es.1) Giustificare $\tau < \infty$ q.c. dove $\tau = \inf\{n : X_n \in \{a, -b\}\}$ con $a, b \in \mathbb{N}^+$

· Analisi di $(X_n)_n$

Forniscono la stessa informazione

a) $X_n \sim \mathcal{F}_n = \sigma(X_1, \dots, X_n) = \sigma(Y_1, Y_1+Y_2, \dots, Y_1+\dots+Y_n) = \sigma(Y_1, \dots, Y_n)$

b) $\forall n \quad \mathbb{E}[|X_n|] = \mathbb{E}[|\sum_{i=1}^n Y_i|] \leq |\mathbb{E}[\sum_{i=1}^n Y_i]| = |\sum_{i=1}^n \mathbb{E}[Y_i]| \underset{(*)}{=} |n \cdot (2p-1)| < \infty \ \forall n$ fix.

(*) $\mathbb{E}[Y_i] = 1 \cdot p + (-1)(1-p) = 2p-1$

c) $\forall n$ ~~...~~ $= \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = \mathbb{E}[Y_{n+1} | \mathcal{F}_n] = \mathbb{E}[Y_{n+1}] = 2p-1 > 0$

$Y_{n+1} \perp\!\!\!\perp Y_i \ \forall i$ — $p > \frac{1}{2}$

$Y_{n+1} \perp\!\!\!\perp \mathcal{F}_n$

$(X_n)_n$ è una sub-martingala.

· Decomposizione di Doob: $X_n = \tilde{M}_n + A_n$, $d_k = X_k - X_{k-1} = Y_k$

$$X_n = \sum_{k=1}^n d_k = \sum_{k=1}^n \left[d_k - \mathbb{E}[d_k | \mathcal{F}_{k-1}]\right] + \sum_{k=1}^n \mathbb{E}[d_k | \mathcal{F}_{k-1}]$$

$$X_n = \sum_{k=1}^n Y_k = \underbrace{\sum_{k=1}^n \left[Y_k - \mathbb{E}[Y_k]\right]}_{\tilde{M}_n} + \underbrace{\sum_{k=1}^n \mathbb{E}[Y_k]}_{A_n} = \left[\sum_{k=1}^n Y_k - n(2p-1)\right] + n(2p-1)$$

· $\tilde{M}_n$ è martingala

Th. : $\forall n \ \tilde{M}_{n+1} - \tilde{M}_n = Y_{n+1} - (2p-1) + \sum_{k=1}^n Y_k - n(2p-1) - \sum_{k=1}^n Y_k + n(2p-1) =$

$$= Y_{n+1} - (2p-1) \leq 1 - 2p + 1 = 2(1-p) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$$

$\Rightarrow \limsup_n (\tilde{M}_n) = +\infty$ (q.c.) su $\{\omega : \tilde{M}_n(\omega) \text{ converge}\}^C$

· L'insieme di convergenza di $\tilde{M}_n$ è $\emptyset$, perché $\forall n \ |\tilde{M}_{n+1} - \tilde{M}_n| = |Y_{n+1} - 2p + 1| \neq 0 \ \forall n$ e $\forall \omega \in \Omega$

$\Rightarrow \limsup_n (\tilde{M}_n) = +\infty$ (q.c.) su $\emptyset^C = \Omega \Rightarrow \limsup_n (\tilde{M}_n) = \infty$ (q.c.)

$\Rightarrow$ (q.c.) $\{\tau < +\infty\}$ perché il processo attraverserà q.c. a.

es.2) $Z_0 = 1$

$$Z_n = \alpha^{X_n} \quad \text{con } \alpha = \frac{q}{p} < 1 \quad , \quad \mathcal{F}_n = \sigma\{Y_1, \dots, Y_n\}$$

Dimostrazione tramite proprietà caratterizzanti:

$\forall n$ fix.

1) $\forall n \; Z_n \sim \mathcal{F}_n$

È noto che $Z_n \sim \sigma(Z_1, \dots, Z_n)$, filtrazione naturale però

$$\sigma(Z_1, Z_2, \dots, Z_n) = \sigma(\alpha^{X_1}, \alpha^{X_2}, \dots, \alpha^{X_n}) = \sigma(\alpha^{Y_1}, \alpha^{Y_1+Y_2}, \dots, \alpha^{Y_1+\dots+Y_n}) =$$

$$= \sigma(Y_1, Y_1+Y_2, \dots, Y_1+\dots+Y_n) = \sigma(Y_1, Y_2, \dots, Y_n) = \mathcal{F}_n$$

[Possiedono la stessa informazione, ovvero, formano la stessa partizione di Ω] $\leftarrow$ [$Y_1^{-1}(1) = (\alpha^{Y_1})^{-1}(\alpha)$; $Y_2^{-1}(1) \mid Y_1+Y_2 = 2) = (\alpha^{Y_1+Y_2})^{-1}(\alpha^2)$]

