Part 1

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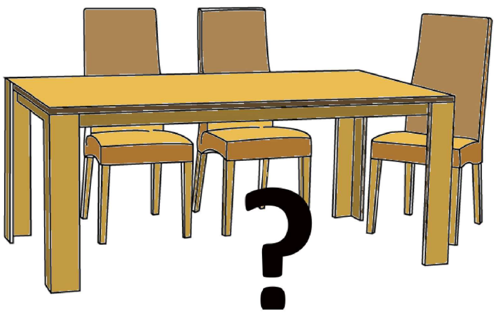
Consider a wood worker making tables and chairs. The wood worker has 15 units of wood. Each table requires 3 units of wood and each chair requires 2 units of wood. The person has 15 hours to spend on making the tables and chairs. Each table requires 2 hours of work and each chair requires 3 hours. The wood worker earns 80 Euro per table and 70 Euro per chair. How many tables and chairs should the wood worker make to maximize his revenue ?



Can we write a mathematical model for the problem ?

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x = number of tables
y = number of chairs

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Resources : wood (15 units) time (15 hours)

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$$
\begin{array}{lll}
\text{wood:} & 3x + 2y & \leq 15 \\
\text{hours:} & 2x + 3y & \leq 15
\end{array}
$$

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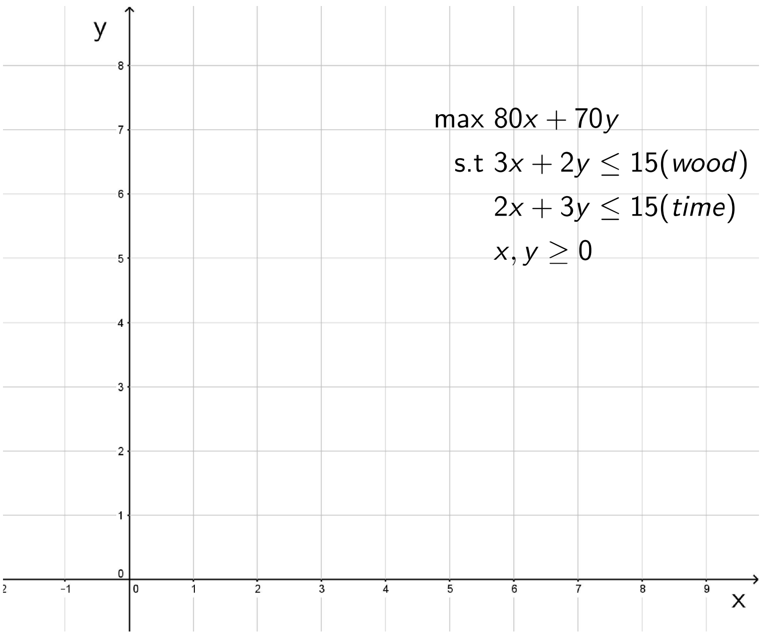
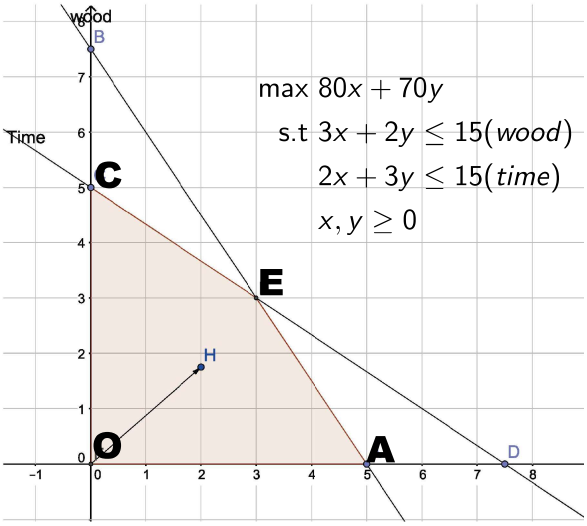
\text{wood:} & 3x + 2y & \leq 15 \\
\text{hours:} & 2x + 3y & \leq 15
\end{array}
$$

Revenue :

$$
\max \quad 80x + 70y
$$



$$
\begin{aligned}
\max\ & 80x + 70y \\
\text{s.t } & 3x + 2y \leq 15(wood) \\
& 2x + 3y \leq 15(time) \\
& x, y \geq 0
\end{aligned}
$$



