where the integral is now over all of N-dimensional space. Then it can be verified that the parabolic equation $u_t = Lu$ transforms to

$$\hat{u}_t(\xi, t) = -\xi^T A\xi\, \hat{u}(\xi, t). \tag{E.40}$$

The requirement that A be positive definite is just what is needed to ensure that all Fourier modes decay, giving a well-posed problem. If $\xi^T A\xi < 0$ for some vector ξ, then it is also negative for any scalar multiple $\alpha\xi$ of this wave vector, and there would be exponential growth of some Fourier modes with arbitrarily large growth rate $\alpha^2\xi^T A\xi$. As observed for the backward heat equation, this would give an ill-posed problem. (For the heat equation with $N = 1$, the matrix A is just the scalar coefficient κ.)

E.3.6 Dispersive waves

Now consider the equation

$$u_t = u_{xxx}. \tag{E.41}$$

Fourier transforming now leads to the ODE

$$\hat{u}_t(\xi, t) = -i\xi^3 \hat{u}(\xi, t),$$

so

$$\hat{u}(\xi, t) = e^{-i\xi^3 t}\hat{\eta}(\xi).$$

This has a character similar to advection problems in that $|\hat{u}(\xi, t)| = |\hat{\eta}(\xi)|$ for all time and each Fourier component maintains its original amplitude. However, when we recombine with the inverse Fourier transform we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi(x-\xi^2 t)}\, d\xi, \tag{E.42}$$

which shows that the Fourier component with wave number ξ is propagating with velocity ξ^2. In the advection equation all Fourier components propagate with the same speed a, and hence the shape of the initial data is preserved with time. The solution is the initial data shifted over a distance at.

With (E.41), the shape of the initial data in general will not be preserved, unless the data is simply a single Fourier mode. This behavior is called *dispersive* since the Fourier components disperse relative to one another. Smooth data typically lead to oscillatory solutions since the cancellation of high wave number modes that smoothness depends on will be lost as these modes shift relative to one another. See, for example, Whitham [102] for an extensive discussion of dispersive waves.

Extending this analysis to an equation of the form

$$u_t + au_x + bu_{xxx} = 0, \tag{E.43}$$

we find that the solution can be written as

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i\xi(x-(a-b\xi^2)t)}\, d\xi,$$

where $\hat{\eta}(\xi)$ is the Fourier transform of the initial data $\eta(x)$. Each Fourier mode $e^{i\xi x}$ propagates at velocity $a - b\xi^2$, called the phase velocity of this wave number. In general the initial data $\eta(x)$ is a linear combination of infinitely many different Fourier modes. For $b \neq 0$ these modes propagate at different speeds relative to one another. Their peaks and troughs will be shifted relative to other modes and they will no longer add up to a shifted version of the original data. The waves are called dispersive since the different modes do not move in tandem. Moreover, we will see below that the "energy" associated with different wave numbers also disperses.

E.3.7 Even- versus odd-order derivatives

Note that odd-order derivatives $\partial_x, \ \partial_x^3, \ \dots$ (as in the advection equation or the dispersive equation (E.41)) have pure imaginary eigenvalues $i\xi, \ -i\xi^3, \ \dots$, which results in Fourier components that propagate with their magnitude preserved. Even-order derivatives, such as the ∂_x^2 in the heat equation, have real eigenvalues ($-\xi^2$ for the heat equation), which results in exponential decay of the eigencomponents. Another such equation is

$$u_t = -u_{xxxx},$$

in which case $\hat{u}(\xi, t) = e^{-\xi^4 t}\hat{\eta}(\xi)$. Solutions to this equation behave much like solutions to the heat equation but with even more rapid damping of oscillatory data.

Another interesting example is

$$u_t = -u_{xx} - u_{xxxx}, \tag{E.44}$$

for which

$$\hat{u}(\xi, t) = e^{(\xi^2 - \xi^4)t}\hat{\eta}(\xi). \tag{E.45}$$

Note that the u_{xx} term has the "wrong" sign—it looks like a backward heat equation and there is exponential growth of some wave numbers. But for $|\xi| > 1$ the fourth order diffusion dominates and $\hat{u}(\xi, t) \to 0$ exponentially fast. For all ξ we have $|\hat{u}(\xi, t)| \le e^{t/4}|\hat{\eta}(\xi)|$ (since $\xi^2 - \xi^4 \le 1/4$ for all ξ) and the equation is well posed.

