Intermediate Macroeconomics
TA II

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Plan for Today

Before we start, do you have questions about the first tutorial or the problem set?

- Derive the **General** Solow growth model (without human capital)
  - Extra material on Canvas $\rightarrow$ derivation of the Solow growth model with human capital

- Derive a model with efficiency wages.

Solow Growth Model - Assumptions

In the last TA, we assumed the following Cobb-Douglas production function:

$$Y_t = B(K_t)^{\alpha}(L_t)^{1-\alpha} \quad \text{with} \quad B > 0 \quad \text{and} \quad 0 < \alpha < 1$$

This week, **one small twist**: B is not a constant, but it evolves over time.

$$Y_t = B_t(K_t)^{\alpha}(L_t)^{1-\alpha} \quad \text{with} \quad 0 < \alpha < 1$$

The remaining *building blocks* of the model are kept the same.

Solow Growth Model - Technology

However, we will rewrite the production function $\rightarrow$ from TFP to a labor-augmenting technology:

$$Y_t = B_t(K_t)^\alpha(L_t)^{1-\alpha} = (K_t)^\alpha(B_t^{1/(1-\alpha)}L_t)^{1-\alpha} = (K_t)^\alpha(A_tL_t)^{1-\alpha}$$

with $0 < \alpha < 1$ and $A_t = B_t^{1/(1-\alpha)}$ (i.e. $B_t = A_t^{1-\alpha}$).

- The evolution of technology is exogenous:

$$A_{t+1} = (1+g)A_t \quad \text{with} \quad g > -1$$

Solow Growth Model - Growth Rates

Again, let's define output per worker as $y_t \equiv \frac{Y_t}{L_t}$ and capital per worker as $k_t \equiv \frac{K_t}{L_t}$.

$$y_t \equiv \frac{Y_t}{L_t} = \frac{1}{L_t}(K_t)^\alpha (A_t L_t)^{1-\alpha} = \left(\frac{K_t}{L_t}\right)^\alpha \left(A_t \frac{L_t}{L_t}\right)^{1-\alpha} = k_t^\alpha A_t^{1-\alpha}$$

From this equation, we can obtain a clean expression for the growth rate of per-worker output by using the properties of the logs.

To do so, let's refresh our knowledge with a short test on menti.com (code **1546 3405**).

Solow Growth Model - Growth Rates (II)

$$y_t = k_t^\alpha A_t^{1-\alpha}$$

Taking the log on both sides for period t and $t-1$ we get:

$$\ln(y_t) = \ln(k_t^\alpha A_t^{1-\alpha}) = \ln(k_t^\alpha) + \ln(A_t^{1-\alpha}) = \alpha \ln(k_t) + (1-\alpha)\ln(A_t)$$

and

$$\ln(y_{t-1}) = \alpha \ln(k_{t-1}) + (1-\alpha)\ln(A_{t-1})$$

Taking the difference side by side:

$$\ln(y_t) - \ln(y_{t-1}) = \alpha(\ln(k_t) - \ln(k_{t-1})) + (1-\alpha)(\ln(A_t) - \ln(A_{t-1}))$$

Solow Growth Model - Growth Rates (III)

$$\ln(y_t) - \ln(y_{t-1}) = \alpha(\ln(k_t) - \ln(k_{t-1})) + (1-\alpha)(\ln(A_t) - \ln(A_{t-1}))$$

The first difference in log terms is a proxy for the growth rate. Indeed:

$$\ln(y_t) - \ln(y_{t-1}) = \ln\left(\frac{y_t}{y_{t-1}}\right) = \ln\left(\frac{(1+g_t^y)y_{t-1}}{y_{t-1}}\right) = \ln(1+g_t^y) \approx g_t^y$$

