Online Appendix to: Business Dynamism, Sectoral Reallocation and Productivity in a Pandemic

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Abstract

This Online Appendix presents the technical details and the derivations used in the main text. It also contains the Appendix included in the main text, presented with a more comprehensive description of the results.

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1 Households

The economy features a continuum of homogeneous households or family of mass one, and markets are complete. For these reasons we consider a representative household from the outset. Each household is populated by a continuum of individuals, who differ in their contagion type. The family is initially of unitary length, while at time t has size $(1-\mathcal{D}_t)$, where $\mathcal{D}_t$ denoted the cumulative number of dead members of the household. The time-t utility of the representative household is:

$$(1-\mathcal{D}_t)\, log\,(c_t) - (1-\mathcal{D}_t)\,\nu \left(\frac{(l_t^s)^{1+\phi}}{1+\phi}\right) - u_d\mathcal{D}_t \tag{1}$$

where c_t is the individual, i.e. of each living household's member, consumption of the final good, l_t^s denotes individual labor supply and u_d is the flow disutility from death, which includes the psychological costs of death on surviving members. The final good is a composite good defined as: $c_t = \left[\chi^{\frac{s}{\eta}} c_t(s)^{\frac{\eta-1}{\eta}} + (1-\chi)^{\frac{1}{\eta}}\, c_t(ns)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$, where $c_t(s)$ and $c_t(ns)$ are the individual consumption levels of the social and the non-social good, respectively. Both $c_t(s)$ and $c_t(ns)$ are defined as aggregators of goods produced in the social and non-social sector, respectively. The parameter χ captures the relative importance of the social good in the consumption basket, while the parameter $\eta > 1$ measures the elasticity of substitution between the social and the non-social goods.

1.1 Income and Investment

In each time period t, agents can purchase any desired state-contingent nominal payment A_{t+1} in period $t+1$ at the dollar cost $E_t\Lambda_{t,t+1}A_{t+1}/\pi_{t+1}$, where $\Lambda_{t,t+1}$ denotes the stochastic discount factor between period t+1 and t, and π_{t+1} denotes the inflation rate over the same period. Households choose consumption, hours of work, and how much to invest in state-contingent assets and in risky stocks $b_{t+1}(q)$. Stock ownership ensures to households a flow of dividend distributed by operative firms. We assume a continuum of differentiated labor inputs indexed by $j \in [0,1]$. Wages are set by labor type specific unions, indexed by $j \in [0,1]$. Given the nominal wage, W_t^j, set by union j, agents stand ready to supply as many hours to labor market j, L_t^j , as required by firms, that is

$$L_t^j = (W_t^j / W_t)^{-\theta_w} L_t^d, \tag{2}$$

where W_t is an aggregate wage index, and L_t^d is aggregate labor demand. The latter can be obtained by integrating firms' individual labor demand over the distribution of idiosyncratic productivities. Formal definitions of the labor demand and of the wage index can be found in the sections devoted to firms. Agents are distributed uniformly across unions, hence aggregate demand for labor type j is spread uniformly across households. Total hours must satisfy the time resource constraint $L_t^s = \int_0^1 L_t^j \, dj$. Combining the latter with equation (2) we obtain

$$L_t^s = L_t^d \int_0^1 (w_t^j / w_t)^{-\theta_w} \, dj$$

where lower case letters denote wages in real terms. The labor market structure rules out differences in labor income between households without the need to resort to contingent markets for hours. The common labor income is given by

$$\int_0^1 (w_t^j L_t^j) = L_t^d \int_0^1 w_t^j (w_t^j / w_t)^{-\theta_w} \, dj.$$

Besides labor income, households enjoy dividend income through stock ownership. The timing of investment in the stock market is as in Bilbiie et al. (2012) and Chugh and Ghironi (2011). At the beginning of period t, the household owns $b_t(q)$ shares of a sector mutual fund that represents the ownership of the $N_t(q)$ incumbents in sector (q) in period t, with $(q) = \{(s), (ns)\}$.

The period-t asset value of the portfolio of firms held in sector (q) can be expressed as the sum of two components. First, the total firms' value in sector (q), which is the product between the average value of a firm $\tilde{v}_t(q)$ and the existing mass of firms $N_t(q)$ in the same sector. Second, following the production and sales of varieties in the imperfectly competitive goods markets, total firms' dividends, distributed only by operative firms. Operative firms in sector (q), that we denote as $N_{o,t}(q)$ and formally define below, are the set of firms that are actively producing in each sector at time t. As shown in Appendix 10, total dividends received by a household in a sector can be written as $N_{o,t}(q)\tilde{e}_t(q)$, where $\tilde{e}_t(q)$ denotes average sectoral dividends, that is the amount of dividends distributed by the firm with average sectoral productivity. To obtain the total value of the portfolio held by households, one needs to sum both components over the two sectoral funds.

During period t, the household purchases $b_{t+1}(q)$ shares in new sectoral funds to be carried to period $t+1$. Since the household does not know which firms will disappear from the market, it finances the continued operations of all incumbent firms as well as those of the new entrants, $N_t^e(q)$, although at the very end of period t a fraction of these firms disappears. The value of total stock market purchases is thus $\sum_{q=s,ns} \tilde{v}_t(q)\left(N_t(q)+N_t^e(q)\right) b_{t+1}(q)$.

A fraction of the resources of household is deposited to financial intermediaries that provide loans to firms. Firms use one-period loans to finance a fraction $\alpha_w \in [0,1]$ of the wage bill in advance of production. In equilibrium, a real amount equal to $\alpha_w w_t L_t^d$ must be gathered for this purpose. The deposit yields a gross interest rate R_t. Interests on deposits are distributed to households at the end of each period t in a lump sum fashion. We can finally write the flow budget constraint of the representative household as:

$$(1-\mathcal{D}_t) \sum_{q=s,ns} \rho_t\left(q\right) c_t\left(q\right) + E_t r_{t,t+1} a_{t+1} + \sum_{q=s,ns} \tilde{v}_t(q)\left(N_t(q)+N_t^e(q)\right) b_{t+1}(q) =$$

$$= L_t^d \int_0^1 w_t^j \left(\frac{w_t^j}{w_t}\right)^{-\theta_w} dj + \frac{a_t}{\pi_t} + \sum_{q=s,ns} \left(N_t(q)\tilde{v}_t(q) + N_{o,t}(q)\tilde{e}_t(q)\right) b_t(q) + \left(R_t - 1\right) \alpha_W w_t L_t^d$$

where $\rho_t\,(q)$ is the price of the good produced in sector (q) expressed in real terms, that we define in the section devoted to firms.

1.2 Utility Maximization

Denoting with V_t the household's value at time t, utility can be written in recursive form as:

$$V_t(\cdot) = (1 - \mathcal{D}_t)\, log\,(c_t) - (1 - \mathcal{D}_t)\, \nu \left(\frac{(l_t^s)^{1+\phi}}{1+\phi}\right) - u_d \mathcal{D}_t + \beta E_t V_{t+1}(\cdot) \qquad (3)$$

The household maximizes (3) with respect to $c_t\,(s)$, $c_t\,(ns)$, l_t^s, $\mathcal{D}_{t+1}$, $\mathcal{I}_{t+1}, \mathcal{S}_{t+1}$, $\mathcal{R}_{t+1}$ (dropped from the maximization as $\mathcal{R}_t = 1 - \mathcal{D}_t - \mathcal{I}_t - \mathcal{S}_t$), a_{t+1}, $b_{t+1}\,(s)$ and $b_{t+1}\,(ns)$ at any t. Constraints to the problem are the household's budget constraint presented above, the time resource constraint $L_t^s = L_t^d \int_0^1 (w_t^j/w_t)^{-\theta_w}\, dj$ and the equations defining contagi-on, presented in the main text. The recursive utility maximization problem reads as:

$$
\begin{aligned}
V_t(\mathcal{S}_t, \mathcal{I}_t, \mathcal{D}_t, a_t, b_t(s), b_t(ns)) = &(1-\mathcal{D}_t)\, log\,(c_t) - (1-\mathcal{D}_t)\,\nu \left(\frac{(l_t^s)^{1+\phi}}{1+\phi} \right) - u_d \mathcal{D}_t + \\
&+ \beta E_t V_{t+1}(\mathcal{S}_{t+1}, \mathcal{I}_{t+1}, \mathcal{D}_{t+1}, a_{t+1}, b_{t+1}(s), b_{t+1}(ns)) \\
&+ \lambda_t \Bigg[L_t^d \int_0^1 w_t^j \left(\frac{w_t^j}{w_t} \right)^{-\theta_w} dj + \frac{a_t}{\pi_t} + \sum_{q=s,ns} \left(N_t(q) \tilde{v}_t(q) + N_{o,t}(q) \tilde{e}_t(q) \right) b_t(q) \\
&+ (R_t - 1)\,\alpha_W w_t L_t^d - (1-\mathcal{D}_t) \sum_{q=s,ns} \rho_t\,(q)\, c_t\,(q) - E_t r_{t,t+1} a_{t+1} + \\
&- \sum_{q=s,ns} \tilde{v}_t(q) \left(N_t(q) + N_t^e(q) \right) b_{t+1}(q) \Bigg] + \\
&+ \frac{\lambda_t w_t}{\tilde{\mu}_t} \left[(1-\mathcal{D}_t)\, l_t^s - L_t^d \int_0^1 \left(\frac{w_t^j}{w_t} \right)^{-\theta_w} dj \right] + \\
&+ \lambda_{\mathcal{T},t} \left[\mathcal{T}_t - \mathcal{S}_t \mathbb{I}_t \pi_1 c_t\,(s)\, C_t\,(s) - -thcal S_t \mathbb{I}_t \pi_2 l_t^s L_t^d - \pi_3 \mathcal{S}_t \mathbb{I}_t \right] + \\
&+ \lambda_{\mathcal{I},t} \left[\mathcal{S}_{t+1} - \mathcal{T}_t - \mathcal{I}_t + (\pi_r + \pi_d)\, \mathcal{I}_t \right] + \\
&+ \lambda_{\mathcal{S},t} \left[\mathcal{S}_{t+1} - \mathcal{S}_t + \mathcal{T}_t \right] + \\
&+ \lambda_{\mathcal{D},t} \left[\mathcal{D}_{t+1} - \mathcal{D}_t - \pi_d \mathcal{I}_t \right]
\end{aligned}
$$

where $\rho_t\,(q) = \frac{P_t(q)}{P_t}$ and $c_t = \left[\chi^{\frac{1}{\eta}} c_t(s)^{\frac{\eta-1}{\eta}} + (1-\chi)^{\frac{1}{\eta}} c_t(ns)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$. The First Order Conditions (FOCs) are the following:[1]

$$
c_t\,(s) : \chi^{\frac{1}{\eta}} \left(\frac{c_t(s)}{c_t} \right)^{\frac{-1}{\eta}} = \lambda_t \rho_t\,(s)\, c_t + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathbb{I}_t}{1-\mathcal{D}_t} \pi_1 C_t\,(s)\, c_t
$$

