International Journal of Modern Physics A

FEYNMAN DIAGRAMS AND DIFFERENTIAL EQUATIONS

MARIO ARGERI

Chemistry Department, Napoli University
Via Cinthia, Napoli, Italy[*]

PIERPAOLO MASTROLIA

Institut für Theoretische Physik, Universität Zürich
CH-8057 Zürich, Switzerland[†]

Received Day Month Year
Revised Day Month Year

We review in a pedagogical way the method of differential equations for the evaluation of D-dimensionally regulated Feynman integrals. After dealing with the general features of the technique, we discuss its application in the context of one- and two-loop corrections to the photon propagator in QED, by computing the Vacuum Polarization tensor exactly in D. Finally, we treat two cases of less trivial differential equations, respectively associated to a two-loop three-point, and a four-loop two-point integral. These two examples are the playgrounds for showing more technical aspects about: Laurent expansion of the differential equations in D (around D=4); the choice of the boundary conditions; and the link among differential and difference equations for Feynman integrals.

Keywords: Feynman Integrals; Differential Equations.

PACS numbers: 02.30.Hq, 02.70.Bf, 11.10-z, 12.20.-m

[*] mario.argeri@unina.it
[†] pierpaolo.mastrolia@physik.unizh.ch

1. Preface

That "*differentiating* is an operation easier than *integrating*" is a statement that should not sound too surprising; while, a more pleasant wonder might result when suitable *differentiations* make us reduce, if not avoid at all, the number of direct *integrations* - of course the two operations, being the inverse of each other, have not to be thought as performed with respect to the same variable!
As paradigmatic example, let us just consider the class of integrals,

$$I_n(\alpha) = \int_0^\infty \mathrm{e}^{-\alpha x^2} \ x^n \ dx \ .$$

For $n = 0$, this is just the Gaussian integral,

$$I_0(\alpha) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \ ;$$

while for $n = 1$, the integrand is integrable by quadrature,

$$I_1(\alpha) = \frac{1}{2 \ \alpha} \ .$$

To compute I_n with $n > 1$, one can use the identity,

$$-\frac{\partial}{\partial\alpha} I_{n-2} = I_n \ .$$

In fact,

- $n = 2s$

$$I_n = \Big(-\frac{\partial}{\partial\alpha} \Big) I_{n-2} = \Big(-\frac{\partial}{\partial\alpha} \Big)^2 I_{n-4} = \ldots = \Big(-\frac{\partial}{\partial\alpha} \Big)^{\frac{n}{2}} I_0 \ ;$$

- $n = 2s + 1$

$$I_n = \Big(-\frac{\partial}{\partial\alpha} \Big) I_{n-2} = \Big(-\frac{\partial}{\partial\alpha} \Big)^2 I_{n-4} = \ldots = \Big(-\frac{\partial}{\partial\alpha} \Big)^{\frac{(n-1)}{2}} I_1 \ .$$

Therefore the infinite set of integrals I_n can be computed without any integration, provided the knowledge of just two basic integrals, namely I_0, and I_1, that in the forthcoming terminology would be defined as the *master integrals* of the class I_n.

The above example was a too lucky one: *i)* the repeated α-derivative did not entangle integrals having even and odd indices, therefore I_0 and I_1 never appear linked by any differential identity; *ii)* the value of the master integrals was known, possibly obtained by direct integrations.

In the more general case masters' are unknown, and their evaluation becomes an open problem. In the following pages, we will see how the exploitation of integration-by-parts not only yields algebraic relations among infinite sets of integrals and their masters', but as well leads to differential equations satisfied by the master integrals themselves. Solving these differential equations becomes a tool for computing master integrals, when their direct integration is not viable.
As it happens to (many) Feynman integrals.

2. Introduction

A perturbative approach to the quantitative description of the scattering of particles in quantum field theory involves the computation of Feynman diagrams. For a given number of external particles - the *legs* of diagram - fixed by the process under study, and a given order in perturbation theory, the skeletons of diagrams are built up by joining the edges of legs and propagators into vertexes, forming *tree* patterns and closed loops.

Beyond the tree level, each Feynman diagram represents an integral which has, in general, a tensorial structure, induced by the tensorial nature of the interacting fields. Therefore, the result of its evaluation must be a linear combination of the tensors provided by the theory and by the kinematics of the process under study. The coefficients of this linear combination, usually called *form factors*, can be always extracted from each Feynman diagram, before performing any evaluation, by means of suitably chosen projectors.

These form factors are *scalar integrals* closely connected to the original Feynman diagram: the numerator of their integrand may contain all the possible scalar products formed by external momenta and loop variables; whereas its denominator is formed by the denominators of propagators present in the diagram itself.

Due to the bad convergence of loop integrals in four dimensions, regularization prescriptions are mandatory. Hereafter the integrals are regularised within the framework of 't Hooft-Veltman continuous-dimensional regularisation scheme[156]. Accordingly, the dimension D of an extended integration space is used as a regulator for both infrared (IR) and ultraviolet (UV) divergences, which finally do appear as poles in $(D-4)$ when D goes to 4 [156].

The aim of a precise calculation is to compute Feynman diagrams for any value of the available kinematic invariants. Except in case of simple configurations (e.g. very few legs and/or few scales), quite generally, approximations have to be taken by limiting the result to a specific kinematics domain, and, thus, looking for a hierarchy among the scales, to get rid of the ones which anyhow would give a negligible contribution in that domain.

The puzzling complexity of the Feynman diagrams calculation arises because of two different sources: either multi-leg or multi-loop processes. In recent years the progress in the evaluation of higher loop radiative corrections in quantum field theory has received a strong boost, due to the optimisation and automatising of various techniques (see refs. in [59,79,77]). In this work we review one of the most effective computational tools which have been developed in the framework of the dimensional regularization: the method of *differential equations for Feynman integrals.*

The method was first proposed by Kotikov[1] in the early nineties, while dealing with the evaluation of 2- and 3-point functions. The basic idea was to consider a given unknown integral as a function of one of the propagator masses, and to write for it a differential equation in that variable. Thus, instead of addressing its direct integration, the value of the integral could be found by solving the differential equation.

The advantages of the novel ideas were soon realized [2,3,4,5,6], and generalised at a later stage by Remiddi [7], who proposed the differentiation with respect to any other available kinematics invariants formed by the external momenta. That enabled the application of the differential equation method also to integral with massless propagators (provided the existence of any other non-trivial scale).

Finally, Remiddi and Gehrmann [8,9,10] fully developed the method by showing its effectiveness through the systematic application to a non-trivial class of two-loop four-point functions, whose result is still considered as state-of-the-art.

From that moment on, the method of differential equations became to be widely used in different contexts [11,12,13,14,15,16,17,18,19,20,21,22]. The lists of unprecedented results obtained through its application spans among multi-loop functions from zero to four external legs, [25,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,137,138,139,140,141] and within the most interesting sectors of particle phenomenology, like jets physics [8,9,10,50,51,52,53,54,55,56,57,58]; QED corrections to lepton form factors [80,81,82]; Bhabha Scattering [109,110,111,112,113,114,115,116,117]; QCD corrections to lepton form factors and top-physics [83,84,85,86,87], forward-backward asymmetry of heavy-quark [95,96,97,98,99]; Higgs Physics [103,104,105,106,107,108]; Electroweak sector [121,122,123,124,125,126]; Sudakov form factors [100,101,102]; semileptonic decay [118,119,120]; static parameters and gauge boson properties [88,89,90,91,92,93,94].

The efforts to achieve analytic solutions of differential equations for Feynman integrals has stimulated new developments on the more mathematical side [127,128], especially concerning the properties of transcendental functions [132,133,134,135,136]. In particular, a novel set of functions that generalize Nielsen polylogarithms, the so called Harmonic Polylogarithms (HPL) [129,130,131], have been found suitable for casting the result in *analytic* form - that means without ambiguities due to zeroes hidden in functional relations, and supplied with series expansions. While HPL's can be considered as iterative integral of rational kernels, recently it has been pointed out that the solution of differential equations for generic integrals with massive loops demands as well for irrational integration kernels, yielding elliptic functions [148,102].

The range of applicability of the differential equations technique is broadened by the possibility of a natural switch toward a semi-numerical approach, since, whenever the analytic integration were not required or not viable, the differential equation(s), analytically obtained, could be solved with numerical techniques [23,24].

Nowadays we are not at the point to have *the* method for evaluating any Feynman integral, but certainly we dispose of several tools [60,61,71,62,65,63,64,67,68,72,73,69,70,74,75,76,77] to attack successfully many problems in perturbation theory, and usually a combination of them is necessary for the achievement of the final answer. Therefore let us discuss in detail how to build ans solve differential equations for integral associated to Feynman graphs.

The computational strategy is twofold.

- In a preliminary stage, by exploiting some remarkable properties of the dimensionally regularised integrals, namely *integration-by-parts identities* (IBP), *Lorentz invariance identities* (LI), and further sets of identities due to kinematic symmetry specific of each diagram, one establishes several relations among the whole set of scalar integrals associated to the original Feynman diagram.

  By doing so, one reduces the result, initially demanding for a large number of scalar integrals (from hundreds to billions, according to the case), to a combination of a minimal set (usually of the order of tens) of independent functions, the so called *master integrals* (MI's).

- The second phase consists of the actual evaluation of the MI's. By using the set of identities previously obtained, it is also possible to write Differential Equations in the kinematic invariants which are satisfied by the MI's themselves. When possible, these equations can be solved exactly in D dimensions. Alternatively, they can be Laurent-expanded around suitable values of the dimensional parameter up to the required order, obtaining a system of chained differential equations for the coefficients of the expansions, which, in the most general case, are finally integrated by Euler's *variation of constants* method.

One of the key advantages of the method is that it yields a clear separation between the merely algebraic part of the work - which is, not surprisingly, always very heavy in multiloop calculations, and can be most conveniently processed by a computer algebra program[152]-, from the actual analytic issues of the problem, which can then be better investigated without the disturbance of the algebraic complexity.
The paper is organised as follows. In section 2, it is described how to reduce a generic (combination of) Feynman integrals to a limited set of MI's and how to write the system of *differential equations* they fulfil. As illustrative applications, respectively in section 3-4, the one-loop and two-loop vacuum polarization functions in QED are explicitly computed. In the further two sections, we discuss some less obvious example of differential equations. In section 5, we describe the solution of a system of three coupled first-order differential equations, to compute three MI's associated to a class of two-loop 3-point functions, addressing as well the problem of finding their boundary conditions. While in section 5, we describe the solution of a fourth-order differential equation involving the MI's of a 4-loop 2-point diagram, and it will be considered the link between differential and *difference equation* for Feynman integrals [65].

3. Reduction to Master Integrals

3.1. *Topologies and Integrals*

Let us consider a Feynman diagram with ℓ loops, g external legs, d internal lines and a given tensorial structure. Such a diagram, when not representing a scalar quantity, can be decomposed as a combination of products of a scalar form factor times a tensor. Thus, computing the contribution of any diagram to a given process is equivalent to the computation of scalar factors, which, after some preliminary algebra (evaluation of Dirac traces and contraction of Lorentz indices) reads as a combination of scalar integrals like,

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\Pi_{i=1}^{N_{sp}} S_i^{n_i}}{\Pi_{j=1}^{d} \mathcal{D}_j}, \tag{1}$$

where

- S_i $(i = 1, \cdots, N_{sp})$ is any scalar product formed by either one of independent external momentum and an internal loop momentum, or by two internal loop momenta (n_i is an integer exponent such that $n_i \geq 0$) and N_{sp} is the total number of such scalar products, given by

$$N_{sp} = \underbrace{\ell(g-1)}_{s.p.\ external-internal} + \underbrace{\frac{\ell(\ell+1)}{2}}_{s.p.\ internal-internal} = \ell\left(g + \frac{\ell}{2} - \frac{1}{2}\right) \tag{2}$$

- $\mathcal{D}_j = q_j^2 + m_j^2$ $(j = 1, \cdots, d)$ is the denominator of the j-th propagator being q_j and m_j, respectively, the corresponding momentum and mass. From now, we call $\mathcal{D}_j$ *propagator*.

The first task relies on a suitable classification of the integrals, in order to minimize the number of the ones which have to be actually integrated.

As a preliminary set up, one is invited to consider not anymore diagrams but *topologies*, that, drawn exactly as scalar Feynman diagrams, contain only and all different propagators (and scalar vertices). In view of settling down a correspondence between a given topology and the class of integrals it represents, one proceeds by simplifying as well the number of scalar products in the numerator. The number N_{sp} is a redundant quantity and can be easily reduced, with a procedure commonly called *trivial tensor reduction*, according to which some (when not all) of the scalar products can be expressed in terms of the denominators of the topology.

In general, if t is the number of propagators of a given topology, we can express t of the N_{sp} scalar products containing the loop momenta in terms of the denominators - which will be later on simplified against the corresponding term in the denominator. Thus the most general integral associated to every topology does contain only the remaining q $(= N_{sp} - t)$ *irreducible* scalar products and reads as

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\Pi_{i=1}^q S_i^{n_i}}{\Pi_{j=1}^t \mathcal{D}_j^{m_j}}, \tag{3}$$

where $n_i \geq 0$ and $m_j \geq 1$.

One can denote with $I_{t,r,s}$ the class of the integrals with: a given set of t denominators; $q = N_{sp} - t$ irreducible scalar products; a total of $r = \sum_i (m_i - 1)$ powers of the t denominators; and $s = \sum_j n_j$ powers of the q scalar products. It can be shown that the number of the integrals belonging to the class $I_{t,r,s}$ is

$$N(I_{t,r,s}) = \binom{r+t-1}{t-1}\binom{s+q-1}{q-1}. \tag{4}$$

In the next sections, we will see that integrals of the type (3), belonging to a given topology, therefore differing for the values of the indices m_i, n_i, are not independent. Algebraic relations among them, can be written in the form of a sum of a finite number of terms set equal to zero, where each term is given by the product of a polynomial (of finite order and with integer coefficients in the variable D, masses and Mandelstam invariants) and one of the integrals belonging to $I_{t,r,s}$. They can be used recursively to express as many as possible integrals of a given class in terms of as few as possible (suitably chosen) ones. The way to generate those kind of relations goes through integration-by-parts, Lorentz invariance and symmetry considerations.

Before going on with the discussion let us see explicitly how the general definition we have introduced do apply in practice.

3.1.1. *Example: from the diagram to the topology*

Let us consider the Feynman diagram showed in Fig.1

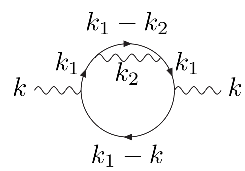


Fig. 1. Feynman diagram with g=2 legs and $\ell = 2$ loops

with $g = 2$ external legs, $\ell = 2$ loops and $d = 5$ internal lines, which gives a number of scalar products amounting to,

$$N_{sp} = 2\left(2 + 1 - \frac{1}{2}\right) = 5 \ .$$

The most general set of scalar integrals possibly arising from its computation has the following representation,

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\Pi_{i=1}^5 S_i^{n_i}}{\Pi_{j=1}^5 \mathcal{D}_j}, \tag{5}$$

where

$$\begin{cases} \mathcal{D}_1 = k_1^2 + m^2 \\ \mathcal{D}_2 = k_2^2 \\ \mathcal{D}_3 = (k_1 - k_2)^2 + m^2 \\ \mathcal{D}_4 = (k_1 - k)^2 + m^2 \\ \mathcal{D}_5 = \mathcal{D}_1 = k_1^2 + m^2 \end{cases} \qquad \begin{cases} S_1 = k_1^2 \\ S_2 = k_2^2 \\ S_3 = k_1 \cdot k_2 \\ S_4 = k \cdot k_1 \\ S_5 = k \cdot k_2 \end{cases}$$

The original diagram contains five internal lines, but two propagators are indeed equal, so there are only four different propagators. Therefore, the integrals (5) actually belong to the *simpler* set,

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\Pi_{i=1}^5 S_i^{n_i}}{\mathcal{D}_1^2 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4}, \tag{6}$$

which can be represented by the topology in Fig. 2.

The trivial tensor reduction of the scalar product can be realized according to the following table, where one chooses to express four (out of five) scalar products in terms of the denominators. In the end, only one of the five

Scalar product S_i	Corresponding propagator	Relationship
$S_1 = k_1^2$	$\mathcal{D}_1 = k_1^2 + m^2$	$k_1^2 = \mathcal{D}_1 - m^2$
$S_2 = k_2^2$	$\mathcal{D}_2 = k_2^2$	$k_2^2 = \mathcal{D}_2$
$S_3 = k_1 \cdot k_2$	$\mathcal{D}_3 = (k_1 - k_2)^2 + m^2$	$k_1 \cdot k_2 = \frac{1}{2}(\mathcal{D}_1 + \mathcal{D}_2 - \mathcal{D}_3)$
$S_4 = k \cdot k_1$	$\mathcal{D}_4 = (k_1 - k)^2 + m^2$	$k \cdot k_1 = \frac{1}{2}(\mathcal{D}_1 - \mathcal{D}_4 + k^2)$

scalar products involving the loop momenta is left over as irreducible, arbitrary chosen to be $k_2 \cdot k$. Therefore, the integrals in (5), represented by the topology in Fig.2, indeed belongs to the class,

$$\mathcal{I}(n_1; m_1, m_2, m_3, m_4) \int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{(k_2 \cdot k)^{n_1}}{\mathcal{D}_1^{m_1} \mathcal{D}_2^{m_2} \mathcal{D}_3^{m_3} \mathcal{D}_4^{m_4}}. \tag{7}$$

The trivial tensor reduction might as well lead to the complete cancellation of some denominator. Should it be the case, the resulting integral can be classified as belonging to the subtopology obtained by pinching the corresponding internal line.

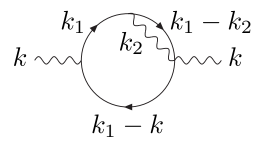


Fig. 2. 4-denominators topology

3.2. *Integration-by-parts Identities*

Integration-by-parts identities (IBP-Id's) are among the most remarkable properties of dimensionally regularized integrals and they were first proposed in the eighties by Chetyrkin and Tkachov [149]. The basic idea underlying IBP-Id's is an extension to D-dimensional spaces of Gauss' theorem. For each of the integrals defined in equation (3) one can write the vanishing of the integral of a divergence given by,

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right\} = 0. \tag{8}$$

In the above identities the index i runs over the number of loops ($i = 1, 2, ..., \ell$), and the vector v_μ can be any of the $(\ell + g - 1)$ independent vectors of the problem:$k_1, \cdots, k_\ell, p_1, \cdots, p_{g-1}$; in such way, for each integrand, $\ell(\ell + g - 1)$ IBP-Id's can be established. When evaluating explicitly the derivatives, one obtains a combination of integrands with a total power of the irreducible scalar products equal to $(s-1)$, s and $(s+1)$ and total powers of the propagators in the denominator equal to $(t+r)$ and $(t+r+1)$, therefore involving, besides the integrals of the class $I_{r,s,t}$, also the classes $I_{t,r,s-1}$, $I_{t,r+1,s}$ and $I_{t,r+1,s+1}$. Simplifications between reducible scalar products and propagators in the denominator may occur, lowering the powers of the propagators. During that simplification, some propagator might disappear, generating an integral belonging to a *subtopology*, with $t - 1$ propagators.

3.3. *Lorentz Invariance Identities*

Another class of identities can be derived by exploiting a general properties of the integrals in (4), namely their nature as Lorentz scalars. If we consider an infinitesimal Lorentz transformation on the external momenta, $p_i \to p_i + \delta p_i$, where $\delta p_i = \omega_{\mu\nu} p_{i,\nu}$ with $\omega_{\mu\nu}$ a totally antisymmetric tensor, we have

$$I(p_i + \delta p_i) = I(p_i). \tag{9}$$

Because of the antisymmetry of $\omega_{\mu\nu}$ and because

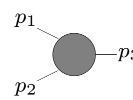
$$I(p_i + \delta p_i) = I(p_i) + \sum_n \frac{\partial I(p_i)}{\partial p_{n,\mu}} = I(p_i) + \omega_{\mu\nu} \sum_n p_{n,\nu} \frac{\partial I(p_i)}{\partial p_{n,\mu}}, \tag{10}$$

we can write the following relation

$$\sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right) I(p_i) = 0. \tag{11}$$

Eq. (11) can be contracted with all possible antisymmetric combinations of the external momenta $p_{i,\mu}p_{j,\nu}$, to obtain other identities for the considered integrals.
In case of integral associated to any *vertex* topologies with two independent external momenta, p_1 and p_2, we can build up the identity

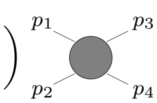
$$\left[(p_1 \cdot p_2)\left(p_{1,\mu}\frac{\partial}{\partial p_{1,\mu}} - p_{2,\mu}\frac{\partial}{\partial p_{2,\mu}} + p_2^2 p_{1,\mu}\frac{\partial}{\partial p_{2,\mu}} - p_1^2 p_{2,\mu}\frac{\partial}{\partial p_{1,\mu}}\right)\right]$$



$$= 0 \tag{12}$$

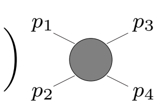
In the case of a richer kinematics, like in the case of integrals associated to *box* topologies with three independent external momenta, p_1, p_2 and p_3, we ca write down three LI-id's

$$(p_{1,\mu}p_{2,\mu} - p_{1,\nu}p_{2,\mu}) \sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right)$$



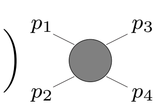
$$= 0, \tag{13}$$

$$(p_{1,\mu}p_{3,\mu} - p_{1,\nu}p_{3,\mu}) \sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right)$$



$$= 0, \tag{14}$$

$$(p_{2,\mu}p_{3,\mu} - p_{2,\nu}p_{3,\mu}) \sum_n \left(p_{n,\nu}\frac{\partial}{\partial p_{n,\mu}} - p_{n,\mu}\frac{\partial}{\partial p_{n,\nu}} \right)$$



$$= 0\ . \tag{15}$$

3.4. *Symmetry relations*

In general, further identities among Feynman integrals can arise when it is possible to redefine the loop momenta so that the value of the integral itself does not change, but the integrand transforms into a combination of different integrands. By imposing the identity of the original integral to the combination of integrals resulting from the change of loop momenta, one obtains a set of identities relating integrals belonging to the same topology.