Part 2

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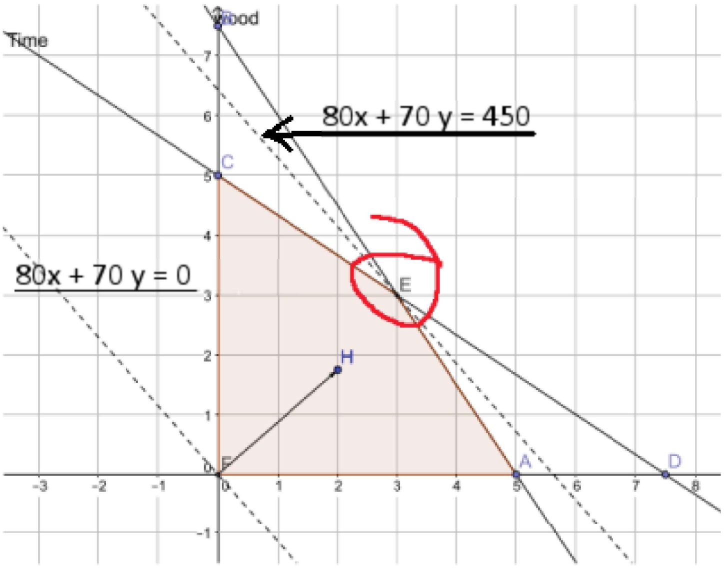

Some mathematics

$z = 80x + 70y$ is a Linear objective function

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The *gradient* of f is $\nabla f = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2, + \cdots + \frac{\partial f}{\partial x_n} e_n$, where e_i is the unit vector in the i-th coordinate direction.
The gradient computed in a point indicates the direction of the greatest rate of increase of f in that point.

Some mathematics

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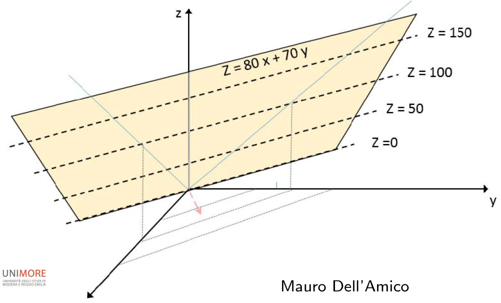
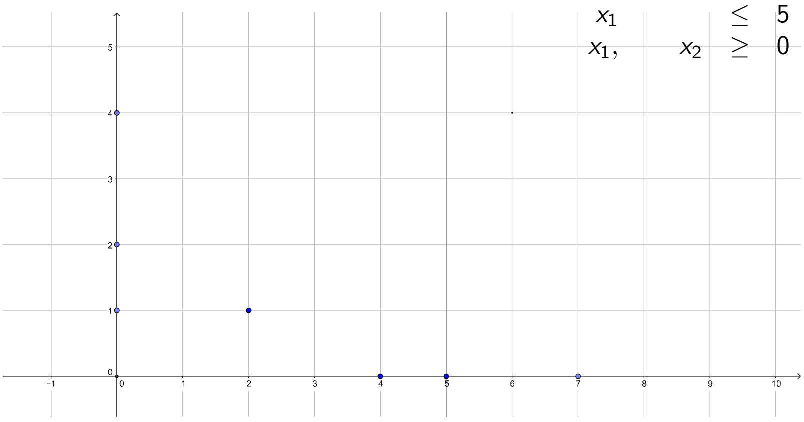
The gradient computed in a point indicates the direction of the greatest rate of increase of f in that point.

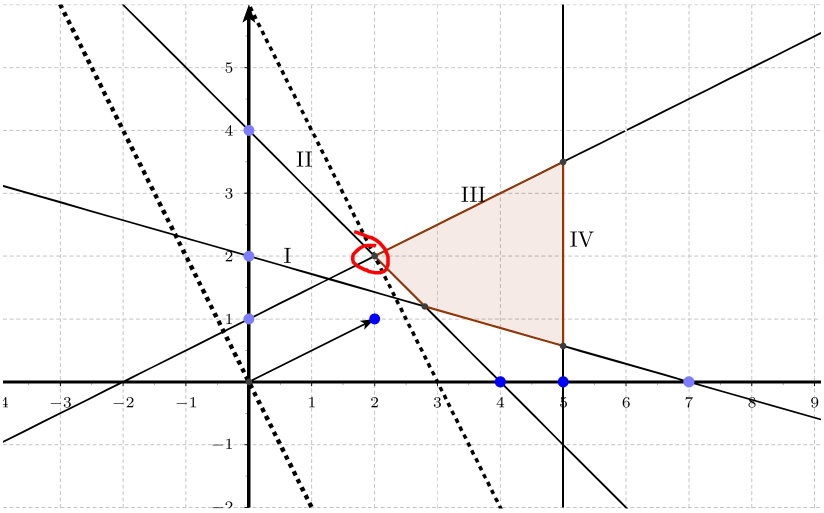
When $f()$ is *Linear* the gradient is given by the *coefficients* of the variables, ex. $f = 80x + 70y$ $\quad \nabla f = (80, 70)$



