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VIX forecasting and variance risk premium: A new GARCH approach

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ABSTRACT

This paper proposes to forecast VIX under GARCH(1,1), GJR, and Heston–Nandi models, and to assess variance risk premium innovatively. The one-day out-of-sample VIXs, computed with traditional empirical GARCH parameters, turn out to be below the market VIXs by roughly 20–30% (10–13%) on average before (after) 22 September 2003. The underestimation is interpreted as a kind of variance risk premium, which for the later part of the data turns out to be significantly smaller. On the other hand, risk-neutral GARCH models can be obtained by calibration against the prior-day market VIX. For the same dataset, the risk-neutral parameters forecast the one-day out-of-sample VIXs with errors within −0.30 to 0.03% on average.

1. Introduction

The CBOE Volatility Index (VIX), proposed by Whaley (1993) and introduced in 1993, has become the standard gauge of the investor fear and market sentiment. It is befittingly referred to as the "fear index" by popular news media, such as the New York Times and the Wall Street Journal. Nowadays, CBOE trades VIX futures, VIX options, VIX Binary Options, and Mini-VIX futures.

Given the importance of VIX as a forecast of the 30-day volatility of the S&P 500 index, the pricing of its derivatives has not been overlooked by academia. VIX futures pricing under various

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continuous-time stochastic variance models was studied extensively (Duan & Yeh, 2010; Lin, 2007; Zhang & Huang, 2010; Zhu & Lian, 2012; Zhang, Shu, & Brenner, 2010; Zhang & Zhu, 2006). Recently, several papers investigated the pricing of VIX options (Cont & Kokholm, 2013; Goard & Mazur, 2013; Lian & Zhu, 2013; Lin, 2013).

As a discrete-time model for volatility, GARCH models seem to be a natural choice for studying VIX. Barone-Adesi, Engle, and Mancini (2008) alluded to the forecasting of VIX briefly. Their idea of path simulations was later applied to VIX with some more details (Byun & Min, 2013). VIX formulas under the empirical measure for five GARCH models were provided by Hao and Zhang (2013). GARCH models utilizing information of VIX for pricing options on the S&P 500 index were suggested by Kanniainen, Lin, and Yang (2014). Thus far though, no research seems to have focused on VIX forecasting using GARCH.

Our paper proposes to forecast VIX using GARCH models. This is initially tried with computing VIX from empirical GARCH parameters. The computed VIXs turn out to under-estimate the market VIXs, which is then suggested to be a new way for assessing variance risk premium. Furthermore, our paper adapts the Barone-Adesi et al. (2008) for obtaining risk-neutral GARCH models. While BEM used the market prices of S&P 500 index options from a previous trading day, our paper simplifies it further by utilizing directly the single prior-day market VIX. Unlike previous studies, our approach neither depends on the average return of the underlying process, nor needs to use the risk-free rate. This is arguably quite an attractive characteristic. Empirically, the risk-neutral models turn out to forecast out-of-sample VIX with reasonable accuracy.

The rest of the paper is organized as follows. For GARCH(1,1), GJR, and Heston–Nandi variance models, closed-form formulas for VIX under empirical measures are derived in Section 2. Next, variance risk premium is discussed and VIX-calibrated risk-neutral GARCH models are suggested. Section 4 then presents extensive empirical investigations. Finally, the paper concludes with comments.

