Operations Research, January 20, 2016

A

First and Last name ______________________________

Exercise 1 (value 8)
Consider the following PLC problem and solve it using the Two Phases Primal Simplex algorithm adding the minimum number of artificial variables.

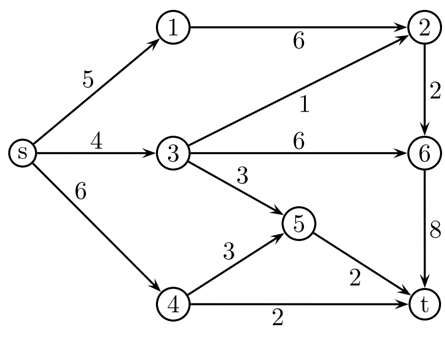
$$\begin{aligned} \min \quad & -3x_1 - 3x_2 \\ & -3x_1 + 2x_2 + x_3 \leq 10 \\ & 6x_1 + x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

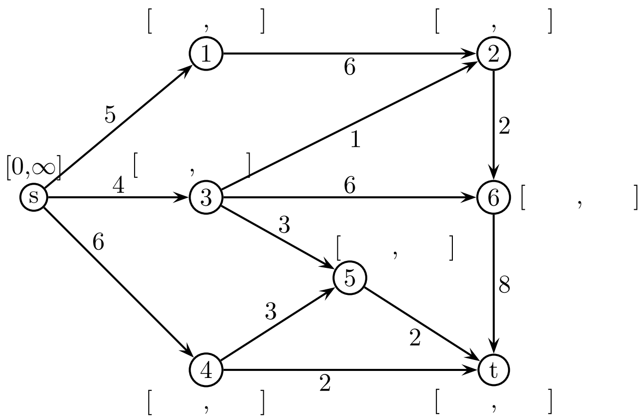
Write the dual problem and find the corresponding optimal solution, if any.

Exercise 2 (value 10).
A manufacturing company has a production process consisting of three stages feed by a unique depot of raw material holding a total quantity of M Kgs. The raw material is first transformed by n departments. All the departments of the first stage produce the same semifinished product, but have some differences in the internal working process, so that they use different quantities of raw material. Let r_i $(i = 1, \ldots, n)$ denote the number of Kgs of raw material that department i uses to produce one unit of the semifinished product. The department i can produce at most Q_i product in a time unit. The semifinished product undergoes a second operation executed by one among m departments. Each department j $(j = 1, \ldots, m)$ of the second stage can produce at most A_j products in the time unit. After the second stage the resulting products can be either sold or sent to a third stage where there is a unique department which can work at most B products in the unit time. A product sold after the second stage gives an income of p_S Euro, while a product sold after the third stage gives an income of p_T Euro. The cost of one Kg of raw material is c_r Euro, the cost of the work of department i of the first stage, to produce a semifinished product is c_i^I Euro, while each product worked by a second stage department j costs c_j^{II} Euro. Finally the cost of one product worked in the third stage is c^{III}.

Write a Linear Programming model to help the company to define the production plan for one time unit that maximizes the total gain (income - costs). Modify the model by adding a constraint imposing that if a second stage department sells products, than it must sell at least θ products.

Exercise 3 (value 7)
Find the maximum flow from s to t, in the graph of the figure. At each iteration, in case of tie, chose the vertex with minimum index for the next labeling phase. Identify on the graph the minimum cut.

