The Greek Letters – Notes

Riccardo Albertini

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What are the Greek letters?

Measures such as *Delta, Gamma, Vega* and *Rho* are collectively referred to as the *Greek letters*. They quantify different aspects of risk in an option position.

What is needed to calculate an option Greek?

To calculate a Greek letter, it is necessary to assume an option pricing model. Traders usually assume the Black-Scholes-Merton (BSM) model for European options and the binomial tree model for American options. (the assumptions of the models are the same)

When calculating Greek letters, traders normally set the volatility equal to the current implied volatility. This approach is often referred to as the "practitioner Black-Scholes model". When volatility is set equal to the implied volatility, the model gives the option price at a particular time as an exact function of the price of the underlying asset, the implied volatility, interest rates and (possibly) dividends.

What is implied volatility?

Volatility is the only parameter in the BSM pricing formulas that cannot be directly observed. It can be estimated from the history of a stock price. However, in practice, traders usually work with implied volatilities, i.e., the volatilities implied by option prices observed in the market. Concretely, the implied volatility is the value σ that, when substituted in the pricing formula, gives the option market price. This value is found through an iterative trial and error procedure.

Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. Whereas historical volatilities are backward looking, implied volatilities are forward looking. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price.

What is Delta and Delta hedging?

The delta (Δ) of an option was initially introduced to explain the binomial pricing model. It is defined as the rate of change of the option price with respect to the underlying asset. It is the slope of the curve that relates the option price to the asset price. Suppose that the delta of a call option is 0.6. This means that when the stock price changes by a small amount, the option price changes by about 60% of that amount.

In general,

$$\Delta = \frac{\delta c}{\delta S}$$

Delta is a value between [0,1] for call options and [0,-1=] for put options.

Example

Suppose that the stock price is \$100 and the option price is \$10. Imagine an investor who has sold call options to buy 2,000 shares of a stock. The investor's position could be hedged by buying 0.6*2,000 = 1,200 shares. The gain (loss) on the stock position would offset the gain (loss) on the stock position. For example, if the stock price goes up by \$1, (producing a gain of \$1,200 on the shares purchased), the option price will tend to go up by 0.6*\$1=\$0.6 (producing a loss of \$1,200 on the options written).

Hence, in this example, the delta of the trader's short position in 2,000 options is

$$0.6 * (-2,000) = -1,200$$

This means that the trader loses 1,220ΔS on the option position when the stock price increases by ΔS. The delta of one share of the stock is 1.0, so that the long position in 1,200 shares has a delta of 1,200. The delta of the trader's overall position is therefore zero. The delta of the stock position offsets the delta of the option position. A position with a delta of zero is referred to as *delta-neutral*

Why does the delta of a call option increase as the underlying price increases?

As a stock moves higher, and a call gets more and more in the money, the more and more that call is going to act like the stock. The opposite is true for a put. The figure below shows the relationship between the stock price and the level of delta.

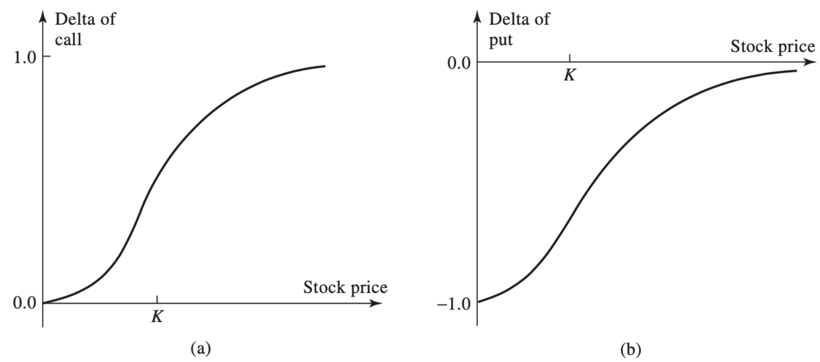


Figure 1: Variation of delta with stock price

What is Theta?

