Written assessment, June 10, 2019

**Last name, First name** ____________________________

**Exercise 1** (value 13)

A multi-utility company is reorganizing working teams to handle customers requests more efficiently. A set $I$ of workers is given and has to be partitioned into teams, so that each worker is assigned to a unique team.

The set $K$ represents the set of proficiencies in different fields related to the multi-utility core business, including proficiencies on water distribution, electric distribution, waste management, etc. The human resource manager (HR) stated that each worker $i \in I$ has a value $a_{ik}$ of proficiency on field $k \in K$, with $a_{ik}$ ranging from 0 (no proficiency) to 5 (full proficiency).

Overall, for each team, at least a value 3 of proficiency for each field $k \in K$ is required.

Moreover, each team must be composed of a number of workers between $l$ and $u$.

The company wants to minimize the number of teams while respecting the given characteristics. HR also reported that each worker have preferences on the other members of the team: in particular, $c_{ih}$ $i \in I, h \in I, i \neq h$ is the preference of worker $i$ to be put in the same team of worker $h$. Among all solutions with the minimum number of teams, select the one with the highest amount of overall preference.

Write a MILP model including all the given specifications.

After checking the first results of the model, HR asks to include the following constraint for a better distribution of proficiencies among the teams: there must be at least one component of the team having at least one proficiency that equals 5, or at least two components having at least a proficiency with value 3. (*Suggestion: consider that set of workers with special skills can be defined by a preprocessor, and used by the model as data.)*

**Exercise 2** (value 9)

Consider the following ILP problem. Solve it with the standard branch-and-bound method. Solve the linear relaxation graphically and perform the first branching on the $x_2$ variable.

$$\begin{aligned}
\max \quad & 2x_1 + 3x_2 \\
& 6x_1 + 17x_2 \leq 102 \\
& 3x_1 + 5x_2 \geq 15 \\
& 3x_1 - 4x_2 \leq 12 \\
& 4x_1 - x_2 \geq 2 \\
& x_1, x_2 \geq 0, integer
\end{aligned}$$

Consider the above model and relax the integrality constraints. Imagine you had to apply the two-phase method to the obtained LP problem:

(i) Write the auxiliary problem of the first phase inserting the **minimum possible number of auxiliary variables**.

(ii) Use the graphic of the ILP to show one possible path of the two-phase algorithm, with respect to the solutions at each iteration, up to the optimal continuous solution.

**Exercise 3** (value 6)
Write a GLPK or XPRESS model corresponding to the following mathematical model

$$\max\ z = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}x_{ij} - \sum_{j=1}^{m} w_j y_j$$

$$\sum_{i=1}^{n} c_i x_{ij} \leq C y_j \qquad j = 1,\dots,m \tag{1}$$

$$\sum_{j=1}^{m} x_{ij} = 1, \qquad i = 1,\dots,m \tag{2}$$

$$\sum_{j \in F_i} x_{ij} = 0, \qquad i = 1,\dots,m \tag{3}$$

$$x_{ij} \in \{0,1\} \qquad i = 1,\dots,n,\ j = 1,\dots,m \tag{4}$$

$$y_j \in \{0,1\} \qquad j = 1,\dots,m \tag{5}$$

**Exercise 1**

## Constants

$T =$ set of all the possible teams ( if no upper bound on the number of teams is known $|T| = |I|$)
$I3 \subseteq I =$ set of workers with at least two proficiencies with value that equal to 3
$I5 \subseteq I =$ set of workers with at least a proficiency equal to 5
$Q = \sum_{i,h \in I, i \neq h} c_{ih}$

## Variables

$x_{ij} = 1$ if worker $i \in I$ is assigned to team $j \in T$, 0 otherwise.
$y_j = 1$ if team $j \in T$ is created, 0 otherwise.
$\theta^j_{ih} = 1$ if both workers, $i \in I$ and $h \in I$ $(i \neq h)$ are assigned to team $j \in T$, 0 otherwise.
$z_j = 1$ if at team $j$ is assigned at least one worker from $I5$, 0 otherwise.
$w_j = 1$ if at team $j$ are assigned at least two workers from $I3$, 0 otherwise.

