Part 6

Mauro Dell'Amico

DISMI, Universitá di Modena e Reggio Emilia
mauro.dellamico{at}unimore.it
www.or.unimore.it

Matrix notation

$$
\begin{array}{llllllll}
\max\ z = & 80x_1 & +70x_2 & & & & \\
\text{s.t} & 3x_1 & +2x_2 & +x_3 & & = & 15 \\
 & 2x_1 & +3x_2 & & +x_4 & = & 15 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
$$

Matrix notation

$$
\begin{array}{llllll}
\max \ z = & 80x_1 & +70x_2 & & & \\
\text{s.t} & 3x_1 & +2x_2 & +x_3 & & = 15 \\
 & 2x_1 & +3x_2 & & +x_4 & = 15 \\
 & x_1, & x_2, & x_3, & x_4 & \geq 0
\end{array}
$$

$$c^T = [80, 70, 0, 0], \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Matrix notation

$$
\begin{array}{llllll}
\max \ z = & 80x_1 & +70x_2 & & & \\
\text{s.t} & 3x_1 & +2x_2 & +x_3 & & = 15 \\
 & 2x_1 & +3x_2 & & +x_4 & = 15 \\
 & x_1, & x_2, & x_3, & x_4 & \geq 0
\end{array}
$$

$$c^T = [80, 70, 0, 0], \ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$z = \sum_{j=1}^{4} c_j x_j = 80_1 x_1 + 70x_2 + 0x_3 + 0x_4 = c^T x$$

$$
\begin{array}{llllllll}
\max \ z = & 80_1 x & +70x_2 & & & & & \\
\text{s.t} & 3x_1 & +2x_2 & +x_3 & & = & 15 \\
 & 2x_1 & +3x_2 & & +x_4 & = & 15 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 15 \\ 15 \end{bmatrix}
$$

$$
\begin{array}{llllllll}
\max \ z = & 80_1 x & +70x_2 & & & & & \\
\text{s.t} & 3x_1 & +2x_2 & +x_3 & & = & 15 \\
 & 2x_1 & +3x_2 & & +x_4 & = & 15 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 15 \\ 15 \end{bmatrix}
$$

first row $\sum_{j=1}^{4} a_{1j} x_j = b_1 \Leftrightarrow 3x_1 + 2x_2 + x_3 (+0x_4) = 15$

second row $\sum_{j=1}^{4} a_{2j} x_j = b_2 \Leftrightarrow 2x_1 + 3x_2 + (0x_3) + x_4 = 15$

$$
\begin{array}{llllll}
\max\ z = & 80_1 x & +70x_2 & & & \\
\text{s.t} & 3x_1 & +2x_2 & +x_3 & & = & 15 \\
 & 2x_1 & +3x_2 & & +x_4 & = & 15 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
$$

$$
A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 15 \\ 15 \end{bmatrix}
$$

$$
\text{first row } \sum_{j=1}^{4} a_{1j}x_j \quad = b_1 \ \Leftrightarrow 3x_1 + 2x_2 + x_3(+0x_4) = 15
$$

$$
\text{second row } \sum_{j=1}^{4} a_{2j}x_j \quad = b_2 \ \Leftrightarrow 2x_1 + 3x_2 + (0x_3) + x_4 = 15
$$

$$
\boxed{\mathrm{Ax} = \mathrm{b}}
$$

A generic PLC problem in standard form in matrix notation is:

$$
\max\{c^T x : Ax = b, x \geq 0\} \ (\text{Let } m = \text{rank}(A))
$$

A generic PLC problem in standard form in matrix notation is:

$$\max\{c^T x : Ax = b, x \geq 0\} \text{ (Let } m = \text{rank}(A))$$

- Basis of A: collection B of m linearly independent columns ;

A generic PLC problem in standard form in matrix notation is:

$$\max\{c^T x : Ax = b, x \geq 0\} \text{ (Let } m = \text{rank}(A))$$

- Basis of A: collection B of m linearly independent columns ;
  basic variables: variables x_j associated with the columns of B
  non basic variables: variables associated to $A \backslash B$

A generic PLC problem in standard form in matrix notation is:

$$\max\{c^T x : Ax = b, x \geq 0\} \text{ (Let } m = \text{rank}(A))$$

- Basis of A: collection B of m linearly independent columns ;
  basic variables: variables x_j associated with the columns of B
  non basic variables: variables associated to $A \backslash B$

$$A = [A_1, \dots, A_n] \Rightarrow A = [B, F] \;\; \text{con} \;\; B = [A_1, \dots, A_m]$$

$$x = \begin{bmatrix} x_B \\ x_F \end{bmatrix}$$

$$Ax = b \;\; \Rightarrow \;\; Bx_B + Fx_F = b \;\; \Rightarrow \;\; x_B = B^{-1}b - B^{-1}Fx_F$$