The theta (Θ) of a portfolio is the rate of change of the portfolio value with respect to the passage of time with all else remaining the same. It is sometimes referred to as the time decay of the portfolio.

How is theta calculated?

For a European call option on a non-dividend paying stock it can be shown from the BSM formula that

$$\Theta(call) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

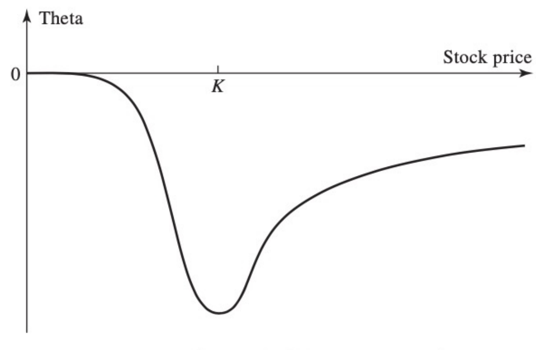
For a European put option on the stock:

$$\Theta(put) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

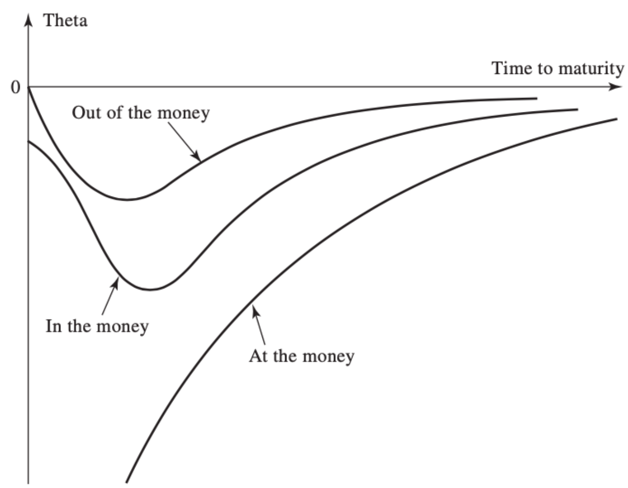
In these formulas theta is measured in years. Usually, when theta is quoted, time is measured in days, so that theta is the change in the portfolio value when 1 day passes.

What is the standard behaviour of theta?

Theta is usually negative for an option. As time passes, with all else remaining the same, the option tends to become less valuable.



(a) Variation of European call option with stock price



(b) Variation of European call option with time to maturity

Figure 2: Typical patterns of theta

From the images above it can be observed that:

1. When the stock price is very low, theta is close to zero.

2. The value of theta is at its highest when an option is at the money, or very near the money. As the underlying security moves further away from the strike price i.e. the option becomes deep in the money or out of the money, the theta value becomes lower.

   A deep-in-the-money or out-of-the-money option would have less extrinsic value to decay because the price would be made up mostly of the intrinsic value. Therefore, the rate of decay would be lower.

   (extrinsic value = option premium - intrinsic value)

3. The length of time until expiry also plays a role as the effect of time decay increases as we come closer to maturity. This means that the Theta value usually gets higher as the maturity comes closer except for deep out-of-the-money options.

What is theta used for?

Theta is not the same hedge parameter as delta. There is uncertainty about the future stock price, but there is no uncertainty about the passage of time. It makes sense to hedge against changes in price of the underlying asset, but it does not make sense to hedge against changes in the passage of time. However, many traders regard theta as a useful descriptive statistic. As we shall see later, in a delta-neutral portfolio theta is a proxy for gamma.

What is Gamma?

The gamma (Γ) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\delta^2 \Pi}{\delta S^2}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made relatively infrequently. If it is highly positive or negative, delta is very sensible to the price of the underlying asset, which makes it risky to leave a delta-neutral portfolio unchanged for a long period of time.

Long options, either calls or puts, have positive gamma [0,1] whereas short options have negative gamma [0,-1].

