

MATHEMATICS OF QUANTUM MECHANICS

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Homework 1

Consider the operator T defined on $\ell^2(\mathbb{N})$ as follows

$$Te_n = e_{2n}, \quad n \in \mathbb{N}$$

where e_n is the n -th vector of the canonical basis.

1. Find the action of the operator on a generic vector $a = \sum_{n \in \mathbb{N}} a_n e_n$, $\{a_n\}_{n \in \mathbb{N}} \subset \ell^2(\mathbb{N})$, i.e., find $(Ta)_n$, $n \in \mathbb{N}$.
2. Prove that the operator is bounded and find its norm.
3. Is $\text{ran}(T)$ dense in $\ell^2(\mathbb{N})$? is the operator invertible?
4. Determine the action of T^* , T^*T and TT^* on a generic sequence $\{a_n\}_{n \in \mathbb{N}} \subset \ell^2(\mathbb{N})$: is the operator T unitary?
5. Find the spectra of both operators $\sigma(T)$ and $\sigma(T^*)$:
 - 5.1. find the eigenvalues of T and the eigenvalues of T^* ;
 - 5.2. find $\sigma_{\text{res}}(T)$;
 - 5.3. recalling that the spectrum is closed, find $\sigma(T)$, $\sigma(T^*)$, $\sigma_{\text{cont}}(T)$, $\sigma_{\text{cont}}(T^*)$.