2. Formulas for VIX under GARCH(1,1) models

Three GARCH(1,1) models can be written as follows under the empirical measure:

$$\begin{aligned} &\ln\left(\frac{S_t}{S_{t-1}}\right) = \mu + \varepsilon_t \\ &v_t = \omega + \beta v_{t-1} + u_{t-1} \end{aligned} \tag{1}$$

where S_t is the time-t price of the "stock," $v_t = \sigma_t^2$ is the variance, and $\varepsilon_t = \sigma_t z_t$ is the innovation for date t. Here z_t is either a standard normal variable or the empirical random variable with a mean of zero and a variance of one. For GARCH(1,1) or G11 hereafter, $u_t = \alpha\varepsilon_t^2$ (Bollerslev, 1986); for asymmetric GJR GARCH(1,1) or GJR hereafter, $u_t = (\alpha + \gamma I\{z_t < 0\})\varepsilon_t^2$, where $I\{z_t < 0\}$ is the indicator function (Glosten, Jagannathan, & Runkle, 1993); for HN GARCH(1,1) or HN hereafter, $u_t = \alpha(z_t - \gamma\sigma_t)^2$ (Heston & Nandi, 2000).

Even though VIX is defined only under the risk-neutral measure, a VIX-like variable under the empirical measure can be imitated, which will be called eVIX hereafter, and estimated by using the following proposition. The derivation of the eVIX formulas is presented in the Appendix.

Proposition. eVIX can be estimated from empirical GARCH(1,1) parameters and the daily empirical variance as follows:

$$eVIX_t^2 = (cv_{t+1} + d) \times 365 \times 100^2 \quad c = \frac{(1 - (105/365)\xi^{20} - (260/365)\xi^{21})}{30(1-\xi)},\ d = V_L\left(\frac{252}{365} - c\right) \tag{2}$$

where the persistency ξ and the long-term average variance V_L are defined, respectively, as:

$$\xi = \alpha + \beta, \quad V_L = \frac{\omega}{(1-\xi)} \qquad (\text{for G11}) \tag{3}$$

$$\xi = \alpha + \beta + 0.5\gamma, \quad V_L = \frac{\omega}{(1-\xi)} \qquad (\text{for GJR}) \tag{4}$$

$$\xi = \alpha\gamma^2 + \beta, \quad V_L = \frac{(\omega + \alpha)}{(1 - \xi)} \qquad \text{(for HN)} \tag{5}$$

Note that Formula (2) has the same linear form as those given by the continuous-time stochastic volatility models (Lin, 2007; Zhu & Lian, 2012; Zhang & Zhu, 2006). Similar but slightly different results for various GARCH models under the risk-neutral measure were previously derived by Hao and Zhang (2013) and Kanniainen et al. (2014). It is worth noting that one nice feature of Eqs. (2)–(5) is their independence of the parameter μ of the "stock" process, while both the results of continuous-time models and the risk-neutral GARCH models in previous literature are not. Arguably, this provides certain justification of our paper for using GARCH models.

3. Variance risk premium and VIX-calibrated GARCH parameters

It is known that empirical GARCH parameters do not price options satisfactorily (Barone-Adesi et al., 2008; Liu & Guo, 2014). Furthermore, VIX is under-estimated consistently in-sample by various GARCH models (Hao & Zhang, 2013).[1] In principle, if the empirical measure were the same as the risk-neutral measure, the computed eVIX should be the same as the market VIX. Apparently then, the one-day out-of-sample difference between eVIX and the market VIX can be regarded and interpreted as an estimation of variance risk premium.

Risk-neutral GJR GARCH parameters calibrated by the market prices of traded options do a much better job in pricing European options (Barone-Adesi et al., 2008) and American options (Liu & Guo, 2014). Further, Barone-Adesi et al. (2008) mentioned briefly that the risk-neutral GARCH parameters can be used to simulate daily variances in order to price VIX. Their idea was adopted by Byun and Min (2013).

