Self consistent models of deformed neutron stars in the framework of general relativity

A numerical project

Riccardo Marchese Konstantinos Topaloglou

14 July, 2025

Table of Contents

Table of Contents

Motivation

Examples

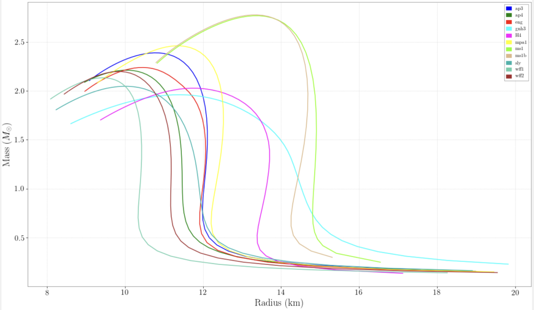


Figure: Examples of solutions of the TOV equation with different equations of state.

- Obtain profile of NS interior
- Extrapolate macroscopical properties (mass-radius relation)

If equipped with an Equation of State (EoS), one can solve the Einstein equations for the NS matter distribution.
Symmetry arguments simplify the problem: staticity and spherical symmetry yield the TOV equations.

The Tolman-Oppenheimer-Volkov equation in GR

The condition of equilibrium on a self-gravitating spherically symmetric and isotropic compact object sourcing a metric

$$ds^2 = e^{\nu} c^2\, dt^2 - \left(1 - \frac{2Gm}{rc^2}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{1}$$

furnishes the classical Tolman-Oppenheimer-Volkov (TOV) equation from the solution of the Einstein equations & energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$.

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\epsilon \left(1 + \frac{P}{\epsilon c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \tag{2}$$

where $\epsilon = \epsilon(P)$ is the energy density and $m = m(r)$ the enclosed mass

$$\frac{dm}{dr} = 4\pi r^2 \epsilon. \tag{3}$$

Table of Contents

Code modelization

The numerical modelization of the classical TOV equation consists of a coupled system of two differential equations:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\epsilon\left(1 + \frac{P}{\epsilon c^2}\right)\left(1 + \frac{4\pi r^3 P}{mc^2}\right)\left(1 - \frac{2Gm}{rc^2}\right)^{-1} \tag{4}$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon \tag{5}$$

and choosing a precise equation of state, the system will be solved with the following initial conditions

$$\begin{aligned} P(r=0) &= P_0\,, \\ m(r=0) &= 0 \end{aligned} \tag{6}$$

until the NS surface is reached, i.e.

$$P(r=R) = 0. \tag{7}$$

Code modelization

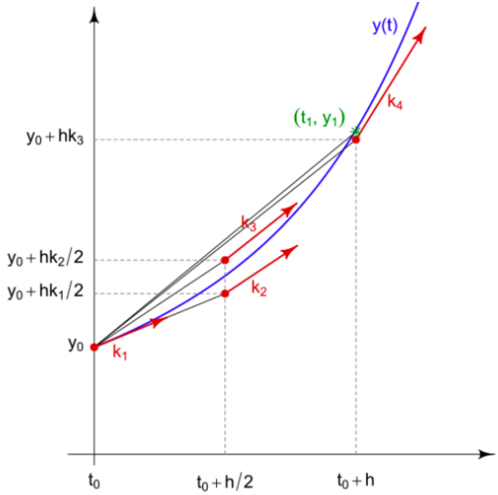
The solution of this given set of ordinary differential equations is direct. In this work, we employ a 4th order Runge Kutta method to solve Eqs. (4), (5).
The algorithm below outlines for obtaining a sequence of compact stars for a given equation of state:

- First, we define a range of central pressures for which the stellar sequence will be calculated for.
- We then set the boundary conditions stated above for $r = 0$.
- Employ 4th order Runge-Kutta methods and solve our system of differential equations. The system is integrated until $P(R) \to 0$, which will define the radius at the surface of the star.
- Repeat steps 2 and 3 until the desired range of central pressures are accounted for. From integrating our system, we obtain the following stellar properties: Mass and Radius.

Numerical methods: 4th order Runge-Kutta

Definition

In numerical analysis, the Runge–Kutta methods are a family of implicit and explicit iterative methods, which include the Euler method, used in temporal discretization for the approximate solutions of simultaneous non-linear equations.