A generic PLC problem in standard form in matrix notation is:

$$\max\{c^T x : Ax = b, x \geq 0\} \text{ (Let } m = \text{rank}(A))$$

- Basis of A: collection B of m linearly independent columns ;
  basic variables: variables x_j associated with the columns of B
  non basic variables: variables associated to $A \backslash B$

$$A = [A_1, \dots, A_n] \Rightarrow A = [B, F] \;\; \text{con} \;\; B = [A_1, \dots, A_m]$$

$$x = \begin{bmatrix} x_B \\ x_F \end{bmatrix}$$

$$Ax = b \;\; \Rightarrow \;\; Bx_B + Fx_F = b \;\; \Rightarrow \;\; x_B = B^{-1}b - B^{-1}Fx_F$$

- Basic solution: $x_F = 0, \quad x_B = B^{-1}b$

  feasible: $\quad x_B = B^{-1}b \geq 0$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_F = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_F = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_F = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

$$B^{-1}b = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$Ax = b \quad \Rightarrow \quad Bx_B + Fx_F = b \quad \Rightarrow \quad B^{-1}Bx_B + B^{-1}Fx_F = B^{-1}b$$

$$x_B = B^{-1}b - B^{-1}Fx_F$$

$$x_B = B^{-1}b - B^{-1}Fx_F$$

$$x_B = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$x_B = B^{-1}b - B^{-1}Fx_F$$

$$x_B = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$\begin{cases} x_1 = & 3 & -\frac{3}{5}x_3 & +\frac{2}{5}x_4 \\ x_2 = & 3 & +\frac{2}{5}x_3 & -\ \frac{3}{5}x_4 \end{cases}$$

Part 7

Mauro Dell'Amico

DISMI, Universitá di Modena e Reggio Emilia
mauro.dellamico{at}unimore.it
www.or.unimore.it

Standard and other forms of an LP

STANDARD

$$\min\{c^T x : Ax = b, x_j \geq 0, j = 1, \dots, n\}$$

CANONICAL

$$\min\{c^T x : Ax \geq b, x_j \geq 0, j = 1, \dots, n\}$$

GENERAL

$$\min\{c^T x : Ax = b, A'x \geq b', x_j \geq 0, j \in J \subset \{1, \dots, n\}\}$$

The three forms are equivalent !

From General to Standard

$$\min\{c^T x : Ax = b, A'x \geq b', x_j \geq 0, j \in J \subset \{1, \dots, n\}\}$$

$$\Downarrow$$

$$\min\{c^T x : Ax = b, x_j \geq 0, j = 1, \dots, n\}$$

From General to Standard

$$\min\{c^T x : Ax = b, A'x \geq b', x_j \geq 0, j \in J \subset \{1, \ldots, n\}\}$$

$$\Downarrow$$

$$\min\{c^T x : Ax = b, x_j \geq 0, j = 1, \ldots, n\}$$

An inequality is transformed into and equation adding a slack variable

From General to Standard

$$\min\{c^T x : Ax = b, A'x \geq b', x_j \geq 0, j \in J \subset \{1, \ldots, n\}\}$$

$$\Downarrow$$

$$\min\{c^T x : Ax = b, x_j \geq 0, j = 1, \ldots, n\}$$

An inequality is transformed into and equation adding a slack variable

An unconstrained variable x_j is substituted by two nonnegative variables $x_j^+, x_j^- \geq 0$

$$x_j = x_j^+ - x_j^-$$

From Standard to Canonical

$$\min\{c^T x : Ax = b, x_j \geq 0, j = 1, \ldots, n\}$$
$$\Downarrow$$
$$\min\{c^T x : \widehat{A}x \geq \widehat{b}, x_j \geq 0, j = 1, \ldots, n\}$$

From Standard to Canonical

$$\min\{c^T x : Ax = b, x_j \geq 0, j = 1, \ldots, n\}$$
$$\Downarrow$$
$$\min\{c^T x : \widehat{A}x \geq \widehat{b}, x_j \geq 0, j = 1, \ldots, n\}$$

An equation is transformed into two inequalities

Part 8

Mauro Dell'Amico

DISMI, Universitá di Modena e Reggio Emilia
mauro.dellamico{at}unimore.it
www.or.unimore.it

