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# Charged Black hole solutions in general relativity from inverse scattering

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# Chapter 1

# Introduction

General relativity is a theory of gravitation developed by Albert Einstein in 1915. Beside its mathematical beauty, General Relativity is a mile stone achievement in the understanding of our universe that changed entirely our perception of the notions of space and time. The theory can be summarize in the phrase attributed to the American physicist John Wheeler "*Space-time tells matter how to move; matter tells space-time how to curve*". This concept can be expressed through a mathematical formulation and indeed is embodied by Einstein's equations of General Relativity:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

These are 10 equations in 4 variables of nonlinear partial differential equations, the task of solving these equation is rather tiresome even for simple cases, to complicate things in general doesn't exist a method to solve non linear partial differential equations hence it becomes cumbersome to solve this equations explicitly even using some ansatz.
One may object that Einstein's equations can be written in the more general form:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Including the cosmological constant $\Lambda$. This equation, even though holds very interesting proprieties, it will not be considered in this thesis, consequently henceforward the cosmological constant will be zero.

Many different methods for generating new solutions without solving explicitly Einstein equations have been implemented. One of these methods is the "inverse scattering generating technique". It consists in the observation that, for Einstein field equations in vacuum or electrovacuum, some integrability condition can be found that simplify incredibly the task of solving Einstein equations, indeed the system of non linear differential equations are transformed in a first order differential equation plus various algebraic transformations.
The inverse scattering technique can be used also to generate a solution from a seed metric in both in the vacuum case and in the case where there is an electromagnetic field. This is done by defining a so called dressing matrix that hold the information about the solution we want to generate inside its poles. The construction of a solution from a seed metric using inverse scattering becomes as easy as following a recipe step-by-step, the only draw back is that the calculations are quite cumbersome, thus most of the calculation done are performed using the software Mathematica.
In this thesis we analyze the inverse scattering generating technique starting from Minkowski solution as seed metric in vacuum and electrovacuum, then we proceed to use Rindler space time as seed metric in the vacuum case. The aim is to generate the most general type D accelerating solution using this technique, after that following the interpretation of the accelerating metric as a pole limiting to infinity we try to generalize the background seed by taking in to account the charged inverse scattering technique with poles at infinity and analyze the solutions obtained. We will use natural units (c=G=1) throughout all this thesis, the signature is everywhere (-,+,+,+) .

# Chapter 2

# Inverse scattering

In this chapter a brief outline of the inverse scattering method is given in the case of non linear-PDE. Some examples called breathers will be analyzed.
Let's take a first order (in time) non linear differential equation of the form

$$u_{,t} = F(u, u_{,z}, u_{,zz}, \dots)$$

Where $F$ is a non linear function containing derivatives of the space coordinate $z$.
Generally this problem can be tackled by means of the common methods for solving DE because of the non-linearity. The first order derivative with respect of time hints of some similarity with Schrödinger equations, indeed by the ISM one considers the scattering problem for the following stationary one-dimensional Schrödinger equation, with $\psi \equiv \psi(t, z, \lambda)$

$$L\psi = \lambda\psi, \quad L = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + u(z,t). \tag{2.1}$$

Hence the unknown $u(z,t)$ plays the role of the potential. This equation is clearly an eigenvalues equation for the linear operator $L$, with eigenvalue $\lambda$ . It's very important to notice that the time plays the role of constant in this equation, indeed is a time independent Schrödinger equation. We require also the condition

$$\lim_{z\to\pm\infty} u(z,t) \approx \frac{1}{z^{2+\varepsilon}}, \quad \varepsilon \in \mathbb{R}^+$$

Hence $u$ vanishes fast enough. Now the big step in the integration scheme is to define the scattering data $S \equiv S(\lambda, t)$, this entity is borrowed from the concept of the scattering matrix in quantum mechanics.
Now that we have two differential equations we have to implement the Cauchy problem and some other condition to make sense of this procedure. Since equation (2.1) is time independent, we can set $t = 0$, hence $u(z, 0)$ is some Cauchy data, now it's interesting to analyze which Scattering data $S(\lambda, 0)$ is produced by this potential. This is very important because $S(\lambda, 0)$ gives us the asymptotic values of the eigenfunctions of $\psi(z, \lambda, 0)$ for $z \to -\infty$ through the given asymptotic values of $\psi(z, \lambda, 0)$ at $z \to +\infty$ for each value of the spectral parameter $\lambda$. This parameter is the energy of the scattered particle and positive values are the continuous spectrum for the problem (2.1). Moreover, a discrete set of negative eigenvalues of $\lambda$ can also enter into the problem corresponding to the bound states of the particle in the potential $u$.Thus, the set $S(\lambda, 0)$ should contain the forward and backward scattering amplitudes for the continuous spectrum.and the negative eigenvalues $\lambda_n$ of the discrete spectrum together with some coefficients, and other data on the asymptotic behavior.
So far we have analyzed the scattering problem for $S$, now we want to recreate the potential $u$ having knowledge of what the scattering data looks like, hence the inverse process as the one described above, therefore we now describe an inverse scattering problem.
It is easy to see that one could solve the Cauchy problem for $u(z, t)$ using this technique. In fact, let us imagine that after constructing the scattering data $S(\lambda, 0)$ corresponding to the potential $u(z, 0)$ at $t = 0$ we could know the time evolution of S and are able to get from the initial values $S(\lambda, 0)$ the scattering data $S(\lambda, t)$ at any arbitrary time $t$. Then we can apply the inverse scattering technique to $S(\lambda, t)$ and reconstruct the potential $u(z, t)$ at any time. This would give the desired solution to the Cauchy problem. This description works if one assumes that the scattering data $S(\lambda, t)$ can be integrated nicely, which is not always the case. Let's look at the possibility of finding the exact solution of the scattering data.
The point is that for the integrable cases of the ISM, the eigenvalues of the

spectral problem (2.1) are, as already remarked, independent of $t$! and the eigenfunctions $\psi$ have obey another PDE which is first order in time, this conclusion is quite intuitive, since otherwise we wouldn't have information on the time evolution of these functions.
This is the key point, since this additional evolution equation for the eigenfunctions allows us to find the exact time dependence of the scattering data. The general form of this equation can be written as

$$\partial_t \psi = A\psi. \tag{2.2}$$

The operator A is a differential operator that depends only on $u$ and its derivatives with respect to $z$.
The set of equations formed by (2.1) and (2.2) are called *Lax pair*,*Lax representation* or *L-A pair*.
A trivial but crucial consideration has to be done $\psi$ has to be solution of both equations, hence some self-consistency conditions for $\psi$ must be satisfied. One can prove that such self-consistency relation is the original equation,

$$u_{,t} = F(u, u_{,z}, u_{,zz}, \dots)$$

Hence this concludes the analysis of the consistency of the problem. Of course, only very special classes of nonlinear differential equations admit L–A pairs and still today there is no general approach on how to find these classes. Despite the existence of a number of powerful techniques each differential system needs individual and, often, sophisticated consideration.
In general the solution for the ISM are not analytical because the problem $S \to u$ usually holds some very nasty integral equations. What is generally done is taking the information out of the limit $t \to \infty$, therefore finding an expression for the asymptotic values of $u(z, t)$.
The case we analyze in this thesis is the case of a special class of solutions $u(z, t)$ for which the direct and inverse scattering problems can be solved exactly in analytic form! These are the so-called *soliton solutions*.

## 2.1 Soliton solutions

A scattering process everybody experiences in daily life is light the scatters on water. Part of the light reflects and the other part is transmitted (refract). There fore the scattering data can be divided in two component, $R(\lambda)$ and $T(\lambda)$.
In the case of soliton solutions it can be shown that the reflection coefficient are identically zero, that is for some initial conditions $u(z,0)$ than the coefficients of $R(\lambda,0)$ are identically zero. In these cases the analytical form of the scattering data is completely determined by the simple poles of the discrete values $\lambda_n$, these poles determine also the eigenfunction spectral problem (2.1) in the $\lambda$-complex plane. We will see by direct applying this method to E-M equations that this form of integrability consists mostly of algebraic manipulations.
Most of the two dimensional equations can be represented as selfconsistency conditions of two matrix equations

$$\psi_{,z} = U^{(1)}\psi, \quad \psi_{,t} = V^{(1)}\psi, \tag{2.3}$$

Where $U^{(1)}$ and $V^{(1)}$ depends on the coordinates z, t and rationally on the complex spectral parameter $\lambda$. $\psi(\lambda,z,t)$ is instead a column matrix.If we differentiate the first equation with respect to t and the second one with respect to z, then the following selfconsistency condition follows

$$U^{(1)}_{,t} - V^{(1)}_{,z} + U^{(1)}V^{(1)} - V^{(1)}U^{(1)} = 0. \tag{2.4}$$

## 2.2 Examples

### 2.2.1 Sine-Gordon

The equation we want to solve is

$$\left(\partial_t^2 - \partial_z^2\right) u(z,t) = \sin\left[u(z,t)\right]$$

Which physically can represent a Thirring model with mass.
Let's now delve into the inverse scattering scheme to generate the spectral

equations.
Now let's take t as a time-like coordinate ans z as a space-like coordinate, we than introduce the null coordinate

$$\zeta = \frac{z+t}{2}, \quad \eta = \frac{z-t}{2}. \tag{2.5}$$

So instead of (2.3) and (2.4) we have

$$\psi,_\zeta = U^{(2)}\psi, \quad \psi,_\eta V^{(2)}\psi, \tag{2.6}$$

$$U^{(2)}_{,\eta} - V^{(2)}_{,\zeta} + U^{(2)}V^{(2)} - V^{(2)}U^{(2)} = 0. \tag{2.7}$$

Where $U^{(2)} = U^{(1)} + V^{(1)}$ and $V^{(1)} = U^{(1)} - V^{(1)}$ If

$$U^{(2)} = i\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & u,_\zeta \\ u,_\zeta & 0 \end{bmatrix}, \quad V^{(2)} = \frac{1}{4i\lambda} \begin{bmatrix} \cos u & -i\sin u \\ i\sin u & -\cos u \end{bmatrix} \tag{2.8}$$

Now from the self consistency equation we obtain

$$\begin{bmatrix} 0 & u,_{\zeta\eta} - \sin u \\ u,_{\zeta\eta} - \sin u & 0 \end{bmatrix} = 0 \tag{2.9}$$

which is the Sine Gordon equation in form

$$,_{\zeta\eta} = \sin u \tag{2.10}$$

Now the spectral equations reads

$$\psi_{1,\zeta} = i\lambda\psi_1 + \frac{i}{2}u,_\eta \psi_2, \tag{2.11}$$

$$\psi_{2,\zeta} = -i\lambda\psi_2 + \frac{i}{2}u,_\zeta \psi_1, \tag{2.12}$$

### 2.2.2 Principal chiral field equation

Now we define the matrix $U^{(2)}$ and $V^{(2)}$ as

$$U^{(2)} = \frac{K}{\lambda - \lambda_0}, \quad V^{(2)} = \frac{L}{\lambda + \lambda_0}, \tag{2.13}$$

therefore this have poles of order one, and K,L are independent of $\lambda$. If we substitute these equations in (2.7) than this equation is satisfied if and only if

$$K,_{\eta} - L,_{\zeta} = 0, \tag{2.14}$$

$$K,_{\eta} + L,_{\zeta} + \frac{1}{\lambda_0}(KL - LK) = 0, \tag{2.15}$$

This equation suggest to write K,L as

$$K = -\lambda_0 g,_{\zeta} g^{-1}, \quad L = \lambda_0 g,_{\eta} g^{-1} \tag{2.16}$$

Where g is some matrix. Hence (2.15) is the integrability condition of the former equation for g, we can then write

$$\left(g,_{\zeta} g^{-1}\right),_{\eta} + \left(g,_{\eta} g^{-1}\right),_{\zeta} = 0 \tag{2.17}$$

which is associated with the model of the principal chiral field.
For any solution of the L-A pair $\psi(\lambda, \rho, z)$, we obtain a solution of the former equation for g, Indeed from (2.7),(2.16)and the definition for $U^{(2)}, V^{(2)}$.

$$\psi,_{\zeta} \psi^{-1} = U^{(2)} = \frac{K}{\lambda - \lambda_0} = -\frac{\lambda_0 g_{\zeta} g^{-1}}{\lambda - \lambda_0} \longrightarrow g_{\zeta} g^{-1}, \tag{2.18}$$

$$\psi,_{\eta} \psi^{-1} = V^{(2)} = \frac{L}{\lambda - \lambda_0} = -\frac{\lambda_0 g_{\eta} g^{-1}}{\lambda + \lambda_0} \longrightarrow g_{\eta} g^{-1}, \tag{2.19}$$

Hence when $\lambda \longrightarrow 0$, we have that

$$g(\zeta, \eta) = \psi(\zeta, \eta, 0). \tag{2.20}$$

If we are interested in the solitonic solutions of (2.17), all we need to know is a particular exact solution $(g_0, \psi_0)$ of (2.17) and (2.6), which we call background solution or seed solution and the number of solitons we want to introduce in this background. We can express the $\psi(\zeta, \eta, \lambda)$ in the form

$$\psi = \chi \psi_0, \tag{2.21}$$

Where $\psi_0(\zeta,\eta,\lambda)$ is a known particular solution, and the new matrix $\chi$ is called dressing matrix. This matrix can be normalized in such a way that it tends to identity $\mathbb{I}$ as $\lambda \to \infty$. Then a suitable form for this matrix is

$$\chi = \sum_{k=1}^{n} \frac{\chi_k}{\lambda - \lambda_k}, \tag{2.22}$$

Where $\lambda_n$ are some constants and the matrices $\chi_k$ do not depend on $\lambda$. The number of poles for this equation is the number of soliton solutions that we added to the seed solution. The neat part is that by substituting the former equation into (2.21) and then into (2.6) we can obtain a new solution algebraically, in particular $\chi_n$.
Then from the last two equations and using $g(\zeta,\eta) = \psi(\zeta,\eta,0)$, we can express $g$ in terms of $g_0$.

$$g(\zeta,\eta) = \psi(\zeta,\eta,0) = \chi(\zeta,\eta,0)g_0 = g_0 - \left(\sum_{k=1}^{n} \frac{\chi_k}{\lambda_k}\right) g_0 \tag{2.23}$$

This procedure is called dressing technique and was first developed by Zakharov and Shabat. For the pure solitonic case it is straightforward to compute the new solutions from a given background solution.