© Wikipedia

Numerical methods: 4th order Runge-Kutta

For a system of first order ordinary differential equations

$$\dot{x}(t) = f(x, y, t), \quad \dot{y}(t) = g(x, y, t) \tag{8}$$

we can integrate from an input (x_0, y_0, t_0) to a subsequent step $(x, y, t_0 + dt)$ as

$$\begin{aligned} x &= x_0 + \frac{dt}{6}(f_1 + 2f_2 + 2f_3 + f_4) \\ y &= y_0 + \frac{dt}{6}(g_1 + 2g_2 + 2g_3 + g_4) \end{aligned} \tag{9}$$

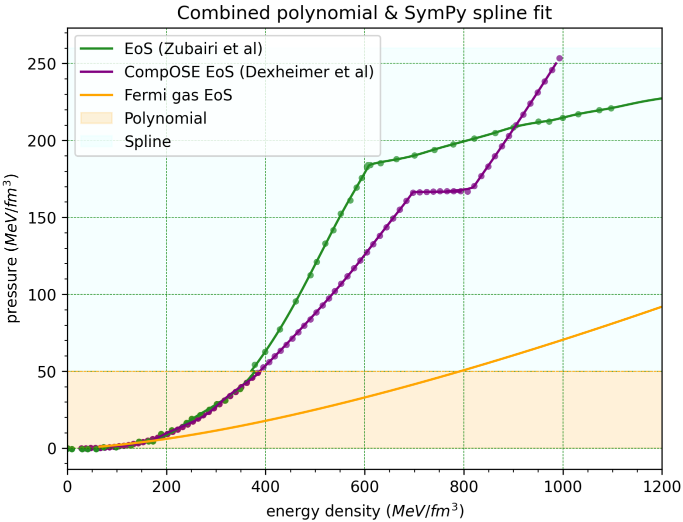
$$f_1 = f(x_0, y_0, t_0) \qquad g_1 = g(x_0, y_0, t_0)$$

$$f_2 = f(x_0 + \frac{dt}{2}f_1, y_0 + \frac{dt}{2}g_1, t_0 + \frac{dt}{2}) \qquad g_2 = g(x_0 + \frac{dt}{2}f_1, y_0 + \frac{dt}{2}g_1, t_0 + \frac{dt}{2})$$

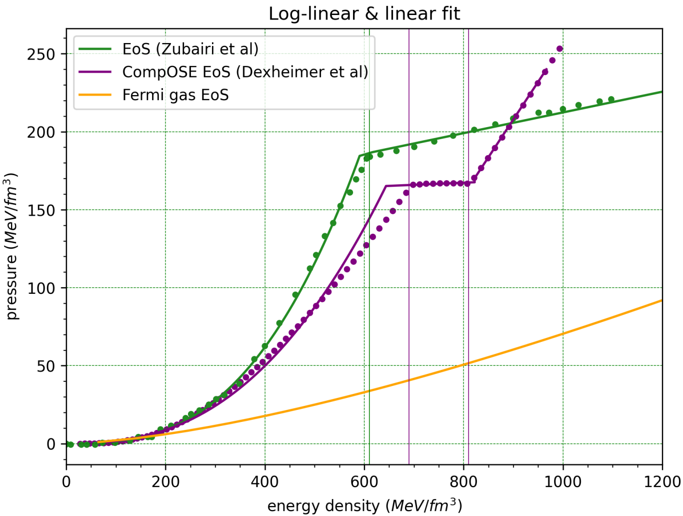
$$f_3 = f(x_0 + \frac{dt}{2}f_2, y_0 + \frac{dt}{2}g_2, t_0 + \frac{dt}{2}) \qquad g_3 = g(x_0 + \frac{dt}{2}f_2, y_0 + \frac{dt}{2}g_2, t_0 + \frac{dt}{2})$$

$$f_4 = f(x_0 + dtf_1, y_0 + dtg_1, t_0 + dt) \qquad g_4 = g(x_0 + dtf_1, y_0 + dtg_1, t_0 + dt)$$

EoS fit approaches



EoS fit approaches



EoS fit approaches

Interpolation (from GitHub PYTOV algorithm): evaluate the EoS function $e(p)$ using *any* EoS dataset at the desired point between $(p_{i-1}, e_{i-1}), (p_i, e_i)$ with a logarithmic interpolation formula

$$\frac{\ln\left(\frac{e(p)}{e_{i-1}}\right)}{\ln\left(\frac{e_{i-1}}{e_i}\right)} = \frac{\ln\left(\frac{p}{p_{i-1}}\right)}{\ln\left(\frac{p_{i-1}}{p_i}\right)}$$