Why do you need to integrate gamma to delta for an optimal hedge?

Figure x illustrates the last sentence of the previous paragraph. When the stock moves from S to S', delta hedging assumes that the option price moves from C to C', when in fact it moves from C to C''. The difference between C' and C'' leads to a hedging error. The size of the error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature.

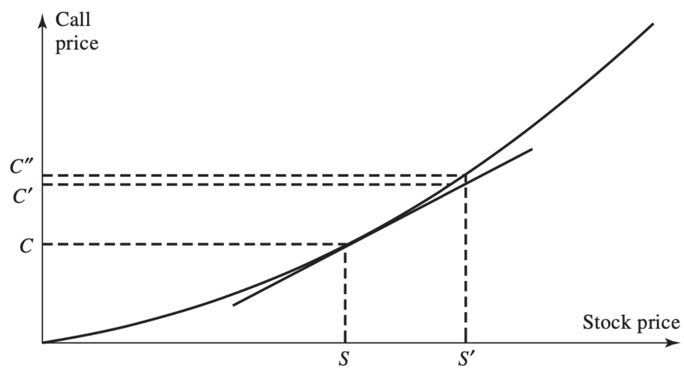


Figure 3: Hedging error introduced by nonlinearity

How can you make a portfolio gamma neutral?

A position in the underlying has a gamma of zero (underlying stock positions will not have gamma because their delta is always 1 or -1) and cannot be used to change the gamma of the portfolio. What is required is a position in an instrument that is not linearly dependent on the underlying asset.

Suppose that a a delta-neutral portfolio has a gamma equal to Γ, and a traded options has a gamma equal to Γ_T. If the number of traded options added to the portfolio is w_T, the gamma of the portfolio is

$$w_t \Gamma_T + \Gamma$$

Hence, the position in the traded option necessary to make the portfolio gamma neutral is $-\Gamma/\Gamma_T$. This move is likely to change the delta of the portfolio, which means that once you become gamma-neutral, you may need to change your underlying asset position to maintain delta-neutrality.

Making a portfolio gamma-neutral as well as delta-neutral can be regarded as a correction for the hedging error illustrated in Figure 3. Delta-neutrality provides protection against relatively small stock price between rebalancing. Gamma-neutrality provides protection against larger movements in this stock price between hedge rebalancing.

Example

Suppose that a portfolio is delta neutral and has a gamma of -3,000. The delta and gamma of a particular traded call option are 0.62 and 1.50 respectively. The portfolio can be made gamma-neutral by including in the portfolio a long position of

$$\frac{3,000}{1.5} = 2,000$$

in the call option. However, the delta of the portfolio will then change from zero to 2,000 x 0,62 = 1,240. Therefore, the 1,240 units of the underlying asset must be sold from the portfolio to keep it delta-neutral.

How is gamma calculated?

For a European call or put option on a non-dividend paying stock, the gamma given by BSM model is

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

As it was said above, the gamma of a long position is always position and varies with S_0. The variation of gamma with time to maturity for OTM, ATM and ITM options can be in the figure below. For a ATM option, gamma increases as time to maturity decreases. Short-life ATM options have very high gammas, which means that the value of the options holder's position is very sensitive to changes in the stock price.

What is the deal with gamma exposure?

Market makers ensure that there is enough liquidity in the market. The buy and sell options from and to traders not for speculative purposes. Hence, they need to hedge their market risk by buying or selling the underlying equities or futures, if they want to avoid going bust. Their delta-hedging process is a non-discretionary operation that occurs regardless of the available liquidity, and that becomes more important as the options market continues to grow. The amount by which market makers need to update their delta-hedging depends on gamma. The overall exposure of market makers to gamma is known as gamma exposure. It informs you how options market makers will likely need to hedge their trades to ensure they are delta-hedged.

What is the relationship between delta, theta and gamma?

