Written assessment, February 19, 2020 **A**

Last name, First name ______________________

Exercise 1 (value 13).
The Mayor of Reggio Emilia wants to decorate the City for the Christmas Holidays. The plan is to decorate exactly T trees of the City. C alternative configurations are possible for each tree. Each configuration $i \in C$ is characterised by b_i^c Christmas balls, l_i^c LED lights, and has an installation cost of c_i^c Euro. The decorations can be bought from the market, and are offered in boxes of different types. Let B the set of box types. A decoration box of type $j \in B$ contains b_j^b Christmas balls, l_j^b LED lights, and is sold at c_j^b Euro. At least F different tree configurations have to be present in the City, in order to have a pleasant variety.

- Write a MILP modelling the problem described, deciding the number of configurations and boxes needed in order to minimise the total cost *(note that there is no need to associate each decoration to a particular tree)*.
- The supplier of decoration boxes $f \in B$ makes a promotion for its products: if at least q_f boxes of type f are ordered, then a 15% discount is applied to the orders relative to boxes $f \in B$. Modify the model to take this new opportunity into account.

Exercise 2 (value 9)
Consider the following PLC problem. Solve it with the simplex method.

$$\begin{aligned} \min \quad & 5x_1 + 3x_2 + 4x_3 \\ & x_1 - 3x_3 = 14 \\ & 3x_1 - 2x_2 + 4x_3 = 15 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Then write the dual of the problem and find the optimal dual solution using the complementary slackness conditions.

Exercise 3 (value 6)
Find the optimal solution of the following knapsack problem using a branch-and-bound method: $n = 4, c = 22$, $(p_j, w_j) = [(100, 8), (140, 15), (160, 14), (50, 3)]$

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Answers A

Exercise 1

Variables

x_i = nr of trees decorated according to configuration $i \in C$.

y_j = nr of decoration boxes of type $j \in B$ ordered.

z_i = 1 if at least one tree is decorated according to configuration $i \in C$, 0 otherwise

γ = 1 if at least q_f boxes of type $f \in B$ have been ordered, 0 otherwise.

w_f = nr of discounted decoration boxes of type f ordered

Constant

M = a sufficiently large number

$$\min z = \sum_{i \in C} c_i^c x_i + \sum_{j \in B} c_j^b y_j - 0.15 c_f^b w_f \tag{1}$$

$$\text{s.t.} \sum_{i \in C} x_i = T \tag{2}$$

$$\sum_{i \in C} b_i^c x_i \le \sum_{j \in B} b_i^b y_j \tag{3}$$

$$\sum_{i \in C} l_i^c x_i \le \sum_{j \in B} l_i^b y_j \tag{4}$$

$$z_i \le x_i \qquad i \in C \tag{5}$$

$$\sum_{i \in C} z_i \ge F \tag{6}$$

$$\gamma \le \frac{y_f}{q_f} \tag{7}$$

$$w_f \le M\gamma \tag{8}$$

$$w_f \le y_f \tag{9}$$

$$x_i \ge 0, \text{integer} \qquad i \in C \tag{10}$$

$$y_j \ge 0, \text{integer} \qquad j \in B \tag{11}$$

$$z_i \in \{0,1\} \qquad i \in C \tag{12}$$

$$\gamma \in \{0,1\} \tag{13}$$

$$w_f \ge 0 \tag{14}$$

Exercise 2

$$\begin{aligned} \min \quad & 5x_1 + 3x_2 + 4x_3 \\ & x_1 - 3x_3 = 14 \\ & 3x_1 - 2x_2 + 4x_3 = 15 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Phase 1

x_1	x_2	x_3	a_1	a_2		
-4	2	-1	0	0	-29	$-\zeta$
1	0	-3	1	0	14	a_1
3	-2	4	0	1	15	a_2

x_1	x_2	x_3	a_1	a_2		
0	-2/3	13/3	0	4/3	-9	$-\zeta$
0	2/3	-13/3	1	-1/3	9	a_1
1	-2/3	4/3	0	1/3	5	x_1

x_1	x_2	x_3	a_1	a_2		
0	0	0	1	1	0	$-\zeta$
0	1	-13/2	3/2	-1/2	27/2	x_2
1	0	-3	1	0	14	x_1

