

Big Data Computing

Master's Degree in Computer Science

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Recap from Last Lectures

- We described linear regression as a powerful technique to predict real-valued function
- Linear regression tries to fit a straight hyperplane between features (i.e., independent variables) and the target (i.e., dependent variable)
- OLS method to easily estimate the parameters of the model
- More advanced techniques may be applied if the relationship between features and the target is not linear (e.g., polynomial regression)

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 - click vs. non-click on a web page or an advertisement

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- **Classification** (as opposed to regression) deals with predicting categorical responses
- Examples:
 - spam vs. non-spam emails
 - click vs. non-click on a web page or an advertisement
- Classification methods may first predict the probability of each category of a qualitative response to make in turn a decision

Why Not Linear Regression, Then?

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- Suppose we want to predict the health condition of a patient arriving in the ER on the basis of her symptoms
- Imagine there are only the following 3 possible diagnoses: **stroke**, **drug overdose**, and **epileptic seizure**
- We may encode the above values as a categorical response variable Y

$$Y = \begin{cases} 1 & \text{if } \text{stroke}; \\ 2 & \text{if } \text{drug overdose}; \\ 3 & \text{if } \text{epileptic seizure}. \end{cases}$$

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- Different (and still legitimate) encodings will produce different models

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 - Predict 1 if the outcome is > 0.5 , 0 otherwise

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- For a binary response with a 0/1 encoding, linear regression by OLS does anyway make sense
 - Predict 1 if the outcome is > 0.5 , 0 otherwise
- Still, it is preferable to use a classification method which works by design

LOGISTIC REGRESSION

Example: Default(Y) vs. Balance(X)

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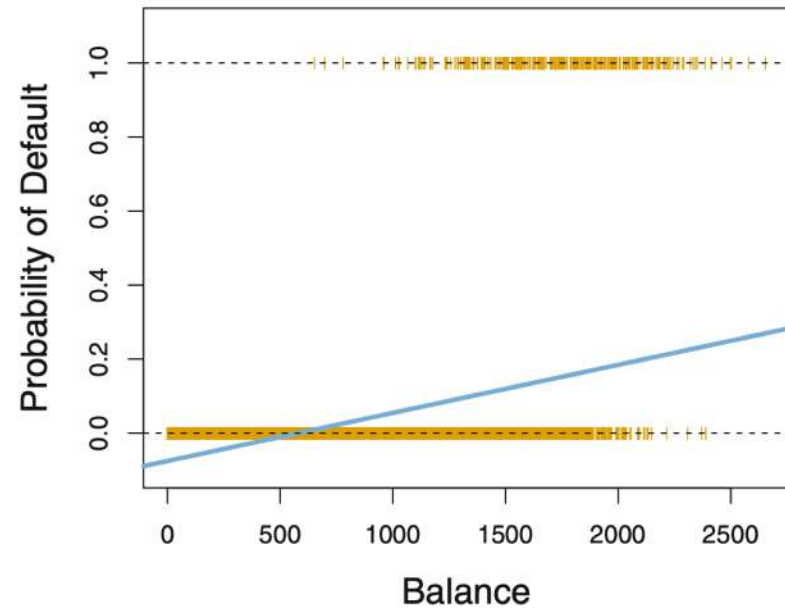
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Suppose we want to predict the value of Y from the value of Balance(X)

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Logistic Regression instead models the **probability** that Y belongs to one of the two possible outcome values

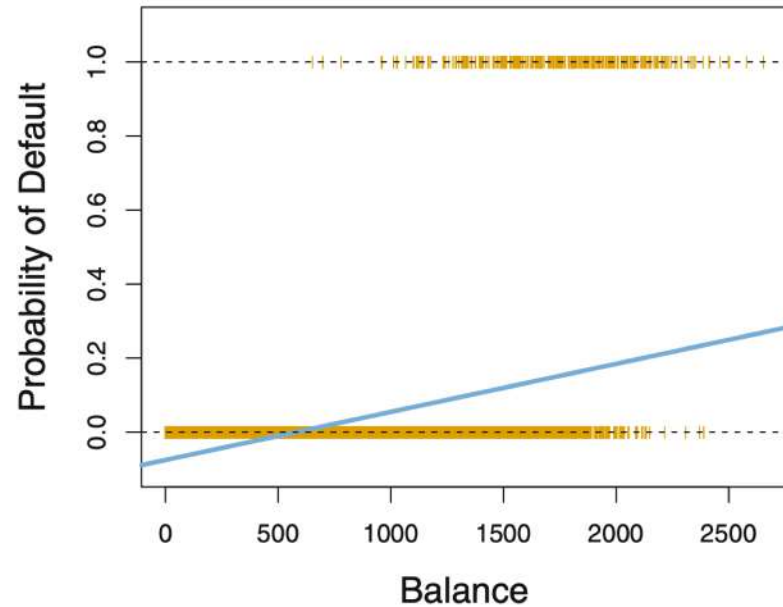
Example: Default(Y) vs. Balance(X)



Predicted probability using **linear regression**
(some estimated probabilities are negative!)

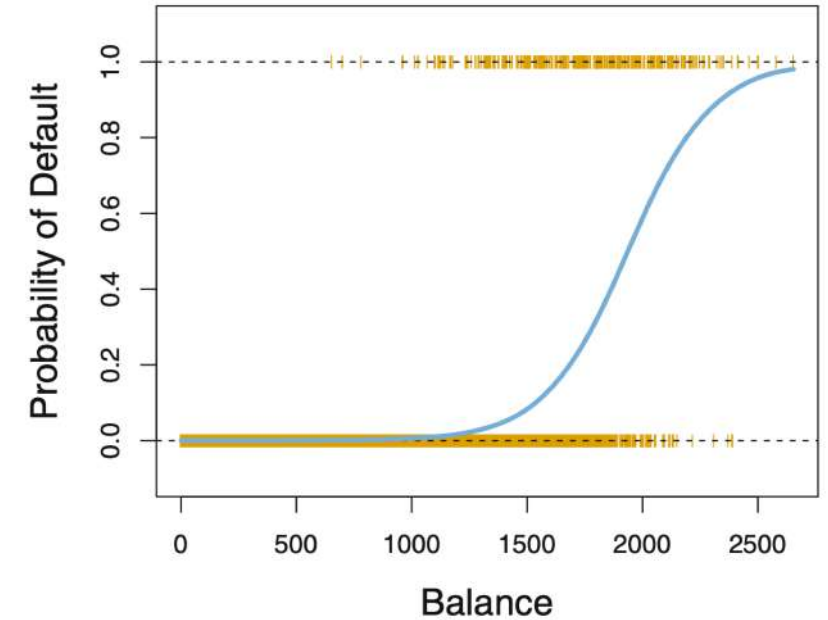
Linear Regression

Example: Default(Y) vs. Balance(X)



Predicted probability using **linear regression**
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Linear Regression



Predicted probability using **logistic regression**
(all probabilities lie between 0 and 1)

Logistic Regression

General Concepts

3 components need to be specified

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Model

Defines the space of
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Error Measure (Cost Function)