More identities, originally found by Larin, may arise as well when there is a sub-loop diagram depending - after its integration - on a specific combination of momenta. By equating the original integral and the one obtained by projecting onto such a combination of momenta, one gets additional relations (see [78] for details).

3.4.1. *Example 1: IBP-id's*

Let us take again the two-loop topology of Fig. 2, and its class of integrals (7). The corresponding IBP-Id's can be established by the vanishing of the following integral of a divergence,

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{(k \cdot k_2)^{n_1}}{\mathcal{D}_1^{m_1}\mathcal{D}_2^{m_2}\mathcal{D}_3^{m_3}\mathcal{D}_4^{m_4}} \right\} = 0 \quad (i = 1, 2), \tag{16}$$

where $v_\mu = k_1, k_2, k$, which amount to $3 \times 2 = 6$ identities.
For simplicity let us choose $n_1 = 0$ and $m_1 = ... = m_4 = 1$. We have

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ \frac{v_\mu}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right\} = 0 \quad (i = 1, 2). \tag{17}$$

The six IBP-Id's are resumed in the following table

derivative $\frac{\partial}{\partial k_{i,\mu}}$	*vector* v_μ	*corresponding* IBP-Id
$\frac{\partial}{\partial k_{1,\mu}}$	$k_{1,\mu}$	$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{1,\mu}} \left(\frac{k_{1,\mu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = 0$
$\frac{\partial}{\partial k_{1,\mu}}$	$k_{2,\mu}$	$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{1,\mu}} \left(\frac{k_{2,\mu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = 0$
$\frac{\partial}{\partial k_{1,\mu}}$	k_{μ}	$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{1,\mu}} \left(\frac{k_{\mu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = 0$
$\frac{\partial}{\partial k_{2,\mu}}$	$k_{1,\mu}$	$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{2,\mu}} \left(\frac{k_{1,\mu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = 0$
$\frac{\partial}{\partial k_{2,\mu}}$	$k_{2,\mu}$	$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{2,\mu}} \left(\frac{k_{2,\mu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = 0$
$\frac{\partial}{\partial k_{2,\mu}}$	k_{μ}	$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{2,\mu}} \left(\frac{k_{\mu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = 0$

Fig. 3. IBP for the topology of Fig.2

Let us perform explicitly the calculation of the last of above identities.
We have

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{2,\mu}} \left(\frac{k_\mu}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = 0 \; . \tag{18}$$

By performing the derivative with respect to $k_{2,\mu}$, we can rewrite the integrand as follows

$$\begin{aligned}
&\frac{\partial}{\partial k_{2,\mu}} \left(\frac{k_\mu}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right) = \frac{\partial k_\mu}{\partial k_{2,\mu}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} + \\
&+ k_\mu \frac{1}{\mathcal{D}_1 \mathcal{D}_4} \frac{\partial}{\partial k_{2,\mu}} \left(\frac{1}{\mathcal{D}_2 \mathcal{D}_3} \right) = k_\mu \frac{1}{\mathcal{D}_1 \mathcal{D}_4} \left[-\frac{2 k_{2,\mu}}{\mathcal{D}_2^2 \mathcal{D}_3} + \frac{2(k_{1,\mu} - k_{2,\mu})}{\mathcal{D}_2 \mathcal{D}_3^2} \right] = \\
&= -\frac{2 k \cdot k_2}{\mathcal{D}_1 \mathcal{D}_2^2 \mathcal{D}_3 \mathcal{D}_4} + \frac{2 k \cdot k_1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3^2 \mathcal{D}_4} - \frac{2 k \cdot k_2}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3^2 \mathcal{D}_4} .
\end{aligned} \tag{19}$$

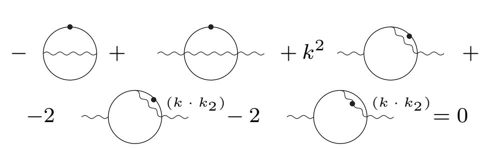
After expressing the scalar product $k \cdot k_1$ in terms of the propagators $\mathcal{D}_1 = k_1^2 + m^2$ and $\mathcal{D}_4 = (k_1 - k)^2 + m^2$

$$2 k \cdot k_1 = \mathcal{D}_1 - \mathcal{D}_4 + k^2, \tag{20}$$

and substituting Eq. (20) in Eq. (19), we obtain,

$$\begin{aligned}
&- \int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3^2} + \int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_2 \mathcal{D}_3^2 \mathcal{D}_4} + \\
&+ k^2 \int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3^2 \mathcal{D}_4} - 2 \int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{k \cdot k_2}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3^2 \mathcal{D}_4} + \\
&- 2 \int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{k \cdot k_2}{\mathcal{D}_1 \mathcal{D}_2^2 \mathcal{D}_3 \mathcal{D}_4} = 0
\end{aligned} \tag{21}$$

Eq. (21) can be pictorially represented by



(22)

where a dot on a propagator line means that the propagator is squared; irreducible scalar products in the numerator are explicitly indicated.

3.4.2. *Example 2: IBP-id's*

Let us consider the case of the two-loop three-leg topology of Fig.4.

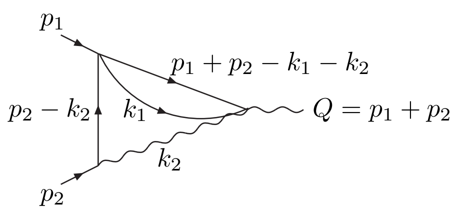


Fig. 4. Example of a two-loop three-leg four-denominators topology: $p_1^2 = p_2^2 = -m^2$, and $(p_1 + p_2)^2 = Q^2 = -s$

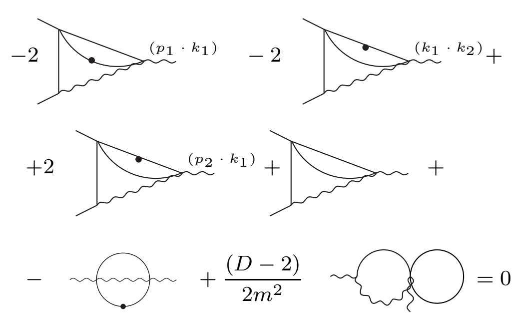
The integrals associated to this topology can have three irreducible scalar products, arbitrary chosen to be $(p_1 \cdot k_1)$, $(p_2 \cdot k_1)$ and $(k_1 \cdot k_2)$. In doing so, Eq. (8), for generic values of the indices m_i and n_i, reads

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{i,\mu}} \left\{ v_\mu \frac{(p_1 \cdot k_1)^{n_1} (p_2 \cdot k_1)^{n_2} (k_1 \cdot k_2)^{n_3}}{\mathcal{D}_1^{m_1} \mathcal{D}_2^{m_2} \mathcal{D}_3^{m_3} \mathcal{D}_4^{m_4}} \right\} = 0 \tag{23}$$

where $\mathcal{D}_1 = k_1^2 + m^2, \mathcal{D}_2 = k_2^2, \mathcal{D}_3 = (p_2 - k_2)^2 + m^2, \mathcal{D}_4 = (p_1 + p_2 - k_1 - k_2)^2 + m^2$ and $i = 1, 2$. By setting, for simplicity, $m_1 = \cdots = m_4 = 1$, $n_1 = \cdots = n_3 = 0$, $v_\mu = p_1$ and $i = 1$, Eq. (23) becomes

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{\partial}{\partial k_{1,\mu}} \left\{ \frac{p_{1,\mu}}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \right\} = 0. \tag{24}$$

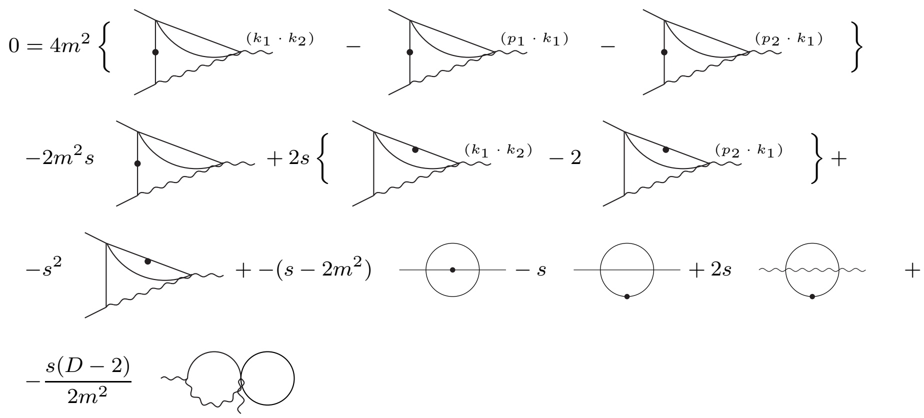
After taking the derivative with respect to k_1 and simplifying the reducible scalar products against the corresponding propagators, one can write Eq.(24) as follows



(25)

3.4.3. *Example: LI-id's*

The Lorentz Invariance identity (12) for the integral of the topology in Fig. 4 with $n_1 = n_2 = n_3 = 0$ and $m_1 = m_2 = m_3 = m_4 = 1$, reads,



(26)

3.4.4. *Example: Symmetry relations*

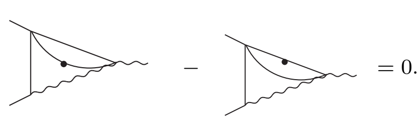
One can exploit the invariance of the integrals belonging to the topology of Fig. 4,

$$\int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{(p_1 \cdot k_1)^{n_1} (p_2 \cdot k_1)^{n_2} (k_1 \cdot k_2)^{n_3}}{\mathcal{D}_1^{m_1} \mathcal{D}_2^{m_2} \mathcal{D}_3^{m_3} \mathcal{D}_4^{m_4}}. \tag{27}$$

under the redefinition of the momenta running in the nested electron-loop. In fact the two denominators with momentum k_1 and $(p_1 + p_2 - k_1 - k_2)$, i.e. $\mathcal{D}_1 = k_1^2 + m^2$ and $\mathcal{D}_4 = (p_1 + p_2 - k_1 - k_2) + m^2$ have the same mass. Therefore the following redefinition of the integration momentum

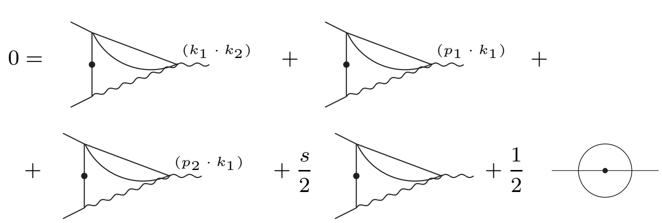
$$k_1 = p_1 + p_2 - k_1' - k_2, \tag{28}$$

that consists in the interchange of the two denominators in the closed electron loop, does not affect the topology of the integral; nevertheless the explicit form of the integrand can change generating non trivial identities. Taking for instance $n_1 = n_2 = n_3 = 0$ and $m_1 = m_2 = m_4 = 1$ and $m_3 = 2$ in Eq. (27), the substitution (28) gives the following very simple relation



(29)

By choosing $n_1 = n_2 = 0$, $n_3 = 1$ and $m_1 = m_2 = m_4 = 1$ and $m_3 = 2$, we get a less trivial identity



(30)

3.4.5. *Master Integrals*

For each of the $N(I_{t,r,s})$ integrals of the class $I_{t,r,s}$ (see Eq.(4)) we dispose of IBP-id's, LI-id's and symmetry relations, involving integrals of the families up to $I_{t,r+1,s+1}$. For $r = s = 0$ the number of all the integrals involved in the identities exceeds the number of equations; but, in writing down systematically all the equations for increasing values of r and s the number of equations grows faster than the number of the integrals, generating an apparently overconstrained system of equations — as realized first by Stefano Laporta [150]. After its generation one is left with the problem of solving such a linear system of identities which is trivial in principle, but algebraically very lengthy, so that some organization is needed for achieving the solution[151].
To this end, one can order the integrals in an appropriate way. In particular a "weight" is assigned to each integral: the weight can be almost any increasing function of the indices n_i and m_j, such that integrals with higher indices have bigger weights. The system can be solved by Gauss' substitution rule, by considering the equations of the system one by one, and using each equation for expressing the integral with the highest weight in terms of the integrals of lower weight. Then the result is substituted in the leftover equations. The algorithm, by now known as *Laporta algorithm*, is straightforward, but its execution requires a great amount of algebra, which has been implemented in several computer codes[152].

One finds that several equations are identically satisfied, and the original unknown integrals are expressed in terms of very few independent integrals, the *master integrals* (MI's). In doing so, the resulting MI's correspond to the integrals of lowest weight; but as the choice of the weight is to a large extent arbitrary, there is also some freedom in the choice of the integrals to pick up as actual MI's (not in their number of course!).
There are several cases in which more than one MI is found for a given topology, while sometimes only one MI is present. It may also happen that no MI for the considered topology is left over in the reduction — *i.e.* all the integrals corresponding to a given topology, with t propagators, can be expressed in terms of the MI's of its subtopologies with $(t-1)$, and/or less, propagators. In such a case one speaks about *reducible topologies*.

Finally, three points are worth mentioning.
First, any given topology with t propagators has $(t-1)$ subtopologies with $(t-1)$ propagators, $(t-1)(t-2)$ subtopologies with $(t-2)$ propagators etc., down to as many propagators as the number of loops. It turns out, however, that many subtopologies are in fact equivalent, up to a translation of the loop variable, and the subtopologies coming from the contribution of different graphs do overlap to a great extent. For these reasons the compilation of the system for the actual number of all *independent* topologies, namely those that cannot be transformed one into the others by a redefinition of internal momenta, is relatively small.

The second remark is about the *independence* of the identities for a given topology. Under an idealistic point of view, it might happen that the infinite set if IBP-id's, obtained by considering the infinite choices of indices of denominators and scalar products for the considered topology, plus the infinite set of IBP-id's of all the *parent* topologies containing it as a subtopology, could include LI-id's and symmetry relations as subset. Certainly, at the practical level, since one is working with finite sets of indices, LI-id's and symmetry relations can be considered as *additional* to the IBP-id's, which surely speed up the solution of the system and work as further checks of the reduction procedure.

Third, we are not able to prove analytically that the MI's one finds are really the minimal set of MI's, *i.e.* that they are independent from each other; in any case the final number of the MI's is quite small, so that reducing the several hundred integrals occurring in a typical calculation to a few of them is after all a great progress. Studies on the *apriori* determination of whether a topology could have or not MI's, have been performed by Baikov [153] and Smirnov [154]. Although a general algorithm is still lacking, we think that the analysis of leading and subleading Landau singularities (see [155] for an extensive treatment) could be related to this issue: since a reducible topology without MI's is expressible in terms of the MI's of its subtopologies, it should mean that the leading singularities of the reducible topology are not independent of the subleading ones, which are leading singularities for the subtopologies.

As a last remark, let us recall that, at the end of the reduction, there is some freedom for choosing the basis of MI's and usually the choice is in general motivated by convenience. For example the behaviour of the functions in the D-to-4 expansion might determine to select: *i)* simpler integrands, in view of a successive analytic computation; *ii)* more complicated integrands, but with a better convergence, should their numerical evaluation be of interest.

4. Differential Equations for Master Integrals

The outcome of the *reduction* procedure, previously discussed, is a collection of identities thanks to which any expression, demanding originally for the evaluation of a very large number of integrals, is simplified and written as linear combination of few MI's with rational coefficients. The completion of the analytic achievement of the result proceeds with the evaluation of the yet unknown MI's. As we will see in a moment, the same collection of identities is as well necessary to write Differential Equations satisfied by the MI's.

4.1. *System of differential equations*

Once all the MI's of a given topology are identified, the problem of their calculation arises. Exactly at this stage of the computation, *differential equations* enter the game. The use of differential equations in one of the internal masses was first proposed out by Kotikov[1], then extended to more general differential equations in any of Mandelstam variables by Remiddi[7].
Let us point out the basic idea of the method. To begin with, consider any scalar integral defined as

$$M(s_1, s_2, \cdots, s_{\mathcal{N}}) = \int \frac{d^D k_1}{(2\pi)^{D-2}} \cdots \frac{d^D k_\ell}{(2\pi)^{D-2}} \frac{S_1^{n_1} \ldots S_q^{n_q}}{\mathcal{D}_1 \ldots \mathcal{D}_t}, \tag{31}$$

where $\{s_1, s_2, \cdots, s_{\mathcal{N}}\}$ is any set of kinematic invariants of the topology and $\mathcal{N}$ is the number of such invariants.

Let us denote the set $\{s_1, s_2, \cdots, s_{\mathcal{N}}\} = \mathbf{s}$ and consider the following quantities

$$O_{jk}(\mathbf{s}) = p_{j,\mu} \frac{\partial M(\mathbf{s})}{\partial p_{k,\mu}} \quad (j, k = 1, 2, \cdots, g-1), \tag{32}$$

where $g-1$ is the number of independent external momenta. By the chain differentiation rule we have

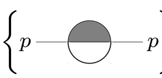
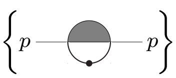
$$O_{jk}(\mathbf{s}) = p_{j,\mu} \cdot \sum_{\alpha=1}^{\mathcal{N}} \frac{\partial s_\alpha}{\partial p_{k,\mu}} \frac{\partial M(\mathbf{s})}{\partial s_\alpha} = \sum_{\alpha=1}^{\mathcal{N}} \left(p_{j,\mu} \cdot \frac{\partial s_\alpha}{\partial p_{k,\mu}} \right) \frac{\partial M(\mathbf{s})}{\partial s_\alpha}. \tag{33}$$

According to the available number of the kinematic invariants, the *r.h.s.* of Eq. (32) and the *r.h.s.* of Eq. (33) may be equated to form the following system

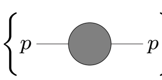
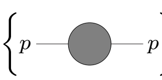
$$\sum_{\alpha=1}^{\mathcal{N}} \left(p_{j,\mu} \cdot \frac{\partial s_\alpha}{\partial p_{k,\mu}} \right) \frac{\partial M(\mathbf{s})}{\partial s_\alpha} = p_{j,\mu} \frac{\partial M(\mathbf{s})}{\partial p_{k,\mu}}, \tag{34}$$

which can be solved in order to express $\frac{\partial M(\mathbf{s})}{\partial s_\alpha}$ in terms of $p_{j,\mu} \frac{\partial M(\mathbf{s})}{\partial p_{k,\mu}}$, so the corresponding identity, can be finally read as a *differential equation* for M.
Examples of such equations are the following.

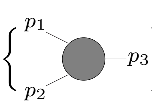
- *2-point case.*
  - Differentiation with respect to a mass

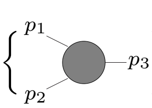
$$\frac{\partial}{\partial m^2} \Big\{$$  $$\Big\} = - \Big\{$$  $$\Big\} \tag{35}$$

where, for simplicity, we assumed there is just one propagator of mass m.
  - Differentiation with respect to the squared momentum

$$p^2 \frac{\partial}{\partial p^2} \Big\{$$  $$\Big\} = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} \Big\{$$  $$\Big\} \tag{36}$$

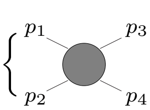
- *3-point case.*

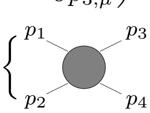
$$P^2 \frac{\partial}{\partial P^2} \Big\{$$  $$\Big\} =$$

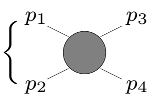
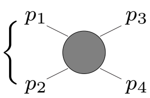
$$= \left[A \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left(p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right] \Big\{$$  $$\Big\}, \tag{37}$$

with $P = p_1 + p_2$ and A, B rational coefficients.

- *4-point case.*

$$P^2 \frac{\partial}{\partial P^2} \Big\{$$  $$\Big\} = \left[C \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + \right.$$

$$\left. + \; E(p_{1,\mu} + p_{3,\mu}) \left(\frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right] \Big\{$$  $$\Big\}, \tag{38}$$

$$Q^2 \frac{\partial}{\partial Q^2} \Big\{$$

$$\Big\} = \left[F \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + G p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + \right.$$
$$\left. + \; H (p_{2,\mu} - p_{1,\mu}) \left(\frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} + \frac{\partial}{\partial p_{3,\mu}} \right) \right] \Big\{$$

$$\Big\}, \tag{39}$$

with $P = p_1 + p_2$, $Q = p_1 - p_3$ and C, D, E, F, G, H rational coefficients.

Equation (34) holds for any function $M(\mathbf{s})$. In particular, let us assume that $M(\mathbf{s})$ is a master integral. We can now substitute the expression of M in the *r.h.s.* of one of the Eqs.(36-39), according to the case, and perform the direct differentiation of the integrand. It is clear that we obtain a combination of several integrals, all belonging to the same topology as M. Therefore, we can use the solutions of the IBP-id's, LI-id's and other identities for that topology and express everything in terms of the MI's of the considered topology (and its subtopologies). If there is more than one MI, the procedure can be repeated for all of them as well. In so doing, one obtains a system of linear differential equations in $\mathbf{s}$ for M and for the other MI's (if any), expressing their $\mathbf{s}$-derivatives in terms of the MI's of the considered topology and of its subtopologies.

The system is formed by a set of *first-order differential equations* (ODE), one for each MI, say M_j, whose general structure reads like the following,

$$\frac{\partial}{\partial s_\alpha} M_j(D, \mathbf{s}) = \sum_k A_k(D, \mathbf{s}) \; M_k(D, \mathbf{s}) + \sum_h B_h(D, \mathbf{s}) \; N_h(D, \mathbf{s}) \tag{40}$$

where $\alpha = 1, \cdots, \mathcal{N}$, is the label of the invariants, and N_k are MI's of the subtopologies. Note that the above equations are exact in D, and the coefficients A_k, B_k are rational factors whose singularities represent the thresholds and the pseudothresholds of the solution.
The system of equations (40) for M_j is not homogeneous, as they may involve MI's of subtopologies. It is therefore natural to proceed bottom-up, starting from the equations for the MI's of the simplest topologies (i.e. with less denominators), solving those equations and using the results within the equations for the MI's of the more complicated topologies with additional propagators, whose non-homogeneous part can then be considered as known.