# Chapter 3

# Inverse scattering in general relativity: the vacuum case

The inverse scattering construction for GR relies on the identification of a linear eigenvalue equation, whose integrability condition corresponds to the non-linear equations one aims to solve. We will describe the procedure developed by Belinski and Zakharov for the Einstein equations in vacuum, so we will work with pure General Relativity with action

$$I = \int R\sqrt{-g}\, d^4x$$

Einstein equation $G_{\mu\nu} = 0$ reduce to

$$R_{\mu\nu} = 0$$

The metric we consider is the one indicated in [1],[2],[3], in Weyl coordinates one has

$$ds^2 = f(\rho, z)\left(d\rho^2 + dz^2\right) + g_{ab}dx^a dx^b.$$

Such a metric is the most general stationary and axisymmetric spacetime: it is written in a form suitable for our purposes, and it contains the Lewis–Weyl–Papapetrou metric as a subcase. $f$ is a real function and $g$ is a $2 \times 2$ symmetric matrix. The lowercase Latin indices take value 0,1 that correspond to $t, \phi$.

Since the metric is invariant under diffeomorfisms, without loss of generality we can impose that

$$\det(g) = -\rho^2 \tag{3.1}$$

It is convenient to rewrite the vacuum Einstein equations in matrix form, in order to apply the inverse scattering formalism. The vacuum equations naturally split into two groups, one for the matrix $g$ and the other for the function $f$ indeed this is a case where the manifold can be expressed using the warp product with warp function $f$ [4]. We start by exploiting the first group.
In vacuum it can be easily shown, for example using Cartan's equation, that

$$R^t_t = 0, \;\; R^{\phi}{}_t = 0, \;\; R^t{}_{\phi} = 0, \tag{3.2}$$

These equations can be written in a more concise form as second order differential equations

$$\partial_\rho \left(\rho \partial_\rho g g^{-1}\right) + \partial_z \left(\rho \partial_z g g^{-1}\right) = 0 \tag{3.3}$$

On the other end equations

$$R_{\rho\rho} - R_{zz} = 0, \quad R_{z\rho},$$

determine the first order differential equations for $f$

$$\begin{aligned} \partial_\rho \log(f) &= -\frac{1}{\rho} + \frac{1}{4\rho} \operatorname{Tr}\left\{U^2 - V^2\right\} \\ \partial_z \log(f) &= \frac{1}{2\rho} \operatorname{Tr}\left\{U^2 - V^2\right\} \end{aligned} \tag{3.4}$$

With the $2 \times 2$ matrices

$$U = \rho(\partial_\rho g)g^{-1}, \;\; V = \rho(\partial_z g)g^{-1} \tag{3.5}$$

With this first step, we managed to rewrite the Einstein equations in matrix form. Such a form is particularly useful to recognize an integrability condition for the equations. The fact that $f$ completely decouples from the other equations means that we can forget about it and work solely on $g$.

## 3.1 Integration

Now we can use (3.3) and the matrices $U$ and $V$ to write

$$\partial_\rho U + \partial_z V = 0, \tag{3.6}$$

The second equation is obtained as the integrability condition of (3.5) with respect to g: with this, we mean a differential equation for U and V which is identically satisfied when it is written in terms of g. Thus we find

$$\partial_z U - \partial_\rho V + \frac{[U,V]}{\rho} + \frac{V}{\rho} = 0, \tag{3.7}$$

by substitution one can verify that the former equation is satisfied for any $g$ given (3.5). The key step of the inverse scattering procedure consists in representing the first order eqs. (3.6) and (3.7) as the compatibility conditions of an over-determined system of matrix equations related to an eigenvalue problem for some linear differential operator. Such a system will depend on a complex spectral parameter $\lambda$, independent of $\rho, z$, and the solutions of the original problem for $g, U$ and $V$ will be determined by the analytic structure of the eigenfunction in the complex $\lambda$-plane.
Now the integration scheme can begin. First we need to introduce the differential operators

$$D_1 = \partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda, \quad D_2 = \partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda, \tag{3.8}$$

These two operators commute, i.e.

$$[D_1, D_2] = 0. \tag{3.9}$$

Now is necessarily to introduce a complex matrix $\psi(\lambda, \rho, z)$, called generating matrix, and consider the system of equations

$$\begin{aligned} D_1\psi &= \frac{\rho V - \lambda U}{\lambda^2+\rho^2}\psi, \\ D_2\psi &= \frac{\rho U + \lambda V}{\lambda^2+\rho^2}\psi, \end{aligned} \tag{3.10}$$

The fundamental property of the system (3.10) is that its compatibility condition coincides exactly with Eqs. (3.6) and(3.7). This can be easily verified by applying $D_2$ to (3.10) and $D_1$ to (3.10), and then by subtracting the results: because of the commutativity of $D_1$ and $D_2$ , we get zero on the left hand side, while the right hand side is a rational function of $\lambda$. Requiring that all the coefficients of the various powers of $\lambda$ vanish, we get exactly Eqs.(3.6) and (3.7).
An important remark is that the system (3.10) gives, when $\lambda = 0$, the system,

$$U = \rho(\partial_\rho\psi)\psi^{-1},\ V = \rho(\partial_z\psi)\psi^{-1}, \tag{3.11}$$

comparing with (3.5) is obvious that $g = \psi$. This observation gives the fundamental property

$$g(\rho, z) = \psi(0, \rho, z), \tag{3.12}$$

Hence a solution of the system (3.10) not only guarantees that the equations satisfied by $U$ and $V$ are true, but also gives a solution $g$ of (3.5).
Now the neat part of the inverse scattering is that it can be used to produce new solution given a known one called "seed solution", we label the elements belonging to the seed solution with a sub script 0, i.e. $g_0$ and $f_0$ are known and represent the seed metric. It comes natural that a seed metric satisfies all the preceding equations.
We look for a solution of the form

$$\psi = \chi\psi_0, \tag{3.13}$$

where $\chi(\lambda, \rho, z)$ is called the dressing matrix. By inserting this ansatz into the eigensystem (3.10), we find the equations

$$\begin{aligned} D_1\chi &= \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho V_0 - \lambda U_0}{\lambda^2 + \rho^2}, \\ D_2\chi &= \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\chi - \chi\frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2}, \end{aligned} \tag{3.14}$$

Other constrains are needed to guarantee that $g$ is real and symmetric, we have to impose that

$$\chi^*(\lambda^*) = \chi(\lambda),\ \psi^*(\lambda^*) = \psi(\lambda), \tag{3.15}$$

so that $g$ is real

$$g = \chi(\lambda) g_0 \chi(-\rho^2/\lambda), \tag{3.16}$$

so that $g$ is symmetric. This condition is not trivial and follows from an invariance property of the system (3.14).Furthermore, we require

$$\chi(\infty) = \mathbb{I}, \tag{3.17}$$

This equation implies that

$$g = \chi(0) g_0, \tag{3.18}$$

So other than equations (3.14) one has to determine the dressing matrix that fulfills these supplementary conditions.
From last equations, given that one has for a general $g$ that $\det g = -\rho^2$, then

$$\det \chi(0) = 1.$$

However, the best strategy is to not take into account this problem during the procedure for the construction of the solution, and simply renormalise the final result in order to obtain the correct functions. We will call such correct functions the physical functions.

## 3.2 Generating a n-soliton solution

The solution for the matrix g that corresponds to the presence of pole singularities in the dressing matrix $\chi(\lambda, \rho, z)$, in the complex plane of the spectral parameters $\lambda$, is called the *n-soliton solution.* We define a genreal dressing matrix as

$$\boxed{\chi = \sum_{k=1}^{n} \frac{R_k(x)}{\lambda - \mu_k} + \sum_{k=0}^{m} \mu^k T^{(k)}(x)} \tag{3.19}$$

Where we will consider $T^{(0)} = \mathbb{I}$ and the other therms in $T$ as zero, these terms are called *poles at infinity* and will be analyzed in the section of the charged inverse scattering in chapter 7. Hence here only simple poles will be considered, and the dressing matrix takes form

$$\boxed{\chi = \mathbb{I} + \sum_{k=1}^{n} \frac{R_k(x)}{\lambda - \mu_k}} \tag{3.20}$$

where the matrices $\boldsymbol{R}_k$ and the functions µk depend only on $\rho$ and $z$. $\mu_k$ are the pole trajectories, i.e. the positions of the poles as functions of $(\rho, z)$.
Now we can substitute (3.20) into (3.10) and impose the constrains on the dressing matrix. These equations completely determines the matrices $\boldsymbol{R}_k$ and the pole trajectories $\mu_k$. The requirement that there must not be poles of order two for $\lambda = \mu_k$ in (3.10), gives the following differential equations.

$$\begin{aligned} \partial_z \mu_k &= -\frac{2\mu_k^2}{\mu_k^2 + \rho^2}, \\ \partial_\rho \mu_k &= \frac{2\rho\mu_k}{\mu_k^2 + \rho^2}, \end{aligned} \tag{3.21}$$

The solutions for this equations are

$$\begin{aligned} \mu_k &= \sqrt{\rho^2 + (z - w_k)^2} - (z - w_k), \\ \bar{\mu}_k &= -\sqrt{\rho^2 + (z - w_k)^2} - (z - w_k), \end{aligned} \tag{3.22}$$

Here $w_k$ are complex constants of integration, the so called *poles*. The first solution is called a *soliton*, the second one is called an *anti-soliton*. In the inverse scattering scheme it doesn't matter which of the two we choose, hence, for definiteness, we will always make use of the solitons.
Now the matrices $\boldsymbol{R}_k$ are degenerate and can be written as

$$(R_k)_{ab} = n_a^{(k)} m_b^{(k)}, \tag{3.23}$$

with $n_a^{(k)}$, $m_b^{(k)}$ as two components vectors. Now equation (3.15) gives the vectors $m_a^{(k)}$ by requiring that the equations are satisfied at the poles $\lambda = \mu_k$ then one finds

$$m_a^{(k)} = m_{0\,b}^{(k)} [\psi_0^{-1}(\mu_k, \rho, z)]_{ba}, \tag{3.24}$$

Where $m^{(k)}_{0\,b}$ are arbitrary constants. These vectors are called BZ vectors.
Now the other constrain (3.16) gives the $n^{(k)}_a$ vectors, which gives an algebraic system of equations

$$\sum_{l=1}^{n} \Gamma_{kl} n^{(k)}_a = \mu_k^{-1} m^{(k)}_c (g_0)_{ca}, \tag{3.25}$$

where $k, l = 1, \ldots, n$. The symmetric matrix $\Gamma_{kl}$ is given by

$$\Gamma_{kl} = \frac{m^{(k)}_c (g_0)_{ca} m^{(k)}_a}{\rho^2 + \mu_k \mu_l} \tag{3.26}$$

This gives the $n^{(k)}_a$ vectors, indeed

$$n^{(k)}_a = \sum_{l=1}^{n} (\Gamma^{-1})_{kl} \mu_l^{-1} m^{(k)}_c (g_0)_{ca}, \tag{3.27}$$

Now we can define

$$L^{(k)}_a = m^{(k)}_c (g_0)_{ca}, \tag{3.28}$$

Now we can compute $g$ indeed if we use the explicit form for $R_k$, and the relation between $g$ and $\psi$ i.e.

$$g = \psi(0) = \chi(0) g_0 = \left( \mathbb{I} - \sum_{k=1}^{n} R_k \mu_k^{-1} \right) g_0 \tag{3.29}$$

Hence the components of $g$ are

$$g_{ab} = (g_0)_{ab} - \sum_{k,l=1}^{n} (\Gamma^{-1})_{kl} \frac{L^{(k)}_a L^{(l)}_b}{\mu_k \mu_l} \tag{3.30}$$

Now $g$ to be a physical quantity the metric has to satisfy the relation

$$\det g = (-1)^n \rho^{2n} \left( \prod_{k=1}^{n} \mu_k^{-2} \right) \det g_0 \tag{3.31}$$

This formula implies that the number of solutions must be even, since $\det g_0 = -\rho^2$. Therefore, the stationary and axisymmetric solutions appear as even-soliton states. Now we impose (3.3) on $g^{(ph)}$, one can verify that

$$g^{(ph)} = \pm \rho \sqrt{-\det g}\, g \tag{3.32}$$

Which satisfies the condition on the determinant and (3.3). Hence by putting toghether the last two equations we get

$$g^{(ph)} = \pm\rho^{-n}\left(\prod_{k=1}^{n}\mu_k^{-2}\right)g \tag{3.33}$$

Now plugging this equation into (3.4), one gets

$$f^{(ph)} = 16C_f f_0 \rho^{-n^2/2}\left(\prod_{k=1}^{n}\mu_k^{-2}\right)^{n+1}\left[\prod_{k>l=1}^{n}(\mu_k-\mu_l)^{-2}\right]\det\Gamma_{kl} \tag{3.34}$$

Where 16 is put for later convenience and $C_f$ is an arbitrary constant.
Now that we have all the ingredients we can construct some soliton solutions given a seed metric: the first step is to plug the components of the seed metric in the system (3.10) and compute $\psi_0$, these are the only differential equations to solve, indeed the remaining computations are purely algebraic as can be seen by (3.33),(3.34). This trait is what gives the inverse scattering such power, it reduces the non linear Einstein equations to a linear system (3.10) and some algebraic computations.