Another example

$$
\begin{array}{lrrcl}
\min z = & 2x_1 & +x_2 & & \\
s.t. & 2x_1 & +7x_2 & \geq & 14 \\
 & x_1 & +x_2 & \geq & 4 \\
 & -x_1 & +2x_2 & \leq & 2 \\
 & x_1 & & \leq & 5 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$





Part 3

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Standard format

- Only equations
- Only non-negative variables

$$\begin{array}{lrrrrcl} \max & 80x & +70y & & & & \\ \text{s.t} & 3x & +2y & +s_1 & & = & 15 \\ & 2x & +3y & & +s_2 & = & 15 \\ & x, & y, & s_1, & s_2 & \geq & 0 \end{array}$$

s_1 and s_2 are called **slack** variables
We have an immediate solution with s_1 and s_2

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s_1 and s_2 are called **slack** variables
We have an immediate solution with s_1 and s_2

$$\begin{array}{|lrrr|} \hline z = & 80x & +70y & \\ \hline s_1 = & -3x & -2y & +15 \\ s_2 = & -2x & -3y & +15 \\ \hline \end{array} \quad (0, 0, 15, 15) \quad z = 0$$

$z =$	$80x$	$+70y$	
$s_1 =$	$-3x$	$-2y$	$+\ 15$
$s_2 =$	$-2x$	$-3y$	$+\ 15$

$(0, 0, 15, 15) \quad z = 0$

It is convenient to increase x ! But how much ? (let $y = 0$)

$z =$	$80x$	$+70y$	
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$(0, 0, 15, 15) \quad z = 0$

It is convenient to increase x ! But how much ? (let $y = 0$)

$$\begin{cases} s_1 = & -3x & +15 & \geq 0 \\ s_2 = & -2x & +15 & \geq 0 \end{cases} \qquad \begin{cases} x & \leq & 5 \\ x & \leq & \frac{15}{2} \end{cases} \Rightarrow x = 5, s_1 = 0$$

$x \uparrow$ enters in the solution, $s_1 \downarrow$ exit

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$$\begin{array}{|llll|}\hline z = & 80x & +70y & \\ \hline s_1 = & -3x & -2y & +15 \\ s_2 = & -2x & -3y & +15 \\ \hline \end{array} \quad (0,0,15,15) \quad z = 0$$

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$$\begin{array}{|llll|}\hline z = & +\frac{50}{3}y & -\frac{80}{3}s_1 & +400 \\ \hline x = & -\frac{2}{3}y & -\frac{1}{3}s_1 & +5 \\ s_2 = & -\frac{5}{3}y & +\frac{2}{3}s_1 & +5 \\ \hline \end{array} \quad (5,0,0,5) z = 400$$

$$\begin{array}{|llll|}\hline z = & +\frac{50}{3}y & -\frac{80}{3}s_1 & +400 \\ \hline x = & -\frac{2}{3}y & -\frac{1}{3}s_1 & +5 \\ s_2 = & -\frac{5}{3}y & +\frac{2}{3}s_1 & +5 \\ \hline \end{array} \quad (5,0,0,5) \quad z = 400$$

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$z =$	$+\frac{50}{3}y$	$-\frac{80}{3}s_1$	$+ 400$
$x =$	$-\frac{2}{3}y$	$-\frac{1}{3}s_1$	$+ 5$
$s_2 =$	$-\frac{5}{3}y$	$+\frac{2}{3}s_1$	$+ 5$

$(5, 0, 0, 5) \quad z = 400$

y should enter the solution (let $s_1 = 0$)

$$\begin{cases} x = -\frac{2}{3}y + 5 \geq 0 \\ s_2 = -\frac{5}{3}y + 5 \geq 0 \end{cases} \qquad \begin{cases} y \leq \frac{15}{2} \\ y \leq 3 \end{cases} \Rightarrow y = 3, s_2 = 0$$

$y \uparrow$ enters in the solution $s_2 \downarrow$ exit.

