Deep Learning & Applied AI

Recap of linear algebra

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Linear algebra is the study of linear maps on finite dimensional vector spaces

Linear algebra is about matrices as much as
astronomy is about telescopes

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- **distributive properties**: $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$

Example: Lists of numbers

$\mathbb{R}^n$ is defined to be the set of all n-long sequences of numbers in $\mathbb{R}$:

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, 2, \ldots, n\}$$

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Addition and multiplication are defined as expected:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

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While the additive identity can be defined as:

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With these definitions, $\mathbb{R}^n$ is a vector space

Example: Functions

Consider the set of all functions $f : [0,1] \to \mathbb{R}$ with the standard definitions for sum and scalar product:

$$(f+g)(x) = f(x) + g(x)$$
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The above forms a vector space. In fact, any set of functions $f : S \to \mathbb{R}$ with $S \neq \emptyset$ (Q: why?) and the definitions above forms a vector space.

Elements of a vector space (called vectors)
are not necessarily lists

A vector space is an abstract entity whose elements
might be lists, functions, or weird objects

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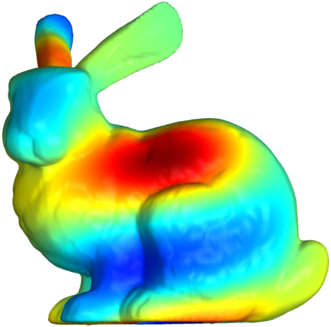

We can still use linear algebra to manipulate functions on surfaces

Subspaces

A subset $U \subset V$ is a subspace of V if it is a vector space (using the same operations defined for V)

In particular:

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- $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$
- The set of piecewise-linear functions on a graph $G = (V, E)$ is a subspace of all functions $f : V \to \mathbb{R}$

Bases

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- $v_1, \ldots, v_n \in V$ are linearly independent if and only if each $v \in \text{span}(v_1, \ldots, v_n)$ has only one representation as a linear combination of $v_1, \ldots, v_n$

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So every vector $v \in V$ can be expressed uniquely as a linear combination

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

You can think of a basis as the minimal set of vectors that generates the entire space

Example: Bases

- $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbb{R}^n$ called the standard basis; its vectors are called the indicator vectors.

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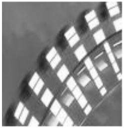
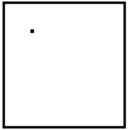
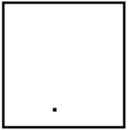
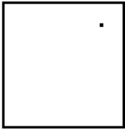
- $(1,2),(3,5.07)$ is a basis of $\mathbb{R}^2$

- $$f_1(x) = \begin{cases} 1 & \text{if } x = x_1 \\ 0 & \text{else} \end{cases}$$
  $$f_2(x) = \begin{cases} 1 & \text{if } x = x_2 \\ 0 & \text{else} \end{cases}$$
  $$\vdots$$

  is the standard basis for the set of functions $f : \mathbb{R} \to \mathbb{R}$; the basis vectors are also called indicator functions

Examples

An image expressed in the standard basis:

 $= \alpha_1$  $+\alpha_2$  $+\alpha_3$  $+\cdots$

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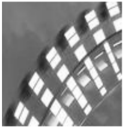
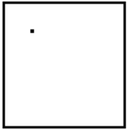
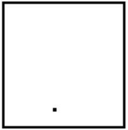
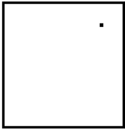
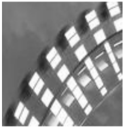
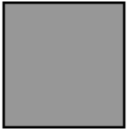
 $= \alpha_1$  $+\alpha_2$  $+\alpha_3$  $+\cdots$

The same image, expressed in terms of a nonlinear map σ:

 $= \sigma($ , ,  $)$

The image is **not** in the span of the three features.

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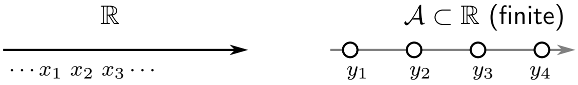

$f : \mathbb{R} \to \mathbb{R}$
infinite dimensional
(functional analysis)

$f : \mathcal{A} \to \mathbb{R}$
finite dimensional
(linear algebra)

Linear maps

A linear map from V to W is a function $T : V \to W$ with the properties:

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- from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined as

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

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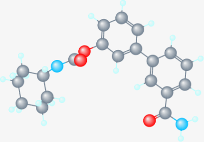
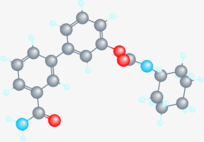
Reflection operation on an image:

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \,, \quad T(x, y) = (-x, y)$$



$\Rightarrow$



Linear maps as a vector space

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If $T : U \to V$ and $S : V \to W$, their product $ST : U \to W$ is defined by

$$(ST)(u) = S(Tu)$$

In other words, ST is just the usual composition $S \circ T$ of two functions

Algebraic properties of products of linear maps

- **associativity**: $(T_1T_2)T_3 = T_1(T_2T_3)$
- **identity**: $TI = IT = T$
- **distributive properties**: $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$

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Keep in mind that composition of linear maps is not commutative, i.e.

$$ST \neq TS$$

in general (although there are special cases)

Example: Take $Sf = f'$ and $(Tf)(x) = x^2 f(x)$

Matrices

Consider a linear map $T : V \to W$, a basis $v_1, \ldots, v_n \in V$ and a basis $w_1, \ldots, w_m \in W$.

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The matrix of T in these bases is the $m \times n$ array of values in $\mathbb{R}$

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Hence each column of $\mathbf{T}$ contains the linear combination coefficients for the image via T of a basis vector from V

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In other words, the matrix encodes how basis vectors are mapped, and this is enough to map all other vectors in their span, since:

$$Tv = T(\sum_j \alpha_j v_j) = \sum_j T(\alpha_j v_j) = \sum_j \alpha_j Tv_j$$

The matrix is a representation for a linear map, and it depends on the choice of bases

Matrix of a vector

Suppose $v \in V$ is an arbitrary vector, while $v_1, \ldots, v_n$ is a basis of V. The matrix of v wrt this basis is the $n \times 1$ matrix:

$$\mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

so that

$$v = c_1 v_1 + \cdots c_n v_n$$

Once again, we see that the matrix depends on the choice of basis for V

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  Q4: do we need the same bases for $S : U \to V$ and $T : V \to W$?

Product of “map matrix” and “vector matrix”

Consider a linear map $T : V \to W$, a basis $v_1, \ldots, v_n \in V$ and a basis $w_1, \ldots, w_m \in W$.

From the definition of matrix product, one can show that it operates on a vector matrix as expected:

$$\mathbf{Tv} = \mathbf{w} \quad \Leftrightarrow \quad Tv = w$$

where $\mathbf{Tv}$ is the matrix product of $\mathbf{T}$ and $\mathbf{v}$, while Tv simply denotes the function evaluation $T(v)$

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Remember: $\mathbf{T}, \mathbf{v}, \mathbf{w}$ must follow a coherent choice of bases in order for the above to make sense. $\mathbf{v}$ can not be expressed in basis $(\tilde{v}_1, \ldots, \tilde{v}_n)$ if $\mathbf{T}$ only knows how to map basis vectors $(v_1, \ldots, v_n)$.

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Because recall that, for bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$:

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We see then that vector $c = \sum_j c_j v_j$ is mapped to $Tc = \sum_j c_j T v_j$.

In other words, matrix product is behaving as expected.

Suggested reading

Sections 1.A – 3.D of the textbook:

S. Axler, "Linear algebra done right – 3rd edition". Springer, 2015