[1]Rearranged from

$$
c_t\,(s) : (1-\mathcal{D}_t) \left[\frac{\eta}{\eta-1} c_t^{\frac{1}{\eta}-1} \chi^{\frac{1}{\eta}} \frac{\eta-1}{\eta} c_t(s)^{\frac{-1}{\eta}} \right] - (1-\mathcal{D}_t)\, \lambda_t \rho_t\,(s) - \lambda_{\mathcal{T},t} \mathcal{S}_t \mathbb{I}_t \pi_1 C_t\,(s) = 0
$$

$$
c_t\,(ns) : (1-\mathcal{D}_t) \left[\frac{\eta}{\eta-1} c_t^{\frac{1}{\eta}-1} (1-\chi)^{\frac{1}{\eta}} \frac{\eta-1}{\eta} c_t(ns)^{\frac{-1}{\eta}} \right] = (1-\mathcal{D}_t)\, \lambda_t \rho_t\,(ns)
$$

$$
l_t^s : -v\,(1-\mathcal{D}_t)\,(l_t^s)^{\phi} + \frac{\lambda_t w_t}{\tilde{\mu}_t} (1-\mathcal{D}_t) - \lambda_{\mathcal{T},t} \mathcal{S}_t \mathbb{I}_t \pi_2 L_t^d = 0
$$

$$c_t\left(ns\right):\left(1-\chi\right)^{\frac{1}{\eta}}\left(\frac{c_t(ns)}{c_t}\right)^{\frac{-1}{\eta}}=\lambda_t\rho_t\left(ns\right)c_t$$

$$l_t^s:\nu\left(l_t^s\right)^{\phi}=\frac{\lambda_t w_t}{\tilde{\mu}_t}-\lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathbb{I}_t}{1-\mathcal{D}_t}\pi_2 L_t^d$$

$$\mathcal{T}_t:\lambda_{\mathcal{T},t}+\lambda_{\mathcal{S},t}-\lambda_{\mathcal{I},t}=0$$

$$\mathcal{I}_{t+1}:\beta E_t V_{\mathcal{I},t+1}+\lambda_{\mathcal{I},t}=0$$

$$\mathcal{S}_{t+1}:\beta E_t V_{\mathcal{S},t+1}+\lambda_{\mathcal{S},t}=0$$

$$\mathcal{D}_{t+1}:\beta E_t V_{\mathcal{D},t+1}+\lambda_{\mathcal{D},t}=0$$

$$a_{t+1}:\beta E_t V_{a,t+1}-\lambda_t E_t r_{t,t+1}=0$$

$$b_{t+1}(s):\beta E_t V_{b(s),t+1}-\lambda_t\tilde{v}_t(s)\left(N_t(s)+N_t^e(s)\right)=0$$

$$b_{t+1}(ns):\beta E_t V_{b(ns),t+1}-\lambda_t\tilde{v}_t(ns)\left(N_t(ns)+N_t^e(ns)\right)=0$$

Finally, the envelope conditions are:

$$V_{\mathcal{I},t}=-\lambda_{\mathcal{I},t}\left(1-\pi_r\right)+\pi_d\left(\lambda_{\mathcal{I},t}-\lambda_{\mathcal{D},t}\right)$$

$$V_{\mathcal{S},t}=\lambda_{\mathcal{T},t}\left[-\mathbb{I}_t\pi_1 c_t\left(s\right)C_t\left(s\right)-\mathbb{I}_t\pi_2 l_t^s L_t^d-\pi_3\mathbb{I}_t\right]-\lambda_{\mathcal{S},t}$$

$$V_{\mathcal{D},t}=-log(c_t)+\nu\left(\frac{\left(l_t^s\right)^{1+\phi}}{1+\phi}\right)-u_d+\lambda_t\left[\rho_t\left(s\right)c_t\left(s\right)+\rho_t\left(ns\right)c_t\left(ns\right)\right]-\frac{\lambda_t w_t}{\tilde{\mu}_t}l_t^s-\lambda_{\mathcal{D},t}$$

$$V_{a,t}=\lambda_t\frac{1}{\pi_t}$$

$$V_{b(s),t}=\lambda_t\left[N_t(s)\tilde{v}_t(s)+N_{o,t}(s)\tilde{e}_t(s)\right]$$

$$V_{b(ns),t} = \lambda_t \left[N_t(ns)\tilde{v}_t(ns) + N_{o,t}(ns)\tilde{e}_t(ns) \right]$$

Jones et al. (2020) point out that there is one externality not internalized by households. When considering current exposure risk in the newly infected equation, households scales it to $\mathbb{I}_t$ and not to $\mathcal{I}_t$. For this reason it neglects the risk of infecting others more tomorrow due to current choices. As a result, their mitigation efforts are lower than what would be socially optimal. Given the available workforce, it follows that aggregate labor supply is $L_t^s = (1 - \mathcal{D}_t)\, l_t^s$. The no arbitrage condition, such that the expected returns on different assets classes are equalized, is $E_t r_{t,t+1} = 1/R_t$.

2 Consumption of Individual Goods and Price Indexes

Let $c_{z,t}(q)$ be the consumption of the good produced by the firm with productivity z in sector (q), or, in short, the consumption of good z in sector (q). Consider minimizing expenditure when buying goods from the social sector, i.e. the sector which entails exposure externalities. The expenditure minimization problem of the household reads as follows. We assume that the levels of consumption involved here are individual, i.e. relative to one member of the household. The problem is:

$$\min_{c_{z,t}(s),} \int_0^\infty N_t(s) p_{z,t}(s)\, c_{z,t}(s)\, g(z) dz$$

such that:

$$c_t(s) = \left(\int_0^\infty N_t(s) c_{z,t}(s)^{\frac{\theta-1}{\theta}} g(z) dz \right)^{\frac{\theta}{\theta-1}}$$

The Lagrangian is:

$$\mathcal{L} = \int_0^\infty N_t(s) p_{z,t}(s)\, c_{z,t}(s)\, g(z) dz + \Theta_t(s) \left[c_t(s) - \left(\int_0^\infty N_t(s) c_{z,t}(s)^{\frac{\theta-1}{\theta}} g(z) dz \right)^{\frac{\theta}{\theta-1}} \right]$$

where $c_{z,t}(s)$ is the consumption of good z in the bundle $c_t(s)$. The F.O.C. is:

$$N_t(s)p_{z,t}(s) - \Theta_t \frac{\theta}{\theta-1}\left(\int_0^\infty N_t(s)c_{z,t}(s)^{\frac{\theta-1}{\theta}} g(z)dz\right)^{\frac{1}{\theta-1}} N_t(s)\,\frac{\theta-1}{\theta} c_{z,t}(s)^{\frac{-1}{\theta}} = 0$$

or

$$p_{z,t}(s) = \Theta_t(s)c_t(s)^{\frac{1}{\theta}}\, c_{z,t}(s)^{-\frac{1}{\theta}}$$

Raise to the power of $1-\theta$, multiply by $N_t(s)$ and integrate over the distribution of the idiosyncratic productivitities

$$\int_0^\infty N_t(s)p_{z,t}(s)^{1-\theta} g(z)dz = \Theta_t(s)^{1-\theta}c_t(s)^{\frac{1-\theta}{\theta}} \int_0^\infty N_t(s)c_{z,t}(s)^{\frac{\theta-1}{\theta}} g(z)dz$$

raise to $\frac{1}{1-\theta}$ to get

$$\left(\int_0^\infty N_t(s)p_{z,t}(s)^{1-\theta} g(z)dz\right)^{\frac{1}{1-\theta}} = \Theta_t(s)c_t(s)^{\frac{1}{\theta}}\left(\int_0^\infty N_t(s)c_{z,t}(1)^{\frac{\theta-1}{\theta}} g(z)dz\right)^{\frac{1}{1-\theta}}$$

Since the term in round brackets in the RHS equals $c_t(s)^{-\frac{1}{\theta}}$ it follows that

$$\left(\int_0^\infty N_t(s)p_{z,t}(s)^{1-\theta} g(z)dz\right)^{\frac{1}{1-\theta}} = \Theta_t(s)$$

$\Theta_t(s)$ represents the cost of relaxing the constraint when the objective is minimized, thus it amounts to the minimum cost of acquiring a unit of $\mathrm{c}_t(s)$, and it can be given the interpretation of the price index in sector s, that is $P_t(s)$, which can thus be written as:

$$P_t(s) = \left(\int_0^\infty N_t(s)p_{z,t}(s)^{1-\theta} g(z)dz\right)^{\frac{1}{1-\theta}}.$$

Substituting the latter into the initial FOC we obtain

$$\frac{c_{z,t}(s)}{c_t(s)} = \left(\frac{p_{z,t}(s)}{P_t(s)}\right)^{-\theta}$$

Given all member of the surviving members of the households have the same consumption, it follows that the latter can be rewritten as

$$\frac{C_{z,t}(s)}{C_t(s)} = \left(\frac{p_{z,t}(s)}{P_t(s)}\right)^{-\theta}$$

where recall that capital letters denote aggregate variables, thus $C_{z,t}(s)$ is the aggregate consumption of good z in sector s, and $C_t(s)$ is the aggregate consumption of bundle s. From the household problem that the individual demand of bundle s is given by:

$$\chi^{\frac{1}{\eta}}\left(\frac{c_t(s)}{c_t}\right)^{\frac{-1}{\eta}} = \left[\lambda_t\rho_t(s) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathbb{I}_t}{1-\mathcal{D}_t}\pi_1 C_t(s)\right]c_t$$

or

$$\left(\frac{c_t(s)}{c_t}\right) = \chi\left[\lambda_t\rho_t(s) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathbb{I}_t}{1-\mathcal{D}_t}\pi_1 C_t(s)\right]^{-\eta}c_t^{-\eta}$$