2) $\forall n \; Z_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$

$Z_n \in L^1$ se $\mathbb{E}[|Z_n|] < \infty$

$$\mathbb{E}[|Z_n|] \overset{Z_n > 0 \, \forall n}{=} \mathbb{E}[Z_n] = \mathbb{E}[\alpha^{Y_1+\dots+Y_n}] \overset{Y_i \perp\!\!\!\perp Y_j \, \forall i \neq j}{=} \prod_{i=1}^{n} \mathbb{E}[\alpha^{Y_i}] \underset{(*)}{=} \prod_{i=1}^{n} 1 = 1 < \infty \quad \forall n$$

$$(*) \; \mathbb{E}[\alpha^{Y_i}] = \alpha \cdot \mathbb{P}[\alpha^{Y_i} = \alpha] + \alpha^{-1} \cdot \mathbb{P}[\alpha^{Y_i} = \alpha^{-1}] = \alpha \mathbb{P}[Y_i = 1] + \alpha^{-1} \mathbb{P}[Y_i = -1]$$

[v.a. discreta; $Im(\alpha^{Y_i}) = \{\alpha, \alpha^{-1}\}$]

$$= \alpha p + \alpha^{-1}(1-p) \overset{\alpha = \frac{1-p}{p}}{=} \frac{(1-p)}{\cancel{p}} \cdot \cancel{p} + \frac{p}{\cancel{(1-p)}} \cancel{(1-p)} = (1-p) + p = 1$$

$\in \Omega$

3) $\forall n \; \mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = Z_n \quad (\Leftrightarrow \mathbb{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] = 0)$

$$\forall n \; \mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\alpha^{Y_1+\dots+Y_n+Y_{n+1}} \mid \mathcal{F}_n] \overset{\alpha^{Y_1+\dots+Y_n} \sim \mathcal{F}_n}{=} \alpha^{Y_1+\dots+Y_n} \mathbb{E}[\alpha^{Y_{n+1}} \mid \mathcal{F}_n] =$$

$$\overset{\alpha^{Y_{n+1}} \perp\!\!\!\perp \mathcal{F}_n}{=} \alpha^{Y_1+\dots+Y_n} \mathbb{E}[\alpha^{Y_{n+1}}] \overset{\mathbb{E}[\alpha^{Y_i}] = 1 \, \forall i}{=} \alpha^{Y_1+\dots+Y_n} = Z_n$$

$\square$

[• $Y_{n+1} \perp\!\!\!\perp Y_i \; \forall i \Rightarrow \alpha^{Y_{n+1}} \perp\!\!\!\perp \alpha^{Y_i} \; \forall i$
$\Rightarrow \alpha^{Y_{n+1}} \perp\!\!\!\perp \alpha^{\sum_{i=1}^{n} Y_i} \Rightarrow \sigma(\alpha^{Y_{n+1}}) \perp\!\!\!\perp \sigma(\alpha^{\sum Y_i})$]

es.3) Calcolare $m_a = \mathbb{P}(X_\tau = a)$

- $(Z_n)_n$ è martingala $\overset{\text{Th. arresto mart.}}{\Rightarrow} (Z_n^\tau)_n$ è martingala, dal punto 2 sappiamo che Z_n è limitata in L^1.
- $Z_n^\tau \overset{\text{definizione}}{=} Z_{n\wedge\tau}$, Z_n limitata in $L^1 \Rightarrow Z_{n\wedge\tau}$ limitata in L^1
- Th. di convergenza di Doob: Z_n^τ è limitata in $L^1 \Rightarrow Z_n^\tau \xrightarrow[n\to\infty]{q.c.} Z_\infty^\tau \in L^1$
- Se $\tau < \infty$ q.c. $\Rightarrow Z_n^\tau \xrightarrow[n\to\infty]{q.c.} Z_\tau \in L^1$
- $|Z_n^\tau| = |\alpha^{X_n^\tau}| \underset{\alpha<1}{\leq} |\alpha^{-b}| = \alpha^{-b} \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \Leftrightarrow Z_n^\tau$ è dominata
- Th. di conv. dominata di Lebesgue: $Z_n^\tau \xrightarrow[n\to\infty]{L^1} Z_\tau \Rightarrow Z_n^\tau$ è una martingala chiusa.
- Th. di arresto di Doob: $\sigma = 0$, $\tau < \infty$ q.c. $\Rightarrow \mathbb{E}[Z_\tau | \mathcal{F}_0] = Z_0$
- $\mathbb{E}[\mathbb{E}[Z_\tau | \mathcal{F}_0]] = \mathbb{E}[Z_\tau] = \mathbb{E}[Z_0] \underset{(Z_0 = 1)}{=} 1$
- $\mathbb{E}[Z_\tau] = \mathbb{E}[\alpha^{X_\tau}] \underset{\text{v.a. discrete}}{=} \alpha^a m_a + \alpha^{-b}(1 - m_a)$
- $\alpha^a m_a - \alpha^{-b} m_a + \alpha^{-b} = 1 \Rightarrow m_a(\alpha^a - \alpha^{-b}) = 1 - \alpha^{-b} \Rightarrow m_a = \dfrac{1 - \alpha^{-b}}{\alpha^a - \alpha^{-b}}$

4) Calcolare $\mathbb{E}[\tau]$. Abbiamo già dimostrato che $\tau < \infty$ q.c.

