Hedging & Portfolio Insurance – Notes

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Cross Hedging and Optimal Hedge Ratio

What is meant by cross hedging?

In the examples of hedging before this topic, the asset underlying the futures contracts were the same as the assets whose prices were being hedged. Cross hedging occurs when the two assets are different. Consider, for example, an airline that is concerned about the future price of jet fuel. Because jet fuel futures are not actively traded, it might choose to use heating oil futures contracts to hedge its exposure.

What is meant by hedge ratio?

The hedge ratio is the ratio of the size of the position taken in futures contracts to the size of the exposure.

When the asset underlying the contract is the same as the asset being hedged, it is natural to use a hedge ratio of 1.0. This is the hedge ratio we have used in the examples considered so far. For instance, in Example 3.2, the hedger's exposure was on 20,000 barrels of oil, and futures contracts were entered into for the delivery of exactly this amount of oil. When cross hedging is used, setting the hedge ratio to 1.0 is not always optimal. The hedger should choose a value for the hedge ratio that minimises the variance of the value of the hedged position.

How do you calculate the minimum variance hedge ratio?

The minimum variance hedge ratio depends on the relationship between changes in the spot price and changes in the futures price. Define:

ΔS: change in spot price, during a period of time equal to the life of the hedge

ΔF: change in futures price, during a period of time equal to then life of the hedge.

If we assume that the relationship between ΔS and ΔF is approximately linear, we can write:

$$\Delta S = a + b\Delta F + \epsilon$$

where a and b are constant and ϵ in an error term. Suppose that the hedge ratio is h, (i.e., a percentage h of the exposure to S is hedged with futures). Then the changes in the value of the position per unit of exposure S is

$$\Delta S - h\Delta F = a + (b - h)\Delta F + \epsilon$$

The standard deviation of this is minimised by setting $h = b$ (so that the second term on the right-hand side disappears).

Denote the minimum variance hedge ratio by h^*. We have shown that $h^* = b$. It follows from the formula for the slope in linear regression that

$$h^* = \rho\frac{\sigma_S}{\sigma_F}$$

where

- ρ is the correlation coefficient between σ_S and σ_F
- σ_S is the standard deviation of ΔS
- σ_F is the standard deviation of ΔF

How do you calculate the optimal number of contracts?

To calculate the number of contracts that should be used in hedging, define

- Q_A: size of position being hedged (units)
- Q_F: size of one futures contract (units)
- N^*: optimal number of futures contracts for hedging.

The futures contracts should be on h^*Q_A units of the asset. The number of futures contracts required is therefore given by
$N^* = \frac{h^*Q_A}{Q_F}$

Portfolio Insurance

What is portfolio insurance and how does it work?

Portfolio managers can use options on a well-diversified index to limit their downside risk.

Suppose that the value of the index today is S_0. Consider a manager in charge of a portfolio whose beta is 1.0. This implies that the returns of the portfolio mirror those of the index. Assuming the dividend yield from the portfolio is the same as the dividend from the index, the percentage changes in the value of the portfolio can be expected to be the same as the percentage changes in the index. Since each contract is 100 times the index, it follows that value of the portfolio is protected against the possibility of the index falling below

K if, for each $100S_0$ dollars in the portfolio, the manager buys one put option contract with strike price K.

Example

Suppose the manager's portfolio is worth $500,00 and the value of the index is currently 1,000. The portfolio is worth 500 times the index. The manager can obtain insurance against the value of the portfolio dropping below 450,000 in the next three months by buying five three-month put options on the index with a strike price of 900.

When the portfolio's beta is not 1.0?

If the portfolio's beta (β) is not 1.0, β put options must be purchased for each $100S_0$ dollars in the portfolio.

Suppose that the $500,000 portfolio considered above has a beta of 2.0 instead of 1.0. We continue to assume the index is 1,000. The number of put options required is

$$2.0 * \frac{500,000}{1,000 * 100} = 10$$

rather than 5 as before.

How do you calculate the strike price of the option that provides you insurance?

To calculate the appropriate strike price, the CAPM can be used.

Suppose that the risk-free rate is 12%, the dividend yield on both the index and the portfolio is 4%, and protection is required against the value of the portfolio dropping below $450,000 in the next three months. Under the CAPM, the expected excess return of a portfolio over the risk-free rate is assumed to equal beta times the excess return of the index portfolio over the risk-free rate.