Hence:

$$g_t^y \approx \alpha g_t^k + (1-\alpha)g$$

The Complete Model

- $Y_t = B_t K_t^{\alpha} L_t^{1-\alpha} = K_t^{\alpha}(A_t L_t)^{1-\alpha}$
- $r_t = \alpha B_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} = \alpha A_t^{1-\alpha} \left(\frac{K_t}{L_t}\right)^{\alpha-1} = \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1}$
- $w_t = (1-\alpha) B_t \left(\frac{K_t}{L_t}\right)^{\alpha} = (1-\alpha) A_t^{1-\alpha} \left(\frac{K_t}{L_t}\right)^{\alpha} = (1-\alpha) A_t \left(\frac{K_t}{A_t L_t}\right)^{\alpha}$
- $S_t = sY_t$
- $K_{t+1} = (1-\delta)K_t + S_t$
- $L_{t+1} = (1+n)L_t$
- $A_{t+1} = (1+g)A_t$

Given L_0, K_0, A_0, α, s, δ, n and g

Solow Growth Model - Dynamics and Steady State

In the last TA, we used per-worker terms to find the steady state where c^*, y^* and k^* were constant.

On the other hand, notice that their aggregate counterparts C_t^*, Y_t^* and K_t^* were growing at a rate n in the steady state.

To show that: $y^* \equiv \frac{Y_t^*}{L_t^*}$. Since L_t^* is growing at a rate n, Y_t^* must grow at a rate n as well (same for consumption and capital).

Today, slightly different approach: we work in effective-worker units.

Solow Growth Model - Dynamics and Steady State (II)

In the general Solow model, it is handy to redefine everything in terms of *effective workers*:

$$\tilde{y}_t = Y_t/\tilde{L}_t = Y_t/(A_t L_t) \quad \text{and} \quad \tilde{k}_t = K_t/\tilde{L}_t = K_t/(A_t L_t)$$

Moreover:

$$\tilde{y}_t = \frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t} K_t^{\alpha} (A_t L_t)^{1-\alpha} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} \left(\frac{A_t L_t}{A_t L_t}\right)^{1-\alpha} = \tilde{k}_t^{\alpha}$$

Divide the intertemporal budget constraint $K_{t+1} = (1-\delta)K_t + sY_t$ by $A_{t+1}L_{t+1} = A_{t+1}(1+n)L_t = (1+g)(1+n)A_t L_t$:

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = s\frac{Y_t}{(1+g)(1+n)A_t L_t} + (1-\delta)\frac{K_t}{(1+g)(1+n)A_t L_t}$$

Solow Growth Model - Dynamics and Steady State (III)

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = s\frac{Y_t}{(1+g)(1+n)A_tL_t} + (1-\delta)\frac{K_t}{(1+g)(1+n)A_tL_t}$$

which gives the **transition equation**:

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)}(s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t)$$

Subtracting $\tilde{k}_t$ from both sides we can obtain the **Solow equation**:

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+g)(1+n)}(s\tilde{k}_t^{\alpha} - (n+g+ng+\delta)\tilde{k}_t)$$

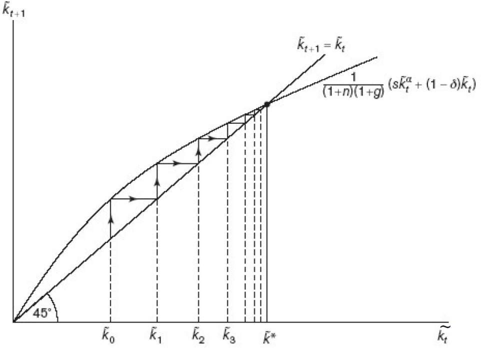
since $-\tilde{k}_t = -\frac{1}{(1+n)(1+g)}(1+n)(1+g)\tilde{k}_t = -\frac{1}{(1+n)(1+g)}(1+n+g+ng)\tilde{k}_t$

Solow Growth Model - Dynamics and Steady State (IV)