# Chapter 4

# Applications of the inverse scattering

## 4.1 Kerr-NUT spacetime

For an example of application of the inverse scattering technique, let's take Minkowski spacetime, in cylindrical coordinates, as a seed:

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2 \tag{4.1}$$

Its easy to see that

$$\begin{aligned} g_0 &= \mathrm{diag}(1, -\rho^2), \\ f_0 &= 1, \end{aligned} \tag{4.2}$$

Thus the condition $\det g_0 = -\rho^2$ is satisfied. From (3.5) one finds that $U_0 = \mathrm{diag}(0, 2)$ and $V_0$ is a null matrix. Now matrix $\psi_0$ can be computed from the eigensystem (3.10), the result is that

$$\psi_0 = \begin{bmatrix} -1 & 0 \\ 0 & \rho^2 - 2\lambda z - \lambda^2 \end{bmatrix}, \tag{4.3}$$

Notice that $\psi_0(0, \rho, z) = g_0$. Now using relation (3.24) and evaluating at $\lambda = \mu_k$, we obtain the components of the BZ vectors

$$m_0^{(k)} = C_0^{(k)}, \quad m_1^{(k)} = C_1^{(k)} \mu_k^{-1} \tag{4.4}$$

Where $C_0^{(k)}$ and $C_1^{(k)}$ are arbitrary constants. Now the components for the $L_a^{(k)}$ vectors can be computed

$$L_0^{(k)} = -C_0^{(k)}, \quad L_1^{(k)} = C_1^{(k)} \mu_k^{-1} \rho^2.$$

Then the components for the $\Gamma$ matrix are:

$$\Gamma_{kl} = \frac{-C_0^{(k)} C_0^{(kl} + C_1^{(k)} C_1^{(l)} \mu_k^{-1} \mu_l^{-1} \rho^2}{\rho^2 + \mu_k \mu_l} \tag{4.5}$$

Now we have all the ingredients to build a solution on top of the Minkowski background. As we have states before we need to produce an even number of solitions (corresponding to the poles $\lambda = \mu_1$ and $\lambda = \mu_2$), in this case two, hence we will have two constants $w_1$ and $w_2$.
The constants can be defined as

$$w_1 = z_1 + \sigma, \;\; w_2 = z_1 - \sigma \tag{4.6}$$

Where $z_1$ represents the position of the Black Hole along the $z$ axis and $\sigma$ will be defined later. Now it's suitable to introduce a change of coordinates in order to simplify the poles, the new coordinates are

$$\rho = \sqrt{(r-m)^2 - \sigma^2} \sin\theta, \quad z = z_1 + (r-m)\cos\theta, \tag{4.7}$$

Where $m$ is a constant that will be specified later. Now we choose the soliton as a solution i.e. $\mu_k$ with the same sign. Note that due to the arbitrariness of $C_f$, both cases lead to the same metric. The two poles take form

$$\mu_1 = (r - m + \sigma)(1 - \cos\theta), \quad \mu_2 = (r - m - \sigma)(1 - \cos\theta), \tag{4.8}$$

Without loss of generality, we can impose the following two conditions on the constants $C_{0,1}^{(k)}$

$$\begin{aligned} C_1^{(1)} C_0^{(2)} - C_0^{(1)} C_1^{(2)} = \sigma, \quad C_1^{(1)} C_0^{(2)} + C_0^{(1)} C_1^{(2)} = -m, \\ C_0^{(1)} C_0^{(2)} - C_1^{(1)} C_1^{(2)} = n, \quad C_1^{(1)} C_1^{(2)} + C_0^{(1)} C_0^{(2)} = -a, \end{aligned} \tag{4.9}$$

The first two relations take advantage of the arbitrariness of the normalization $C_{0,1}^{(k)} \to \gamma^{(k)} C_{0,1}^{(k)}$. After such a transformation the metric $g$ does not change, hence the metric does not depend on $\gamma^{(k)}$. The relations involving $m$ is just the definition of the constant $m$. Physically $m$ is the "mass parameter" of the black hole. The other two relations define the new parameters $n$ and $a$. Physically $n$ is the "Taub-NUT parameter" of the black hole, while $a$ is the "rotational parameter". One can easily see by combining the previous equations one obtains

$$\sigma^2 = m^2 - a^2 + n^2, \tag{4.10}$$

which explicitly proves that we are introducing only three new parameters into the physical metric.
Now we have everything we need to compute the elements of the physical metric, the element $f^{(ph)}$ takes form

$$f^{(ph)} = C_f \frac{-r^2 + 2an\cos\theta - n^2(1+\cos^2\theta) + (\sigma^2 - m^2)\cos^2\theta}{(m-r)^2 - \sigma^2\cos^2\theta} \tag{4.11}$$

The change of coordinates reads

$$d\rho^2 + dz^2 = ((r-m)^2 - \sigma^2\cos^2\theta)([(r-m)^2 - \sigma^2]dr^2 + d\theta^2) \tag{4.12}$$

It's convenient to define a new pair of functions to simplify the metric

$$\Sigma = r^2 + (n - a\cos\theta)^2, \quad \Delta = r^2 - 2mr + a^2 - n^2. \tag{4.13}$$

The component for $g^{(ph)}$ are

$$g^{(ph)}_{tt} = -1 + \frac{2(n^2 + mr - an\cos\theta)}{\Sigma} \tag{4.14}$$

$$g^{(ph)}_{t\phi} = \frac{-2a(r-m)r\sin^2\theta + 2\cos\theta(-n + a\cos\theta)\Delta}{\Sigma} \tag{4.15}$$

$$\begin{aligned} g^{(ph)}_{\phi\phi} =& [2\Delta[2n^2 + r(m+3r) + an\cos\theta(-3+\cos^2\theta) + \cos^2\theta(r(r-m)+\Delta)] \\ &+ 4r^2\sin^2\theta\sigma^2 - (5+\cos^2\theta)\Sigma\Delta]/\Sigma \end{aligned} \tag{4.16}$$

Now this solution is not yet the well known Kerr-NUTT solution indeed one has to choose $C_f = -1$ and execute the transformation $t \to t + 2a\phi$, then one has Kerr-NUTT in Boyer-Lindquist coordinate:

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{\Delta(a\sin^2\theta + 2n\cos\theta)^2 - \sin^2\theta(r^2+a^2+n^2)^2}{\Sigma}d\phi^2 + \frac{4\Delta n\cos\theta - 4a\sin^2\theta(mr+n^2)}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \tag{4.17}$$

## 4.2 Schwarzschild black hole

To obtain Schwarzschild black hole, the static and stationary spherically symmetric black hole, we can use what we have done for the Kerr-NUT black hole, switching off the "rotation parameter" and "NUT parameter": $a = b = 0$. Then $\sigma = m$, $\rho$ and $z$ coordinates, expressed in the polar ones are

$$\rho = \sqrt{r^2 - 2mr}\sin\theta, \quad z = z_1 + (r-m)\cos\theta. \tag{4.18}$$

And the solitons take form

$$\mu_1 = r(1-\cos\theta), \quad \mu_1 = (r-2m)(1-\cos\theta). \tag{4.19}$$

Now the constants are

$$\begin{aligned} C_1^{(1)}C_0^{(2)} - C_0^{(1)}C_1^{(2)} &= m, \quad C_1^{(1)}C_0^{(2)} + C_0^{(1)}C_1^{(2)} = m, \\ C_0^{(1)}C_0^{(2)} - C_1^{(1)}C_1^{(2)} &= 0, \quad C_1^{(1)}C_1^{(2)} + C_0^{(1)}C_0^{(2)} = 0, \end{aligned} \tag{4.20}$$

Now since these equations are scale invariant we can choose

$$\begin{aligned} C_1^{(1)}C_0^{(2)} - C_0^{(1)}C_1^{(2)} &= 1, \quad C_1^{(1)}C_0^{(2)} + C_0^{(1)}C_1^{(2)} = -1, \\ C_0^{(1)}C_0^{(2)} - C_1^{(1)}C_1^{(2)} &= 0, \quad C_1^{(1)}C_1^{(2)} + C_0^{(1)}C_0^{(2)} = 0, \end{aligned} \tag{4.21}$$

It follows that $C_0^{(2)} = C_1^{(1)} = 0$ and $C_1^{(2)} = 1$, $C_0^{(1)} = -1$

$$L_0^{(1)} = -1, \quad L_0^{(2)} = 0, \quad L_1^{(1)} = 0, \quad L_1^{(2)} = -\frac{\rho^2}{\mu_2} \tag{4.22}$$

Now using (3.34) and using $(r,\theta)$ coordinates

$$f^{(ph)}(r,\theta) = -C_f\frac{r^2}{(r-m)^2 - m^2\cos^2\theta} \tag{4.23}$$

The change of coordinates in the metric is like the one in the previous section but with the new parameters, multiplying it by $f^{(ph)}$, one has

$$f^{(ph)}(d\rho^2 + dz^2) = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2, \tag{4.24}$$

as before $C_f$=-1, now using (3.33)

$$g_{tt}^{(ph)} = \left(1 - \frac{2m}{r}\right) \tag{4.25}$$

$$g_{\phi t}^{(ph)} = 0 \tag{4.26}$$

$$g_{\phi\phi}^{(ph)} = r^2 \sin^2\theta \tag{4.27}$$

Hence we have back Scharzschild black hole solution

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4.28}$$

# Chapter 5

# Rindler and accelerating spacetimes

## 5.1 Rindler spacetime

Given Minkowski spacetime with the coordinates that takes real values from $(-\infty, \infty)$.

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + d\zeta^2 \tag{5.1}$$

in the region $z > |t|$ we can applying the following coordinate transformation

$$\begin{cases} \tau = z \sinh t \\ \zeta = z \cosh t \end{cases} \tag{5.2}$$

The metric we obtain in a Rindler metric of the form

$$ds^2 = -z^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{5.3}$$

with $z \in (0, +\infty)$ and $t \in (-\infty, \infty)$

To cover the region $z < |t|$ it's we can use

$$\begin{cases} \tau = -z \sinh t \\ \zeta = -z \cosh t \end{cases} \tag{5.4}$$

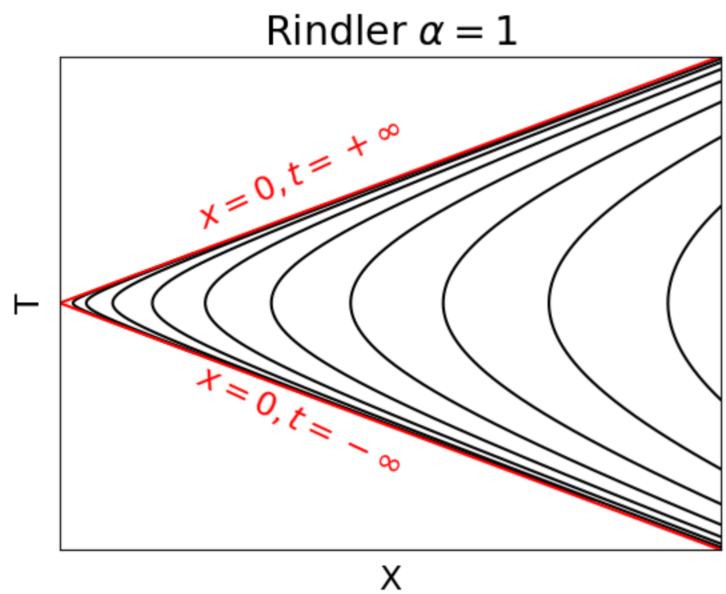


Figure 5.1: Right Rindler Wedge for the region $z > |t|$, each hyperbole (black) corresponds to an equation of the form $z^2 - t^2 = const.$, the red line represent the limit as time approaches infinity.

The curves with constant $x, y, z$ are hyperbole on which $z^2 - t^2$ =const. for each region, while curves with constant $t$ are straight lines passing through the origin of the axes.
Rindler space time is just a different representation of the parts of Minkowski space time with $|z| > |t|$. Curve on which the coordinates $x, y, z$ are constants are time like therefore we can choose $z = \alpha^{-1}$. The trajectories followed by these particles are

$$\tau = \pm\frac{1}{\alpha}\sinh t, \quad \zeta = \pm\frac{1}{\alpha}\cosh t$$

where one has to take same sign for both coordinates. Introducing the proper time $s = \alpha^{-1}t$ the 4-velocity of these test particles is $u^\mu = (\pm\cosh t, 0, 0 \pm \sinh t)$ such that $u_\mu u^\mu = -1$, the 4-acceleration is then, $a^\mu = du^\mu/ds = (\pm\sinh t, 0, 0, \alpha\cosh t)$, then $a^\mu a_\mu = \alpha^2$.
Hence such particles have constant acceleration $\alpha$ in the positive or negative z direction, as illustrated in figure 5.1, even if points with different values of z have a different values of their accelerations. The metric (5.3) is known as uniformly accelerated metric or, as already said, as Rindler metric. Rindler spacetime metric will be often used as background starting metric in the cases where we will want to generate black hole solutions of Einstein equation with an acceleration parameter.

Another way to find this solution is a Schwarzischild near horizon approximation. In the situation in which we apply the symmetry $\theta \longrightarrow \theta - \pi$ so that we can approximate $\sin\theta \approx 1$, we also apply the near horizon condition i.e. $r \approx 2m$ so that we can write the metric coefficient in Laurent series in r, i.e.

$$ds^2 = -\frac{r - 2m}{2m}dt^2 + \frac{2m}{r - 2m}dr^2 + 4m^2 d\Omega^2_{\theta=\pi/2} \tag{5.5}$$

Now after the coordinate transformation

$$\begin{cases} t = \alpha 4m^2 t' \\ \rho = \sqrt{2m}\sqrt{r - 2m} \end{cases} \tag{5.6}$$

The metric becomes clearly a Rindler type solution

$$ds^2 = -\alpha^2\rho^2 dt'^2 + d\rho^2 + 4m^2\left(d\theta^2 + d\phi^2\right) \tag{5.7}$$

if we re-scale the coordinates $\theta \to \theta/2m, \phi \to \phi/2m$, the resembles becomes uncanny.

$$ds^2 = -\alpha^2 \rho^2 dt'^2 + d\rho^2 + d\theta^2 + d\phi^2 \tag{5.8}$$

Therefore the situation in which a massive test particle finds itself near the horizon of a Schwarschild black hole, the metric can be approximated to a Rindler spacetime. In this case the particle is doomed to fall into the black hole, therefore undergoes an acceleration towards the event horizon and the interpretation of Rindler's metric is clear: it describes an accelerating observer.

