RICERCA OPERATIVA

Esercises 1, Summer 2022 **A**

Last name, First name ____________________

Exercise 1 (value 12)

Market basket analysis is a techniques used by the manager of large stores (supermarket and hypermarket) to identify the choices of the customers, in terms of groups of products often bought together. Let N be the set of the n items available in a store. Each item $i \in N$ belongs to a class $c_i \in C$. The $n \times n$ matrix F reports, for each pair of items (i, j), the frequency the pair is present in a basket of a customer (higher is the value F_{ij}, higher is the probability that the customer that buy i also buy j).

The manager of the store wants to identify the subset of items that maximizes the total frequency (i.e., the sum of the frequencies of all pairs in the selected subset). Write a linear mathematical model to help the manager to find such a subset.
(*Suggestion: use two sets of boolean variables: one to know if an item is selected and one to know if a pair of items is selected.*)

Update the above model by adding the following constraint: if an item of a class $c \in C$ is selected, at most 3 items from the class must be selected, independently of the frequencies.

Answer

$y_i = 1$ if item i is selected, 0 otherwise
$x_{ij} = 1$ if both i and j are selected, 0 otherwise

$$
\begin{array}{llll}
\max & \sum_{i \in N} \sum_{j \in N} F_{ij} x_{ij} & & \\
& 2x_{ij} & \leq y_i + y_j & i, j \in N \\
& \sum_{i \in N : c_i = c} y_i & \leq 3 & c \in C \\
& y_i & \in \{0, 1\} & i \in N \\
& x_{ij} & \in \{0, 1\} & i, j \in N
\end{array}
$$

Exercise 2 (value 8)
Consider the following LP problem:

$$\begin{aligned} \min \quad & 12x_1 - 8x_2 + 14x_3 \\ & 4x_1 + 4x_2 + 2x_3 \leq 3 \\ & -x_1 - 2x_2 + x_3 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Write the corresponding dual problem without changing the direction of the inequalities.
- Write the complementary slackness conditions.
- Given the dual solution $\overline{u} = (0, 14)$, verify if it is the optimal solution by using the complementary slackness. Motivate your answer.

Answer

$$\begin{aligned} \max \quad & 3u_1 + 8u_2 \\ & 4u_1 - u_2 \leq 12 \\ & 4u_1 - 2u_2 \leq -8 \\ & 2u_1 + u_2 = 14 \\ & u_1 \leq 0 \\ & u_2 \geq 0 \end{aligned}$$

$$\left\{\begin{aligned} (4x_1 + 4x_2 + 2x_3 - 3)u_1 &= 0 \\ (-x_1 - 2x_2 + x_3 - 8)u_2 &= 0 \\ (4u_1 - u_2 - 12)x_1 &= 0 \\ (4u_1 - 2u_2 + 8)x_2 &= 0 \\ (2u_1 + u_2 - 14)x_3 &= 0 \end{aligned}\right. \qquad \left\{\begin{array}{l} (4x_1 + 4x_2 + 2x_3 - 3)0 = 0 \\ x_3 = 8 \\ x_1 = 0 \\ x_2 = 0 \\ (0)x_3 = 0 \end{array}\right. \qquad \left\{\begin{array}{ll} -- & \\ x_3 = & 8 \\ x_1 = & 0 \\ x_2 = & 0 \\ -- & \end{array}\right.$$

The dual solution is not the optimal solution since the corresponding primal solution does not satisfy the first constraint.

Exercise 3 (value 7)
Find the optimal solution of the following knapsack problem using a Dynamic Programming method, without reordering the items:
$n = 4, c = 7$, $(p_j, w_j) = [(18, 7), (22, 4), (12, 2), (18, 4)]$

Answer

	0	1	2	3	4	5	6	7
f_0	0	0	0	0	0	0	0	0
f_1	0	0	0	0	0	0	0	18
f_2	0	0	0	0	22	22	22	22
f_3	0	0	12	12	22	22	34	34
f_4	0	0	12	12	22	22	34	34

	0	1	2	3	4	5	6	7
J_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
J_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{1\}$
J_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
J_3	$\emptyset$	$\emptyset$	$\{3\}$	$\{3\}$	$\{2\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$
J_4	$\emptyset$	$\emptyset$	$\{3\}$	$\{3\}$	$\{2\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$

The optimal solution $x = \{2, 3\}$ has profit 34 and weight 6.