RICERCA OPERATIVA

Written assessment, January 8, 2020

Last name, First name ____________________

Exercise 1 (value 13)
A Company operating in the Home Health Care sector has to prepare the hiring plan for the next $T = \{1, 2, \ldots, m\}$ months. At the beginning of the planning period, K *trained* nurses work for the Company. Every trained nurse can produce H_t hours of operational work per month and gets a salary of S_t Euro. Every month $i \in T$ it is possible to permanently hire *new* nurses. Each new nurse gets a salary of S_n Euro during the first month but is not able to operate autonomously: she/he has to works shoulder to shoulder with a trained nurse for training purposes. Every trained nurse employed in such training tasks is only able to produce $\overline{H}_t$ hours of operational work. After the training month, newly employed nurses become trained nurses, with standard working capabilities and salary. To cover peak requests, it is also possible to hire *external* nurses. Each external nurse is already trained and is hired for a fixed term of 3 months. She/he can work for H_e hours per month and has a monthly cost of S_e Euro. External nurses cannot be hired in the first 2 and in the last 2 months of period T. The number of required working hours for each month i in T months is given by W_i.

- Write a MILP modelling the problem described, minimising the personnel costs for the given period.
- Due to the workload on other activities, the Human Resources Department of the Company can handle new employment procedures (for new nurses) only in one among months $f, g, h \in T$. Modify the model consequently.

Exercise 2 (value 9)
Consider the following problem

$$
\begin{aligned}
\min \quad & -5x_1 + 2x_2 + x_3 \\
& -2x_1 + x_2 + x_3 = 4 \\
& 3x_1 + 2x_2 \leq 10 \\
& x_1 - x_2 \leq 3 \\
& x_1, x_2, x_3 \geq 0, \text{integer}
\end{aligned}
$$

and solve it using the standard branch-and-bound method for PLI. Perform the first branching on the basic variable with smallest value. Explore a maximum of 4 nodes of the decision tree.

Exercise 3 (value 6)
In the rear of the sheet.

Exercise 3 (value 6)
Give a digraph $G = (V, A)$ with arc costs given by the following matrix, find the shortest path from vertex 1 to vertex 7.

	1	2	3	4	5	6	7	8
1	-	2	-	3	-	7	-	4
2	1	-	3	4	6	-	-	2
3	1	2	-	4	8	6	1	-
4	1	-	3	-	2	6	7	-
5	1	2	-	-	-	-	7	8
6	-	-	3	4	1	-	7	-
7	-	-	-	-	-	6	-	2
8	-	5	6	4	-	6	7	-

Answer in this table

S	L_i								$pred_i$							
	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8

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Answers

Exercise 1

Variables

$$
\begin{aligned}
t_i &= \text{trained nurses available at month } i \in T \cup \{0\}.\\
n_i &= \text{new nurses employed at month } i \in T \cup \{0\}.\\
e_i &= \text{external nurses hired at month } i \in T.\\
o_i &= 1 \text{ if external nurses hiring happen in month } i,\ 0 \text{ otherwise.} \quad i \in \{f,g,h\}
\end{aligned}
$$

Constant

$M =$ a big number, e.g. $\sum_{i \in T} W_i$

Model

$$\min z = S_t \sum_{i \in T} t_i + S_n \sum_{i \in T} n_i + 3S_e \sum_{i \in T} e_i \tag{1}$$

$$\text{s.t. } t_0 = K \tag{2}$$

$$e_0 = e_1 = e_{m-1} = e_m = n_0 = 0 \tag{3}$$

$$n_i \le t_i \qquad i \in T \tag{4}$$

$$t_i = t_{i-1} + n_{i-1} \qquad i \in T \tag{5}$$

$$H_i t_i + H_e \sum_{j=\max(i-2,1)}^{i} e_j - (H_t - \overline{H_t}) n_i \ge W_i \qquad i \in T \tag{6}$$