4.2. *Boundary conditions*

The coefficients of the differential equations (40) are in general singular at some kinematic points (thresholds and pseudothresholds), and correspondingly, the solutions of the equations can show singular behaviours in those points, while the unknown integral might have not. The boundary conditions for the differential equations are found by exploiting the known analytical properties of the MI's under consideration, imposing the regularity or the finiteness of the solution at the *pseudo-thresholds* of the MI. This qualitative information is sufficient for the quantitative determination of the otherwise arbitrary integration constants, which naturally arise when solving a system of differential equations.

4.3. *Laurent expansion around $D = 4$*

The system of differential equations (40) is exact in D, and, when possible, its solutions could be found for arbitrary value of the dimensional parameter. Very often, the result of the computation will have to be anyway expanded around some typical value of the space-time dimension. Indeed, in what follows, we will discuss the Laurent expansion of the solutions around $D = 4$, though the procedure can be applied equivalently for other values of D. In general, we use the ansatz

$$M_j(D, \mathbf{s}) = \sum_{k=-n_0}^{n} (D-4)^k \; M_j^{(k)}(\mathbf{s}) + \mathcal{O}\Big((D-4)^{(n+1)}\Big) \,, \tag{41}$$

where n_0 (positive) corresponds to the highest pole, and n is the required order in $(D-4)$.

When expanding systematically in $(D-4)$ all the MI's (including those appearing in the non-homogeneous part) and all the D-dependent terms of (41), one obtains a system of chained equations for the Laurent coefficients $M_j^{(k)}$ of (41). The first equation, corresponding to the highest pole involves only the coefficient $M_i^{(-n_0)}$ as unknown; the next equation, corresponding to the next pole in $(D-4)$, involves the $M_i^{(-n_0+1)}$ as unknown, but it may involve $M_i^{(-n_0)}$ in the non-homogeneous term (if it appears as multiplied by any power of $(D-4)$); but such a term, $M_i^{(-n_0)}$, has to be considered known once the equation for the highest pole has been solved. For the subsequent equations we have the same structure: at a given order k in $(D-4)$, the equations involve $M_j^{(k)}$ as unknown, and previous coefficients $M_j^{(\ell)}$ $(-n_0 \leq \ell < k)$ as known non-homogeneous terms.

Let us note that the homogeneous part of all the equations arising from the $D \to 4$ expansion of (41) is always the same and obviously identical to the homogeneous part of Eq.(40) read at $D = 4$. It is, therefore, natural to look for the solutions of the chained non-homogeneous equations by means of Euler's method of the *variation of the constants*, using repeatedly the solutions of the homogeneous equation as integration kernel, as we will see in the following chapters.

General algorithms for the solution of the homogeneous equations are not available; it turns out however that in all the considered cases the homogeneous equations at $D = 4$ have (almost trivial) solutions, so that Euler's formula can immediately be written. With suitable changes of variable, according to the kinematics of the problem, all integrations can further be carried out in closed analytic form, by exploiting the shuffle algebra induced by integration-by-parts on the nested integral representation of the solution.

5. Laplace Transform of Difference Equations

Difference Equations are functional relations among values of functions shifted by integers, and can be considered a "discretization" of differential equations. The set of identities used to derive Differential Equations in the external invariants of Feynman integral, can be used as well to derive Difference Equation in one of the denominator powers, as discussed by Laporta[65,66]. Alternatively, the shifted dimension relations among scalar integrals of Tarasov[26] naturally yield equations where the integer variable is the dimensional parameter D. We briefly describe the former kind of difference equation and their link to differential equations.

Let us consider the difference equation

$$\sum_{k=0}^{N} p_k(n)\ U(n+k) = 0\ , \tag{42}$$

where N is the equation order, and $p_k(n)$ are polynomial in n of maximum degree P, whose generic structure can be parametrised as follows,

$$p_k(n) = A_{k0} + \sum_{i=1}^{P} A_{ki} \prod_{j=0}^{i-1} (n+k+j)\ . \tag{43}$$

The Laplace transform method consists in the substitution

$$U(n) = \int_\gamma dt\ t^{n-1}\ v(t)\ , \tag{44}$$

where γ is a suitable integration path, and whose effect can be shown to be [65] the translation of the difference equation (42), into a differential equation for $v(t)$,

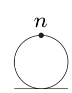
$$\sum_{i=0}^{P} \Phi_i(t)\ (-t)^i\ \frac{d^{(i)}}{dt^i} v(t) = 0\ , \tag{45}$$

with

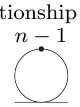
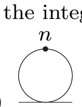
$$\Phi_i(t) = \sum_{k=0}^{N} A_{ki}\ t^k\ . \tag{46}$$

The solution $v(t)$ of Eq.(45), still carry unknown integration constants which will be present as well in $U(n)$, reconstructed by integrating Eq.(44). Finally, the value of the yet undetermined integration constants can be fixed by imposing boundary conditions in the large-n regime of $U(n)$, easily found by direct inspection of its original representation as loop integral[a].

5.0.1. *Example: one-loop Tadpole*

$$U(n) = \pi^{-\frac{D}{2}} \int \frac{d^D k}{(k^2+1)^n} =$$  $\ . \quad (47)$

By means IBP Id's, one can write the following relationship between the integrals of family (47)

$$-\left(n-1-\frac{D}{2}\right)$$  $+ (n-1)$  $= 0, \quad (48)$

which, after renaming $n \to n+1$, can be put in the form (42), and read as a first-order difference equation for the function $U(n)$,

$$-\left(n-\frac{D}{2}\right) U(n) + n\ U(n+1) = 0. \tag{49}$$

[a] In the rest of the paper, for computational convenience, we adopt two different definitions of the loop measure: *(i)* $d^D k/(2\pi)^{D-2}$, for integrals treated with the Differential Equations method; *(ii)* $d^D k/(\pi)^{D/2}$, in case of use of Difference Equations method.

The boundary condition is determined by the asymptotic behaviour ($n \to \infty$) of the integral (47), which reads

$$\pi^{-\frac{D}{2}} \int \frac{d^D k}{(k^2+1)^n} \approx \pi^{-\frac{D}{2}} \int d^D k e^{-nk^2} = n^{-\frac{D}{2}}. \tag{50}$$

One way to solve equation (49) is to seek for the solution as a *factorial series* (see [65,66] for details). Equivalently, one can convert the difference equation into a differential equation, by means of the Laplace transform of $U(n)$. Accordingly, with the ansatz

$$U(n) = \int_0^1 dt\ t^{n-1}\ v(t)\ , \tag{51}$$

and

$$\begin{aligned} A_{00} &= \frac{D}{2}\ , & A_{01} &= -1\ , \\ A_{10} &= -1\ , & A_{11} &= 1\ , \end{aligned} \tag{52}$$

equation (49) becomes a differential equation in the form of (45) for the function $v(t)$,

$$-\,t(t-1)v'(t) + \left(\frac{D}{2} - t\right)v(t) = 0. \tag{53}$$

The solution is,

$$v(t) = v_0\ t^{-D/2}\ (1-t)^{D/2-1}\ , \tag{54}$$

where v_0 is the yet unknown integration constant. With the above solution, Eq.(51) can be integrated and $U(n)$ reads,

$$U(n) = v_0 \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)}\ . \tag{55}$$

By comparing its large-n expansion,

$$U(n) \overset{n\to\infty}{=} v_0\ n^{-D/2}\left(\Gamma\left(\frac{D}{2}\right) + \mathcal{O}\left(\frac{1}{n}\right)\right) \tag{56}$$

with the asymptotic limit explicitly computed in Eq.(50), one finally determines the value of the constant,

$$v_0 = \frac{1}{\Gamma\left(\frac{D}{2}\right)}\ . \tag{57}$$

6. One-Loop Vacuum Polarization in QED

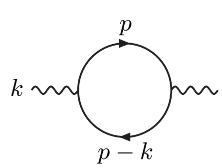


Fig. 5. 1-loop vacuum polarization diagram.

As an application, we calculate the one-loop correction to the photon propagator: the so called Vacuum Polarization tensor. The only contributing diagram is shown in Fig. 5, and corresponds to the expression

$$i\left(\frac{\alpha}{\pi}\right)\Pi_{\mu\nu}(k) = (-ie)^2(-1)\int \frac{d^Dp}{(2\pi)^D}\mathrm{Tr}\left\{\frac{\gamma_\mu(-i\not{p}+m)\gamma_\nu[-i(\not{p}-\not{k})+m]}{(p^2+m^2)[(p-k)^2+m^2]}\right\}. \tag{58}$$

$\Pi_{\mu\nu}$ is a tensor which depends only on the external momentum k_μ and the metric tensor $\delta_{\mu\nu}$ and can be decomposed as a sum of two contributions

$$\Pi_{\mu\nu}(k) = \Pi(D,k^2)k_\mu k_\nu + \Delta(D,k^2)\delta_{\mu\nu}, \tag{59}$$

where $\Pi(D,k^2)$ and $\Delta(D,k^2)$ are scalar functions. The Ward identity (current conservation) tells us that $k_\mu \Pi_{\mu\nu}(k) = 0$, hence

$$k_\mu \Pi_{\mu\nu}(k) = 0 \Longrightarrow k_\nu k^2 \Pi(D,k^2) + k_\nu \Delta(D,k^2) = 0 \Longrightarrow \Delta(D,k^2) = -k^2 \Pi(D,k^2), \tag{60}$$

and we can rewrite Eq. (59) as

$$\Pi_{\mu\nu}(k) = \Pi(D,k^2)(k_\mu k_\nu - k^2 \delta_{\mu\nu}) \; . \tag{61}$$

Let us extract the scalar function $\Pi(D,k^2)$ taking the trace of the (61)

$$\Pi_{\mu\mu}(D,k) = \Pi(D,k^2)(k^2 - k^2 D) \Longrightarrow \Pi(D,k^2) = \frac{1}{k^2(1-D)} \Pi_{\mu\mu}(k). \tag{62}$$

From (58), setting $\alpha = \frac{e^2}{4\pi}$, we have

$$i\Pi_{\mu\nu}(k) = \int \frac{d^D p}{(2\pi)^{D-2}} \frac{\mathrm{Tr}(-\gamma_\mu \not{p} \gamma_\nu \not{q} + m^2 \gamma_\mu \gamma_\nu)}{(p^2+m^2)[(p-k)^2+m^2]}; \tag{63}$$

after some operations within the Dirac algebra, our expression becomes

$$\begin{aligned}
&\mathrm{Tr}(-\gamma_\mu \not{p} \gamma_\nu \not{q} + m^2 \gamma_\mu \gamma_\nu) = \\
&\; = -p_\alpha q_\beta \mathrm{Tr}(\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta) + m^2 \mathrm{Tr}(\gamma_\mu \gamma_\nu) = \\
&\; = -\mathrm{Tr}(\mathbf{I})_D \; p_\alpha q_\beta (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) + \mathrm{Tr}(\mathbf{I})_D \; m^2 \delta_{\mu\nu} \\
&\; = -\mathrm{Tr}(\mathbf{I})_D \; (p_\mu q_\nu - p\cdot q \delta_{\mu\nu} + p_\nu q_\mu - m^2 \delta_{\mu\nu}) = \\
&\; = -\mathrm{Tr}(\mathbf{I})_D \; [2p_\mu p_\nu - p_\mu k_\nu - p_\nu k_\mu - (p^2 - p\cdot k + m^2)\delta_{\mu\nu}],
\end{aligned} \tag{64}$$

where $\mathrm{Tr}(\mathbf{I})_D$ is the trace of the unit matrix in D dimensions (this is, in general, different from D). However, $\lim_{D\to 4} \mathrm{Tr}(\mathbf{I})_D \; = 4$. Substituting (64) in (63) and putting $\mu = \nu$ we obtain

$$\begin{aligned}
&\Pi_{\mu\mu}(k) = \\
&\; = \mathrm{Tr}(\mathbf{I})_D \; i \int \frac{d^D p}{(2\pi)^{D-2}} \frac{2p^2 - 2p\cdot k - D(p^2 - p\cdot k + m^2)}{(p^2+m^2)[(p-k)^2+m^2]} = \\
&\; = \mathrm{Tr}(\mathbf{I})_D \; i \int \frac{d^D p}{(2\pi)^{D-2}} \frac{(D-2)(p\cdot k) - (D-2)p^2 - D\, m^2}{(p^2+m^2)[(p-k)^2+m^2]}.
\end{aligned} \tag{65}$$

We can define,

$$\begin{aligned}
\mathcal{D}_1 &= p^2 + m^2 \\
\mathcal{D}_2 &= (p-k)^2 + m^2
\end{aligned}$$

so that after the trivial tensor reduction, according to

$$p^2 = D_1^2 - m^2 \tag{66}$$

$$p \cdot k = \frac{1}{2}(\mathcal{D}_1 - \mathcal{D}_2 + k^2) \tag{67}$$

Eq. (65) becomes

$$\begin{aligned}
&\Pi_{\mu\mu}(k) = \\
&= \mathrm{Tr}(\mathbf{I})_D \; i \int \frac{d^D p}{(2\pi)^{D-2}} \frac{(D-2)(p\cdot k) - (D-2)p^2 - D\,m^2}{\mathcal{D}_1\mathcal{D}_2} = \\
&= \mathrm{Tr}(\mathbf{I})_D \; i \int \frac{d^D p}{(2\pi)^{D-2}} \frac{(D-2)[(\mathcal{D}_1 - \mathcal{D}_2 + k^2)/2] - (D-2)(\mathcal{D}_1 - m^2) - D\,m^2}{\mathcal{D}_1\mathcal{D}_2} = \\
&= \mathrm{Tr}(\mathbf{I})_D \; i \left\{ \frac{(2-D)}{2} \int \frac{d^D p}{(2\pi)^{D-2}} \left(\frac{1}{\mathcal{D}_1} + \frac{1}{\mathcal{D}_2} \right) + \left[\frac{k^2(D-2)}{2} - 2m^2 \right] \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1\mathcal{D}_2} \right\} = \\
&= \mathrm{Tr}(\mathbf{I})_D \; i \left\{ \frac{(2-D)}{2}(J_{10} + J_{01}) + \left[\frac{k^2(D-2)}{2} - 2m^2 \right] J_{11} \right\},
\end{aligned} \tag{68}$$

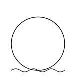
where all the integrals can be seen as belonging to the class,

$$J_{m_1 m_2} = \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1^{m_1}\mathcal{D}_2^{m_2}} \; . \tag{69}$$

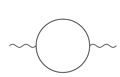
Finally, the scalar vacuum polarization function can be written as

$$\begin{aligned}
&\Pi(D, k^2) = \\
&= \frac{\mathrm{Tr}(\mathbf{I})_D \; i}{2k^2(1-D)} \left\{ (2-D)(J_{10} + J_{01}) + [k^2(D-2) - 4m^2] J_{11} \right\} = \\
&= \frac{\mathrm{Tr}(\mathbf{I})_D \; i}{k^2(1-D)} \left\{ (2-D) T(D, m^2) + \frac{1}{2}[k^2(D-2) - 4m^2] J(D, k^2) \right\}.
\end{aligned} \tag{70}$$

One may notice the result is expressed in terms of two integrals: the *massive Tadpole* [7]

$$T(D, m^2) = J_{10} = J_{01} =$$  $$= \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1} = C(D) \frac{m^{D-2}}{(D-2)(D-4)} \; , \tag{71}$$

and the *1-loop 2-point function*

$$J(D, k^2) = J_{11} =$$  $$= \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1\mathcal{D}_2} \; , \tag{72}$$

where

$$C(D) = (4\pi)^{\frac{4-D}{2}} \Gamma(3 - D/2) \; , \tag{73}$$

is an overall loop factor whose value in 4 dimension is $C(4) = 1$. In the following pages, when not ambiguous, we will give as understood a factor $C(D)$ for every loop.

6.1. *Reduction to master integrals*

We have seen that the scalar vacuum polarization function $\Pi(D,k^2)$ is reducible to the sum of a Tadpole and an integral belonging to the 1-loop 2-point topology, the most general integral of this class being

$$J_{m_1 m_2} = \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1^{m_1} \mathcal{D}_2^{m_2}} \; . \tag{74}$$

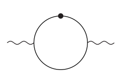
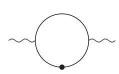
By means of IBP id's, we can reduce every integral of the type (74) to

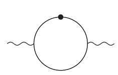
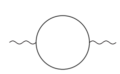
$$J_{11} = J(D,k^2) = \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2}, \tag{75}$$

which is, actually, the only MI of this topology, and to the tadpole J_{10}, which is the MI of the only possible subtopology.
Let us write explicitly some useful IBP id's for the integral (75). we have

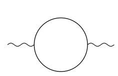
$$\begin{cases} \int \frac{d^D p}{(2\pi)^{D-2}} \frac{\partial}{\partial p_\mu} \left(\frac{k_\mu}{\mathcal{D}_1 \mathcal{D}_2} \right) = 0 \\ \int \frac{d^D p}{(2\pi)^{D-2}} \frac{\partial}{\partial p_\mu} \left(\frac{p_\mu}{\mathcal{D}_1 \mathcal{D}_2} \right) = 0 \end{cases} , \tag{76}$$

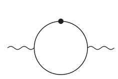
where $\mathcal{D}_1 = p^2 + m^2$ and $\mathcal{D}_2 = (p-k)^2 + m^2$. After the trivial tensor reduction we obtain the identities

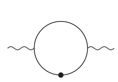
 $=$ 

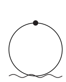
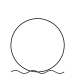
 $= -\frac{(D-3)}{(4m^2+k^2)}$  $+ \frac{1}{(4m^2+k^2)}$  , (77)

with

 $= \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2}$, (78)

 $= \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1^2 \mathcal{D}_2}$ (79)

 $= \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2^2}$, (80)

 $= \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1^2} = -\frac{(D-2)}{2\ m^2}$  (81)

where the last equation has been obtained as IBP identity for the tadpole [7].

6.2. *Differential equation for* $J(D, k^2)$

The master integral $J(D,k^2)$ is an analytic function of the argument k^2 and it can be viewed as the solution of a suitable differential equation. Let us see how to build and solve such an equation. For $J(D,k^2)$ the following trivial identity holds,

$$\frac{\partial J}{\partial k_\mu} = \frac{\partial J}{\partial k^2}\frac{\partial k^2}{\partial k_\mu} = 2k_\mu \frac{\partial J}{\partial k^2}. \tag{82}$$

By contracting (82) with the vector k_μ we have

$$k_\mu \frac{\partial J}{\partial k_\mu} = 2k^2 \frac{\partial J}{\partial k^2}. \tag{83}$$

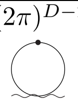
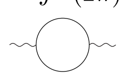
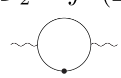
On the other hand

$$\frac{\partial J}{\partial k_\mu} = \int \frac{d^D p}{(2\pi)^{D-2}} \frac{\partial}{\partial k_\mu}\left(\frac{1}{\mathcal{D}_1 \mathcal{D}_2}\right) = \int \frac{d^D p}{(2\pi)^{D-2}} \frac{2(p_\mu - k_\mu)}{\mathcal{D}_1 \mathcal{D}_2^2}, \tag{84}$$

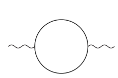
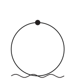
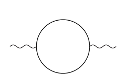
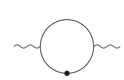
so

$$k_\mu \frac{\partial J}{\partial k_\mu} = \int \frac{d^D p}{(2\pi)^{D-2}} \frac{2(p\cdot k - k^2)}{\mathcal{D}_1 \mathcal{D}_2^2} \underbrace{=}_{2p\cdot k = \mathcal{D}_1 - \mathcal{D}_2 + k^2}$$

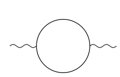
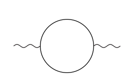
$$= \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_2^2} - \int \frac{d^D p}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2} - \int \frac{d^D p}{(2\pi)^{D-2}} \frac{k^2}{\mathcal{D}_1 \mathcal{D}_2^2} =$$

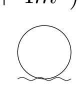
$=$  $-$  $- k^2$  (85)

By substituting Eq. (85) in Eq. (83) we have

$\frac{d}{dk^2}$  $= \frac{1}{2k^2}$  $- \frac{1}{2k^2}$  $- \frac{1}{2}$  , (86)

which is rewritten, thanks to the second identity of the (77) and to (81), as a non-homogeneous first-order differential equation for $J(D,k^2)$

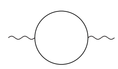
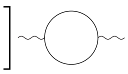
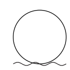
$\frac{d}{dk^2}$  $+ \frac{1}{2}\left[\frac{1}{k^2} - \frac{(D-3)}{(k^2+4m^2)}\right]$ 

$= -\frac{(D-2)}{4m^2}\left[\frac{1}{k^2} - \frac{1}{(k^2+4m^2)}\right]$  . (87)

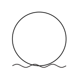
Eq. (87) contains the boundary condition for the solution. In fact, thanks to the analytic properties of Feynman integrals, we know that $J(D,k^2)$ must be a regular function in $k^2 = 0$, that is

$$\lim_{k^2 \to 0} k^2 \frac{dJ}{dk^2} = 0. \tag{88}$$

By multiplying Eq. (87) by k^2 and taking the limit $k^2 \to 0$, we have

$$\lim_{k^2 \to 0} k^2 \frac{d}{dk^2}$$  $$+ \lim_{k^2 \to 0} \frac{1}{2} \left[1 - \frac{(D-3)k^2}{(k^2+4m^2)} \right]$$  $$= - \lim_{k^2 \to 0} \frac{(D-2)}{4m^2} \left[1 - \frac{k^2}{(k^2+4m^2)} \right]$$  $$, \tag{89}$$

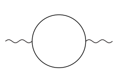
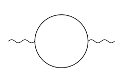
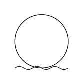
out of which one has,

$$\lim_{k^2 \to 0} J(D, k^2) = J(D, 0) = -\frac{(D-2)}{2m^2}$$  $$\tag{90}$$

Eq. (87) with the condition (90) constitutes the Cauchy problem we have to solve.