## 5.2 C-Metric

From Rindler space time one can show that it's possible to create a generalization of the Schwarzschild black hole with an extra parameter related to the acceleration of the black hole. The metric is diagonal and quite simple, the line element takes form

$$ds^2 = \frac{1}{\alpha(x+y)^2}\left(-F dt^2 + \frac{dy^2}{F} + \frac{dx^2}{G} + G d\phi^2\right) \tag{5.9}$$

with

$$G = \left(1 - x^2\right)(1 + 2\alpha m x), \quad F = -\left(1 - y^2\right)(1 - 2\alpha m y) \tag{5.10}$$

This solution is a simplification given by Hong and Theo in [5] To maintain a Lorentzian signature of(5.9) , it is necessary that $G > 0$, i.e. that the coordinate $x$ must lie between appropriate roots of $G$, these roots are $x = \pm 1, 1/2m\alpha$. Instead, there is no constraint on the sign of $F$ . It can be seen from that $x + y = 0$ corresponds to conformal infinity, hence physical spacetime must satisfy either $x + y < 0$ or $x + y > 0$, in any case we will consider the case in which $x + y > 0$, since in this case the metric describes a space time with accelerating black holes.
This solution is quite peculiar because of the so called C-singularities, that can be found using spherical coordinates, i.e. by the change of coordinates

$$x = \cos\theta, \quad y = \frac{1}{\alpha r} \tag{5.11}$$

with $\theta \in [0, \pi]$ and $t \rightarrow \alpha t$. The metric than becomes

$$ds^2 = \frac{1}{(1 - \alpha r \cos\theta)^2}\left(-Qdt^2 + \frac{dr^2}{Q} + \frac{r^2}{P}d\theta^2 + Pr^2 \sin^2\theta d\phi^2\right), \tag{5.12}$$

With

$$P = 1 - 2\alpha m \cos\theta, \quad Q = \left(1 - \frac{2m}{r}\right)(1 - \alpha^2 r^2) \tag{5.13}$$

It's clear from this interpretation that the metric reduces to Schwarzschild when $\alpha = 0$. Also for this form of the C-metric it cam be seen that there is a singularity for $r = 2m$. We still have to interpret the parameter $\alpha$, assumed to be positive, and the horizon at $r = \alpha^{-1}$ . It will be argued that this solution describes an accelerating black hole in which $\alpha$ can be interpreted as the acceleration.

Now considering the regularity around the axis of symmetry and a small circle around the poles i.e. $\theta = 0, \pi$, one gets

$$\lim_{\theta \to 0} \frac{1}{\theta}\sqrt{\frac{g_{\phi\phi}}{g_{\theta\theta}}} = 2\pi C(1 - 2\alpha m) \neq 2\pi C(1 + 2\alpha m) = \lim_{\theta \to \pi} \frac{1}{\pi - \theta}\sqrt{\frac{g_{\phi\phi}}{g_{\theta\theta}}} \tag{5.14}$$

Since these limits are not $2\pi$, we say that there is a *conical singularity*, it's clear that by choosing the parameter C adequately one of the two angles can be normalized to $2\pi$ but the other one will remain different from this value. An embedding can be seen in figure

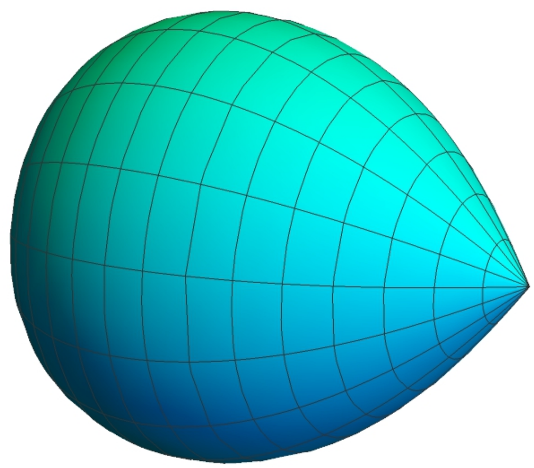

Figure 5.2: Embedding of the C-Metric Black hole in $\mathbb{E}^3$ with $t, r$ constants, the sharp edge correspondent to the conical singularity at $\theta = 0$ can clearly be seen, while for $\theta = \pi$ is regular, this is a consequence of the normalization of the C constant given in (5.14)

# Chapter 6

# Generating accelerating black holes

The question arises, can the C-metric be generalized? The answer is positive and can be done with the means of the inverse scattering technique, indeed has already been shown that by taking a peculiar form of Rindler space time one can build the C-metric and a rotating C-metric as shown in [7].

## 6.1 Rotating C-metric

The procedure starts by using Rindler as a background in cylindrical coordinates as in [6]. The metric takes form

$$ds^2 = -\mu_0 dt^2 + \frac{\mu_0}{\rho^2 + \mu_0^2}\left(d\rho^2 + dz^2\right) + \frac{\rho^2}{\mu_0}d\phi^2 \tag{6.1}$$

$\mu_0$ is the solition associated to the acceleration parameter $\alpha$. It's easy to identify the components of the metric with as $g_0 = \mathrm{diag}(-\mu_0, \rho^2/\mu_0)$, $\det g_0 = -\rho^2$ and $f_0 = \frac{\mu_0}{\rho^2+\mu_0^2}$.
Now the important part is to find the equation for $\psi_0$, by using (3.10) the explicit solution can be found to be

$$\psi_0 = \begin{bmatrix} \lambda - \mu_0 & 0 \\ 0 & \lambda + \frac{\rho^2}{\mu_0} \end{bmatrix}, \tag{6.2}$$

The BZ vectors are

$$m^{(k)} = \left( \frac{C_0^{(k)}}{\mu_k - \mu_0}, C_1^{(k)} \frac{\mu_0}{\rho^2 + \mu_k \mu_0} \right) \tag{6.3}$$

here $C_0^{(k)}$ and $C_1^{(k)}$ are constants. Now we proceed to add two soliton to the Rindler background and recover a rotating C-metric as in [5]. The cumbersome part is to choose an appropriate set of $C_0^{(k)}$ and $C_1^{(k)}$, or the solution can become quite involved. By choosing

$$\begin{aligned} &C_1^{(1)} C_0^{(2)} - C_0^{(1)} C_1^{(2)} = -\sqrt{m^2 - a^2}, \quad C_1^{(1)} C_0^{(2)} + C_0^{(1)} C_1^{(2)} = m \frac{1 - a^2\alpha^2}{1 + a^2\alpha^2}, \\ &C_0^{(1)} C_0^{(2)} - C_1^{(1)} C_1^{(2)} = \frac{2am\alpha}{1 + a^2\alpha^2}, \quad C_1^{(1)} C_1^{(2)} + C_0^{(1)} C_0^{(2)} = a, \end{aligned} \tag{6.4}$$

The solution simplifies quite a lot, and by taking the limit for $\alpha \to 0$, the form of the equations for the constants is equal to the one for Kerr-NUT. Another problem is given by the coordinates $(\rho, z)$ that give quite a difficult solution to read, in [7], a change of coordinates is given to recover a rotating C-metric in the form [5]. Indeed is useful to firstly use a set of coordinates $(u, v)$ defined by

$$\rho = \frac{2\kappa\sqrt{(1 - u^2)(v^2 - 1)(1 + \nu u)(1 + \nu v)}}{(u - v)^2}, \quad z = \frac{\kappa(1 - uv)(2 + \nu u + \nu v)}{(u - v)^2} \tag{6.5}$$

with poles

$$w_1 = -\nu\kappa^2, \quad w_2 = \nu\kappa^2, \quad w_0 = \kappa^2, \tag{6.6}$$

with the parameters $\nu, \kappa$ that have to respect the inequalities $w_1 < w_2 < w_0$, then to work with the coordinates $(x, y)$, we perform the Möbius transformation

$$u = \frac{x + d}{1 + dx}, \quad v = \frac{y + d}{1 + dy}, \quad \nu = \frac{c - d}{1 - cd}, \tag{6.7}$$

Where $c, d$ are two new real parameters. Now to have a solution as in [5] we choose

$$c = \alpha \left( m + \sqrt{m^2 - a^2} \right), \quad d = \alpha \left( m - \sqrt{m^2 - a^2} \right) \tag{6.8}$$

$$\kappa^2 = \frac{1 - a^2\alpha^2}{2\alpha^2}, \quad C_f = -\frac{1 + a^2\alpha^2}{\alpha^6}. \tag{6.9}$$

with these coordinate the metric becomes

$$\begin{aligned} ds^2 =& \frac{1}{\alpha^2(x-y)^2}\Big[\frac{G(y)}{1+a^2\alpha^2x^2y^2}(dt - a\alpha x^2 d\phi)^2 - \frac{1+a^2\alpha^2x^2y^2}{G(y)}dy^2 \\ &+ \frac{1+a^2\alpha^2x^2y^2}{G(x)}dx^2 + \frac{G(x)}{1+a^2\alpha^2x^2y^2}(d\phi + a\alpha y^2 dt)^2\Big] \end{aligned} \tag{6.10}$$

using

$$G(\xi) = \left(1-\xi^2\right)(1+r_+\alpha\xi)(1+r_-\alpha\xi) \tag{6.11}$$

Where $r_\pm$ are the usual Kerr horizons, i.e. $r_\pm = m \pm \sqrt{m^2 - a^2}$.
Now from this solution, by setting $a = 0$, the C-Metric can be retrieved in the same form as in (5.9). By the change of coordinates

$$x = \cos\theta, \quad y = \frac{1}{\alpha r} \tag{6.12}$$

and by setting $\alpha = 0$ Kerr solution is found.
So far there is no trace of the NUT parameter in the generation of the solution, indeed, to our knowledge, there isn't yet an accelerating solution, containing NUT parameter, generated by the inverse scattering method. This solution would be quite mesmerizing because it would be a type-D solution belonging to the Plebański-Demiański class [8], which would be a novelty, because so far all accelerating solution with NUT parameter belong to the type I-G [9] [10], these solutions contain some extra features in the background metric, such as charge or rotation, but they have some peculiar behavior.

## 6.2 Accelerating Kerr-NUT

The inverse scattering method reviewed so far is quite useful to generate solution, the only drawback is that the constants $C_0^{(k)}$, $C_1^{(k)}$ and the coordinates have to be chosen carefully or the solution becomes unreadable. To generate a general type of accelerating black hole, we start again from Rindler metric

$$g_0(\rho,z) = \begin{bmatrix} -\mu_0 & 0 \\ 0 & \rho^2/\mu_0 \end{bmatrix}, \quad f_0(\rho,z) = \mu_0/(\rho^2+\mu_0^2) \tag{6.13}$$

Then the components of the new metric can be obtained by (3.33) and (3.34). The BZ vectors read

$$m^{(k)} = \left( \frac{C_0^{(k)}}{\mu_k - \mu_0}, \frac{C_1^{(k)}}{\mu_k + \rho^2/\mu_0} \right) \tag{6.14}$$

Now the constants that we chose are

$$C_0^{(1)} = 1, \qquad C_1^{(1)} = \frac{m - \alpha m^2 + \sigma + \alpha\sigma^2}{(a+n)(1+\alpha(m+\sigma))}, \tag{6.15}$$

$$C_0^{(2)} = \frac{(a+n)(1+\alpha(m+\sigma))}{2(1+a^2\alpha^2(m^2-\sigma^2))}, \; C_1^{(2)} = \frac{1}{2}\left(-\sigma + \frac{m(1-\alpha^2(m^2-\sigma^2))}{1+\alpha^2(m^2-\sigma^2)}\right), \tag{6.16}$$

The task to find these constant is quite cumbersome because there is no defined method to find them, so one has to try the combination that best suits them in order to find the desired solution in a decent form. Another problem is that of defining the coordinates that best describes the solution, to do so we perform a coordinate change similar to the one done in the previous section, with $r$ as a radial coordinate and $x = \cos\theta$, with $\theta$ as the polar angle, the coordinate transformation is

$$\rho = \frac{\sqrt{\Delta_r \Delta_x}}{(1+\alpha r x)^2}, \quad z = \frac{(\alpha r + x)[(r-m)(1+\alpha m x) + \alpha x \sigma^2]}{(1+\alpha r x)^2} \tag{6.17}$$

With

$$\Delta_r = (1-\alpha^2 r^2)[(r-m)^2 - \sigma^2], \quad \Delta_x = (1-x^2)[(1+\alpha m x)^2 - \alpha^2 x^2 \sigma^2] \tag{6.18}$$

The poles take form

$$w_1 = -\sigma, \quad w_2 = \sigma, \quad w_0 = \frac{1-\alpha^2(m^2-\sigma^2)}{2\alpha}, \quad C_f = -\frac{1-\alpha^2(n^2-a^2)}{\alpha^3(1+\alpha^2 a^2)} \tag{6.19}$$

The solution has now form

$$ds^2 = -f(r,z)[dt - \omega(r,x)d\phi]^2 + \frac{1}{f(r,z)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{\Delta_x}\right) + \rho(r,x)d\phi^2\right], \tag{6.20}$$

With

$$\Omega = 1 + \alpha x(\alpha x(r^2 - 2a^2) - 4r) \tag{6.21}$$

$$f = \frac{(1+\alpha r x)^{-2}\{[1+\alpha^2(n^2-a^2)x^2]^2\Delta_r - [a+2\alpha n r + a\alpha^2 r^2]^2\Delta_x\}}{\{\alpha^2 n^4 x^2 + 2an(\alpha r - x)(1-\alpha r x) + (1+\alpha^2 a^2)(r^2+a^2x^2) + n^2\Omega\}} \tag{6.22}$$

$$\omega = \frac{(a - 2nx + ax^2)[1-(b^2-a^2)x^2]\Delta_r + (r^2+n^2-a^2)(a+2\alpha n r + \alpha^2 a r^2)\Delta_x}{[1+\alpha^2(n^2-a^2)x^2]^2\Delta_r - [a+2\alpha n r + a\alpha^2 r^2]^2\Delta_x}, \tag{6.23}$$

$$\gamma = \frac{1}{2}\log\left[C_f \frac{\left(1+\alpha^2(n^2-a^2)x^2\right)^2\Delta_r - \left(a+2\alpha n r + a\alpha^2 r^2\right)^2\Delta_x}{(1+\alpha r x)^4}\right]. \tag{6.24}$$

By using Mathematica this solution can be verified to be indeed of type D, a brief introduction to Petrov types and how to do the calculations can be seen in A. Notably this solutions allows to get clear limits to all the possible physical sub-cases, this is a strength compared to the type I solution in [8], notably including the type D accelerating Taub-NUT, spacetime, which it was previously unknown. Indeed if we take set the parameter $a = 0$ we remain with a C-Metric-NUT. Which is also a Type D solution. Even more, if all the parameters are set to zero except for the acceleration one gets Rindler spacetime, this is something that doesn't happened with type I solutions as they have an extra charge on the background,see [11].