The latter represents the demand of good s by an individual member of the household. Aggregate consumption levels are:

$$C_t(s) = (1-\mathcal{D}_t)\, c_t(s)$$

and

$$C_t = (1-\mathcal{D}_t)\, c_t$$

As a result, the aggregate consumption of bundle s is

$$\left(\frac{C_t(s)}{C_t}\right) = \chi\left[\lambda_t\rho_t(s) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathbb{I}_t}{1-\mathcal{D}_t}\pi_1 C_t(s)\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta}$$

or

$$C_t(s) = (1-\mathcal{D}_t)^{\eta}\chi\left[\lambda_t\rho_t(s) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathbb{I}_t}{1-\mathcal{D}_t}\pi_1 C_t(s)\right]^{-\eta}(C_t)^{1-\eta}$$

Turning to the consumption of bundle ns, from the household problem we get:

$$(1-\chi)^{\frac{1}{\eta}}\left(\frac{C(ns)}{C_t}\right)^{\frac{-1}{\eta}}=\lambda_t\rho_t\left(ns\right)\frac{C_t}{(1-\mathcal{D}_t)}$$

or

$$C_t(ns)=(1-\chi)\left(1-\mathcal{D}_t\right)^{\eta}\left[\lambda_t\rho_t\left(ns\right)\right]^{-\eta}C_t^{1-\eta}$$

Note that both demand functions simplify to standard CES demands if there is no pandemic.

3 Aggregate Price Index

The aggregate price index must be such that

$$P_tC_t=P_t\left(s\right)C_t\left(s\right)+P_t\left(ns\right)C_t\left(ns\right)$$

substituting for the demand functions:

$$P_tC_t=P_t\left(s\right)\chi\left(1-\mathcal{D}_t\right)^{\eta}\left[\lambda_t\rho_t\left(s\right)+\lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathbb{I}_t}{1-\mathcal{D}_t}\pi_1C_t\left(s\right)\right]^{-\eta}C_t^{1-\eta}+$$

$$+P_t\left(ns\right)\left(1-\chi\right)\left(1-\mathcal{D}_t\right)^{\eta}\left(\lambda_t\rho_t\left(ns\right)\right)^{-\eta}C_t^{1-\eta}$$

or

$$P_t=\left(\frac{C_t}{1-\mathcal{D}_t}\right)^{-\eta}\left\{P_t\left(s\right)\chi\left[\lambda_t\rho_t\left(s\right)+\lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathbb{I}_t}{1-\mathcal{D}_t}\pi_1C_t\left(s\right)\right]^{-\eta}+P_t\left(ns\right)\left(1-\chi\right)\left(\lambda_t\rho_t\left(ns\right)\right)^{-\eta}\right\}$$

Notice that when $\mathbb{I}_t=0$, $\mathcal{D}_t=0$ and, thus, $\lambda_t=\frac{1}{C_t}$:

$$P_t=\chi P_t(s)^{1-\eta}P_t^{\eta}+\left(1-\chi\right)P_t(ns)^{1-\eta}P_t^{\eta}$$

$$P_t=\left[\chi P_t(s)^{1-\eta}+\left(1-\chi\right)P_t(ns)^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

which is the traditional price index under CES production function.

4 Firms and Cost Minimization

Each sector (q) is populated by a mass $N_t(q)$ of atomistic firms. Once upon entry, firms draw a time invariant idiosyncratic productivity level, denoted by z, from a known distribution function, $g(z)$, which is identical across sectors and has a positive support. Within their sector of operation, the only source of heterogeneity across firms is the idiosyncratic productivity level, so that we can can index firms within a sector with z. Firms compete monopolistically within the sector and are subject to entry and exit. Each firm produces an imperfectly substitutable good $y_{z,t}(q)$, which is an input to the production of a sectoral bundle $Y_t(q)$ by a sectoral good producer. The latter adopts a CES production function defined as:

$$Y_t\left(q\right) = \left(\int_0^{\infty} N_t(q) y_{z,t}(q)^{\frac{\theta-1}{\theta}} g(z) dz\right)^{\frac{\theta}{\theta-1}} \tag{4}$$

where $\theta > 1$ is the degree of substitution between sectoral goods. The production function of individual goods producers is a constant return to scale Cobb-Douglas function, with parameter $0 \leq \alpha \leq 1$. The two inputs are labor, $l_{z,t}(q)$, and an intermediate input, $X_{z,t}(q)$. The latter is a composite of all the goods in the economy, i.e. we define a roundabout in production. The individual production function reads as:

$$y_{z,t}(q) = Z_t z l_{z,t}(q)^{1-\alpha} X_{z,t}(q)^{\alpha}$$

where the variable Z_t is an exogenous, and common to all firms, level of productivity. The labor input is defined as a CES aggregator of differentiated labor inputs indexed by $j \in [0, 1]$, defined as:

$$l_{z,t}(q) = \left(\int_0^1 (l^j_{z,t}(q))^{\frac{\theta_w-1}{\theta_w}} dj\right)^{\frac{\theta_w}{\theta_w-1}} \tag{5}$$

where $\theta_w > 1$ is the degree of substitution between labor inputs. The minimization of total labor costs, $\int_0^1 W_t^j l_{z,t}^j(q)dj$, delivers firm z's demand of labor input j and the definition of the aggregate nominal wage index, which are respectively:

$$l_{z,t}^j(q) = \left(\frac{w_t^j}{w_t}\right)^{-\theta_w} l_{z,t}(q) \tag{6}$$

and

$$w_t = \left(\int_0^1 \left(w_t^j\right)^{1-\theta_w} dj\right)^{\frac{1}{\theta_w - 1}} \tag{7}$$

where W_t^j (w_t^j) is the nominal (real) wage of labor input j, and $l_{z,t}(q)$ denotes the demand of the labor bundle by firm z.

Our analysis encompasses the case where firms have to partially pay their workers before production takes place. In this case, firms finance a fraction $0 \leq \alpha_W \leq 1$ of their wage bill resorting to loans from financial intermediaries. Loans are reimbursed at the end of the period at the gross risk-free interest rate R_t. When $\alpha_W = 1$ the entire wage bill must be borrowed in advance of production, while when $\alpha_W = 0$ firms do not borrow at all.[2] Additionally, firms face fixed costs of production $f_{x,t}$, defined in terms of the final good.

Before maximizing profits, firms choose the optimal levels of labor and intermediate input to minimize the costs of production for a given level of idiosyncratic output. The minimization of the costs of production for a firm is:

$$\min_{l_{z,t}(q), X_{z,t}(q)} \left(\alpha_W R_t + 1 - \alpha_W\right) W_t l_{z,t}(q) + P_t X_{z,t}(q) + P_t f_{x,t}$$

subject to the definition of the production function:

$$y_{z,t}(q) = Z_t z l_{z,t}(q)^{1-\alpha} X_{z,t}(q)^{\alpha} \tag{8}$$

Note that $f_{x,t}P_t$ are the nominal fixed costs of production (since $f_{x,t}$ are the real

[2] The latter is assumed in the benchmark specification.

fixed production costs). The Lagrangian is:

$$\mathcal{L} = (\alpha_W R_t + 1 - \alpha_W) W_t l_{z,t}(q) + P_t X_{z,t}(q) + P_t f_{x,t} + \lambda_{z,t}(q) \left[y_{z,t}(q) - Z_t z l_{z,t}(q)^{1-\alpha} X_{z,t}(q)^{\alpha} \right]$$

The F.O.C. with respect to $l_{z,t}(q)$ is:

$$(\alpha_W R_t + 1 - \alpha_W) W_t = \lambda_{z,t}(q) (1 - \alpha) Z_t z l_{z,t}(q)^{-\alpha} X_{z,t}(q)^{\alpha} \tag{9}$$

while the F.O.C. with respect to $X_{z,t}(q)$ is:

$$P_t = \lambda_{z,t}(q) \alpha Z_t z l_{z,t}(q)^{1-\alpha} X_{z,t}(q)^{\alpha-1} \tag{10}$$

Combining the two F.O.C.s we get the optimal ratio between the two inputs, which does not depend on idiosyncratic variables nor on sectoral quantities:

$$\frac{X_{z,t}(q)}{l_{z,t}(q)} = \frac{\alpha}{1-\alpha} \frac{W_t}{P_t} (\alpha_W R_t + 1 - \alpha_W)$$

because of this reason, this optimal condition holds in both sectors and for all firms.

Moreover, it is easy to show that $\lambda_{z,t}(q)$ is the marginal cost. First, substitute (9) and (10) in the cost function

$$\begin{aligned}(\alpha_W R_t + 1 - \alpha_W) W_t l_{z,t}(q) + P_t X_{z,t}(q) + f_{x,t} P_t &= \lambda_{z,t}(q) (1 - \alpha) Z_t z l_{z,t}(q)^{-\alpha} X_{z,t}(q)^{\alpha} l_{z,t}(q) + \\ &= + \lambda_{z,t}(q) \alpha Z_t z l_{z,t}(q)^{1-\alpha} X_{z,t}(q)^{\alpha-1} X_{z,t}(q) + f_{x,t} P_t = \\ &= \lambda_{z,t}(q) Z_t z l_{z,t}(q)^{1-\alpha} X_{z,t}(q)^{\alpha} + f_{x,t} P_t = \lambda_{z,t}(q) y_{z,t}(q) + f_{x,t} P_t\end{aligned}$$

Hence the cost function is linear in the output (CRTS) and $\frac{\partial TC_{z,t}(q)}{\partial y_{z,t}(q)} = MC_{z,t}(q) = \lambda_{z,t}(q)$.