6.3. *Exact solution in D dimensions*

Equation (87) can be solved exactly in D dimensions. Let us introduce the dimensionless variable $x = \frac{k^2}{4m^2}$, obviously $\frac{d}{dk^2} = \frac{dx}{dk^2}\frac{d}{dx} \to \frac{d}{dk^2} = \frac{1}{4m^2}\frac{d}{dx}$, and the equation (87) can be rewritten as follows

$$\frac{d}{dx}$$  $$+ \frac{1}{2} \left[\frac{1}{x} - \frac{(D-3)}{(1+x)} \right]$$  $$= -\frac{(D-2)}{4m^2} \left[\frac{1}{x} - \frac{1}{(1+x)} \right]$$  $$. \tag{91}$$

The general solution of Eq. (91) is the sum of solution of the homogeneous equation, say J_0, and particular solution of the complete equation, say J^*. The homogeneous equation is

$$\frac{dJ_0}{dx} = -\frac{1}{2} \left[\frac{1}{x} - \frac{(D-3)}{(1+x)} \right] J_0 \tag{92}$$

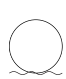
with solution,

$$J_0(D, x) = A x^{-\frac{1}{2}} (1+x)^{\frac{D-3}{2}}. \tag{93}$$

To find a particular solution, $J^*(D, x)$ of the complete equation, we use Euler's variation of constants method, and write

$$J^*(D, x) = x^{-\frac{1}{2}} (1+x)^{\frac{(D-3)}{2}} \phi(x), \tag{94}$$

where $\phi(x)$ is an unknown function to be found by imposing J^* fulfills Eq.(91). In so doing, we obtain the following equation for $\phi(x)$,

$$\frac{d\phi}{dx} = -\frac{(D-2)}{4m^2}$$  $$[x^{-\frac{1}{2}} (1+x)^{\frac{3-D}{2}} - x^{\frac{1}{2}} (1+x)^{\frac{1-D}{2}}], \tag{95}$$

hence

$$\phi(x) = -\frac{(D-2)}{4m^2} \left[\int x^{-\frac{1}{2}} (1+x)^{-\frac{D-3}{2}} dx - \int x^{\frac{1}{2}} (1+x)^{-\frac{D-1}{2}} dx \right]. \tag{96}$$

The integrals in Eq.(96) are a representation of hypergeometric functions

$$\begin{aligned} &\textstyle\int x^{-\frac{1}{2}} (1+x)^{-\frac{D-3}{2}} dx = 2x^{\frac{1}{2}} \ {}_2F_1\left(\frac{1}{2}, \frac{D-3}{2}; \frac{3}{2}; -x\right) \\ &\textstyle\int x^{\frac{1}{2}} (1+x)^{-\frac{D-1}{2}} dx = \frac{2}{3} x^{\frac{3}{2}} \ {}_2F_1\left(\frac{3}{2}, \frac{D-1}{2}; \frac{5}{2}; -x\right), \end{aligned} \tag{97}$$

hence

$$\begin{aligned} J(D,x) &= J_0 + J^* = \\ &= A x^{-\frac{1}{2}} (1+x)^{\frac{(D-3)}{2}} - \frac{(D-2)}{4m^2} (1+x)^{\frac{D-3}{2}} \times \\ &\times \left[2\, {}_2F_1\left(\frac{1}{2}, \frac{D-3}{2}; \frac{3}{2}; -x\right) - \frac{2}{3} x\, {}_2F_1\left(\frac{3}{2}, \frac{D-1}{2}; \frac{5}{2}; -x\right) \right]. \end{aligned} \tag{98}$$

By imposing the boundary condition, we see that the constant $A = 0$, because the term $x^{-\frac{1}{2}}(1+x)^{\frac{(D-3)}{2}}$ is singular for $x \to 0$, while the MI is regular by inspection.

Finally, the full solution of our Cauchy problem is

$$\begin{aligned} J(D,x) &= -\frac{(D-2)}{2m^2} (1+x)^{\frac{D-3}{2}} \times \\ &\times \left[{}_2F_1\left(\frac{1}{2}, \frac{D-3}{2}; \frac{3}{2}; -x\right) - \frac{x}{3}\, {}_2F_1\left(\frac{3}{2}, \frac{D-1}{2}; \frac{5}{2}; -x\right) \right] = \\ &= -\frac{(D-2)}{2m^2} (1+x)^{\frac{D-3}{2}} \ {}_2F_1\left(\frac{D-1}{2}, \frac{1}{2}; \frac{3}{2}; -x\right) = \\ &= -\frac{(D-2)}{2m^2} \ {}_2F_1\left(\frac{4-D}{2}, 1; \frac{3}{2}; -x\right) \end{aligned} \tag{99}$$

6.4. *Renormalized vacuum polarization function*

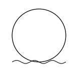
Now we can write the exact D-dimensional expression for the renormalized vacuum polarization function. This is

$$\Pi(D, k^2)_R = \Pi(D, k^2) - \Pi(D, 0), \tag{100}$$

where $\Pi(D, 0)$ is the function evaluated at zero momentum transfer

$$\Pi(D,0) = \frac{\mathrm{Tr}(\mathbf{I})_D \ i}{(1-D)} \frac{(2-D)}{4m^2} \lim_{x\to 0} \frac{1}{x} + \frac{D-2}{2} J(D,0) - \frac{1}{2} \lim_{x \to 0} \frac{J(D,x)}{x}. \tag{101}$$

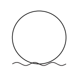
We see that

$$\lim_{x\to 0}\frac{J(D,x)}{x} = J(D,0)\lim_{x\to 0}\frac{1}{x} + J'(D,0) = \frac{(2-D)}{2m^2}$$  $$\lim_{x\to 0}\frac{1}{x} + J'(D,0)\ , \tag{102}$$

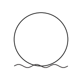
hence

$$\Pi(D,0) = \frac{\mathrm{Tr}(\mathbf{I})_D\ i}{(1-D)}\left\{\frac{D-2}{2}J(D,0) - \frac{1}{2}J'(D,0)\right\}. \tag{103}$$

Finally, one has the renormalization counterterm,

$$\Pi(D,0) = \mathrm{Tr}(\mathbf{I})_D\ i\ \frac{(D-2)}{6m^2}$$  $$\tag{104}$$

yielding the renormalised form factor,

$$\Pi_R(D,k^2) = \Pi(D,k^2) - \Pi(D,0) = \frac{\mathrm{Tr}(\mathbf{I})_D\ i}{(1-D)}\frac{(2-D)}{4m^2}$$  $$\times$$

$$\times\left\{\frac{1}{x} + \left[(D-2) - \frac{1}{x}\right]{}_2F_1\left(\frac{4-D}{2},1;\frac{3}{2};-x\right) + \frac{2}{3}(1-D)\right\}. \tag{105}$$

which can be ultimately expanded around $D=4$ up to the finite order in terms of HPL's

$$\Pi_R(D,k^2) = i\left(\frac{5}{9} - \frac{4}{3\,(1-z)^2} + \frac{4}{3\,(1-z)} + \frac{(1+z)\,\left(1-4\,z+z^2\right)\,H(0,z)}{3\,(1-z)^3}\right) + \mathcal{O}((D-4)). \tag{106}$$

with the spacelike variable z defined as

$$z = \frac{\sqrt{-x+1}-\sqrt{-x}}{\sqrt{-x+1}+\sqrt{-x}} = \frac{\sqrt{-k^2+4m^2}-\sqrt{-k^2}}{\sqrt{-k^2+4m^2}+\sqrt{-k^2}} \tag{107}$$

7. Two-Loop Vacuum Polarization in QED

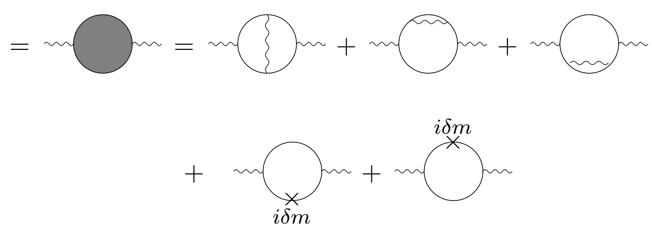
The two-loop corrections to the vacuum polarization function was first calculated in 1955 by Källen and Sabry[142]. At that time, it was one of the first applications of perturbative QED, soon after the theory was definitely laid out. The exact expression in dimensional regularization was calculated in 1993 by Broadhurst, Fleischer and Tarasov[144]. The result was expressed in terms of generalized Hypergeometric functions ${}_2F_1\left(\frac{k^2}{4m^2}\right)$ and ${}_3F_2\left(\frac{k^2}{4m^2}\right)$, where k is the momentum of the external photon and m the mass of the internal fermion.

In the following sections we will derive the same result by means of the method of differential equations. First of all, we will show that the use of IBP allows one to reduce the whole problem to five MI's, out of which three are product of one-loop integrals (known from the previous section), and two are actual two-loop integrals, as yet unknown. Then, we will write for them a system of two first order differential equations in the square of the external momentum. We will see that the system is equivalent to a pair of second order differential equations, one for each MI. These equations can be solved exactly in D dimensions and the solution can be written in terms of generalized Hypergeometric functions.

7.1. *Diagrams*

The *one particle irreducible* contributions to the two-loop vacuum polarization are showed in Eq. (108). They consist of three bare diagrams, plus the corresponding counterterms, namely two fermion self-energy counterterms and two vertex corrections counterterms. Due to Ward identities of QED, some of these counterterms cancel each other: the vertex subtractions cancel exactly the wavefunction renormalization counterterms; so the only effective contribution comes from the fermion mass counterterm, as shown in Eq. (108). The amplitude reads

$$i\left(\frac{\alpha}{\pi}\right)^2 \Pi_{\mu\nu}(k) =$$

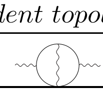
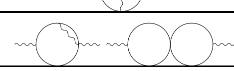
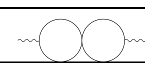
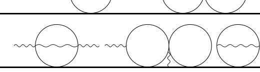
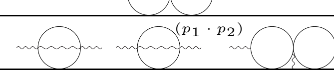
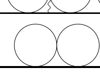
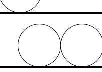


(108)

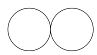
7.2. *Topologies and Master Integrals*

From the above Feynman diagrams, one can identify seven *independent* topologies, which cannot be related to each other by a transformation of the internal momenta, as shown in the second column of Tab. 1.

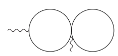
By the systematic application of the reduction algorithm, one can express all the needed integrals as combination of just five MI's, depicted in the last column of Tab. 1: one MI with four denominators; three with three; and one with two.

denominators	*Independent topologies*	MI
5		completely reducible
4		
3		
2		

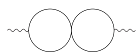
By giving a close look at them, one soon realizes that three of them are indeed product of the one-loop MI's we met in the previous section, namely,



$$= T^2(D, m^2) \ , \tag{109}$$



$$= J(D, k^2) \times T(D, m^2) \ , \tag{110}$$

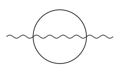


$$= J^2(D, k^2) \ , \tag{111}$$

where $T(D, m^2)$ and $J(D, k^2)$ were given respectively in Eqs. (71, 99).

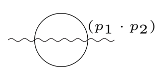
The two yet unknown MI's, both belonging to the same topology, are,

$$J_1(k^2) =$$



$$= \int \frac{d^D p_1}{(2\pi)^{D-2}} \int \frac{d^D p_2}{(2\pi)^{D-2}} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3} \tag{112}$$

$$J_2(k^2) =$$



$$= \int \frac{d^D p_1}{(2\pi)^{D-2}} \int \frac{d^D p_2}{(2\pi)^{D-2}} \frac{(p_1 \cdot p_2)}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3}, \tag{113}$$

where $\mathcal{D}_1 = p_1^2 + m^2, \mathcal{D}_2 = p_2^2, \mathcal{D}_3 = (k - p_1 - p_2)^2 + m^2$, p_1 and p_2 being the two loop momenta and k being the external momentum. Let's discuss their evaluation.

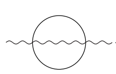
7.3. *Differential equations for $J_1(k^2)$ and $J_2(k^2)$.*

The functions $J_1(k^2)$ and $J_2(k^2)$ are found to fulfill a system of first-order differential equations in the external momentum squared that reads as

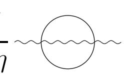
$$\begin{cases} \frac{dJ_1}{dk^2} = \left[\frac{(D-3)}{2k^2} + \frac{(3D-7)}{2(k^2+4m^2)}\right] J_1 - \frac{3(D-2)}{2m^2} \left[\frac{1}{k^2} - \frac{1}{k^2+4m^2}\right] J_2 - \frac{(D-2)}{2m^2} \left[\frac{1}{k^2} - \frac{1}{k^2+4m^2}\right] T^2 \\ \frac{dJ_2}{dk^2} = \frac{(D-2)}{4} \ J_1 - \frac{(D-2)}{2k^2} J_2 - \frac{(D-2)}{4k^2} \ T^2 \end{cases} \tag{114}$$

where $T(D, m^2)$ is the massive tadpole.
Such a system is equivalent to a second-order differential for J_1

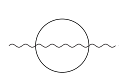
$$\eta^2 \ (\eta + 4m^2) \frac{d^2}{d\eta^2}$$



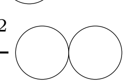
$$- \eta \left[\frac{3}{2} (D-4) \eta - 6m^2\right] \frac{d}{d\eta}$$



$$+ \ + \ (D-3) \left[\eta \frac{(D-4)}{2} - (D-2)m^2\right]$$



$$- \frac{(D-2)^2}{2}$$



$$= 0, \tag{115}$$

where $\eta = k^2$.

Differentiating once more with respect to η, we obtain a third order differential equation for J_1

$$\begin{aligned} &\eta^2 (\eta + 4m^2) \frac{d^3 J_1}{d\eta^3} - \eta \left[\frac{3}{2} (D-6) \eta - 14m^2\right] \frac{d^2 J_1}{d\eta^2} + \\ &+ \left[\frac{(D-4)(D-9)}{2} \eta - D(D-5)m^2\right] \frac{dJ_1}{d\eta} + \frac{(3-D)(4-D)}{2} J_1 = 0, \end{aligned} \tag{116}$$

which, introducing the dimensionless variable $x = \frac{\eta}{4m^2}$, becomes

$$x^2(1+x)\frac{d^3J_1}{dx^3} - x\left[\frac{3(D-6)}{2}x - \frac{7}{2}\right]\frac{d^2J_1}{dx^2} + \\ + \left[\frac{(D-4)(D-9)}{2}x - \frac{D(D-5)}{4}\right]\frac{dJ_1}{dx} + \frac{(3-D)(4-D)}{2}J_1 = 0. \tag{117}$$

7.4. *Exact solution in D dimensions.*

Quite in general, a differential equation like

$$x^2(1+x)\frac{d^3\psi}{dx^3} + x[x(a_1+a_2+a_3+3) + (b_1+b_2+1)]\frac{d^2\psi}{dx^2} + \\ +[x(a_1a_2+a_1a_3+a_2a_3+a_1+a_2+a_3+1) + b_1b_2]\frac{d\psi}{dx} + \\ +(a_1a_2a_3)\psi = 0, \tag{118}$$

where a_1, a_2, a_3, b_1, b_2 are parameters, is classified as hypergeometric and its solution space is spanned by the functions

$$\psi_1(x) = {}_3F_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; -x\right]$$

$$\psi_2(x) = (-x)^{1-b_1}\,{}_3F_2\left[\begin{matrix} 1+a_1-b_1, 1+a_2-b_1, 1+a_3-b_1 \\ 2-b_1, 1+b_2-b_1 \end{matrix}; -x\right] \tag{119}$$

$$\psi_3(x) = (-x)^{1-b_2}\,{}_3F_2\left[\begin{matrix} 1+a_1-b_2, 1+a_2-b_2, 1+a_3-b_2 \\ 2-b_2, 1+b_1-b_2 \end{matrix}; -x\right].$$

We can easily see that Eq. (117) is hypergeometric if we identify the parameters as follows:

$$\begin{cases} a_1 = 1 \\ a_2 = 3-D \\ a_3 = (4-D)/2 \\ b_1 = D/2 \\ b_2 = (5-D)/2\,. \end{cases} \tag{120}$$

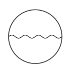
With the above choice, the three independent solutions of Eq. (117) become

$$\psi_1(x) = {}_3F_2\left[\begin{matrix} (3-D), (4-D)/2, 1 \\ D/2, (5-D)/2 \end{matrix}; -x\right] \\ \psi_2(x) = (-x)^{-\frac{(D-2)}{2}}\,{}_3F_2\left[\begin{matrix} (8-3D)/2, (3-D), (4-D)/2 \\ (4-D)/2, (7-2D)/2 \end{matrix}; -x\right] \\ \psi_3(x) = (-x)^{\frac{(D-3)}{2}}\,{}_3F_2\left[\begin{matrix} (3-D), 1/2, (D-1)/2 \\ (2D-3)/2, (D-1)/2 \end{matrix}; -x\right] \tag{121}$$

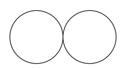
Finally, the general solution of Eq. (117) can be written as a linear combination of $\psi_1(x), \psi_2(x)$ e $\psi_3(x)$

$$J_1(x) = A\psi_1(x) + B\psi_2(x) + C\psi_3(x). \tag{122}$$

Now we can choose the value of the three integration constants A, B and C by means of suitable boundary conditions. By inspection we see that for $x \to 0$ the MI J_1 becomes

$J_1(0) =$  $=$

$$= \int \frac{d^D p_1}{(2\pi)^{D-2}} \int \frac{d^D p_2}{(2\pi)^{D-2}} \frac{1}{(p_1^2+m^2)p_2^2[(p_1-p_2)^2+m^2)} =$$

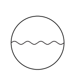
$= -\frac{(D-2)}{2m^2(D-3)}$  , (123)

where the last line has been obtained by means of IBP-id's. If we study the behaviour of (123) in the UV and IR limits, we see that it is regular for $2 < D < 4$. This means that $B = C = 0$. If this was not the case, in fact, terms such as $(-x)^{-\frac{(D-2)}{2}}$ and $(-x)^{\frac{(D-3)}{2}}$ would give rise, in the abovementioned range of D, to a divergent behaviour, and this would not be compatible with the finite result of (123).

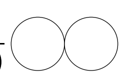
Furthermore, (123) allows one to choose once and for all the constant A

$$J_1(0) = A\psi_1(0) = A, \tag{124}$$

because, like all Hypergeometric functions, $\psi_1(0) = 1$. Comparing (124) with (123), we have

$A =$  ,

hence

$J_1(x) = -\frac{(D-2)}{2m^2(D-3)}$  ${}_3F_2\left[\begin{matrix}(3-D), (4-D)/2, 1\\ D/2, (5-D)/2\end{matrix}; -x\right].$ (125)

Now we can obtain the expression for $J_2(x)$ by simply substituting $J_1(x)$ and $\frac{dJ_1}{dx}$ in the first equation of (114). To do so, we have to know the expression for the first order derivative of the Hypergeometric function ${}_3F_2(x)$.
Quite in general, a function like

$${}_3F_2\left[\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; -x\right]$$

has the following series representation

$${}_3F_2\left[\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; x\right] = \sum_{n=0}^{+\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{(-x)^n}{n!}, \tag{126}$$

where $(\xi)_n$ is the *Pochhammer symbol*, defined as

$$(\xi)_n = \frac{\Gamma(\xi+n)}{\Gamma(\xi)}.$$

By differentiating Eq. (126) with respect to x, we obtain

$$\begin{aligned}
&\frac{d}{dx}\,{}_3F_2\left[\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix};x\right]= \\
&=\frac{d}{dx}\sum_{n=0}^{+\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}\frac{(-x)^n}{n!}=-\sum_{n=0}^{+\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}n\frac{(-x)^{n-1}}{n!}= \\
&=-\sum_{n=1}^{+\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}\frac{(-x)^{n-1}}{(n-1)!}= \\
&=-\sum_{k=0}^{+\infty}\frac{(a_1)_{k+1}(a_2)_{k+1}(a_3)_{k+1}}{(b_1)_{k+1}(b_2)_{k+1}}\frac{(-x)^k}{k!}.
\end{aligned} \tag{127}$$

Now, we see that

$$(\xi)_{k+1} = \frac{\Gamma(\xi+k+1)}{\Gamma(\xi)} = x\frac{\Gamma(\xi+1+k)}{\Gamma(\xi+1)} = \xi(\xi+1)_k,$$

hence

$$\begin{aligned}
&\frac{d}{dx}\,{}_3F_2\left[\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix};-x\right]= \\
&=-\frac{(a_1)(a_2)(a_3)}{(b_1)(b_2)}\sum_{k=0}^{+\infty}\frac{(a_1+1)_k(a_2+1)_k(a_3+1)_k}{(b_1+1)_k(b_2+1)_k}\frac{(-x)^k}{k!}= \\
&=-\frac{(a_1)(a_2)(a_3)}{(b_1)(b_2)}\,{}_3F_2\left[\begin{matrix}a_1+1,a_2+1,a_3+1\\ b_1+1,b_2+1\end{matrix};-x\right]
\end{aligned} \tag{128}$$

Solving (114) with respect to J_2 we have

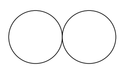
$$\begin{aligned}
J_2(x) = {}&\frac{a}{3(D-2)}[2(2D-5)x+(D-3)]J_1(x)+ \\
&-\frac{2m^2}{3(D-2)}x(x+1)\frac{dJ_1(x)}{dx}-\frac{1}{3}T^2(D,m^2).
\end{aligned} \tag{129}$$

Substituting (125) in (129) and because of

$$\frac{d}{dx}\,{}_3F_2\left[\begin{matrix}(3-D),(4-D)/2,1\\ D/2,(5-D)/2\end{matrix};-x\right]=\frac{2(D-3)(D-4)}{D(D-5)}\,{}_3F_2\left[\begin{matrix}(4-D),(6-D)/2,2\\ (D+2)/2,(7-D)/2\end{matrix};-x\right], \tag{130}$$

we can write down finally the exact expression in D for $J_2(x)$

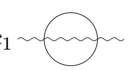
$$J_2(z) = \Bigg\{ -\frac{\Big[2(2D-5)x + (D-3)\Big]}{6(D-3)} \, {}_3F_2\left[\begin{matrix}(3-D),(4-D)/2,1\\ D/2,(5-D)/2\end{matrix};-x\right] + \\ +\frac{2(D-4)}{3D(D-5)} x(x+1) {}_3F_2\left[\begin{matrix}(4-D),(6-D)/2,2\\ (D+2)/2,(7-D)/2\end{matrix};-x\right] + \\ -\frac{1}{3}\Bigg\} \tag{131}$$

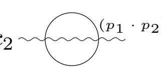


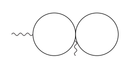
7.5. *Renormalized vacuum polarization function*

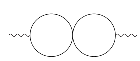
By means of the integrals' reduction, the two-loop Π_{2R}, with the one-loop subdivergences already subtracted off according to Eq.(108), admits the following decomposition in terms of MI's,

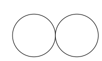
$$\Pi_{2L,1R}(D,k^2) = c_1 \quad + c_2 \quad + c_3 \quad + c_4 \quad + c_5 \quad , \tag{132}$$











with

$$c_1 = \frac{-2\,(-2+D)}{(-12+19D-8\,D^2+D^3)\;k^2\,m^2\;(k^2+4\,m^2)}\bigg(\left(-16+18D-7D^2+D^3\right)k^4 \\ +4\left(-14+22D-9D^2+D^3\right)k^2m^2+16\,(-3+D)^2\,m^4\bigg)\;; \tag{133}$$

$$c_2 = \frac{12\,(-2+D)^2\;\left(\left(8-5\,D+D^2\right)\;k^2+4\;\left(10-7\,D+D^2\right)\;m^2\right)}{(-12+19\,D-8\,D^2+D^3)\;k^2\,m^2\;(k^2+4\,m^2)}\;; \tag{134}$$

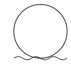
$$c_3 = \frac{-2\;(-2+D)}{(-12+19\,D-8\,D^2+D^3)\;k^2\,m^2\;(k^2+4\,m^2)}\bigg(\left(-16+18\,D-7\,D^2+D^3\right)\;k^4 \\ +4\,(-4+D)^2\;(-1+D)\;k^2\,m^2-32\;(-3+D)\;m^4\bigg)\;; \tag{135}$$

$$c_4 = \frac{1}{(4-5\,D+D^2)\;k^2\;(k^2+4\,m^2)}\bigg(-2\;\left(-32+30\,D-9\,D^2+D^3\right)\;k^4 \\ -8\;\left(-40+38\,D-11\,D^2+D^3\right)\;k^2\,m^2+64\,m^4\bigg)\;; \tag{136}$$

$$c_5 = \frac{6\,(-2+D)^2\;\left(\left(8-5\,D+D^2\right)\;k^2+4\;\left(10-7\,D+D^2\right)\;m^2\right)}{(-12+19\,D-8\,D^2+D^3)\;k^2\,m^2\;(k^2+4\,m^2)} \tag{137}$$

and where we used the mass counterterm defined, in terms of the one-loop tadpole, as,

$$\delta m = \frac{(D-1)(D-2)}{2\;m\;(D-3)} \tag{138}$$



The completion of the two-loop renormalization procedure requires the subtraction of the value of $\Pi_{(2L,1R)}$ at zero momentum, which can be obtained from the above expression,

$$\Pi_{2L,1R}(D,0) = \frac{34 - D\ (41 + (-12 + D)\ D)}{2\ (-5 + D)\ (-4 + D)\ (-3 + D)\ D} \tag{139}$$

The two-loop renormalized expression

$$\Pi_{2R}(D,k^2) = \Pi_{2L,1R}(D,k^2) - \Pi_{2L,1R}(D,0) \tag{140}$$

agrees with the result in literature [144].