Unfortunately, it is quite expensive computationally to obtain the risk-neutral GARCH parameters by simulating variance paths, pricing the options, and minimizing pricing errors (Barone-Adesi et al., 2008; Byun & Min, 2013). Furthermore, since it is computed from the prices of traded options, the market VIX can be used instead to obtain risk-neutral GARCH models. Nowadays, VIX is well-established. Therefore, this paper proposes to calibrate the risk-neutral model by the market VIX via Formula (6) as follows:

$$(VIX_t)^2 = (av_{t+1} + b) \times 365 \times 100^2, \quad a = \frac{1 - \xi^{*30}}{30(1 - \xi^*)}, \; b = V_L^*(1 - a) \tag{6}$$

where ξ^* and V_L^* are related via Eqs. (3)–(5) to the risk-neutral GARCH parameters ω^*, α^*, β^* and γ^*. The saving of doing this is twofold. First, prices of traded options are not needed. Second, price paths and daily variances do not have to be simulated. Furthermore, the proposed new scheme obtains the risk-neutral measure directly through Formula (6), without having to use either the prices of options or the risk-free interest rate. This makes the new approach simpler additionally. Note that the derivation of Formula (6) is identical to that of Formula (2).

As was done in Eq. (12) of Barone-Adesi et al. (2008), the risk-neutral parameters in Formula (6) are based on calendar days, while Formula (2) is on trading days. Since VIX is defined in calendar days and the risk-neutral GARCH model is to be calibrated by the market VIX, a convention of calendar days could be employed in Formula (6). This is quite similar to the computation of the implied volatility based on calendar days via the market price of traded options.

With the VIX-calibrated risk-neutral GARCH parameters, VIX can then be forecasted out-of-sample by utilizing Formula (6). Once again, daily variances do not have to be simulated.

[1] Only a single set of parameters was used to compute the in-sample GARCH implied VIXs for all the days in the sample by Hao and Zhang (2013), which is markedly different from our approach of one set of parameters for each day and computed out-of-sample VIXs.

Table 1
Summary statistics of the market VIX.

	Phase I	Phase II
Minimum	12.00	9.89
Maximum	45.74	80.86
Mean	23.58	21.08
Standard deviation	5.73	10.51
Skewness	0.97	2.09
Excess kurtosis	1.06	5.59
No. of observation	1942	2106

Phase I: 2 January 1996–19 September 2003. Phase II: 22 September 2003–31 January 2012.

4. Empirical study

4.1. Data description

Since CBOE does not provide historical data for the S&P 500 index, both daily VIX and the S&P 500 index from Yahoo!Finance are used in this study. A comparison of VIX from CBOE and Yahoo shows only very minor differences for a few days. It seems that the quality of Yahoo!Finance data should not be a problem for our purpose.

On 22 September 2003, CBOE modified the method for computing VIX. Accordingly, this paper divides the time series of VIX into two sub-periods, since it could be interesting to see whether VIX is affected by this change. Phase I is between 2 January 1996[2] and 19 September 2003.[3] Phase II goes from 22 September 2003 to 31 January 2012.[4] Table 1 provides a summary description of the VIX data.

Apparently, the long-term average of VIX is somewhat stable, but VIX shows more pronounced variation in Phase II. Both the skewness and excess kurtosis for the two phases are positive, but the asymmetry and peakedness are more pronounced for Phase II. The more drastic deviation of the second phase from normality is perhaps a result of the inclusion of the 2008 financial crisis in the sample period.

4.2. Computational details and empirical analyses

Assume t is the forecast date, and $t-1$ is the calibration date. Following Barone-Adesi et al. (2008), the paper utilizes 3500 daily returns of the S&P 500 index between $t-3500$ and $t-1$ to obtain the optimal set of empirical GARCH parameters (for date t). The negative of the maximum-likelihood function (Hull, 2012), $\sum_{i=1}^{3500}(\ln v_{t-i} + \varepsilon_{t-i}^2/v_{t-i})$, is minimized via the Nelder–Mead algorithm (Press, Teukolsky, Vetterling, & Flannery, 2002) with the constraint of the stable condition $\xi < 1$. The procedure is then repeated for other dates in a moving window fashion. The reported parameters averaged over the trading days within the two phases are quite close, as Table 2 shows.