Dynamics to the steady state:

$$\tilde{k}_{t+1} = \frac{1}{(1+n)(1+g)}\left(s\tilde{k}_t^{\alpha} + (1-\delta)\tilde{k}_t\right).$$

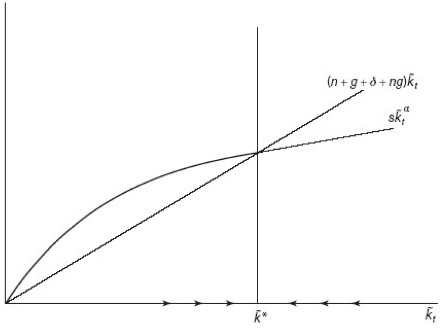
Transition diagram



Dynamics to the steady state:

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{1}{(1+n)(1+g)}\left(s\tilde{k}_t^{\alpha} - (n+g+\delta+ng)\tilde{k}_t\right)$$

Solow diagram



Solow Growth Model - Dynamics and Steady State (VI)

The steady state exists and it is *unique*. We can solve for it starting from the Solow equation, by imposing $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}^*$:

- $\tilde{k}_{t+1} - \tilde{k}_t = \tilde{k}^* - \tilde{k}^* = 0 = \frac{1}{(1+g)(1+n)}(s(\tilde{k}^*)^\alpha - (n+g+gn+\delta)\tilde{k}^*)$

hence:

- $s(\tilde{k}^*)^\alpha = (n+g+gn+\delta)\tilde{k}^* \rightarrow s(\tilde{k}^*)^{\alpha-1} = (n+g+gn+\delta) \rightarrow$ $\tilde{k}^* = \left(\frac{n+g+gn+\delta}{s}\right)^{\frac{1}{\alpha-1}}$

Finally:

- $\tilde{k}^* = \left(\frac{s}{n+g+gn+\delta}\right)^{\frac{1}{1-\alpha}}$
- $\tilde{y}^* = (\tilde{k}^*)^\alpha = \left(\frac{s}{n+g+gn+\delta}\right)^{\frac{\alpha}{1-\alpha}}$
- $\tilde{c}^* = (1-s)\tilde{y}^* = (1-s)\left(\frac{s}{n+g+gn+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

Solow Growth Model - Results

What about **per-worker** steady state variables? Since $\tilde{k}_t \equiv K_t / A_t L_t$

$$A_t \tilde{k}_t = A_t \frac{K_t}{A_t L_t} = \frac{K_t}{L_t} = k_t$$

- $k_t^* = A_t \tilde{k}^* = A_t \left(\frac{s}{n+g+gn+\delta} \right)^{\frac{1}{1-\alpha}}$
- $y_t^* = A_t \tilde{y}^* = A_t \left(\frac{s}{n+g+gn+\delta} \right)^{\frac{\alpha}{1-\alpha}}$
- $c_t^* = A_t \tilde{c}^* = A_t (1-s) \left(\frac{s}{n+g+gn+\delta} \right)^{\frac{\alpha}{1-\alpha}}$

Note that:

$$A_t = (1+g) A_{t-1} = (1+g)(1+g) A_{t-2} = (1+g) \ldots (1+g) A_0 = (1+g)^t A_0$$

Hence, consumption, output and capital per-worker grow at a rate g in the steady state.

Solow Growth Model - Results (II)

Furthermore:

- $r_t^* = \alpha \left(\frac{K_t^*}{A_t^* L_t^*}\right)^{\alpha-1} = \alpha(\tilde{k}^*)^{\alpha-1} = \alpha \left(\frac{s}{n+g+gn+\delta}\right)^{\frac{\alpha-1}{1-\alpha}} = \alpha \left(\frac{s}{n+g+gn+\delta}\right)^{-1} = r^*$
- $w_t^* = (1-\alpha)A_t^* \left(\frac{K_t^*}{A_t^* L_t^*}\right)^{\alpha} = (1-\alpha)A_t^*(\tilde{k}^*)^{\alpha} = (1-\alpha)A_t \left(\frac{s}{n+g+gn+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

Steady state **wages are growing** as well, while return on capital is **constant**.