Second, note that from (9) and (10) we can get the expression for the marginal cost which is given by $\lambda_{z,t}(q)$

$$\left(\alpha_W R_t + 1 - \alpha_W\right) W_t = \lambda_{z,t}(q)\left(1-\alpha\right) Z_t z \left(\frac{X_{z,t}(q)}{l_{z,t}(q)}\right)^{\alpha}$$

or

$$\left(\alpha_W R_t + 1 - \alpha_W\right) W_t = \lambda_{z,t}(q)\left(1-\alpha\right) Z_t z \left(\frac{\lambda_{z,t}(q)\alpha Z_t z}{P_t}\right)^{\frac{\alpha}{1-\alpha}} =$$

$$= \lambda_{z,t}(q)^{\frac{1}{1-\alpha}}\left(1-\alpha\right)\left(Z_t z\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{P_t}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{\left(\alpha_W R_t + 1 - \alpha_W\right) W_t}{1-\alpha}\left(\frac{P_t}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} = \left[\lambda_{z,t}(q)\left(Z_t z\right)\right]^{\frac{1}{1-\alpha}}$$

$$\frac{1}{Z_t z}\left[\frac{\left(\alpha_W R_t + 1 - \alpha_W\right) W_t}{1-\alpha}\right]^{1-\alpha}\left(\frac{P_t}{\alpha}\right)^{\alpha} = \lambda_{z,t}(q) \tag{11}$$

Thus:

$$MC_{z,t}(q) = MC_{z,t} = \frac{1}{Z_t z}\left[\frac{\left(\alpha_W R_t + 1 - \alpha_W\right) W_t}{1-\alpha}\right]^{1-\alpha}\left(\frac{P_t}{\alpha}\right)^{\alpha}. \tag{12}$$

Marginal costs are affected by both the idiosyncratic productivity level, z, and by aggregate productivity, Z_t. The aggregate price level, P_t, appears in the definition of marginal costs because it represents the cost of one unit of the intermediate input $X_{z,t}(q)$. Real profits of firm z in sector (q) read as :

$$e_{z,t}(q) = p_{z,t}(q)y_{z,t}(q) - \left(\alpha_W R_t + 1 - \alpha_W\right) W_t l_{z,t}(q) - P_t X_{z,t}(q) - P_t f_{x,t} \tag{13}$$

Profits maximization is presented in the following Appendix.

5 Profits Maximization

Firms maximize their per-period nominal profits by choosing the optimal price $p_{z,t}(q)$. Maximization is, thus:

$$\max_{p_{z,t}(q)} \quad p_{z,t}(q)y_{z,t}(q) - (\alpha_W R_t + 1 - \alpha_W) W_t l_{z,t}(q) - P_t X_{z,t}(q) - P_t f_{x,t} \quad q = 1, 2$$

Subject to the definition of production (8), to the F.O.C. from costs minimization, and to the demand constraint:

$$y_{z,t}(q) = \left(\frac{p_{z,t}(q)}{P_t(q)}\right)^{-\theta} Y_t(q)$$

which derives from the costs minimization of the sectoral bundlers. By combining the optimality condition from the costs minimization and the definition of the Cobb-Douglas, nominal profits can be rewritten as:

$$p_{z,t}(q)y_{z,t}(q) - y_{z,t}(q)\frac{1}{Z_t z}\left(\frac{(\alpha_W R_t + 1 - \alpha_W) W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{P_t}{\alpha}\right)^{\alpha} - f_{x,t}P_t$$

Using the demand function, profits can be rewritten as:

$$p_{z,t}(q)^{1-\theta}\left(\frac{1}{P_t(q)}\right)^{-\theta} Y_t(q) - p_{z,t}(q)^{-\theta}\left(\frac{1}{P_t(q)}\right)^{-\theta} Y_t(q) MC_{z,t} - f_{x,t}P_t$$

Under monopolistic competition, firm z takes sectoral and aggregate variables as given. As a result, the first order condition for profit maximization with respect to $p_{z,t}(q)$ reads as

$$(1-\theta) p_{z,t}(q)^{-\theta}\left(\frac{1}{P_t(q)}\right)^{-\theta} Y_t(q) + \theta p_{z,t}(q)^{-1-\theta}\left(\frac{1}{P_t(q)}\right)^{-\theta} Y_t(q) MC_{z,t} = 0$$

Under monopolistic competition, the optimal real price $\rho_{z,t}(q) = p_{z,t}(q)/P_t$ satisfies:

$$\rho_{z,t}(q) = \frac{\theta}{\theta-1}\frac{MC_{z,t}}{P_t} = \frac{\theta}{\theta-1}\frac{1}{Z_t z}\left(\frac{(\alpha_W R_t + 1 - \alpha_W)\, w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}$$

Note that the firm-z real price is solely determined by the idiosyncratic productivity level z. The term $\theta/(\theta-1)$ represents the standard time and firm invariant constant markup over marginal costs under monopolistic competition.

Individual real profits read as:

$$e_{z,t}(q) = \frac{p_{z,t}(q)}{P_t}y_{z,t}(q) - y_{z,t}(q)\frac{MC_{z,t}}{P_t} - f_{x,t} \tag{14}$$

$$e_{z,t}(q) = \rho_{z,t}(q)y_{z,t}(q) - \frac{\theta-1}{\theta}\rho_{z,t}(q)y_{z,t}(q) - f_{x,t}$$

$$e_{z,t}(q) = \frac{1}{\theta}\rho_{z,t}(q)y_{z,t}(q) - f_{x,t} = \frac{1}{\theta}\rho_{z,t}(q)\left(\frac{p_{z,t}\left(q\right)}{P_t\left(q\right)}\right)^{-\theta} Y_t\left(q\right) - f_{x,t}$$

The sectoral quantity shares the same aggregator and demand constraint of the sectoral consumption, which, thus, entails the internalization of the exposure to contagion:[3]

$$\left(\frac{Y_t(s)}{Y_t}\right) = \chi\left[\lambda_t \rho_t\left(s\right) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t \mathbb{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\left(s\right)\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta}$$

While the demand function of the non-social good, that is good ns, reads as:

$$\left(\frac{Y_t(ns)}{Y_t}\right) = (1-\chi)\left[\lambda_t \rho_t\left(ns\right)\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta}$$

Using the demand functions just provided, individual profits is sector $(q) = (s)$ can be rewritten as:

$$e_{z,t}(s) = \frac{1}{\theta}\rho_{z,t}(s)^{1-\theta}\rho_t(s)^{\theta}\chi\left[\lambda_t \rho_t\left(s\right) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t \mathbb{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\left(s\right)\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t - f_{x,t}$$

[3]See Appendix 7.

while in sector $(q) = (ns)$ they are:

$$e_{z,t}(ns) = \frac{1}{\theta}\rho_{z,t}(ns)^{1-\theta}\rho_t(ns)^{\theta}(1-\chi)\left[\lambda_t\rho_t\left(ns\right)\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t - f_{x,t}$$

In the next sections we describe the endogenous exit margin and we compute the cut-off productivity levels, used in the main text.

6 Productivity Cut-off

Firms turn inactive when, by producing, they would make negative profits. Using this, we can define a cut-off productivity level, one for each sector, below which firms become idle. Setting equilibrium real profits equal to zero we get:

$$f_{x,t} = \frac{1}{\theta}\rho_{zc,t}(q)^{1-\theta}\rho_t(q)^{\theta}Y_t\left(q\right)$$

or:

$$\left(\frac{f_{x,t}}{\rho_t(q)^{\theta}Y_t(q)}\right)^{\frac{1}{1-\theta}}\theta^{\frac{1}{1-\theta}} = \rho_{zc,t}(q)$$

substituting the real price $\rho_{z,t}$, evaluated at the cut-off zc:

$$\frac{\theta}{\theta-1}\frac{1}{Z_t z_t^c(q)}\left(\frac{\left(\alpha_W R_t + 1 - \alpha_W\right)w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha} = \left(\frac{f_{x,t}}{\rho_t(q)^{\theta}Y_t(q)}\right)^{\frac{1}{1-\theta}}\theta^{\frac{1}{1-\theta}}$$

Solving for the sectoral cut-off productivity $z_t^c(q)$:

$$z_t^c(q) = \frac{\theta^{\frac{\theta}{\theta-1}}}{\theta-1}\frac{1}{Z_t}\left(\frac{\left(\alpha_W R_t + 1 - \alpha_W\right)w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\left(\frac{f_{x,t}}{\rho_t(q)^{\theta}Y_t(q)}\right)^{\frac{1}{\theta-1}}$$

which is the formula we use in the main text.

7 Fictitious Bundler of Y_t

The demand functions for the sectoral outputs $Y_t(q)$ can also be obtained from a fictitious bundler that maximizes, in real terms:

$$Y_t - \rho_t(s)Y_t(s) - \rho_t(ns)Y_t(ns)$$

subject to:

$$Y_t = \left[\chi^{\frac{1}{\eta}} Y_t(s)^{\frac{\eta-1}{\eta}} + (1-\chi)^{\frac{1}{\eta}} Y_t(ns)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

and to:

$$\mathcal{T}_t = \mathcal{S}_t \mathbb{I}_t \pi_1 c_t\left(s\right) C_t\left(s\right) + \mathcal{S}_t \mathbb{I}_t \pi_2 l_t^s L_t^d + \pi_3 \mathcal{S}_t \mathbb{I}_t$$

The Lagrangian is:

$$\begin{gathered}\mathbb{L} = Y_t - \rho_t(s)Y_t(s) - \rho_t(ns)Y_t(ns) + \\ +\bar{\lambda}_t\left(\left[\chi^{\frac{1}{\eta}} Y_t(s)^{\frac{\eta-1}{\eta}} + (1-\chi)^{\frac{1}{\eta}} Y_t(ns)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} - Y_t\right) + \\ +\bar{\lambda}_{t,t}\left(\mathcal{T}_t - \mathcal{S}_t \mathbb{I}_t \pi_1 c_t\left(s\right) C_t\left(s\right) - \mathcal{S}_t \mathbb{I}_t \pi_2 l_t^s L_t^d - \pi_3 \mathcal{S}_t \mathbb{I}_t\right)\end{gathered}$$

The first order condition with respect to $Y_t(s)$ is:

$$-\rho_t(s) + \bar{\lambda}_t \frac{\eta}{1-\eta} Y_t^{\frac{1}{\eta}} \chi^{\frac{1}{\eta}} \frac{\eta-1}{\eta} Y_t(s)^{\frac{-1}{\eta}} - \bar{\lambda}_{t,t} \mathcal{S}_t \mathbb{I}_t \pi_1 c_t\left(s\right) \frac{\delta C_t\left(s\right)}{\delta Y_t(s)} = 0$$