Geometry of the LP

Let $x \in \Re^n$

Hyperplane : $\alpha^T x = \alpha_0$

Halfspace : $\alpha^T x \leq \alpha_0$

Geometry of the LP

Let $x \in \Re^n$

Hyperplane : $\alpha^T x = \alpha_0$

Halfspace : $\alpha^T x \leq \alpha_0$

- *Hyperplanes and halfspaces are convex sets*
- *The intersection of a finite number of hyperplanes and halfspaces is a convex sets*

$$\Downarrow$$

The solution space of an LP is a convex set

Polyhedron : intersection of a finite number of hyperplanes and halfspaces
$P = \{x \in \Re^n : Ax = b, A'x \geq b'\}$

Polytope : a bounded polyhedron
$(\exists M > 0 : \|x\| \leq \ M \ \forall x \in P)$

Vertex : a point x of a polyhedron P such that $\nexists x^1, x^2 \in P$ with $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$

Polyhedron : intersection of a finite number of hyperplanes and halfspaces
$P = \{x \in \Re^n : Ax = b, A'x \geq b'\}$

Polytope : a bounded polyhedron
$(\exists M > 0 : \|x\| \leq \ M \ \forall x \in P)$

Vertex : a point x of a polyhedron P such that $\nexists x^1, x^2 \in P$ with $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$

A polyhedron has a finite number of vertices.

Theorem

Any point of a polytope can be obtained as a convex combination of its vertices (Minkowski-Weyl)

Theorem

Given a PLC problem $\min\{c^T x \ : \ x \in P\}$*, where* P *is a (finite) polytope, then there is at least a vertex of* P *giving and optimal solution.*

Proof.

Theorem

Any point of a polytope can be obtained as a convex combination of its vertices (Minkowski-Weyl)

Theorem

Given a PLC problem $\min\{c^T x \ : \ x \in P\}$, *where* P *is a (finite) polytope, then there is at least a vertex of* P *giving and optimal solution.*

Proof. Let $x^1, \dots, x^k$ be the vertices of P, let $y \in P$ be any point of P and set $z^* := \min\{c^T x^i : i = 1, \dots, k\}$

$$y \in P \Rightarrow \exists \lambda_1, \dots, \lambda_k \geq 0, \sum_{i=1}^{k} \lambda_i = 1 : y = \sum_{i=1}^{k} \lambda_i x^i (\text{Minkowski-Weyl})$$

Theorem

Any point of a polytope can be obtained as a convex combination of its vertices (Minkowski-Weyl)

Theorem

Given a PLC problem $\min\{c^T x \ : \ x \in P\}$, *where* P *is a (finite) polytope, then there is at least a vertex of* P *giving and optimal solution.*

Proof. Let $x^1, \dots, x^k$ be the vertices of P, let $y \in P$ be any point of P and set $z^* := \min\{c^T x^i : i = 1, \dots, k\}$

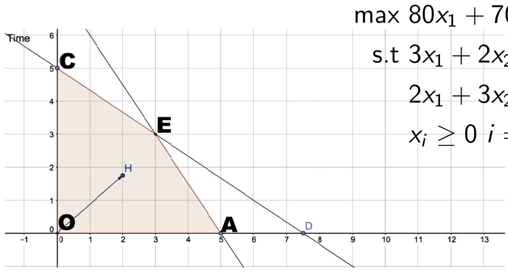
$$y \in P \Rightarrow \exists \lambda_1, \dots, \lambda_k \geq 0, \sum_{i=1}^{k} \lambda_i = 1 : y = \sum_{i=1}^{k} \lambda_i x^i (\text{Minkowski-Weyl})$$

$$c^T y = c^T \sum_{i=1}^{k} \lambda_i x^i = \sum_{i=1}^{k} \lambda_i (c^T x^i) \geq \sum_{i=1}^{k} \lambda_i z^* = z^* \quad \square$$

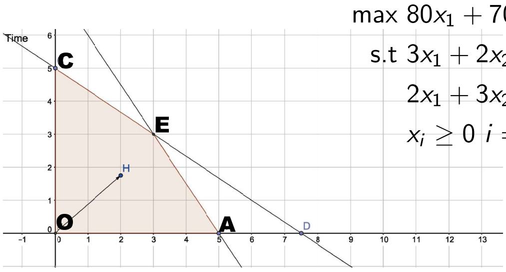
UNIMORE Mauro Dell'Amico

Theorem

Given a PLC problem $\max\{c^T x : Ax = b, x \geq 0\}$ *and a basis* B*, if the basic solution* $x_B = B^{-1}b$*,* $x_F = 0$ *is feasible, then it defines a vertex of* $P := \{x : Ax = b, x \geq 0\}$

