Exercise 1 (value 8)
Consider the following PLC problem, write the dual of the problem, draw the feasible region.

$$\begin{array}{ll} \max & 6x_1 + x_2 \\ & 2x_1 + x_2 + 2x_3 \le 3 \\ & -3x_1 + 2x_3 \ge 1 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

[a] Consider the slack variables of the dual solutions and using this information identify the primal variables that must be null in the optimal primal solution (explain the reasoning you use).
[b] Find the optimal dual solution with the graphical method. Compute a primal solution using the following conditions (in a system of equations): (i) the results of point [a], (ii) the strong duality theorem, and (iii) the equation $2x_1 + x_2 + 2x_3 = 3$. Verify if this solution is feasible and in this case explain if it is optimal or not.

Exercise 2 (value 14).
A company owns a biomass energy generator whose forecasted production if b_t MW in each period $t \in T$ (where $T = \{1, \dots, |T|\}$ is the considered time horizon). The energy demand in MW for each period t is d_t. When the biomass production is not sufficient to cover the demand, energy can be bought externally at a price of e_t euro per MW. Biomass overproduction and not used bought energy can be stored in some battery storage power stations. Each power station has a total capacity of W MW and a cost of C_{PS} euro. Batteries are bough at the beginning of the periods and, once bought, are available for all the periods. No energy is stored at period 0. Overproduction or stored energy can be sold to the public energy company for a price of f_t euro per MW, depending on the period t. If the company buys energy in a period $t \in T$, no energy can be sold in that period.
Write a linear model to help the company to plan the energy usage in order to satisfy the demand and minimize overall cost.

Modify the above model considering that the company bought a new generator for internal energy production. The cost of producing a MW with the generator is C. For each period the minimum and maximum production in MW, if the generator is active, are $\underline{M}$ and $\overline{M}$, respectively. The generator can be activated at most once in T and cannot be deactivated. The activation has a cost of F. The company wants to decide if and when to activate the generator and how much it should produce in MW in each period, while minimizing the costs.

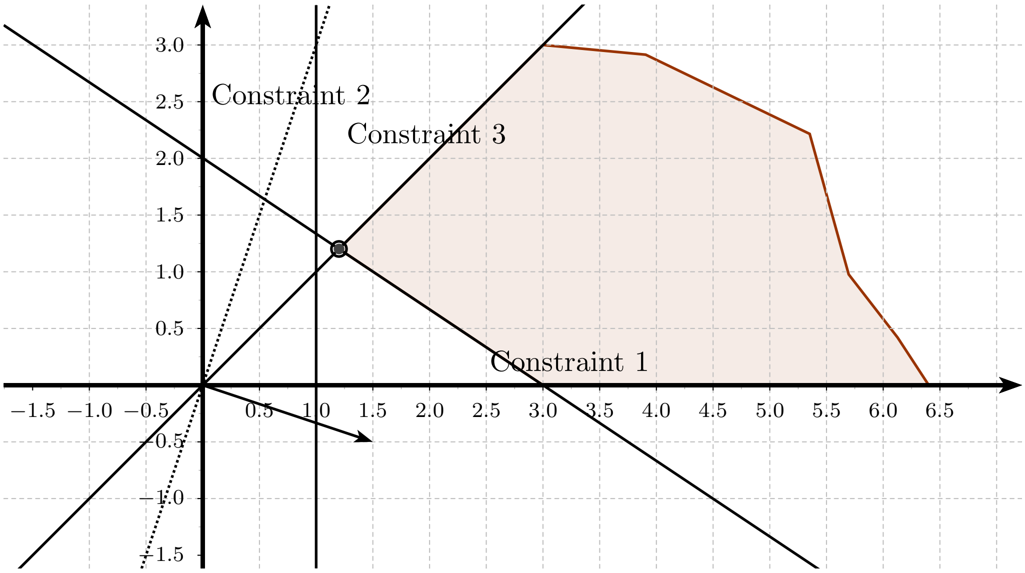
Exercise 3 (value 6).
Consider the following ILP and solve it using the cutting plane method. Use the Bland's rule when applying the simplex algorithm. Generate the Gomory's cut from the first (top) admissible row. Perform at most two Gomory's iterations.

$$\begin{array}{rcl} \min z & = & -4x_1 - 3x_2 \\ & & 5x_1 + 7x_2 \le 14 \\ & & 2x_1 + 3x_2 \le 8 \\ & & x_1, x_2 \ge 0 \text{ integer} \end{array}$$

Draw in the grid in the next page: (a) the feasible region, (b) the added Gomory's cuts, (c) the optimal LP and ILP solutions.