Phase 2

x_1	x_2	x_3		
0	0	77/2	-221/2	$-z$
0	1	-13/2	27/2	x_2
1	0	-3	14	x_1

$$x = (14, 27/2, 0), z_P = 221/2$$

Dual:

$$\begin{aligned} \max \quad & 14u_1 + 15u_2 \\ & u_1 + 3u_2 \leq 5 \\ & -2u_2 \leq 3 \\ & -3u_1 + 4u_2 \leq 4 \end{aligned}$$

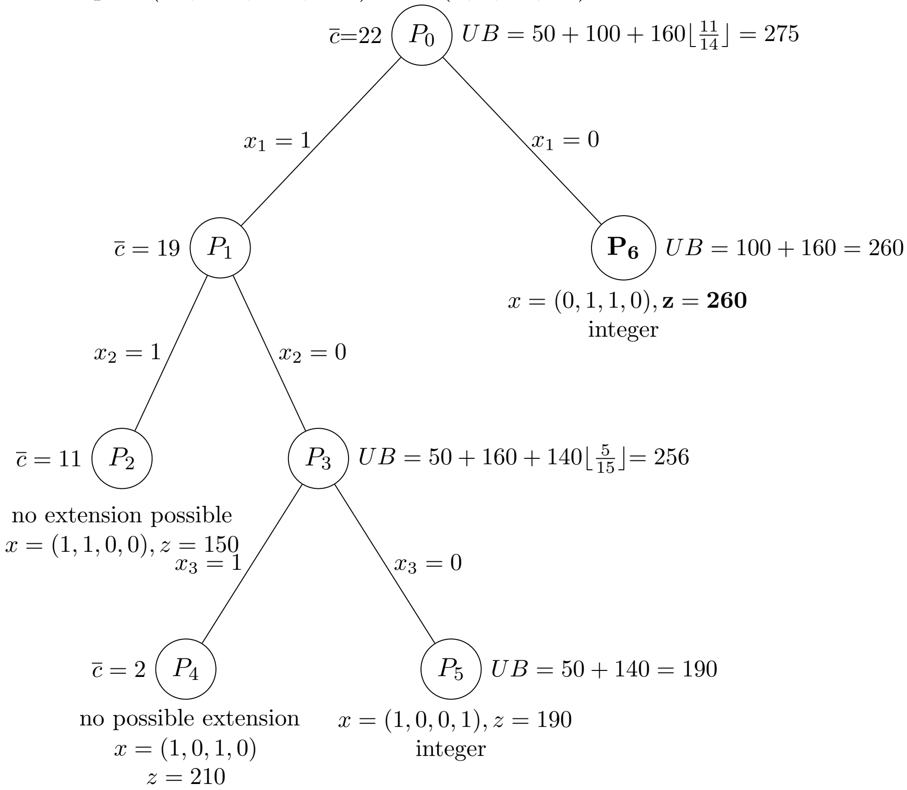
Dual solution:

$$\begin{cases} (u_1 + 3u_2 - 5)x_1 = 0 \\ (-2u_2 - 3)x_2 = 0 \\ (-2u_1 + 4u_2 - 4)x_3 = 0 \end{cases} \rightarrow \begin{cases} (u_1 + 3u_2 - 5)14 = 0 \\ (-2u_2 - 3)\frac{27}{2} = 0 \\ (-2u_1 + 4u_2 - 4)0 = 0 \end{cases} \rightarrow \begin{cases} u_1 + 3u_2 = 5 \\ -2u_2 = 3 \end{cases} \rightarrow \begin{cases} u_1 = \frac{133}{14} \\ u_2 = -\frac{3}{2} \end{cases}$$

$u = (133/14, -3/2, 0), z_D = 221/2$

Exercise 3

Sorted items $p = (50, 100, 160, 140)$ $w = (3, 8, 14, 15)$



Optimum: $z = 260, x = (0, 1, 1, 0)$

Exercise 1 (value 13).
A retired engineer wants to reorganise his garden for the coming Spring. The garden is divided into L lots. The engineer selected C possible configurations (combinations of flowers and plants) and each lot has to be prepared according to one of these configurations (the possible configurations are the same for all the lots). Each configuration $c \in C$ is composed by f^r_c flowers and p^r_c plants. Flowers and plants can be bought from the local florist, and are offered in bundles (boxes) of different types (note that the compositions of the bundles do not match lots' configurations). Let B the set of the available bundle types. A bundle of type $b \in B$ contains f^s_b flowers and p^s_b plants, and is sold at c_b Euro. At least D different lot configurations have to be present in the garden, in order to have a pleasant variety.