Another feature is that the C-Metric can be clearly obtained as the limit for $n \to 0$ and $a \to 0$. KerrNUT can be obtained as the limit $\alpha \to 0$ up to an adjustment of the frame of reference described by a trivial change $t \to t{+}2a\phi$. An important remark to be made is that all the limits of the parameters that we do in this metric are commutative, for example one can take the limit as $\alpha \to 0$ first and then $n \to 0$ or vice versa. This is not the case in general for accelerating spacetimes, in fact the accelerating Kerr-NUT metrics known so far in the literature, this was not the case. For instance turning off the angular momentum in previous metrics would also turn off the acceleration, but the vice-versa was not true. That was the reason why it was not possible to obtain the accelerating-Taub-NUT spacetime from previous accelerating Kerr-NUT solutions.

In [11] it's shown that this solution can be found by generating a double Kerr-NUT solution and taking the limit of a pole to infinity much like what is done

to find Rindler spacetime or the C-Metric. The idea, is indeed the same, in the case of two black hole we have 4 poles $w_1, w_2, w_3, w_4$, now keeping $w_1, w_2$ as they are and taking carefully the limit $w_4 \to \infty$ while keeping $w_3$ as a constant, this limit to infinity actually pose a pole to infinity.
The question arises, can this background be generate by merely modifying the dressing matrix so that it has a pole at infinity? Are there more general solutions for a background? What it's physical significance?

# Chapter 7

# Charged inverse scattering

## 7.1 General form of the dressing matrix

In order to look for this poles at infinity the most general and suitable technique to be used is charged inverse scattering with a modified dressing matrix. We will follow the path laid by [12] and [13], and give a brief layout of what their method consists of.

First off we have to introduce a general form for L-A pairs for Einstein-Maxwell equations.

The eigenfunction for the ISM is now called $\psi$ and is a $3\times 3$ matrix and the operator $A$ of chapter 2 is represented by $\Lambda_\mu{}^\nu U_\nu$, hence

$$\partial_\mu \varphi = \Lambda_\mu{}^\nu U_\nu \varphi \tag{7.1}$$

Now we can exploit the method of generating a new solution given a seed metric. Using the dressing method $\varphi = \chi\varphi^{(0)}$, in particular Alekseev dressing matrix

$$\boxed{\chi = \sum_{i=1}^{n} \frac{R_i(x)}{\omega - \omega_i} + \sum_{k=0}^{m} \omega^k A^{(k)}(x)} \tag{7.2}$$

We will than focus on the monopole expansion

$$\boxed{\chi = \frac{R_1(x)}{\omega - \omega_1} + A^{(0)}(x) + \omega A^{(1)}(x)} \tag{7.3}$$

The first part of the procedure is quite general and consists in the analysis of the Einstein-Maxwell equations. The second part is specific to the type of dressing matrix that is being used.

### 7.1.1 Analysis of Einstein-Maxwell equations

The metric is of the form

$$ds^2 = f\eta_{\mu\nu}dx^\mu\, dx^\nu + g_{ab}dx^a dx^b \tag{7.4}$$

where the coefficients depend only on the coordinates $x^\mu \equiv (\rho, z)$ and the metric components $\eta_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Another condition is that

$$\det(g_{ab}) = -\alpha^2\,(x^\mu)$$

For the electromagnetic potential we have that

$$A_\mu = 0, \quad A_{a,b} = 0, \quad A_{a,\mu} \neq 0$$

Given the special form of (7.4) Einstein-Maxwell equations can be divided in three parts

$$\begin{aligned} R_a^b &= 2\left(F_{\lambda a}F^{\lambda b} - \frac{1}{2}\delta_a^b F_{\lambda c}F^{\lambda c}\right), \\ R_\nu^\mu &= 2\left(F_{\nu c}F^{\mu c} - \frac{1}{2}\delta_\nu^\mu F_{\lambda c}F^{\lambda c}\right), \\ (f\alpha F^{\mu a})_{,\mu} &= 0. \end{aligned} \tag{7.5}$$

The Greek indices equations hold interesting properties, indeed by contracting the second equation above we obtain

$$R_\mu^\mu = 0, \quad R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R_\lambda^\lambda = 2\left(F_{\nu c}F^{\mu c} - \frac{1}{2}\delta^m u_n u F_{\lambda c}F^{\lambda c}\right).$$

By direct calculation the first equation holds

$$\eta^{\mu\nu}\left((\ln f)_{,\mu\nu} + (\ln\alpha)_{,\mu\nu} + \frac{1}{4}g^{ab}g^{cd}g_{bc,\mu}g_{da,\nu}\right) = 0$$

This equation will be satisfied by Bianchi identity so in not important and we can forget about it from now on.
The second equation is quite cumbersome and it is convenient to write it down later, the peculiarity of this equation is that it doesn't have second derivatives of $f$. So these two equation are used to calculate $f$ by quadrature, indeed there equations are obtained from constrains applied to the field equations. The task is now to find the other metric coefficients and the electromagnetic potentials. Thus, we turn our attention to the remaining equations of (7.5), this system of equation does not contain $f$. The explicit form of these equations is

$$\begin{aligned} &\eta^{\mu\nu}\alpha^{-1}\left(\alpha g^{bc}g_{ac,\mu}\right)_{,\nu} = \eta^{\mu\nu}\left(-4g^{bc}A_{a,\mu}A_{c,\nu} + 2\delta^b_a g^{cd}A_{c,\mu}A_{d,\nu}\right), \\ &\eta^{\mu\nu}\left(\alpha g^{ac}A_{c,\mu}\right)_{,\nu} = 0. \end{aligned} \tag{7.6}$$

By contracting the indices $a$ and $b$ is easy to obtain the following equation

$$\eta^{\mu\nu}\alpha_{,\mu\nu} = 0.$$

Of course $\alpha$ is a given solution of this equation. The second functionally independent solution can be obtained by quadrature of

$$\beta_{,\mu} = \eta_{\mu\rho}\varepsilon^{\rho\sigma}\alpha_{,\sigma} \tag{7.7}$$

with

$$\varepsilon^{\mu\nu} = \varepsilon_{\mu\nu} = \varepsilon^{ab} = \varepsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

To simplify our calculation is useful to define an auxiliary potential $B_a$ that does not appear in the final results.

$$B_{a,\mu} = -\alpha^{-1}\eta_{\mu\nu}\varepsilon^{\nu\lambda}g_{ab}\varepsilon^{bc}A_{c,\lambda}$$

This equation satisfies $\varepsilon^{\mu\nu} B_{a,\mu\nu} = 0$, that is the integrability condition, and coincides with the second equation of (7.6) i.e. Maxwell equation. The inverse relation of the previous equation is simple to obtain

$$A_{a,\mu} = \alpha^{-1} \eta_{\mu\nu} \varepsilon^{\nu\lambda} g_{ab} \varepsilon^{bc} B_{c,\lambda}$$

These two equations combined form the imaginary and real part, respectively, of the derivative a complex potential

$$\Phi_a = A_a + i B_a. \tag{7.8}$$

i.e.

$$\Phi_{a,\mu} = -i\alpha^{-1} \eta_{\mu\nu} \varepsilon^{\nu\lambda} g_{ab} \varepsilon^{bc} \Phi_{c,\lambda} \tag{7.9}$$

From which the Maxwell equation (last equation of (7.5)) for $\mathbf{\Phi}$ can be obtained

$$\eta^{\mu\nu} \left( \alpha g^{ab} \Phi_{b,\mu} \right)_{,\nu} = 0 \tag{7.10}$$

By direct calculation the first of equations (7.5) can be written as

$$\eta^{\mu\nu} \alpha^{-1} \left( \alpha g^{bc} g_{ac,\mu} \right)_{,\nu} = -2\eta^{\mu\nu} g^{bc} \Phi^*_{a,\mu} \Phi_{c,\nu} \tag{7.11}$$

Where $*$is the complex conjugation. It is obvious from the definition of $B_a$ that the r.h.s. of the previous equation is real. Now given $g_{ab}$ one has that $A_a = \mathrm{Re}(\Phi_a)$ is a solution of the E-M equations.

Now E-M equations can be written in a more compact form using 3-dimensional matrices, the components of these matrices will be enumerated using Latin capital letters, i.e. A,B,... each with index value 1,2,3.

The basic object for such representation is an Hermitian matrix $\mathbf{G}$, of the form

$$\boldsymbol{G} = \begin{bmatrix} 4g_{22} + 4\Phi_2\Phi_2^* & -4g_{12} - 4\Phi_2\Phi_1^* & 2\Phi_2 \\ -4g_{12} - 4\Phi_1\Phi_2^* & 4g_{11} + 4\Phi_1\Phi_1^* & -2\Phi_1 \\ 2\Phi_2^* & -2\Phi_1^* & 1 \end{bmatrix} \tag{7.12}$$

The determinant of this matrix is $\det \mathbf{G} = 16(g_{11}g_{22} - g_{12}^2)$. The inverse is

$$\boldsymbol{G}^{-1} = \begin{bmatrix} \frac{g^{22}}{4} & -\frac{g^{12}}{4} & -\frac{\Phi^2}{2} \\ -\frac{g^{12}}{4} & \frac{g^{11}}{4} & \frac{\Phi^1}{2} \\ -\frac{\Phi^{*2}}{2} & \frac{\Phi^{*1}}{2} & 1 + \Phi^{*a}\Phi_a \end{bmatrix} \tag{7.13}$$

Now is useful to introduce a matrix

$$Y = \begin{bmatrix} 1 & 0 & 2\Phi_2 \\ 0 & 1 & -2\Phi_1 \\ 0 & 0 & 2 \end{bmatrix}$$

So that the Einstein-Maxwell equations (7.5), together with condition (7.9) give a simple matrix form

$$\eta^{\mu\nu}(\alpha \boldsymbol{Y}\boldsymbol{G}_{,\mu}\boldsymbol{G}^{-1})_{,\nu} = 0 \tag{7.14}$$

Then by defining

$$K_\mu = (\alpha Y G_{,\mu} G^{-1}) \tag{7.15}$$

the equations (even the one in f) acquire a simple form

$$\eta^{\mu\nu}\boldsymbol{K}_{\mu,\nu} = 0 \tag{7.16}$$

$$\ln(f)_{,\mu} = \ln\left(\frac{D}{\alpha}\right)_{,\mu} + \frac{\ln(\alpha)_{,\lambda}}{4D}\left[2\eta^{\lambda\sigma}\,\mathrm{Tr}\left\{\boldsymbol{K}_\mu\boldsymbol{K}_\sigma\right\} - \delta^\lambda_\mu\eta^{\rho\sigma}\,\mathrm{Tr}\left\{\boldsymbol{K}_\rho\boldsymbol{K}_\sigma\right\}\right] \tag{7.17}$$

where $D = \eta^{\mu\nu}\alpha_{,\mu}\alpha_{,\nu}$, this is the second equation for $f$ that we mentioned before. The integrability of the last equation i.e. $\varepsilon^{\mu\nu}(\ln f)_{,\mu\nu} = 0$ is satisfied automatically if the previous equations for the metric coefficients $g_{ab}$ and potentials $\boldsymbol{\Psi}_a$ are satisfied.

### 7.1.2 Building the solution

From the last paragraph we see that the matrix $\boldsymbol{G}$ holds all the information about a solution of the Einstein-Maxwell equations, this matrix satisfies not only (7.14), but also has to be hermitian. Correspondingly we need to construct a three-dimensional linear spectral problem the self-consistency conditions of which provide for both of these properties. This task is resolved by the following spectral equation for three-dimensional matrix $\varphi$ that satisfies

$$\varphi_{,\mu} = \frac{(w+\beta)\delta^\nu_\mu - \alpha\eta_{\mu\rho}\epsilon^{\rho\nu}}{2i((w+\beta)^2+\alpha^2)}U_\nu\varphi, \tag{7.18}$$

where $w$ plays the role of a complex spectral parameter independent of the coordinates. This equation is first derived in [13], in the same paper is defined another auxiliary $3\times3$ matrix $U_\mu$, depending only on the two coordinates $x^\mu$ that helps to define the spectral equations. This matrix takes form

$$U_\mu = -i\alpha\eta_{\nu\rho}\varepsilon^{\rho\sigma}G^{-1}G_{,\sigma}Y^\dagger - 4\left(\alpha^2 G^{-1}\right)_{,\nu}\Omega, \tag{7.19}$$

The matrix $\Omega$ is defined as

$$\Omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.20}$$

By direct calculation one can show that the integrability conditions for the equation (7.18) i.e. $\varphi_{,\mu\nu} = \varphi_{,\nu\mu}$ coincide with equation (7.14), then, as already said, due to the structure of $G$ and $Y$, this equation gives back Einstein-Maxwell equations.

Then, for how the problem has been described, any solution for $\varphi$ automatically contains solution for Einstein-Maxwell equations encoded in matrix $G$. Now it's mandatory to find a way to derive $G$ from $\varphi$.

The inverse scattering dressing procedure implies that we need to know some seed solution $g_{ab}^{(0)}$, $\Phi_a^{(0)}$of the Einstein-Maxwell equations for metric coefficients and electromagnetic potentials from which we can get $G^{(0)}$ from (7.12). Then using the relation for $U_\mu$ (7.19), we can find the background matrices$U_\mu^{(0)}$, then by solving (7.18) the background spectral matrix $\varphi^{(0)}$ can be found. This is the only integration of differential equations that we need to do the other steps consists only in algebraic manipulations and simple quadrature.