The Kuramoto–Sivashinsky equation (11.13) involves terms of this form, and the exponential growth of some wave numbers leads to chaotic behavior and interesting pattern formation.

E.3.8 The Schrödinger equation

The discussion of the previous section supposed that $u(x, t)$ is a real-valued function. The vacuum Schrödinger equation for a complex wave function $\psi(x, t)$ has the form (dropping some physical constants)

$$i\psi_t(x, t) = -\psi_{xx}(x, t). \tag{E.46}$$

This involves a second derivative, but note the crucial fact that ψ_t is multiplied by i. Fourier transforming thus gives

$$i\hat{\psi}_t(\xi, t) = \xi^2\hat{\psi}(\xi, t),$$

so

$$\hat{\psi}(\xi, t) = e^{-i\xi^2 t}\hat{\psi}(\xi, 0)$$

and

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\psi}(\xi,0) e^{i\xi(x-(a-\xi)t)}\, d\xi.$$

Hence the Schrödinger equation has dispersive wavelike solutions in spite of the even-order derivative.

E.3.9 The dispersion relation

Consider a general real-valued PDE of the form

$$u_t + a_1 u_x + a_3 u_{xxx} + a_5 u_{xxxxx} + \cdots = 0 \tag{E.47}$$

that contains only odd-order derivative in x. The Fourier transform $\hat{u}(\xi,t)$ satisfies

$$\hat{u}_t(\xi,t) + a_1 i\xi \hat{u}(\xi,t) - a_3 i\xi^3 \hat{u}(\xi,t) + a_5 i\xi^5 \hat{u}(\xi,t) + \cdots = 0,$$

and hence

$$\hat{u}(\xi,t) = e^{-i\omega t}\hat{\eta}(\xi),$$

where

$$\omega = \omega(\xi) = a_1\xi - a_3\xi^3 + a_5\xi^5 - \cdots. \tag{E.48}$$

The solution can thus be written as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i(\xi x - \omega(\xi)t)}\, d\xi. \tag{E.49}$$

The relation (E.48) between ξ and ω is called the *dispersion relation* for the PDE. Once we've gone through this full Fourier analysis a couple times we realize that since the different wave numbers ξ decouple, the dispersion relation for a linear PDE can be found simply by substituting a single Fourier mode of the form

$$u(x,t) = e^{-i\omega t} e^{i\xi x} \tag{E.50}$$

into the PDE and canceling the common terms to find the relation between ω and ξ. This is similar to what is done when applying von Neumann analysis for analyzing finite difference methods (see Section 9.6). In fact, there is a close relation between determining the dispersion relation and doing von Neumann analysis, and the dispersion relation for a finite difference method can be defined by an approach similar to von Neumann analysis by setting $U_j^n = e^{-i\omega nk} e^{i\xi jh}$, i.e., using $e^{-i\omega k}$ in place of g.

Note that this same analysis can be done for equations that involve even-order derivatives, such as

$$u_t + a_1 u_x + a_2 u_{xx} + a_3 u_{xxx} + a_4 u_{xxxx} + \cdots = 0,$$

but then we find that

$$\omega(\xi) = a_1\xi + i a_2 \xi^2 - a_3\xi^3 - i a_4 \xi^4 - \cdots.$$

The even-order derivatives give imaginary terms in $\omega(\xi)$ so that

$$e^{-i\omega t} = e^{(a_2\xi^2 - a_4\xi^4 + \cdots)t} e^{i(a_1\xi - a_3\xi^3 + \cdots)t}.$$

The first term gives exponential growth or decay, as we expect from Section E.3.3, rather than dispersive behavior. For this reason we call the PDE (purely) dispersive only if $\omega(\xi)$ is real for all $\xi \in \mathbb{R}$. Informally we also speak of an equation like $u_t = u_{xx} + u_{xxx}$ as having both a diffusive and a dispersive term.