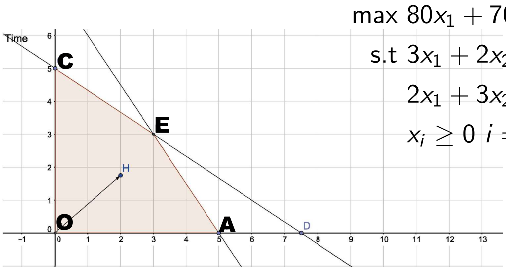


$$
\begin{aligned}
\max\ & 80x_1 + 70x_2 \\
\text{s.t}\ & 3x_1 + 2x_2 + x_3 \qquad = 15 \\
& 2x_1 + 3x_2 \qquad + x_4 = 15 \\
& x_i \geq 0 \;\; i = 1, \dots, 4
\end{aligned}
$$



$$\max\ 80x_1 + 70x_2$$
$$\text{s.t } 3x_1 + 2x_2 + x_3 \qquad = 15$$
$$2x_1 + 3x_2 \qquad + x_4 = 15$$
$$x_i \geq 0 \ i = 1, \ldots, 4$$

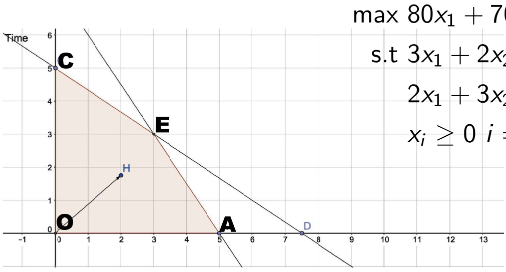
Vertex **O**: $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$



$$\max\ 80x_1 + 70x_2$$
$$\text{s.t } 3x_1 + 2x_2 + x_3 \qquad = 15$$
$$2x_1 + 3x_2 \qquad + x_4 = 15$$
$$x_i \geq 0 \ i = 1, \ldots, 4$$

Vertex **O**: $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$

Vertex **A**: $B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ $x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ -2/3 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$



$$
\begin{aligned}
\max\ & 80x_1 + 70x_2 \\
\text{s.t } & 3x_1 + 2x_2 + x_3 \qquad = 15 \\
& 2x_1 + 3x_2 \qquad + x_4 = 15 \\
& x_i \geq 0 \ i = 1, \ldots, 4
\end{aligned}
$$

Vertex **O**: $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$

Vertex **A**: $B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$ $x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ -2/3 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

Vertex **E**: $B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ $x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/5 & -2/5 \\ -2/5 & 3/5 \end{bmatrix} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Vertex **C**...

Part 9

Mauro Dell'Amico

DISMI, Universitá di Modena e Reggio Emilia
mauro.dellamico{at}unimore.it
www.or.unimore.it

Simplex algorithm

Simplex algorithm: (first version - maximization)
Find a basis B giving a basic feasible solution x
while ("x is not optimal and not unlimited")
begin
    transform the current basis B into a new basis
    by changing one column,
    and so that the objective function increases
end

- The algorithm terminates in a finite number of iterations:

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

Optimality condition

Maximization problem

$$\max\{c^T x : Ax = b, x \geq 0\}$$
$$x_B = B^{-1}b - B^{-1}Fx_F, \quad B^{-1}b \geq 0$$

Optimality condition

Maximization problem

$$\max\{c^T x : Ax = b, x \geq 0\}$$

$$x_B = B^{-1}b - B^{-1}Fx_F, \quad B^{-1}b \geq 0$$

$$\begin{aligned} z = c^T x &= [c_B^T, c_F^T] \begin{bmatrix} x_B \\ x_F \end{bmatrix} = c_B^T(B^{-1}b - B^{-1}Fx_F) + c_F^T x_F \\ &= c_B^T B^{-1}b + (c_F^T - c_B^T B^{-1}F)x_F = c_B^T B^{-1}b + \bar{c}_F^T x_F \end{aligned}$$

Optimality condition

Maximization problem

$$\max\{c^T x : Ax = b, x \geq 0\}$$

$$x_B = B^{-1}b - B^{-1}Fx_F, \quad B^{-1}b \geq 0$$

$$\begin{aligned} z = c^T x &= [c_B^T, c_F^T] \begin{bmatrix} x_B \\ x_F \end{bmatrix} = c_B^T(B^{-1}b - B^{-1}Fx_F) + c_F^T x_F \\ &= c_B^T B^{-1}b + (c_F^T - c_B^T B^{-1}F)x_F = c_B^T B^{-1}b + \bar{c}_F^T x_F \end{aligned}$$

$\boxed{\bar{c}^T = c^T - c_B^T B^{-1} A}$: *reduced costs* with respect to basis B

▷ The cost of the current basic solution is $c_B^T B^{-1} b \quad (x_F = 0)$

Selection of the column entering the basis

Theorem

A basic feasible solution of a PLC problem in maximization form is optimal if the reduced costs are non-positive ($\overline{c}^T \leq 0$).