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```
graph TD; A[3 components need to be specified] -- blue arrow --> B[Model]; A -- green arrow --> C[Error Measure (Cost Function)]; A -- orange arrow --> D[Learning Algorithm];
```

Model

Defines the space of representable hypotheses

Error Measure (Cost Function)

Measures the price of misclassification errors

Learning Algorithm

Picks the *best* hypothesis exploring search space

MODEL

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$$\mathcal{F} = \{f_{\boldsymbol{\theta}} : \mathbb{R}^{d+1} \mapsto \mathbb{R} \mid f_{\boldsymbol{\theta}}(\mathbf{x}) = \boldsymbol{\theta}^T \mathbf{x} = \sum_{i=0}^d \theta_i x_i\}$$

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- Each function in F outputs a real number (i.e., a scalar) as a linear combination of the input \mathbf{x} with the parameters $\boldsymbol{\theta}$
- $f_{\boldsymbol{\theta}}(\mathbf{x})$ is referred to as **(linear) signal**

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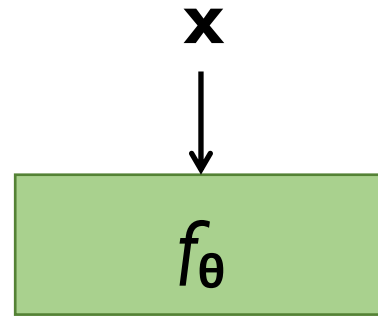
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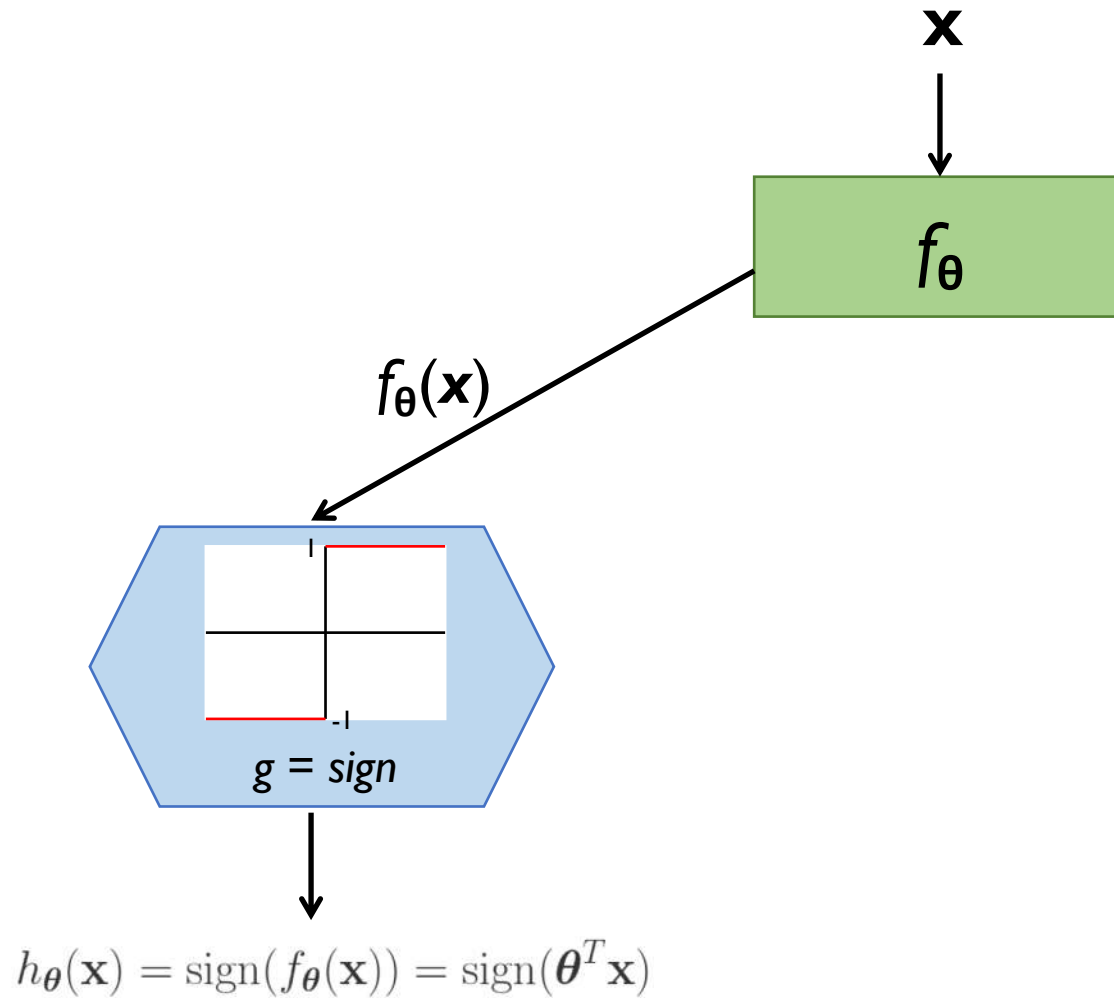
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The set of possible hypotheses H changes depending on the parametric model (f_{θ}) and on the **thresholding function** (g)

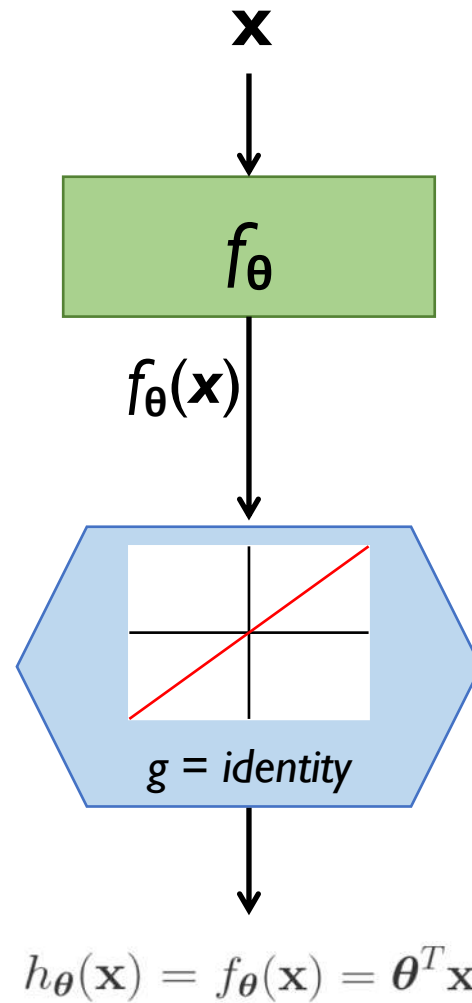
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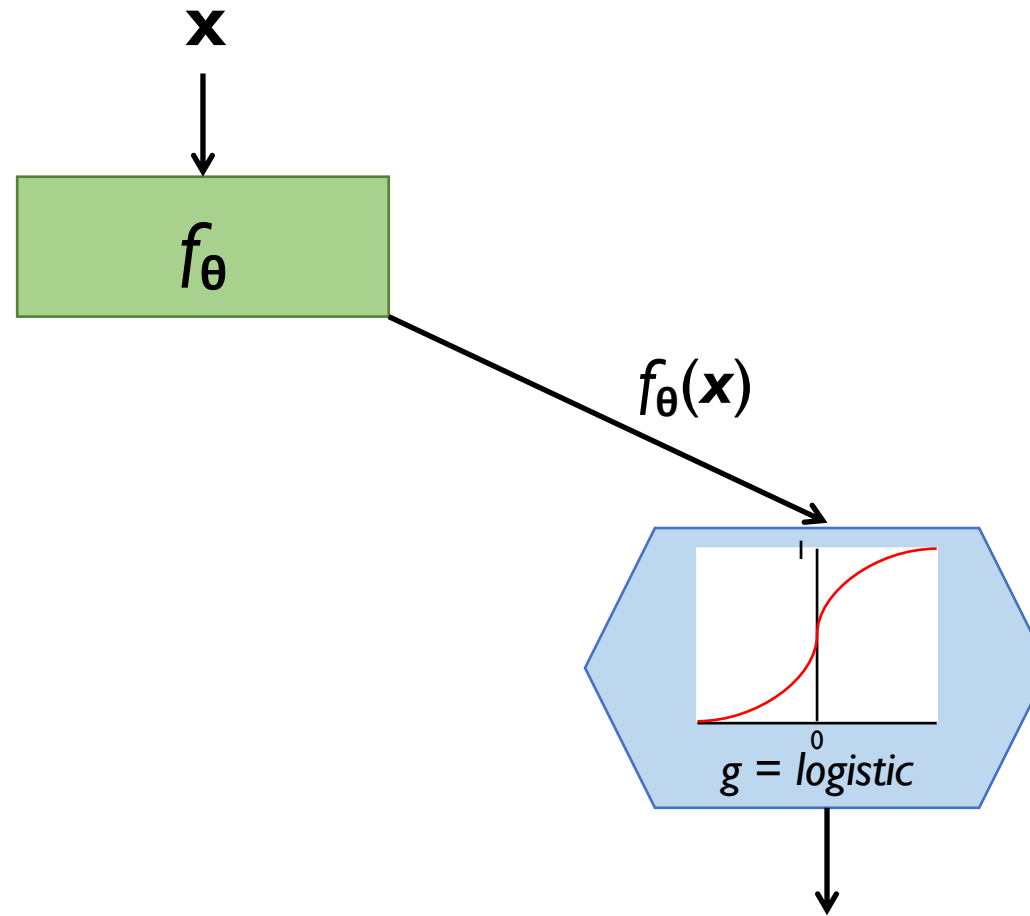
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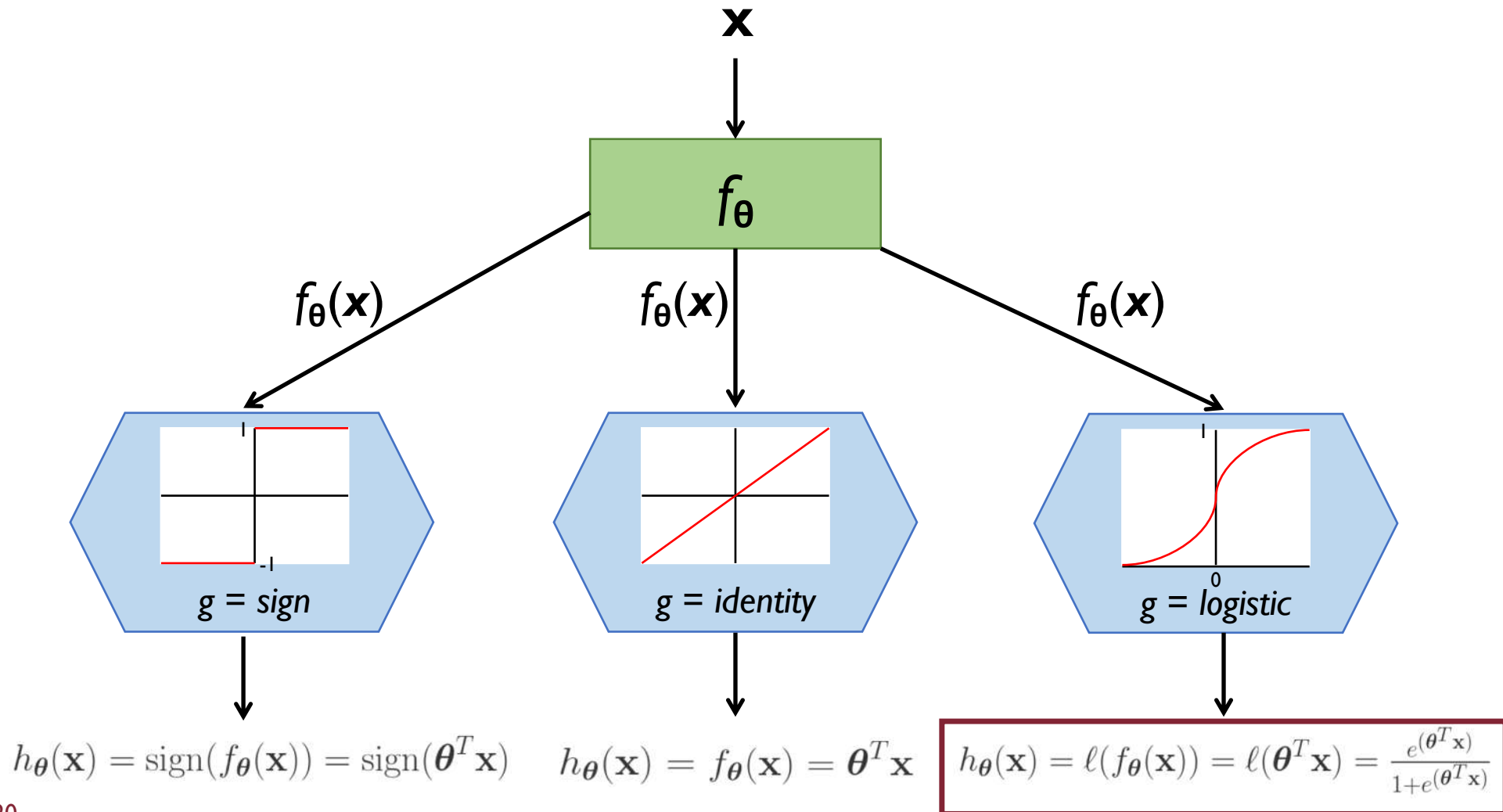


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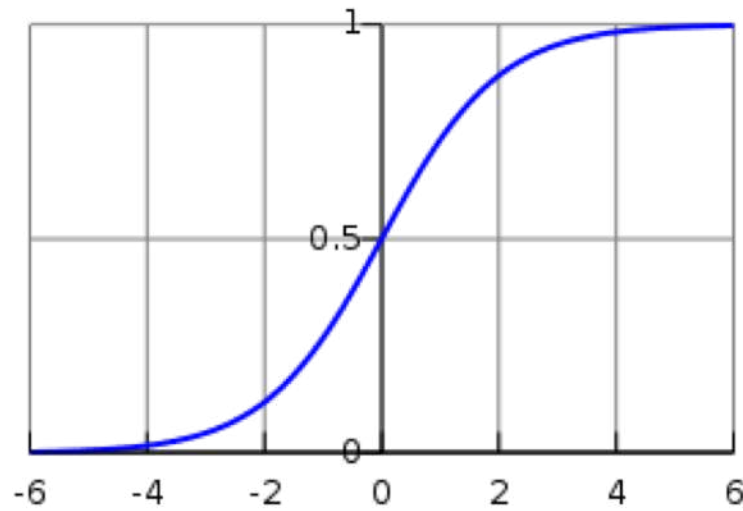


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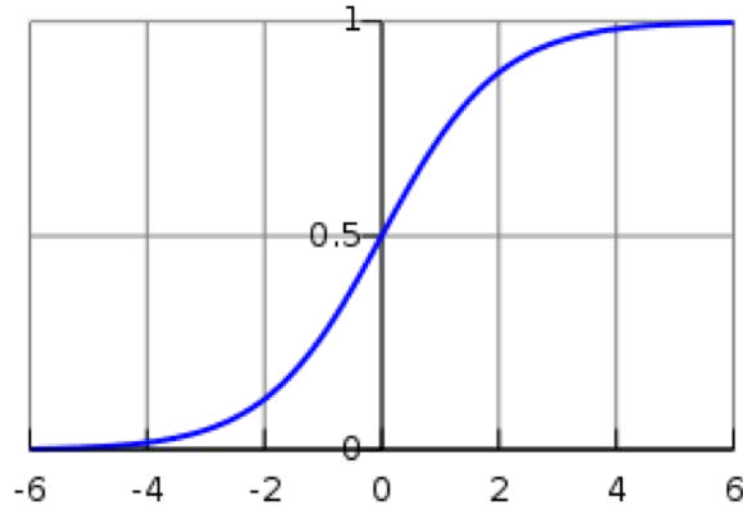


The Logistic Function



$$l(z) = \frac{e^z}{1+e^z}$$

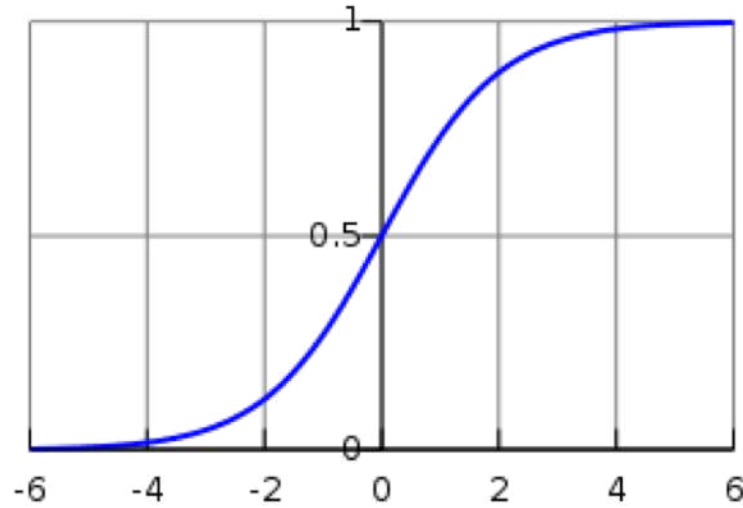
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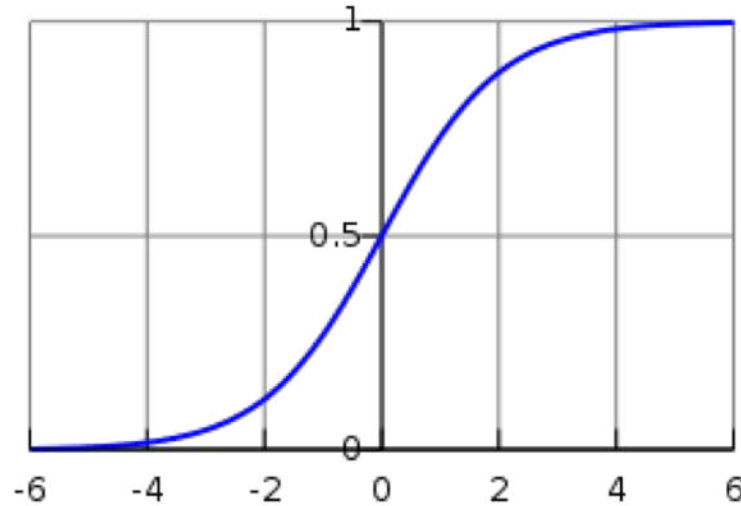
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- Output can be *genuinely* interpreted as a probability value

Probabilistic Interpretation

$$h_{\boldsymbol{\theta}}(\mathbf{x}) = \ell(f_{\boldsymbol{\theta}}(\mathbf{x})) = \ell(\boldsymbol{\theta}^T \mathbf{x}) = \frac{e^{(\boldsymbol{\theta}^T \mathbf{x})}}{1 + e^{(\boldsymbol{\theta}^T \mathbf{x})}}$$

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- Other functions may have the same property [e.g., $1/\pi \arctan(x) + 1/2$]

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- The key points here are:
 - the output of the logistic function can be interpreted as a probability even during learning
 - the logistic function is mathematically convenient!

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- $\text{odds}(\text{failure}) = q/p = 1/p/q = 1/\text{odds}(\text{success})$
- $\text{logit}(p) = \ln(\text{odds}(\text{success})) = \ln(p/q) = \ln(p/1-p) = \ln(p) - \ln(1-p)$

Odds

Logistic Regression is in fact an ordinary linear regression where the logit is the response variable!

$$\text{logit}(p) = \ln\left(\frac{p}{1-p}\right) = \theta_0 + \theta_1 x_1 + \dots + \theta_d x_d = \boldsymbol{\theta}^T \mathbf{x}$$

The coefficients of logistic regression are expressed in terms of the natural logarithm of odds

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Probabilities are only defined on the range $[0, 1]$

It would need very complicated constraints on the regression coefficients to work with probability

From Odds to Probability

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$$e^{\text{logit}(p)} = e^{\ln\left(\frac{p}{1-p}\right)} = \frac{p}{1-p} = e^{(\boldsymbol{\theta}^T \mathbf{x})}$$

$$p = e^{(\boldsymbol{\theta}^T \mathbf{x})}(1 - p) = e^{(\boldsymbol{\theta}^T \mathbf{x})} - e^{(\boldsymbol{\theta}^T \mathbf{x})}p$$

$$p + e^{(\boldsymbol{\theta}^T \mathbf{x})}p = e^{(\boldsymbol{\theta}^T \mathbf{x})}$$

$$p(1 + e^{(\boldsymbol{\theta}^T \mathbf{x})}) = e^{(\boldsymbol{\theta}^T \mathbf{x})}$$

