Let $a \in \ell^2(\mathbb{N})$, $a = \sum_{n\in\mathbb{N}} a_n e_n$, where e_n is the canonical basis of ℓ^2. The action of the operator T can be described as follows:

$$Ta = \sum_{n\in\mathbb{N}} a_n T e_n = \sum_{n\in\mathbb{N}} a_n e_{2n},$$

i.e.

$$\{a_1, a_2, \ldots\} \mapsto \{0, a_1, 0, a_2, 0, \ldots\}$$

Consider now $\phi \in \ell^2(\mathbb{N})$,

$$||T\phi||^2 = \langle \sum_{n\in\mathbb{N}} \phi_n e_{2n} | \sum_{m\in\mathbb{N}} \phi_m e_{2m} \rangle = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \phi_n^* \phi_m \langle e_{2n} | e_{2m} \rangle = \sum_{n\in\mathbb{N}} \phi_n^* \phi_n = ||\phi||^2$$

Thus, T is bounded and

$$||T||_{\mathcal{L}} = \sup_{\phi\in\ell^2(\mathbb{N}), ||\phi||=1} ||T\phi|| = ||\phi|| = 1$$

The operator T is trivially not dense in $\ell^2(\mathbb{N})$. In fact, $\nexists \{\psi_n\} \subset ran(T)$ such that

$$\psi_n \to \{1, 0, 0, 0, \ldots\} \in \ell^2(\mathbb{N})$$

By inversion theorem, T is not invertible.

We already now that $\mathcal{D}(T) = \ell^2(\mathbb{N})$. We want to find $\mathcal{D}(T^*)$. Indeed, being T bounded:

$$\begin{aligned}\mathcal{D}(T^*) &= \{\phi \in \ell^2(\mathbb{N}) : \sup_{\psi\in\mathcal{D}(T), ||\psi||=1} |\langle \phi | T\psi \rangle| < \infty\} \\ &= \ell^2(\mathbb{N})\end{aligned}$$

Furthermore, for $\phi, \psi \in \ell^2$:

$$\begin{aligned}\langle \phi | T\psi \rangle = \langle \sum_{n\in\mathbb{N}} \phi_n e_n | \sum_{m\in\mathbb{N}} \psi_m e_{2m} \rangle &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \phi_n^* \psi_m \langle e_n | e_{2m} \rangle \\ &= \sum_{m\in\mathbb{N}} \phi_{2m}^* \psi_m = \langle T^* \phi | \psi \rangle\end{aligned}$$

Hence, for $a \in \ell^2 = \mathcal{D}(T^*)$:

$$T^*a = \sum_{n\in\mathbb{N}} a_n e_{n/2}$$

i.e.

$$\{a_1, a_2, a_3, \dots\} \mapsto \{a_2, a_4, a_6, \dots\}$$

We now compute the following actions:

$$T^*Ta = T^*\{0, a_1, 0, a_2, 0, \dots\} = \{a_1, a_2, \dots\} = a$$

Namely,

$$T^*T = \mathbb{I}_{\ell^2}$$

On the other hand,

$$TT^* = T\{a_2, a_4, a_6, \dots\} = \{0, a_2, 0, a_4, 0, \dots\} = \sum_{n\in\mathbb{N}} a_{2n}e_{2n} \neq a$$

Thus, $TT^* \neq T^*T$ and T is not unitary.

Remark: Another way to show that T is not unitary is to observe that it is not surjective or to observe that T is not invertible.

In order to determine the spectra of T and T^*, we start by searching for their eigenvalues.

Remark: $||T|| = ||T^*|| = 1$.

Take $\psi_\lambda \in \mathcal{D}(T)$ and $\lambda \in \mathbb{C}$, $\lambda \leq 1$ (since $radius(T) \leq ||T||$) such that

$$T\psi_\lambda = \lambda\psi_\lambda$$

Namely,

$$\begin{cases} 0 = \lambda\psi_1 \\ \psi_1 = \lambda\psi_2 \\ 0 = \lambda\psi_3 \\ \vdots \end{cases}$$

For which the only solution admissible is the trivial one, i.e. the null vector. Hence,

$$\sigma_{pp}(T) = \emptyset$$

On the other hand,

$$T^*\psi_\lambda = \lambda\psi_\lambda \iff \begin{cases} \psi_2 = \lambda\psi_1 \\ \psi_4 = \lambda\psi_2 \\ \psi_6 = \lambda\psi_3 \\ \psi_8 = \lambda\psi_4 \\ \quad\vdots \end{cases}$$

We can rewrite the last system of equations as follows:

$$\begin{cases} \psi_{2^k} = \lambda^k\psi_1 \\ \psi_{3\cdot 2^k} = \lambda^k\psi_3 \\ \psi_{5\cdot 2^k} = \lambda^k\psi_5 \\ \qquad\vdots \\ \psi_{(2n+1)\cdot 2^k} = \lambda^k\psi_{2n+1} \\ \qquad\vdots \end{cases}$$

where $k \geq 0$.

The eigenvector ψ_λ belongs to ℓ^2 if and only if

$$\sum_{k\in\mathbb{N}} |\psi_{2^k}|^2 + \sum_{k\in\mathbb{N}} |\psi_{3\cdot 2^k}|^2 + \ldots + \sum_{k\in\mathbb{N}} |\psi_{(2n+1)\cdot 2^k}|^2 + \ldots < \infty$$

where $n \in \mathbb{N}$.

Clearly, this is true if and only if

$$\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} |\psi_{(2n+1)\cdot 2^k}|^2 < \infty \iff \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} |\psi_{2n+1}|^2|\lambda|^{2k} < \infty$$

$$\iff \sum_{n\in\mathbb{N}} |\psi_{2n+1}|^2 \sum_{k\in\mathbb{N}} |\lambda|^{2k} < \infty$$

by Fubini-Tonelli.

Remark: We suppose that $\psi_\lambda \in \ell^2$, this guarantees that Fubini-Tonelli can be applied. In fact, we are going to see that this assumption fails for all $\lambda : |\lambda| \geq 1$.

Indeed, we want $\psi_\lambda \in \ell^2$. This would imply that every sub-sequence of ψ still belongs to ℓ^2. Thus, the previous inequality holds whenever $|\lambda| < 1$, which implies the convergence of the second factor of the first term (it is a geometric series). Hence,

$$\sigma_{pp}(T^*) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

and

$$radius(T) \leq 1$$

As an intermediate step, recall that, for $A \in \mathcal{L}(\mathcal{H})$,

$$\overline{\lambda} \in \sigma_{pp}(A^*) \implies \lambda \in \sigma_{pp}(A) \vee \lambda \in \sigma_{res}(A)$$

And that

$$\lambda \in \sigma_{res}(A) \implies \overline{\lambda} \in \sigma_{pp}(A^*)$$

Being $T \in \mathcal{L}(\ell^2(\mathbb{N})$ such that $\sigma_{pp}(T) = \emptyset$ and

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} = \{\overline{\lambda} \in \mathbb{C} : |\overline{\lambda}| < 1\}$$

we obtain

$$\sigma_{res}(T) = \sigma_{pp}(T^*)$$

Finally, since both $\sigma(T)$ and $\sigma(T^*)$ must be closed sets, we conclude that

$$\sigma_c(T) = \sigma_c(T^*) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$