(*In a minimization problem the solution is optimal if* $\overline{c}^T \geq 0$)

- To increase the current solution value select a variable of x_F with positive reduced cost.

Selection of the column leaving the basis

Let $\overline{A} = B^{-1}A$, $\overline{F} = B^{-1}F$ and $\overline{b} = B^{-1}b$.

A basic solution is $x_B = B^{-1}b - B^{-1}Fx_F = \overline{b} - \overline{F}x_F$

Let $x_B = [x_{[1]}, \ldots, x_{[m]}]^T$. If x_h (column h) enters the basis:

Selection of the column leaving the basis

Let $\overline{A} = B^{-1}A$, $\quad \overline{F} = B^{-1}F$ and $\overline{b} = B^{-1}b$.

A basic solution is $x_B = B^{-1}b - B^{-1}Fx_F = \overline{b} - \overline{F}x_F$

Let $x_B = [x_{[1]}, \dots, x_{[m]}]^T$. If x_h (column h) enters the basis:

$$\left\{ \begin{array}{rcl} x_{[1]} & = & \overline{b}_1 - \overline{a}_{1h}x_h \geq 0 \\ & & \dots \\ x_{[i]} & = & \overline{b}_i - \overline{a}_{ih}x_h \geq 0 \\ & & \dots \\ x_{[m]} & = & \overline{b}_m - \overline{a}_{mh}x_h \geq 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{c} \overline{a}_{1h}x_h \leq \overline{b}_1 \\ \dots \\ \overline{a}_{ih}x_h \leq \overline{b}_i \\ \dots \\ \overline{a}_{mh}x_h \leq \overline{b}_m \end{array} \right.$$

- $\overline{a}_{ih} \leq 0 \quad \Rightarrow \quad$ no constraint for x_h
- $\overline{a}_{ih} > 0 \quad \Rightarrow \quad x_h \leq \overline{b}_i / \overline{a}_{ih}$

$$x_h \leq \min \left\{ \frac{\overline{b}_i}{\overline{a}_{ih}} : \overline{a}_{ih} > 0, i = 1, \dots, m \right\} \qquad (1)$$

Let t be the row giving $\min \left\{ \frac{\overline{b}_i}{\overline{a}_{ih}} : \overline{a}_{ih} > 0, i = 1, \dots, m \right\}$

Since $x_{[t]} = \overline{b}_t - \overline{a}_{th}x_h$

$$x_h = \frac{\overline{b}_t}{\overline{a}_{th}} \Rightarrow x_{[t]} = 0 \;\; \text{leaves the basis}$$

Let t be the row giving $\min\left\{\frac{\overline{b}_i}{\overline{a}_{ih}} : \overline{a}_{ih} > 0, i = 1, \dots, m\right\}$

Since $x_{[t]} = \overline{b}_t - \overline{a}_{th} x_h$

$$x_h = \frac{\overline{b}_t}{\overline{a}_{th}} \Rightarrow x_{[t]} = 0 \ \text{ leaves the basis}$$

..if instead...

$$\boxed{\overline{a}_{ih} \leq 0, \ \forall i = 1, \dots, m}$$

x_h can increase indefinitely while $Ax = b, x \geq 0$ remains satisfied

$$\Downarrow$$

$$\boxed{\text{The problem is unlimited}}$$

$$\begin{aligned}
\max\ & 80x_1 + 70x_2 \\
\text{s.t } & 3x_1 + 2x_2 + x_3 \qquad = 15 \\
& 2x_1 + 3x_2 \qquad + x_4 = 15 \\
& x_i \geq 0 \ i = 1, \dots, 4
\end{aligned}$$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1/3 & 0 \\ -2/3 & 1 \end{bmatrix}$$

$$
\begin{aligned}
\max\ & 80x_1 + 70x_2 \\
\text{s.t}\ & 3x_1 + 2x_2 + x_3 \qquad\quad = 15 \\
& 2x_1 + 3x_2 \qquad\quad + x_4 = 15 \\
& x_i \geq 0 \ \ i = 1, \ldots, 4
\end{aligned}
$$

$$
A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \ B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \ B^{-1} = \begin{bmatrix} 1/3 & 0 \\ -2/3 & 1 \end{bmatrix}
$$

$$
\overline{A} = \begin{bmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & 5/3 & -2/3 & 1 \end{bmatrix} \ \overline{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \overline{c}^T = [0, 50/3, -80/3, 0]
$$

$$
\begin{aligned}
\max\ & 80x_1 + 70x_2 \\
\text{s.t}\ & 3x_1 + 2x_2 + x_3 \qquad\quad = 15 \\
& 2x_1 + 3x_2 \qquad\quad + x_4 = 15 \\
& x_i \geq 0 \ \ i = 1, \ldots, 4
\end{aligned}
$$

$$
A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \ B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \ B^{-1} = \begin{bmatrix} 1/3 & 0 \\ -2/3 & 1 \end{bmatrix}
$$

$$
\overline{A} = \begin{bmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & 5/3 & -2/3 & 1 \end{bmatrix} \ \overline{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \overline{c}^T = [0, 50/3, -80/3, 0]
$$