$$e_i \le M o_i \qquad i \in \{f,g,h\} \tag{7}$$

$$\sum_{i \in \{f,g,h\}} o_i \le 1 \tag{8}$$

$$t_i \ge 0 \qquad i \in T \cup \{0\} \tag{9}$$

$$n_i, e_i \ge 0 \qquad i \in T \tag{10}$$

$$\delta \in \{0,1\} \tag{11}$$

$$o_i \in \{0,1\} \qquad i \in \{f,g,h\} \tag{12}$$

Exercise 2

Let's write the continuous relaxation of the model in standard form:

$$
\begin{aligned}
\min \quad & -5x_1 + 2x_2 + x_3 \\
& -2x_1 + x_2 + x_3 = 4 \\
& 3x_1 + 2x_2 + x_4 = 10 \\
& x_1 - x_2 + x_5 = 3 \\
& x_1, \dots, x_5 \geq 0
\end{aligned}
$$

x_1	x_2	x_3	x_4	x_5		
-3	1	0	0	0	-4	$-z$
-2	1	1	0	0	4	x_3
3	2	0	1	0	10	x_4
1	-1	0	0	1	3	x_5

x_1	x_2	x_3	x_4	x_5		
0	-2	0	0	3	5	$-z$
0	-1	1	0	2	10	x_3
0	5	0	1	-3	1	x_4
1	-1	0	0	1	3	x_1

x_1	x_2	x_3	x_4	x_5		
0	0	0	$\frac{2}{5}$	$\frac{9}{5}$	$\frac{27}{5}$	$-z$
0	0	1	$\frac{1}{5}$	$\frac{7}{5}$	$\frac{51}{5}$	x_3
0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$	$\frac{1}{5}$	x_2
1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{16}{5}$	x_1

$LB = \lceil -27/5 \rceil = -5$, $x = (16/5, 1/5, 51/5, 0, 0)$ branch on x_2

First branch $x_2 \leq 0$

x_1	x_2	x_3	x_4	x_5	x_6		
0	0	0	$\frac{2}{5}$	$\frac{9}{5}$	0	$\frac{27}{5}$	$-z$
0	0	1	$\frac{1}{5}$	$\frac{7}{5}$	0	$\frac{51}{5}$	x_3
0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$	0	$\frac{1}{5}$	x_2
1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{16}{5}$	x_1
0	1	0	0	0	1	0	x_6

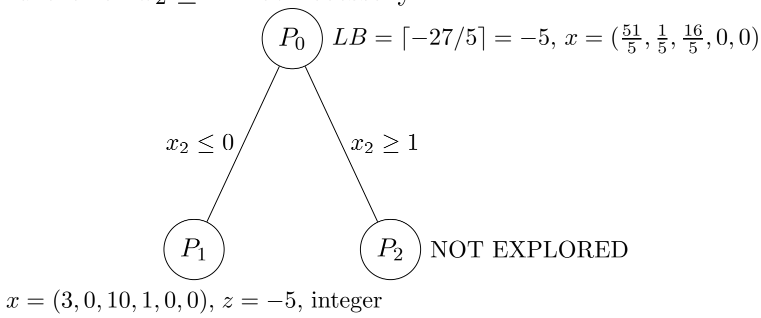
x_1	x_2	x_3	x_4	x_5	x_6		
0	0	0	$\frac{2}{5}$	$\frac{9}{5}$	0	$\frac{27}{5}$	$-z$
0	0	1	$\frac{1}{5}$	$\frac{7}{5}$	0	$\frac{51}{5}$	x_3
0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$	0	$\frac{1}{5}$	x_2
1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{16}{5}$	x_1
0	0	0	$-\frac{1}{5}$	$\frac{3}{5}$	1	$-\frac{1}{5}$	x_6