The calculations are the following

Value of the index in 3 months	1,040	960
return from change in index	4%	-4%
dividends from index	1% per 3mo	1% per 3mo
total return from index	5% per 3mo	-3% per 3mo
risk-free rate	3% per 3mo	3% per 3mo
excess return over index	2%	-6%
expected excess return over risk-free	2 (beta) x 2% = 4%	-12%
expected return from portfolio	4% + 3% = 7%	-9%
dividends from portfolio	1%	1%
expected increase in portfolio	7% - 1% = 6%	-10%
expected value of the portfolio	$500,000 x 1.06 - $530,000	$500,000 x 0.9 = $450,000

To illustrate how the insurance works, consider what happens if the value of the index falls to $880. This makes the value of the portfolio equal to $370,000. The put options pay off (960 - 880) x 10 x 100 = $80,000, and this is exactly what is necessary to move the total value of the portfolio manager's position up from $370,000 to $450,000.

What is an alternative to buy put options on a market index?

A portfolio manager is often interested in acquiring a put option on his or her portfolio. This provides protection against market declines while preserving the potential for a gain if the market does well. An alternative to the method used above consists of creating the options synthetically rather than buy them. Creating an option synthetically involves maintaining a position in the underlying asset (or futures on the underlying asset) so that the delta of the position is equal to the delta of the required option. The position necessary to create an option synthetically is the reverse of that necessary to hedge it. This is because the procedure for hedging an option involves the creation of an equal and opposite option synthetically.

If we write down the BSM formula for a put

$$p = Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$$

we can see that Ke^{-rT} is, in the end, a discounted K, which looks like the price of a bond that would pay K at time T. So you can replicate (synthesise) a put by buying a bond that pays out K at T in the quantity $N(-d_2)$ units and selling (there is a minus sign) a stock in the quantity $e^{-qT}N(-d_1)$ (i.e., the delta). Hence, you can you replicate the put by selling delta units of the stock and depositing the money you receive.

There are two reasons why it may be more attractive for the portfolio manager to create the required put option synthetically than to buy it in the market. First, option markets do not always have the liquidity to absorb the trades required by managers of large funds. Secondly, fund managers often require strike prices and exercise dates that are different from those available in exchange-traded options markets.

The synthetic option can be created from trading the portfolio of from trading in index futures contracts. We first examine the creation of a put option by trading the portfolio.

The delta of a European put on the portfolio is

$$\Delta = e^{-qT}[N(d_1) - 1]$$

where, with the usual notation,

$$d_1 = \frac{ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{(T)}}$$

The volatility of the portfolio can be assumed to be its beta times the volatility of the market index. To create a put option synthetically, the fund manager should ensure that at any given time a proportion

$$e^{-qT}[1 - N(d_1)]$$

of the stocks in the original portfolio has been sold and the proceeds invested in risk-less assets. As the value of the original portfolio declines, the delta of the put given by the first equation above becomes more negative and the proportion of the original portfolio sold must be increased.

Using this strategy to create portfolio insurance means that at any given time funds are divided between the stock portfolio on which insurance is required and risk-less assets. As the value of the stock portfolio increases, risk-less assets are sold and the position in the stock portfolio is increased.

The cost of the insurance arises from the fact that the portfolio manager is always selling after a decline in the market and buying after a rise in the market.

Example

A portfolio is worth $90 million. To protect against market downturns the manager of the portfolio requires a 6-month European put option on the portfolio with a strike price of $87 million. The risk-free rate is 9%, the dividend yield is 3% and the volatility of the portfolio is estimated as 25% per annum. A stock index stands at 900. The portfolio is considered to mimic the index fairly closely. One alternative discussed in Section 17.1 is to buy 1,000 put option contracts ($\frac{90m}{900*100}$) on the index with a strike price of 870.

Another alternative is to create the required option synthetically. In this case, $S_0 = 87$ million, r=0,09, q=0.03, $\sigma = 0.25$ and T=0.5, so that

$$d_1 = \frac{ln(90/87) + (0.09 - 0.03 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = -.4499$$

and the delta of the required option is

$$e^{-qT}[N(d_1) - 1] = -0.3215$$

This shows that -32.15% of the portfolio should be sold initially and invested risk-free assets to match the delta of the required option. The amount of the portfolio sold must be monitored frequently. For example, if the value of the original portfolio reduced to $88 million after 1 day, the delta of the required option changes to 0.3679 and a further 4.64% of the original portfolio should be sold and invested in risk-free assets.

