Politecnico di Milano – a.a. 2023/24
MSc in Mathematical Engineering

Mathematics of Quantum Mechanics

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Homework 3

Consider an hydrogen atom immersed in a magnetic field $\mathbf{B}(\mathbf{x})$. Neglecting the interaction between the electron spin and the magnetic field, the Hamiltonian in the center of mass reference frame reads (with $r = |\mathbf{x}|$)

$$H_{\mathbf{A}} = \frac{\hbar}{2m}\left(-i\nabla + \gamma\mathbf{A}\right)^2 - \frac{e^2}{r},$$

where $\gamma > 0$ is the coupling constant and the vector potential $\mathbf{A} \in C_0^\infty(\mathbb{R}^3)$ is real and such that $\mathbf{B}(\mathbf{x}) = \nabla \wedge \mathbf{A}(\mathbf{x})$ and $\nabla \cdot \mathbf{A} = 0$ (Coulomb gauge).

1. Prove that the operator $H_{\mathbf{A}}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ and bounded from below.
2. Prove that $\sigma_{\mathrm{ess}}(H_{\mathbf{A}}) = \mathbb{R}^+$.
3. Prove that $\sigma_{\mathrm{disc}}(H_{\mathbf{A}}) \neq \emptyset$ and find upper and lower bounds for $E_1 := \inf \sigma(H_{\mathbf{A}})$ matching to leading order in γ as $\gamma \to 0$.