Financial Engineering - Interest Rates

RA

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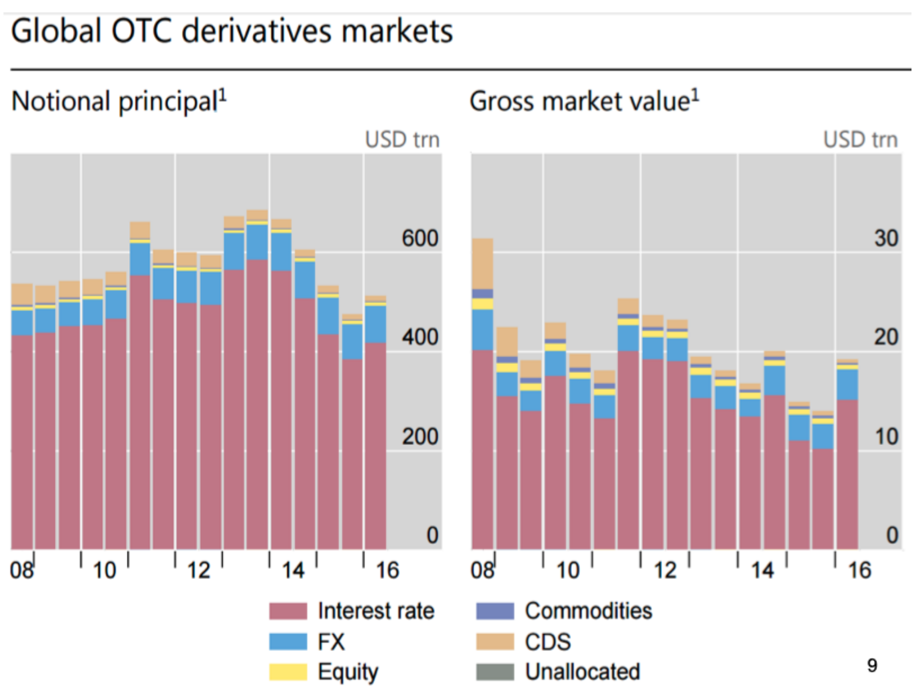
1 Why do derivatives matter?

Whenever there is a tradable asset. there is a derivatives market, which is sometimes bigger and more liquid than the underlying market. The derivative market allows (a) investors to get exposure to the underlying risk without having to buy the underlying asset, and (b) hedgers to eliminate an existing exposure to the underlying asset.

2 Why do interest rates matter?



3 What is the origin of fixed income derivatives?

The creation of fixed income derivatives can probably be traced back to the large interest rates fluctuations of the early seventies. During this period, the market witnessed two oil crises, which caused instability in prices and expectations. Initially, fixed income derivatives were OTC only (today most is still OTC), i.e., bilateral agreements that allow for for customisation (e.g., date, underlying asset). Later, they became standardised and traded in exchanges. To be able to understand interest rates derivatives, it is necessary to understand interest rates and bonds, because it is important to understand the factors that cause them to fluctuate.

4 What is the deal with the Treasury market?

Government are long-term entities that need to invest (e.g., infrastructure) for citizens. They borrow money using the Treasury bond market. They can be thought as investors whose payoff consists of the collection of taxes.

What rate can the treasury borrow at? The decision depends on two dimensions:

1. likelihood that the state will default. The rate is low because the lender is sure to get the loan back.

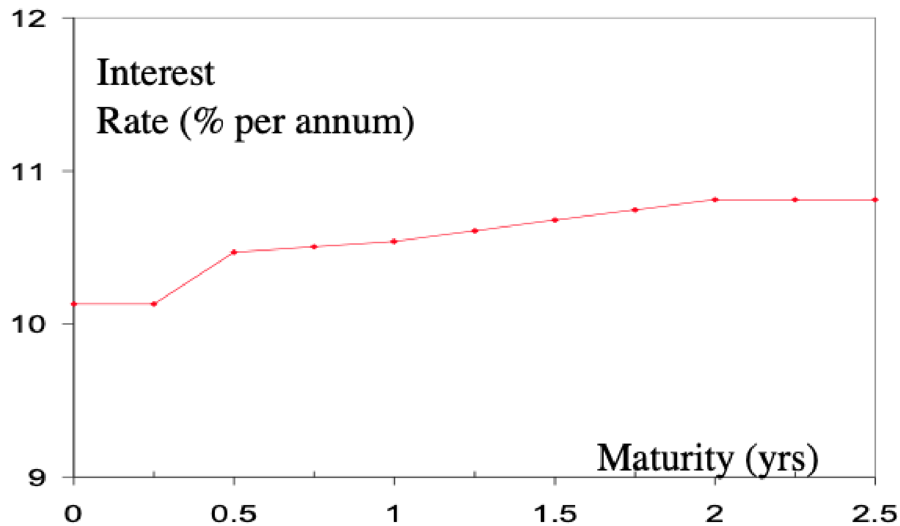
2. how long does the state borrow for and what might happen during this period.

5 What are zero rates?

The n-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for n years. All the interest and principal is realised at the end of n years. Hence, there are no intermediate payments. It is also referred to as the n-year spot rate, n-year zero-rate, or simply n-year zero.