$$p = \frac{e^{(\boldsymbol{\theta}^T \mathbf{x})}}{1 + e^{(\boldsymbol{\theta}^T \mathbf{x})}} = \frac{1}{e^{-(\boldsymbol{\theta}^T \mathbf{x})} + 1}$$

Odds Ratio

Using (log) odds rather than actual probabilities provides an easier interpretation of the model's coefficients learned

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Suppose we want to measure the effect of a unit increase in one of the predictors to the output response

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Let's measure the ratio between the odds computed at a certain input \mathbf{x} and the odds computed at a different point \mathbf{x}'

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$$\mathbf{X} = x_1 + \dots + x_i + \dots + x_d$$

$$\mathbf{x}' = x_1 + \dots + (x_i + 1) + \dots + x_d$$

\mathbf{x}' is just the same as \mathbf{x} where the i -th predictor/feature is increased by 1 unit

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$$\frac{e^{\theta^T \mathbf{x}'}}{e^{\theta^T \mathbf{x}}}$$

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The ratio of the odds for 1-unit increase in x_i

or

θ_i is the ratio of the natural log(odds) for 1-unit increase in x_i

Why Odds Ratio?

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Example

An odds ratio of 1.08 will give an 8% increase in the odds at *any* value of x_i

Probabilistically-Generated Data

As with any other supervised learning problem we are given a finite set D of m i.i.d. labelled examples which we can try to learn from

$$\mathcal{D} = \{(\mathbf{x}_1, y_1)\}, \dots, (\mathbf{x}_m, y_m)\}$$

where each y_i is a binary variable taking on two values (e.g., $\{-1, +1\}$)

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The data we observe from D is actually generated by an *underlying and unknown probability function* (**noisy target**) which we want to estimate

$$P(y|\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) & \text{if } y = +1 \\ 1 - \phi(\mathbf{x}) & \text{if } y = -1 \end{cases}$$

Deterministic vs. Noisy Target

- Deterministic function: given \mathbf{x} as input it always outputs either $y = +1$ or $y = -1$ (mutually exclusive)

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Goal

$\phi: \mathbb{R}^{d+1} \rightarrow [0,1]$ is the unknown noisy target which generates our examples, our aim is to find an estimate ϕ^* which best approximates ϕ

Estimating Noisy Target

$$P(y|\mathbf{x}) = \begin{cases} \phi^*(\mathbf{x}) & \text{if } y = +1 \\ 1 - \phi^*(\mathbf{x}) & \text{if } y = -1 \end{cases}$$

Estimating Noisy Target

$$P(y|\mathbf{x}) = \begin{cases} \phi^*(\mathbf{x}) & \text{if } y = +1 \\ 1 - \phi^*(\mathbf{x}) & \text{if } y = -1 \end{cases}$$

We claim that the best estimate ϕ^* of ϕ is $h_{\theta}^*(\mathbf{x})$ which in turn is picked from the set of hypotheses defined by logistic function

$$\phi^*(\mathbf{x}) = h_{\theta}^*(\mathbf{x}) = \ell(\boldsymbol{\theta}^T \mathbf{x}) \approx \phi(\mathbf{x})$$

Hypothesized Noisy Target

- How do we estimate $h^*_{\theta}(\mathbf{x})$?

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- How do we estimate $h^*_{\theta}(\mathbf{x})$?
- We will use the same general framework introduced for the supervised learning problem!
- We already fixed the set of hypothesis function to select from
- We still need:
 - A training set D
 - An error measure (cost function) to minimize

COST FUNCTION

Finding The Best Hypothesis

If the hypothesis space H is made of a family of parametric models,
 $h_{\theta}^*(\mathbf{x})$ can be picked as:

$$h_{\theta}^* = \operatorname{argmax}_{h_{\theta} \in \mathcal{H}} P(h_{\theta} \mid \mathcal{D})$$

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That is, we want to maximize the probability of the chosen hypothesis
given the data D we observed

Flipping the Coin: The Likelihood Function

We measure the error we are making by assuming that $h^*_{\theta}(\mathbf{x})$ approximates the true noisy target ϕ

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Flipping the Coin: The Likelihood Function