## Model

$$\min Q \sum_{j \in T} y_j - \sum_{i \in I} \sum_{h \in I, i \neq h} \sum_{j \in T} c_{ih} \theta^j_{ih}$$

$$\sum_{j \in T} x_{ij} = 1 \quad i \in I$$

$$\sum_{i \in I} a_{ik} x_{ij} \geq 3 y_j \quad k \in K, j \in T$$

$$\sum_{i \in I} x_{ij} \leq u y_j \quad j \in T$$

$$\sum_{i \in I} x_{ij} \geq l y_j \quad j \in T$$

$$(x_{ij} + x_{hj}) \geq 2\theta^j_{ih} \quad i, h \in I, i \neq h, j \in T$$

$$(x_{ij} + x_{hj}) \leq \theta^j_{ih} + 1 \quad i, h \in I, i \neq h, j \in T$$

$$\sum_{i \in I5} x_{ij} \geq z_j \quad j \in T$$

$$\sum_{i \in I3} x_{ij} \geq 2 w_j \quad j \in T$$

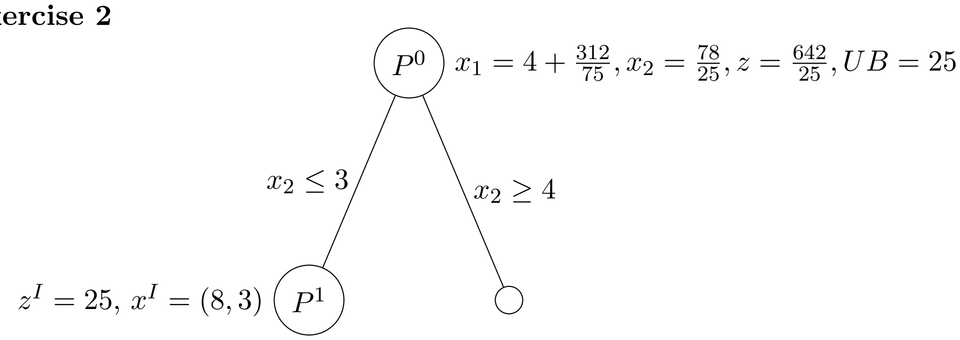
$$z_j + w_j = y_j \quad j \in T$$

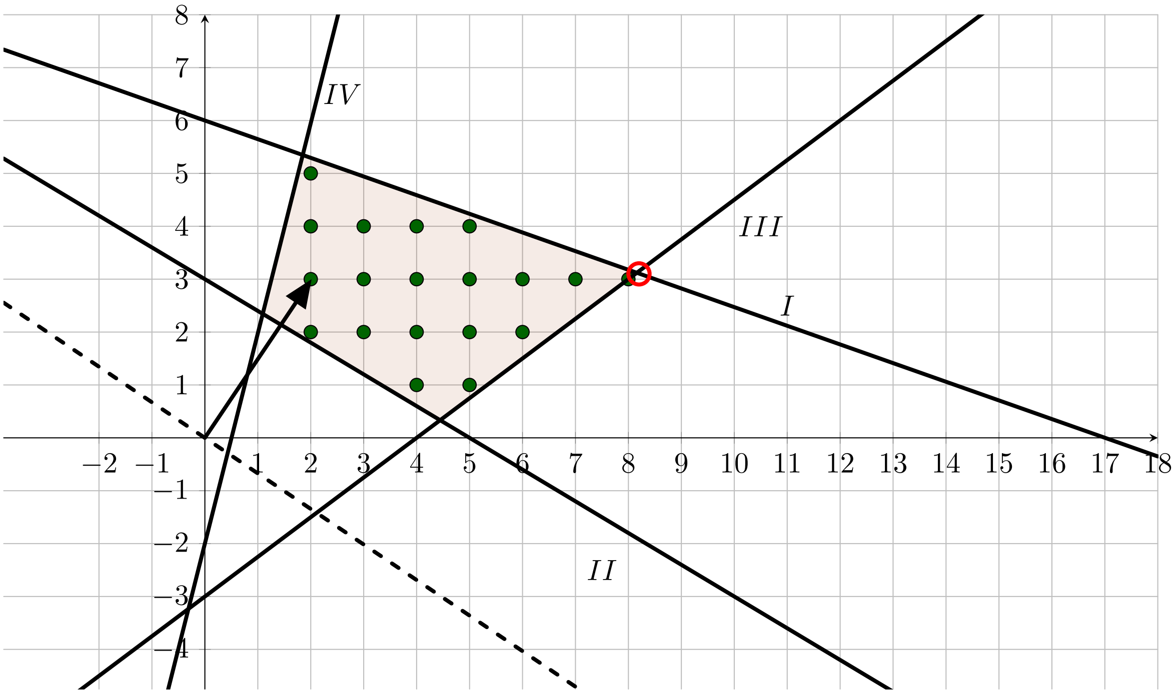
$$x_{ij} \in \{0, 1\} \quad i \in I, j \in T$$