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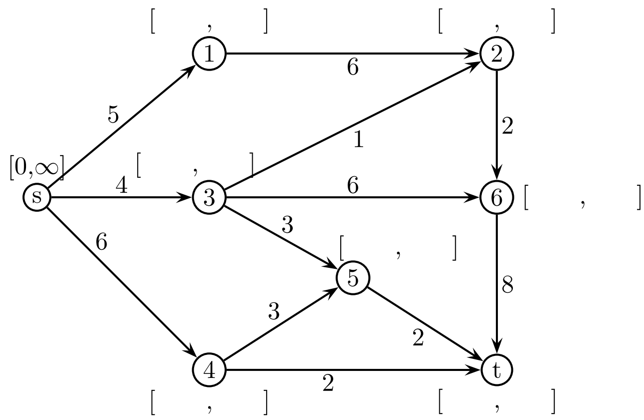
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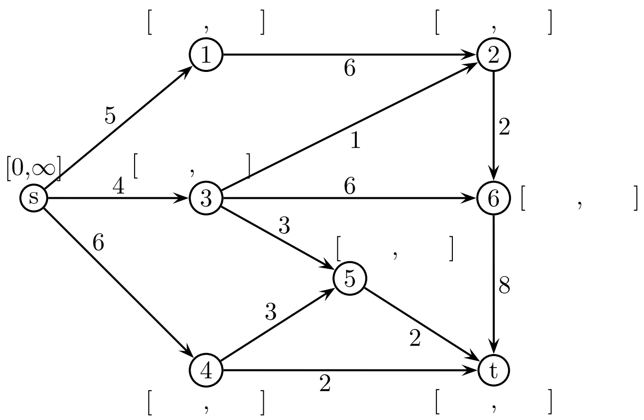
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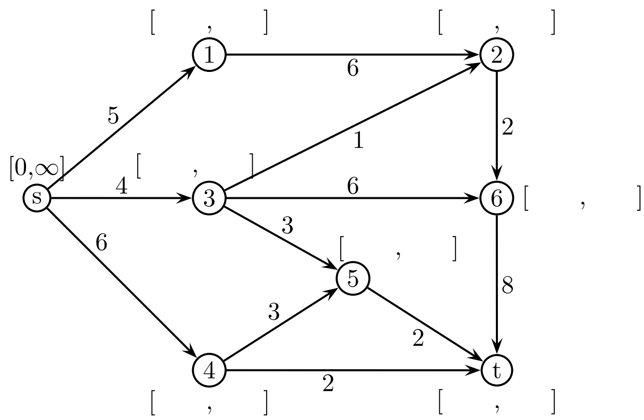
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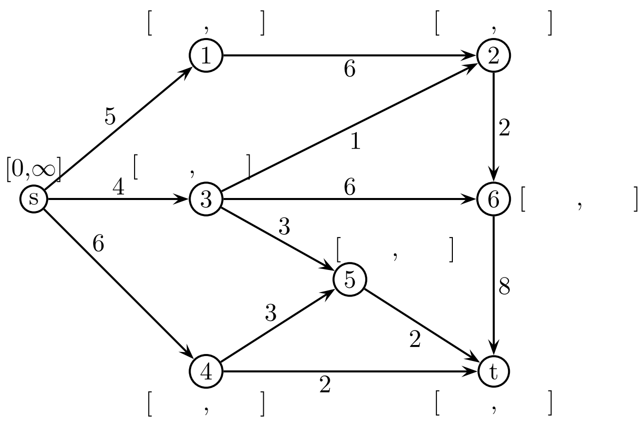
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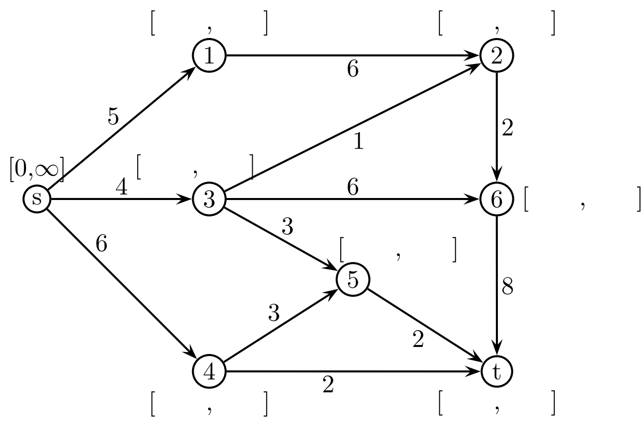
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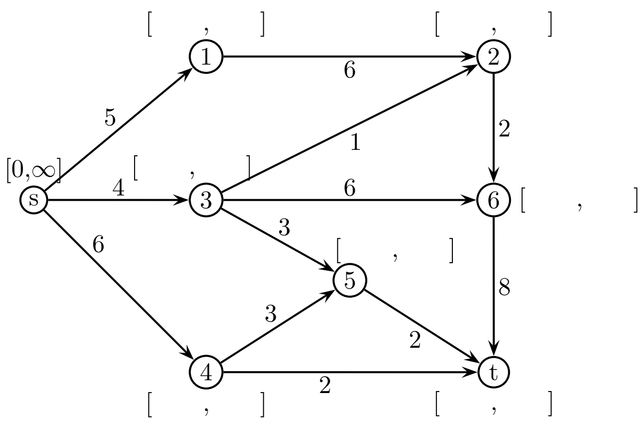
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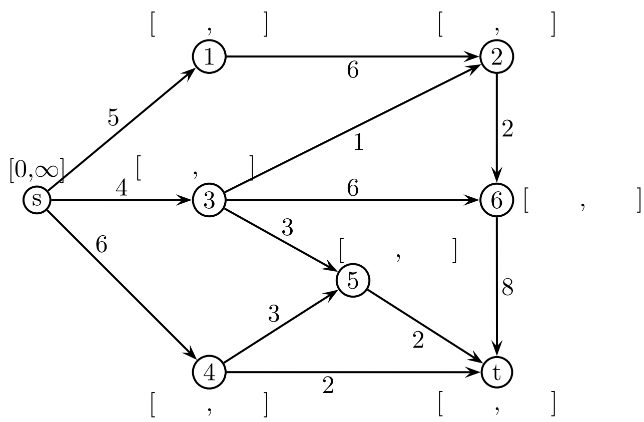
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Exercise 1