Exercises

1. A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth $360 million. The value of the S&P 500 is 1200 and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next six months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum. One index option contract is on 100 times the index (i.e. contract size = 100).

   (a) If the fund manager buys traded European put options, how much would the insurance cost?

Determine the number of contracts needed

$$\frac{360m}{1200 * 100} = 3,000$$

How do we find the price of the insurance? We determine how much it costs us to buy 3,000 options. We do that using BSM.

- $S_0 = 1,200$
- $K = 1,140$
- $r = 0.06$
- $q = 0.03$
- $\sigma = 0.3$
- $T = 0.5$

Hence,

$$\begin{aligned} d_1 &= \frac{ln(1,200/1,140) + (0.06 - 0.03 + \frac{1}{2}0.3^2)0.5}{0.3\sqrt{0.5}} \\ &= 0.4186 \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.2064 \\ N(d_1) &= 0.6622; N(d2) = 0.5818; \\ N(-d_1) &= 0.3378; N(-d_2) = 0.4182 \end{aligned}$$

The value of one put option is

$$\begin{aligned} p &= Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1) \\ &= 1140e^{-0.06*0.5} * 0.4182 - 1200e^{-0.03*0.5} * 0.3378 \\ &= 63.40 \end{aligned}$$

The cost of the insurance results in

$$63.40\$ * 3,000 * 100 = \$19,020,000$$

(b) Explain carefully alternative strategies open to the fund manager involving traded European call options, and show that they lead to the same result.

From put-call parity

$$\begin{aligned} S_0e^{-qT} + p &= c + Xe^{-rT} \\ p &= c + Xe^{-rT} - S_0e^{-qT} \end{aligned}$$

This shows that a put option can be created bu shorting quantity e^{-qt} of the index, buying a call option a investing the remainder at the risk-free rate. Applying this to the situation under consideration, the fund manager should:

- Sell $360e^{-0.03*0.5} = \$354.64$ million of stock.
- Buy 300,000 call options on the S&P 500 with exercise price 1140 and maturity at 6 months.
- Invest the remaining cash at the risk free interest rate of 6% p.a. for six months.

This strategy gives the same results as buying put options directly.

(c) If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?

Here, instead of holding put option on index, we sell part of the equity portfolio to create the same amount of delta produced by the put options. In other words, we create a synthetic put option written on the portfolio by selling part of the equity portfolio and investing the proceeds in the risk free rate.

The delta of a put written on the equity portfolios:$\Delta^{Portfolio} = e^{-qT}[N(d_1) - 1]$. The following information can be inferred from the question; $\sigma p = \sigma m = 0.3, q_p = q_m = 0.03$. The delta of one put option is

$$\begin{aligned} d_1 &= \frac{ln(360/342) + (0.06 - 0.03 + \frac{1}{2}0.3^2)0.5}{0.3\sqrt{0.5}} \\ &= 0.4186 \\ N(d_1) &= 0.6622 \end{aligned}$$

$$\Delta^{Portfolio} = e^{-qT}[N(d_1) - 1] = e^{-0.03*0.5}[0.6622 - 1] = -0.3327$$

This indicates that 33.27% of the portfolio should be sold initially (i.e., \$119.77 million) and invested in risk-free securities.

Note: The portfolio beta equals one, i.e. the portfolio tracks the index. Hence, the synthetic portfolio put written on the portfolio is "equivalent" to the synthetic put option written on the index. This also means that the delta of a put written on the index is the same to the delta written on the portfolio. This is why d_1 and $N(d_1)$ calculated above are equal to those calculated in part (a).

(d) If the fund manager decides to provides insurance by using nine-month index futures, what should the initial position be? Each index futures is on \$250 times the index.

Hence, the position in futures is

$$\frac{-0.3253 * 360m}{1200 * 250} = 390$$

2. A fund manager has a well-diversified portfolio that is worth \$360 million. The value of the S&P 500 is 1200 and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next six months. The risk-free interest rate is 6% per annum. The dividend yield on the S&P 500 is 3%, and the volatility of the index is 30% per annum. The portfolio has a beta of 1.5. Assume that the dividend yield on the portfolio is 4% per annum. One index option contract is on 100 times the index (i.e. contract size = 100)

   Here, we need to work out the corresponding change in S&P 500 that coincides with a 5% drop in portfolio value.

