

Homework 1, Mathematics of Quantum Mechanics

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Let $a \in \ell^2(\mathbb{N})$, $a = \sum_{n \in \mathbb{N}} a_n e_n$, where e_n is the canonical basis of ℓ^2 . The action of the operator T can be described as follows:

$$Ta = \sum_{n \in \mathbb{N}} a_n T e_n = \sum_{n \in \mathbb{N}} a_n e_{2n},$$

i.e.

$$\{a_1, a_2, \dots\} \mapsto \{0, a_1, 0, a_2, 0, \dots\}$$

Consider now $\phi \in \ell^2(\mathbb{N})$,

$$\|T\phi\|^2 = \left\langle \sum_{n \in \mathbb{N}} \phi_n e_{2n} \middle| \sum_{m \in \mathbb{N}} \phi_m e_{2m} \right\rangle = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \phi_n^* \phi_m \langle e_{2n} | e_{2m} \rangle = \sum_{n \in \mathbb{N}} \phi_n^* \phi_n = \|\phi\|^2$$

Thus, T is bounded and

$$\|T\|_{\mathcal{L}} = \sup_{\phi \in \ell^2(\mathbb{N}), \|\phi\|=1} \|T\phi\| = \|\phi\| = 1$$

The operator T is trivially not dense in $\ell^2(\mathbb{N})$. In fact, $\nexists \{\psi_n\} \subset \text{ran}(T)$ such that

$$\psi_n \rightarrow \{1, 0, 0, 0, \dots\} \in \ell^2(\mathbb{N})$$

By inversion theorem, T is not invertible.

We already know that $\mathcal{D}(T) = \ell^2(\mathbb{N})$. We want to find $\mathcal{D}(T^*)$. Indeed, being T bounded:

$$\begin{aligned} \mathcal{D}(T^*) &= \{\phi \in \ell^2(\mathbb{N}) : \sup_{\psi \in \mathcal{D}(T), \|\psi\|=1} |\langle \phi | T\psi \rangle| < \infty\} \\ &= \ell^2(\mathbb{N}) \end{aligned}$$

Furthermore, for $\phi, \psi \in \ell^2$:

$$\begin{aligned}\langle \phi | T \psi \rangle &= \left\langle \sum_{n \in \mathbb{N}} \phi_n e_n \middle| \sum_{m \in \mathbb{N}} \psi_m e_{2m} \right\rangle = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \phi_n^* \psi_m \langle e_n | e_{2m} \rangle \\ &= \sum_{m \in \mathbb{N}} \phi_{2m}^* \psi_m = \langle T^* \phi | \psi \rangle\end{aligned}$$

Hence, for $a \in \ell^2 = \mathcal{D}(T^*)$:

$$T^* a = \sum_{n \in \mathbb{N}} a_n e_{n/2}$$

i.e.

$$\{a_1, a_2, a_3, \dots\} \mapsto \{a_2, a_4, a_6, \dots\}$$

We now compute the following actions:

$$T^* T a = T^* \{0, a_1, 0, a_2, 0, \dots\} = \{a_1, a_2, \dots\} = a$$

Namely,

$$T^* T = \mathbb{I}_{\ell^2}$$

On the other hand,

$$T T^* = T \{a_2, a_4, a_6, \dots\} = \{0, a_2, 0, a_4, 0, \dots\} = \sum_{n \in \mathbb{N}} a_{2n} e_{2n} \neq a$$

Thus, $T T^* \neq T^* T$ and T is not unitary.

Remark: Another way to show that T is not unitary is to observe that it is not surjective or to observe that T is not invertible.

In order to determine the spectra of T and T^* , we start by searching for their eigenvalues.

Remark: $\|T\| = \|T^*\| = 1$.

Take $\psi_\lambda \in \mathcal{D}(T)$ and $\lambda \in \mathbb{C}$, $\lambda \leq 1$ (since $\text{radius}(T) \leq \|T\|$) such that

$$T \psi_\lambda = \lambda \psi_\lambda$$

Namely,

$$\begin{cases} 0 = \lambda \psi_1 \\ \psi_1 = \lambda \psi_2 \\ 0 = \lambda \psi_3 \\ \vdots \end{cases}$$

For which the only solution admissible is the trivial one, i.e. the null vector.
Hence,

$$\sigma_{pp}(T) = \emptyset$$

On the other hand,

$$T^* \psi_\lambda = \lambda \psi_\lambda \iff \begin{cases} \psi_2 = \lambda \psi_1 \\ \psi_4 = \lambda \psi_2 \\ \psi_6 = \lambda \psi_3 \\ \psi_8 = \lambda \psi_4 \\ \vdots \end{cases}$$

We can rewrite the last system of equations as follows:

$$\begin{cases} \psi_{2^k} = \lambda^k \psi_1 \\ \psi_{3 \cdot 2^k} = \lambda^k \psi_3 \\ \psi_{5 \cdot 2^k} = \lambda^k \psi_5 \\ \vdots \\ \psi_{(2n+1) \cdot 2^k} = \lambda^k \psi_{2n+1} \\ \vdots \end{cases}$$

where $k \geq 0$.

The eigenvector ψ_λ belongs to ℓ^2 if and only if

$$\sum_{k \in \mathbb{N}} |\psi_{2^k}|^2 + \sum_{k \in \mathbb{N}} |\psi_{3 \cdot 2^k}|^2 + \dots + \sum_{k \in \mathbb{N}} |\psi_{(2n+1) \cdot 2^k}|^2 + \dots < \infty$$

where $n \in \mathbb{N}$.

Clearly, this is true if and only if

$$\begin{aligned} \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} |\psi_{(2n+1) \cdot 2^k}|^2 < \infty &\iff \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} |\psi_{2n+1}|^2 |\lambda|^{2k} < \infty \\ &\iff \sum_{n \in \mathbb{N}} |\psi_{2n+1}|^2 \sum_{k \in \mathbb{N}} |\lambda|^{2k} < \infty \end{aligned}$$

by Fubini-Tonelli.

Remark: We suppose that $\psi_\lambda \in \ell^2$, this guarantees that Fubini-Tonelli can be applied. In fact, we are going to see that this assumption fails for all $\lambda : |\lambda| \geq 1$.

Indeed, we want $\psi_\lambda \in \ell^2$. This would imply that every sub-sequence of ψ still belongs to ℓ^2 . Thus, the previous inequality holds whenever $|\lambda| < 1$, which implies the convergence of the second factor of the first term (it is a geometric series). Hence,

$$\sigma_{pp}(T^*) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

and

$$\text{radius}(T) \leq 1$$

As an intermediate step, recall that, for $A \in \mathcal{L}(\mathcal{H})$,

$$\bar{\lambda} \in \sigma_{pp}(A^*) \implies \lambda \in \sigma_{pp}(A) \vee \lambda \in \sigma_{res}(A)$$

And that

$$\lambda \in \sigma_{res}(A) \implies \bar{\lambda} \in \sigma_{pp}(A^*)$$

Being $T \in \mathcal{L}(\ell^2(\mathbb{N}))$ such that $\sigma_{pp}(T) = \emptyset$ and

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} = \{\bar{\lambda} \in \mathbb{C} : |\bar{\lambda}| < 1\}$$

we obtain

$$\sigma_{res}(T) = \sigma_{pp}(T^*)$$

Finally, since both $\sigma(T)$ and $\sigma(T^*)$ must be closed sets, we conclude that

$$\sigma_c(T) = \sigma_c(T^*) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$