Dual simplex

x_1	x_2	x_3	x_4	x_5	x_6		
0	0	0	0	3	2	5	$-z$
0	0	1	0	2	0	10	x_3
0	1	0	0	0	0	0	x_2
1	0	0	0	1	0	3	x_1
0	0	0	1	-3	-5	1	x_6

$z = -5$, $x = (3, 0, 10, 1, 0, 0)$

Second branch $x_2 \geq 1$: not necessary



Exercise 3 Using the Dijkstra algorithm

	1	2	3	4	5	6	7	8
1	-	2	-	3	-	7	-	4
2	1	-	3	4	6	-	-	2
3	1	2	-	4	8	6	1	-
4	1	-	3	-	2	6	7	-
5	1	2	-	-	-	-	7	8
6	-	-	3	4	1	-	7	-
7	-	-	-	-	-	6	-	2
8	-	5	6	4	-	6	7	-

S	L_i								$pred_i$							
	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
1	-	2	-	3	-	7	-	4	-	1	-	1	-	1	-	1
1,2	-		5	3	8	7	-	4			2	1	2	1	-	1
1,2,4	-		5		5	7	10	4			2		4	1	4	1
1,2,4,8	-		5		5	7	10				2		4	1	4	
1,2,4,8,3	-				5	7	6						4	1	3	
1,2,4,8,3,5	-					7	6							1	3	
1,2,4,8,3,5,7	-					7			-	1	2	1	4	1	3	1

RICERCA OPERATIVA

Written assessment, January 8, 2020 **OLD**

Last name, First name ____________________

Exercise 1 (value 13)
A Company operating in the Home Health Care sector has to prepare the hiring plan for the next $T = \{1, 2, \ldots, m\}$ months. At the beginning of the planning period, K *trained* nurses work for the Company. Every trained nurse can produce H_t hours of operational work per month and gets a salary of S_t Euro. Every month $i \in T$ it is possible to permanently hire *new* nurses. Each new nurse gets a salary of S_n Euro during the first month but is not able to operate autonomously: she/he has to works shoulder to shoulder with a trained nurse for training purposes. Every trained nurse employed in such training tasks is only able to produce $\overline{H}_t$ hours of operational work. After the training month, newly employed nurses become trained nurses, with standard working capabilities and salary. To cover peak requests, it is also possible to hire *external* nurses. Each external nurse is already trained and is hired for a fixed term of 3 months. She/he can work for H_e hours per month and has a monthly cost of S_e Euro. External nurses cannot be hired in the first 2 and in the last 2 months of period T. The number of required working hours for each month i in T months is given by W_i.

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Exercise 2 (value 9)
Consider the following problem

$$\begin{aligned}
\min \quad & -5x_1 + 2x_2 + x_3 \\
& -2x_1 + x_2 + x_3 = 4 \\
& 3x_1 + 2x_2 \leq 10 \\
& x_1 - x_2 \leq 3 \\
& x_1, x_2, x_3 \geq 0, \text{integer}
\end{aligned}$$

and solve it using the standard branch-and-bound method for PLI. Perform the first branching on the basic variable with smallest value. Explore a maximum of 4 nodes of the decision tree.

Exercise 3 (value 6) Implement the model below in GLPK or XPRESS

$$\min z = \sum_{i=1}^{n} \sum_{j=1}^{m} (c_{ij} x_{ij} + 50 y_{ij}) \tag{13}$$

$$\text{s.t.} \sum_{j=1}^{n} \alpha_j x_{ij} \leq b_i \qquad i = 1, \ldots, n \tag{14}$$

$$x_{ij} \leq L y_{ij} \qquad i = 1, \ldots, n, j \in A \tag{15}$$

$$\sum_{j \in B} x_{ij} \leq b_i / 3 \qquad i = 1, \ldots, n \tag{16}$$

$$x_{ij} \geq 0 \text{ integer} \qquad i = 1, \ldots, n, j = 1, \ldots, m \tag{17}$$

$$y_{ij} \in \{0, 1\} \qquad i = 1, \ldots, n, j = 1, \ldots, m \tag{18}$$

$$\tag{19}$$

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Answers

Exercise 1

Variables

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\begin{aligned}
t_i &= \text{trained nurses available at month } i \in T \cup \{0\}. \\
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o_i &= 1 \text{ if external nurses hiring happen in month } i,\ 0 \text{ otherwise.} \quad i \in \{f, g, h\}
\end{aligned}
$$