6 Term structure

By plotting the interest rates for different maturities, we can build up the term structure.



7 What is the deal with interest rate risk?

Interest rates are volatile, they are subject to shocks. If interest rates rise the government will have to pay more interest on its debt. Hence the need to manage the possibility of interest rate change.

8 What is the LIBOR?

London Interbank Offered Rate. It is/was the interbank market reference interest rate reflecting the average cost for a bank to borrow. More precisely, LIBOR is the interest rate that banks in London are willing to lend Eurodollars to another London bank. Eurodollars are dollar deposits outside the U.S. LIBOR is the average rate quoted from the two middle quartiles of quoting banks. Rates for various maturities are quoted noon each day. From 01/01/2022 24 of the 35 LIBOR settings, which relate to specific currencies and time periods, are no longer available. This is because of the misreporting of rates. Some banks under-reported their borrowing costs during the financial crisis.

8.1 Will this rate be above or below the treasury rate?

It will be above because the lender is taking a (small) risk that the bank will go bust and will be unable to pay back the loan. The bank therefore has to pay a higher rate of interest than the government to attract lenders. Consequently, the term-structure for banking institutions will be similar to that of the government, but with higher rates.

9 How do you price a bond?

The theoretical value of a bond can be calculated as the present value of all the cash flows that will be received by the owner of the bond. Sometimes bond traders use the same discount rate for all the cash flows underlying a bond, but a more accurate approach is to use a different zero for each cash flow.

9.1 Bond Yield

A bond's yield is the single discount rate that, when applied to all cash flows, gives a bond price equal to its market price. Suppose that the theoretical price of a $98.39 bond is also its market value. If y is the yield on the bond, expressed with continuous compounding, it must be true that:

$$3e^{-y*0.5} + 3e^{-y*1} + 3e^{-y*1.5} + 103e^{-y*2.0} = 98.39$$

This equation can be solved iteratively and gives y = 6.76% (the zero rates used to)

9.2 Par Yield

The par yield for a certain bond maturity is the coupon rate that causes the bond price to equal its par value (i.e., its principal value). Usually the bond is assumed to pay semiannual coupons. Suppose that the coupon on. a 2-year bond is c per annum (or $\frac{1}{2}c$ per 6 months). Using the zero rates available, the value of the bond is equal to its par value of 100 when:

$$\frac{c}{2}e^{-0.05*0.5} + \frac{c}{2}e^{-0.058*1} + \frac{c}{2}e^{-0.064*1.5} + (100 + \frac{c}{2})e^{-0.068*2.0} = 100$$

This equation can be solved in a straightforward way to give c = 6.87. The 2-year par yield is therefore 6.87% per annum.

How do you determine the zero rates?

A method that can be used to determine zero rates is the bootstrap method. It can also be used to for coupon-bearing bonds.

Bond principal	Time to mat. (years)	Annual coupon ($)	Bond price ($)	Bond yield (%)
100	0.25	0	99.6	1.6064
100	0.5	0	99.0	2.0202
100	1.00	0	97.8	2.2495
100	1.5	4	102.5	2.2949
100	2.00	5	105.0	2.4238

Because the first three bonds pay no coupons, the zero rates corresponding to the maturities of these bonds can easily be calculated. The 3-month bond has the effect of turning an investment of 99.6 into 100 in 3 months. Hence the continuously compounded rate must satisfy

$$100 = 99.6e^{R*0.25}$$

That results in 1.603% per annum ($\frac{\ln(\frac{100}{99.6})}{0.25}$). The same procedure can be applied to the 6-month and 1-year bonds, which result in 2.010% and 2.225% rates per annum. The fourth bond lasts 1.5 years and pays coupons semiannually. From our earlier calculations, we know that the discount rate for the payments at the end of 6 months and 1 year. Hence, we can solve for the remaining unknown rate at 1.5 years.

$$2e^{-0.02010*0.5} + 2e^{-0.02225*1.0} + 102e^{-R*1.5} = 102.5$$

This reduces to

$$e^{-1.5R} = 0.96631$$

or

$$R = -\frac{ln(0.96631)}{1.5} = 0.02284$$

The 1.5-year zero is therefore 2.284%. A chart showing the zero rate as a function of maturity is known as zero curve. A common assumption is that the zero curve is linear between the points determined using the bootstrap method (i.e., the 1.25-year zero rate is 0.5x2.225 + 0.5x2.284)

What are Forward rates?

Forward rates are the rates of interest implied by current zero rates for periods of time in the future. Suppose a zero rate for a 1-year investment in 3%, and a zero rate for a 2-year investment is 4%. The forward interest rate in this example is the rate of interest the is implied by the zero rates for the period between the end of the first year and the end of the second year. It is the rate of interest for year 2 that, when combined with 3% per annum for year 1, gives 4% overall for year 2. In this case it is 5% because

$$100e^{0.03*1}e^{0.05*1} = \$108.33 = 100e^{0.04*2}$$

What are Forward rate agreements?

