Politecnico di Milano – a.a. 2023/24
MSc in Mathematical Engineering

Mathematics of Quantum Mechanics

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Homework 2

Consider the operator $H = -\Delta$ defined on $C_0^\infty((0,2\pi)) \subset L^2((0,2\pi), \mathrm{d}x)$.

1. Show that the sets $C^2([0,2\pi])$ and
$$H^2((0,2\pi)) = \left\{ \Psi = \sum_{n\in\mathbb{Z}} \psi_n e_n(x) \;\middle|\; \left\{n^2\psi_n\right\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\},$$
are both contained in the domain of the adjoint.

2. Prove that H is not essentially self-adjoint on $C_0^\infty((0,2\pi))$ but it admits self-adjoint extensions.

3. Determine the number of parameters characterizing the families of self-adjoint extensions.

4. Prove that $H_\mathrm{N} = H_\mathrm{D} = -\Delta$ with domains
$$\mathscr{D}(H_\mathrm{N}) = \left\{ \Psi \in H^2((0,2\pi) \;\middle|\; \Psi'(0) = \Psi'(2\pi) = 0 \right\},$$
$$\mathscr{D}(H_\mathrm{D}) = \left\{ \Psi \in H^2((0,2\pi) \;\middle|\; \Psi(0) = \Psi(2\pi) = 0 \right\},$$
belong to the family.

5. Rewrite the domains $\mathscr{D}(H_\mathrm{N})$ and $\mathscr{D}(H_\mathrm{D})$ in terms of the Fourier coefficients and prove that for any $a > 0$ there exists $b < +\infty$ such that
$$\|\Psi\|_\infty \leqslant a \left\|H_{\mathrm{N/D}}\right\|_2 + b \|\Psi\|_2 .$$

6. Let $V(x) = x^{-1/4}$, prove that $H_\mathrm{N} + V$ and $H_\mathrm{D} + V$ are self-adjoint and find the self-adjointness domains.