Constant

$M =$ a big number, e.g. $\sum_{i \in T} W_i$

Model

$$
\begin{aligned}
& \min\ z = S_t \sum_{i \in T} t_i + S_n \sum_{i \in T} n_i + 3S_e \sum_{i \in T} e_i && && (20) \\
& \text{s.t. } t_0 = K && && (21) \\
& \quad e_0 = e_1 = e_{m-1} = e_m = n_0 = 0 && && (22) \\
& \quad n_i \le t_i && i \in T && (23) \\
& \quad t_i = t_{i-1} + n_{i-1} && i \in T && (24) \\
& \quad H_i t_i + H_e \sum_{j=\max(i-2,1)}^{i} e_j - (H_t - \overline{H_t}) n_i \ge W_i && i \in T && (25) \\
& \quad e_i \le M o_i && i \in \{f, g, h\} && (26) \\
& \quad \sum_{i \in \{f,g,h\}} o_i \le 1 && && (27) \\
& \quad t_i \ge 0 && i \in T \cup \{0\} && (28) \\
& \quad n_i, e_i \ge 0 && i \in T && (29) \\
& \quad \delta \in \{0, 1\} && && (30) \\
& \quad o_i \in \{0, 1\} && i \in \{f, g, h\} && (31)
\end{aligned}
$$

Exercise 2

Let's write the continuous relaxation of the model in standard form:

$$
\begin{aligned}
\min \quad & -5x_1 + 2x_2 + x_3 \\
& -2x_1 + x_2 + x_3 = 4 \\
& 3x_1 + 2x_2 + x_4 = 10 \\
& x_1 - x_2 + x_5 = 3 \\
& x_1, \dots, x_5 \geq 0
\end{aligned}
$$

x_1	x_2	x_3	x_4	x_5		
-3	1	0	0	0	-4	$-z$
-2	1	1	0	0	4	x_3
3	2	0	1	0	10	x_4
1	-1	0	0	1	3	x_5

x_1	x_2	x_3	x_4	x_5		
0	-2	0	0	3	5	$-z$
0	-1	1	0	2	10	x_3
0	5	0	1	-3	1	x_4
1	-1	0	0	1	3	x_1

x_1	x_2	x_3	x_4	x_5		
0	0	0	$\frac{2}{5}$	$\frac{9}{5}$	$\frac{27}{5}$	$-z$
0	0	1	$\frac{1}{5}$	$\frac{7}{5}$	$\frac{51}{5}$	x_3
0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$	$\frac{1}{5}$	x_2
1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{16}{5}$	x_1

$LB = \lceil -27/5 \rceil = -5$, $x = (16/5, 1/5, 51/5, 0, 0)$ branch on x_2

First branch $x_2 \leq 0$

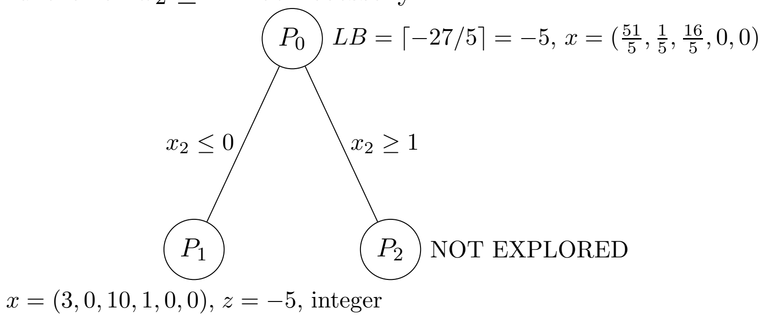
x_1	x_2	x_3	x_4	x_5	x_6		
0	0	0	$\frac{2}{5}$	$\frac{9}{5}$	0	$\frac{27}{5}$	$-z$
0	0	1	$\frac{1}{5}$	$\frac{7}{5}$	0	$\frac{51}{5}$	x_3
0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$	0	$\frac{1}{5}$	x_2
1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{16}{5}$	x_1
0	1	0	0	0	1	0	x_6

