Written assessment, January 15, 2024

Last name, First name ________________________________

Exercise 1 (value 8)

A company that produces m types of cakes can produce between L_j and U_j boxes of cake of type j, $j \in \{1, 2, \ldots, m\}$, with the additional constraint that if at least T_A boxes of cake A are produced, then no more than T_B boxes of B can be produced.

A set of n ingredients is used for the preparation of the cakes. To prepare a box of cake j, the company uses q_{ij} gram of ingredient i. The company has in stock Q_i gram of each ingredient i. If the stock is not enough, the company can buy ingredients on the market with a cost of c_i per gram for each ingredient i. Producing a box of cake j generates an income of r_j.

Write a model to help the company to design am optimized production plan that maximized the difference between incomes and costs.

Exercise 2 (value 12)

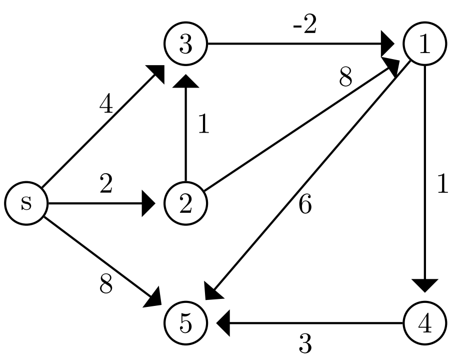
Consider the following PLC problem.

$$\begin{aligned} \min \quad & 12x_1 + 6x_2 - 2x_3 \\ & x_1 \qquad\quad + 3x_3 = 15 \\ & 3x_1 - 2x_2 + 5x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- Solve the with the simplex method
- Write the dual of the problem
- Find the optimal dual solution using the complementary slackness conditions
- What happens to the optimal (primal) basis if the objective function coefficient of variable x_1 becomes 14?

Exercise 3 (value 7)

Consider the following graph $G = (V, A)$. Find the shortest path from s to 5 using an algorithm of your choice among those seen during the course. Clearly state the name of the algorithm used, show the solving procedure and the final solution with the associated cost.



RICERCA OPERATIVA

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Solution sketch

Exercise 1

Variables

- x_j: boxes of cake j produced
- y_i: quantity of ingredient i bought from the market
- δ: takes value 1 if at least T_A boxes of A are produced, 0 otherwise

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{m} r_j x_j - \sum_{i=1}^{n} c_i y_i \\
& \sum_{j=1}^{m} q_{ij} x_j \le Q_i + y_i \quad i \in \{1, 2, \dots, n\} \\
& x_A \le T_A + (U_A - T_A)\delta \\
& x_B \le T_B + (U_B - T_B)(1 - \delta) \\
& L_j \le x_j \le U_j \text{ integer} \quad j \in \{1, 2, \dots, m\} \\
& y_i \ge 0 \quad i \in \{1, 2, \dots, n\} \\
& \delta \in \{0, 1\}
\end{aligned}
$$

Exercise 2

Phase 1

x_1	x_2	x_3	s_1	a_1		
-1	0	-3	0	0	-15	$-z$
1	0	3	0	1	15	a_1
3	-2	5	1	0	18	s_1

x_1	x_2	x_3	s_1	a_1		
0	-2/3	-4/3	1/3	0	-9	$-z$
0	2/3	4/3	-1/3	1	9	a_1
1	-2/3	5/3	1/3	0	6	x_1

x_1	x_2	x_3	s_1	a_1		
0	0	0	0	1	0	$-z$
0	1	2	-1/2	3/2	27/2	x_2
1	0	3	0	1	15	x_1

Phase 2

x_1	x_2	x_3	s_1		
12	6	-2	0	0	$-z$
0	1	2	-1/2	27/2	x_2
1	0	3	0	15	x_1

x_1	x_2	x_3	s_1		
0	0	-50	3	-261	$-z$
0	1	2	-1/2	27/2	x_2
1	0	3	0	15	x_1

x_1	x_2	x_3	s_1		
50/3	0	0	3	-11	$-z$
-2/3	1	0	-1/2	7/2	x_2
1/3	0	1	0	5	x_3

$x = (0, 7/2, 5), z_P = 11$

Dual:

$$\begin{aligned} \max \quad & 15u_1 + 18u_2 \\ & u_1 + 3u_2 \leq 12 \\ & -2u_2 \leq 6 \\ & 3u_1 + 5u_2 \leq -2 \\ & u_2 \leq 0 \end{aligned}$$

Dual solution through slackness conditions:

$$\begin{cases} (x_1 + 3x_3 - 15)u_1 = 0 \\ (3x_1 - 2x_2 + 5x_3 - 18)u_2 = 0 \\ (u_1 + 3u_2 - 12)x_1 = 0 \\ (-2u_2 - 6)x_2 = 0 \\ (3u_1 + 5u_2 + 2)x_3 = 0 \end{cases} \rightarrow \begin{cases} -2u_2 = 6 \\ 3u_1 + 5u_2 = -2 \end{cases} \rightarrow \begin{cases} u_2 = -3 \\ 3u_1 = 13 \end{cases} \rightarrow \begin{cases} u_2 = -3 \\ u_1 = 13/3 \end{cases}$$

$u = (13/3, -3), z_D = 11$

The variable x_1 is not in the optimal basis. If c_1 increases from 12 to 14, x_1 becomes less attractive for the optimization and it remains out of the basis. The basis, in turn, does not change.

Exercise 3

Since we have arcs with negative costs, we use the Belmann-Ford algorithm.

Iter	$f(j)$						pred					
	s	1	2	3	4	5	s	1	2	3	4	5
0	0	-	-	-	-	-	s	-	-	-	-	-
1	0	-	2	4	-	8	s	-	s	s	-	s
2	0	2	2	3	-	8	s	3	s	2	-	s
3	0	1	2	3	3	8	s	3	s	2	1	s
4	0	1	2	3	2	6	s	3	s	2	1	4
5	0	1	2	3	2	5	s	3	s	2	1	4

Optimal path: $s \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 4 \rightarrow 5$. Cost: 5.