Since $Y_t(s) = C_t(s) + X_t(s) + f_{x,t} N_{o,t}(s) + f_{e,t} N_t^e(s)$ and $c_t(s)(1-\mathcal{D}_t) = C_t(s)$, this can be written as:

$$\bar{\lambda}_t Y_t^{\frac{1}{\eta}} \chi^{\frac{1}{\eta}} Y_t(s)^{\frac{-1}{\eta}} = \rho_t(s) + \bar{\lambda}_{t,t} \frac{\mathcal{S}_t \mathbb{I}_t}{1-\mathcal{D}_t} \pi_1 C_t\left(s\right)$$

The Lagrange multiplier $\bar{\lambda}_t$ represents the real value in terms of increased revenues of an extra unit of Y_t (indeed, without contagion, the condition would be real marginal cost of $Y_t(s)$, i.e. $\rho_t(s)$, equal to the marginal benefit of $Y_t(s)$, which is the marginal benefit of Y_t times the marginal product of $Y_t(s)$, i.e. $\frac{\delta Y_t}{\delta Y_t(s)}$). The first is equal to 1. For the household, the value of one extra unit of Y_t is hence $1 \cdot \frac{\delta C_t}{\delta Y_t} \frac{\delta c_t}{\delta C_t} \frac{\delta U}{\delta c_t}$, which is $\frac{1-\mathcal{D}_t}{C_t}$.

On the other hand, the Lagrange multiplier $\bar{\lambda}_{t,t}$ represents the costs in terms of newly infected of having one extra unit of $Y_t(s)$. This is equal to $\lambda_{t,t}\frac{\delta C_t(s)}{\delta Y_t(s)} = \lambda_{\mathcal{T},t}$. Finally, since the household owns the bundler, we rescale both multipliers by $1/\lambda_t$ to express everything in terms of utility. Thus:

$$\frac{1-\mathcal{D}_t}{C_t}\frac{1}{\lambda_t} Y_t^{\frac{1}{\eta}} \chi^{\frac{1}{\eta}} Y_t(s)^{\frac{-1}{\eta}} = \rho_t(s) + \frac{\lambda_{\mathcal{T},t}}{\lambda_t} \frac{\mathcal{S}_t \mathbb{I}_t}{1-\mathcal{D}_t} \pi_1 C_t(s)$$

This gives:

$$Y_t^{\frac{1}{\eta}} \chi^{\frac{1}{\eta}} Y_t(s)^{\frac{-1}{\eta}} = \left[\lambda_t \rho_t(s) + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathbb{I}_t}{1-\mathcal{D}_t} \pi_1 C_t(s)\right] \frac{C_t}{1-\mathcal{D}_t}$$

Raising to the power of $-\eta$:

$$\frac{Y_t(s)}{Y_t} = \chi \left[\lambda_t \rho_t(s) + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathbb{I}_t}{1-\mathcal{D}_t} \pi_1 C_t(s)\right]^{-\eta} \left(\frac{C_t}{1-\mathcal{D}_t}\right)^{-\eta}$$

With the same reasoning, and knowing that $C_t(ns)$ does not impact the endogenous contagion rate, the first order condition for $Y_t(ns)$ is:

$$\frac{Y_t(ns)}{Y_t} = (1-\chi) \left[\lambda_t \rho_t(ns)\right]^{-\eta} \left(\frac{C_t}{1-\mathcal{D}_t}\right)^{-\eta}$$

Those are the two demand constraints used in the main text.

8 Labor Unions

Nominal wage rigidities are modeled according to the Calvo (1983) mechanism. In each period, a union faces a constant probability $(1-\alpha^*)$ of re-optimizing the wage. Due to symmetry, we denote the optimal real wage chosen at time t, as w_t^*. This wage is chosen to maximize the relevant part of agents' lifetime utilities. We assume that wages that are not re-optimized are not index to inflation, i.e. $w_{t+s} = \frac{w_t^*}{\prod_{k=1}^{s} \pi_{t+k}}$

Then, the maximization problem of the union can be written as follows:

$$E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k}\right)^{\theta_w} L_{t+s}^d (w_{t+s})^{\theta_w} \lambda_{t+s} \left\{ (w_t^*)^{1-\theta_w} \left(\prod_{k=1}^{s} \frac{1}{\pi_{t+k}}\right) - \left(\frac{w_{t+s}}{\tilde{\mu}_{t+s}}\right) (w_t^*)^{-\theta_w} \right\}$$

The FOCs with respect to w_t^* reads as: -

$$E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k}\right)^{\theta_w} L_{t+s}^d (w_{t+s})^{\theta_w} \lambda_{t+s} \left[\begin{array}{c} (1-\theta_w)(w_t^*)^{-\theta_w} \left(\prod_{k=1}^{s} \frac{1}{\pi_{t+k}}\right) + \\ +\theta_w \left(\frac{w_{t+s}}{\tilde{\mu}_{t+s}}\right) (w_t^*)^{-\theta_w - 1} \end{array} \right] = 0$$

or

$$E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k}\right)^{\theta_w} L_{t+s}^d \left(\frac{w_t^*}{w_{t+s}}\right)^{-\theta_w} \lambda_{t+s} \left[\begin{array}{c} w_t^* \frac{(\theta_w - 1)}{\theta_w} \left(\prod_{k=1}^{s} \frac{1}{\pi_{t+k}}\right) + \\ -\left(\frac{w_{t+s}}{\tilde{\mu}_{t+s}}\right) \end{array} \right] = 0$$

For simplicity, define:

$$\left(\frac{w_{t+s}}{\tilde{\mu}_{t+s}}\right) = \chi_{t+s}^*$$

such that:

$$E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k}\right)^{\theta_w} L_{t+s}^d (w_{t+s})^{\theta_w} \lambda_{t+s} \left[w_t^* \frac{(\theta_w - 1)}{\theta_w} \left(\prod_{k=1}^{s} \frac{1}{\pi_{t+k}}\right) - \chi_{t+s}^* \right] = 0$$

The latter is equivalent to:

$$
\begin{aligned}
&\frac{(\theta_w - 1)}{\theta_w} w_t^* E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k} \right)^{\theta_w - 1} L_{t+s}^d (w_{t+s})^{\theta_w} \lambda_{t+s} \\
= \quad & E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k} \right)^{\theta_w} L_{t+s}^d (w_{t+s})^{\theta_w} \lambda_{t+s} \chi_{t+s}^*
\end{aligned}
$$

Define, following Schmitt-Grohé and Uribe (2005):

$$
f_t^1 = E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k} \right)^{\theta_w - 1} L_{t+s}^d (w_{t+s})^{\theta_w} \lambda_{t+s} \tag{15}
$$

and

$$
f_t^2 = E_t \sum_{s=0}^{\infty} (\beta\alpha^*)^s \left(\prod_{k=1}^{s} \pi_{t+k} \right)^{\theta_w} L_{t+s}^d (w_{t+s})^{\theta_w} \lambda_{t+s} \chi_{t+s}^*
$$

The first order condition for wage setting is thus:

$$
w_t^* = \frac{\theta_w}{(\theta_w - 1)} \frac{f_t^2}{f_t^1}
$$

where w_t^* is the newly reset wage in real terms, and f_t^1 and f_t^s are recursively defined as:

$$
f_t^1 = L_t^d (w_t)^{\theta_w} \lambda_t + \alpha^* \beta E_t \pi_{t+1}^{\theta_w - 1} f_{t+1}^1
$$

and

$$
f_t^2 = L_t^d (w_t)^{\theta_w} \left(\nu (l_t^s)^{\phi} + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathbb{I}_t}{1 - \mathcal{D}_t} \pi_2 L_t^d \right) + \alpha^* \beta E_t \pi_{t+1}^{\theta_w} f_{t+1}^2
$$

Since:

$$
\lambda_t \chi_t^* = \lambda_t \frac{w_t}{\tilde{\mu}_t} = \nu (l_t^s)^{\phi} + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathbb{I}_t}{1 - \mathcal{D}_t} \pi_2 L_t^d
$$

9 Aggregation

For the law of large number, in each period t the wage is optimally reset in a fraction $1-\alpha^*$ of the labor markets. Demand of hours in each of those markets is:

$$l_t^* = \left(\frac{W_t^*}{W_t}\right)^{-\theta_w} L_t^d$$

As a result, total demand of hours in market where the wage has been newly reset is : $L_t^* = (1-\alpha^*)\, l_t^*$. In markets where the wage was last reset τ periods ago the demand of hours is:

$$L_{t,\tau} = (1-\alpha^*)\,(\alpha^*)^{\tau} \left(\frac{W_{t,t-\tau}^*}{W_t}\right)^{-\theta_w} L_t^d$$

Summing across all possible τ we obtain:

$$L_{t,t-\tau} = (1-\alpha^*) \sum_{\tau=1}^{\infty} (\alpha^*)^{\tau} \left(\frac{W_{t,t-\tau}^*}{W_t}\right)^{-\theta_w} L_t^d$$

Combining these definitions we can write:

$$L_t^s = L_t^* + L_{t,t-\tau} = (1-\alpha^*) \sum_{\tau=0}^{\infty} (\alpha^*)^{\tau} \left(\frac{W_{t,t-\tau}^*}{W_t}\right)^{-\theta_w} L_t^d = \tau_t^* L_t^d \tag{16}$$

where τ_t^* measures the resource cost due to wage dispersion. The latter entails an inefficiently large labor supply with respect to the the one that is required for production. The variable τ_t^* can be written recursively as:

$$\tau_t^* = (1-\alpha^*)\left(\frac{w_t^*}{w_t}\right)^{-\theta_w} + \alpha^* \left(\frac{w_{t-1}}{w_t}\right)^{-\theta_w} \pi_t^{\theta_w} \tau_{t-1}^* \tag{17}$$