The dressing procedure for construction a new solutions from the given background one means that the new spectral matrix $\varphi$ can be represented as "dressed" value of its seed solution $\varphi^{(0)}$, it takes form

$$\varphi = \chi\varphi^{(0)} \tag{7.21}$$

Where matrix $\chi$ play a role of relative exact perturbation of the background and depends on all three variables $x^\mu$ and $w$.

Substituting the former equation in (7.18) creates an equation for the dressing matrix $\chi$

$$\chi_{,\mu} = \frac{(w+\beta)\delta_\mu^\nu - \alpha\eta_{\mu\rho}\epsilon^{\rho\nu}}{2i((w+\beta)^2+\alpha^2)}\left(U_\nu\chi - \chi U_\nu^{(0)}\right). \qquad (7.22)$$

Now using the special structure of $G$ and the former equation a new important algebraic relation follows, as shown in [3] and in [13], the relation is

$$G + 4i(w+\beta)\Omega = \left(\chi^{-1}\right)^\dagger \left[G^{(0)} + 4i(w+\beta)\Omega\right]\chi^{-1} \qquad (7.23)$$

which is satisfied everywhere in the complex plane of parameter $w$.
From this equation follows matrix $G$ and as a consequence the metric and electromagnetic potentials in terms of the background solution if also the dressing matrix $\chi$ can be found in terms of the background quantities. However, precisely this miraculous possibility is provided by the inverse scattering method in case of the meromorphic structure of the dressing matrices $\chi$ and $\chi^{-1}$ with respect to the parameter $w$. In this way the final expressions for the metric and electromagnetic potentials can be obtained in exact analytical form.

## 7.2 Dressing matrix with poles at infinity

As said at the end of the previous chapter we want to explore poles at infinity because they might produce an accelerating background and even generalize it. To do so so we can rely on the charged inverse scattering and solely modify the dressing matrix. This was done in [13] and [12].Now we give a review of the method.

### 7.2.1 Construction of the solution and analysis

The dressing matrix and its inverse now take form

$$\chi = Aw + P, \quad \chi^{-1} = Bw + Q \qquad (7.24)$$

Here $A, P, B, Q$ do not depend on $w$ and $P$ has inverse $P^{-1}$. A complete set of equations to determine the explicit form of the dressing matrix can be derives from the relations $\chi\chi^{-1} = \mathbb{I}$ , $\chi^{-1}\chi = \mathbb{I}$ and relations (7.23) (7.22). Each of these can be represented as zero condition for some polynomial (cubic or second order) with respect to the parameter $w$. The vanishing conditions for all coefficients of the second order $w$-polynomials $\chi\chi^{-1} = \mathbb{I}$ , $\chi^{-1}\chi = \mathbb{I}$, define matrices $B$ and $Q$ as

$$Q = P^{-1}, \quad B = -P^{-1}AP^{-1}, \tag{7.25}$$

Plus the requirement for $A$

$$AP^{-1}A = 0, \tag{7.26}$$

Now using equation (7.23) (7.22), two additional demands for the matrix $A$ follows

$$\begin{aligned} &A,_{\mu} = 0, \\ &A^{\dagger}\Omega A = 0, \end{aligned} \tag{7.27}$$

And basic expression for the matrices $G$ and $U_{\mu}$

$$G + 4i\beta\Omega = \left(P^{\dagger}\right)^{-1}\left[G^{(0)} + 4i(\beta)\Omega\right]P^{-1}, \tag{7.28}$$

$$U_{\mu} = 2i\left(\beta\delta_{\mu}^{\nu} + \alpha\eta_{\mu\rho}\varepsilon^{\rho\nu}\right)P,_{\nu}P^{-1} + PU_{\mu}^{(0)}P^{-1} \tag{7.29}$$

Using these last three conditions and also following (7.22)and (7.23) the following relations follows

$$2iP^{-1}P,_{\mu} - 2i\left(\beta\delta_{\mu}^{\nu} + \alpha\eta_{\mu\rho}\varepsilon^{\rho\nu}\right)P^{-1}P,_{\nu}P^{-1}A = U_{\mu}^{(0)}P^{-1}A - P^{-1}AU_{\mu}^{(0)}, \tag{7.30}$$

$$\left(P^{-1}A\right)^{\dagger}\left(G^{(0)} + 4i\beta\Omega\right)P^{-1}A + 4i\left(P^{\dagger}\Omega A + A^{\dagger}\Omega P\right) = 0, \tag{7.31}$$

$$\left(G^{(0)} + 4i\beta\Omega\right)P^{-1}A + \left(P^{-1}A\right)^{\dagger}\left(G^{(0)} + 4i\beta\Omega\right) + 4iP^{\dagger}\Omega P = 4i\Omega. \tag{7.32}$$

The constrains on $A$ and $P$ plus these last three equations represent the complete self-consistent system from which matrices $A$ and $P$ can be found in terms of the background matrices $G^{(0)}$ and $U_{\mu}^{(0)}$ and given (chosen from the outset) functions $\alpha$,$\beta$.

In [13] it's shown that this system can be solved exactly, which is quite a remarkable feature ,and it has a unique solution up to the three arbitrary real constants, which represent free parameters of the dressed solution generated by the pole at infinity of the $w$-plane.
After we retrieve matrix P we can than generate $G$ and $U_\mu$ from the former three equations.
Calculations show that $A$ is a degenerate matrix of the form:

$$A = \begin{bmatrix} 8ik_1k_2 & -8ik_1^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.33}$$

where $k_1$ and $k_2$ are two arbitrary real constants and matrix $P$ has the following structure:

$$P = \begin{bmatrix} 2k_3N - k_2\kappa(L+2QN) - \kappa^{-1} & k_1\kappa(L+2QN) & -2k_1N \\ -k_2k_1^{-1}\kappa & \kappa & 0 \\ k_3k_1^{-1} - k_2k_1^{-1}\kappa Q & \kappa Q & -1 \end{bmatrix} \tag{7.34}$$

Where $k_3$ is a third arbitrary constant parameter which is a complex number of the unit modulus, in other words

$$k_3 = e^{i\gamma} \tag{7.35}$$

with $\gamma$ a real arbitrary constant. The explicit form of $P^{-1}$ is quite simple

$$P^{-1} = \begin{bmatrix} -\kappa & k_1\kappa L & 2k_1\kappa N \\ -k_2k_1^{-1}\kappa & k_2\kappa L + \kappa^{-1} & 2k_2\kappa N \\ -k_3k_1^{-1}\kappa Q & k_3\kappa L + Q & 2k_3\kappa N - 1 \end{bmatrix} \tag{7.36}$$

Moreover

$$\det P = \mathbb{I} \tag{7.37}$$

It is convenient to organize the three constants $k_1, k_2, k_3$ in a 3 dimensional array

$$k_A = (k_1, k_2, k_3), \tag{7.38}$$

No matrix P depends of four complex functions, namely $\kappa, N, Q, L$ that do not depend on $w$. These function can be obtained in terms of the arbitrary constants $k_A$, the background quantities $G^{(0)}$,$U_\mu^{(0)}$ and the given functions $\alpha$ and $\beta$. The equations for $\kappa$ are quite involved and it's convenient to consider the inverse i.e. $\kappa^{-1}$. Now we can divide into real and imaginary parts. For the real part of the function $\kappa^{-1}$

$$\kappa^{-1} + \bar{\kappa}^{-1} = 2\bar{k}_A k_B \left(G^{(0)}\right)^{AB} \tag{7.39}$$

and its imaginary part should be calculated from the quadrature

$$\left(\kappa^{-1} + \bar{\kappa}^{-1}\right),_\mu = -4\bar{k}_C k_B \left[\Omega^{CA} \left(U_\mu^{(0)}\right)_A^{\ B} - \Omega^{BA} \left(\bar{U}_\mu^{(0)}\right)_A^{\ C}\right], \tag{7.40}$$

now by using the definition of $U^{(0)}$ in the last two equations, the real part of $\kappa^{-1}$ can be defined by quadrature

$$\kappa^{-1},_\mu = -k_B \bar{k}_C \Omega^{CA} \left(U_\mu^{(0)}\right)_A^{\ C} \tag{7.41}$$

The function $N$ can be expressed in a simple form

$$N = \bar{k}_A \left(G^{(0)}\right)^{A3}, \tag{7.42}$$

function $Q$ can be obtained by quadrature

$$Q,_\mu = 4k_C \left[k_3 (U^{(0)})_1^{\ C} - k_1 \left(U^{(0)}\right)_3^{\ C}\right], \tag{7.43}$$

L takes form

$$L = -8ik_1\beta\kappa^{-1} - 2\kappa^{-1}\bar{k}_C \left(G^{(0)}\right)^{C2} - 2NQ + R, \tag{7.44}$$

function $R$ ha to be calculated by quadrature

$$R,_\mu = 4\kappa^{-1}\bar{k}_A \left(\bar{U}_\mu^{(0)}\right)_1^{\ A} + 4\bar{\kappa}^{-1}\bar{k}_A \left(U_\mu^{(0)}\right)_1^{\ A} \tag{7.45}$$

## 7.3 General solution with background

Now that we have $A, P$ and function $\kappa, Q, L, R$ using equation (7.28), The matrix $G$ can be found, that is the final expression for the metric coefficients $g_{ab}$ and the electromagnetic potentials $\Phi_a$ in terms of the background solutions of the Einstein-Maxwell equations. For $g_{ab}$ the results are

$$g_{11} = (\kappa\bar{\kappa})^{-1} g_{11}^{(0)} + 2R\left(k_1 g_{12}^{(0)} - k_2 g_{11}^{(0)}\right) + R^2 \kappa\bar{\kappa}\left(k_2^2 g_{11}^{(0)} + k_1^2 g_{22}^{(0)} - k_1 k_2 g_{12}^{(0)}\right) \tag{7.46}$$

$$g_{12} = k_1^{-1}\left(k_1 g_{12}^{(0)} - k_2 g_{11}^{(0)}\right) + k_1^{-1} R\kappa\bar{\kappa}\left(k_2^2 g_{11}^{(0)} + k_1^2 g_{22}^{(0)} - k_1 k_2 g_{12}^{(0)}\right) \tag{7.47}$$

$$g_{22} = \kappa\bar{\kappa} k_1^{-2}\left(k_2^2 g_{11}^{(0)} + k_1^2 g_{22}^{(0)} - 2k_1 k_2 g_{12}^{(0)}\right) \tag{7.48}$$

For the electro magnetic potentials

$$\Phi_1 = \frac{1}{2}\bar{Q} - \frac{\kappa R}{2}\left(2k_1\Phi_2^{(0)} - 2k_2\Phi_1^{(0)} + \bar{k}_3\right) - \frac{\Phi_1^{(0)}}{\bar{\kappa}} \tag{7.49}$$

$$\Phi_2 = -\frac{\kappa}{2k_1}\left(2k_1\Phi_2^{(0)} - 2k_2\Phi_1^{(0)} + \bar{k}_3\right) \tag{7.50}$$

having all the components, matrices $G$ and $Y$ can be obtained, after which we can calculate matrices $K_\mu$ from (7.15)and then $f$ can be calculated explicitly using (7.17).

$$f = c_0(\kappa\bar{\kappa})^{-1} f^{(0)}. \tag{7.51}$$

Where $c_0$ is an arbitrary real constant and, of course, $f^{(0)}$ represents the component $f$ of the back ground solution.
Now that we have all the ingredients we can choose a background metric and following the steps laid before. As background Minkowski can be choose since it's clearly a solution of Einstein equations. The components of the metric are

$$g^{(0)} = \begin{bmatrix} -1 & 0 \\ 0 & \rho^2 \end{bmatrix} \tag{7.52}$$

Writing the metric in this form implies that $\alpha = \rho$ hence we can choose using (7.7) $\beta = -z$. the $f$ component reads

$$f^{(0)} = c_1 \tag{7.53}$$

Where $c_1$ is an arbitrary real constant. Since there is no electromagnetic field in the background:

$$\Phi_a^{(0)} = 0 \tag{7.54}$$

We can now find $\kappa, Q$ and $R$ but before that we need $G^{(0)}, Y^{(0)}, U_\mu^{(0)}$. These can be found using (7.14), (7.19) and (7.1.1) respectively

$$G^{(0)} = \begin{bmatrix} 4\rho^2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; U_\mu^{(0)} = \begin{bmatrix} -2i\beta,_\mu & 0 & 0 \\ -2\rho\rho,_\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; Y^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{7.55}$$

These results can be plugged in equations (7.41),(7.43) and (7.45), the remaining calculations are some simple quadrature, the results are

$$\kappa^{-1} = 4k_1^2\rho^2 - 4k_2^2 + 1 + i(c2 - 8k_1k_2\beta), \tag{7.56}$$

$$Q = -8ik_1k_3\beta + c_3 + ic_4, \tag{7.57}$$

$$R = -16ik_1c_2\beta + 64k_1^2k_2\beta^2 + c5, \tag{7.58}$$

Where $c_2, c_3, c_4, c_5$ are now arbitrary real constants. The potentials $\Phi_a$ take a very simple form since there is no electromagnetic field in the background

$$\Phi_1 = \frac{1}{2}\bar{Q} - \frac{1}{2}\bar{k}_3\kappa R, \tag{7.59}$$

$$\Phi_2 = -\frac{1}{2k_1}\bar{k}_3\kappa. \tag{7.60}$$

The physical potentials are defined as the real parts of $A_a = \operatorname{Re}\Phi_a$, evaluating this relations one has

$$A_1 = \frac{1}{4}(Q + \bar{Q}) + \frac{R}{4}(\bar{k}_3\kappa + k_3\bar{\kappa}), \tag{7.61}$$

$$A_2 = -\frac{1}{4k_1}(\bar{k}_3\kappa + k_3\bar{\kappa}), \tag{7.62}$$

The components of the metric and physical potentials can be now be make explicit, by using the relations found for the functions $Q, R, \kappa^{-1}$,explicitly

$$g_{11} = -(\kappa\bar{\kappa})^{-1} + 2k_2 R + \kappa\bar{\kappa}(-k_2^2 + k_1^2\rho^2)R^2, \tag{7.63}$$

$$g_{12} = k_2 k_1^{-1} + \kappa\bar{\kappa}k_1^{-1}(\rho^2 k_1^2 - k_2^2)R, \tag{7.64}$$

$$g_{22} = \kappa\bar{\kappa}k_1^{-2}(\rho^2 k_1^2 - k_2^2), \tag{7.65}$$

$$f = c_0 c_1 (\kappa\bar{\kappa})^{-1}, \tag{7.66}$$

$$A_2 = -\frac{\kappa\bar{\kappa}}{2k_1}\left[\left(4k_1^2\rho^2 - 4k_2^2 + 1\right)\cos\gamma - (c_2 - 8k_1 k_2 \beta)\sin\gamma\right], \tag{7.67}$$

$$A_1 = 4k_1\beta\sin\gamma + \frac{c_3}{2} + k_1 R A_2, \tag{7.68}$$

With

$$\kappa\bar{\kappa} = \left[(4k_1^2\rho^2 - 4k_2^2 + 1)^2 + (c_2 - 8k_1 k_2 \beta)^2\right]^{-1}, \tag{7.69}$$