We measure the error we are making by assuming that $h^*_{\theta}(\mathbf{x})$ approximates the true noisy target ϕ

How likely is that the observed data D have been generated by our selected hypothesis $h^*_{\theta}(\mathbf{x})$?

Find the hypothesis which maximizes the probability of the observed data D given a particular hypothesis

$$h^*_{\theta} = \operatorname{argmax}_{h_{\theta} \in \mathcal{H}} P(\mathcal{D} | h_{\theta})$$

The Likelihood Function

Given the generic training example (\mathbf{x}, y) and assuming it has been generated by a hypothesis $h_{\theta}(\mathbf{x})$ the likelihood function is:

$$P(y|\mathbf{x}) = \begin{cases} h_{\theta}(\mathbf{x}) & \text{if } y = +1 \\ 1 - h_{\theta}(\mathbf{x}) & \text{if } y = -1 \end{cases}$$

where ϕ has been replaced with our hypothesis

The Likelihood Function

If we assume the hypothesis is the logistic function

$$h_{\boldsymbol{\theta}}(\mathbf{x}) = \ell(\boldsymbol{\theta}^T \mathbf{x})$$

The Likelihood Function

If we assume the hypothesis is the logistic function

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And by noticing that logistic function is symmetric, i.e. $\ell(-z) = 1 - \ell(z)$, the likelihood for a single example is:

$$P(y \mid \mathbf{x}) = \ell(y\boldsymbol{\theta}^T \mathbf{x})$$

The Likelihood Function

Having access to a full set of m i.i.d. training examples D

$$\mathcal{D} = \{(\mathbf{x}_1, y_1)\}, \dots, (\mathbf{x}_m, y_m)\}$$

The overall likelihood function is computed as:

$$\prod_{i=1}^m P(y_i \mid \mathbf{x}_i) = \prod_{i=1}^m \ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i)$$

Why Does Likelihood Make Sense?

How does the likelihood $\ell(y_i \theta^T \mathbf{x}_i)$ changes w.r.t. the sign of y_i and $\theta^T \mathbf{x}_i$?

	$\theta^T \mathbf{x}_i > 0$	$\theta^T \mathbf{x}_i < 0$
$y_i > 0$	≈ 1	≈ 0
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If the label is **concordant** with the signal (either positively or negatively)
then $\ell(y_i \theta^T \mathbf{x}_i)$ approaches to 1

prediction agrees with the true label

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prediction disagrees with the true label

Maximum Likelihood Estimate (MLE)

Find the vector of parameters $\boldsymbol{\theta}$ such that the likelihood function is maximum

$$\operatorname{argmax}_{\boldsymbol{\theta}} \left(\prod_{i=1}^m P(y_i \mid \mathbf{x}_i) \right) = \operatorname{argmax}_{\boldsymbol{\theta}} \left(\prod_{i=1}^m \ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i) \right)$$

From MLE to In-Sample Error

Given a hypothesis h_θ and a training set D of m labelled samples we are interested in measuring the "in-sample" (i.e. *training*) error

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$$E_{\text{in}}(\boldsymbol{\theta}) = \frac{1}{m} \sum_{i=1}^m e(h_{\boldsymbol{\theta}}(\mathbf{x}_i), y_i)$$

where $e()$ measures how "far" the chosen hypothesis is from the true observed value

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How we can "transform" MLE to the "in-sample" error above?

Negative Log-Likelihood

$$\operatorname{argmax}_{\boldsymbol{\theta}} \left(\prod_{i=1}^m \ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i) \right)$$

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$$= \operatorname{argmin}_{\boldsymbol{\theta}} \left(-\frac{1}{m} \ln \left(\ell(y_1 \boldsymbol{\theta}^T \mathbf{x}_1) \right) - \dots - \frac{1}{m} \ln \left(\ell(y_m \boldsymbol{\theta}^T \mathbf{x}_m) \right) \right)$$

as $k \ln(a \cdot b) = k(\ln(a) + \ln(b)) = k \ln(a) + k \ln(b)$.

Negative Log-Likelihood

$$\operatorname{argmax}_{\boldsymbol{\theta}} \left(\prod_{i=1}^m \ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i) \right) \qquad \operatorname{argmax}_{\boldsymbol{\theta}} \left(\frac{1}{m} \ln \left(\prod_{i=1}^m \ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i) \right) \right)$$

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$$= \operatorname{argmin}_{\boldsymbol{\theta}} \left(\frac{1}{m} \sum_{i=1}^m -\ln(\ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i)) \right)$$