$z =$	$-20s_1$	$-10s_2$	$+ 450$
$x =$	$-\frac{3}{5}s_1$	$+\frac{2}{5}s_2$	$+ 3$
$y =$	$+\frac{2}{5}s_1$	$-\frac{3}{5}s_2$	$+ 3$

$(3, 3, 0, 0) \quad z = 450$

No increase is possible: optimal solution !

Solutions explored:

$z =$	$80x$	$+70y$	
$s_1 =$	$-3x$	$-2y$	$+ 15$
$s_2 =$	$-2x$	$-3y$	$+ 15$

$(0, 0, 15, 15) \quad z = 0$

$z =$	$+\frac{50}{3}y$	$-\frac{80}{3}s_1$	$+ 400$
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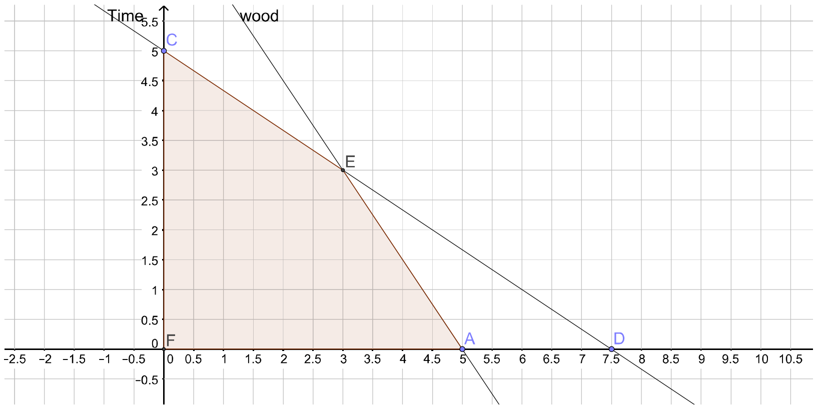

Part 4

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Tableau

$$
\begin{array}{lllllll}
\max & 80x & +70y & & & \\
 & 3x & +2y & +s_1 & & = 15 \\
 & 2x & +3y & & +s_2 & = 15
\end{array}
$$

x	y	s_1	s_2		
80	70	0	0	0	$-z$
3	2	1	0	15	s_1
2	3	0	1	15	s_2

$$
\begin{array}{lllll}
3\cdot 0 & +2\cdot 0 & +1\cdot s_1 & +0\cdot s_2 & =15 \\
2\cdot 0 & +3\cdot 0 & +0\cdot s_1 & +1\cdot s_2 & =15
\end{array}
$$

x	y	s_1	s_2		
80	70	0	0	0	$-z$
3	2	1	0	15	s_1
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$x \uparrow$ enters in the solution, $s_1 \downarrow$ exit

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$z =$	$+\frac{50}{3}y$	$-\frac{80}{3}s_1$	$+\ 400$
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$s_2 =$	$-\frac{5}{3}y$	$+\ \frac{2}{3}s_1$	$+5$

When x enters in a solution it must appear in ONE equation
x ha coefficient 1 in one equation
x ha coefficient 0 in all other equations, including the obj. function

	x	y	s_1	s_2		
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x	y	s_1	s_2		
0	$\frac{50}{3}$	$-\frac{80}{3}$	0	-400	$-z$
1	$\frac{2}{3}$	$\frac{1}{3}$	0	5	x
0	$\frac{5}{3}$	$-\frac{2}{3}$	1	5	s_2

Pivot

Pivot= transform the system of equations in an equivalent system with exactly one '**1**' in the column of the entering variable

x	y	s_1	s_2		
0	$\frac{50}{3}$	$-\frac{80}{3}$	0	-400	$-z$
1	$\frac{2}{3}$	$\frac{1}{3}$	0	5	x
0	$\frac{5}{3}$	$-\frac{2}{3}$	1	5	s_2