$$\begin{aligned} \min \quad & -3x_1 - 3x_2 \\ & -3x_1 + 2x_2 + x_3 \leq 10 \\ & 6x_1 + x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

FASE I

x_1	x_2	x_3	x_4	x_5	x_6		
-6	-1	-1	0	1	0	0	$-z$
-3	2	1	1	0	0	10	x_4
6	1	1	0	-1	1	6	x_6

x_1	x_2	x_3	x_4	x_5	x_6		
0	0	0	0	0	1	6	$-z$
0	$\frac{5}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	13	x_4
1	$\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	1	x_1

FASE II

x_1	x_2	x_3	x_4	x_5		
0	$-\frac{5}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-z$
0	$\frac{5}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	13	x_4
1	$\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	1	x_1

PROBLEM UNLIMITD

$$\begin{aligned} \min \quad & -3x_1 - 3x_2 \\ & -3x_1 + 2x_2 + x_3 \leq 10 \\ & 6x_1 + x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & -10u_1 + 6u_2 \\ & 3u_1 + 6u_2 \leq -3 \\ & -2u_1 + u_2 \leq -3 \\ & -u_1 + u_2 \leq 0 \\ & u_1, u_2 \geq 0 \end{aligned}$$

The dual problem has no solution. Consider, e.g., the first constraint: it cannot be satisfied by non negative values of the variables.

Exercise 2

$$\begin{aligned}
x_i &= \text{products worked by dep. } i \text{ of the first stage}\\
y_{ij} &= \text{products transferred from dep. } i \text{ (first stage) to dep. } j \text{ (second stage)}\\
w_j &= \text{product of dep. } j \text{ sold after second stage}\\
z_j &= \text{products sent to the third stage from dep. } j \text{ of second stage}\\
\\
\delta_j &= 1 \text{ if department } j \text{ sells products; 0 otherwise}
\end{aligned}$$

$$\max \sum_{j=1}^{m}(p_S w_j + p_T z_j) - \sum_{i=1}^{n}(c_r r_i + c^I)x_i - \sum_{j=1}^{m} c^{II}(w_j + z_j) - c^{III}\sum_{j=1}^{m} z_j \tag{1}$$

$$\sum_{i=1}^{n} r_i x_i \leq M \tag{2}$$

$$\sum_{j=1}^{m} y_{ij} = x_i \quad i = 1, \ldots, n \tag{3}$$

$$\sum_{i=1}^{n} y_{ij} = w_j + z_j \quad j = 1, \ldots, m \tag{4}$$

$$w_j + z_j \leq A_j \quad j = 1, \ldots, m \tag{5}$$

$$\sum_{j=1}^{m} z_j \leq B \tag{6}$$

$$w_j \geq \theta\delta_j \quad j = 1, \ldots, m \tag{7}$$

$$w_j \leq A_j\delta_j \quad j = 1, \ldots, m \tag{8}$$

$$0 \leq x_i \leq Q_i, \text{ integer} \quad i = 1, \ldots, n \tag{9}$$

$$y_{ij} \geq 0, \text{ integer} \quad i = 1, \ldots, n; j = 1, \ldots, m \tag{10}$$

$$w_j, z_j \geq 0, \text{ integer } j = 1, \ldots, m \tag{11}$$

$$\tag{12}$$