Where R is defined as above.
The solution can then be written as

$$ds^2 = g_{11}dt^2 + g_{22}dtd\phi + f(d\rho^2 + d\beta^2) + g_{22}d\phi^2, \tag{7.70}$$

This solution has a lot of parameters we notice that $c_0$ and $c_1$ compare always coupled i.e. as $c_0 c_1$, while the parameter $\gamma$ comes from the $U(1)$ symmetry of the electromagnetic potential and cane be used to rotate electric and magnetic potentials.
The solution even though quite general, it doesn't seem to have an accelerating parameter in it. This can be cause by the fact that even though the charged inverse scattering uses only one (complex) pole, it acts like a double pole at infinity, while the accelerating background can be see as only one pole to infinity.
None the less this solution holds some peculiar solutions inside of it. The fact that this solution has 9 parameters, actually 8 if we count that $c_0$ and $c_1$ are never independent, makes it very difficult to work with and to understand

it's physical significance, in fact so far is just an offspring of a marvelous mathematical procedure.
In [12] it has been shown that by setting most of the parameters to zero i.e.

$$k_2 = \gamma = c_2 = c_3 = c_4 = c_5 = 0, \tag{7.71}$$

note that $\gamma = 0$, means $k_3 = 1$, and by setting $k_1 = -\frac{1}{4}B_0$ one retrieves Melvin metric.
In coordinates $\rho, \beta$ as defined above.

$$ds^2 = \left(1 + \frac{B^2}{4}\rho^2\right)^2 (d\rho^2 + d\beta^2 - dt^2) + \left(1 + \frac{B^2}{4}\rho^2\right)^{-2} \rho^2 d\phi^2 \tag{7.72}$$

With potential

$$A_t = 0,\ A_\phi = -\frac{2}{B}\left(1 + \frac{B^2}{4}\rho^2\right)^{-1} \tag{7.73}$$

Here the constant $c_0, c_1$ are equal to one.
Now this solution has been show to be a limiting case for a double extremal Reissner-Nordström black hole (clarifying the nature of the poles stated above). In this cylindrical symmetric spacetime, the magnetic field, represented by $B$, is parallel to the regular axis of symmetry. It's easy to see that the magnetic field strength monotonically decreases away from the symmetry axis, thus $B$ is understood as the maximum magnetic field strength in the Melvin universe. Following this intuition we have speculated that the swirling universe [15] and [16] might be part of this solution. This is the case. Indeed by taking (7.63),(7.64),(7.65),(7.66), and setting the parameters as

Unless for a gauge translation in the potential and a trivial rescaling of the cyclic coordinates one obtains

$$ds^2 = \left( \frac{\rho^2(4j\beta)^2}{\left(\frac{B^2\rho^2}{4} + 1\right)^2 + j^2\rho^4} - \frac{1}{16}\left(B^2\rho^2 + 4\right)^2 - j^2\rho^4 \right) dt^2 - \frac{\rho^2(4j\beta)}{\left(\frac{B^2\rho^2}{4} + 1\right)^2 + j^2\rho^4} dt d\phi$$
$$+ \left( \left(\frac{B^2\rho^2}{4} + 1\right)^2 + j^2\rho^4 \right)(d\rho^2 + \beta^2) + \frac{\rho^2}{\left(\frac{B^2\rho^2}{4} + 1\right)^2 + j^2\rho^4} d\phi^2, \tag{7.74}$$

by using (7.68) and (7.67) one has

$$A_t = \frac{8\beta B j \rho^2 \left(B^2\rho^2 + 4\right)}{\rho^4 \left(B^4 + 16j^2\right) + 8B^2\rho^2 + 16}, \tag{7.75}$$

$$A_\phi = -\frac{2B\rho^2 \left(B^2\rho^2 + 4\right)}{\rho^4 \left(B^4 + 16j^2\right) + 8B^2\rho^2 + 16} \tag{7.76}$$

But this is in fact Melvin universe with a rotational components $j$ as in [16], and by taking the limit as $B \to 0$ one retrieves one again swirling universe. This solution is quite interesting, per se the Melvin universe, i.e. our magnetic background, can be interpreted as two infinitely distant (along the symmetry axis) Reissner-Nordström black holes at infinity with opposite magnetic charges. While the rotating part of the metric, determined by $j$, can be interpreted as two counter rotating black holes at infinity, such as two black holes with opposite NUT charges. While the nature of Melvin spacetime is clear, the nature of the rotating back ground is still unclear, indeed we know that is associated to the NUT charge, but it's unclear if it's the only type of rotating universe with source at infinity i.e. if a universe with Kerr black holes at infinity can exist as a standalone solution or is already included in our rotational background.

These cases underline the fact that even if there is no accelerating background as we wished this solution has quite a few physical interpretations. One thing to notice is that in all these cases the parameter $k_2 = 0$, this parameter is associated with a rotation of the space time, indeed if we consider metric (7.70) and evaluate the frame dragging of the whole spacetime after a trivial rescaling $t \longrightarrow t_0 t$ this gives

$$\begin{aligned} \omega =& \frac{k_1 t_0 \big(k_2 \left(c_2^2 + 16k_1^4 \left(4\beta^2\rho^2 + \rho^4\right) + 8k_1^2\rho^2 + 1\right) + k_1^2\rho^2 (c_5 - 16\beta c_2 k_1)}{k_2^2 - k_1^2\rho^2} \\ &+ \frac{-c_5 k_2^2 - 8k_2^3 \left(4k_1^2\rho^2 + 1\right) + 16k_2^5\big)}{k_2^2 - k_1^2\rho^2} \end{aligned} \tag{7.77}$$

hence the frame dragging oh the space time is zero if $k_1 = 0$ unless we choose $t_0 \to 1/k_1$, in this case a series expansion show that the above equation is divergent around $\rho = k_2/k_1$ and diverges as

$$\omega \approx -\frac{(c_2 - 8\beta k_1 k_2)^2 + 1}{2k_1\left(\rho - \frac{k_2}{k_1}\right)} \tag{7.78}$$

Confirming the divergence at this point. This divergence for $\rho$ disappears whenever $k_1 = 0$ or $k_2 = 0$, underling that there has to be some sort of interaction between the two.
To discover more about this divergence we look for CTCs, indeed since the the angular velocity grows unbounded: this would lead to the conclusion that there exist superluminal observers, since the value of the gravitational dragging can easily exceed 1 (i.e. the speed of light, in our units) and, then, it would violate causality. In this perspective, let us study the possible occurrence of closed timelike curves (CTCs): considering curves in which $t, \rho, \beta$ are constant are characterized by

$$ds^2_{t,\rho,\beta} = g_{\phi\phi} d\phi^2 \tag{7.79}$$

with

$$g_{\phi\phi} = \frac{\left(k_1^2\rho^2 - k_2^2\right)}{k_1^2\left((c_2 - 8\beta k_1 k_2)^2 + 8k_1^2\left(1 - 4k_2^2\right)\rho^2 + 16k_1^4\rho^4 + 16k_2^4 - 8k_2^2 + 1\right)} \tag{7.80}$$

It can be easily seen that such intervals are not always positive and can become zero or even negative.Therefore there are CTCs and there are related causality issues. Even though this is the case in general, for a wise choice of the parameters as seen in the examples before, where there is no trace of the superluminar observer as explained in [15].
Both Melvin and Swirling universe can be mapped into a Flat extremal Reissner-Nordström and to a flat Taub-NUT, since both this solution can be retrieved, as shown, from (7.70), then it comes natural to ask if one can map (7.70) to a more general flat metric, for example the one given in [18] with g=0 and $\aleph = 0$, the answer is positive and it can be done though a double wick rotation (i.e. $t \to i\phi$, $\phi \to it$) of metric (7.70) and then by

performing the change of coordinates

$$\rho = \frac{\sqrt{a^2 + 2mr + P^2 + Q^2}}{2\sqrt{P^2 + Q^2}}, \quad \beta = \alpha u. \tag{7.81}$$

and then by choosing the parameters as

$$\begin{aligned} k_2 &= \frac{a}{\sqrt{P^2 + Q^2}}, \; k_1 = 1, \; c_2 = n, \; \tan\gamma = -\frac{P}{Q}, \\ c_5 &= 0, \; c_0 = \frac{16}{c_1(P^2 + Q^2)}, \; \alpha = \frac{\sqrt{P^2 + Q^2}}{4}. \end{aligned} \tag{7.82}$$

$c_3$ is proportional to the gauge constant of the electromagnetic potential and can be chose to be zero. Then, beside some trivial rescaling of the cyclic coordinates, flat Kerr-Newmann Nut is retrieved, with the constrain $2m = \sqrt{P^2 + Q^2}$

$$\begin{aligned} ds^2 = &-\frac{2mr + P^2 + Q^2}{(n + au)^2 + r^2} dt^2 + \frac{\left(au^2 + 2nu\right)\left(a^2 + 2mr + P^2 + Q^2\right) + a\left(n^2 + r^2\right)}{(n + au)^2 + r^2} dt d\phi + \\ &+ \frac{(n + au)^2 + r^2}{a^2 + 2mr + P^2 + Q^2} dr^2 + \left((n + au)^2 + r^2\right) du^2 + \\ &+ \frac{\left(-a^2u^4 - 4anu^3 - 4n^2u^2\right)\left(a^2 + 2mr + P^2 + Q^2\right) + \left(n^2 + r^2\right)^2}{(n + au)^2 + r^2} d\phi^2 \end{aligned} \tag{7.83}$$

The two potentials are

$$A_t = -\frac{aPu + nP + Qr}{(au + n)^2 + r^2}, \; A_\phi = \frac{u\left(anPu + aQru + n^2P + 2nQr - Pr^2\right)}{(au + n)^2 + r^2} \tag{7.84}$$

Two main features can be seen here, $c_2 = n$ hence this parameter is directly proportional to the Taub-NUT parameter $n$, which is compatible with the Swirling-Melvin universe where $c_2 \propto j$, and is speculated that the swirling parameter is generated by Taub-NUT parameter. What's even more striking is that $k_2$ seems to be directly proportional to $a$ that is the rotational parameter of the space time. indeed when $k_2 = 0$ would imply that $a = 0$ on

one would get back the relation Swirling-Melvin → flat Reissner-Nordström-NUT. Hence there seems to be rotational universe without a swirling parameter $j$ but with a parameter associated to $a$.
Indeed this is the case as we'll show. If we set the parameters as follows

$$c_0 = c_1 = 4,\ c_2 = 0,\ c_3 = 0,\ c_5 = 0,\ k_1 = -\frac{B}{4},\ k_2 = \frac{K_2}{4}. \tag{7.85}$$

The following metric is obtained

$$\begin{aligned} ds^2 = &-\frac{4\beta^4 B^4 K_2^2 \rho^2 + \left(B^2\rho^2 - K_2^2 + 4\right)^4}{4\beta^2 B^2 K_2^2 + \left(B^2\rho^2 - K_2^2 + 4\right)^2} dt^2 - \\ &- \frac{4K_2\left(4\beta^2 B^4 \rho^2 + \left(B^2\rho^2 - K_2^2 + 4\right)^2\right)}{B\left(4\beta^2 B^2 + \left(B^2\rho^2 - K_2^2 + 4\right)^2\right)} dt d\phi + \\ &+ \left(4\beta^2 B^2 K_2^2 + \left(B^2\rho^2 - K_2^2 + 4\right)^2 1\right)\left(d\rho^2 + dz^2\right) \\ &- \frac{16\left(K_2^2 - B^2\rho^2\right)}{B^2\left(4\beta^2 B^2 K_2^2 + \left(B^2\rho^2 - K_2^2 + 4\right)^2\right)} d\phi^2 \end{aligned} \tag{7.86}$$

with electromagnetic potentials

$$A_t = -\frac{4\beta B\left(B^2\rho^2 - K_2^2 + 4\right)\left(\sin(\gamma)\left(B^2\rho^2 - K_2^2 + 4\right) + 2\beta B K_2 \cos(\gamma)\right)}{K_2^2\left(B^2\left(4\beta^2 - 2\rho^2\right) - 8\right) + \left(B^2\rho^2 + 4\right)^2 + K_2^4} \tag{7.87}$$

$$A_\phi = \frac{2}{K_2^2 - 4} - \frac{8\left(\cos(\gamma)\left(-B^2\rho^2 + K_2^2 - 4\right) + 2\beta B K_2 \sin(\gamma)\right)}{B\left(K_2^2\left(B^2\left(4\beta^2 - 2\rho^2\right) - 8\right) + \left(B^2\rho^2 + 4\right)^2 + K_2^4\right)} \tag{7.88}$$