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0	$\frac{50}{3}$	$-\frac{80}{3}$	0	-400	$-z$
1	$\frac{2}{3}$	$\frac{1}{3}$	0	5	x
0	$\frac{5}{3}$	$-\frac{2}{3}$	1	5	s_2

x	y	s_1	s_2		
0	0	- 20	-10	-450	$-z$
1	0	$\frac{3}{5}$	$-\frac{2}{5}$	3	x
0	1	$-\frac{2}{5}$	$\frac{3}{5}$	3	y

$$
\begin{array}{llllll}
\min z= & x_1 & -2x_2 & -6x_3 & & \\
\text{s.t.} & x_1 & & & \leq & 2 \\
 & & x_2 & & \leq & 3 \\
 & & & x_3 & \leq & 3 \\
 & x_1 & +x_2 & +x_3 & \leq & 4 \\
 & & & x_1, x_2, x_3 & \geq & 0
\end{array}
\qquad
\begin{array}{llllllllll}
\min z= & x_1 & -2x_2 & -6x_3 & & & & & & \\
\text{s.t.} & x_1 & & & +x_4 & & & & = & 2 \\
 & & x_2 & & & +x_5 & & & = & 3 \\
 & & & x_3 & & & +x_6 & & = & 3 \\
 & x_1 & +x_2 & +x_3 & & & & +x_7 & = & 4 \\
 & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & & \geq & 0
\end{array}
$$

$$
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\min z= & x_1 & -2x_2 & -6x_3 & & \\
\text{s.t.} & x_1 & & & \leq & 2 \\
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 & & & x_3 & \leq & 3 \\
 & x_1 & +x_2 & +x_3 & \leq & 4 \\
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\qquad
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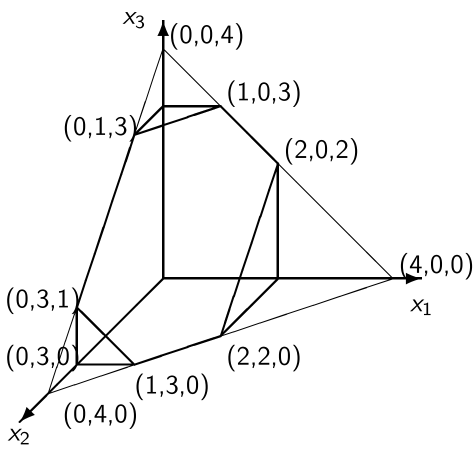
\min z= & x_1 & -2x_2 & -6x_3 & & & & & & \\
\text{s.t.} & x_1 & & & +x_4 & & & & = & 2 \\
 & & x_2 & & & +x_5 & & & = & 3 \\
 & & & x_3 & & & +x_6 & & = & 3 \\
 & x_1 & +x_2 & +x_3 & & & & +x_7 & = & 4 \\
 & & & & & & x_1, x_2, x_3, x_4, x_5, x_6, x_7 & & \geq & 0
\end{array}
$$



x_1	x_2	x_3	x_4	x_5	x_6	x_7	
1	-2	-6	0	0	0	0	0
1	0	0	1	0	0	0	2
0	$\boxed{1}$	0	0	1	0	0	3
0	0	1	0	0	1	0	3
1	1	1	0	0	0	1	4

(0,0,0,
2,3,3,4)

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
1	-2	-6	0	0	0	0	0
1	0	0	1	0	0	0	2
0	$\boxed{1}$	0	0	1	0	0	3
0	0	1	0	0	1	0	3
1	1	1	0	0	0	1	4

(0,0,0,
2,3,3,4)

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
1	0	-6	0	2	0	0	6
1	0	0	1	0	0	0	2
0	1	0	0	1	0	0	3
0	0	1	0	0	1	0	3
1	0	$\boxed{1}$	0	-1	0	1	1

(0,3,0,
2,0,3,1)

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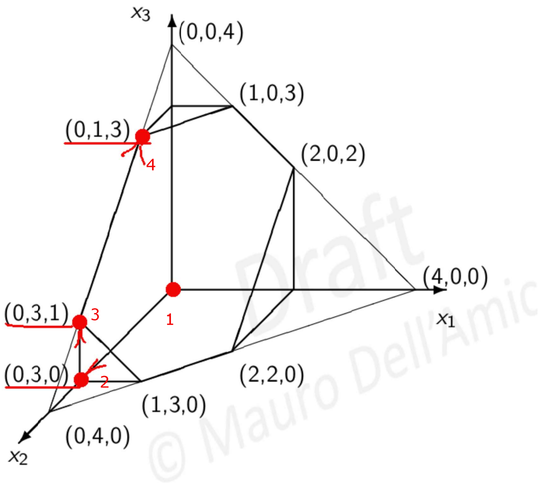
x_1	x_2	x_3	x_4	x_5	x_6	x_7		
1	-2	-6	0	0	0	0	0	
1	0	0	1	0	0	0	2	
0	$\boxed{1}$	0	0	1	0	0	3	(0,0,0,
0	0	1	0	0	1	0	3	2,3,3,4)
1	1	1	0	0	0	1	4	