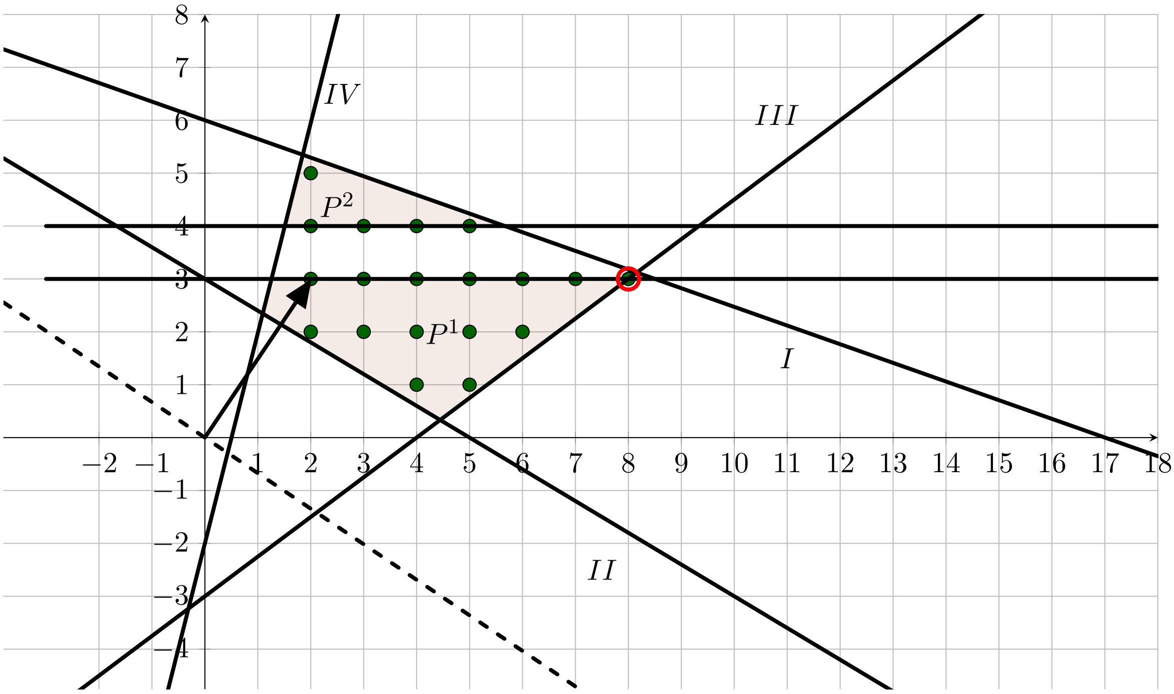
$$y_j, z_j, w_j \in \{0, 1\} \quad j \in T$$

$$\theta^j_{ih} \in \{0, 1\} \quad i, h \in I, i \neq h, j \in T$$

**Exercise 2**


$P^0$
$x_1 = 4 + \frac{312}{75}, x_2 = \frac{78}{25}, z = \frac{642}{25}, UB = 25$
$x_2 \leq 3$
$x_2 \geq 4$
$z^I = 25,\ x^I = (8,3)$
$P^1$


$IV$
$III$
$I$
$II$


$IV$
$III$
$P^2$
$P^1$
$I$
$II$

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/* Exercise 3, 2016 09 12 */

param n, integer, > 0;
param m, integer, > 0;
set I := 1..n;
set J := 1..m;
set F{i in I};

param p{i in I, j in J}, >= 0;
param c{i in I}, >= 0;
param w{j in J}, >= 0;
param C, >= 0;

var  x{i in I, j in J},  binary;
var  y{j in J},  binary;

maximize z: sum{i in I, j in J} p[i,j]*x[i,j]-sum{j in J} w[j]*y[j];

s.t.  cap{j in J}:  sum{i in I} c[i]*x[i,j] <=   C*y[j];
   one{i in I}: sum{j in J} x[i,j] = 1;
   forbidden{i in I}: sum{j in F[i]} x[i,j] = 0;
solve;

printf "\n";
for{i in I} {
   printf "\n%1d)",i;
   printf{j in J} "%5d ", x[i,j];
}
printf "\n\n-----z = %g\n\n",z;
printf "\n\n  ";

data;

param n := 6;
param m := 3;

param C := 50;
param c:= [1] 10 [2] 22 [3] 5 [4] 14 [5] 9 [6] 11;
param w:= [1] 20 [2] 30 [3] 15 ;
param p :  1  2 3  :=
1  5  2 3
2  6  2 4
3  2  5 1
4  2  3 1
5  4  2 9
6  3  8 4;
set F[1] := 1 ;
set F[2] := 2;
set F[3] := 1 3;
set F[4] := 3;
set F[5] := 3;
set F[6] := 2 3;
end;
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