- $X_n = \tilde{M}_n + A_n$ con $\tilde{M}_n$ martingala $\tilde{M}_n = \sum_{i=1}^{n} Y_i - n(2p-1)$, $\overset{\ge 0}{A_n} = n(2p-1)$

$\downarrow$

- X_n^τ è ancora sub-martingala. $X_n^\tau = \tilde{M}_n^\tau + A_n^\tau$ $\begin{cases} \to X_n & \text{per } n \le \tau \\ \to X_\tau & \text{per } n > \tau \end{cases}$

- $|X_n^\tau| \le a \vee b \Rightarrow \mathbb{E}[|X_n^\tau|] \le a \vee b \Rightarrow$ (X_n^τ è limitato in L^1, $X_n^\tau \in L^1$ e dominato)

- $|X_n^\tau| = |\tilde{M}_n^\tau + A_n^\tau| \to |\tilde{M}_n^\tau| \le |\tilde{M}_n^\tau + \underbrace{A_n^\tau}_{\ge 0}| \le a \vee b \Rightarrow$ (M_n^τ è limitato in L^1 e $M_n^\tau \in L^1$ ed è anche dominata in L^1)

- $\tilde{M}_n^\tau$ limitato in $L^1 \Rightarrow \tilde{M}_n^\tau \xrightarrow[n\to\infty]{q.c.} \tilde{M}_\tau$

- $\underbrace{X_n^\tau}_{\in L^1} - \underbrace{M_n^\tau}_{\in L^1} = A_n^\tau \in L^1$

- (Th. conv. dominata di Lebesgue) $\tilde{M}_n^\tau \xrightarrow[n\to\infty]{L^1} \tilde{M}_\tau \Rightarrow \tilde{M}_n^\tau$ è una martingala chiusa

- (Th. di arresto di Doob) $\tau < \infty$ q.c., $\sigma = 0$, $\mathbb{E}[\tilde{M}_\tau \mid \mathcal{F}_0] = \tilde{M}_0 \Rightarrow \mathbb{E}[\tilde{M}_\tau] = \mathbb{E}[\tilde{M}_0] = 0$

- $\overset{Sx}{X_{n\wedge\tau}} = \tilde{M}_{n\wedge\tau} + \overset{Dx}{A_{n\wedge\tau}}$

(Sx) $X_{n\wedge\tau} \xrightarrow[n\to\infty]{q.c.} X_\tau \Rightarrow$ (Th. conv. dominata di Lebesgue) $\lim_n \mathbb{E}[X_{n\wedge\tau}] = \mathbb{E}[\lim_n X_{n\wedge\tau}] = \mathbb{E}[X_\tau]$

$|X_{n\wedge\tau}| < a \vee b$

(Dx) $A_{n\wedge\tau} = (n\wedge\tau)(2p-1) \xrightarrow{q.c.} \tau(2p-1)$

$\Rightarrow$ (Th. di convergenza monotona) $\lim_n \mathbb{E}[A_{n\wedge\tau}] =$

$0 \le (n\wedge\tau)(2p-1) \uparrow$

$= \mathbb{E}[\lim_n A_{n\wedge\tau}] = \mathbb{E}[A_\tau] = \mathbb{E}[\tau(2p-1)] =$

$= (2p-1)\mathbb{E}[\tau]$

- $A_n^\tau \in L^1$ e $\mathbb{E}[|A_n^\tau|] = \mathbb{E}[A_n] = (2p-1)\mathbb{E}[\tau] < \infty \Rightarrow \mathbb{E}[\tau] < \infty$

- $\mathbb{E}[X_{n\wedge\tau}] = \mathbb{E}[\tilde{M}_{n\wedge\tau}] + \mathbb{E}[A_{n\wedge\tau}]$

$\lim_{n\to\infty}$ $\downarrow$ q.c.

$\mathbb{E}[X_\tau] = \overset{=0}{\cancel{\mathbb{E}[\tilde{M}_\tau]}} + \mathbb{E}[A_\tau]$, $\mathbb{E}[X_\tau] = a\, m_a + (-b)(1-m_a)$

con $m_a = \dfrac{1-\alpha^{-b}}{\alpha^{a} - \alpha^{-b}}$

$$\mathbb{E}[X_\tau] = \mathbb{E}[A_\tau] \Rightarrow a\, m_a + (-b)(1-m_a) = (2p-1)\, \mathbb{E}[\tau]$$

$$\Rightarrow \mathbb{E}[\tau] = \frac{a\, m_a + (-b)(1-m_a)}{2p-1}$$