8. System of Three Differential Equations

After having shown in its completeness the calculation of the vacuum polarization at 1- and 2-loop in QED, we proceed with the discussion of a less trivial case of differential equations whose solutions [83] are required to compute a set of integrals which enter the 2-loop QCD corrections to the heavy-quark form factors [84,85,86,87,106] We are going to describe directly the evaluation of three master integrals belonging to the same topology. In this case we give as understood *i)* the reduction to the master integrals, namely assuming that all the MI's have been already identified; and *ii)* the knowledge of the MI belonging to the subtopologies which enters the non-homogeneous term of our differential equations.

We will see, in this example, the technical details related to the Laurent expansion of the system of equations around specific values of the dimensional parameter ($D \to 4$), and to the choice of the boundary conditions.

According to our assumptions, we have got the table of identities for reducing all the integrals belonging to the topology in Fig.6 and its subtopologies.

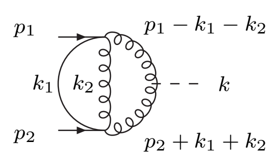
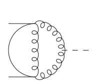
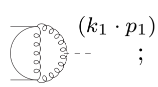


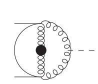
Fig. 6. Two-loop vertex diagram: $p_1^2 = p_2^2 = -m^2$; $k = p_1 + p_2$; $k_{1,2}$ are loop variables; a curly line stands for massless propagator; a solid line, for propagator of mass m.

The scalar integrals represented by the topology in Fig.6 are

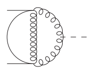
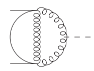
$$\mathcal{J}(n_1,n_2,n_3;m_1,m_2,m_3,m_4) = \int \frac{d^D k_1}{(2\pi)^{D-2}} \frac{d^D k_2}{(2\pi)^{D-2}} \frac{(k_1 \cdot p_1)^{n_1} (k_2 \cdot p_1)^{n_2} (k_2 \cdot p_1)^{n_3}}{\mathcal{D}_1^{m_1} \mathcal{D}_2^{m_2} \mathcal{D}_3^{m_3} \mathcal{D}_4^{m_4}}, \tag{141}$$

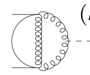
where $\mathcal{D}_1 = k_1^2 + m^2, \mathcal{D}_2 = k_2^2, \mathcal{D}_3 = (p_1 - k_1 - k_2)^2, \mathcal{D}_4 = (p_2 + k_1 + k_2)^2$.
At the end of the reduction, one is left over with three master integrals, $\Phi_i (i = 1,2,3)$, which we choose to be

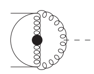
$\Phi_1(D,k^2) =$  ; $\Phi_2(D,k^2) =$ 

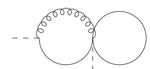
; $\Phi_3(D,k^2) =$  . (142)

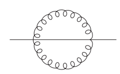
These MI are found to fulfills the following system of first-order ODE, in the variable k^2, corresponding to the momentum transfer,

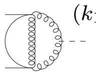
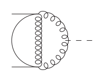
$$\frac{d}{dk^2}$$  $$= -\frac{\left(22 - 13\,D + 2\,D^2\right)}{2\,k^2}$$ 

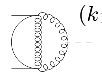
$$-\frac{(-2+D)\;(-7+2\,D)\;(-8+3\,D)}{(-3+D)\;k^2\;(k^2+4\,m^2)}$$  $$(k_1 \cdot p_1)$$

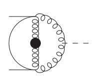
$$+\frac{(-4+D)\;(-5+2\,D)\;m^2}{2\;(-3+D)\;(k^2+4\,m^2)}$$ 

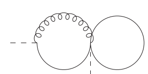
$$+\frac{(-4+D)\;(-2+D)\;(-5+2\,D)}{2\;(-3+D)\;k^2\;(k^2+4\,m^2)}$$ 

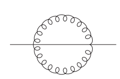
$$-\frac{(-4+D)\;(-5+2\,D)\;(-10+3\,D)\;(-8+3\,D)}{4\;(-3+D)\;(-7+2\,D)\;k^2\;(k^2+4\,m^2)}$$  $$; \qquad (143)$$

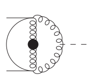
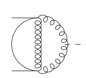
$$\frac{d}{dk^2}$$  $$(k_1 \cdot p_1) = \frac{(-3+D)\;\left((-4+D)\,k^2 + 4(-3+D)\,m^2\right)}{8\,k^2}$$ 

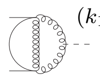
$$+\frac{\left((20-16\,D+3\,D^2)\,k^2 + (56-56\,D+12\,D^2)\,m^2\right)}{4\,k^2\;(k^2+4\,m^2)}$$  $$(k_1 \cdot p_1)$$

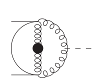
$$-\frac{(-4+D)\;m^2}{8}$$ 

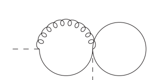
$$-\frac{\left(10-6\,D+D^2\right)}{8\,k^2}$$ 

$$+\frac{\left(-236+244\,D-82\,D^2+9\,D^3\right)}{16\;(-7+2\,D)\;k^2}$$  $$; \qquad (144)$$

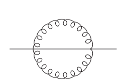
$$\frac{d}{dk^2}$$  $$= -\frac{(-4+D)\;(-3+D)\;\left((-2+D)\;k^2 + 4\;(-3+D)\;m^2\right)}{4\,k^4\,m^2}$$ 

$$-\frac{(-4+D)\;(-2+D)\;(-8+3\,D)}{2\,k^4\,m^2}$$  $$(k_1 \cdot p_1)$$

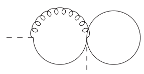
$$+\frac{\left((4-6\,D+D^2)\;k^2 + 4\;\left(-2-4\,D+D^2\right)\;m^2\right)}{4\,k^2\;(k^2+4\,m^2)}$$ 

$$+\frac{(-2+D)\;\left(\left(14-8\,D+D^2\right)\;k^2 + 4\;\left(8-6\,D+D^2\right)\;m^2\right)}{4\,k^4\,m^2\;(k^2+4\,m^2)}$$ 

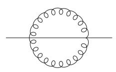
$$-\frac{(-8+3\,D)}{8\;(-7+2\,D)\;k^4\,m^2\;(k^2+4\,m^2)}\Big\{\left(-164+136\,D-36\,D^2+3\,D^3\right)\;k^2 + $$

$$+4\;\left(-104+98\,D-30\,D^2+3\,D^3\right)\;m^2\Big\}$$  $$. \qquad (145)$$

The non-homogeneous terms of the above equations contain the integrals,



$$= J(D, k^2) \times T(D, m^2) \; , \tag{146}$$



$$= -\frac{(m^2)^{D-3}}{8\,(D-4)} \frac{\Gamma(-D-5)\Gamma^2((D-2)/2)\Gamma(2D-5)}{\Gamma(-(D-6)/2)\Gamma(D-2)\Gamma((3D-9)/2)} \; , \tag{147}$$

which are MI's of the non-vanishing subtopologies - any other subdiagrams would contain a massless tadpole, vanishing in dimensional regularization. The former is the product of the massive tadpole $T(D, m^2)$, given in Eq. (71), and the 1-loop 2-point function $J(D, k^2)$, in Eq. (99); while the latter, known as 2-loop *sunrise*, could be easily evaluated by direct parametric integration.

In principle, solving a system of three first-order ODE's is equivalent to solve a single third-order ODE for one of the three MI, say Φ_1. But instead of writing directly the third-order ODE, one can observe that: in Eqs.(143,144), Φ_3 appears to be multiplied by $(D-4)$; and in Eq.(145), Φ_1 and Φ_2 are multiplied by $(D-4)$. This features means that, after expanding around $D = 4$, the original system of three coupled equations, get simplified: it decouples in a system of two coupled equations for Φ_1 and Φ_2, plus a single equation for Φ_3.

It is important to remark that the shape of the differential equations depends strongly on the choice of the MI: a different choice for $\Phi_i (i = 1, 2, 3)$ could lead to a system which does not get simplified after the Laurent expansion either. In all the problems we studied, given the arbitrariness of the choice, the practical criterion of having to deal with a simpler system of differential equations has determined which integrals had to be picked up as master ones - though there is no apriori assurance to find any simplification at all.

8.1. *Laurent expansion in $(D-4)$*

Indeed, one rearranges the system of equations (143,144,145) as follows.

- Writes a second-order ODE for Φ_1, from Eqs.(143,144).
- Take the first-order ODE Eq.(145) for Φ_3.
- Perform the change of variable

$$k^2 \to x = \frac{\sqrt{-k^2} - \sqrt{-k^2 + m^2}}{\sqrt{-k^2} + \sqrt{-k^2 + m^2}} \; . \tag{148}$$

- Finally, expands both equations around $D = 4$, with a Laurent ansatz for the solutions,

$$\Phi_1(D, x) = \sum_{k=-2}^{\infty} (D-4)^k \; \Phi_1^{(k)}(x) \tag{149}$$

$$\Phi_2(D, x) = \sum_{k=-2}^{\infty} (D-4)^k \; \Phi_2^{(k)}(x) \tag{150}$$

$$\Phi_3(D, x) = \sum_{k=-2}^{\infty} (D-4)^k \; \Phi_3^{(k)}(x) \; . \tag{151}$$

After the Laurents expansion, the first-order ODE for $\Phi_3(D,x)$ will induce, order-by-order in $(D-4)$, a system of chained first-order ODE for the Laurent coeffiecients, $\Phi_3^{(k)}(x)$, which at the first two orders, $k=-2,-1$, reads,

$$0 = \left\{ \frac{\left(1+x^2\right)}{(-1+x)\ x\ (1+x)} + \frac{d}{dx} \right\} \Phi_3^{(-2)}(x) \ ; \tag{152}$$

$$\begin{aligned} 0 = {} & \frac{1+x}{8\,(-1+x)^3} + \frac{H(0,x)}{8\ (-1+x)\ (1+x)} + \frac{H(1,x)}{4\ (-1+x)\ (1+x)} + \\ & + \frac{(1+x)\ \left(1+x^2\right)\ \Phi_1^{(-2)}(x)}{2\,(-1+x)^3\,x} + \frac{4\ (1+x)\ \Phi_2^{(-2)}(x)}{(-1+x)^3} - \frac{\left(1+6\,x+x^2\right)\ \Phi_3^{(-2)}(x)}{2\ (-1+x)\ x\ (1+x)} + \\ & + \left\{ \frac{\left(1+x^2\right)}{(-1+x)\ x\ (1+x)} + \frac{d}{dx} \right\} \Phi_3^{(-1)}(x) \ . \end{aligned} \tag{153}$$

We remark that the non-homogeneous term of (153), the equation for $\Phi_3^{(-1)}(x)$, demands for the knowledge of the previous order solutions, $\Phi_1^{(-2)}(x)$, $\Phi_2^{(-2)}(x)$, and $\Phi_3^{(-2)}(x)$.

Analogously, for the second-order ODE for $\Phi_1(D,x)$, the system of chained second-order ODE just for the first two coefficients of the Laurent series of $\Phi_1(D,x)$, for $k=-2,-1$ reads,

$$0 = \frac{1}{4\,(-1+x)^2\,x} - \left\{ \frac{2}{(-1+x)^2\,x} - \frac{2}{1+x}\frac{d}{dx} - \frac{d^2}{dx^2} \right\} \Phi_1^{(-2)}(x) \ ; \tag{154}$$

$$\begin{aligned} 0 = {} & \frac{-1}{8\,(-1+x)^2\,x} - \frac{H(0,x)}{4\,(-1+x)^2\,x} - \frac{H(1,x)}{2\,(-1+x)^2\,x} + \frac{\Phi_3^{(-2)}(x)}{2\,x^2} + \\ & - \left[\frac{\left(1+x^2\right)}{(-1+x)^2\,x^2} - \frac{\left(1+6\,x+x^2\right)}{2\,x-2\,x^3}\frac{d}{dx} \right] \Phi_1^{(-2)}(x) + \\ & - \left\{ \frac{2}{(-1+x)^2\,x} - \frac{2}{1+x}\frac{d}{dx} - \frac{d^2}{dx^2} \right\} \Phi_1^{(-1)}(x) \ . \end{aligned} \tag{155}$$

In this case, the non-homogeneous term of (155), the equation for $\Phi_1^{(-1)}(x)$, requires the knowledge of the previous order solutions, $\Phi_1^{(-2)}(x)$ and $\Phi_3^{(-2)}(x)$.

Therefore, by looking at the Eqs.(152,154) for $k=-2$, and at the structure of the non-homogeneous term of Eq.(153,155) for $k=-1$, the computational strategy is soon outlined:

- $k=-2$.
  (1) solve Eq.(152) to find $\Phi_3^{(-2)}(x)$;
  (2) solve Eq.(154) to find $\Phi_1^{(-2)}(x)$;
  (3) invert the $D \to 4$ expansion Eq.(143), and substitute the expressions of $\Phi_3^{(-2)}(x)$ and $\Phi_1^{(-2)}(x)$ in it, to find $\Phi_2^{(-2)}(x)$.
- $k=-1$.
  (1) plug the results of the previous order, $\Phi_1^{(-2)}(x)$, $\Phi_2^{(-2)}(x)$ and $\Phi_3^{(-2)}(x)$, in the non-homogeneous term of Eqs.(153,155);

(2) solve Eq.(153) to find $\Phi_3^{(-1)}(x)$;
(3) solve Eq.(155) to find $\Phi_1^{(-1)}(x)$;
(4) invert the $D \to 4$ expansion Eq.(143), and substitute the expressions of $\Phi_3^{(-1)}(x)$ and $\Phi_1^{(-1)}(x)$ in it, to find $\Phi_2^{(-1)}(x)$.

The construction of Laurent coefficients for $k \geq 0$ goes on by repeating the steps (1–4),

- $k = j, \ j \geq 0$.
  (1) plug the results of the previous orders, $\Phi_1^{(i)}(x)$, $\Phi_2^{(i)}(x)$ and $\Phi_3^{(i)}(x)$ $(-2 \leq i < j)$, in the non-homogeneous term of the equations;
  (2) solve the first order equation for $\Phi_3^{(j)}(x)$;
  (3) solve the second order equation for $\Phi_1^{(j)}(x)$;
  (4) invert Eq.(143) to find $\Phi_2^{(j)}(x)$.

Let us see in detail the case $k = -2$.

8.2. *Homogeneous differential equations*

The homogeneous differential equation is the same at every order in $(D-4)$, as one can realize by looking at the operator in the curly brakets in Eqs.(152,153) for $\Phi_3^{(k)}(x)$ and, correspondingly, at Eqs.(154,155) for $\Phi_1^{(k)}(x)$.

The solution $\phi_3(x)$ of the first-order homogeneous equation is

$$\phi_3(x) = -\frac{x}{(1-x)\,(1+x)} \tag{156}$$

The two solutions $\phi_{1,1}(x), \phi_{1,2}(x)$ of the 2nd order homogeneous equation are

$$\phi_{1,1}(x) = \frac{x}{(1-x)\,(1+x)} \tag{157}$$

$$\phi_{1,2}(x) = \frac{x}{(1-x)\,(1+x)}\left(-\frac{1}{x} + x - 2\,H(0,x)\right) \tag{158}$$

whose Wronskian reads

$$W(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = -(1+x)^{-2} \ . \tag{159}$$

8.3. *Solution of the chained equations*

Let us discuss the solutions of the system of equations at order $k = -2$, according to the above strategy.

(1) One easily solves Eq. (152), finding

$$\Phi_3^{(-2)}(x) = -\frac{x}{1-x^2}\Phi_3^{(1,-2)} \tag{160}$$

where $\Phi_3^{(1,-2)}$ is an integration constant to be later fixed by imposing the boundary conditions.

(2) Afterwards, one solves Eq. (154) by Euler variation of constants,

$$\begin{aligned}\Phi_1^{(-2)}(x) = \phi_{1,1}(x) \left[\Phi_1^{(1,-2)} + \int_0^x \frac{dx'}{W(x')} \phi_{1,2}(x') K^{(-2)}(x')\right] \\ +\phi_{1,2}(x) \left[\Phi_i^{(2,-2)} - \int_0^x \frac{dx'}{W(x')} \phi_{1,1}(x) K^{(-2)}(x')\right]\end{aligned} \tag{161}$$

with $\Phi_1^{(i,-2)}, (i = 1, 2)$ being integration constants, and $K^{(-2)}(x)$ the non-homogeneous term of Eq.(154), thus obtaining,

$$\begin{aligned}\Phi_1^{(-2)}(x) = \frac{x\,(1 + H(0,x))}{4\,(-1+x)\,(1+x)} - \frac{x}{(-1+x)\,(1+x)} \Phi_1^{(1,-2)} + \\ -\frac{\left(-1 + x^2 - 2\,x\,H(0,x)\right)}{(-1+x)\,(1+x)} \Phi_1^{(2,-2)}\end{aligned} \tag{162}$$

(3) Then, the knowledge of $\Phi_3^{(-2)}(x)$ and $\Phi_1^{(-2)}(x)$, and of theirs derivatives, allows the determination of the the coefficient of Laurent expansion of the third MI, by inverting the $D \to 4$ expansion of Eq.(143),

$$\begin{aligned}\Phi_2^{(-2)}(x) = \frac{-\,(1+x)\,(1 + H(0,x))}{32\,(-1+x)} + \frac{(1+x)}{8\,(-1+x)} \Phi_1^{(1,-2)} + \\ +\frac{(1+x)\,\left(-1 + x^2 - 2\,x\,H(0,x)\right)}{8\,(-1+x)\,x} \Phi_1^{(2,-2)}\ .\end{aligned} \tag{163}$$

At the end of this three steps, one knows the expressions of the $1/(D-4)^2$-coefficient of the three MI, $\Phi_i(D,x), (i = 1, 3)$ up to the determination of the *real* integration constants, $\Phi_3^{(1,-2)}, \Phi_1^{(1,-2)}, \Phi_1^{(2,-2)}$.