Using the wage index $W_t = \left[\int_0^1 \left(W_t^j\right)^{1-\theta_w} dj\right]^{1/(1-\theta_w)}$ one can show that:

$$w_t^{1-\theta_w} = (1-\alpha^*)\,(w_t^*)^{1-\theta_w} + \alpha^* \left(\frac{w_{t-1}}{\pi_t}\right)^{1-\theta_w} \tag{18}$$

In equilibrium, the representative family holds the entire portfolio of firms and the trade of state-contingent asset trade is nil. As a result, $b_{t+1}(q) = b_t(q) = 1$, and $a_{t+1} = a_t = 0$, and the following resource constraint follows:

$$C_t + N_t^e(s)\tilde{v}_t(s) + N_t^e(ns)\tilde{v}_t(ns) = (\alpha_W R_t + 1 - \alpha_W)\, w_t L_t^d + N_{o,t}(s)\tilde{e}_t(s) + N_{o,t}(ns)\tilde{e}_t(ns) \tag{19}$$

By definition, Y_t is either consumed, used as intermediate input in the production process or used to cover fixed costs of production and of entry, thus:

$$Y_t = C_t + X_t + (N_{o,t}(s) + N_{o,t}(ns))\, f_{x,t} + N_t^e(s) f_{e,t}(s) + N_t^e(ns) f_{e,t}(ns) \tag{20}$$

Finally, since the mass of households has unitary measure:

$$\mathbb{I}_t = \mathcal{I}_t, \quad \mathbb{S}_t = \mathcal{S}_t, \quad \mathbb{D}_t = \mathcal{D}_t.$$

10 Aggregation Details

Following Melitz (2003), we assume that the distribution function $g(z)$ follows a Pareto distribution with parameters $zmin$ (minimum) and κ (tail). We then define $\tilde{z}_t(q)$ as a special average productivity in each sector (q). This productivity summarizes all the relevant information within a sector, as the industry is isomorphic to one populated by identical $N_{o,t}(q)$ firms endowed with productivity $\tilde{z}_t(q)$, as we show below.

Thanks to the properties of the Pareto distribution, we can write $\tilde{z}_t(q)$ as a function of the cut-off productivity, $z_t^c(q)$, as follows:

$$\tilde{z}_t(q) = \left[\frac{1}{1 - G\left(z_t^c(q)\right)} \int_{z_t^c(q)}^{\infty} z^{\theta-1} g(z) dz \right]^{\frac{1}{\theta-1}} = \Gamma z_t^c(q) \tag{21}$$

where $\Gamma = \left[\frac{\kappa}{\kappa-(\theta-1)}\right]^{\frac{1}{\theta-1}}$ and, again due to the properties of the Pareto distribution,

$1-G\left(z_t^c(q)\right)=\left(\frac{zmin}{z_t^c(q)}\right)^{\kappa}$.

In what follows, tilded variables refer to firms characterized by the special average productivity. Given that only some firms are active in each sector, the sectoral price $P_t(q)$ can be written as:

$$P_t(q)=\left[\frac{1}{1-G\left(z_t^c(q)\right)}\int_{z_t^c(q)}^{\infty}p_{z,t}(q)^{1-\theta}N_{o,t}(q)g(z)dz\right]^{\frac{1}{1-\theta}}$$

Substituting the optimal individual price, $p_{z,t}(q)$, one obtains:

$$P_t(q)=N_{o,t}(q)^{\frac{1}{1-\theta}}\left(\frac{W_t\left(\alpha_W R_t+1-\alpha_W\right)}{(1-\alpha)}\right)^{1-\alpha}\left(\frac{P_t}{\alpha}\right)^{\alpha}\frac{\theta}{\theta-1}\frac{1}{Z_t}$$

$$\left[\frac{1}{1-G\left(z_t^c(q)\right)}\int_{z_t^c(q)}^{\infty}z^{\theta-1}g(z)dz\right]^{\frac{1}{1-\theta}}$$

By using the definition of the special average productivity $\tilde{z}_t(q)$ this becomes:

$$P_t(q)=N_{o,t}(q)^{\frac{1}{1-\theta}}\left(\frac{W_t\left(\alpha_W R_t+1-\alpha_W\right)}{(1-\alpha)}\right)^{1-\alpha}\left(\frac{P_t}{\alpha}\right)^{\alpha}\frac{\theta}{\theta-1}\frac{1}{Z_t\tilde{z}_t(q)}$$

or, by using the definition of the optimal individual price, $p_{z,t}(q)$:

$$P_t(q)=N_{o,t}(q)^{\frac{1}{1-\theta}}\tilde{p}_t(q)$$

The latter also implies that the ratio $\rho_t(q)=\frac{P_t(q)}{P_t}$ equals

$$\rho_t(q)=N_{o,t}(q)^{\frac{1}{1-\theta}}\tilde{\rho}_t\left(q\right)$$

We can use this result to substitute out $\rho_t(q)$ from the equilibrium conditions regarding profits and cut-off productivities.

Moreover, by definition $w_tL_t(q)=\frac{1}{1-G(z_t^c(q))}\int_{z_t^c(q)}^{\infty}w_tl_{t,z}(q)N_{o,t}(q)g(z)dz$ and the same holds for $X_t(q)$ and $\Omega_t(q)$, where $L_t(q)$ is the total labor demanded in sector

(q), $X_t(q)$ is the total intermediate input demanded in sector (q) and $\Omega_t^e(q)$ are the total dividends of sector (q). Following the steps above, namely by substituting for $l_{t,z}(q)$, $X_{t,z}(q)$ and $e_{t,z}(q)$ a function of z only, one can show that:

$$L_t(q) = N_{o,t}(q)\tilde{l}_t(q), \quad X_t(q) = N_{o,t}(q)\tilde{X}_t(q) \quad \text{and} \quad \mathit{\Omega}_t(q) = N_{o,t}(q)\tilde{e}_t(q)$$

These conditions complete the aggregation from the sectors to the whole economy. Indeed, from the market clearing condition we get:

$$L_t^d = L_t(s) + L_t(ns)$$

and

$$X_t = X_t(s) + X_t(ns)$$

and

$$\Omega_t = \Omega_t(s) + \Omega_t(ns)$$

Note that the same results hold also for the sectoral and aggregate firm value as a function of $\tilde{v}_t(q)$. However, in this case we must multiply by $N_t(q)$ instead of $N_{o,t}$, as also inactive firms have a non-zero value due to the possibility of became active in the future upon survival.

From the definition of aggregate price:

$$P_t = \left(\frac{C_t}{1-\mathcal{D}_t}\right)^{-\eta} \left\{P_t\left(s\right)\chi\left[\lambda_t\rho_t\left(s\right) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\left(s\right)\right]^{-\eta} + P_t\left(ns\right)\left(1-\chi\right)\left(\lambda_t\rho_t\left(ns\right)\right)^{-\eta}\right\}$$

considering that $P_t\left(q\right) = N_{o,t}(q)^{\frac{1}{1-\theta}}\tilde{p}_t(q)$

$$P_t = \left(\frac{C_t}{1-\mathcal{D}_t}\right)^{-\eta} \{N_{o,t}(s)^{\frac{1}{1-\theta}}\tilde{p}_t(s)\chi\left[\lambda_t\rho_t\left(s\right) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\left(s\right)\right]^{-\eta} +$$

$$+N_{o,t}(ns)^{\frac{1}{1-\theta}}\tilde{p}_t(ns)\left(1-\chi\right)\left(\lambda_t\rho_t\left(ns\right)\right)^{-\eta}\}$$

dividing through by P_t and considering that $\rho_t(q) = N_{o,t}(q)^{\frac{1}{1-\theta}}\tilde{\rho}_t(q)$ we get

$$1 = \left(\frac{C_t}{1-\mathcal{D}_t}\right)^{-\eta}\{N_{o,t}(s)^{\frac{1}{1-\theta}}\tilde{\rho}_t(s)\chi\left[\lambda_t N_{o,t}(s)^{\frac{1}{1-\theta}}\tilde{\rho}_t(s) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t(s)\right]^{-\eta} +$$

$$+N_{o,t}(ns)^{\frac{1}{1-\theta}}\tilde{\rho}_t(ns)(1-\chi)\left(\lambda_t N_{o,t}(ns)^{\frac{1}{1-\theta}}\tilde{\rho}_t(ns)\right)^{-\eta}\}$$

which gives:

$$\left(\frac{C_t}{1-\mathcal{D}_t}\right)^{\eta} = \{N_{o,t}(s)^{\frac{1}{1-\theta}}\tilde{\rho}_t(s)\chi\left[\lambda_t N_{o,t}(s)^{\frac{1}{1-\theta}}\tilde{\rho}_t(s) + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t(s)\right]^{-\eta} +$$

$$+N_{o,t}(ns)^{\frac{1-\eta}{1-\theta}}\tilde{\rho}_t(ns)^{1-\eta}(1-\chi)\lambda_t^{-\eta}\}$$

Notice that when $\mathcal{I}_t = 0$, $\mathcal{D}_t = 0$ and, thus, $\frac{1}{C_t} = \lambda_t$ the condition reduces to:

$$1 = \chi\left[N_{o,t}(s)^{\frac{1}{1-\theta}}\tilde{\rho}_t(s)\right]^{1-\eta} + (1-\chi)\left(N_{o,t}(ns)^{\frac{1}{1-\theta}}\tilde{\rho}_t(ns)\right)^{1-\eta}$$

which is the usual price index with two sectors and CES demand.