In order to obtain the risk-neutral GARCH parameters for date t, the objective function $[100^2 \times 365(av_t + b) - (VIX_{t-1}^m)^2]^2$ is minimized via the Nelder–Mead algorithm with the constraint of $\xi^* < 1$.[5] Here v_t (and ε_t as well) is computed via Eq. (1) under the empirical models. The risk-neutral parameters are markedly different from their corresponding empirical ones for G11 and GJR (Table 3). Interestingly, the objective function is a maximum-likelihood estimator with a single market VIX of

[2] This choice makes the two parts have roughly the same number of trading days.
[3] 31 January 1997 and 26 November 1997 are missing from the downloaded VIX dataset.
[4] 11 June 2004 and 2 January 2007 are missing from both the index and VIX datasets.
[5] Nelder–Mead is sensitive to the initial choice of the variables' characteristic lengths, which are obtained through extensive testing of the three GARCH models and different for estimating the empirical GARCH and calibrating the risk-neutral parameters. Note that the minimization may fail to converge within the allowed 15,000 iterations. There are 322 such failures for G11, 360 for GJR, and 355 for HN in Phase I, and 99, 93, and 234 in Phase II, respectively. On the other hand, the minimization for the empirical GARCH models has only four days with failed convergence.

Table 2
GARCH parameters under the empirical measure.

G11		$\omega \times 10^6$	$\alpha \times 10^2$	β		ξ
Phase I	Mean	1.543	8.175	0.9056		0.9874
	Std Dev	0.629	1.416	0.0188		0.0074
Phase II	Mean	1.193	8.279	0.9104		0.9932
	Std Dev	0.396	0.719	0.0087		0.0032
GJR		$\omega \times 10^6$	$\alpha \times 10^3$	β	γ	ξ
Phase I	Mean	2.343	7.761	0.9062	0.1185	0.9732
	Std Dev	0.731	2.291	0.0156	0.0226	0.0111
Phase II	Mean	1.741	8.550	0.9077	0.1389	0.9857
	Std Dev	0.450	2.764	0.0100	0.0165	0.0040
HN		$\omega \times 10^{14}$	$\alpha \times 10^6$	β	γ	ξ
Phase I	Mean	5.310	5.145	0.8772	116.5	0.9444
	Std Dev	2.642	1.625	0.0273	26.8	0.0145
Phase II	Mean	5.220	5.521	0.8046	162.8	0.9548
	Std Dev	2.293	1.054	0.0540	25.6	0.0094

Phase I: 2 January 1996–19 September 2003. Phase II: 22 September 2003–31 January 2012. The GARCH models are $\ln(S_t/S_{t-1}) = \mu + \varepsilon_t$, $v_t = \omega + \beta v_{t-1} + u_{t-1}$, where $v_t = \sigma_t^2$, $\varepsilon_t = \sigma_t z_t$, and z_t is the empirical random variable with a mean of zero and a variance of one. For G11, $u_t = \alpha\varepsilon_t^2$, $\xi = \alpha + \beta$; for GJR, $u_t = (\alpha + \gamma I\{z_t < 0\})\varepsilon_t^2$, where $I\{z_t < 0\}$ is the indicator function, $\xi = \alpha + \beta + 0.5\gamma$; for HN, $u_t = \alpha(z_t - \gamma\sigma_t)^2$, $\xi = \alpha\gamma^2 + \beta$. Parameters are estimated in a moving window fashion by minimizing $\sum_{i=1}^{3500} (\ln v_{t-i} + \varepsilon_{t-i}^2/v_{t-i})$ with 3500 daily returns.

the prior day. The proposed simple two-step approach, following Barone-Adesi et al. (2008) for pricing options, is quite different from the joint maximum-likelihood method (Hao & Zhang, 2013; Kanniainen et al., 2014).

With the risk-neutral parameters, VIX for the forecast date t can then be computed via Formula (6). Note that v_{t+1} is once again computed via Eq. (1) using the empirical parameters. Since the data

Table 3
Risk-neutral GARCH parameters.