8.4. *Boundary conditions*

Usually boundary conditions for Feynman integrals can be read at special values of the kinematic variables, and they do correspond to integrals belonging to subdiagrams, therefore to simpler functions. That is true when the limit to such a specific values is smooth, as it happens around the pseudo-thresholds of the corresponding diagrams. In our case the value of $x = -1$, meaning $k^2 = 4m^2$, is a pseudo-threshold. To reach the point $x = -1$, being x a space-like variable, as defined in Eq.(148), one need an analytic continuation to the region, $x \to y = -x + i\epsilon$, where the MI's develop an imaginary part,

$$\Phi_3^{(-2)}(y) = -\frac{y}{(-1+y)\,(1+y)} \Phi_3^{(1,-2)} \tag{164}$$

$$\begin{aligned}\Phi_1^{(-2)}(y) = \frac{-y(1 + H(0,y))}{4\,(-1+y)\,(1+y)} + \frac{y}{(-1+y)\,(1+y)} \Phi_1^{(1,-2)} + \\ -\frac{\left(-1 + y^2 + 2\,y\,H(0,y)\right)}{(-1+y)\,(1+y)} \Phi_1^{(2,-2)} - i\pi \frac{y\,\left(1 + 8\Phi_1^{(2,-2)}\right)}{4\,(-1+y)\,(1+y)}\end{aligned} \tag{165}$$

Their expansion around $y = 1$ reads,

$$\Phi_3^{(-2)}(y) = \frac{1}{2\,(1-y)}\Phi_3^{(1,-2)} + \mathcal{O}\Big((1-y)^0\Big) \tag{166}$$

$$\Phi_3^{(-2)}(y) = \frac{1}{(1-y)}\left(\frac{1}{8} - \frac{\Phi_1^{(1,-2)}}{2} + i\,\pi\left(\frac{1}{8} + \Phi_1^{(2,-2)}\right)\right) + \mathcal{O}\Big((1-y)^0\Big) \tag{167}$$

Finally, the three conditions needed to fix the value of the arbitrary constants, order by order in $D-4$, are: *i)* the regularity of $\Phi_3^{(k)}(y)$ as $y \to 1$; *ii)* the regularity of $\Phi_1^{(k)}(y)$ as $y \to 1$, meaning the vanishing of the $1/(1-y)$-coefficients in both cases; *iii)* the realness of the constants, meaning that real and imaginary part of the $1/(1-y)$-coefficient must vanish separately. From the above equations, that translates to,

$$\Phi_3^{(1,-2)} = 0; \quad \Phi_1^{(1,-2)} = \frac{1}{4}; \quad \Phi_1^{(2,-2)} = -\frac{1}{8}. \tag{168}$$

We have all the ingredient to go up in the chain of Laurent coefficients, and repeat the previous steps for the case $k = -1$. The iterative structure of the method yields a bottom-up reconstruction of the three master integrals, $\Phi_1(D,x)$, $\Phi_2(D,x)$, and $\Phi_3(D,x)$, around $D = 4$.

9. System of Four Differential Equations

This section is devoted to the analytic evaluation of the MI's associated to the 4-loop sunrise graph with two massless lines, two massive lines of equal mass M, another massive line of mass m, with $m \neq M$, and the external invariant timelike and equal to m^2, as depicted in Fig. 7 [147].

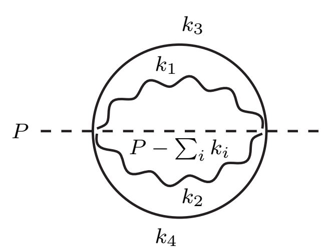


Fig. 7. Four-loop sunrise diagram: $P^2 = -m^2$; $k_{1,2,3,4}$ are loop variables; a wavy line stands for massless propagator; a solid line, for propagator of mass M; a dashed line, for propagator of mass m;

In this case, the heart of the analytic calculation is the study of a homogeneous fourth order differential equation, whose solutions turned out to be, in a remarkably simple way, either a rational fraction or repeated quadratures of rational fractions. The required four-loop integral could then be obtained almost mechanically by repeated quadratures in terms of HPL's.

Following the reduction algorithm - by now sounding familiar to the reader -, we identify the MI's of the problem; write the system of differential equations in $x = m/M$ satisfied by the MI's; convert it into a higher order differential equation for a single MI; Laurent-expand it around $D = 4$; solve the associated homogeneous equation at $D = 4$ and then use recursively Euler's method of the variation of the constants for obtaining the coefficient of the $(D-4)$-expansion in closed analytic form. The result involves HPL's of argument x and weight increasing with the order in $(D-4)$. The integration constants are fixed at $x=0$. After solving the differential equations for arbitrary x, we will show how to compute, independently, the numerical value of the solution at $x=1$ by using the Finite Difference method of Laporta [65], discussed in Sec.5, to show the relation among Differential and Difference Equations for Feynman integrals.

9.1. *Master Integrals and Differential Equations*

We find that the problem has 5 MI's, which we choose to be

$$F_i(D) = \int \frac{d^D k_1 \ldots d^D k_4}{(2\pi)^{4(D-2)}} \frac{N_i}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5} \,, \tag{169}$$

where the numerators N_i are $(M^2, k_1\cdot k_3, p\cdot k_3, k_1\cdot k_2, p\cdot k_2)$ and the denominators $\mathcal{D}_1 = k_1^2, \mathcal{D}_2 = k_2^2, \mathcal{D}_3 = k_3^2 + M^2, \mathcal{D}_4 = k_4^2 + M^2, \mathcal{D}_5 = (P - k_1 - k_2 - k_3 - k_4) + m^2$. In terms of the dimensionless variable $x = m/M$ and putting $P = Mp$ one can introduce 5 dimensionless functions $\Phi_i(d,x)$ through

$$F_i(D) = M^{4D-8}\ C^4(D)\ \Phi_i(D,x) \,, \tag{170}$$

where $C(D) = (4\pi)^{\frac{4-D}{2}}\Gamma(3-D/2)$ is an overall loop normalization factor, with the limiting value $C(4) = 1$ at $D = 4$.

The derivation of the system of differential equations is straightforward; the derivatives of the MI's, *i.e.* of the 5 functions $\Phi_i(D,x)$, with respect to x are easily carried out in their representation as loop-integrals Eq.(169); when the result is in turn expressed in terms of the same MI's, one obtains the following linear system of first order differential equations in x

$$\begin{aligned}\frac{d\Phi_1(D,x)}{dx} &= \left\{\frac{(3D-7)}{x} + \frac{3\ (D-2)}{2\ (1-x)} - \frac{3\ (D-2)}{2\ (1+x)}\right\}\Phi_1(D,x) - \left\{\frac{3(D-2)}{x} + \frac{3\ (D-2)}{2\ (1-x)}\right. \\ &\quad \left. -\frac{3\ (D-2)}{2\ (1+x)}\right\}\Big(3\Phi_2(D,x) - 3\Phi_3(D,x) + \Phi_4(D,x) - 3\Phi_5(D,x)\Big)\,,\end{aligned} \tag{171}$$

$$\frac{d\Phi_2(D,x)}{dx} = -\frac{(D-2)}{x}\Big(\Phi_2(D,x) - 2\Phi_3(D,x) + \Phi_4(D,x) - 2\Phi_5(D,x)\Big)\,, \tag{172}$$

$$\begin{aligned}\frac{d\Phi_3(D,x)}{dx} &= -\left\{\frac{3\ (D-2)}{2\ (1-x)} - \frac{3\ (D-2)}{2\ (1+x)}\right\}\Big(\Phi_1(D,x) - 3\Phi_2(D,x) - \Phi_4(D,x)\Big) \\ &\quad -\left\{\frac{3(D-2)}{x} + \frac{9\ (D-2)}{2\ (1-x)} - \frac{9\ (D-2)}{2\ (1+x)}\right\}\Big(\Phi_3(D,x) + \Phi_5(D,x)\Big)\,,\end{aligned} \tag{173}$$

$$\frac{d\Phi_4(D,x)}{dx} = \frac{2(D-2)}{x}\Big(\Phi_2(D,x) + \Phi_4(D,x)\Big)\,, \tag{174}$$

$$\frac{d\Phi_5(D,x)}{dx} = \frac{2(D-2)}{x}\Big(\Phi_3(D,x) + \Phi_5(D,x)\Big)\,, \tag{175}$$

The system is homogeneous: indeed, quite in general the non homogeneous terms are given by the MI's of the subtopologies of the considered graph, obtained by shrinking to a point any of its propagator lines. When that is done for the parent topology with five propagators of Fig.(7), one obtains the product of 4 tadpoles; but as the considered graph has two massless propagators, at least one massless tadpole is always present in the product; as in the D-dimensional regularization massless tadpoles vanish, the product of the 4 tadpoles is always equal to zero – and therefore the differential equations are homogeneous.

By inspection, one sees that $\Phi_3(D,x), \Phi_5(D,x)$ appear in the r.h.s. of Eq.s(171-175) only in the combination

$$\Psi_3(D,x) = \Phi_3(D,x) + \Phi_5(D,x) \; ; \tag{176}$$

the other linearly independent combination of the two MIs, say

$$\Psi_5(D,x) = \Phi_3(D,x) - \Phi_5(D,x) \; , \tag{177}$$

decouples and can be expressed in terms of the other integrals by means of the trivial 1st order differential equation

$$\begin{aligned} \frac{d\Psi_5(D,x)}{dx} = &-\left\{ \frac{3\;(D-2)}{2\;(1-x)} - \frac{3\;(D-2)}{2\;(1+x)} \right\} \Big(\Phi_1(D,x) - 3\Phi_2(D,x) - \Phi_4(D,x) \Big) \\ &-\left\{ \frac{5(D-2)}{x} + \frac{9\;(D-2)}{2\;(1-x)} - \frac{9\;(D-2)}{2\;(1+x)} \right\} \Psi_3(D,x) \end{aligned} \tag{178}$$

As $\Psi_5(D,x)$ does not enter in the r.h.s. of Eq.s(171-175), the 4 linear equations for $\Phi_1(D,x)$, $\Phi_2(D,x)$, $\Psi_3(D,x)$, and $\Phi_4(D,x)$ can be written as a fourth order equation for $\Phi_1(x)$, which will be called simply $\Phi(D,x)$ from now on, and which is therefore equal to

$$\Phi(D,x) = \frac{C^{-4}(D)}{(2\pi)^{4(D-2)}} \int \frac{d^Dk_1\; d^Dk_2\; d^Dk_3\; d^Dk_4}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_4\mathcal{D}_5} \;,\; (p^2=-x^2) \; . \tag{179}$$

One obtains for $\Phi(D,x)$ the following fourth-order ODE

$$\begin{aligned} & x^3(1-x^2)\frac{d^4\Phi(D,x)}{dx^4} + x^2\Big\{ 1+5x^2-3(D-4)(1-3x^2) \Big\} \frac{d^3\Phi(D,x)}{dx^3} \\ & -x\Big\{ 12+6x^2+(D-4)(13+32x^2)+(D-4)^2(1+26x^2) \Big\} \frac{d^2\Phi(D,x)}{dx^2} \\ & +\Big\{ 12-18x^2+(D-4)(25-2x^2)+8(D-4)^2(2+5x^2)+ \\ & \qquad +3(D-4)^3(1+8x^2) \Big\} \frac{d\Phi(D,x)}{dx} \\ & +4x\Big\{ +12+29(D-4)+23(D-4)^2+6(D-4)^3 \Big\} \Phi(D,x) = 0 \\ & . \end{aligned} \tag{180}$$

9.2. *Behaviour of $\Phi(D, x)$ in the limit $x \to 0$*

By inspection, one finds that the most general solution of Eq.(180) can be expanded for $x \to 0$ in the form

$$\Phi(D, x) = \sum_{i=1}^{4} x^{\alpha_i} \left(\sum_{n=0}^{\infty} A_n^{(i)}(D) x^{2n} \right) , \tag{181}$$

where the values of the 4 exponent α_i are

$$\begin{aligned} \alpha_1 &= 0 \ , \\ \alpha_2 &= (D-2) \ , \\ \alpha_3 &= -(D-2) \ , \\ \alpha_4 &= (3D-7) \ ; \end{aligned} \tag{182}$$

the $A_0^{(i)}(D)$ are the 4 integration constants, and all the other coefficients $A_n^{(i)}(D)$ for $n > 0$ are determined by the differential equation Eq.(180), once the integration constants are fixed.

A qualitative inspection of the integrals which one tries to evaluate by means of the differential equation Eq.(179) shows that it is finite (just finite, not analytic!) for $x \to 0^+$ and $(D-2) > 0$; that is sufficient to rule out from their expression as solutions of the differential equation the terms with the behaviour of the third and the fourth exponent (which is negative when D is just above 2).

In the current case, as the equation for $\Phi(D, x)$ is homogeneous, the only information one gets out is that $A_0^{(3)}(D)$ and $A_0^{(4)}(D)$ are both equal to zero, due to the finiteness for $x \to 0^+$; by substituting the *ansatz* Eq.(181) in Eq.(180) and dropping $A_0^{(3)}(D), A_0^{(4)}(D)$, one finds for $\Phi(D, x)$ Eq.(179) the $x \to 0$ expansion

$$\begin{aligned} \Phi(D, x) = A_0^{(1)}(D) &\left(1 - \frac{2(2D-5)(3D-8)}{3D(D-4)} \, x^2 + O(x^4) \right) \\ &+ A_0^{(2)}(D) \, x^{D-2} \left(1 + \frac{(D-3)(D-4)(3D-8)}{2D(2D-7)} \, x^2 + O(x^4) \right) \ . \end{aligned} \tag{183}$$

The expansion depends on the two as yet unspecified integrations constants $A_0^{(1)}(D), A_0^{(2)}(D)$. To fix them, one has to provide some independent information, such as the value of the required Feynman integral and of its first derivative at $x = 0$. Those value can be provided by an explicit conventional calculation, say in parameter space, which is in any case much easier than a calculation for non-zero

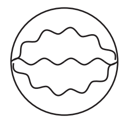


Fig. 8. Four-loop watermelon diagram: a wavy line stands for massless propagator; a solid line, for propagator of mass M.

values of the variable x [b]. That is done explicitly in [147], and the results are

$$
\begin{aligned}
A_0^{(1)} &= -\frac{3D-11}{8(D-2)(D-3)(D-4)^3(2D-5)(2D-7)(3D-8)(3D-10)} \\
&\quad \times \frac{\Gamma(1-(D-4))\Gamma(1-2(D-4))\Gamma^2\left(1+\frac{1}{2}(D-4)\right)\Gamma^2\left(1-\frac{3}{2}(D-4)\right)}{\Gamma^4\left(1-\frac{1}{2}(D-4)\right)\Gamma(1-3(D-4))} \\
A_0^{(2)} &= -\frac{2\,(2D-7)}{3(D-2)^2(D-3)(D-4)^4(3D-8)(3D-10)} \\
&\quad \times \frac{\Gamma\left(1+\frac{1}{2}(D-4)\right)\Gamma\left(1-\frac{3}{2}(D-4)\right)\Gamma^2(1-(D-4))}{\Gamma^2\left(1-\frac{1}{2}(D-4)\right)\Gamma(1-2(D-4))} \ , \qquad (184)
\end{aligned}
$$

where the term $A_0^{(1)}$ is the value of the vacuum graph in Fig.8.

9.3. *Expansion around D = 4 and homogeneous equation*

The Laurent's expansion in $(D-4)$ of $\Phi(D,x)$ Eq.(179) is

$$\Phi(D,x) = \sum_{n=-4}^{\infty} (D-4)^n \Phi^{(n)}(x) \ , \qquad (185)$$

as it is known on general grounds that it develops at most a fourth order pole in $(D-4)$. By substituting in Eq.(180) one obtains a system of inhomogeneous, chained equations for the coefficients $\Phi^{(n)}(x)$ of the expansion in $(D-4)$; the generic equation reads

$$\mathcal{D}\Phi^{(n)}(x) = K^{(n)}(x) \ , \qquad (186)$$

with

$$\mathcal{D} = \left[x^3(1-x^2)\frac{d^4}{dx^4} + x^2(1+5x^2)\frac{d^3}{dx^3} - 6x(2+x^2)\frac{d^2}{dx^2} + 6(2-3x^2)\frac{d}{dx} + 48x\right] \ , \qquad (187)$$

[b] In the present case the knowledge of the regularity of the solution at $x = 1$ does not provide any additional information.

and

$$
\begin{aligned}
K^{(n)}(x) = & \left\{ 24x + \left(3 + 24x^2\right) \frac{d}{dx} \right\} \Phi^{(n-3)}(x) \\
& + \left\{ 92x + \left(16 + 40x^2\right) \frac{d}{dx} - \left(x + 26x^3\right) \frac{d^2}{dx^2} \right\} \Phi^{(n-2)}(x) \\
& + \left\{ 116x + \left(25 - 2x^2\right) \frac{d}{dx} - \left(13x + 32x^3\right) \frac{d^2}{dx^2} - \left(3x^2 - 9x^4\right) \frac{d^3}{dx^3} \right\} \Phi^{(n-1)}(x) \\
, &
\end{aligned} \tag{188}
$$

which shows that the equation at a given order n for $\Phi^{(n)}(x)$ involves in the inhomogeneous term the coefficients $\Phi^{(k)}(x)$ (and their derivatives) with $k < n$ (obviously $\Phi^{(k)}(x) = 0$ when $k < -4$). Such a structure calls for an algorithm of solution bottom-up, i.e. starting from the lowest value of n (which is $n = -4$) and proceeding recursively to the next $n + 1$ value up to the required order.

The Eq.s(186) have all the same associated homogeneous equation, independent of n,

$$
\left[x^3(1 - x^2) \frac{d^4}{dx^4} + x^2(1 + 5x^2) \frac{d^3}{dx^3} - 6x(2 + x^2) \frac{d^2}{dx^2} + 6(2 - 3x^2) \frac{d}{dx} + 48x \right] \phi(x) = 0 \ ; \tag{189}
$$

once the solutions of Eq.(189) are known, all the Eq.s(186) can be solved by the method of the variation of the constants of Euler.

To our (pleasant) surprise, the solutions of Eq.(189) are almost elementary. By trial and error, a first solution is found to be

$$
\phi_1(x) = x^2 \ . \tag{190}
$$

We then substitute $\phi(x) = \phi_1(x)\xi(x)$ in Eq.(189), obtaining the following 3rd order equation for the derivative of $\xi(x)$

$$
\left[x^3(1 - x^2) \frac{d^3}{dx^3} + 3x^2(3 - x^2) \frac{d^2}{dx^2} + 6x(1 + 2x^2) \frac{d}{dx} - 6(5 + 2x^2) \right] \xi'(x) = 0 \ , \tag{191}
$$

and find that it admits the solution

$$
\xi_2'(x) = \frac{1}{x^3}(1 - x^2 + x^4) \ . \tag{192}
$$

Substituting $\xi'(x) = \xi_2'(x)\chi(x)$ in Eq.(191) we obtain the following 2nd order equation for $\chi'(x)$

$$
\left[x^2(1 - x^2)(1 - x^2 + x^4) \frac{d^2}{dx^2} + 6x^5(2 - x^2) \frac{d}{dx} - 6(2 - 2x^4 + x^6) \right] \chi'(x) = 0 \ , \tag{193}
$$

which admits as solution

$$
\chi_3'(x) = \frac{1}{x^3}(1 - x^2)^4 \ \frac{5 - 2x^2 + 5x^4}{(1 - x^2 + x^4)^2} \ . \tag{194}
$$

Finally, substituting $\chi'(x) = \chi_3'(x)\tau(x)$ in Eq.(193), we obtain the equation

$$
\begin{aligned}
& \left[x(1 - x^2)^5(1 - x^2 + x^4)(5 - 2x^2 + 5x^4) \frac{d}{dx} \right. \\
& \left. -2(1 - x^2)^4(15 - 12x^2 + 11x^4 + 30x^6 - 24x^8 + 20x^{10}) \right] \ \tau'(x) = 0 \ ,
\end{aligned} \tag{195}
$$

which has the solution

$$\tau_4'(x) = \frac{x^6}{(1-x^2)^5} \, \frac{1-x^2+x^4}{5-2x^2+5x^4} \; . \tag{196}$$

By repeated quadratures in x and multiplications by the previous solutions we obtain the explicit analytic expressions of the 4 solutions of Eq.(189); the nasty denominators $(1-x^2+x^4)$ and $(5-2x^2+5x^4)$ disappear in the final results, while the repeated integrations of the terms with denominators $x, (1+x)$ and $(1-x)$ generate, almost by definition, HPL's of argument x and weight up to 3.

$$\phi_2(x) = -\frac{1}{2}(1-x^4) - H(0,x)x^2 \; , \tag{197}$$

$$\phi_3(x) = \frac{(5+18x^2+14x^6+5x^8)}{8\;x^2} + \frac{1}{2}(12+x^2-12x^4)H(0;x) + 12\;x^2 H(0,0;x) \; , \tag{198}$$

$$\begin{aligned}\phi_4(x) = {} & \frac{(1+x^2)(15+182x^2+15x^4)}{65536\;x} + \frac{3(1-x^2)^2(5+x^2)(1+5x^2)}{131072\;x^2}\Big[H(-1;x)+H(1;x)\Big] \\ & -\frac{9(1-x^4)}{8192}\Big[H(0,-1;x)+H(0,1;x)\Big] - \frac{9x^2}{4096}\Big[H(0,0,-1;x)+H(0,0,1;x)\Big] \; . \end{aligned} \tag{199}$$

The corresponding Wronskian has the remarkably simple expression

$$W(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \phi_4(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \phi_4'(x) \\ \phi_1''(x) & \phi_2''(x) & \phi_3''(x) & \phi_4''(x) \\ \phi_1'''(x) & \phi_2'''(x) & \phi_3'''(x) & \phi_4'''(x) \end{vmatrix} = \frac{(1-x^2)^3}{x} \; , \tag{200}$$

in agreement (of course) with the coefficients of the 4^{th} and 3^{rd} x-derivative of $\phi(x)$ in Eq.(189).

9.4. *Solutions of the chained equations*

With the results established in the previous Section one can use Euler's method of the variation of the constants for solving Eq.s(186) recursively in n, starting from $n=-4$. We write Euler's formula as

$$\Phi^{(n)}(x) = \sum_{i=1}^{4} \; \phi_i(x) \left[\Phi_i^{(n)} + \int_0^x \frac{dx'}{W(x')} M_i(x') K^{(n)}(x')\right] \; , \tag{201}$$

where the $\phi_i(x)$ are the solutions of the homogeneous equation given in Eq.s(190,199), the $\Phi_i^{(n)}$ are the as yet undetermined integration constants, the Wronskian $W(x)$ can be read from Eq.(200), the $M_i(x)$ are the minors of the $\phi_i^{'''}(x)$ in the determinant Eq.(200), and the $K^{(n)}(x)$ are the inhomogeneous terms of Eq.(188). The constants $\Phi_i^{(n)}$ are then fixed by comparing the expansion in

x for $x \to 0$ of Eq.(201) with the expansion in $(D-4)$ for $D \to 4$ of Eq.(183). Explicitly, we find, for the first two coefficients of the Laurent expansion in Eq.(185),

$$\Phi^{(-4)}(x) = -\frac{1}{64}x^2 \ , \tag{202}$$

$$\Phi^{(-3)}(x) = -\frac{1}{384} + \frac{9}{256}x^2 - \frac{1}{192}x^4 - \frac{1}{48}x^2 H(0;x) \ . \tag{203}$$

$$\tag{204}$$

The full results become quickly too lengthy to be reported explicitly here.

