20210216A

1. **MM**

   A company that produces hand sanitizer want to prepare 1000 liters of sanitizer of type A and 800 liters of type B. To produce a liter of A it must blend 200 ml of semifinished product 1 and 500 ml semifinished product 2, while to produce a liter of B the recipe is 600 ml of 1 and 100 ml of 2. The company has a current stock of semifinished products that equals K_1 and K_2 ml, respectively. However, if the stock is not enough, the company can either buy more semifinished products or to produce less liters of A and B. Buying a liter of semifinished products 1 and 2 cost, repectively, c_1 and c_2 Euro. Producing one less liter of A or B corresponds to a loss of l_A an l_B Euro, respectively. Minimize the costs respecting the production constraints.

   Notes: (not included in XML)

   - Solution:
     - x_A, x_B: liters of sanitizer of type A and B produced
     - s_A, s_B: liters of sanitizer of type A and B not produced
     - y_1, y_2: liters of semifinished product 1 and 2 bought

   $$\begin{array}{rrcl} \min & c_1y_1 + c_2y_2 + l_As_A + l_Bs_B & & \\ & x_A + s_A & = & 1000 \\ & x_B + s_B & = & 800 \\ & 200x_A + 600x_B & \leq & K_1 + 1000y_1 \\ & 500x_A + 100x_B & \leq & K_2 + 1000y_2 \\ & x_A, x_B, s_A, s_B, y_1, y_2 & \geq & 0 \end{array}$$

2. **PLC**

   Consider the following LP problem:

   $$\begin{array}{lll} \max & -5x_1 + 5x_2 + 13x_3 & \\ & -x_1 + x_2 + 3x_3 + x_4 & = 20 \\ & 12x_1 + 4x_2 + 10x_3 + x_5 & = 90 \\ & x_1, x_2, x_3, x_4, x_5 & \geq 0 \end{array}$$

   The optimal solution is $x = (0, 20, 0, 0, 10)$ with value 100.
   Given that $B^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$, answer the following questions. Provide motivations to support your arguments:

- what happens to the optimal base if b_1, the right hand side of the first constraint, becomes 24?
- what happens to the optimal base if c_3, the coefficient of variable x_3 becomes 10?

The question is whether the current solution changes, we are not interested in eventual new optimal solutions.

Notes: (not included in XML)

- Solution:
  If b_1 changes, we need to understand if the current base is still optimal. This happens when:
  $B^{-1}(b + \Delta b) \geq 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 20 + \Delta b_1 \\ 90 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -20 \leq \Delta b_1 \leq 2.5 \Rightarrow 0 \leq b_1 \leq 22.5$. Therefore if b_1 becomes 24 then the optimal base changes.
  The variable x_3 is not in the optimal base. If c_3 decreased its coefficient from 13 to 10, x_3 becomes even less attractive for the optimization, and therefore the solution does not change.

3. **DP**

Use Dynamic Programming to solve the following Knapsack Problem: weight w_j = (4, 2, 3), profit p_j = (2, 1, 2), and capacity $C = 5$. Use a Dynamic Programming where the **states** correspond to the possible **profit levels**. (Tip: you can use M for infinity and E for empty set). Report all the iterations giving the states and their values. Report the optimal solution.

Notes: (not included in XML)

- Solution:
  Upper bound on profit: $P = \sum_j p_j = 5$

	0	1	2	3	4	5
f^0	0	M	M	M	M	M
f^1	0	M	4	M	M	M
f^2	0	2	4	6	M	M
f^3	0	2	3	**5**	7	9

	0	1	2	3	4	5
J^0	E	E	E	E	E	E
J^1	E	E	1	E	E	E
J^2	E	2	1	$1, 2$	E	E
J^3	E	2	3	**2,3**	$1, 3$	$1, 2, 3$

Optimal solution: $\{2, 3\}$ with profit 3.