- Write a MILP modelling the problem described, deciding the number of bundles to buy and the configuration to use so to minimise the total cost *(Note that the association between lot and configuration is irrelevant at this stage).*
- The local florist makes the following promotion: if at least q_i boxes of bundle $i \in B$ are ordered, then a 15% discount is applied to the total bill (not only to boxes of type i). Modify the model to take this new opportunity into account.

Exercise 2 (value 9)
Consider the following PLC problem. Solve it with the simplex method.

$$\begin{aligned} \min \quad & 10x_1 + 4x_2 - x_3 \\ & x_1 + 2x_3 = 14 \\ & 3x_1 - 2x_2 + 4x_3 = 15 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Then write the dual of the problem and find the optimal dual solution using the complementary slackness conditions.

Exercise 3 (value 6)
Find the optimal solution of the following knapsack problem using a branch-and-bound method: $n = 4, c = 22$, $(p_j, w_j) = [(50, 3), (100, 8), (210, 20), (140, 15)]$

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Answers B

Exercise 1

Variables

x_c = nr of lots according to configuration $c \in C$.

y_b = nr of bundles of type $b \in B$ bought.

z_c = 1 if at least one lot is organised according to configuration $c \in C$, 0 otherwise

δ = 1 if at least q_i of type i have been ordered, 0 otherwise.

w_b = nr of discounted decoration boxes of type $b \in B$ ordered

Constant

M = a sufficiently large number

$$\min\ z = \sum_{b \in B} c_b y_b - 0.15 \sum_{b \in B} c_b w_b \tag{15}$$

$$\text{s.t.} \quad \sum_{c \in C} x_c = L \tag{16}$$

$$\sum_{c \in C} f_c^r x_c \le \sum_{b \in B} f_b^s y_b \tag{17}$$

$$\sum_{c \in C} p_c^r x_c \le \sum_{b \in B} p_b^s y_b \tag{18}$$

$$z_c \le x_c \qquad c \in C \tag{19}$$

$$\sum_{c \in C} z_c \ge D \tag{20}$$

$$\delta \le \frac{y_i}{q_i} \tag{21}$$

$$w_b \le M\delta \qquad b \in B \tag{22}$$

$$w_b \le y_b \qquad b \in B \tag{23}$$

$$x_c \ge 0, \text{integer} \qquad c \in C \tag{24}$$

$$y_b \ge 0, \text{integer} \qquad b \in B \tag{25}$$

$$z_c \in \{0,1\} \qquad c \in C \tag{26}$$

$$\delta \in \{0,1\} \tag{27}$$

$$w_b \ge 0 \qquad b \in B \tag{28}$$

Exercise 2

$$\begin{array}{ll} \min & 10x_1 + 4x_2 - x_3 \\ & x_1 + 2x_3 = 14 \\ & 3x_1 - 2x_2 + 4x_3 = 15 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Phase 1

x_1	x_2	x_3	a_1	a_2		
-4	2	-6	0	0	-29	$-\zeta$
1	0	2	1	0	14	a_1
3	-2	4	0	1	15	a_2

x_1	x_2	x_3	a_1	a_2		
1/2	-1	0	0	3/2	-13/2	$-\zeta$
-1/2	1	0	1	-1/2	13/2	a_1
3/4	-1/2	1	0	1/4	15/4	x_3

x_1	x_2	x_3	a_1	a_2		
0	0	0	1	1	0	$-\zeta$
-1/2	1	0	1	-1/2	13/2	x_2
1/2	0	1	1/2	0	7	x_3

Phase 2

x_1	x_2	x_3		
25/2	0	0	-19	$-z$
-1/2	1	0	13/2	x_2
1/2	0	1	7	x_3

$x = (0, 23/2, 7), z_P = 19$

Dual:

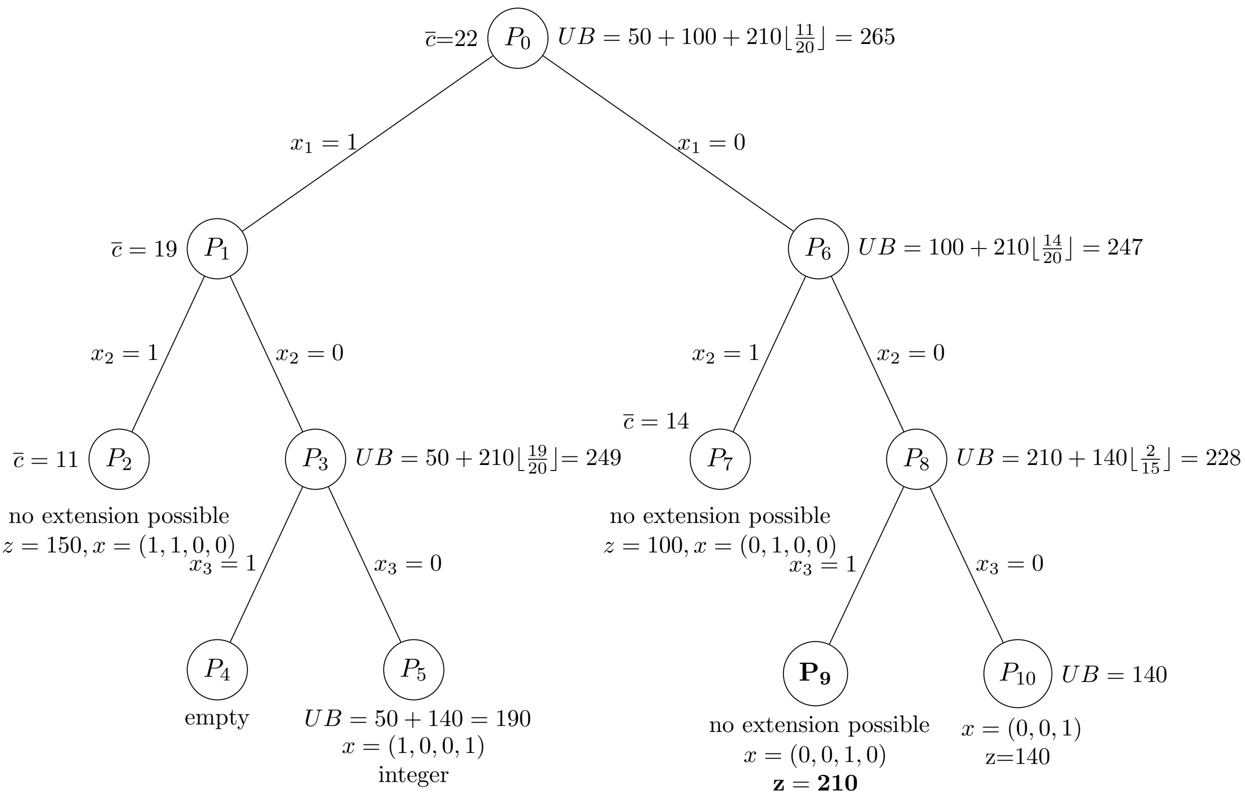
$$\begin{array}{ll} \max & 14u_1 + 15u_2 \\ & u_1 + 3u_2 \leq 10 \\ & -2u_2 \leq 4 \\ & 2u_1 + 4u_2 \leq -1 \end{array}$$

Dual solution:

$$\begin{cases} (u_1 + 3u_2 - 10)x_1 = 0 \\ (-2u_2 - 4)x_2 = 0 \\ (2u_1 + 4u_2 + 1)x_3 = 0 \end{cases} \rightarrow \begin{cases} (u_1 + 3u_2 - 10)0 = 0 \\ (-2u_2 - 4)\frac{23}{2} = 0 \\ (2u_1 + 4u_2 + 1)7 = 0 \end{cases} \rightarrow \begin{cases} -u_2 = 2 \\ 2u_1 + 4u_2 = -1 \end{cases} \rightarrow \begin{cases} u_2 = -2 \\ u_1 = \frac{7}{2} \end{cases}$$

$u = (7/2, -2, 0), z_D = 19$

Exercise 3



Optimum: $z = 210, x = (0, 0, 1, 0)$