9.5. *Value at $x = 1$ from Difference Equations*

In this section we will calculate $\Phi(D, x = 1)$ by using the Difference Equation technique described by Laporta in [65], which is a formidable way to get numerical results with high accuracy. We will see in this example, the link between Differential Equations and Difference Equations for Feynman integral.

Having set $\mathbb{D}_1 = (k_1^2+1), \mathbb{D}_2 = ((k_2-k_1)^2+1), \mathbb{D}_3 = ((k_3-k_2)^2+1), \mathbb{D}_4 = (k_4-k_3)^2, \mathbb{D}_5 = (p-k_4)^2$, we define

$$I_5(D,n) = \pi^{-2D} \int \frac{d^Dk_1 \ d^Dk_2 \ d^Dk_3 \ d^Dk_4}{\mathbb{D}_1^n \mathbb{D}_2 \mathbb{D}_3 \mathbb{D}_4 \mathbb{D}_5} \ , \quad p^2 = -1 \ , \tag{205}$$

so that $I_5(D, 1)$ is equal to $\Phi(D, x)$ of Eq.(179) at $x = 1$ up to a known multiplicative factor

$$I_5(D,1) = [4\Gamma(1+\epsilon)]^4 \Phi(D = 4-2\epsilon, x = 1) \ . \tag{206}$$

By combining identities obtained by integration by parts one finds that $I_5(D, n)$ satisfies the third-order difference equation

$$\begin{aligned}
&32(n-1)(n-2)(n-3)(n-3D+5) \quad I_5(D,n) \\
&-4(n-2)(n-3)\Big[15n^2 + (39-50D)n + 27D^2 + 5D - 54\Big] \ I_5(D,n-1) \\
&+2(n-3)\Big[12n^3 - (38D+24)n^2 + (23D^2+133D-84)n \\
&\qquad\qquad +9D^3 - 141D^2 + 134D - 24\Big] \ I_5(D,n-2) \\
&+(n-D-1)(n-2D+1)(2n-3D)(2n-5D+4) \ I_5(D,n-3) = 0 \qquad .
\end{aligned} \tag{207}$$

We will solve this difference equation by using the Laplace *ansatz*

$$I_5(D,n) = \int_0^1 t^{n-1} v_5(D,t) \ dt \ , \tag{208}$$

giving for $v_5(D,t)$ the fourth-order differential equation

$$\begin{aligned}
&4t^4(8t+1)(t-1)^2 \ \tfrac{d^4}{dt^4}v_5(D,t) \\
&+4t^3(t-1)\Big[24(D+1)t^2+(18-26D)t-7D+12\Big] \ \tfrac{d^3}{dt^3}v_5(D,t) \\
&t^2\Big[(576(D-1)t^3+(-108D^2-420D+648)t^2 \\
&+(46d^2+38D-144)t+71D^2-284D+288\Big] \ \tfrac{d^2}{dt^2}v_5(D,t) \\
&+t\Big[(576D-960)t^3+(-216D^2+360D)t^2 \\
&+(-18D^3+190D^2-496D+384)t+77D^3-533D^2+1236D-960)\Big] \ \tfrac{d}{dt}v_5(D,t) \\
&+(D-3)(2D-5)(3D-8)(5D-12)\ v_5(D,t)=0 \ . \qquad (209)
\end{aligned}$$

We will look for the solution of Eq.(209) in the form of a power series expansions, which, inserted in Eq.(208) and integrated term by term, will provide very accurate values of $I_5(D,n)$. As the convergence is faster for larger n, we will consider large enough values of the index n (see below); the repeated use top-down of Eq.(207) (*i.e.* using it for expressing $I_5(D,n-3)$ in terms of the $I_5(k)$ with $k=n,n-1,n-2$) will give the values corresponding to smaller indices, till $I_5(D,1)$ is eventually obtained. To go on with this program, initial conditions for $v_5(D,t)$ are needed.

From the definition Eq.(205), and introducing spherical coordinates in D-dimension for the loop momentum k_1, $d^Dk_1=k_1^{D-1}dk_1d\Omega(D,\hat{k}_1)$ one has

$$I_5(D,n)=\frac{1}{\Gamma\left(\frac{D}{2}\right)}\int_0^\infty \frac{(k_1^2)^{D/2-1}\,dk_1^2}{(k_1^2+1)^n}f_5(D,k_1^2)\ , \qquad (210)$$

$$f_5(D,k_1^2)=\int\frac{d\Omega(D,\hat{k}_1)}{\Omega(D)}I_4(D,1,(p-k_1)^2)\ , \qquad (211)$$

where $\Omega(D)$ is the D-dimensional solid angle, and $I_4(D,1,(p-k_1)^2)$ is the 3-loop (off mass-shell) sunrise integral

$$I_4(D,n,(p-k_1)^2)=\pi^{-3D/2}\int\frac{d^Dk_2\ d^Dk_3\ d^Dk_4}{\mathbb{D}_2^n\mathbb{D}_3\mathbb{D}_4\mathbb{D}_5}\ . \qquad (212)$$

By the change of variable $1/(k_1^2+1)=t$, $\quad k_1^2=(1-t)/t$, one finds

$$I_5(D,n)=\frac{1}{\Gamma\left(\frac{D}{2}\right)}\int_0^1 t^{n-1}(1-t)^{\frac{D}{2}-1}t^{-\frac{D}{2}}f_5\left(D,\frac{1-t}{t}\right)\ dt\ , \qquad (213)$$

from which one gets the relation between $v_5(D,t)$ and $f_5(D,(1-t)/t)$

$$v_5(D,t)=\frac{1}{\Gamma\left(\frac{D}{2}\right)}(1-t)^{\frac{D}{2}-1}t^{-\frac{D}{2}}f_5\left(D,\frac{1-t}{t}\right)\ dt\ . \qquad (214)$$

From that relation we see that we can derive boundary conditions for $v_5(D,t)$ in the $t\to1$ limit from the expansion of $f_5(D,k_1^2)$ in the $k_1\to0$ limit, which is easy to obtain. Indeed, only the first denominator of Eq.(212) depends on k_1; expanding for small k_1 and performing the angular integration one gets

$$\begin{aligned}\int \frac{d\Omega(D,\hat{k}_1)}{\Omega(D)} \frac{1}{(k_2-k_1)^2+1} =& \int \frac{d\Omega(D,\hat{k}_1)}{\Omega(D)} \left(\frac{1}{k_2^2+1} - \frac{k_1^2 - 2k_1\cdot k_2}{(k_2^2+1)^2} + \frac{(k_1^2-2k_1\cdot k_2)^2}{(k_2^2+1)^3} + \dots \right) \\ =& \frac{1}{k_2^2+1} + k_1^2 \left(-\frac{1}{(k_2^2+1)^2} + \frac{4}{D}\frac{k_2^2}{(k_2^2+1)^3} \right) + \dots .\end{aligned} \tag{215}$$

The above result gives the expansion of $f_5(D,k_1^2)$ at $k_1^2=0$:

$$\begin{aligned} f_5(D,k_1^2) &= f_5^{(0)}(D) + f_5^{(1)}(D)\ k_1^2 + O(k_1^4)\ , \\ f_5^{(0)}(D) &= I_4(D,1,p^2) \\ f_5^{(1)}(D) &= -\ I_4(D,2,p^2) + \frac{4}{D}\left[I_4(D,2,p^2) - I_4(D,3,p^2) \right]\ . \end{aligned} \tag{216}$$

Note that $f_5(D,k_1^2)$ is regular in the origin.

By inspecting the differential equation (209) one finds that the behaviour at $t=1$ of the 4 independent solution is $\sim (1-t)^{\alpha_i}$, with $\alpha_1 = D/2-1$, $\alpha_2 = D/2$, $\alpha_3 = 0$, and $\alpha_4 = 1$; for comparison with Eq.(214) the behaviours $\alpha_3 = 0$, and $\alpha_4 = 1$ are ruled out and the expansion reads

$$v_5(D,t) = (1-t)^{\frac{D}{2}-1} \left(v_5^{(0)}(D) + v_5^{(1)}(D)\ (1-t) + O(1-t)^2 \right)\ ; \tag{217}$$

by comparison with Eq.(216) ($t=1$ corresponds to $k_1^2=0$), one obtains

$$\begin{aligned} v_5^{(0)}(D) =& \frac{1}{\Gamma\left(\frac{D}{2}\right)} f_5^{(0)}(D)\ , \\ v_5^{(1)}(D) =& \frac{1}{\Gamma\left(\frac{D}{2}\right)} \left[\frac{D}{2} f_5^{(0)}(D) + f_5^{(1)}(D) \right]\ . \end{aligned} \tag{218}$$

The values $I_4(D,n)$ of $I_4(D,n,p^2)$ at $p^2=-1$ are therefore required

$$I_4(D,n) \equiv I_4(D,n,p^2) = \pi^{-3D/2} \int \frac{d^Dk_2\ d^Dk_3\ d^Dk_4}{(k_2^2+1)^n \mathbb{D}_3\mathbb{D}_4\mathbb{D}_5}, \quad p^2 = -1\ . \tag{219}$$

The problem of evaluating the $I_4(D,n)$ is similar to the original problem of evaluating the $I_5(D,n)$, but in fact it is much simpler, as the $I_4(D,n)$ involve one less loop and one less propagator. As above, by using integration-by-parts identities one finds that $I_4(D,n)$ satisfies the third-order difference equation

$$\begin{aligned} & 6(n-1)(n-2)(n-3)I_4(D,n) \\ & -(n-2)(n-3)(10n-7D-10)I_4(D,n-1) \\ & +(n-3)(2n^2+(2D-18)n-7D^2+29D-8)I_4(D,n-2) \\ & +(n-D-1)(n-2D+1)(2n-3D)I_4(D,n-3) = 0\ . \end{aligned} \tag{220}$$

We solve the difference equation by using again the Laplace *ansatz*

$$I_4(D,n) = \int_0^1 t^{n-1} v_4(D,t)\ dt\ , \tag{221}$$

where $v_4(D,t)$ satisfies the differential equation

$$
\begin{aligned}
&2t^3(3t+1)(t-1)^2\frac{d^3}{dt^3}v_4(d,t)\\
&+t^2(t-1)(36t^2+(6-7D)t-9D+18)\frac{d^2}{dt^2}v_4(D,t)\\
&+t(36t^3-14Dt^2+(-7D^2+33D-36)t+13D^2-61D+72)\frac{d}{dt}v_4(D,t)\\
&+(D-3)(2D-5)(3D-8)v_4(D,t)=0\,. \qquad (222)
\end{aligned}
$$

Following the procedure used above, we write

$$
\begin{aligned}
I_4(D,n) &= \frac{1}{\Gamma\left(\frac{D}{2}\right)}\int_0^\infty \frac{(k_2^2)^{D/2-1}\,dk_2^2}{(k_2^2+1)^n}f_4(k_2^2)\,, && (223)\\
f_4(k_2^2) &= \int \frac{d\Omega(D,\hat{k}_2)}{\Omega(D)}I_3(D,(p-k_2)^2)\,, && (224)\\
I_3(D,(p-k_2)^2) &= \pi^{-D}\int \frac{d^Dk_3\;d^Dk_4}{((k_3-k_2)^2+1)(k_4-k_3)^2(p-k_4)^2}\,, && (225)\\
v_4(D,t) &= \frac{1}{\Gamma\left(\frac{D}{2}\right)}(1-t)^{\frac{D}{2}-1}t^{-\frac{D}{2}}f_4\left(D,\frac{1-t}{t}\right)\,. && (226)
\end{aligned}
$$

At variance with the previous case, the function $f_4(D,k_2^2)$ is *not* regular for $k_2\to 0$, as at $k_2=0$ the value of the external momentum squared $(p-k_2)^2$ becomes the threshold of the 2-loop sunrise graph associated to $I_3(D,p^2)$. But it is not difficult to evaluate analytically $I_3(D,q^2)$ for generic off-shell q^2 by using Feynman parameters:

$$
I_3(D,q^2)=\frac{2\Gamma(5-D)\Gamma\left(3-\frac{D}{2}\right)\Gamma^2\left(\frac{D}{2}-1\right)}{(D-4)^2(3-D)\Gamma\left(\frac{D}{2}\right)}\,{}_2F_1\left(3-D,2-\frac{D}{2};\frac{D}{2};-q^2\right)\,,
$$

where ${}_2F_1$ is the Gauss hypergeometric function. The expansion of $I_3(D,q^2)$ in $q^2=-1$ consists of the sum of two series,

$$
I_3(D,q^2)=a_0(D)\Big[1+O(q^2+1)\Big] \;+\; b_0(D)\;(q^2+1)^{2D-5}\Big[1+O(q^2+1)\Big]\,, \qquad (227)
$$

$$
\begin{aligned}
a_0(D) \;=\; I_3(D,-1) \;&=\frac{2\Gamma(5-D)\Gamma\left(3-\frac{D}{2}\right)\Gamma^2\left(\frac{D}{2}-1\right)\Gamma(2D-5)}{(4-D)^2(3-D)\Gamma\left(\frac{3}{2}D-3\right)\Gamma(D-2)}\,,\\
b_0(D) \;&=\Gamma^2\left(\frac{D}{2}-1\right)\Gamma(5-2D)\,.
\end{aligned}
$$

Inserting Eq.(227) into Eq.(224), setting $q=p-k_2$ and performing the angular integration over $\hat{k}_2$ by means of the formula (see Eq.(88) of Ref.[65]) valid for $k_2\to 0$

$$
\frac{1}{\Omega(D)}\int\frac{d\Omega(D,\hat{k}_2)}{((p-k_2)^2+1)^N}\approx(k_2^2)^{-\frac{N}{2}}\frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{N}{2}\right)}{2\Gamma(N)\Gamma\left(\frac{1}{2}(D-N)\right)}\,,\quad k_2\to 0\,; \qquad (228)
$$

with $N = -(2D-5)$, as in the term $(q^2+1)^{2D-5}$ of Eq.(227), one obtains

$$f_4(D,k_2^2) = a_0(D)\left[1+O(k_2^2)\right] + \frac{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{5}{2}-D\right)}{2\Gamma(5-2D)\Gamma\left(\frac{1}{2}(3D-5)\right)}\, b_0(D)\,(k_2^2)^{\frac{1}{2}(2D-5)}\left[1+O(k_2^2)\right]. \quad (229)$$

Using the variable $1/(k_2^2+1) = t$ in Eq.(229) and inserting it in Eq.(226) one gets the initial condition for $v_4(D,t)$ at the singular point $t = 1$

$$\begin{aligned} v_4(D,t) = {} & \frac{a_0(D)}{\Gamma\left(\frac{D}{2}\right)}(1-t)^{\frac{D}{2}-1}\left[1+O(1-t)\right] \\ & + \frac{\Gamma\left(\frac{5}{2}-D\right)}{2\Gamma(5-2D)\Gamma\left(\frac{1}{2}(3D-5)\right)} b_0(D)(1-t)^{\frac{1}{2}(3D-7)}\left[1+O(1-t)\right]. \end{aligned} \quad (230)$$

By inspecting the equation (222) one gets that the behaviour at $t = 0$ of $v_4(D,t)$ is

$$v_4(D, t\to 0) \approx c_4^{(1)}(D)\; t^{-D+3} + c_4^{(2)}(D)\; t^{-2D+5} + c_4^{(3)}(D)\; t^{\frac{1}{2}(-3D+8)}\,, \quad (231)$$

so that for $D \to 4$ the integral (221) is convergent for $n \geq 4$.

All the quantities depending on D are then systematically expanded in $(D-4)$; and the series are truncated at some fixed number of terms. We solve finally the differential equation (222) with the initial condition (230) by a first expansions in series at $t = 1$; due to the presence in Eq.(222) of a singular point at $t = -1/3$, in order to have fast convergence till $t = 0$, we switch to the subsequent series expansions at the intermediate points 1/2, 1/4, 1/8 and 0; then we calculate the integral (221) for $n = 4, 5, 6, 7, 8$ by integrating the series term by term By applying repeatedly top-down the recurrence relation (220) to $I_4(D,8)$, $I_4(D,7)$, $I_4(D,6)$, we obtain $I_4(D,5)$ and $I_4(D,4)$, $I_4(D,3)$, $I_4(D,2)$ and $I_4(D,1)$. Those values of $I_4(D,n)$ are used to determine the initial condition for $v_5(D,t)$, Eq.s(217,218,216). The game must be repeated again for $v_4(d,t)$. In fact, we solve the differential equation (209) by expansions in series centered in the points $t = 1$, 1/2, 1/4, 1/8, 1/16 and 0 (as above, this subdivision is due to the presence of a singular point at $t = -1/8$). By inspecting the equation (209) one gets that the behaviour at $t = 0$ of $v_4(d,t)$ is

$$v_5(D, t\to 0) \approx c_5^{(1)}(D)t^{-D+3} + c_5^{(2)}(D)t^{-2D+5} + c_5^{(3)}(D)t^{(-3D+8)/2} + c_5^{(4)}(D)t^{(-5D+12)/2}\,, \quad (232)$$

so that, when $D \to 4$, the integral (208) is surely convergent for $n \geq 5$; then we calculate the integral (208) for $n = 5, 6, 7, 8, 9$ by integrating the series term by term. By using repeatedly top-down the recurrence relation (220) starting from $n = 9$, we finally obtain $I_5(D,6)$, $I_5(D,5)$, $I_5(D,4),\ldots$, $I_5(D,1)$. By taking into account the normalization (206) one finds complete agreement with the value at $x = 1$ of solution of the differential equation computed in the previous section.

10. Conclusions

The evaluation of multiloop Feynman diagrams in the last years has received a strong boost, thanks to the ability of turning generic properties of scalar integrals in dimensional regularization into tools for computing them. Integration-by-parts, Lorentz invariance, and kinematic symmetries have been exploited to establish infinite sets of relations among integrals sharing (partially) common integrands. The Laporta algorithm systematizes the by now standard reduction to Master Integrals, that is the algebraic procedure for expressing any Feynman integral as a linear combination of few basic integrals with the simplest integrands.

The completion of the computational task, consisting in the actual evaluation of the Master Integrals can be as well afforded by employing the same set of identities. In fact, by combining the differentiation of Master Integrals with respect to their Mandelstam invariants, and the reduction of the new born integrals, it is possible to derive a system of non-homogeneous first order differential equations fulfilled by the Master Integrals themselves.

Solving such a system of differential equation amounts to the evaluation of the Master Integrals, alternatively to their direct parametric integration.

We have reviewed the method of differential equations by its direct application, trying to follow a didactical path. We discussed the reduction algorithm plus the general derivation of differential equations for Feynman integrals. Successively, the calculation of Master Integrals in the context of the evaluation of the one- and two-loop corrections to the photon propagator in QED; whereas, in the last two sections, we presented two cases of less trivial differential equations, to show more technical aspects related to the solution of homogeneous equations and to the choice of the boundary conditions.

In general, solving a system of first order differential equations for more than one Master Integral is equivalent to solving a higher order equation for a single Master Integral. Despite to the lack of a theoretical procedure for solving differential equations in the most general case, Euler's variation of constants offers a viable procedure. Accordingly, the solution of the non-homogeneous equation is obtained by quadrature, using as a kernel the Wronskian of the associated homogeneous equation – whose solution can be preliminarily found by constants' variation as well.

The main achievement is the integration of the differential equation for exact value of the parameters (Mandelstam invariants and dimensional-parameter). When that is not possible, one can Laurent expand the equation, which then becomes a chained system of equations for the Laurent coefficients of the solution, suitable for a bottom-up solving algorithm, starting from the lowest Laurent coefficient.

As a natural feature of Euler's variation of constants, the solution manifests an analytic integral representation, generic of transcendental functions: a flexible nested structure of multiple integrations (or equivalently, iterative summations) which benefits of the shuffle algebras induced by the integration-by-parts. Within this framework, the actual efforts required by the computation are the finding out of the homogeneous solutions, and the definition of new occurring functions, ordered according to their increasing trascendentality – as required by the iterative fulfilment of the non-homogeneous equation.

Boundary conditions are found by imposing the regularity or the finiteness of the solution at the pseudo-thresholds of the Master Integrals. This qualitative information is sufficient for the quantitative determination of the arbitrary integration constants. At the diagrammatic level, boundary conditions usually correspond to integrals related to simpler diagrams.

The use of Differential Equation in the external invariants is a very powerful tool for computing Feynman (master) integrals. Dimensional regularization was fundamental for the derivation of the differential equations we discussed in this review.

In principle differential identities for integral functions can be derived whenever it is allowed by the algebra of the integral representation under use - as induced by integration-by-parts. And their use is not limited to the perturbative description of Feynman diagrams. Therefore we like to conclude by remarking that the use of Differential Equations for integrals' evaluation, is not just a technique, but a *point of view* from which any integral is seen under a new *perspective*, where there appear, explicitly exposed, its analytic structure, its singularities which finally determine its value.

11. Acknowledgment

We wish to thank Roberto Bonciani, Thomas Gehrmann, and Ettore Remiddi, for their careful reading and comments on the manuscript, but especially for their invaluable collaboration during these years. The research of PM was supported by Marie-Curie-EIF under the contract MEIF-CT-2006-024178. MA wishes to anknowledge the Institute for Theoretical Physics of Zürich for the kind hospitality while part of this work was performed, with the support of the abovementioned Marie-Curie-EIF founds.