20210216B

1. **MM**

   A company that produces cookies wants to prepare 50 Kg of cookies of type A and 60 Kg cookies of type B. To produce one Kg of A it must blend 600g of flour 300g of sugar and 3 eggs, while to produce one Kg of B the recipe requires 400g of flour, 400g of sugar and 4 eggs. The company has a current stock of F Kg of flour, S Kg of sugar and E eggs. However, if the stock is not enough, the company can buy more ingredients or can decide to produce less cookies. Buying flour or sugar costs c_1 and c_2 per Kg, respectively, while each egg cost c_3 Euro. Producing one less Kg of cooky of type A and B induces a loss of l_A and l_B Euro, respectively. Minimize the costs respecting the production constraints.

   Notes: (not included in XML)

   - Solution:
     - x_A, x_B: number of Kgs of cookies of type A and B produced
     - s_A, s_B: number of Kgs of cookies of type A and B not produced
     - y_1, y_2, y_3: Kgs of flour and sugar bought, and number of eggs bought

$$
\begin{array}{lrcl}
\min \quad z = & \sum_{i=1}^{3} c_i y_i + l_A s_A + l_B s_B & & \\
 & x_A + s_A & = & 50 \\
 & x_B + s_B & = & 60 \\
 & 0.6x_A + 0.4x_B & \leq & F + y_1 \\
 & 0.3x_A + 0.4x_B & \leq & S + y_2 \\
 & 3x_A + 4x_B & \leq & E + y_3 \\
 & x_A, x_B, s_A, s_B, y_1, y_2 & \geq & 0 \\
 & y_3 & \geq & 0 \text{ integer}
\end{array}
$$

2. **PLC**

   Consider the following LP problem:

$$
\begin{array}{lll}
\max & -5x_1 + 5x_2 + 3x_3 & \\
 & -x_1 + x_2 + 3x_3 + x_4 & = 20 \\
 & 12x_1 + 2x_2 + 10x_3 + x_5 & = 30 \\
 & x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{array}
$$

and the solution $x = (0, 15, 0, 5, 0)$ with associated $B^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}$, answer the following questions. Provide motivations to support your arguments:

- what happens to the optimal basis if b_1, the right hand side of the first constraint, becomes 15?
- what happens to the optimal basis if c_1, the coefficient of variable x_1, becomes -10?

The question is whether the current basis changes, we are not interested in eventual new optimal basis.

Notes: (not included in XML)

- Solution:
  If b_1 changes, we need to understand if the current basis is still optimal. This happens when:
  $B^{-1}(b+\Delta b) \geq 0 \Rightarrow \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 20 + \Delta b_1 \\ 30 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \Delta b_1 \geq -5 \Rightarrow$ $15 \leq b_1 \leq +\infty$. Therefore if b_1 becomes 15 then the optimal basis does not change.
  The variable x_1 is not in the optimal basis. If c_1 decreased its coefficient from -5 to -10, x_1 becomes even less attractive for the optimization, and therefore the basis does not change.

3. **DP**

Use Dynamic Programming to solve the following Knapsack Problem: weight $w_j = (4,\ 2,\ 3)$, profit $p_j = (1,\ 3,\ 1)$, and capacity $C = 5$. Use a Dynamic Programming where the **states** correspond to the possible **profit levels**. (Tip: you can use M for infinity and E for empty set). Report all the iterations giving the states and their values. Report the optimal solution.

Notes: (not included in XML)

- Solution:

Upper bound on profit: $P = \sum_j p_j = 5$

	0	1	2	3	4	5
f^0	0	M	M	M	M	M
f^1	0	4	M	M	M	M
f^2	0	4	M	2	6	M
f^3	0	3	7	2	**5**	9

	0	1	2	3	4	5
J^0	E	E	E	E	E	E
J^1	E	1	E	E	E	E
J^2	E	1	E	E	$1,2$	E
J^3	E	3	$1,3$	2	$2,3$	$1,2,3$

Optimal solution: $\{2,3\}$ with profit 4.