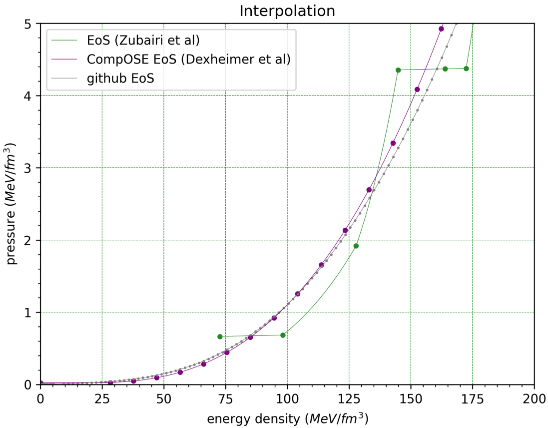


Figure: Interpolation applied to the Zubairi and Dexheimer EoS data points, and to the PYTOV EoS data points. The graph is focused on the low energy region.

Table of Contents

Riccardo Marchese & Konstantinos Topaloglou

The TOV equation in the deformed model

"The first principle is that you must not fool yourself — and you are the easiest person to fool."

— *Richard P. Feynman*

The TOV equation in the deformed model

Non-rotating neutron stars are generally assumed to be perfect spheres, whose stellar properties such as masses and radii are described in the framework of general relativity by the Tolman- Oppenheimer-Volkoff (TOV) equation.

Definition

The assumption of perfect spherical symmetry may not always be correct. Magnetic fields are present inside of neutron stars. In particular, if the magnetic field is strong (up to around 10^{18} Gauss in the core) and/or the pressure of the matter in the cores of neutron stars is non-isotropic, then a deformation of neutron stars can occur.

The TOV equation in the deformed model

One way of investigating deformity of neutron stars is by applying a parametrization on the polar radius in terms of the equatorial radius along with a deformation constant γ on the metric, which reads as:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-\gamma} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \tag{10}$$

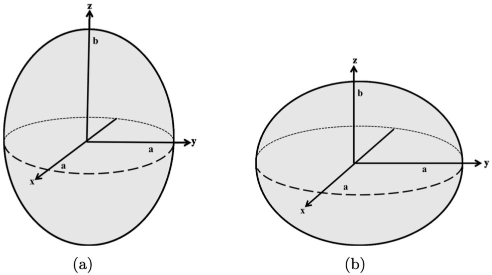


Figure: Illustration of the geometry of an prolate (a) and oblate (b) spheroid.

The TOV equation in the deformed model

The parameter γ determines the degree of deformation either in the oblate ($\gamma < 1$) or prolate case ($\gamma > 1$), where the parametrization is described by $z = \gamma r$. Using Eq. (10), a parametrized TOV equation is then obtained. The gravitational mass of a deformed neutron star parametrized with the deformation constant γ:

Definition of the 1D problem

$$\frac{dP_{\parallel}}{dr} = -\frac{\left(\epsilon + P_{\parallel}\right)\left(\frac{1}{2}r + 4\pi r^3 P_{\parallel} - \frac{1}{2}r\left(1 - \frac{2m}{r}\right)^{\gamma}\right)}{r^2\left(1 - \frac{2m}{r}\right)^{\gamma}} . \tag{11}$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon\gamma \, , \tag{12}$$

where the total mass M, of a deformed neutron star with an equatorial radius R is obtained as the result of the differential equation.

The TOV equation in the deformed model

In order to investigate the deformation, we use the parametrization of the polar coordinate in terms of the radial coordinate $z = \gamma r$ and assuming the radial and polar step sizes are small (on the order of meters), we can look at the deformation constant γ as a differential ratio of the polar and radial step sizes described by $\gamma = \frac{dz}{dr}$.

Now apply a transformation on Eq. (11) to obtain a relationship between pressure and the polar direction

$$\frac{dP_\perp}{dz} = -\frac{(\epsilon + P_\perp)\left(\frac{z}{2\gamma} + 4\pi\left(\frac{z}{\gamma}\right)^3 P_\perp - \frac{z}{2\gamma}\left(1 - \frac{2m\gamma}{z}\right)^\gamma\right)}{\frac{z^2}{\gamma^3}\left(1 - \frac{2m\gamma}{z}\right)^\gamma}, \tag{13}$$

The TOV equation in the deformed model

The mass functions considered in this model are essentially given by

$$\frac{dm}{dr} = 4\pi r^2 \epsilon(P_{\parallel})\gamma \tag{14}$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon(P_{\perp})\gamma \tag{15}$$

Examples

Let's take a total mass for a given spheroid, which could be expressed as

$$M_{total}(r, z) = \frac{4}{3}\pi\epsilon(r, z)r^2 z = \frac{4}{3}m(r, z)\,.$$

This is not the case, since integrating one of the two expressions, the energy density still depends on the radius. However this expression is true only if the energy density is constant.