$h = 2,\ x_h \leq \min\{\frac{5}{2/3}, \frac{5}{5/3}\} = 3 \Rightarrow t = 2$

$$\begin{aligned} \max\ & 80x_1 + 70x_2 \\ \text{s.t}\ & 3x_1 + 2x_2 + x_3 \qquad = 15 \\ & 2x_1 + 3x_2 \qquad + x_4 = 15 \\ & x_i \geq 0 \ i = 1, \dots, 4 \end{aligned}$$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \ B = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \ B^{-1} = \begin{bmatrix} 1/3 & 0 \\ -2/3 & 1 \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 1 & 2/3 & 1/3 & 0 \\ 0 & 5/3 & -2/3 & 1 \end{bmatrix} \ \overline{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \overline{c}^T = [0, 50/3, -80/3, 0]$$

$h = 2$, $x_h \leq \min\{\frac{5}{2/3}, \frac{5}{5/3}\} = 3 \Rightarrow t = 2$

The second basic variable $(x_{[2]}) = x_4$ leaves the basis

Part 10

Mauro Dell'Amico

DISMI, Universitá di Modena e Reggio Emilia
mauro.dellamico{at}unimore.it
www.or.unimore.it

Simplex algorithm summary

optimality non-positive reduced costs (maximization)

entering variable a non-basic variable x_h with $\overline{c}_h > 0$

pivot row $t = \text{argmin}\{\overline{b}_i/\overline{a}_{ih} : \overline{a}_{ih} > 0\}$ (basic variable $x_{[t]}$ exit the basis)

unlimited problem $\overline{a}_{ih} \leq 0 \ \forall i$

Symplex algorithm (second version - maximization)

Find a feasible basis $B = [A_{[1]}, \dots, A_{[m]}]$
unlimited := FALSE; optimal := FALSE;
while (optimal = FALSE **and** unlimited = FALSE) **do**
 Compute B^{-1} and set $u^T := c_B^T B^{-1}$;
 Compute reduced costs $\overline{c}_h = c_h - u^T A_h, \ \forall x_h : h \in A \backslash B$
 if ($\overline{c}_h \leq 0 \ \forall x_h$) **then** optimal := TRUE
 else
  Choose a non basic variable x_h such that $\overline{c}_h > 0$;
  Compute $\overline{b} := B^{-1} b$ and $\overline{A}_h := B^{-1} A_h$;
  if‘ ($\overline{a}_{ih} \leq 0, i = 1, \dots, m$) **then** unlimited:= TRUE
  else
   Find $t := \text{argmin}\{\overline{b}_i/\overline{a}_{ih}, i = 1, \dots, m : \overline{a}_{ih} > 0\}$;
   Update the basis setting $[t] := h$;
  endif
 endif
endwhile

Part 11

Mauro Dell'Amico

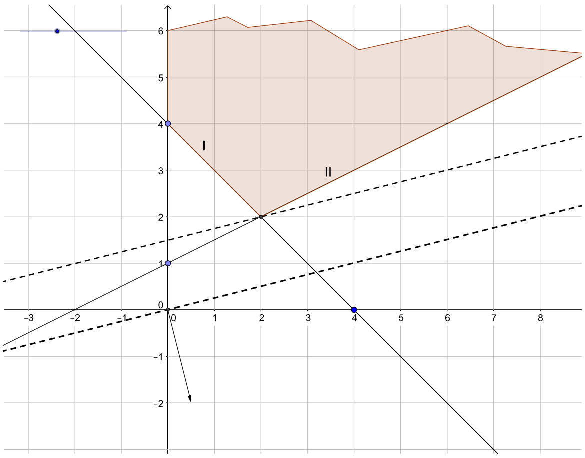
DISMI, Universitá di Modena e Reggio Emilia
mauro.dellamico{at}unimore.it
www.or.unimore.it

Finding a initial solution

$$\begin{array}{llllll} \max & 80x_1 & +70x_2 & & \\ \text{s.t} & 3x_1 & +2x_2 & \leq & 15 \\ & 2x_1 & +3x_2 & \leq & 15 \\ & x_1, & x_2 & \geq & 0 \end{array} \qquad \begin{array}{llllllll} \max & 80x_1 & +70x_2 & & & & \\ \text{s.t} & 3x_1 & +2x_2 & +x_3 & & = & 15 \\ & 2x_1 & +3x_2 & & +x_4 & = & 15 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

