$ such that P(k) holds which means $\gcd(f_{k+1}, f_k) = 1$.

To complete the proof, I must show that P(k + 1) holds.

P(k + 1): I need to show that $\gcd(f_{k+1}, f_{k+2}) = 1$.

$f_{k+1} = f_k + f_{k-1}$ and $f_{k+2} = f_{k+1} + f_k = f_k + f_{k-1} + f_k = 2f_k + f_{k-1}$.

Since for all $k \geq 1$, $f_{k+2} > f_{k+1}$, this means that $\gcd(f_{k+1}, f_{k+2}) = \gcd(f_{k+1}, f_{k+2} - f_{k+1})$.

$f_{k+2} - f_{k+1} = 2f_k + f_{k-1} - f_k + f_{k-1} = f_k$.

Therefore $\gcd(f_{k+1}, f_{k+2}) = \gcd(f_{k+1}, f_k) = 1$ and P(k + 1) holds and the proof is complete.

Therefore, $\forall n \in N$, $n \geq 1$, $\gcd(f_n, f_{n+1}) = 1$.

2.)

Let N denote the set of all natural numbers.

$a_1 = 1$, $a_2 = 1$. $\forall n \in N$, $n \geq 3$, $a_n = a_{n-1} + a_{n-2} + 1$.

P(n): $\forall n \in N$, $n \geq 1$, $a_n = 2f_n - 1$.

The base case for P(n) is when n = 1.

P(1): $a_1 = 2f_1 - 1 = 2(1) - 1 = 2 - 1 = 1$. Therefore P(1) holds.

Suppose $\forall j \in N$, $\exists k \in N$, $k \geq j \geq 1$, $P(j)$ holds which means $a_j = 2f_j - 1$.

To complete the proof , I must show that $P(k + 1)$ holds.

$P(k + 1)$: I need to show that $a_{k+1} = 2f_{k+1} - 1$.

$a_k = a_{k-1} + a_{k-2} + 1$

$a_{k+1} = a_k + a_{k-1} + 1$

$a_{k+1} = 2f_k - 1 + 2f_{k-1} - 1 + 1$

$a_{k+1} = 2(f_k + f_{k-1}) - 1$ * from problem 1, I know that $f_k + f_{k-1} = f_{k+1}$

$a_{k+1} = 2f_{k+1} - 1$

Therefore $a_{k+1} = 2f_{k+1} - 1$ and $P(k + 1)$ holds and the proof is complete.

Therefore, $\forall n \in N$, $n \geq 1$, $a_n = 2f_n - 1$.

3.)

Let N denote the set of all natural numbers.

Let P denote the set of all prime numbers.

$P(n)$: $\forall n \in N$, $n > 1$, $\exists p \in P$, $p \mid n$.

Assume $\neg P(n)$ which means $\exists n \in N$, $n > 1$, $\forall p \in P$, $p \nmid n$.

Let $S = \{k \in N \mid \neg P(k)\}$.

Then $S = \{k \in N \mid k > 1, \forall p \in P, p \nmid k\}$.

Then $S \neq \varnothing$ and $S \subset N$.

By the well ordering principle, S has a minimum element c and by the definition of S this means that c is not divisible by a prime and $\forall d \in N$, if $1 < d < c$ then d is divisible by a prime number.

There are two cases to consider, when c is a prime number or when c is not a prime number.

Case 1: c is a prime number

c is obviously divisible by itself which makes it divisible by a prime number which means P(c) holds and this means $c \notin S$ which is a contradiction.

Case 2: c is not a prime number

In this case c is a composite number. So $\exists a \in N$, $\exists b \in N$ such that $c = ab$ with $1 < a < c$ and $1 < b < c$. Since $1 < a < c$ this means $\exists p \in P$ such that $p \mid a$. Since $a \mid c$, and $p \mid a$, this implies that $p \mid c$ which means c is divisible by a prime number which means P(c) holds and this means $c \notin S$ which is a contradiction.

Since both cases have been proven this means the proof is complete and it is true that

$\forall n \in N, n > 1, \exists p \in P, p \mid n$.

4.)

Let N denote the set of all natual numbers.

Let R denote the set of all real numbers.

$h_0 = 1$, $h_1 = 2$, $h_2 = 3$. $\forall k \in N, k \geq 3,\ h_k = h_{k-1} + h_{k-2} + h_{k-3}$.

Suppose that $\forall s \in R, s^3 \geq s^2 + s + 1, s > 1.83$.

P(n): $\forall n \in N, n \geq 2, h_n \leq s^n$.

The base case for P(n) is when $n = 2$.

P(2): $h_2 = 3$, $s > 1.83$ so $s^2 > 3.3489$ and $3 < 3.3489$ which means $h_2 \leq s^2$ and therefore P(2) holds.

Suppose $\forall j \in N, \exists m \in N, m \geq j \geq 2$, P(j) holds which means $h_j \leq s^j$.

To complete the proof, I must show that P(m + 1) holds.

P(m + 1): I need to show that $h_{m+1} \leq s^{m+1}$.

$h_m = h_{m-1} + h_{m-2} + h_{m-3}$

$h_{m+1} = h_m + h_{m-1} + h_{m-2} \leq s^m + s^{m-1} + s^{m-2}$

$h_{m+1} = h_m + h_{m-1} + h_{m-2} \leq s^{m-2}(s^2 + s + 1)$ * $s^3 \geq s^2 + s + 1$

$h_{m+1} = h_m + h_{m-1} + h_{m-2} \leq s^{m-2}(s^3)$

$h_{m+1} = h_m + h_{m-1} + h_{m-2} \leq s^{m+1}$

$h_{m+1} \leq s^{m+1}$

Therefore P(m + 1) holds and the proof is complete which means $\forall n \in N, n \geq 2, h_n \leq s^n$.