Table of Contents

Code modelization

The set of ordinary differential equations for our two parametrized model for deformed neutron stars are given by these four of differential equations:

$$\frac{dP_{\|}}{dr} = -\frac{(\epsilon + P_{\|})\left[\frac{1}{2}r + 4\pi r^3 P_{\|} - \frac{1}{2}r\left(1 - \frac{2m(r,z)}{r}\right)^{\gamma}\right]}{r^2\left(1 - \frac{2m(r,z)}{r}\right)^{\gamma}}, \tag{16}$$

$$\frac{dP_{\perp}}{dz} = -\frac{(\epsilon + P_{\perp})\left[\frac{z}{2\gamma} + 4\pi\left(\frac{z}{\gamma}\right)^3 P_{\perp} - \frac{z}{2\gamma}\left(1 - \frac{2m(r,z)\gamma}{z}\right)^{\gamma}\right]}{\frac{z^2}{\gamma^3}\left(1 - \frac{2m(r,z)\gamma}{z}\right)^{\gamma}}, \tag{17}$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon(P_{\|})\gamma \tag{18}$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon(P_{\perp})\gamma \tag{19}$$

Code modelization

The boundary conditions for this set of equations are as follows

$$\begin{aligned} P_{\|}(r=0) &= P_{\|0}\,, \quad & P_{\perp}(z=0) &= P_{\perp 0}\,, \\ P_{\|}(r=R) &= 0\,, \quad & P_{\perp}(z=Z) &= 0\,, \\ m_{\|}(r=0) &= 0\,, \quad & m_{\perp}(z=0) &= 0\,. \end{aligned} \tag{20}$$

Equations of state

We use two different equation of state, one fitted by data found in the CompOSE database (DS (CMF)-6 Hybrid with crust, by V. Dexheimer et al.) and another fitted from the equation of state in the paper by Zubairi and Weber (2017 J. Phys.: Conf. Ser. 845 012005).

Code modelization

We employ another 4th order Runge Kutta method designed to solve the four equations simultaneously. The algorithm below outlines the necessary steps:

- First, we define a range of central pressures for which the stellar sequence will be calculated for.
- We then set the boundary conditions stated above for $r, z = 0$.
- Define a range of values of deformation, from which the differential equations will be solved.
- Employ 4th order Runge-Kutta methods and solve our system of differential equations. The system is integrated until $P_{\parallel}(R) \rightarrow 0$ and $P_{\perp}(R) \rightarrow 0$, which will define the radius at the surface of the star.
- Repeat steps 2 and 3 until the desired range of central pressures are accounted for. From integrating our system, we obtain the following stellar properties: Mass and equatorial and axial radius.

Results from the modified TOV equation

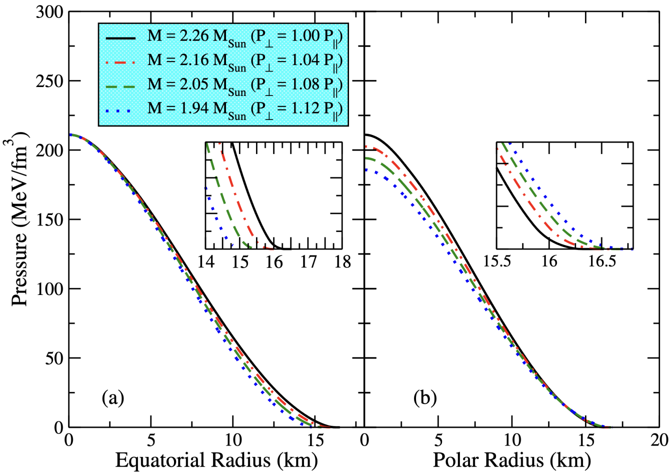


Figure: Pressure profiles for both the equatorial (a) and polar (b) directions for pressure gradients. The mass decreases with prolateness.