$$= \operatorname{argmin}_{\boldsymbol{\theta}} \left(\frac{1}{m} \sum_{i=1}^m \ln \left(\frac{1}{\ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i)} \right) \right)$$

$$\text{as } -\ln(a) = \ln\left(\frac{1}{a}\right).$$

Cross-Entropy Error

$$\operatorname{argmin}_{\theta} \left(\frac{1}{m} \sum_{i=1}^m \ln \left(\frac{1}{\ell(y_i \boldsymbol{\theta}^T \mathbf{x}_i)} \right) \right)$$

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By noticing that logistic function can be rewritten as follows:

$$\ell(z) = \frac{e^z}{1+e^z} = \frac{1}{e^{-z}+1}$$

We can finally write the "in-sample" error to be minimized:

$$E_{\text{in}}(\boldsymbol{\theta}) = \frac{1}{m} \sum_{i=1}^m \ln(e^{-y_i \boldsymbol{\theta}^T \mathbf{x}_i} + 1)$$

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Cross-Entropy Error

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$$y = \{-1, +1\}$$

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$$y = \{-1, +1\}$$

$$-\frac{1}{m} \sum_{i=1}^m y_i \ln(p) + (1 - y_i) \ln(1 - p)$$

$$p = \frac{e^{\theta^T \mathbf{x}}}{e^{\theta^T \mathbf{x}} + 1} = \frac{1}{1 + e^{-\theta^T \mathbf{x}}}$$

$$y = \{0, 1\}$$

Cross-Entropy (a.k.a. Log-Loss) Formulations

$$Y = \{0, 1\}$$
$$Y \sim \text{Bernoulli}(p)$$

$$f_Y(y; p) = L_Y(p; y) = \begin{cases} p & \text{if } y = 1 \\ q = 1 - p & \text{if } y = 0 \end{cases}$$

Probability density function of a Bernoulli-distributed random variable with known parameter p

Likelihood of an observed Bernoulli-distributed random variable (parameter p is unknown)

Likelihood Function

Likelihood function of m **i.i.d.** observations of Y

$$L_Y(p; y_1 \dots y_m) = \prod_{i=1}^m p^{y_i} (1-p)^{(1-y_i)}$$

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$$L_Y(p; y_1 \dots y_m) = \prod_{i=1}^m p^{y_i} (1 - p)^{(1-y_i)}$$

Here the unknown is the parameter p and we use the observations y_1, \dots, y_m to find p so as to maximize the likelihood

$$p^* = \operatorname{argmax}_p \left\{ \prod_{i=1}^m p^{y_i} (1 - p)^{(1-y_i)} \right\}$$

Negative Log-Likelihood Function

$$p^* = \operatorname{argmin}_p \left\{ -\ln \left[\prod_{i=1}^m p^{y_i} (1-p)^{(1-y_i)} \right] \right\}$$

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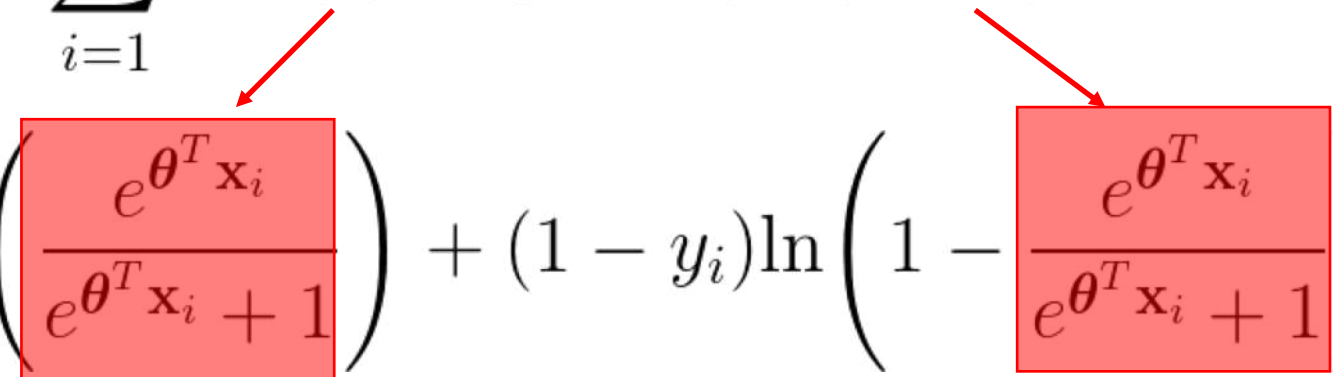
$$p^* = \operatorname{argmin}_p \left\{ - \sum_{i=1}^m y_i \ln(p) + (1-y_i) \ln(1-p) \right\}$$

Except for the $1/m$ factor this is exactly the second formulation we gave for the cross-entropy error

Substituting p

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Substituting p

$$\begin{aligned} & - \sum_{i=1}^m y_i \ln(p) + (1 - y_i) \ln(1 - p) \\ & - \sum_{i=1}^m y_i \ln \left(\frac{e^{\theta^T \mathbf{x}_i}}{e^{\theta^T \mathbf{x}_i} + 1} \right) + (1 - y_i) \ln \left(1 - \frac{e^{\theta^T \mathbf{x}_i}}{e^{\theta^T \mathbf{x}_i} + 1} \right) \\ & - \sum_{i=1}^m y_i [\ln(e^{\theta^T \mathbf{x}_i}) - \ln(e^{\theta^T \mathbf{x}_i} + 1)] + (1 - y_i) [\ln(1) - \ln(e^{\theta^T \mathbf{x}_i} + 1)] \end{aligned}$$

Substituting p

$$-\sum_{i=1}^m y_i [\ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i}) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)] + (1 - y_i) [\ln(\cancel{1}) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)]$$

0

Substituting p

$$\begin{aligned}
& - \sum_{i=1}^m y_i [\ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i}) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)] + (1 - y_i) [\ln(1) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)] \\
& - \sum_{i=1}^m y_i \boldsymbol{\theta}^T \mathbf{x}_i - y_i \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1) + y_i \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)
\end{aligned}$$

Substituting p

$$- \sum_{i=1}^m y_i [\ln(e^{\theta^T \mathbf{x}_i}) - \ln(e^{\theta^T \mathbf{x}_i} + 1)] + (1 - y_i) [\cancel{\ln(1)} - \ln(e^{\theta^T \mathbf{x}_i} + 1)]$$
$$- \sum_{i=1}^m y_i \theta^T \mathbf{x}_i - \cancel{y_i \ln(e^{\theta^T \mathbf{x}_i} + 1)} - \ln(e^{\theta^T \mathbf{x}_i} + 1) + \cancel{y_i \ln(e^{\theta^T \mathbf{x}_i} + 1)}$$