G11		$\omega \times 10^6$	$\alpha \times 10^2$	β		ξ
Phase I	Mean	4.129	4.753	0.9182		0.9658
	Std Dev	0.531	0.419	0.0229		0.0239
Phase II	Mean	4.306	4.586	0.8908		0.9367
	Std Dev	0.722	0.516	0.0406		0.0427
GJR		$\omega \times 10^6$	$\alpha \times 10^3$	β	γ	ξ
Phase I	Mean	4.138	4.613	0.8971	0.1379	0.9707
	Std Dev	0.219	0.199	0.0238	0.0071	0.0213
Phase II	Mean	4.232	4.451	0.8647	0.1403	0.9393
	Std Dev	0.272	0.295	0.0397	0.0074	0.0395
HN		$\omega \times 10^{14}$	$\alpha \times 10^6$	β	γ	ξ
Phase I	Mean	5.430	5.546	0.7379	201.5	0.9631
	Std Dev	0.137	0.597	0.0457	5.7	0.0272
Phase II	Mean	5.440	5.682	0.6926	197.3	0.9136
	Std Dev	0.118	0.509	0.0492	8.6	0.0549

Phase I: 2 January 1996–19 September 2003. Phase II: 22 September 2003–31 January 2012. Parameters are estimated by minimizing $\{\frac{365,000}{3}[30V_L + (v_t - V_L)(1-\xi^{30})/(1-\xi)] - (VIX_{t-1}^m)^2\}^2$, where VIX_{t-1}^m is the market VIX for the calibration date, $\xi = \alpha + \beta$, $V_L = \omega/(1-\xi)$ for G11, $\xi = \alpha + \beta + 0.5\gamma$, $V_L = \omega/(1-\xi)$ for GJR, and $\xi = \alpha\gamma^2 + \beta$, $V_L = (\omega + \alpha)/(1-\xi)$ for HN. Further, v_t is computed by using the empirical GARCH parameters via $\ln(S_t/S_{t-1}) = \mu + \varepsilon_t$, $v_t = \omega + \beta v_{t-1} + u_{t-1}$, where $v_t = \sigma_t^2$, $\varepsilon_t = \sigma_t z_t$, $u_t = \alpha\varepsilon_t^2$ for G11, $u_t = (\alpha + \gamma I\{z_t < 0\})\varepsilon_t^2$ (where $I\{z_t < 0\}$ is the indicator function) for GJR, and $u_t = \alpha(z_t - \gamma\sigma_t)^2$ for HN.

Table 4
Out-of-sample one-day forecasting errors of VIX by GARCH.

	Empirical			Risk-neutral		
	MFE (%)	MAE (%)	RMSE	MFE (%)	MAE (%)	RMSE
Phase I						
G11	−19.91	21.63	5.972	−0.30	4.39	1.534
GJR	−25.46	25.81	6.887	−0.21	3.54	1.254
HN	−29.57	29.62	8.305	−0.11	3.03	0.898
Phase II						
G11	−12.54	15.27	4.374	−0.08	4.49	1.926
GJR	−12.99	15.47	4.489	0.03	3.53	1.395
HN	−10.11	16.96	7.072	−0.01	2.99	0.978
HAR				0.19	4.74	1.960