Results from the modified TOV equation

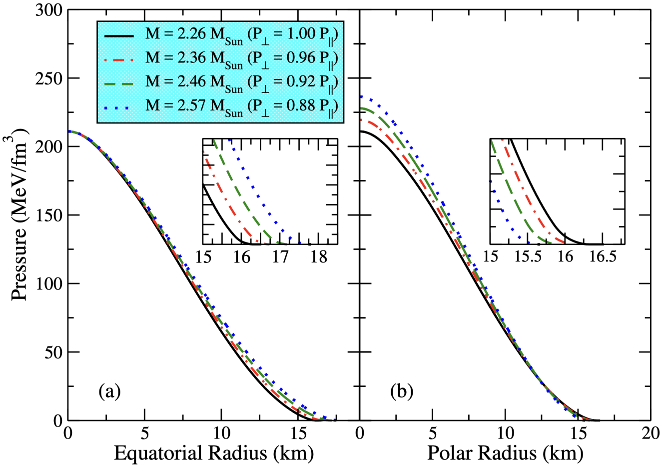


Figure: Pressure profiles for both the equatorial (a) and polar (b) directions for pressure gradients. The mass increases with oblateness.

Results from the modified TOV equation

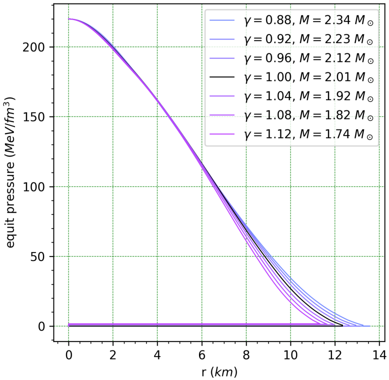


Figure: Pressure profiles for the equatorial directions for pressure gradients (the grey curves).

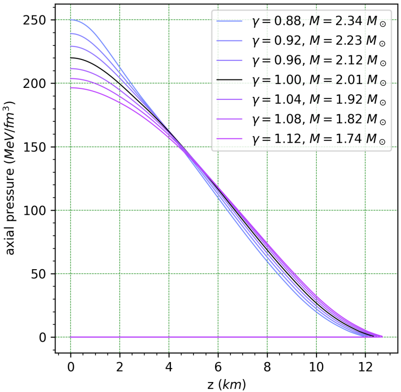


Figure: Pressure profiles for the polar directions for pressure gradients.

Mass-radius function

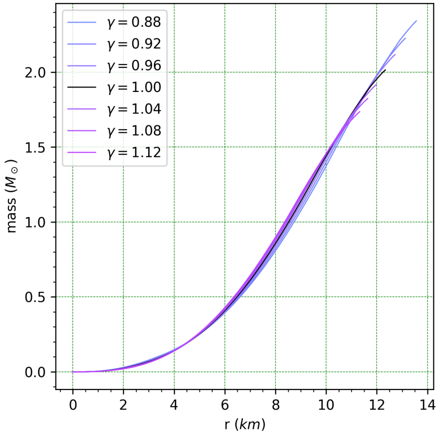


Figure: Mass-radius integration for the EoS proposed by Zubairi et al compared with published results.

Mass-radius function

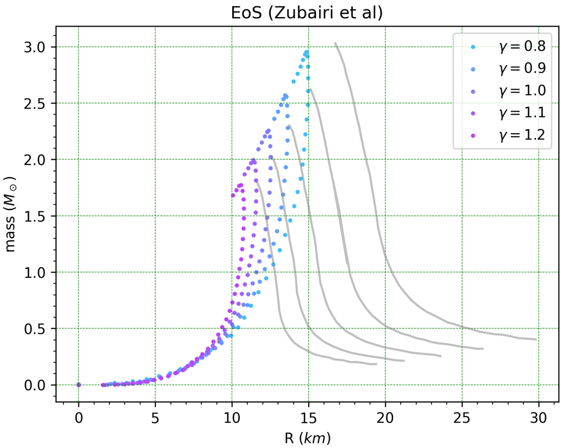


Figure: Mass-radius relation for the EoS proposed by Zubairi et al compared with published results.

Results from the modified TOV equation

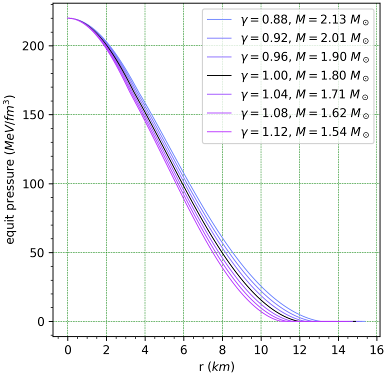


Figure: Pressure profiles for the equatorial directions for pressure gradients (from 11).