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$$- \sum_{i=1}^m y_i [\ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i}) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)] + (1 - y_i) [\cancel{\ln(1)} - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)]$$

0

$$- \sum_{i=1}^m y_i \boldsymbol{\theta}^T \mathbf{x}_i - \cancel{y_i \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)} - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1) + \cancel{y_i \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)}$$

$$- \sum_{i=1}^m y_i \boldsymbol{\theta}^T \mathbf{x}_i - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)$$

Equivalence Between 2 Formulations

We want to show the 2 formulations below lead to the same function to be minimized

$$\sum_{i=1}^m \ln(e^{-y_i \boldsymbol{\theta}^T \mathbf{x}_i} + 1)$$

$$y = \{-1, +1\}$$

$$-\sum_{i=1}^m y_i \boldsymbol{\theta}^T \mathbf{x}_i - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i} + 1)$$

$$y = \{0, 1\}$$

Equivalence Between 2 Formulations

We want to show the 2 formulations below lead to the same function to be minimized

$$\sum_{i=1}^m \ln(e^{\theta^T \mathbf{x}_i} + 1) \quad y = -1 \quad = \quad \sum_{i=1}^m \ln(e^{\theta^T \mathbf{x}_i} + 1) \quad y = 0$$

Equivalence Between 2 Formulations

We want to show the 2 formulations below lead to the same function to be minimized

$$\sum_{i=1}^m \ln(e^{-\theta^T \mathbf{x}_i} + 1)$$

$$y = 1$$

$\stackrel{?}{=}$

$$-\sum_{i=1}^m \theta^T \mathbf{x}_i - \ln(e^{\theta^T \mathbf{x}_i} + 1)$$

$$y = 1$$

Equivalence Between 2 Formulations

$$\boxed{\sum_{i=1}^m \ln(e^{-\boldsymbol{\theta}^T \mathbf{x}_i} + 1)} = \sum_{i=1}^m \ln\left(\frac{1}{e^{\boldsymbol{\theta}^T \mathbf{x}_i}} + 1\right) = \sum_{i=1}^m \ln\left(\frac{1 + e^{\boldsymbol{\theta}^T \mathbf{x}_i}}{e^{\boldsymbol{\theta}^T \mathbf{x}_i}}\right)$$

Equivalence Between 2 Formulations

$$\boxed{\sum_{i=1}^m \ln(e^{-\boldsymbol{\theta}^T \mathbf{x}_i} + 1)} = \sum_{i=1}^m \ln\left(\frac{1}{e^{\boldsymbol{\theta}^T \mathbf{x}_i}} + 1\right) = \sum_{i=1}^m \ln\left(\frac{1 + e^{\boldsymbol{\theta}^T \mathbf{x}_i}}{e^{\boldsymbol{\theta}^T \mathbf{x}_i}}\right)$$
$$= \sum_{i=1}^m \ln(1 + e^{\boldsymbol{\theta}^T \mathbf{x}_i}) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i})$$

Equivalence Between 2 Formulations

$$\begin{aligned} \boxed{\sum_{i=1}^m \ln(e^{-\boldsymbol{\theta}^T \mathbf{x}_i} + 1)} &= \sum_{i=1}^m \ln\left(\frac{1}{e^{\boldsymbol{\theta}^T \mathbf{x}_i}} + 1\right) = \sum_{i=1}^m \ln\left(\frac{1 + e^{\boldsymbol{\theta}^T \mathbf{x}_i}}{e^{\boldsymbol{\theta}^T \mathbf{x}_i}}\right) \\ &= \sum_{i=1}^m \ln(1 + e^{\boldsymbol{\theta}^T \mathbf{x}_i}) - \ln(e^{\boldsymbol{\theta}^T \mathbf{x}_i}) \\ &= \boxed{-\sum_{i=1}^m \boldsymbol{\theta}^T \mathbf{x}_i - \ln(1 + e^{\boldsymbol{\theta}^T \mathbf{x}_i})} \end{aligned}$$

LEARNING ALGORITHM

Picking the Best Hypothesis

- So far, we have defined:
 - The model (logistic function)
 - The error measure (cross-entropy)

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 - The model (logistic function)
 - The error measure (cross-entropy)

To actually select the best hypothesis, we have to pick the vector of parameters $\boldsymbol{\theta}$ so that the error measure is minimized

$$E_{\text{in}}(\boldsymbol{\theta}) = \frac{1}{m} \sum_{i=1}^m \ln(e^{-y_i \boldsymbol{\theta}^T \mathbf{x}_i} + 1)$$

Mean Squared Error vs. Cross-Entropy

In the case of linear regression we have a similar expression for the error measure, i.e. Mean Squared Error (MSE)

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Minimising MSE through Ordinary Least Squares (OLS) leads to a **closed-form solution** often referred to as the OLS estimator for $\boldsymbol{\theta}$

$$\hat{\boldsymbol{\theta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

Mean Squared Error vs. Cross-Entropy

The problem is that using Cross-Entropy as error measure we cannot find a closed-form solution to the minimization problem