Phase I: 2 January 1996–19 September 2003. Phase II: 22 September 2003–31 January 2012. VIX is computed via the empirical formula $\{\frac{365{,}000}{3}[\frac{252}{365}\times 30V_L+(v_{t+1}-V_L)(1-\frac{105}{365}\xi^{20}-\frac{260}{365}\xi^{21})/(1-\xi)]\}^{0.5}$ or the risk-neutral formula $\left\{\frac{365{,}000}{3}[30V_L+(v_t-V_L)(1-\xi^{30})/(1-\xi)]\right\}^{0.5}$, where $\xi=\alpha+\beta$, $V_L=\omega/(1-\xi)$ for G11, $\xi=\alpha+\beta+0.5\gamma$, $V_L=\omega/(1-\xi)$ for GJR, and $\xi=\alpha\gamma^2+\beta$, $V_L=(\omega+\alpha)/(1-\xi)$ for HN. Further, v_{t+1} is obtained by using the empirical GARCH parameters via $\ln(S_t/S_{t-1})=\mu+\varepsilon_t$, $v_t=\omega+\beta v_{t-1}+u_{t-1}$, where $v_t=\sigma_t^2$, $\varepsilon_t=\sigma_t z_t$, $u_t=\alpha\varepsilon_t^2$ for G11, $u_t=(\alpha+\gamma I\{z_t<0\})\varepsilon_t^2$ (where $I\{z_t<0\}$ is the indicator function) for GJR, and $u_t=\alpha(z_t-\gamma\sigma_t)^2$ for HN. HAR: the heterogeneous autoregressive model (Corsi, 2009) with the coefficients estimated in a moving window fashion from 3500 market VIXs. MFE, mean of forecasting errors; MAE, mean of absolute forecasting errors; RMSE, root mean of square forecasting errors.

of their previous days are missing, 3 February 1997, 28 November 1997, 14 June 2004, and 3 January 2007 are excluded from forecasting. Further, 17 September 2001 is not included due to the closing of markets over 9/11. The remaining out-of-sample one-day forecasting results are summarized in Table 4.

Table 4 reports three error measures. The mean of forecasting errors (MFE) is defined as $\sum_{j=1}^{N}(VIX_j^c/VIX_j^m-1)/N$, where VIX_j^c is the computed VIX and VIX_j^m the market VIX for date j. The mean of absolute forecasting errors (MAE) is obtained via $\sum_{j=1}^{N}\left|VIX_j^c/VIX_j^m-1\right|/N$. The root mean of square forecasting errors (RMSE) is computed by $[\sum_{j=1}^{N}(VIX_j^c-VIX_j^m)^2/N]^{0.5}$.

Let's first look at the results under the empirical measure. MFE errors are rather large and negative for all three GARCH models. This implies that on average the empirical GARCH underestimates VIX. This out-of-sample underestimation of the market VIX by eVIX can be regarded as a kind of variance risk premium, for which is not accounted by the observed prices of the underlying S&P 500 index.

Interestingly, the MFE error for Phase I is around −19.91 to −29.57%, but only about −10.11 to −12.99% for Phase II. This seems to suggest that the variance risk premium becomes smaller in Phase II. Actually, the average GARCH forecasting error is model dependent and quite different between GJR (Fig. 1) and HN (Fig. 2) for Phase II.

Surprisingly, HN over-estimates VIX by about 12% (a negative variance risk premium) between 23 November 2004 and 21 June 2007. For this same period, the underestimation by GJR is about 4% (a small variance risk premium). Therefore, one has to be cautious about reading too much into the quantitative results of variance risk premiums from empirical GARCH, which may be both model and time dependent.

Now it is time to focus on errors under the calibrated risk-neutral measure. Compared with the empirical measure, all the out-of-sample one-day forecasting errors are quite small for both phases. MFEs are very close to zero, while MAEs (RMSEs) are less than 4.5% (2.0) and much smaller than those under the empirical measure. This implies that on average the errors cancel out. Remarkably, all three GARCH models forecast out-of-sample VIX with reasonable accuracy, and outperform significantly the in-sample joint return-VIX estimation of Hao and Zhang (2013).

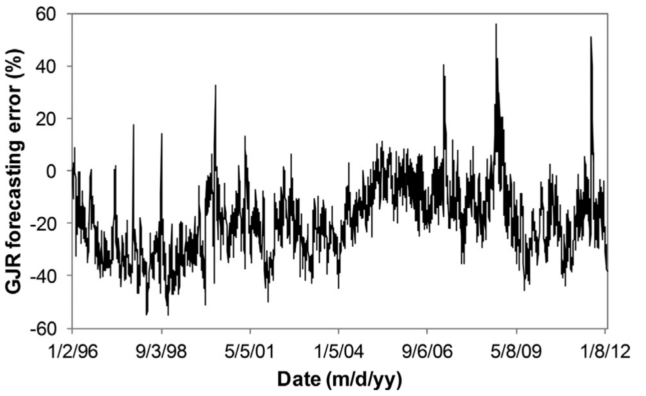

Fig. 1. One-day out-of-sample forecasting errors of the computed VIX with GJR under the empirical measure.