Exercise 1

$$\begin{array}{ll} \max & 6x_1 + x_2 \\ & 2x_1 + x_2 + 2x_3 \leq 3 \\ & -3x_1 + 2x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array} \qquad \begin{array}{ll} \max & 6x_1 + x_2 \\ & 2x_1 + x_2 + 2x_3 \leq 3 \\ & 3x_1 - 2x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{array} \qquad \begin{array}{ll} \min & 3u_1 - u_2 \\ & 2u_1 + 3u_2 \geq 6 \\ & u_1 \geq 1 \\ & 2u_1 - 2u_2 \geq 0 \\ & u_1, u_2 \geq 0 \end{array}$$



[a] The second constraint is redundant, so the associated slack variable must be always non-null. Therefore the complementary slackness conditions $(u^T A_j - c^T) * x = 0$ indicate that the second primal variable is always null: $x_2 = 0$.

[b] Optimal dual solution $u_1 = u_2 = 6/5$, $z_D = 12/5$

$$\begin{cases} x_2 = 0 \\ 6x_1 + x_2 = 12/5 \\ 2x_1 + x_2 + 2x_3 = 3 \end{cases} \qquad \begin{cases} x_2 = 0 \\ x_1 = 2/5 \\ x_3 = 11/10 \end{cases}$$

The primal solution obtained using the three given conditions satisfies both primal constraints and gives a primal solution with the same value of the optimal dual solution, so it is feasible and optimal.

Exercise 2

Variables

$$
\begin{aligned}
x_t &= \text{amount of energy bought in period } t \in T. \\
y_t &= \text{amount of energy sold in period } t \in T. \\
\gamma &= \text{number of power stations bought} \\
I_t &= \text{amount of MW stored in period } t \in T \cup \{0\}. \\
\delta_t &= \text{1 if energy is bought in period } t \in T,\ \text{0 otherwise.} \\
r_t &= \text{amount of energy produced by the new generator in period } t \in T. \\
\alpha_t &= \text{1 if generator is activated at the beginning of period } t \in T,\ \text{0 otherwise.}
\end{aligned}
$$

Constant

$$M = \text{a big number, e.g. } \sum_{t \in T} d_t$$

$$
\begin{aligned}
\min\ z = \sum_{t \in T} (e_t x_t - f_t y_t) + C_{PS}\gamma + \sum_{t \in T} (C r_t + F\alpha_t) & & (25) \\
\text{s.t. } x_t + b_t + I_{t-1} + r_t = I_t + d_t + y_t & \quad t \in T & (26) \\
I_0 = 0 & & (27) \\
I_t \leq W\ \gamma & \quad t \in T & (28) \\
x_t \leq M\delta_t & \quad t \in T & (29) \\
y_t \leq M(1 - \delta_t) & \quad t \in T & (30) \\
\sum_{t \in T} \alpha_t \leq 1 & \quad t \in T & (31) \\
r_t \leq \overline{M}\ \sum_{i=0}^{t} \alpha_i & \quad t \in T & (32) \\
r_t \geq \underline{M}\ \sum_{i=0}^{t} \alpha_i & \quad t \in T & (33) \\
x_t, y_t, r_t \geq 0 & \quad t \in T & (34) \\
I_t \geq 0 & \quad t \in T \cup \{0\} & (35) \\
\gamma \geq 0 \text{ integer} & & (36) \\
\delta_t, \alpha_t, \in \{0, 1\} & \quad t \in T. & (37)
\end{aligned}
$$

Exercise 3

x_1	x_2	x_3	x_4		
-4	-3	0	0	0	$-z$
5	7	1	0	14	x_3
2	3	0	1	8	x_4

x_1	x_2	x_3	x_4		
0	$\frac{13}{5}$	$\frac{4}{5}$	0	$\frac{56}{5}$	$-z$
1	$\frac{7}{5}$	$\frac{1}{5}$	0	$\frac{14}{5}$	x_1
0	$\frac{1}{5}$	$-\frac{2}{5}$	1	$\frac{12}{5}$	x_4

Gomory's cut : $\frac{2}{5}x_2 + \frac{1}{5}x_3 \geq \frac{4}{5}$

x_1	x_2	x_3	x_4	x_5		
0	$\frac{13}{5}$	$\frac{4}{5}$	0	0	$\frac{56}{5}$	$-z$
1	$\frac{7}{5}$	$\frac{1}{5}$	0	0	$\frac{14}{5}$	x_1
0	$\frac{1}{5}$	$-\frac{2}{5}$	1	0	$\frac{12}{5}$	x_4
0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	1	$-\frac{4}{5}$	x_5

x_1	x_2	x_3	x_4	x_5		
0	1	0	0	4	8	$-z$
1	1	0	0	1	2	x_1
0	1	0	1	-2	4	x_4
0	2	1	0	-5	4	x_3

Gomory's cut : $\frac{2}{5}x_2 + \frac{1}{5}x_3 \geq \frac{4}{5} \rightarrow \frac{2}{5}x_2 + \frac{1}{5}(14 - 5x_1 - 7x_2) \geq \frac{4}{5} \rightarrow 5x_1 + 5x_2 \leq 10$