80	70	0	0	0
3	2	1	0	15
2	3	0	1	15

$$\begin{array}{lrrcl}
\max z = & x_1 & -4x_2 & & \\
s.t. & x_1 & +x_2 & \geq & 4 \\
& -x_1 & +2x_2 & \geq & 2 \\
& x_1 & x_2 & \geq & 0
\end{array}$$



$$\begin{array}{lrrrrcl}
\max z = & x_1 & -4x_2 & & & & \\
s.t. & x_1 & +x_2 & -x_3 & & = & 4 \\
& -x_1 & +2x_2 & & -x_4 & = & 2 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

x_1	x_2	x_3	x_4	
1	-4	0	0	0
1	1	-1	0	4
-1	2	0	-1	2

$$
\begin{array}{lrrrrcl}
\max z = & x_1 & -4x_2 & & & & \\
s.t. & x_1 & +x_2 & -x_3 & & = & 4 \\
 & -x_1 & +2x_2 & & -x_4 & = & 2 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}
$$

x_1	x_2	x_3	x_4	
1	-4	0	0	0
1	1	-1	0	4
-1	2	0	-1	2

No basis is immediately available

Add a dummy basis giving a small value to the obj. function

artificial variables

x_1	x_2	x_3	x_4	a_1	a_2	
1	-4	0	0	ε	ε	0
1	1	-1	0	1	0	4
-1	2	0	-1	0	1	2

Add a dummy basis giving a small value to the obj. function

artificial variables

x_1	x_2	x_3	x_4	a_1	a_2	
1	-4	0	0	ε	ε	0
1	1	-1	0	1	0	4
-1	2	0	-1	0	1	2

Numerical problems !

Define a new *minimization* problem where the obj. functions. ask to have the artificial variable not in the optimal basis

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	1	-1	0	1	0	4
-1	2	0	-1	0	1	2

Define a new *minimization* problem where the obj. functions. ask to have the artificial variable not in the optimal basis

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	1	-1	0	1	0	4
-1	2	0	-1	0	1	2

Transform the tableau in basis form

x_1	x_2	x_3	x_4	a_1	a_2	
0	-3	1	1	0	0	-6
1	1	-1	0	1	0	4
-1	$\boxed{2}$	0	-1	0	1	2

Define a new *minimization* problem where the obj. functions. ask to have the artificial variable not in the optimal basis

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	1	-1	0	1	0	4
-1	2	0	-1	0	1	2

Transform the tableau in basis form

x_1	x_2	x_3	x_4	a_1	a_2	
0	-3	1	1	0	0	-6
1	1	-1	0	1	0	4
-1	$\boxed{2}$	0	-1	0	1	2

x_1	x_2	x_3	x_4	a_1	a_2	
-3/2	0	1	-1/2	0	3/2	-3
3/2	0	-1	1/2	1	-1/2	3
-1/2	1	0	-1/2	0	1/2	1

x_1	x_2	x_3	x_4	a_1	a_2	
-3/2	0	1	-1/2	0	3/2	-3
$\boxed{3/2}$	0	-1	1/2	1	-1/2	3
-1/2	1	0	-1/2	0	1/2	1

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	0	-2/3	1/3	2/3	-1/3	2
0	1	-1/3	-1/3	1/3	1/3	2

x_1 and x_2 in the solution a_1 and a_2 outside

x_1	x_2	x_3	x_4	a_1	a_2	
-3/2	0	1	-1/2	0	3/2	-3
$\boxed{3/2}$	0	-1	1/2	1	-1/2	3
-1/2	1	0	-1/2	0	1/2	1

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	0	-2/3	1/3	2/3	-1/3	2
0	1	-1/3	-1/3	1/3	1/3	2

x_1 and x_2 in the solution a_1 and a_2 outside

remove last two columns and restore the original obj. function

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	0	-2/3	1/3	2/3	-1/3	2
0	1	-1/3	-1/3	1/3	1/3	2

remove last two columns and restore the original obj. function

NB original problem in *maximization* form

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	0	-2/3	1/3	2/3	-1/3	2
0	1	-1/3	-1/3	1/3	1/3	2

remove last two columns and restore the original obj. function

NB original problem in *maximization* form

x_1	x_2	x_3	x_4	
1	-4	0	0	0
1	0	-2/3	1/3	2
0	1	-1/3	-1/3	2

x_1	x_2	x_3	x_4	a_1	a_2	
0	0	0	0	1	1	0
1	0	-2/3	1/3	2/3	-1/3	2
0	1	-1/3	-1/3	1/3	1/3	2