x_1	x_2	x_3	x_4	x_5	x_6	x_7		
1	0	-6	0	2	0	0	6	
1	0	0	1	0	0	0	2	
0	1	0	0	1	0	0	3	(0,3,0,
0	0	1	0	0	1	0	3	2,0,3,1)
1	0	$\boxed{1}$	0	-1	0	1	1	

x_1	x_2	x_3	x_4	x_5	x_6	x_7		
7	0	0	0	-4	0	6	12	
1	0	0	1	0	0	0	2	
0	1	0	0	1	0	0	3	(0,3,1,
-1	0	0	0	$\boxed{1}$	1	-1	2	2,0,2,0)
1	0	1	0	-1	0	1	1	

x_1	x_2	x_3	x_4	x_5	x_6	x_7		
7	0	0	0	-4	0	6	12	
1	0	0	1	0	0	0	2	
0	1	0	0	1	0	0	3	(0,3,1,
-1	0	0	0	$\boxed{1}$	1	-1	2	2,0,2,0)
1	0	1	0	-1	0	1	1	

3	0	0	0	0	4	2	20	
1	0	0	1	0	0	0	2	
1	1	0	0	0	-1	1	1	(0,1,3,
-1	0	0	0	1	1	-1	2	2,2,0,0)
0	0	1	0	0	1	0	3	*ottimo*



Part 5

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$$
\begin{array}{llrrrl}
\max & z = & x_1 & +3x_2 & & \\
\text{s.t} & & x_1 & -2x_2 & \leq & 4 \\
 & & x_1 & -x_2 & \leq & 8 \\
 & & x_1, & x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{llrrrrrl}
\max & z = & x_1 & +3x_2 & & & & \\
\text{s.t} & & x_1 & -2x_2 & +x_3 & & = & 4 \\
 & & x_1 & -x_2 & & +x_4 & = & 8 \\
 & & x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
$$

x_1	x_2	x_3	x_4		
1	3	0	0	0	$-z$
1	-2	1	0	4	x_3
1	-1	0	1	8	x_4

$$
\begin{array}{llrrrl}
\max & z = & x_1 & +3x_2 & & \\
\text{s.t} & & x_1 & -2x_2 & \leq & 4 \\
 & & x_1 & -x_2 & \leq & 8 \\
 & & x_1, & x_2 & \geq & 0
\end{array}
\qquad
\begin{array}{llrrrrrl}
\max & z = & x_1 & +3x_2 & & & & \\
\text{s.t} & & x_1 & -2x_2 & +x_3 & & = & 4 \\
 & & x_1 & -x_2 & & +x_4 & = & 8 \\
 & & x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
$$

x_1	x_2	x_3	x_4		
1	3	0	0	0	$-z$
1	-2	1	0	4	x_3
1	-1	0	1	8	x_4

$$
\begin{cases} x_3 = -x_1 + 4 \geq 0 \\ x_4 = -x_1 + 8 \geq 0 \end{cases}
\qquad
\begin{cases} x_1 \leq 4 \\ x_1 \leq 8 \end{cases} \Rightarrow x_1 = 4, x_3 = 0
$$

$x_1 \uparrow$ enters in the solution, $x_3 \downarrow$ exit

$$\begin{array}{llrrl} \max & z = & x_1 & +3x_2 & \\ \text{s.t} & & x_1 & -2x_2 & \leq 4 \\ & & x_1 & -x_2 & \leq 8 \\ & & x_1, & x_2 & \geq 0 \end{array} \qquad \begin{array}{llrrrrl} \max & z = & x_1 & +3x_2 & & & \\ \text{s.t} & & x_1 & -2x_2 & +x_3 & & = 4 \\ & & x_1 & -x_2 & & +x_4 & = 8 \\ & & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

x_1	x_2	x_3	x_4		
1	3	0	0	0	$-z$
1	-2	1	0	4	x_3
1	-1	0	1	8	x_4

$$\begin{cases} x_3 = -x_1 + 4 \geq 0 \\ x_4 = -x_1 + 8 \geq 0 \end{cases} \qquad \begin{cases} x_1 \leq 4 \\ x_1 \leq 8 \end{cases} \Rightarrow x_1 = 4, x_3 = 0$$