The price of a single derivative dependent on a non-dividend paying stock that follows a geometric Brownian motion (GBM) process must satisfy the BSM differential equation. It follows that the value of Π of a portfolio of such derivatives also satisfies the differential equation

$$\frac{\delta \Pi}{\delta t} + rS\frac{\delta \Pi}{\delta S} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 \Pi}{\delta S^2} = r\Pi$$

Since

$$\Theta = \frac{\delta \Pi}{\delta t}, \Delta = \frac{\delta \Pi}{\delta S}, \Gamma = \frac{\delta^2 \Pi}{\delta S^2}$$

it follows that

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

For a delta-neutral portfolio, $\Delta = 0$ and

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

This shows that, when Θ is large and positive, gamma of a portfolio tends to be large and negative, and vice versa. (remember that gamma is always positive when you buy an option with theta acting negatively when buying options ; and gamma is always negative when selling an option as theta acts positively in case of sale)

What is Vega?

When Greek letters are calculated the volatility of the asset is in practice set equal to its implied volatility. The BSM model assumes that the volatility of an asset underlying an option is constant. This means that the implied volatility of all options on the asset are constant and equal to this assumed volatility.

But in practice the volatility of an asset changes over time. As a result, the value of an option is liable to changes because of movements in volatility as well as because of changes in the asset price and the passage of time. The vega, ν, of an option is the rate of change in its value with respect to the volatility of the underlying asset:

$$\nu = \frac{\delta f}{\delta \sigma}$$

where f is the option price and σ is usually the option's implied volatility. When vega is highly positive or highly negative, there is a high sensitivity to changes in volatility. If the vega of an option position is close to zero, volatility changes have very little effect on the value of the position.

A position in the underlying asset has zero vega. Vega therefore cannot be changed by taking a position in the underlying asset. In this respect vega is like gamma. A complication is that different options in a portfolio have different implied volatilities. If all implied volatilities are assumed to change by the same amount during any short period of time, vega can be trated like gamma and the vega risk can be hedged by taking a position in a single option. If ν is the vega of a portfolio and ν_T is the vega of a traded option, a position of $-\nu/\nu_T$ in the traded options makes the portfolio instantaneously vega-neutral. Unfortunately, a portfolio that is gamma neutral will not in general be vega neutral, and vice versa. If a hedger requires a portfolio to be both gamma and vega neutral, at least two traded options dependent on the underlying asset must be used

What is Rho?

The rho (ρ) of a portfolio of options is the rate of change of the portfolio wrt the interest rate:

$$\frac{\delta \Pi}{\delta r}$$

It measures the sensitivity of the value of a portfolio to a change in the interest rate when all else remains the same. For a European call option on a non-dividend-paying stock,

$$rho(call) = KTe^{-rT}N(d_2)$$

For a European put,

$$rho(put) = -KTe^{-rT}N(-d_2)$$

Exercises

1. A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?

   (a) A virtually constant spot rate.

   (b) Wild movements in the spot rate.

   Explain your answer.

   [**Ans:**] A long position in either a put or a call option has a positive gamma. When gamma is positive the hedger gains from a large change in the stock price and loses from a small change in the stock price. Hence the hedger will fare better in case (b).

2. A bank's position in options on the dollar–euro exchange rate has a delta of 30,000 and a gamma of -80,000. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

   [**Ans:**] The delta indicates that when the value of the euro exchange rate increases by \$0.01, the value of the bank's position increases by $0.01 \times 30,000 = \$300$. The gamma indicates that when the euro exchange rate increases by \$0.01 the delta of the portfolio decreases by $0.01 \times 80{,}000 = \$800$. For delta neutrality 30,000 euros should be shorted. When the exchange rate moves up to 0.93, we expect the delta of the portfolio to decrease by $0.03 \times 80{,}000 = 2{,}400$ so that it becomes 27,600. To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position 2,400 euros so that a net 27,600 have been shorted. As shown in the figure below (y-axis is the change in portfolio value Π and the x-axis is the change in S), when a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset price. We can conclude that the bank is likely to have lost money.