remove last two columns and restore the original obj. function

NB original problem in *maximization* form

x_1	x_2	x_3	x_4	
1	-4	0	0	0
1	0	-2/3	1/3	2
0	1	-1/3	-1/3	2

x_1	x_2	x_3	x_4	
0	0	-2/3	-5/3	6
1	0	-2/3	1/3	2
0	1	-1/3	-1/3	2

O

x_1	x_2	x_3	x_4	a_1	a_2	
0	-3	1	1	0	0	-6
1	1	-1	0	1	0	4
-1	2	0	-1	0	1	2

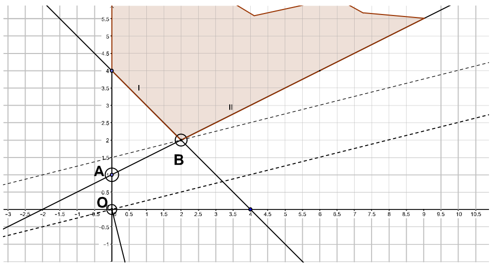
A

x_1	x_2	x_3	x_4	a_1	a_2	
-3/2	0	1	-1/2	0	3/2	-3
3/2	0	-1	1/2	1	-1/2	3
-1/2	1	0	-1/2	0	1/2	1

B

x_1	x_2	x_3	x_4	
0	0	-2/3	-5/3	6
1	0	-2/3	1/3	2
0	1	-1/3	-1/3	2

N.B. Not in all cases we arrive immediately at the optimal solution



$$
\begin{array}{rrrrl}
\max z = & -2x_1 & +x_2 & & \\
& x_1 & -2x_2 & \geq & 5 \\
& 2x_1 & +5x_2 & = & 6 \\
& x_1, & x_2 & \geq & 0
\end{array}
$$

Phase I

$$
\begin{array}{rrrrrrrl}
\min w = & & & & a_1 & +a_2 & & \\
& x_1 & -2x_2 & -s_1 & +a_1 & & = & 5 \\
& 2x_1 & +5x_2 & & & +a_2 & = & 6 \\
& x_1, & x_2, & s_1, & a_1, & a_2 & \geq & 0
\end{array}
$$

x_1	x_2	s_1	a_1	a_2	
0	0	0	1	1	0
1	-2	-1	1	0	5
2	5	0	0	1	6

-3	-3	1	0	0	-11
1	-2	-1	1	0	5
$\boxed{2}$	5	0	0	1	6

x_1	x_2	s_1	a_1	a_2	
-3	-3	1	0	0	-11
1	-2	-1	1	0	5
$\boxed{2}$	5	0	0	1	6

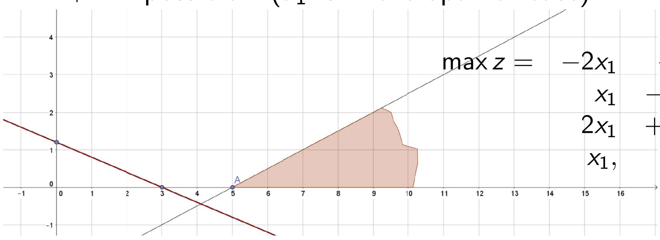
0	9/2	1	0	3/2	-2
0	-9/2	-1	1	-1/2	2
1	5/2	0	0	1/2	3

$w = +2$ Impossible ! (a_1 is in the optimal base)

x_1	x_2	s_1	a_1	a_2	
-3	-3	1	0	0	-11
1	-2	-1	1	0	5
$\boxed{2}$	5	0	0	1	6

0	9/2	1	0	3/2	-2
0	-9/2	-1	1	-1/2	2
1	5/2	0	0	1/2	3

$w = +2$ Impossible ! (a_1 is in the optimal base)



$$
\begin{array}{rrrcl}
\max z = & -2x_1 & +x_2 & & \\
& x_1 & -2x_2 & \geq & 5 \\
& 2x_1 & +5x_2 & = & 6 \\
& x_1, & x_2 & \geq & 0
\end{array}
$$

Two phases method: summary

Let (x^*, a^*) be the optimal solution of an artificial problem (Phase I), and let w^* be its value

$w^* > 0$: No solution exists in which all the artificial variables are outside the basis: UNFEASIBLE

$w^* = 0$: and all artificial variables outside the basis: x^* defines an optimal basis for the original problem

$w^* = 0$: and an artificial variable a_h is in the basis

a) if the row of the tableau having coefficient 1 in column h is zero elsewhere : delete the row (linearly dependent in the original problem)

b) if the row of the tableau having coefficient 1 in column h has another nonzero value : pivot on this element (also if it is negative)