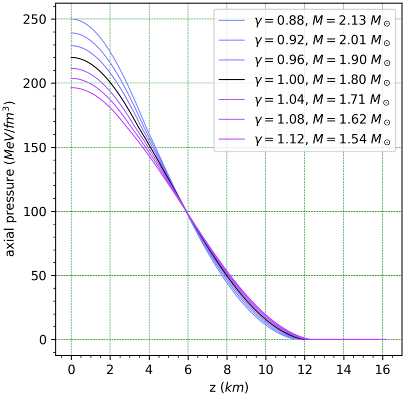


Figure: Pressure profiles for the polar directions for pressure gradients .

Mass-radius function

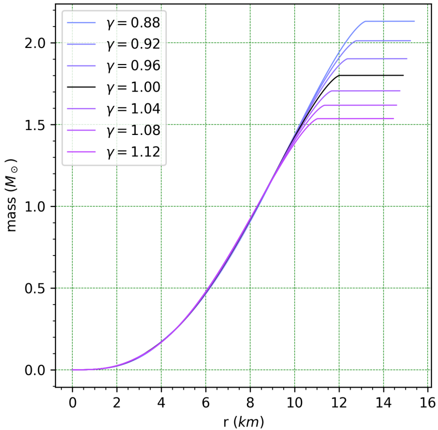


Figure: Mass-radius integration for the CompOSE EoS (Dexheimer et al).

Mass-radius function

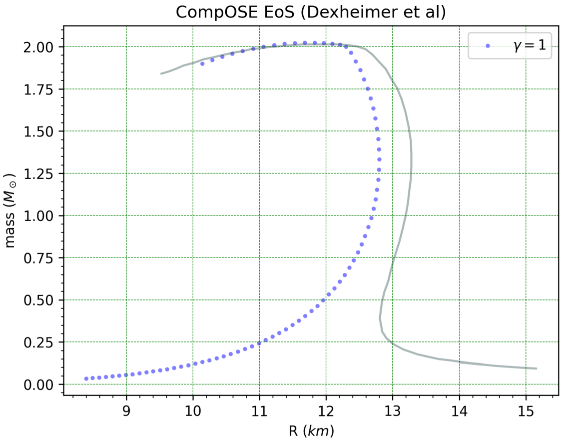


Figure: Mass-radius relation for the CompOSE EoS (Dexheimer et al) compared with published results (the grey curve).

EoS merging

In order to get an accurate low-energy behaviour we use a different set of data coming from a GitHub repository (PYTOV). This provides a smooth approximation of our EoS to use inside the TOV solver as a low-energy complement, giving us a method for getting energy in function of the pressure $\epsilon(P)$, even in cases where the EoS data resolution is insufficient in this key region.

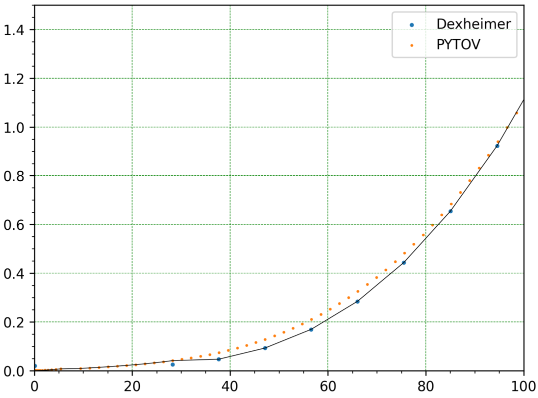


Figure: Merging of the Dexheimer (high energy) and PYTOV (low energy) EoS. The black line represents the product of the data merging, with the switch occurring at $e \approx 30$.

Mass-radius function

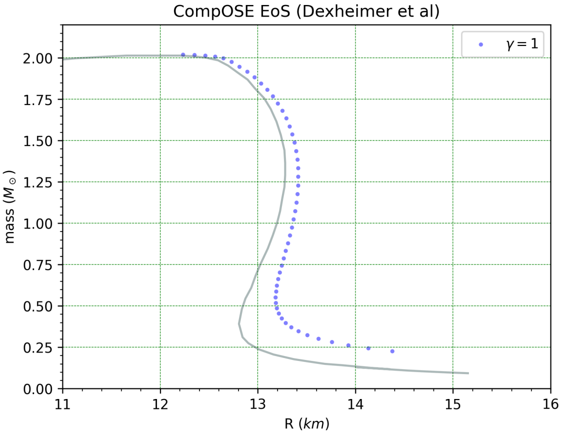


Figure: Mass-radius relation for the CompOSE EoS (Dexheimer et al), after the merge with the EoS given in PYTOV for lower energies