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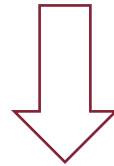
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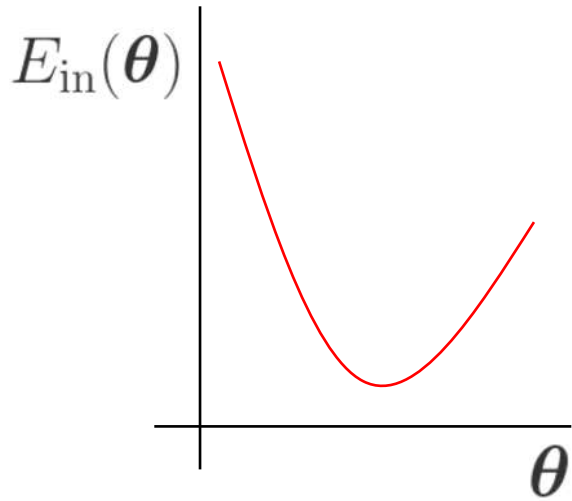
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Iterative Solution

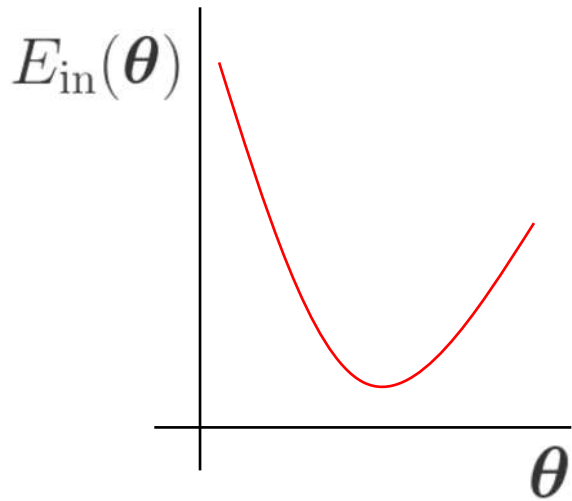
(Batch) Gradient Descent

General iterative method for any nonlinear optimization



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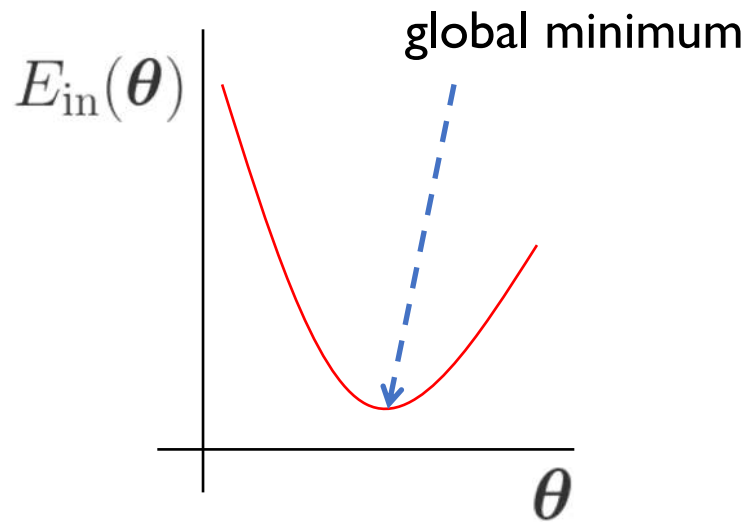


The method **guarantees the convergence to a local minimum**

(Under specific assumptions on the objective function and learning rate)

(Batch) Gradient Descent

General iterative method for any nonlinear optimization



The method **guarantees the convergence to a local minimum**

(Under specific assumptions on the objective function and learning rate)

If the objective function is **convex** (like cross-entropy) then the local minimum is also the **global minimum**

Gradient Descent: The Main Idea

- I. At $t=0$ initialize the (guessed) vector of parameters θ to $\theta(0)$

Gradient Descent: The Main Idea

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 - a. Update the current vector of parameters $\theta(t)$ by taking a "step" along the "steepest" slope: $\theta(t+1) = \theta(t) + \eta \mathbf{v}$
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How do we determine the direction \mathbf{v} ?

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We want ΔE_{in} to be **as negative as possible**, which means that we are actually reducing the error w.r.t. the previous iteration $t-1$

Gradient Descent: The Direction \mathbf{v}

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Let's first assume we are in the **univariate** case, i.e. $\boldsymbol{\theta} = \theta$ in \mathbb{R}

$$f = E_{\text{in}}$$

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$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \approx \frac{\delta f}{\delta x}$$

Gradient Descent: The Direction \mathbf{v}

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First-order Taylor approximation Second-order error term

To summarize and generalize to the multivariate case of $\boldsymbol{\theta}$:

$$\delta f = f(x) - f(x_0) = \Delta E_{\text{in}} = \eta \nabla E_{\text{in}}(\boldsymbol{\theta}(t-1))^T \mathbf{v} + O(\eta^2)$$

The greek letter *nabla* indicates the gradient

Gradient Descent: The Direction \mathbf{v}

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The unit vector \mathbf{v} only contributes to the **direction** and not to the magnitude of the iterative step

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The unit vector \mathbf{v} only contributes to the **direction** and not to the magnitude of the iterative step

The second-order approximation term is negligible
(when the step size is small)

Gradient Descent: The Direction \mathbf{v}

$$\nabla E_{\text{in}}(\boldsymbol{\theta}(t-1))^T = \mathbf{u}$$
$$\Delta E_{\text{in}} = \eta \mathbf{u} \cdot \mathbf{v}$$

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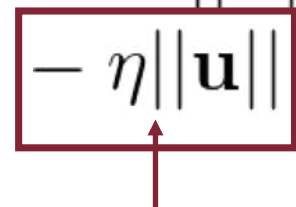
The most **positive** ΔE_{in} when $\cos(\alpha) = 1$ (i.e., $\alpha = 0^\circ$)

Both error and step vectors have the same direction

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The most **negative** ΔE_{in} when $\cos(\alpha) = -1$ (i.e., $\alpha = 180^\circ$)

The error and step vectors have opposite direction

Gradient Descent: The Direction \mathbf{v}

At each iteration t , we want the unit vector \mathbf{v} which makes exactly **the most negative** ΔE_{in}

$$\eta \mathbf{u} \cdot \mathbf{v} = -\eta ||\mathbf{u}||$$

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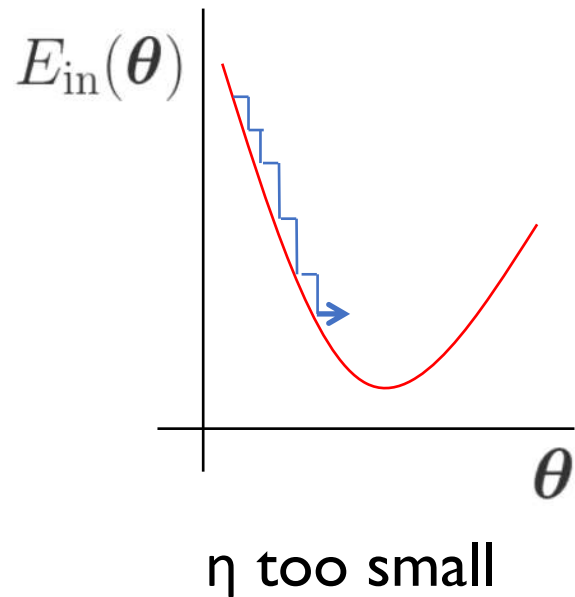
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Gradient Descent: The Step η

How the step magnitude η affects the convergence?

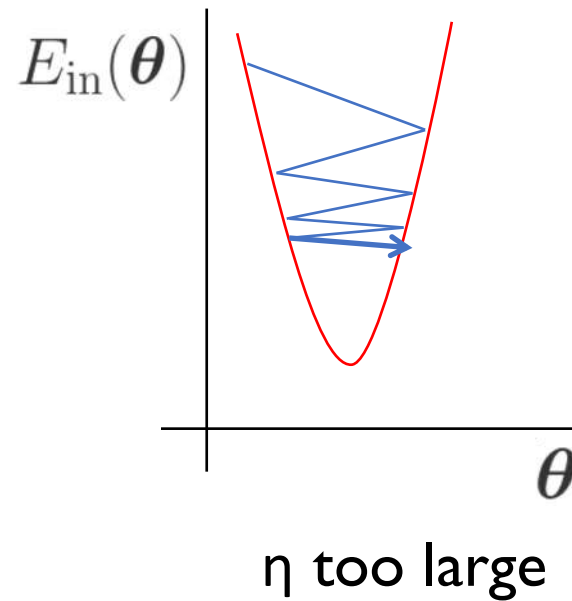
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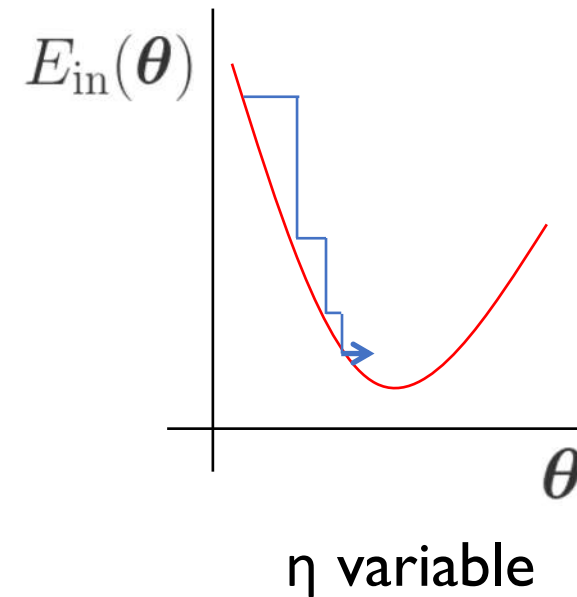
