Written assessment, January 30, 2024

Last name, First name ______________________________

Exercise 1 (value 9)

A Company has to program the production for the next period. A total of n types of articles can be produced, using m types of raw materials. Article i ($i = 1, \dots, n$) uses a_{ij} units of the raw material j ($j = 1, \dots, m$). Each raw material j is available in a maximum of Q_j units. Every unit of article i produced generates a revenue of r_i Euros, but to start the production of i a set-up cost of s_i Euros has to be paid.
Moreover, if the first article is produced, its quantity has to be higher than or equal to that of the second article (note that the second article can be produced even if the first article is not). Write a linear programming model to help the Company to choose the production that maximises the total income (revenues-costs).

Exercise 2 (value 9)

Given the following LP

$$
\begin{aligned}
z = \min \quad & 2x_1 + x_2 + 2x_3 \\
& x_1 - x_2 + x_4 = -10 \\
& 2x_1 + 3x_2 - x_3 + x5 = 20 \\
& x_1, \dots, x_5 \geq 0
\end{aligned}
$$

- Solve it with a simplex method, reporting all the tableaus.
- Consider a variation of the r.h.s. of the first constraint and calculate the range for which the basis found in the previous point remains optimal.

Exercise 3 (value 9)

Consider a Knapsack problem with four objects characterized by the following profits $p = (10, 5, 8, 7)$ and weights $w = (2, 3, 4, 3)$, and with a container of capacity $c = 8$. Calculate the optimal solution using dynamic programming. Write which method you are using and motivate the choice. Report all the steps of the algorithm together with the final solution and its cost.

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Solution sketch

Exercise 1

Variables

$$x_i = \text{ number of products of type } i$$
$$y_i = \text{ 1 if one or more units of product } i \text{ are made, zero otherwise}$$

Constants

M a very large number, e.g., $\max_{j=1,\dots,m} Q_j$

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} r_i x_i - \sum_{i=1}^{n} s_i y_i \\
& \sum_{i=1}^{n} a_{ij} x_i \le Q_j & j = 1,\dots,m \\
& x_i \le M y_i & i = 1,\dots,n \\
& x_1 \ge x_2 - M(1 - y_1) \\
& x_i \ge 0 \text{ integer} & i = 1,\dots,n \\
& y_i \in \{0,1\} & i = 1,\dots,n
\end{aligned}$$

Exercise 2

x_1	x_2	x_3	x_4	x_5		
2	1	2	0	0	0	$-z$
1	-1	0	1	0	-10	x_4
2	3	-1	0	1	20	x_5

x_1	x_2	x_3	x_4	x_5		
3	0	2	1	0	-10	$-z$
-1	1	0	-1	0	10	x_2
5	0	-1	3	1	-10	x_5

x_1	x_2	x_3	x_4	x_5		
13	0	0	7	2	-30	$-z$
-1	1	0	-1	0	10	x_2
-5	0	1	-3	-1	10	x_3

$$B = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} -1 & 0 \\ -3 & -1 \end{bmatrix}$$

$$B^{-1}(b + \Delta b) \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -10 + \Delta \\ 20 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 - \Delta \\ 30 - 3\Delta \end{bmatrix} \ge \begin{bmatrix} 0 \\ 20 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \Delta \le 10 \\ \Delta \le 10/3 \end{cases} \Rightarrow \text{The base changes once } b_1 \text{ increases of } 10/3 \text{ units.}$$

The solution remains the same when $-\infty \le b_1 \le -20/3$

Exercise 3

f									J								
0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
0	0	10	10	10	10	10	10	10	–	–	1	1	1	1	1	1	1
0	0	10	10	10	15	15	15	15	–	–	1	1	1	1, 2	1, 2	1, 2	1, 2
0	0	10	10	10	15	18	18	18	–	–	1	1	1	1, 2	1, 3	1, 3	1, 3
0	0	10	10	10	17	18	18	22	–	–	1	1	1	1, 4	1, 3	1, 3	1, 2, 4

Optimal solution : $\{1, 2, 4\}$, profit $= 22$