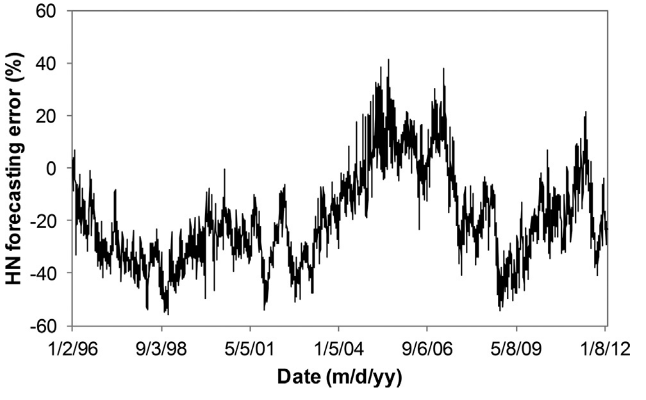

Fig. 2. One-day out-of-sample forecasting errors of the computed VIX with Heston–Nandi under the empirical measure.

As a benchmark, the heterogeneous autoregressive (HAR) model (Corsi, 2009) is shown in Table 4 with out-of-sample one-day forecasting errors. The HAR model for VIX is as follows (Fernandes, Medeiros, & Scharth, 2014):

$$y_t = \beta_0 + \beta_1 y_{t-1}^{(1)} + \beta_2 y_{t-1}^{(5)} + \beta_3 y_{t-1}^{(10)} + \beta_4 y_{t-1}^{(22)} + \beta_5 y_{t-1}^{(66)} + \varepsilon_t$$

where $y_t = \ln(VIX_t)$ and $y_t^{(h)} = \sum_{s=1}^{h} y_{t-s+1}/h$. In other words, the VIX process is determined by its previous daily, weekly, biweekly, monthly, and quarterly averages. The regression coefficients for date t were estimated from market VIXs before date t. To be consistent with the empirical GARCH, the moving window has a size of 3500 VIXs.[6] The dataset covers the period between 27 February 2004 and 31 January 2012. It is assuring to note that in terms of the three error measures, GARCH is at least as good as HAR.

Among the three GARCH models, G11 (HN) has the smallest errors for forecasting VIX under the empirical measure in Phase I (II). G11 shows the biggest forecasting errors under the risk-neutral measure in both phases. This seems to be understandable, since G11 does not take asymmetric information into consideration.

[6] Fernandes et al. (2014) used windows of size 1000.

5. Conclusions

This paper proposes a simple two-step GARCH approach to forecasting out-of-sample VIX and estimating variance risk premium. Closed-form formulas are provided for computing VIX based on symmetric GARCH(1,1), asymmetric GJR GARCH(1,1), and asymmetric Heston–Nandi GARCH(1,1) models. The GARCH models can be either estimated from the returns of the S&P 500 index or calibrated by the market VIX.

The empirical GARCH parameters are estimated solely from 3500 daily returns of the S&P 500 index. The computed one-day out-of-sample VIX under-estimates the market VIX on average by 19.91–29.57% for the period of 2 January 1996–19 September 2003 and by 10.11–12.99% for the period of 22 September 2003–31 January 2012. The out-of-sample underestimation could be referred to as variance risk premium.

The risk-neutral GARCH models are calibrated against the market VIX of the previous trading day. They reduce dramatically the out-of-sample one-day forecasting errors to be within ±0.30% on average. Importantly, the differences among the three GARCH models and between the two sub-periods are almost negligible.