In the purely dispersive case (E.47) the single Fourier mode (E.50) can be written as

$$u(x,t) = e^{i\xi(x-(\omega/\xi)t)}$$

and so a pure mode of this form propagates at velocity ω/ξ. This is called the *phase velocity* for this wave number,

$$c_p(\xi) = \frac{\omega(\xi)}{\xi}. \tag{E.51}$$

Most physical problems have data $\eta(x)$ that is not simply sinusoidal for all $x \in (-\infty, \infty)$ but instead is concentrated in some restricted region, e.g., a Gaussian pulse as in (E.29),

$$\eta(x) = e^{-\beta x^2}. \tag{E.52}$$

The Fourier transform of this function is a Gaussian in ξ, (E.30),

$$\hat{\eta}(\xi) = \frac{1}{\sqrt{2\beta}} e^{-\xi^2/4\beta}. \tag{E.53}$$

Note that for β small, $\eta(x)$ is a broad and smooth Gaussian with a Fourier transform that is sharply peaked near $\xi = 0$. In this case $\eta(x)$ consists primarily of low wave number smooth components. For β large $\eta(x)$ is sharply peaked while the transform is broad. More high wave number components are needed to represent the rapid spatial variation of $\eta(x)$ in this case.

If we solve the dispersive equation with data of this form, then the different modes propagate at different phase velocities and will no longer sum to a Gaussian, and the solution evolves as shown in Figure E.1, forming "dispersive ripples." Note that for large times it is apparent that the wave length of the ripples is changing through this wave and that the energy associated with the low wave numbers is apparently moving faster than the energy associated with larger wave numbers. The propagation velocity of this energy is not, however, the phase velocity $c_p(\xi)$. Instead it is given by the *group velocity*

$$c_g(\xi) = \frac{d\omega(\xi)}{d\xi}. \tag{E.54}$$

For the advection equation $u_t + au_x = 0$ the dispersion relation is $\omega(\xi) = a\xi$ and the group velocity agrees with the phase velocity (since all waves propagate at the same velocity a), but more generally the two do not agree. For the dispersive equation (E.43), $\omega(\xi) = a\xi - b\xi^3$ and we find that

$$c_g(\xi) = a - 3b\xi^2,$$

whereas

$$c_p(\xi) = a - b\xi^2.$$

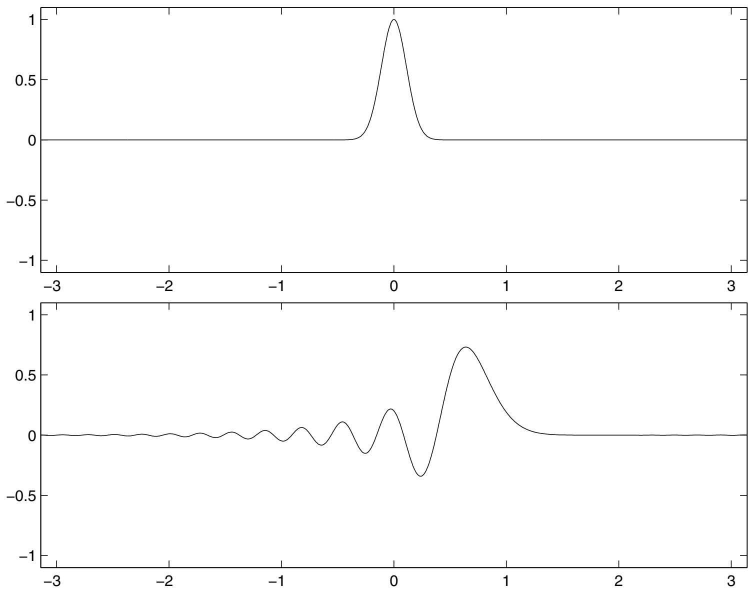

Figure E.1. *Gaussian initial data propagating with dispersion.*

E.3.10 Wave packets

The notion and importance of group velocity is easiest to appreciate by considering a "wave packet" with data of the form

$$\eta(x) = e^{i\xi_0 x} e^{-\beta x^2} \tag{E.55}$$

or the real part of such a wave,

$$\eta(x) = \cos(\xi_0 x) e^{-\beta x^2}. \tag{E.56}$$

This is a single Fourier mode modulated by a Gaussian, as shown in Figure E.2.