In order to close the model, we must write the idiosyncratic $\tilde{X}_t(q)$ as functions of aggregate variables and productivity levels to be able to solve for the equilibrium. In sector (q), from the individual demand, given the relation for the sectoral price obtained above, we get:

$$\tilde{y}_t(q) = N_{o,t}(q)^{\frac{\theta}{1-\theta}}Y_t(q)$$

Recall that the cost minimization problem implies

$$\frac{X_{z,t}(q)}{l_{z,t}(q)} = \frac{\alpha w_t(\alpha_W R_t + 1 - \alpha_W)}{1-\alpha}$$

From the definition of the Cobb-Douglas and the optimality condition from costs minimization we can write:

$$\tilde{y}_t(q) = Z_t\tilde{z}_t(q)\tilde{X}_t(q)^{\alpha}\tilde{l}_t(q)^{1-\alpha} = Z_t\tilde{z}_t(q)\tilde{X}_t(q)\left(\frac{\alpha w_t(\alpha_W R_t + 1 - \alpha_W)}{(1-\alpha)}\right)^{\alpha-1}$$

Thus:

$$N_{o,t}(q)^{\frac{\theta}{1-\theta}}Y_t(q) = Z_t\tilde{z}_t(q)\tilde{X}_t(q)\left(\frac{\alpha w_t\left(\alpha_W R_t + 1 - \alpha_W\right)}{(1-\alpha)}\right)^{\alpha-1}$$

which gives:

$$\tilde{X}_t(q) = \frac{Y_t\left(q\right)N_{o,t}(q)^{\frac{\theta}{1-\theta}}}{Z_t\tilde{z}_t(q)}\left(\frac{\alpha w_t\left(\alpha_W R_t + 1 - \alpha_W\right)}{(1-\alpha)}\right)^{1-\alpha}$$

Hence, the sectoral variables are:

$$X_t(s) = N_{o,t}(s)\tilde{X}_t(s) \quad \text{and} \quad X_t(ns) = N_{o,t}(ns)\tilde{X}_t(ns)$$

Finally:

$$X_t = X_t(s) + X_t(ns)$$

11 List of Equilibrium Conditions

A competitive equilibrium is a set of processes for the following 37 variables $\{\mathcal{T}_t, \mathcal{S}_t, \mathcal{I}_t, \mathcal{D}_t, l_t^s, L_t^d, C_t\left(s\right), C_t(ns), C_t, w_t, \tilde{\mu}_t, \lambda_t, \lambda_{\mathcal{S},t}, \lambda_{\mathcal{I},t}, \lambda_{\mathcal{D},t}, \lambda_{\mathcal{T},t}, z_t^c\left(s\right), N_t\left(s\right),$ $\tilde{\rho}_t\left(s\right), z_t^c\left(ns\right), N_t\left(ns\right), \tilde{\rho}_t\left(ns\right), \tilde{e}_t\left(s\right), \tilde{e}_t\left(ns\right), R_t, \pi_t, w_t^*, f_t^1, f_t^2, X_t, N_t^e\left(s\right), N_t^e\left(ns\right),$ $\tau_t^*, Y_t, Z_t, f_{e,t}(s), f_{e,t}(ns)\}$ that satisfy the 37 equilibrium conditions reported below.

In addition we have 29 parameters $\{\pi_1, \pi_2, \pi_3, \pi_d, \pi_r, \phi, \beta, \eta, \chi, \kappa, \theta, \delta, \theta_w, \alpha^*, \alpha, \alpha_w,$ $\varphi_\pi, \varphi_Y, \varphi_R, \Gamma, f_x, \nu, \psi_0, \psi_1, \gamma, \rho_Z, \varepsilon_Z, zmin, u_d\}$, where Γ is a convolution of other parameters.

Households and SIR

$$1) \quad \mathcal{T}_t = \frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\left(s\right)^2 + \mathcal{S}_t\mathcal{I}_t\pi_2 l_t^s L_t^d + \pi_3\mathcal{S}_t\mathcal{I}_t$$

$$2) \quad \mathcal{I}_{t+1} = \mathcal{T}_t + \mathcal{I}_t - \left(\pi_r + \pi_d\right)\mathcal{I}_t$$

$$3) \quad \mathcal{S}_{t+1} = \mathcal{S}_t - \mathcal{T}_t$$

$$4)\quad \mathcal{D}_{t+1} = \mathcal{D}_t + \pi_d \mathcal{I}_t$$

$$5)\quad \nu \left(l_t^s\right)^{\phi} = \frac{\lambda_t w_t}{\tilde{\mu}_t} - \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathcal{I}_t}{1-\mathcal{D}_t} \pi_2 L_t^d$$

$$6)\quad \lambda_{\mathcal{T},t} = \lambda_{\mathcal{I},t} - \lambda_{\mathcal{S},t}$$

$$7)\quad \lambda_{\mathcal{I},t} = -\beta E_t \left[-\lambda_{\mathcal{I},t+1} \left(1-\pi_r\right) + \pi_d \left(\lambda_{\mathcal{I},t+1} - \lambda_{\mathcal{D},t+1}\right)\right]$$

$$8)\quad \lambda_{\mathcal{S},t} = -\beta E_t \left[\lambda_{\mathcal{T},t+1} \left(-\frac{\mathcal{I}_{t+1}}{1-\mathcal{D}_{t+1}} \pi_1 C_{t+1}\left(s\right)^2 - \mathcal{I}_{t+1} \pi_2 l_{t+1}^s L_{t+1}^d - \pi_3 \mathcal{I}_{t+1} \right) - \lambda_{\mathcal{S},t+1} \right]$$

$$9)\quad \lambda_{\mathcal{D},t} = -\beta E_t \cdot$$

$$\left[-log\left(\frac{C_{t+1}}{1-\mathcal{D}_{t+1}}\right) + \nu \left(\frac{\left(l_{t+1}^s\right)^{1+\phi}}{1+\phi} \right) - u_d + \lambda_{t+1} \frac{C_{t+1}}{1-\mathcal{D}_{t+1}} - \frac{\lambda_{t+1} w_{t+1}}{\tilde{\mu}_{t+1}} l_{t+1}^s - \lambda_{\mathcal{D},t+1} \right]$$

$$10)\quad C_t = C_t\left(s\right)^{\frac{1}{1-\eta}} \left(1-\mathcal{D}_t\right)^{\frac{\eta}{\eta-1}} \chi^{\frac{1}{\eta-1}} \left[\lambda_t \left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s) \right]^{\frac{1}{1-\theta}} \tilde{\rho}_t(s) + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathcal{I}_t}{1-\mathcal{D}_t} \pi_1 C_t\left(s\right) \right]^{\frac{\eta}{1-\eta}}$$

$$11)\quad C_t(ns) = \left(1-\chi\right)\left(1-\mathcal{D}_t\right)^{\eta} \left[\lambda_t \left[\left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns) \right]^{\frac{1}{1-\theta}} \tilde{\rho}_t(ns) \right]^{-\eta} C_t^{1-\eta}$$

$$12)\quad f_{e,t}(s) = \beta\left(1-\delta\right) E_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t}\right) \left(f_{e,t+1}(s) + \left(\frac{zmin}{z_{t+1}^c(s)}\right)^{\kappa} \tilde{e}_{t+1}(s) \right) \right]$$

$$13)\quad f_{e,t}(ns) = \beta\left(1-\delta\right) E_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t}\right) \left(f_{e,t+1}(ns) + \left(\frac{zmin}{z_{t+1}^c(ns)}\right)^{\kappa} \tilde{e}_{t+1}(ns) \right) \right]$$

$$14)\quad f_{e,t}(s) = \psi_0 + \psi_1 \left(N_t^e(s)\right)^{\gamma}$$

$$15)\quad f_{e,t}(ns) = \psi_0 + \psi_1 \left(N_t^e(ns)\right)^{\gamma}$$

$$16)\quad 1 = \beta E_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t}\right) \frac{R_t}{\pi_{t+1}} \right]$$

Unions

$$17)\quad w_t^* = \frac{\theta_w}{\left(\theta_w - 1\right)} \frac{f_t^2}{f_t^1}$$

$$18)\quad f_t^1 = L_t^d w_t^{\theta_w} \lambda_t + \alpha^* \beta E_t \pi_{t+1}^{\theta_w - 1} f_{t+1}^1$$

$$19)\quad f_t^2 = L_t^d w_t^{\theta_w} \left(\nu \left(l_t^s\right)^{\phi} + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathcal{I}_t}{1-\mathcal{D}_t} \pi_2 L_t^d \right) + \alpha^* \beta E_t \pi_{t+1}^{\theta_w} f_{t+1}^2$$

Firms

$$20)\quad \tilde{\rho}_t(s) = \frac{\theta}{\theta-1}\frac{1}{Z_t\Gamma z_t^c(s)}\left(\frac{(\alpha_W R_t + 1-\alpha_W)\,w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}$$

$$21)\quad \tilde{\rho}_t(ns) = \frac{\theta}{\theta-1}\frac{1}{Z_t\Gamma z_t^c(ns)}\left(\frac{(\alpha_W R_t + 1-\alpha_W)\,w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}$$

$$22)\quad \tilde{e}_t(s) = \frac{1}{\theta}\tilde{\rho}_t(s)\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{\theta}{1-\theta}}\chi\cdot$$

$$\left[\lambda_t\tilde{\rho}_t(s)\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{1}{1-\theta}} + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\,(s)\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t - f_x$$

$$23)\quad \tilde{e}_t(ns) = \frac{1}{\theta}\tilde{\rho}_t(ns)\left[\left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\right]^{\frac{\theta}{1-\theta}}(1-\chi)\cdot$$

$$\left[\lambda_t\tilde{\rho}_t(ns)\left[\left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\right]^{\frac{1}{1-\theta}}\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t - f_x$$

Entry and Exit

$$24)\quad N_{t+1}(s) = (1-\delta)\left(N_t(s) + N_t^e(s)\right)$$

$$25)\quad N_{t+1}(ns) = (1-\delta)\left(N_t(ns) + N_t^e(ns)\right)$$

$$26)\quad z_t^c\,(s) = \frac{\theta^{\frac{\theta}{\theta-1}}}{\theta-1}\frac{1}{Z_t}\left(\frac{(\alpha_W R_t + 1-\alpha_W)\,w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\chi^{\frac{1}{1-\theta}}\cdot$$

$$\left(\frac{f_x}{\tilde{\rho}_t(s)^{\theta}\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{\theta}{1-\theta}}\left[\lambda_t\tilde{\rho}_t(s)\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{1}{1-\theta}} + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\,(s)\right]^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t}\right)^{\frac{1}{\theta-1}}$$

$$27)\quad z_t^c\,(ns) = \frac{\theta^{\frac{\theta}{\theta-1}}}{\theta-1}\frac{1}{Z_t}\left(\frac{(\alpha_W R_t + 1 - \alpha_W)\,w_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}(1-\chi)^{\frac{1}{1-\theta}}$$

$$\left(\frac{f_x}{\tilde{\rho}_t(ns)^{\theta-\eta}\left[\left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\right]^{\frac{\theta-\eta}{1-\theta}}\lambda_t^{-\eta}\left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t}\right)^{\frac{1}{\theta-1}}$$