   - $T = 0.5$
   - $r_P = -5\%$
   - $q_P = 4\% * \frac{1}{2} = 2\%$
   - $R_P = r_p + q_p = -3\%$
   - $R_m = \frac{R_P - R_f}{\beta} + R_F = \frac{-3\%P - 3\%}{1.5} + 3\% = -1\%$
   - $q_m = 3\% * \frac{1}{2} = 1.5\%$
   - $r_m = R_m - q_m = -1\% - 1.5\% = -2.5\%$

   The strike price of the option to give a 2.5% protection on S&P 500 is

   $$1200 * (1 - 0.025) = 1170$$

   and the number of put contracts is

   $$\beta\frac{P}{A} = 1.5 * \frac{360m}{1200 * 100} = 4,500$$

   which equals 4,500 contracts or 450,000 options.

   (a) If the fund manager buys traded European put options, how much would the insurance cost?

   $S_0 = 1200, X = 1170, r = 0.06, \sigma = 0.30, T = 0.50$ and $q = 0.03$. Hence

   $$\begin{aligned} d_1 &= \frac{ln(1,200/1,170) + (0.06 - 0.03 + \frac{1}{2}0.3^2)0.5}{0.3\sqrt{0.5}} = 0.2961 \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.0840 \\ N(d_1) &= 0.6164; N(d2) = 0.5335; \\ N(-d_1) &= 0.3836; N(-d_2) = 0.4665; \end{aligned}$$

The value of the option is

$$p = Xe^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$$

$$= 1170e^{-0.06*0.5} * 0.4665 - 1200e^{-0.03*0.5} * 0.3836 = 76.28$$

The total cost of the insurance is therefore

$$4500 * 100 * \$76.28 = \$34,326,000.$$

Note that this is significantly greater than the cost of the insurance in problem 13.16 previously.

(b) Can the fund manager obtain the insurance by buying traded call options?

The fund manager can

i. sell $354.64 million of stock.
ii. buy 450,000 call options on the S&P 500 with exercise price 1170 and exercise date in six months and
iii. invest the remaining cash at the risk-free interest rate.

(c) If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?

The portfolio is 50% more volatile than the S&P 500. When the insurance is considered as an option on the portfolio, the parameters are as follows: $S0 = 360, r = 0.06, T = 0.50, q = 0.04$ and

$$X = 0.95 * 360 = 342,$$

$$\sigma = 1.5 * 0.30 = 0.45.$$

Hence

$$\begin{aligned} d_1 &= \frac{ln(360/342) + (0.06 - 0.04 + \frac{1}{2}0.45^2)0.5}{0.45\sqrt{0.5}} \\ &= 0.3517 \\ N(d_1) &= 0.6375 \\ \Delta^{Portfolio} = e^{-qT}[N(d_1) - 1] &= e^{-0.04*0.5}[0.6375 - 1] \\ &= -0.3553 \end{aligned}$$

This indicates that in order to replicate the put option, 35.5% of the portfolio (i.e. $127.9 million) should be sold and invested in riskless securities.

Note: The portfolio beta does not equals one, i.e. the portfolio does not tracks the index. Hence, the synthetic portfolio put written on the portfolio is not "equivalent" to the synthetic put option written on the index. This also means that the delta of a put written on the index is not the same to the delta written on the portfolio. This is why d_1 and $N(d_1)$ calculated above are not equal to those calculated in part (a).

(d) If the fund manager decides to provides insurance by using nine-month index futures, what should the initial position be? Each index futures is on $250 times the index.

Here, we return to the situation considered in (a) where put options in the index are used to hedge the equity portfolio. Now, instead of selling the stock portfolio, we will sell index futures. The delta of a put option written on index futures is: $\Delta^{Futures} = e^{-(r-q)^{T^*}} * e^{-qT}[N(d_1)-1]$ where T^* is the time to maturity of the index futures and T is the maturity of the option.

When the strike price of the put option is $X = 1170$ we have:

$$\begin{aligned} d_1 &= \frac{ln(1200/1170) + (0.06 - 0.03 + \frac{1}{2}0.3^2)0.5}{0.3\sqrt{0.5}} \\ &= 0.2961 \\ N(d_1) &= 0.6164 \end{aligned}$$

When the strike price of the put option is $X = 1170$ we have:

$$\Delta^{Futures} = e^{-(0.06-0.03)\frac{9}{12}} * e^{-0.03\frac{6}{12}} * [0.6164-1] = \frac{-0.3379}{1.023} = -0.3695$$

Hence, the position in futures is

$$1.5 * \frac{-0.3695 * 360m}{1200 * 250} = 665$$