The Fourier transform of (E.55) is

$$\hat{\eta}(\xi) = \frac{1}{\sqrt{2\beta}} e^{-(\xi-\xi_0)^2/4\beta}, \tag{E.57}$$

a Gaussian centered about $\xi - \xi_0$. If the packet is fairly broad (β small), then the Fourier transform is concentrated near $\xi = \xi_0$ and hence the propagation properties of the wave packet are well approximated in terms of the phase velocity $c_p(\xi)$ and the group velocity $c_g(\xi)$. The wave crests propagate at the speed $c_p(\xi_0)$, while the envelope of the packet propagates at the group velocity $c_g(\xi_0)$.

To get some idea of why the packet propagates at the group velocity, consider the expression (E.49),

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\eta}(\xi) e^{i(\xi x - \omega(\xi)t)}\, d\xi.$$

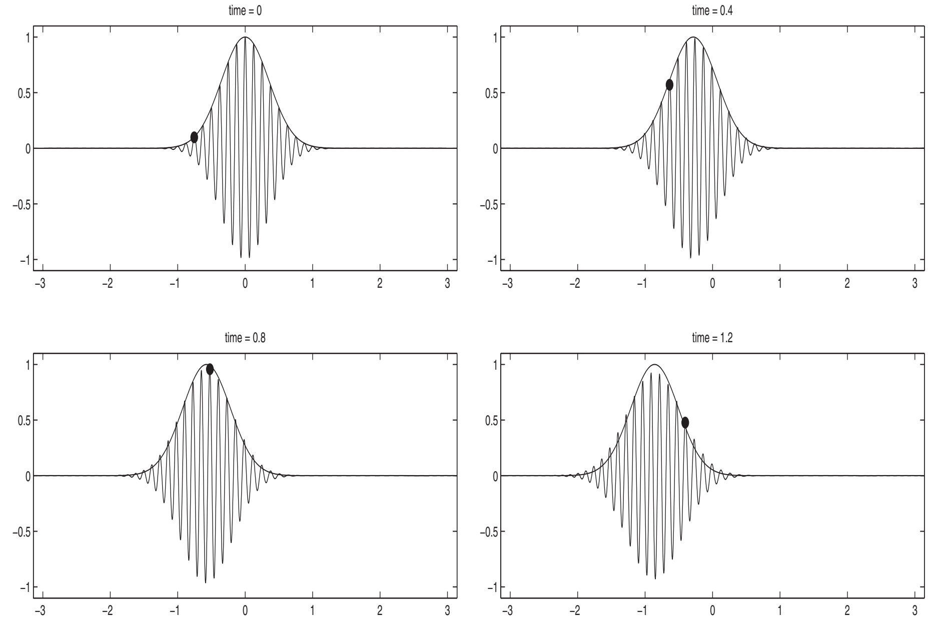

Figure E.2. *The oscillatory wave packet satisfies the dispersive equation $u_t + au_x + bu_{xxx} = 0$. Also shown is a black dot attached to one wave crest, translating at the phase velocity $c_p(\xi_0)$, and a Gaussian that is translating at the group velocity $c_g(\xi_0)$. Shown for a case in which $c_g(\xi_0) < 0 < c_p(\xi_0)$.*

For a concentrated packet, we expect $u(x,t)$ to be very close to zero for most x, except near some point ct, where c is the propagation velocity of the packet. To estimate c we will ask where this integral could give something nonzero. At each fixed x the integral is a Gaussian in ξ (the function $\hat{\eta}(\xi)$) multiplied by an oscillatory function of ξ (the exponential factor). Integrating this product will given essentially zero at a particular x provided the oscillatory part is oscillating rapidly enough in ξ that it averages out to zero, although it is modulated by the Gaussian $\hat{\eta}(\xi)$. This happens provided the function $\xi x - \omega(\xi)t$ appearing as the phase in the exponential is rapidly varying as a function of ξ at this x. Conversely, we expect the integral to be significantly different from zero only near points x where this phase function is stationary, i.e., where

$$\frac{d}{d\xi}(\xi x - \omega(\xi)t) = 0.$$

This occurs at

$$x = \omega'(\xi)t,$$

showing that the wave packet propagates at the group velocity $c_g = \omega'(\xi)$. This approach to studying oscillatory integrals is called the "method of stationary phase" and is useful in other applications as well. See, for example, [55], [102] for more on dispersive waves.