$x_1 \uparrow$ enters in the solution, $x_3 \downarrow$ exit

x_1	x_2	x_3	x_4		
0	5	-1	0	-4	$-z$
1	-2	1	0	4	x_1
0	1	-1	1	4	x_4

x_1	x_2	x_3	x_4		
0	5	-1	0	-4	$-z$
1	-2	1	0	4	x_1
0	1	-1	1	4	x_4

$$\begin{cases} x_1 = 2x_2 + 4 \geq 0 \\ x_4 = -x_2 + 4 \geq 0 \end{cases} \qquad \begin{cases} \text{always true} \\ x_2 \leq 4 \end{cases} \Rightarrow x_2 = 4, \quad x_4 = 0$$

$x_2 \uparrow$ enters in the solution, $x_4 \downarrow$ exit

x_1	x_2	x_3	x_4		
0	5	-1	0	-4	$-z$
1	-2	1	0	4	x_1
0	1	-1	1	4	x_4

$$\begin{cases} x_1 = 2x_2 + 4 \geq 0 \\ x_4 = -x_2 + 4 \geq 0 \end{cases} \quad \begin{cases} \text{always true} \\ x_2 \leq 4 \end{cases} \Rightarrow x_2 = 4, \quad x_4 = 0$$

$x_2 \uparrow$ enters in the solution, $x_4 \downarrow$ exit

x_1	x_2	x_3	x_4		
0	0	4	-5	-24	$-z$
1	0	-1	2	12	x_1
0	1	-1	1	4	x_2

x_1	x_2	x_3	x_4		
0	5	-1	0	-4	$-z$
1	-2	1	0	4	x_1
0	1	-1	1	4	x_4

$$\begin{cases} x_1 = 2x_2 + 4 \geq 0 \\ x_4 = -x_2 + 4 \geq 0 \end{cases} \quad \begin{cases} \text{always true} \\ x_2 \leq 4 \end{cases} \Rightarrow x_2 = 4, \quad x_4 = 0$$

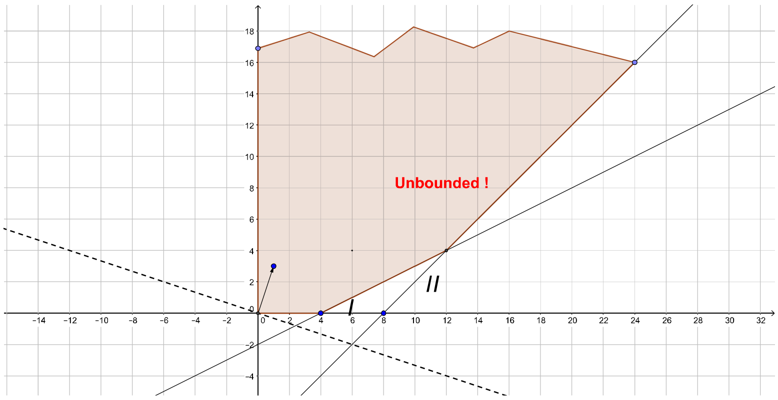
$x_2 \uparrow$ enters in the solution, $x_4 \downarrow$ exit

x_1	x_2	x_3	x_4		
0	0	4	-5	-24	$-z$
1	0	-1	2	12	x_1
0	1	-1	1	4	x_2

$$\begin{cases} x_1 = x_3 + 12 \geq 0 \\ x_2 = x_3 + 4 \geq 0 \end{cases} \quad \begin{cases} \text{always true} \\ \text{always true} \end{cases}$$

??

$$
\begin{array}{llrcl}
\max\ z = & x_1 & +3x_2 & & \\
\text{s.t} & x_1 & -2x_2 & \leq & 4 \\
 & x_1 & -x_2 & \leq & 8 \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$



Pivot: rules

Column:

select a column with **positive** obj. coeff. (maximization)
select a column with **negative** obj. coeff. (minimization)

Row:

select the row with minimum ratio (rhs)/(column coeff.)
for **positive** coefficients
⇒ if no positive coefficient exists the problem is **unbounded**