x_1	x_2	x_3	x_4	x_5	x_6			
0	0	0	$\frac{2}{5}$	$\frac{9}{5}$	0	$\frac{27}{5}$	$-z$	
0	0	1	$\frac{1}{5}$	$\frac{7}{5}$	0	$\frac{51}{5}$	x_3	
0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$	0	$\frac{1}{5}$	x_2	Dual simplex
1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{16}{5}$	x_1	
0	0	0	$-\frac{1}{5}$	$\frac{3}{5}$	1	$-\frac{1}{5}$	x_6	

x_1	x_2	x_3	x_4	x_5	x_6		
0	0	0	0	3	2	5	$-z$
0	0	1	0	2	0	10	x_3
0	1	0	0	0	0	0	x_2
1	0	0	0	1	0	3	x_1
0	0	0	1	-3	-5	1	x_6

$z = -5$, $x = (3, 0, 10, 1, 0, 0)$

Second branch $x_2 \geq 1$: not necessary



Exercise 3

```
/* index */
param n integer > 0 ;
param m integer > 0 ;
param K integer > 0 ;
set rows := 1..n ;
set cols := 1..m ;

/* constants */
set A;
set B;
param alpha {j in cols } >= 0 ;
param b {i in rows} >= 0 ;
param c{i in rows, j in cols } >= 0;
param L  >= 0 ;

/* dvariables */
var x {i in rows, j in cols } >= 0, integer ;
var y {i in rows, j in cols } >= 0, binary ;
/* objective function */
minimize z :  sum{i in rows, j in cols } (c[i,j] * x[i,j] + 50 * y[i,j]);
/* constraints */
s.t. makeAll { j in cols } : sum{i in rows} x[i,j]= 1 ;
C1 { i in rows} : sum{j in cols} alpha[j]*x[i,j] <= b[i] ;
C2 { i in rows, j in A } : x[i,j] <= L*y[i,j];
C3 { i in rows } : sum{j in B} x[i,j] <= b[i]/3;

solve ;
printf "\n\Results\nz=%f\n\n",z;
for {i in rows, j in cols: x[i,j] > 0}{
printf "x %3d %3d  %d\n",i,j,x[i,j];
}
data;
param n:=10;
param m:=8;
set A := 1  3 5;
set B := 2 4 6 8;
param alpha := [1] 8 [2] 4 [3] 5 [4] 6 [5] 5 [6] 9 [7] 10 [8] 11 ;
param b := [1] 80 [2] 30 [3] 50 [4] 46 [5] 35 [6] 90 [7] 100 [8] 91 [9] 30 [10]
14 [11] 60 [12] 80 [13] 80 [14] 13 [15] 12 [16] 20 [17] 40 [18] 70 [19] 90 [20] 18;
param c : 1 2 3 4 5 6 7 8 :=
1  2 3 2 4 1 4 5 7
2  2 3 2 4 1 4 5 7
3  2 3 2 4 1 4 5 7
4  2 3 2 4 1 4 5 7
5  2 3 2 4 1 4 5 7
6  2 3 2 4 1 4 5 7
7  2 3 2 4 1 4 5 7
8  2 3 2 4 1 4 5 7
9  2 3 2 4 1 4 5 7
10 2 3 2 4 1 4 5 7;
param L := 50;
end;
```