Workers are getting more productive over time, even in the steady state, hence the higher wages (workers are paid the marginal productivity of labor) and higher output per worker.

Model with efficiency wages - Firms

We have n firms producing differentiated products under monopolistic competition. The demand function Y_i for variety i is:

$$D(P_i) = Y_i = \left(\frac{P_i}{P}\right)^{-\sigma} \frac{Y}{n} \quad \text{with} \quad \sigma > 1$$

where Y is aggregate demand, P_i is the price of good i, P is the price index and σ is the elasticity of substitution between goods.

Firms internalize the demand function when maximizing profits and have market power $\rightarrow$ here firms are **price makers**, not price takers as in the Solow model.

Model with efficiency wages - Firms (II)

Same production function for each firm/good i:

$$Y_i = a_i L_i$$

where the *efficiency* function $a(\cdot)$ is given by:

$$a_i = (w_i - \nu)^\eta \quad \text{with} \quad 0 < \eta < 1$$

Note that (w_i) W_i is the (real) wage paid by firm i, where $w_i \equiv W_i/P$, and ν is the outside option of the worker.

Model with efficiency wages - Firms (III)

Subject to the constraints above, the maximization of firm i is:

$$\max_{p_i, w_i} \frac{P_i Y_i}{P} - \frac{W_i L_i}{P} = Y_i p_i - w_i \frac{Y_i}{a_i} = Y_i \left(p_i - \frac{w_i}{(w_i - \nu)^\eta} \right) =$$

$$= p_i^{-\sigma} \frac{Y}{n} \left(p_i - \frac{w_i}{(w_i - \nu)^\eta} \right)$$

Remember that $D[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. Thus, the F.O.C. with respect to the real price p_i is:

$$-\sigma p_i^{-\sigma-1} \frac{Y}{n} \left(p_i - \frac{w_i}{(w_i - \nu)^\eta} \right) + p_i^{-\sigma} \frac{Y}{n} = 0$$

Simplifying:
$\sigma \left(1 - \frac{w_i}{p_i (w_i - \nu)^\eta} \right) = 1 \rightarrow \sigma - 1 = \frac{\sigma w_i}{p_i (w_i - \nu)^\eta} \rightarrow p_i = \frac{\sigma}{\sigma - 1} \frac{w_i}{(w_i - \nu)^\eta}$

Model with efficiency wages - Firms (IV)

$$p_i^{-\sigma} \frac{Y}{n} \left(p_i - \frac{w_i}{(w_i - \nu)^\eta} \right)$$

Remember that $D\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$. Thus, the F.O.C. with respect to the real wage w_i is:

$$-p_i^{-\sigma} \frac{Y}{n} \left(\frac{(w_i - \nu)^\eta - w_i \eta (w_i - \nu)^{\eta - 1}}{(w_i - \nu)^{2\eta}} \right) = 0$$

Simplifying:
$\frac{1 - w_i \eta (w_i - \nu)^{-1}}{(w_i - \nu)^\eta} = 0 \rightarrow 1 - w_i \eta (w_i - \nu)^{-1} = 0 \rightarrow (w_i - \nu) = w_i \eta$

which gives:

$$w_i = \frac{\nu}{1 - \eta}$$

There is no *true* heterogeneity: same price and same wage for each firm i.