With $\frac{2}{K_2^2-4}$ as a gauge constant of the potential.
It is clear that this metric contains a rotational parameter $K_2$, indeed if this parameter is set to zero the metric becomes Melvin magnetic universe.
In this case we have only two parameters $B$ representing a magnetic field and $K_2$, if we were able to map this metric to a metric like a flat Kerr-Newman, the nature of $K_2$ would be even more clear. The flat Kerr-Newman metric is

$$
\begin{aligned}
ds^2 =& -\frac{-2mr+p^2+q^2}{r^2+x^2a^2}dt^2 + a\left(k + \frac{x^2\left(-2mr+p^2+q^2\right)}{r^2+x^2a^2}\right)dtd\phi + \\
&+ (r^2+x^2a^2)/(p^2+q^2-2mr+ka^2)dr^2 + \frac{r^2+x^2a^2}{k}dx^2 + \\
&+ \frac{kr^4 - x^4a^2\left(ka^2-2mr+p^2+q^2\right)}{r^2+x^2a^2}d\phi^2
\end{aligned} \tag{7.89}
$$

Wit potentials

$$
A_t = \frac{qr+pax}{r^2+a^2x^2},\ A_\phi = \frac{rx(qxa-pr)}{r^2+a^2x^2}, \tag{7.90}
$$

The mapping can be done by taking the double wick rotation of the metric (7.89) and by changing $p \to ip$, $q \to iq$ in order to conserve the reality of the electromagnetic potentials, then by taking(7.86) ad choosing the following parameters

$$
K_2 = \frac{a}{2B},\ m = -2B^2,\ B = \frac{\sqrt{p^2+q^2}}{4},\ \tan\gamma = \frac{\sqrt{16B^2-q^2}}{q} \tag{7.91}
$$

And change of coordinates

$$
\rho = \frac{\sqrt{r-4+K_2^2}}{B},\ \beta = x \tag{7.92}
$$

One maps the two metric into each other. From (7.91) the nature of $B$ is clear it's associated to the electromagnetic charges of a flat Kerr-Newmann space time. It is also clear that, as pointed out before, $K_2 \propto a$ therefore $K_2$ is a parameter proportional to the rotational parameter $a$.
The solution is interesting because it generates a rotational-Melvin spacetime that is different from Swirling-Melvin, that we introduced before. As a matter of fact if we consider the gravitational dragging along the $\beta$-axis, we have

$$
\omega\Big|_{\rho=0} = -d_1\frac{B\left(K_2^2-4\right)^2}{4K_2} + \omega_0. \tag{7.93}
$$

Where $\omega_0$ is an integration constant and $d_1$ is a real constant that comes form a trivial rescaling of the cyclic coordinates, and can be chosen to be one. Thus the we find a constant gravitational dragging, while in (7.74) the dragging was proportional to $\beta$, another interesting feature of this metric is that it can have ergoregions, indeed the equation $g_{tt} = 0$ holds the relation

$$\beta = \pm \frac{B^2\rho^2 - K_2^2 + 4}{2B^{3/2}\sqrt{|K_2|\rho}} \tag{7.94}$$

That gives the ergoregions, in figures 7.1,7.2 we give a visual example of these regions. In the first figure 7.1 the picture presents a cavity much like the one in [15], but the two ergoregions are clearly different. To our knowledge this solution is a novelty.

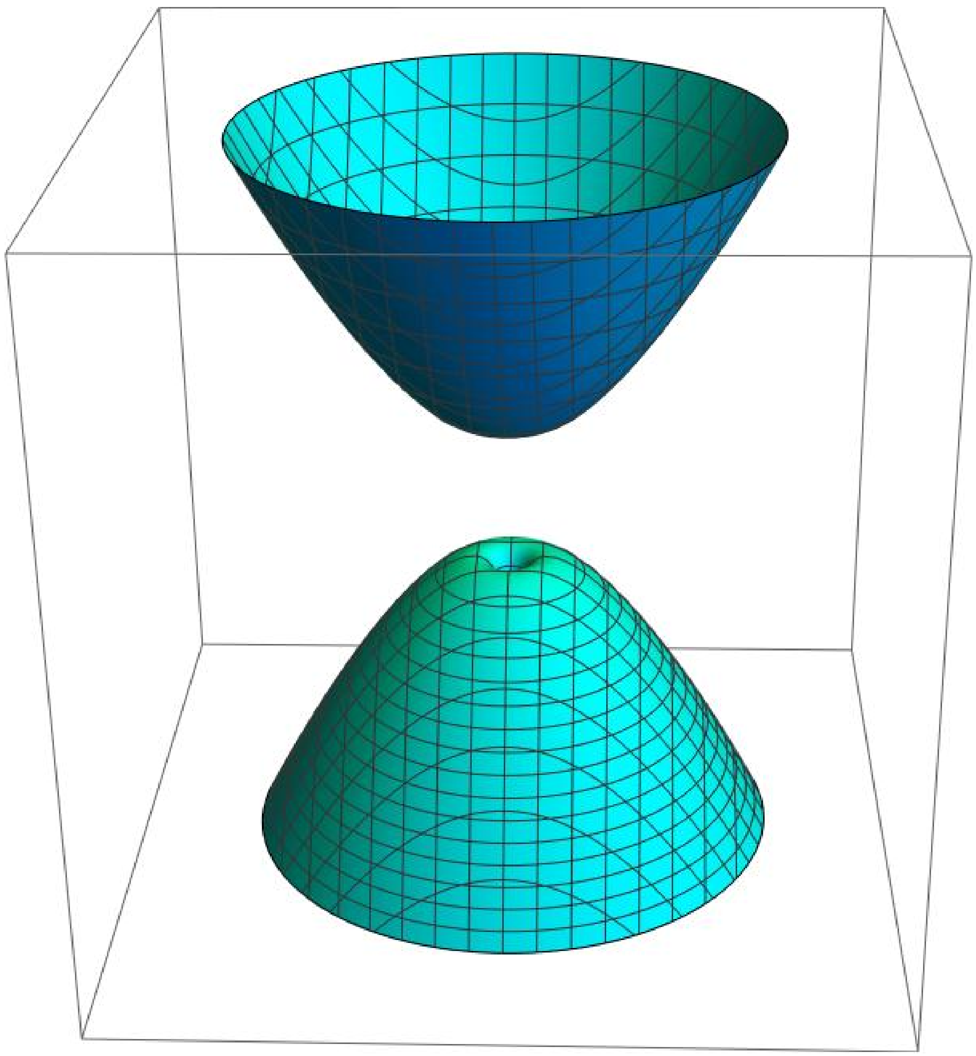

Figure 7.1: Figure describing the ergoregions of (7.86), with $\boldsymbol{B} = 1$ and $\boldsymbol{K}_2 = 1$. It's evident that there is a cavity at the center of the ergoregion.

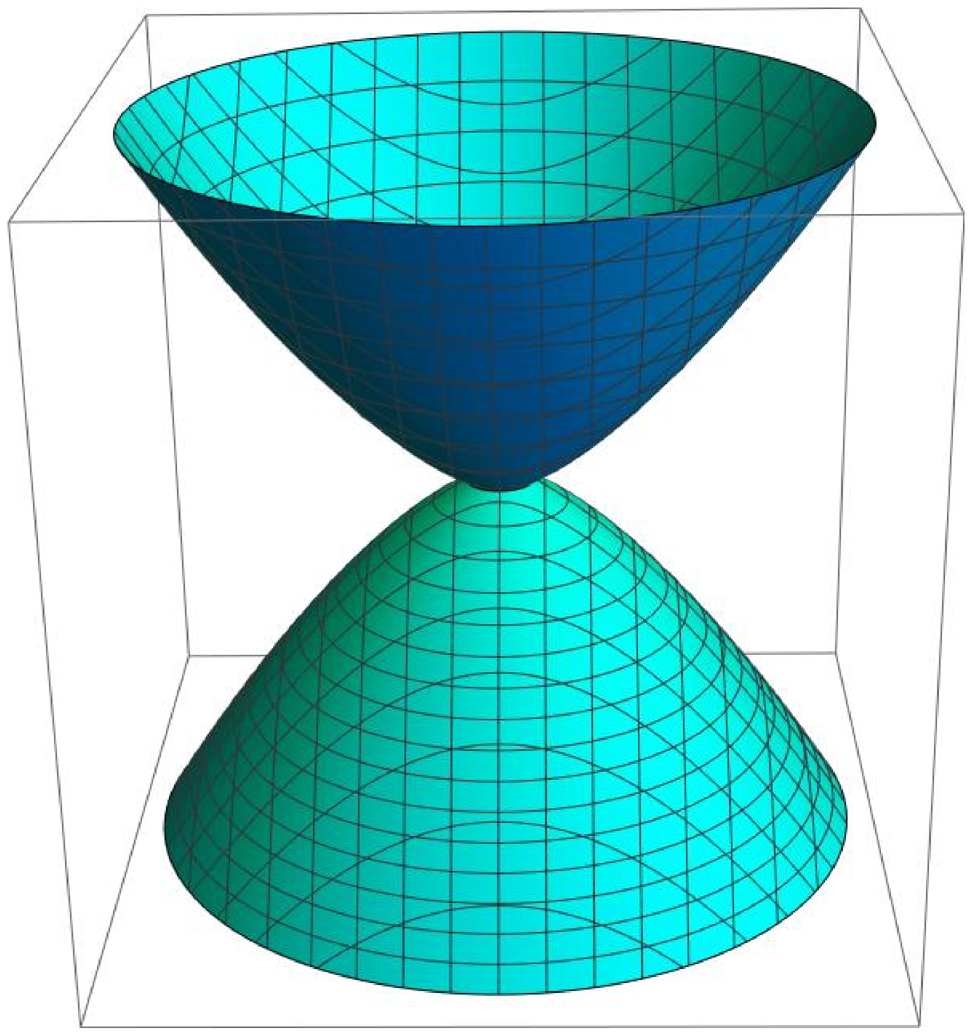

Figure 7.2: Figure describing the ergoregions of (7.86), with $B = 1$ and $K_2 = 2.2$, it's clear that the overlapping of the two regions creates a big ergoregion

# Chapter 8

# Conclusions

The inverse scattering technique in vacuum is a marvelous tool to generate new solution of Einstein equations given a known background as seen in previous chapters. Indeed by choosing the simplest background, that is Minkowski one can generate with ease Kerr-NUT, once we have all the information on the background one needs only to perform algebraic manipulations to obtain the solution.

Once the background gets more complicated the techniques requires more calculations, but the results can be surprising as seen in the case of using Rindler spacetime as seed metric. In this case the application of the inverse scattering technique has produced an accelerating Kerr-NUT spacetime of Petrov Type-D and an accelerating NUT spacetime as a sub case, which was quite an elusive solution, indeed all accelerating NUT solutions known so far where of type I.

The charged inverse scattering as seen in the previous chapter is a technique that is formally different than the one in the vacuum case, but it still quite nifty, and in this case has been used to produce a general background solution. Even though we haven't found a generalization of the accelerating spacetime, quite a few results have been proposed with the comparison to a topological flat spacetime. On this note, it might be possible that a more general accelerating spacetime can be identified by a comparison with a topological non flat spacetime, but that remains a mystery.

# Appendix A

# Appendix A

## A.1 Petrov Type

In this appendix is explained how to calculate the Petrov type, for all calculation a Mathemtica package called "EDCRGTCcode.m" has been used. Pertov classifications is based on scalar invariants and in particular of Newman-Penrose scalars $\Psi_i$ that will be defined below. Since we have to describe a type D solution we focus on the scalar invariant

$$I^3 - 27J^2 \, , \tag{A.1}$$

where

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2 \qquad , \qquad J = \det\begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix} \, . \tag{A.2}$$

If the quantity in (A.1) is null the spacetime is algebraically special or type D. After evaluating the scalar invariant, we conclude that the spacetime generated in chapter 6 is, generically, not of type I, but algebraically special, that is of Petrov Type D. In fact we have that both $I$ and $J$ are not zero. and they are different.

$$
\begin{aligned}
I \neq J &\neq 0 \ , \\
K = \Psi_1 \Psi_4^2 - 3\Psi_4 \Psi_3 \Psi_2 + 2\Psi_3^3 &\neq 0 \ , \\
N = 12L^2 - \Psi_4^2 I &\neq 0 \ ,
\end{aligned}
\tag{A.3}
$$

with

$$
L = \Psi_2 \Psi_4 - \Psi_3^2 \neq 0 \ . \tag{A.4}
$$

The definitions of the Newman-Penrose scalars $\Psi_i$ necessary to compute the above scalar invariants can be found below

$$
\begin{aligned}
\Psi_0 &= C_{\mu\nu\sigma\rho} k^\mu m^\nu k^\sigma m^\rho \ , \\
\Psi_1 &= C_{\mu\nu\sigma\rho} k^\mu l^\nu k^\sigma m^\rho \ , \\
\Psi_2 &= C_{\mu\nu\sigma\rho} k^\mu m^\nu \bar{m}^\sigma l^\rho \ , \\
\Psi_3 &= C_{\mu\nu\sigma\rho} l^\mu k^\nu l^\sigma \bar{m}^\rho \ , \\
\Psi_4 &= C_{\mu\nu\sigma\rho} l^\mu \bar{m}^\nu l^\sigma \bar{m}^\rho \ .
\end{aligned}
\tag{A.5}
$$

These five complex scalar functions characterize the Weyl tensor. They can be explicitly computed after defining a null Newman-Penrose tetrad. We have chosen the following tetrad

$$
\begin{aligned}
\mathbf{k} &= \left( \frac{1}{\sqrt{-2g_{tt}}} \ \partial_t + \frac{1}{\sqrt{2g_{xx}}} \ \partial_x \right) \ , \\
\mathbf{l} &= \left( \frac{1}{\sqrt{-2g_{tt}}} \ \partial_t - \frac{1}{\sqrt{2g_{xx}}} \ \partial_x \right) \ , \\
\mathbf{m} &= \left( \frac{g_{t\varphi}}{\sqrt{2Dg_{tt}}} \ \partial_t + \frac{i}{\sqrt{2g_{rr}}} \ \partial_r + \sqrt{\frac{g_{tt}}{2D}} \ \partial_\varphi \right) \ ,
\end{aligned}
\tag{A.6}
$$

where

$$
D = g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2 \ .
$$

The non null scalar products between these vectors are just $k_\mu l^\mu = -1$ and $m_\mu \bar{m}^\mu = 1$.

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