June, 8, 2022 **A**

Last name, First name ____________________________

Exercise 1 (value 12)
Due to the **Energy Crisis** Europe is looking for a plan to gradually replace the supplies of gas from Russia with alternative sources. Today Europe buys, each month, Q billions of cubic meters from Russia. In the next T months this quantity must be reduced to q billions with a linear decrease. At the same time the quantity non bought from Russia must be bought from other suppliers. There are n possible suppliers. In month $t = 1, \dots, T$ each supplier i can sell v_{it} billions of cubic meters of gas at a price of c_{it} for billion. Moreover, if the overall gas bought in a month is more than the planned reduction, there is the possibility to store the exceeding cubic meters at cost of s_t for billion/month. The maximum storage for a month is S billions. At current month (0) no storage exists, but at the end of month T a storage of $\overline{S} \leq S$ billions must be guarantee.

Write a linear mathematical model to help Europe to define the cheaper way to buy gas in the next T months, while reducing the quantities bought from Russia.

Modify the above model by adding the following constraint: to avoid dependence from a single supplier, Europe decided to buy, in the T months, from at least 3 suppliers.

Answer
Variables

$$
\begin{aligned}
x_{it} &= \text{ billions of cubic meters of gas bought from supplier } i \text{ in month } t\\
I_t &= \text{ billions of cubic meters of gas stored at the end of month } t\\
y_i &= \text{ 1 if supplier } i \text{ is used}
\end{aligned}
$$

Constants

$$
\begin{array}{ll}
\Delta & = \frac{1}{T}(Q-q) \text{ monthly reduction of gas from Russia}\\
M & \text{a big number}
\end{array}
$$

$$
\min \quad \sum_{i=1}^{n}\sum_{t=1}^{T} c_{it}x_{it} + \sum_{t=1}^{T} s_t I_t
$$

$$
\begin{array}{rll}
I_{t-1} + \sum_{i=1}^{n} x_{it} & = t\Delta + I_t & t = 1, \dots, T\\
I_0 & = 0 & \\
I_T & \geq \overline{S} & \\
\sum_{i=1}^{n} y_i & \geq 3 & \\
y_i & \leq M \sum_{t=1}^{T} x_{it} & i = 1, \dots, n\\
0 \leq I_t & \leq S & t = 1, \dots, T\\
0 \leq x_{it} & \leq v_{it} & i = 1, \dots, n, t = 1, \dots, T
\end{array}
$$

Exercise 2 (value 8)
Consider the following LP problem.

$$
\begin{array}{rl}
\min & 4x_1 + 5x_2 + 3x_3\\
& 2x_1 + 3x_2 + x_3 \geq 3\\
& 3x_1 + 5x_2 + 4x_3 \geq 1\\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

- Write the corresponding tableau.
- Manipulate the tableau, if necessary, and solve the problem with the most convenient (for this case) simplex method, following the Bland's rule.

- Expose your motivations for the chosen method.
- Write the optimal solution and its cost.

Answer

x_1	x_2	x_3	y_1	y_2	$-z$	
4	5	3	0	0	0	
2	3	1	-1	0	3	y_1
3	5	4	0	-1	1	y_2

Since all the reduced costs are positive, it is worth to multiply both constraints by -1 and apply the Dual simplex method.

x_1	x_2	x_3	y_1	y_2	$-z$	
4	5	3	0	0	0	
-2	$-\mathbf{3}$	-1	1	0	-3	y_1
-3	-5	-4	0	1	-1	y_2

x_1	x_2	x_3	y_1	y_2	$-z$	
$\frac{2}{3}$	0	$\frac{4}{3}$	$\frac{5}{3}$	0	-5	
$\frac{2}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	1	x_2
$\frac{1}{3}$	0	$-\frac{7}{3}$	$-\frac{5}{3}$	1	4	y_2

The optimal solution is $x = (0, 1, 0)$ with cost 5.

Exercise 3 (value 7)
Find the optimal solution of the following knapsack problem using a Dynamic Programming method, without reordering the items: $n = 4, c = 32$, $(p_j, w_j) = [(1, 18), (2, 25), (3, 14), (2, 3)]$

Answer
$P = 1 + 2 + 3 + 2 = 8$

	0	1	2	3	4	5	6	7	8
f_0	0	∞	∞	∞	∞	∞	∞	∞	∞
f_1	0	18	∞	∞	∞	∞	∞	∞	∞
f_2	0	18	25	43	∞	∞	∞	∞	∞
f_3	0	18	25	14	32	39	57	∞	∞
f_4	0	18	3	14	28	$\boxed{17}$	35	42	60

	0	1	2	3	4	5	6	7	8
J_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
J_1	$\emptyset$	$\{1\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
J_2	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
J_3	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\emptyset$	$\emptyset$
J_4	$\emptyset$	$\{1\}$	$\{4\}$	$\{3\}$	$\{2,4\}$	$\boxed{\{3,4\}}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$

The optimal solution $x = \{3, 4\}$ has profit 5 and weight 17.