Model with efficiency wages - Equilibrium

Given the market structure, and that, by symmetry, $w = w_i$, we can define the outside option as:

$$\nu \equiv ub + (1-u)w$$

where u is the unemployment rate/probability and b the unemployment benefit. Hence:

$$w_i = w = \frac{\nu}{1-\eta} = \frac{ub + (1-u)w}{1-\eta}$$

Solving for w:

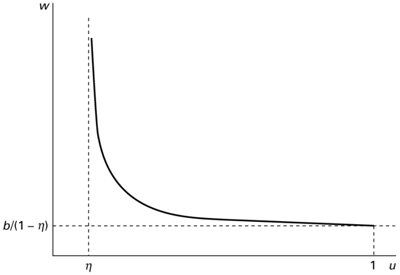
$w\left(1 - \frac{1-u}{1-\eta}\right) = w\frac{u-\eta}{1-\eta} = \frac{ub}{1-\eta} \rightarrow w = \frac{u}{u-\eta}b \rightarrow w = \frac{1}{1-\frac{\eta}{u}}b$ with $u > \eta$.

Model with efficiency wages - Equilibrium (II)

Equilibrium wage, i.e. **wage equation**:

$$w = \frac{1}{1 - \frac{\eta}{u}} b$$

The wage is increasing in b and in η, while **decreasing** in u.



Indeed, as $u \uparrow$, $\nu \downarrow$, so $w \downarrow$ to compensate.

Model with efficiency wages - Equilibrium (III)

Due to symmetry, all firms set the same price P_i. Hence, $P = P_i$, since the price index is *an* average of the idiosyncratic prices.

The real price $p_i = P_i/P = 1$ is:

$$p_i = 1 = \frac{\sigma}{\sigma - 1}\frac{w}{(w-\nu)^\eta} = m\frac{w}{(w-\nu)^\eta}$$

Thus:

$$mw = (w-\nu)^\eta$$

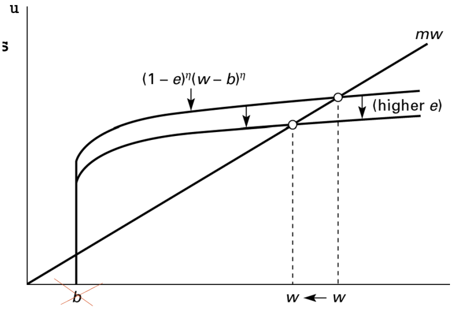
Plugging in the definition of ν:

$$mw = (w - ub - (1-u)w)^\eta = (uw - ub)^\eta = [u(w-b)]^\eta$$

Model with efficiency wages - Equilibrium (IV)

This gives the **price equation**:

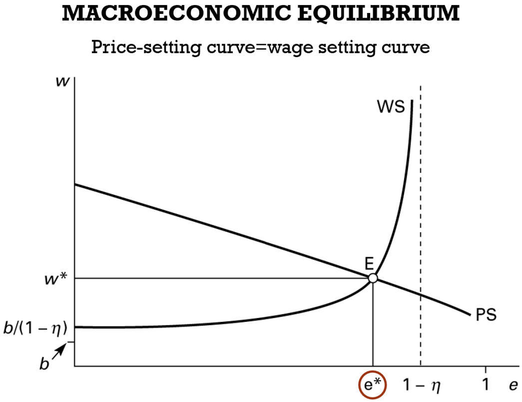
$$mw = \left[u\left(w-b\right)\right]^{\eta} = \left(1-e\right)^{\eta}\left(w-b\right)^{\eta}$$



Indeed, if $u\downarrow$ (or $e\uparrow$), then $\nu\uparrow$, $a\downarrow$, $P\uparrow$ (since $\frac{W}{a}\uparrow$). Hence, by the definition of the real wage, $w\downarrow$.

Model with efficiency wages - Equilibrium (V)

Putting the two curves together, we find the equilibrium wage and employment rate:



Model with efficiency wages - GE

To close the model, the supply of labor is inelastic at L, employment is thus e^*L.

In every firm, the labor employed is then e^*L/n.

Individual productivity is:

$$a^* = (w^* - \nu)^\eta = (w^* - (1 - e^*)b - e^*w^*)^\eta = [(w^* - b)(1 - e^*)]^\eta$$

Hence, the individual production is a^*e^*L/n.

Finally, total production Y^* is a^*e^*L.