References

1. A. V. Kotikov, Phys. Lett. B **254** (1991) 158.
2. E. E. Boos and A. I. Davydychev, Theor. Math. Phys. **89** (1991) 1052 [Teor. Mat. Fiz. **89** (1991) 56].
3. A. V. Kotikov, ITF-91-26E
4. A. V. Kotikov, Phys. Lett. B **259** (1991) 314.
5. F. A. Lunev, Phys. Rev. D **50** (1994) 6589 [arXiv:hep-th/9407174].
6. R. Scharf, PhD theis, Wurzburg 1994.
7. E. Remiddi, Nuovo Cim. A **110** (1997) 1435 [arXiv:hep-th/9711188].
8. T. Gehrmann and E. Remiddi, Nucl. Phys. B **580**, 485 (2000) [arXiv:hep-ph/9912329].
9. T. Gehrmann and E. Remiddi, Nucl. Phys. B **601**, 248 (2001) [arXiv:hep-ph/0008287].
10. T. Gehrmann and E. Remiddi, Nucl. Phys. B **601**, 287 (2001) [arXiv:hep-ph/0101124].
11. C. Ford and D. R. T. Jones, Phys. Lett. B **274** (1992) 409 [Erratum-ibid. B **285** (1992) 399].
12. M. Caffo, H. Czyz, S. Laporta and E. Remiddi, Acta Phys. Polon. B **29**, 2627 (1998) [arXiv:hep-th/9807119].
13. M. Caffo, H. Czyz, S. Laporta and E. Remiddi, Nuovo Cim. A **111**, 365 (1998) [arXiv:hep-th/9805118].
14. R. Bonciani, Acta Phys. Polon. B **30** (1999) 3463.
15. A. V. Kotikov, arXiv:hep-ph/0102178.
16. M. Czachor and H. Czyz, Acta Phys. Polon. B **32**, 3823 (2001) [arXiv:hep-ph/0110351].
17. M. Caffo, H. Czyz and E. Remiddi, Nucl. Instrum. Meth. A **502**, 613 (2003) [arXiv:hep-ph/0211171].
18. M. Caffo, H. Czyz and E. Remiddi, Nucl. Phys. Proc. Suppl. **116**, 422 (2003) [arXiv:hep-ph/0211178].
19. P. Mastrolia and E. Remiddi, Nucl. Phys. Proc. Suppl. **116**, 412 (2003) [arXiv:hep-ph/0211210].
20. U. Aglietti, arXiv:hep-ph/0408014.
21. R. Bonciani, Acta Phys. Polon. B **35**, 2587 (2004) [arXiv:hep-ph/0410210].
22. V. A. Smirnov, "Feynman integral calculus," *Berlin, Germany: Springer (2006) 283 p*
23. M. Caffo, Acta Phys. Polon. B **34**, 5357 (2003) [arXiv:hep-ph/0311052].
24. M. Caffo, H. Czyz, A. Grzelinska and E. Remiddi, Nucl. Phys. B **681**, 230 (2004) [arXiv:hep-ph/0312189].
25. J. A. Gracey, Phys. Lett. B **277** (1992) 469.
26. O. V. Tarasov, Phys. Rev. D **54** (1996) 6479 [arXiv:hep-th/9606018];
Nucl. Phys. Proc. Suppl. **89** (2000) 237 [arXiv:hep-ph/0102271].
27. O. V. Tarasov, Nucl. Phys. B **502** (1997) 455 [arXiv:hep-ph/9703319].
28. J. Fleischer, A. V. Kotikov and O. L. Veretin, Phys. Lett. B **417** (1998) 163 [arXiv:hep-ph/9707492].

29. F. A. Berends, A. I. Davydychev and N. I. Ussyukina, Phys. Lett. B **426** (1998) 95 [arXiv:hep-ph/9712209].
30. J. M. Chung and B. K. Chung, Phys. Rev. D **59** (1999) 105014 [arXiv:hep-ph/9805432].
31. J. Fleischer, A. V. Kotikov and O. L. Veretin, Nucl. Phys. B **547**, 343 (1999) [arXiv:hep-ph/9808242].
32. A. I. Davydychev and V. A. Smirnov, Nucl. Phys. B **554** (1999) 391 [arXiv:hep-ph/9903328].
33. J. Fleischer, M. Y. Kalmykov and A. V. Kotikov, arXiv:hep-ph/9905379.
34. J. Fleischer, M. Y. Kalmykov and A. V. Kotikov, Phys. Lett. B **462** (1999) 169 [arXiv:hep-ph/9905249].
35. J. Fleischer and M. Y. Kalmykov, Comput. Phys. Commun. **128** (2000) 531 [arXiv:hep-ph/9907431].
36. J. Fleischer and M. Y. Kalmykov, Phys. Lett. B **470** (1999) 168 [arXiv:hep-ph/9910223].
37. J. Fleischer, O. V. Tarasov and M. Tentyukov, Nucl. Phys. Proc. Suppl. **89** (2000) 112.
38. V. A. Smirnov, arXiv:hep-ph/0209177.
39. V. A. Smirnov, Nucl. Phys. Proc. Suppl. **116**, 417 (2003) [arXiv:hep-ph/0209295].
40. A. Onishchenko and O. Veretin, Phys. Atom. Nucl. **68** (2005) 1405 [Yad. Fiz. **68** (2005) 1461] [arXiv:hep-ph/0207091].
41. S. P. Martin, Phys. Rev. D **68**, 075002 (2003) [arXiv:hep-ph/0307101].
42. E. Remiddi, Acta Phys. Polon. B **34** (2003) 5311 [arXiv:hep-ph/0310332].
43. E. Remiddi, Nucl. Phys. Proc. Suppl. **135** (2004) 247.
44. S. Actis, A. Ferroglia, G. Passarino, M. Passera and S. Uccirati, Nucl. Phys. B **703**, 3 (2004) [arXiv:hep-ph/0402132].
45. T. G. Birthwright, E. W. N. Glover and P. Marquard, JHEP **0409**, 042 (2004) [arXiv:hep-ph/0407343].
46. B. A. Kniehl and A. V. Kotikov, arXiv:hep-ph/0508238.
47. A. V. Bogdan and R. N. Lee, Nucl. Phys. B **732**, 169 (2006) [arXiv:hep-ph/0509181].
48. B. A. Kniehl, A. V. Kotikov, A. Onishchenko and O. Veretin, Nucl. Phys. B **738**, 306 (2006) [arXiv:hep-ph/0510235].
49. O. V. Tarasov, Phys. Lett. B **638** (2006) 195 [arXiv:hep-ph/0603227].
50. C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J. B. Tausk, Nucl. Phys. B **580**, 577 (2000) [arXiv:hep-ph/0003261].
51. T. Gehrmann and E. Remiddi, Nucl. Phys. Proc. Suppl. **89**, 251 (2000) [arXiv:hep-ph/0005232].
52. V. A. Smirnov, Phys. Lett. B **500**, 330 (2001) [arXiv:hep-ph/0011056].
53. T. Gehrmann and E. Remiddi, in *Proc. of the 5th International Symposium on Radiative Corrections (RADCOR 2000)* ed. Howard E. Haber, arXiv:hep-ph/0101147.
54. T. Gehrmann, arXiv:hep-ph/0107090.
55. W. Giele *et al.*, arXiv:hep-ph/0204316.
56. S. Frixione, Nucl. Phys. Proc. Suppl. **117**, 222 (2003) [arXiv:hep-ph/0211434].
57. J. B. Tausk, Phys. Lett. B **469** (1999) 225 [arXiv:hep-ph/9909506];
    Z. Bern, L.J. Dixon and A. Ghinculov, Phys. Rev. D **63** (2001) 053007 [hep-ph/0010075];
    C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. B **601** (2001) 318 [hep-ph/0010212]; **601** (2001) 347 [hep-ph/0011094]; **605** (2001) 486 [hep-ph/0101304];
    E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. **605** (2001) 467 [hep-ph/0102201];
    C. Anastasiou, E.W.N. Glover and M.E. Tejeda-Yeomans, Nucl. Phys. B **629** (2002) 255 [hep-ph/0201274];
    E.W.N. Glover and M.E. Tejeda-Yeomans, JHEP **0306** (2003) 033 [hep-ph/0304169];
    E.W.N. Glover, JHEP **0404** (2004) 021 [hep-ph/0401119];
    Z. Bern, A. De Freitas and L.J. Dixon, JHEP **0109** (2001) 037 [hep-ph/0109078]; JHEP **0203** (2002) 018 [hep-ph/0201161]; JHEP **0306** (2003) 028 [hep-ph/0304168];
    A. De Freitas and Z. Bern, JHEP **0409** (2004) 039 [hep-ph/0409007];
    Z. Bern, A. De Freitas, L.J. Dixon, A. Ghinculov and H.L. Wong, JHEP **0111** (2001) 031 [hep-ph/0109079];
    T. Binoth, E.W.N. Glover, P. Marquard and J.J. van der Bij, JHEP **0205** (2002) 060 [hep-ph/0202266];

58. L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis and E. Remiddi, Nucl. Phys. B **627** (2002) 107 [hep-ph/0112081] and **642** (2002) 227 [hep-ph/0206067]. T. Gehrmann and E. Remiddi, Nucl. Phys. B **640** (2002) 379 [hep-ph/0207020].

59. C. Buttar *et al.*, arXiv:hep-ph/0604120.
60. A. I. Davydychev, Acta Phys. Polon. B **28** (1997) 841 [arXiv:hep-ph/9610510].
61. C. Anastasiou, E. W. N. Glover and C. Oleari, Nucl. Phys. B **572** (2000) 307 [arXiv:hep-ph/9907494].
62. A. V. Kotikov, arXiv:hep-ph/0102177.
63. G. Passarino, Nucl. Phys. B **619**, 257 (2001) [arXiv:hep-ph/0108252];
G. Passarino and S. Uccirati, Nucl. Phys. B **629**, 97 (2002) [arXiv:hep-ph/0112004];
A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, Nucl. Phys. B **680**, 199 (2004) [arXiv:hep-ph/0311186];
S. Actis, A. Ferroglia, G. Passarino, M. Passera and S. Uccirati, Nucl. Phys. B **703**, 3 (2004) [arXiv:hep-ph/0402132]
G. Passarino and S. Uccirati, Nucl. Phys. B **747**, 113 (2006) [arXiv:hep-ph/0603121].
64. A. V. Kotikov, arXiv:hep-ph/0112347.
65. S. Laporta, Int. J. Mod. Phys. A **15**, 5087 (2000) [arXiv:hep-ph/0102033].
66. S. Laporta, Acta Phys. Polon. B **34** (2003) 5323 [arXiv:hep-ph/0311065].
67. S. Laporta, Phys. Lett. B **549**, 115 (2002) [arXiv:hep-ph/0210336].
68. A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, Nucl. Phys. B **650** (2003) 162 [arXiv:hep-ph/0209219].
69. A. T. Suzuki, E. S. Santos and A. G. M. Schmidt, J. Phys. A **36** (2003) 11859 [arXiv:hep-ph/0309080].
70. V. A. Smirnov, Nucl. Phys. Proc. Suppl. **135**, 252 (2004) [arXiv:hep-ph/0406052].
71. T. Binoth and G. Heinrich, Nucl. Phys. B **585** (2000) 741 [arXiv:hep-ph/0004013].
72. T. Binoth and G. Heinrich, Nucl. Phys. B **680** (2004) 375 [arXiv:hep-ph/0305234].
73. G. Duplancic and B. Nizic, Eur. Phys. J. C **35**, 105 (2004) [arXiv:hep-ph/0303184].
74. M. Czakon, arXiv:hep-ph/0511200.
75. C. Anastasiou and A. Daleo, arXiv:hep-ph/0511176.
76. V. A. Smirnov, Nucl. Phys. Proc. Suppl. **157**, 131 (2006) [arXiv:hep-ph/0601268].
77. S. Weinzierl, arXiv:hep-ph/0604068.
78. A. G. Grozin, Int. J. Mod. Phys. A **19** (2004) 473 [arXiv:hep-ph/0307297].
79. V. A. Smirnov, Springer Tracts Mod. Phys. **211** (2004) 1.
80. R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B **661**, 289 (2003) [Erratum-ibid. B **702**, 359 (2004)] [arXiv:hep-ph/0301170].
81. R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B **676**, 399 (2004) [arXiv:hep-ph/0307295].
82. R. Bonciani, Nucl. Phys. Proc. Suppl. **152**, 168 (2006) [arXiv:hep-ph/0410092].
83. R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B **690**, 138 (2004) [arXiv:hep-ph/0311145].
84. W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia and E. Remiddi, Nucl. Phys. B **706**, 245 (2005) [arXiv:hep-ph/0406046].
85. W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia and E. Remiddi, Nucl. Phys. B **712**, 229 (2005) [arXiv:hep-ph/0412259].
86. W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber and E. Remiddi, Nucl. Phys. B **723**, 91 (2005) [arXiv:hep-ph/0504190].
87. W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia and E. Remiddi, Phys. Rev. Lett. **95** (2005) 261802 [arXiv:hep-ph/0509341].
88. A. V. Kotikov, A. V. Lipatov, G. Parente and N. P. Zotov, Eur. Phys. J. C **26** (2002) 51 [arXiv:hep-ph/0107135].
89. A. V. Kotikov and L. N. Lipatov, Nucl. Phys. B **661** (2003) 19 [Erratum-ibid. B **685** (2004) 405] [arXiv:hep-ph/0208220].
90. F. Jegerlehner and M. Y. Kalmykov, Nucl. Phys. B **676** (2004) 365 [arXiv:hep-ph/0308216].

91. C. Anastasiou, L. J. Dixon, K. Melnikov and F. Petriello, Phys. Rev. D **69** (2004) 094008 [arXiv:hep-ph/0312266].
92. K. G. Chetyrkin, J. H. Kuhn, P. Mastrolia and C. Sturm, Eur. Phys. J. C **40** (2005) 361 [arXiv:hep-ph/0412055].
93. Y. Schroder and A. Vuorinen, JHEP **0506**, 051 (2005) [arXiv:hep-ph/0503209].
94. M. Faisst, P. Maierhoefer and C. Sturm, arXiv:hep-ph/0611244.
95. M. Neubert, Phys. Rept. **245** (1994) 259 [arXiv:hep-ph/9306320].
96. A. Juste *et al.*, arXiv:hep-ph/0601112.
97. W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia and E. Remiddi, PoS **HEP2005**, 229 (2006) [arXiv:hep-ph/0601207].
98. W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, T. Leineweber, P. Mastrolia and E. Remiddi, arXiv:hep-ph/0604031.
99. R. Bonciani, arXiv:hep-ph/0607037.
100. U. Aglietti and R. Bonciani, Nucl. Phys. B **668**, 3 (2003) [arXiv:hep-ph/0304028].
101. U. Aglietti and R. Bonciani, Nucl. Phys. B **698**, 277 (2004) [arXiv:hep-ph/0401193].
102. U. Aglietti, R. Bonciani, L. Grassi and E. Remiddi, arXiv:0705.2616 [hep-ph].
103. P. N. Maher, L. Durand and K. Riesselmann, Phys. Rev. D **48** (1993) 1061 [Erratum-ibid. D **52** (1995) 553] [arXiv:hep-ph/9303233].
104. U. Aglietti, R. Bonciani, G. Degrassi and A. Vicini, Phys. Lett. B **595**, 432 (2004) [arXiv:hep-ph/0404071].
105. U. Aglietti, R. Bonciani, G. Degrassi and A. Vicini, Phys. Lett. B **600**, 57 (2004) [arXiv:hep-ph/0407162].
106. W. Bernreuther, R. Bonciani, T. Gehrmann, R. Heinesch, P. Mastrolia and E. Remiddi, Phys. Rev. D **72**, 096002 (2005) [arXiv:hep-ph/0508254].
107. C. Anastasiou, S. Beerli, S. Bucherer, A. Daleo and Z. Kunszt, arXiv:hep-ph/0611236.
108. U. Aglietti, R. Bonciani, G. Degrassi and A. Vicini, arXiv:hep-ph/0611266.
109. V. A. Smirnov, Phys. Lett. B **524**, 129 (2002) [arXiv:hep-ph/0111160].
110. R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi and J. J. van der Bij, Nucl. Phys. B **681**, 261 (2004) [Erratum-ibid. B **702**, 364 (2004)] [arXiv:hep-ph/0310333].
111. R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi and J. J. van der Bij, Nucl. Phys. B **701**, 121 (2004) [arXiv:hep-ph/0405275].
112. G. Heinrich and V. A. Smirnov, Phys. Lett. B **598**, 55 (2004) [arXiv:hep-ph/0406053].
113. R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi and J. J. van der Bij, Nucl. Phys. B **716**, 280 (2005) [arXiv:hep-ph/0411321].
114. M. Czakon, J. Gluza and T. Riemann, Phys. Rev. D **71**, 073009 (2005) [arXiv:hep-ph/0412164].
115. R. Bonciani and A. Ferroglia, Phys. Rev. D **72**, 056004 (2005) [arXiv:hep-ph/0507047].
116. R. Bonciani and A. Ferroglia, Nucl. Phys. Proc. Suppl. **157**, 11 (2006) [arXiv:hep-ph/0601246].
117. M. Czakon, J. Gluza, K. Kajda and T. Riemann, Nucl. Phys. Proc. Suppl. **157**, 16 (2006) [arXiv:hep-ph/0602102].
118. D. Seidel, Phys. Rev. D **70**, 094038 (2004) [arXiv:hep-ph/0403185].
119. K. Melnikov and A. Mitov, Phys. Lett. B **620**, 69 (2005) [arXiv:hep-ph/0505097].
120. H. M. Asatrian, T. Ewerth, A. Ferroglia, P. Gambino and C. Greub, Nucl. Phys. B **762** (2007) 212 [arXiv:hep-ph/0607316].
121. F. De Fazio, arXiv:hep-ph/0010007.
122. F. Jegerlehner, M. Y. Kalmykov and O. Veretin, Nucl. Phys. B **658** (2003) 49 [arXiv:hep-ph/0212319].
123. M. Awramik, M. Czakon, A. Freitas and G. Weiglein, Phys. Rev. Lett. **93**, 201805 (2004) [arXiv:hep-ph/0407317].
124. M. Awramik, M. Czakon, A. Freitas and G. Weiglein, Nucl. Phys. Proc. Suppl. **135**, 119 (2004) [arXiv:hep-ph/0408207].
125. A. Freitas, M. Awramik and M. Czakon, arXiv:hep-ph/0507159.

126. M. Awramik, M. Czakon and A. Freitas, arXiv:hep-ph/0605339.
127. D. J. Broadhurst, arXiv:hep-th/9806174.
128. A. I. Davydychev and R. Delbourgo, Acta Phys. Polon. B **29**, 2891 (1998) [arXiv:hep-th/9806248].
129. Remiddi, E. and Vermaseren, J. A. M., Int. J. Mod. Phys. A **15** (2000) 725-754 [hep-ph/9905237].
Gehrmann, T. and Remiddi, E., Comput. Phys. Commun. **141** (2001) 296 [hep-ph/0107173] ; Comput. Phys. Commun. **144** (2002) 200 [hep-ph/0111255]; Nucl. Phys. B **640** (2002) 379 [hep-ph/0207020].
130. D. Maitre, Comput. Phys. Commun. **174** (2006) 222 [arXiv:hep-ph/0507152].
131. D. Maitre, arXiv:hep-ph/0703052.
132. Moch, S. and Uwer, P. and Weinzierl, S., J. Math. Phys. **43** (2002) 3363 [hep-ph/0110083].
133. Weinzierl, S., Comput. Phys. Commun. **145** (2002) 357 [math-ph/0201011]
134. Blumlein, J., (2003) hep-ph/0311046.
135. M. Y. Kalmykov, Nucl. Phys. Proc. Suppl. **135**, 280 (2004) [arXiv:hep-th/0406269].
136. M. Y. Kalmykov, B. F. L. Ward and S. Yost, arXiv:hep-th/0612240.
137. G. V. Dunne, JHEP **0402**, 013 (2004) [arXiv:hep-th/0311167].
138. S. P. Martin, Phys. Rev. D **70**, 016005 (2004) [arXiv:hep-ph/0312092].
139. S. P. Martin and D. G. Robertson, Comput. Phys. Commun. **174**, 133 (2006) [arXiv:hep-ph/0501132].
140. G. V. Dunne and M. Krasnansky, JHEP **0604** (2006) 020 [arXiv:hep-th/0602216].
141. J. A. Gracey, arXiv:hep-th/0605037.
142. G. Källen, A. Sabry, *Da. Mat. Fys. Medd.* 29, No.17 (1955) 1.
143. A. Djouadi, P.Gambino *Phys. Rev.* **D49** (1994) 3499
144. D.J. Broadhurst, J. Fleischer, O.V. Tarasov, *Z. für Physik* **C60** (1993) 287 .
145. M. Argeri, P. Mastrolia and E. Remiddi, Nucl. Phys. B **631** (2002) 388 [arXiv:hep-ph/0202123].
146. P. Mastrolia and E. Remiddi, Nucl. Phys. B **657** (2003) 397 [arXiv:hep-ph/0211451].
147. S. Laporta, P. Mastrolia and E. Remiddi, Nucl. Phys. B **688** (2004) 165 [arXiv:hep-ph/0311255].
148. S. Laporta and E. Remiddi, Nucl. Phys. B **704** (2005) 349 [arXiv:hep-ph/0406160].
149. Chetyrkin, K. G. and Tkachov, F. V., Nucl. Phys. B **192** (1981) 159.
Tkachov, F. V., Phys. Lett. B **100** (1981) 65.
150. S. Laporta and E. Remiddi, Phys. Lett. B **379** (1996) 283 [arXiv:hep-ph/9602417].
151. P. Mastrolia and E. Remiddi, Nucl. Phys. Proc. Suppl. **89** (2000) 76.
152. `SOLVE`, by E. Remiddi, available from the author.
`SYS`, by S. Laporta, [65] .
`IdSolver`, by M. Czakon.
`AIR`, by C. Anastasiou and A. Lazopoulos, JHEP **0407** (2004) 046 [arXiv:hep-ph/0404258].
153. P. A. Baikov, Phys. Lett. B **634** (2006) 325 [arXiv:hep-ph/0507053].
154. A. V. Smirnov and V. A. Smirnov, JHEP **0601** (2006) 001 [arXiv:hep-lat/0509187].
155. R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne "The Analytic S-Matrix," *Cambridge University Press (1966) 287 p*
156. t Hooft, G. and Veltman, M. J. G., Nucl. Phys.", B **44** (1972) 189-213.
Bollini, C. G. and Giambiagi, J. J., Phys. Lett. B **40** (1972) 566-568; Nuovo Cim. B **12** (1972) 20-25.
Ashmore, J. F., Lett. Nuovo Cim. 4 (1972) 289-290.
Cicuta, G. M. and Montaldi, E., Nuovo Cim. Lett. 4 (1972) 329-332.
Gastmans, R. and Meuldermans, R., Nucl. Phys. **B** 63 (1973) 277-284.
Collins, J. C., *Renormalization*, "Cambridge University Press", (1987).