Taylor Rule

$$28)\quad \left(\frac{R_t}{R}\right) = \left[\left(\frac{\pi_t}{\pi}\right)^{\varphi_\pi}\left(\frac{Y_t}{Y}\right)^{\varphi_Y}\right]^{1-\varphi_R}\left(\frac{R_{t-1}}{R}\right)^{\varphi_R}$$

Aggregation and Market Clearing

$$29)\quad C_t + N_t^e(s) f_{e,t}(s) + N_t^e(ns) f_{e,t}(ns) =$$

$$= (\alpha_W R_t + 1 - \alpha_W)\, w_t L_t^d + \left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\tilde{e}_t(s) + \left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\tilde{e}_t(ns)$$

$$30)\quad 1 = (1-\mathcal{D}_t)^{\eta}\left\{\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{1}{1-\theta}}\tilde{\rho}_t(s)\chi\left[\lambda_t\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{1}{1-\theta}}\tilde{\rho}_t(s) + \right.\right.$$

$$\left.\left. + \lambda_{\mathcal{T},t}\frac{\mathcal{S}_t\mathcal{I}_t}{1-\mathcal{D}_t}\pi_1 C_t\,(s)\right]^{-\eta} + \left[\left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\right]^{\frac{1-\eta}{1-\theta}}\tilde{\rho}_t(ns)^{1-\eta}\,(1-\chi)\,\lambda_t^{-\eta}\right\} C_t^{-\eta}$$

$$31)\quad C_t = \left[\chi^{\frac{1}{\eta}} C_t(s)^{\frac{\eta-1}{\eta}} + (1-\chi)^{\frac{1}{\eta}} C_t(ns)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$32)\quad Y_t = C_t + X_t + \left(\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s) + \left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\right) f_x +$$

$$+ N_t^e(ns) f_{e,t}(ns) + N_t^e(ns) f_{e,t}(ns)$$

$$33) \quad X_t = \frac{\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{1}{1-\theta}}}{Z_t \Gamma z_t^c(s)} \left(\frac{\alpha w_t\left(\alpha_W R_t + 1 - \alpha_W\right)}{(1-\alpha)}\right)^{1-\alpha} \chi \cdot$$

$$\left[\lambda_t \left(\left[\left(\frac{zmin}{z_t^c(s)}\right)^{\kappa} N_t(s)\right]^{\frac{1}{1-\theta}} \tilde{\rho}_t(s)\right) + \lambda_{\mathcal{T},t} \frac{\mathcal{S}_t \mathcal{I}_t}{1-\mathcal{D}_t} \pi_1 C_t(s)\right]^{-\eta} \left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t +$$

$$+ \frac{\left[\left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\right]^{\frac{1}{1-\theta}}}{Z_t \Gamma z_t^c(ns)} \left(\frac{\alpha w_t\left(\alpha_W R_t + 1 - \alpha_W\right)}{(1-\alpha)}\right)^{1-\alpha} (1-\chi) \cdot$$

$$\left[\lambda_t \left(\left[\left(\frac{zmin}{z_t^c(ns)}\right)^{\kappa} N_t(ns)\right]^{\frac{1}{1-\theta}} \tilde{\rho}_t(ns)\right)\right]^{-\eta} \left(\frac{C_t}{(1-\mathcal{D}_t)}\right)^{-\eta} Y_t$$

$$34) \quad w_t^{1-\theta_w} = (1-\alpha^*)(w_t^*)^{1-\theta_w} + \alpha^* \left(\frac{w_{t-1}}{\pi_t}\right)^{1-\theta_w}$$

$$35) \quad (1-\mathcal{D}_t)\, l_t^s = \tau_t^* L_t^d$$

$$36) \quad \tau_t^* = (1-\alpha^*)\left(\frac{w_t^*}{w_t}\right)^{-\theta_w} + \alpha^* \left(\frac{w_{t-1}}{w_t}\right)^{-\theta_w} \pi_t^{\theta_w} \tau_{t-1}^*$$

Exogenous Process for Aggregate Productivity

$$37) \quad Z_t = \rho_Z Z_{t-1} + \varepsilon_Z$$

12 Analytical Derivation of Aggregate Productivity in the Simplified Model

The first step in our derivation is to show that the aggregate price index can be written as $P_t = N_{o,t}^{1/(1-\theta)} \tilde{P}_t$, where $N_{o,t} \equiv N_{o,t}(s) + N_{o,t}(ns)$ and $\tilde{P}_t$ is an average of producers' prices. When $\pi_1 = 0$ the price index equation reduces to:

$$1 = (1-\mathcal{D}_t)^{\eta} \left\{ \chi \lambda_t^{-\eta} \left[[N_{o,t}(s)]^{\frac{1}{1-\theta}} \tilde{\rho}_t(s) \right]^{1-\eta} + [N_{o,t}(ns)]^{\frac{1-\eta}{1-\theta}} \tilde{\rho}_t(ns)^{1-\eta} (1-\chi)\, \lambda_t^{-\eta} \right\} C_t^{-\eta}$$

or, in nominal terms using that $\tilde{\rho}_t(q) = \tilde{p}_t(q)/P_t$:[4]

$$P_t = \left\{ \chi \left[N_{o,t}(s)\right]^{\frac{1-\eta}{1-\theta}} \tilde{p}_t(s)^{1-\eta} + (1-\chi) \left[N_{o,t}(ns)\right]^{\frac{1-\eta}{1-\theta}} \tilde{p}_t(ns)^{1-\eta} \right\}^{\frac{1}{1-\eta}}$$

Using the definition of $N_{o,t}$ provided above we get:

$$P_t = N_{o,t}^{\frac{1}{1-\theta}} \left\{ \chi \omega_s^{\frac{1-\eta}{1-\theta}} \tilde{p}_t(s)^{1-\eta} + (1-\chi)(1-\omega_s)^{\frac{1-\eta}{1-\theta}} \tilde{p}_t(ns)^{1-\eta} \right\}^{\frac{1}{1-\eta}} = N_{o,t}^{\frac{1}{1-\theta}} \tilde{P}_t$$

where $\omega_s = N_{o,t}(s)/N_{o,t}$. Note that $\tilde{P}_t$ is a form of weighted average of the average producers' prices in the two sectors $\tilde{p}_t(S)$ and $\tilde{P}_t(ns)$. Given the price index, we aggregate the production function to obtain a notion of aggregate labor productivity. From the sectoral demand constraint and the definition of sectoral production we get:

$$Y_t(s) = \chi \rho_t(s)^{-\eta} Y_t = N_{o,t}(s)^{\frac{1}{\theta-1}} Z_t \tilde{z}_t(s) L_t(s)$$

and similar results for the non-social sector. Solving for $L_t(q)$ we get:

$$\chi \rho_t(s)^{-\eta} Y_t N_{o,t}(s)^{\frac{1}{1-\theta}} \frac{1}{Z_t \tilde{z}_t(s)} = L_t(s)$$

and

$$(1-\chi)\rho_t(ns)^{-\eta} Y_t N_{o,t}(ns)^{\frac{1}{1-\theta}} \frac{1}{Z_t \tilde{z}_t(ns)} = L_t(ns)$$

Summing side by side:

$$\frac{Y_t}{Z_t} \left(\chi \rho_t(s)^{-\eta} N_{o,t}(s)^{\frac{1}{1-\theta}} \frac{1}{\tilde{z}_t(s)} + (1-\chi)\rho_t(ns)^{-\eta} N_{o,t}(ns)^{\frac{1}{1-\theta}} \frac{1}{\tilde{z}_t(ns)} \right) = L_t^d$$

[4]Provided that $(1-\mathcal{D}_t)/(\lambda_t C_t)$ is equal to 1. This can be proven under mild assumptions or it can be obtained by re-scaling the Lagrange multiplier of the household. Thus, in the following we omit this term from the derivations.

Using again the definition of $N_{o,t}$ and of ω_s this becomes:

$$\frac{Y_t}{Z_t} N_{o,t}^{\frac{1}{1-\theta}} \left(\chi \omega_s^{\frac{1}{1-\theta}} \rho_t(s)^{-\eta} \frac{1}{\tilde{z}_t(s)} + (1-\chi)(1-\omega_s)^{\frac{1}{1-\theta}} \rho_t(ns)^{-\eta} \frac{1}{\tilde{z}_t(ns)} \right) = L_t^d$$

Solving for Y_t:

$$Y_t = N_{o,t}^{\frac{1}{\theta-1}} Z_t \left(\chi \omega_s^{\frac{1}{1-\theta}} \rho_t(s)^{-\eta} \frac{1}{\tilde{z}_t(s)} + (1-\chi)(1-\omega_s)^{\frac{1}{1-\theta}} \rho_t(ns)^{-\eta} \frac{1}{\tilde{z}_t(ns)} \right)^{-1} L_t^d = N_{o,t}^{\frac{1}{\theta-1}} Z_t \tilde{Z}_t L_t^d$$

The aggregate labor productivity $\tilde{Z}_t$ is a weighted armonic average of the average sectoral productivities $\tilde{z}_t(s)$ and $\tilde{z}_t(ns)$. The statistics we compute and we use for the comparison with the aggregate labor productivity in the data is:

$$\frac{\left(\frac{Y_t P_t}{\tilde{P}_t}\right)}{L_t^d} = Z_t \tilde{Z}_t$$

The reason is the following: in the data, real variables in units of consumption are obtained by deflating the nominal quantities with price deflators as the CPI. However, these deflators, by being based on averages of producers' prices over a semi-fixed bundle of goods, are conceptually more similar to the average producer price $\tilde{P}_t$ than to the consumer welfare-based price index P_t.[5] For this reason, *real* variables in the data do not correspond to $P_t X_t / P_t = X_t$ in the model but to $P_t X_t / \tilde{P}_t$, and these are the statistics we use. Note that this allows us to *correct* for the presence of love for variety in the model and directly use $\tilde{Z}_t$ and $\tilde{z}_t(q)$ as measures of aggregate and sectoral productivity, respectively.

[5]For a deeper discussion on the topic, see Ghironi and Melitz (2005) and Bilbiie et al. (2012).

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