A forward rate agreement (FRA) is an agreement to exchange a predetermined fixed rate for a reference rate that will be observed in the market at a future time. Both rates are applied to a specified principal, but the principal itself is not exchanged. Historically, the reference rate has usually been LIBOR. Consider an agreement to exchange 3% for three-month LIBOR in two years with both rates being applied to a principal of \$100 million. One side (Party A) would agree to pay LIBOR and receive the fixed rate of 3%. The other side (Party B) would agree to receive LIBOR and pay the fixed rate of 3%. If three-month LIBOR proved to be 3.5% in two years, Party A would receive (assume all rates are compounded quarterly)

$$\$100m * (0.035 - 0.03) * 0.25 = \$125.000$$

What is the deal with Duration and Modified duration?

Duration is defined as the average time it takes to receive all the cash flows of a bond, weighted by the present value of each of the cash flows. Essentially, it is the payment-weighted point in time at which an investor can expect to recoup his or her original investment. In 1938, economist Frederick Macaulay suggested duration as a way of determining the price volatility of bonds. 'Macaulay duration' is now the most common duration measure. Until the 1970s, few people paid attention to duration due to the relative stability of interest rates. When interest rates began to rise dramatically, investors became very interested in a tool that would help them assess the price volatility of their fixed income investments. During this period, the concept of 'modified duration' was developed, which offered a more precise calculation of the change in bond prices given varying coupon payment schedules. Duration can help predict the likely change in the price of a bond given a change in interest rates. As a general rule, for every 1% increase or decrease in interest rates, a bond's price will change approximately 1% in the opposite direction for every year of duration. For example, if a bond has a duration of 5 years, and interest rates increase by 1%, the bond's price will decline by approximately 5%. Conversely, if a bond has a duration of 5 years and interest rates fall by 1%, the bond's price will increase by approximately 5%.

9.3 Calculate duration

Suppose that a bond provides the holder with cash flows c_i at time t_i ($1 \leq$ i $\leq$ n). The bond price B and bond yield y (continuously compounded) are related by

$$B = \sum_{i=1}^{n} c_i e^{-yt_i}$$

The duration of the bond, D, is defined as

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i}}{B}$$

This can be written

$$D = \sum_{i=1}^{n} t_i [\frac{c_i e^{-yt_i}}{B}]$$

The term in square brackets is the ratio of the present value of the cash flow at time t_i to the bond price. The bond price is the present value of all payments. The duration is therefore a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i.

9.4 Example

A two-year bond with a \$1000 face value and one coupon payment every six months of \$50, the duration (calculated in years) is:

$$0.5y(\frac{50}{1200}) + 1y(\frac{50}{1200}) + 1.5y(\frac{50}{1200}) + 2y(\frac{1000}{1200}) = 1.87$$

years The negative relationship between B and y causes bond prices to decrease as yields increase. The key duration relationship results in:

$$\Delta B = -BD\Delta y$$

This can be written as

$$\frac{\Delta B}{B} = -D\Delta y$$

This is an approximate relationship between percentage changes in a bond price and changes in its yield.

9.5 Modified duration

If y is expressed with annual compounding instead of continuous, the approximate relationship changes to

$$\Delta B = \frac{-BD\Delta y}{1 + y/m}$$

where m is the frequency of compounding. A variable D*, defined by

$$D^* = \frac{D}{1 + y/m}$$

is sometimes referred to as modified duration, because it allows the duration relationship to be simplified to

$$\Delta B = -BD^*\Delta y$$

9.6 Rules of thumb for duration

- The duration of any bond that pays a coupon will be less than its maturity, because some amount of coupon payments will be received before the maturity date.

- The lower a bond's coupon, the longer its duration, because proportionately less payment is received before final maturity. The higher a bond's coupon, the shorter its duration, because proportionately more payment is received before final maturity.

- Because zero coupon bonds make no coupon payments, a zero coupon bond's duration will be equal to its maturity.

- The longer a bond's maturity, the longer its duration, because it takes more time to receive full payment. The shorter a bond's maturity, the shorter its duration, because it takes less time to receive full payment when y is expressed with a compounding frequency of m times per year.

What is the deal with Convexity?

The duration relationship above applies only to small changes in yields. This is shown in the figure below, which shows the relationship between the percentage change in value and change in yield for two bond portfolios having the same duration. The gradients of the two curves are the same at the origin. This means that both portfolios change in value by the same percentage for a small change in yield. The same cannot be said about large yield changes. Portfolio X has more curvature than Y. Convexity measures this curvature and can improve the duration relationship. A measure of convexity is

$$C = \frac{1}{B}\frac{d^2B}{dy^2} = \frac{\Sigma_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B}$$

From Taylor series expansion, it is possible to obtain a more accurate expression for the ΔB relationship than the one presented above through

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}\Delta y^2$$

This leads to

$$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

For a portfolio with a particular duration, the convexity of a bond portfolio tends to be greatest when the portfolio provides payments evenly over a long period of time. It is least when the payments are concentrated around one particular point in time. By choosing a portfolio with a net duration of zero and a net convexity of zero, a financial institution can make itself immune to relatively large parallel shifts in the zero curve. However, it is still exposed to non-parallel risks.