In summary, the proposed two-step GARCH method leads to an ease estimation of variance risk premium, and can forecast out-of-sample VIX with reasonable accuracy. Further, the new approach, with closed-form analytic formulas, is also computationally efficient. Finally, the forecasting method could in principle be extended and applied to the pricing of VIX futures, VIX options, and even S&P 500 index options.

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Appendix.

By definition, z_t and σ_t are independent in Eq. (1), and the conditional expectations are $E_t[z_{t+1}]=0$, $E_t[z_{t+1}^2]=1$, $E_t[z_{t+1}\sigma_{t+1}]=0$, and $E_t[v_{t+1}]=v_{t+1}$. Further, $E_t[z_{t+1}^2 I\{z_{t+1}<0\}]=0.5$. Therefore using GJR as an example, one has:

$$\begin{aligned} E_t v_{t+k} &= \omega + E_t[\beta v_{t+k-1} + (\alpha + \gamma I\{z_{t+k-1} < 0\})\varepsilon_{t+k-1}^2] \\ &= \omega + E_t[(\beta + E_{t+k-2}[(\alpha + \gamma I\{z_{t+k-1} < 0\})z_{t+k-1}^2])v_{t+k-1}] = \omega + \xi E_t v_{t+k-1} \end{aligned}$$

where $\xi=\alpha+\beta+0.5\gamma$ is the persistency. Hereafter it is assumed that the stable condition $\xi<1$ is satisfied.

Let $\omega=(1-\xi)V_L$, where V_L represents the long-term average variance. Then from the above expression it is easy to show (following Hull (2012)) that:

$$E_t[v_{t+k} - V_L] = \xi E_t[v_{t+k-1} - V_L] = \xi^{k-1}(v_{t+1} - V_L)$$

The future variance can thus be forecasted as:

$$E_t[v_{t+k}] = V_L + \xi^{k-1}(v_{t+1} - V_L)$$

The CBOE 30-day volatility index VIX is expressed in calendar days. Since the empirical GARCH parameters are estimated from trading days, an average of 30 variances in calendar days is actually an average of $K = 252 \times \frac{30}{365} = 20 + \frac{260}{365}$ variances in trading days. Therefore, the computed VIX under the empirical measure can be defined as:

$$
\begin{aligned}
\frac{\overline{VIX}_t^2}{100^2} &= \frac{1}{K}\left(\sum_{k=1}^{20} E_t[252v_{t+k}] + \frac{260}{365}E_t[252v_{t+21}]\right) \\
&= \tfrac{252}{K}\left(20V_L + (v_{t+1}-V_L)\sum_{k=1}^{20}\xi^{k-1} + \tfrac{260}{365}[V_L + \xi^{20}(v_{t+1}-V_L)]\right) \\
&= \frac{252}{K}\left(\left(20+\frac{260}{365}\right)V_L + (v_{t+1}-V_L)\left[\frac{1-\xi^{20}}{1-\xi} + \frac{260}{365}\xi^{20}\right]\right) \\
&= \left(252V_L + 365(v_{t+1}-V_L)\frac{(1-\frac{105}{365}\xi^{20} - \frac{260}{365}\xi^{21})}{30(1-\xi)}\right)
\end{aligned}
$$

On the other hand, VIX under the risk-neutral measure can be computed according to Eq. (12) of Barone-Adesi et al. (2008) as:

$$
\frac{VIX_t^2}{100^2} = \frac{365}{30}\left(\sum_{k=1}^{30} E_t[v_{t+k}]\right) = \left(V_L^* + (v_{t+1}-V_L^*)\frac{1-\xi^{*30}}{30(1-\xi^*)}\right) \times 365
$$

where ξ^* and V_L^* are defined by risk-neutral parameters ω^*, α^*, β^* and γ^*. Finally, note that results for G11 and HN can be obtained accordingly.

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