Written assessment, June 19, 2023

Last name, First name ____________________

Exercise 1 (value 8)

The nurses of a hospital have to deliver medical treatment to n patients during a given day. One nurse is required for each treatment, and the treatment for patient i ($i = 1, \ldots, n$) requires t_i minutes. There are m nurses, and nurse j ($j = 1, \ldots, m$) has a working contract specifying that she can work maximum s_j minutes per the day. The hospital aims for a fair allocation of the treatment tasks to the nurses, i.e., each nurse should have a "similar" proportional workload. To achieve this, the target is to maximize the minimum *percentage of working time* among all the nurses, where the *percentage of working time* is *the ratio between the time worked and the maximum working time allowed by the contract.* Help the hospital to find an optimal solution by writing an Integer Linear Program.

In a second phase, each nurse also expresses preferences among the patients: p_{ij} is the preference level for patient i according to nurse j. The sum of the preferences of the patients treated by each nurse j cannot be below a given threshold P (we assume all the nurses will work). Write the appropriate constraint for the model.

Exercise 2 (value 11)

Consider the following LP problem.

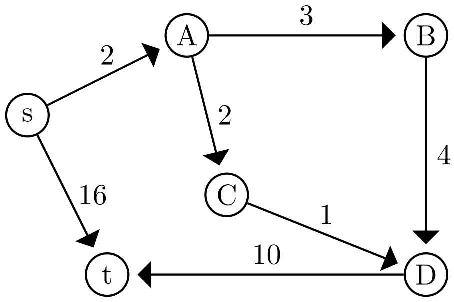
$$
\begin{aligned}
z = \min \quad & 2x_1 - x_2 - 2x_3 \\
& x_1 + x_2 + x_3 \leq 30 \\
& 2x_1 + x_3 \leq 28 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Knowing that in the optimal solution the non-zero variables are x_2 and x_3:

- Calculate the value of the variables and the value of the objective function without running the simplex algorithm at this stage (show the procedure);
- What is the optimal solution if the first r.h.s. (i.e. 30) is decreased by 3 units? And what is its cost?

Exercise 3 (value 8)

Consider the following graph $G = (V, A)$ and find the shortest path from s to t using the Bellman-Ford method. Table the values of $f(j)$ and *pred* for all the steps of the algorithm.



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Solution sketch

Exercise 1

Variables

$x_{ij} = 1$ if patient i is treated by nurse $j; 0$ otherwise
$z =$ minimum percentage of working time

$$
\begin{array}{llll}
\max & z & & \\
& \displaystyle\sum_{j=1}^{m} x_{ij} & = 1 & i = 1, \dots, n \\
& \displaystyle\sum_{i=1}^{n} t_i x_{ij} & \leq s_j & j = 1, \dots, m \\
& z & \leq \left(\displaystyle\sum_{i=1}^{n} t_i x_{ij} \right) / s_j & j = 1, \dots, m \\
& \displaystyle\sum_{i=1}^{n} p_{ij} x_{ij} & \geq P & j = 1, \dots, m \text{ (phase 2)} \\
& x_{ij} & \in \{0, 1\} & i = 1, \dots, n; j = 1, \dots, m \\
& z & \geq 0 &
\end{array}
$$

Exercise 2

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 2 \\ 28 \end{bmatrix} \Rightarrow z = -58$$

$$B^{-1}(b + \Delta b) \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 30 + \Delta b_1 \\ 28 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 + \Delta b_1 \\ 28 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \Delta b_1 \geq -2 \\ -- \end{cases} \Rightarrow \text{The base changes for } b_1 \text{ decreasing of 3 units.}$$

We need to solve a simplex to find the new optimum.

$$B^{-1}b = \begin{bmatrix} -1 \\ 28 \end{bmatrix} \Rightarrow z = 54$$

$$B^{-1}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 1 & -1 \\ 2 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\bar{c}_F^T = c_F^T - c_B^T B^{-1} F = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -3 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \end{bmatrix}$$

x_1	x_2	x_3	y_1	y_2	$-z$	
5	0	0	1	3	54	
-1	1	0	1	**-1**	-1	x_1
2	0	1	0	1	28	x_2

x_1	x_2	x_3	y_1	y_2	$-z$	
4	1	0	2	0	54	
1	-1	0	-1	1	1	y_1
1	1	1	1	0	27	x_2

The optimal solution is $x = (0, 0, 27)$ with value -54.

Exercise 3

Iter	f(j)						pred					
	s	A	B	C	D	t	s	A	B	C	D	t
0	0	-	-	-	-	-	s	-	-	-	-	-
1	0	2	-	-	-	16	s	s	-	-	-	s
2	0	2	5	4	-	16	s	s	A	A	-	s
3	0	2	5	4	5	16	s	s	A	A	C	s
4	0	2	5	4	5	15	s	s	A	A	C	D
5	0	2	5	4	5	15	s	s	A	A	C	D

Optimal path $s \to t$: cost = 15, path = (s, A, C, D, t).