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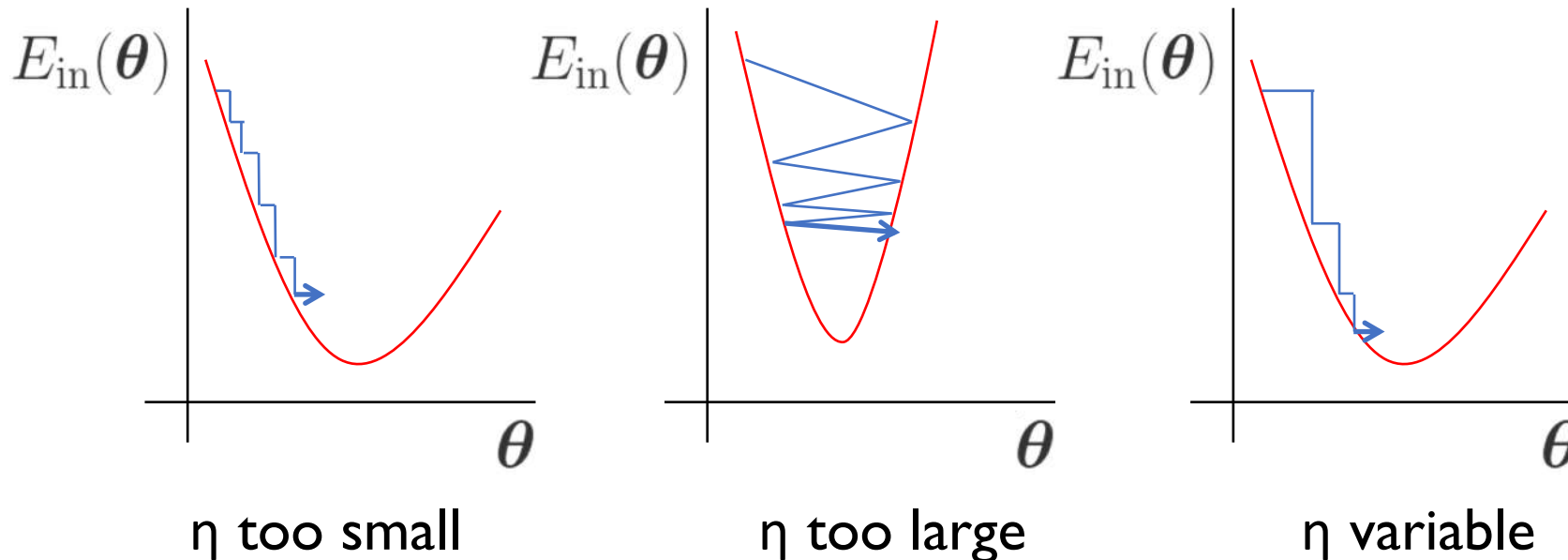
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Rule of thumb

Dynamically change η proportionally to the gradient!

Gradient Descent: The Step η

Remember that at each iteration the update strategy is:

$$\boldsymbol{\theta}(t + 1) = \boldsymbol{\theta}(t) + \eta \mathbf{v}$$

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At each iteration t , the step η is fixed

$$\boldsymbol{\theta}(t + 1) = \boldsymbol{\theta}(t) - \eta \frac{\nabla E_{\text{in}}(\boldsymbol{\theta}(t))}{\|\nabla E_{\text{in}}(\boldsymbol{\theta}(t))\|}$$

Gradient Descent: The Step η

Instead of having a fixed η at each iteration, use a variable η_t as function of η

$$\boldsymbol{\theta}(t + 1) = \boldsymbol{\theta}(t) + \eta_t \mathbf{v} \qquad \eta_t = \eta k$$

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Computing the Gradient of Cross-Entropy

$$\nabla E_{\text{in}}(\boldsymbol{\theta}) = \nabla \left[\frac{1}{m} \sum_{i=1}^m \ln(e^{-y_i \boldsymbol{\theta}^T \mathbf{x}_i} + 1) \right]$$

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chain rule of derivative

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Gradient Descent: The Algorithm

I. At $t=0$ initialize the (guessed) vector of parameters θ to $\theta(0)$

Gradient Descent: The Algorithm

1. At $t=0$ initialize the (guessed) vector of parameters $\boldsymbol{\theta}$ to $\boldsymbol{\theta}(0)$
2. For $t = 0, 1, 2, \dots$ until stop:

a. Compute the gradient of the cross-entropy error $E_{\text{in}}(\boldsymbol{\theta}) = \frac{1}{m} \sum_{i=1}^m \ln(e^{-y_i \boldsymbol{\theta}^T \mathbf{x}_i} + 1)$

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2. For $t = 0, 1, 2, \dots$ until stop:

a. Compute the gradient of the cross-entropy error $E_{\text{in}}(\boldsymbol{\theta}) = \frac{1}{m} \sum_{i=1}^m \ln(e^{-y_i \boldsymbol{\theta}^T \mathbf{x}_i} + 1)$

$$\nabla E_{\text{in}}(\boldsymbol{\theta}(t)) = -\frac{1}{m} \sum_{i=1}^m \frac{y_i \mathbf{x}_i}{1 + e^{y_i \boldsymbol{\theta}(t)^T \mathbf{x}_i}}$$

b. Update the vector of parameters: $\boldsymbol{\theta}(t+1) = \boldsymbol{\theta}(t) - \eta \nabla E_{\text{in}}(\boldsymbol{\theta}(t))$

c. Return to 2.

3. Return the final vector of parameters $\boldsymbol{\theta}(\infty)$

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- In general, we may get to the local minimum nearest to $\theta(0)$

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- GD can still be used to try to optimize **non-convex** objectives
- Problem: non-convex functions may have several local minima
- A bad initialization might cause GD to end up into a "bad" local minimum and miss "better" ones (or even the global if it exists)
- Solution (heuristic): repeating GD 100÷1,000 times each time with a different $\theta(0)$ may reduce the chance the above issue occurs

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- In general, we don't know if eventually the gradient gets to 0 therefore we can use several criteria of termination:
 - stop whenever the difference between two iterations is "small enough" \rightarrow may converge "prematurely"
 - stop when the error equals to $\varepsilon \rightarrow$ may not converge if the target error is not achievable
 - stop after T iterations
 - combinations of the above in practice works...

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- Stochastic Gradient Descent (SGD)
 - At each iteration, compute the gradient only from one sample (not the full dataset)
- Regularization
 - Include the L1- or L2-norm of the vector of parameters θ in the cross-entropy error to avoid overfitting

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- It fits a regression line between input (features) and output (logarithm of the odds), assuming probability takes the form of a sigmoid function
- Parameter estimation is typically done via MLE (i.e., by minimizing Cross-Entropy error)
- No closed-form solution → iterative Gradient Descent