RC2 Project

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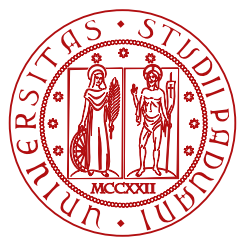
April 4, 2025



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Problem Description and Motivations

For the unicycle presented in class:

- design one or more controllers to follow a chosen desired trajectory
- avoid obstacles during the path
- at the end of the maneuver park in a chosen position with a specific orientation

Motivation:

- search and rescue vehicles
- automated warehouses
- farming automation (autonomous tractors)
- ...

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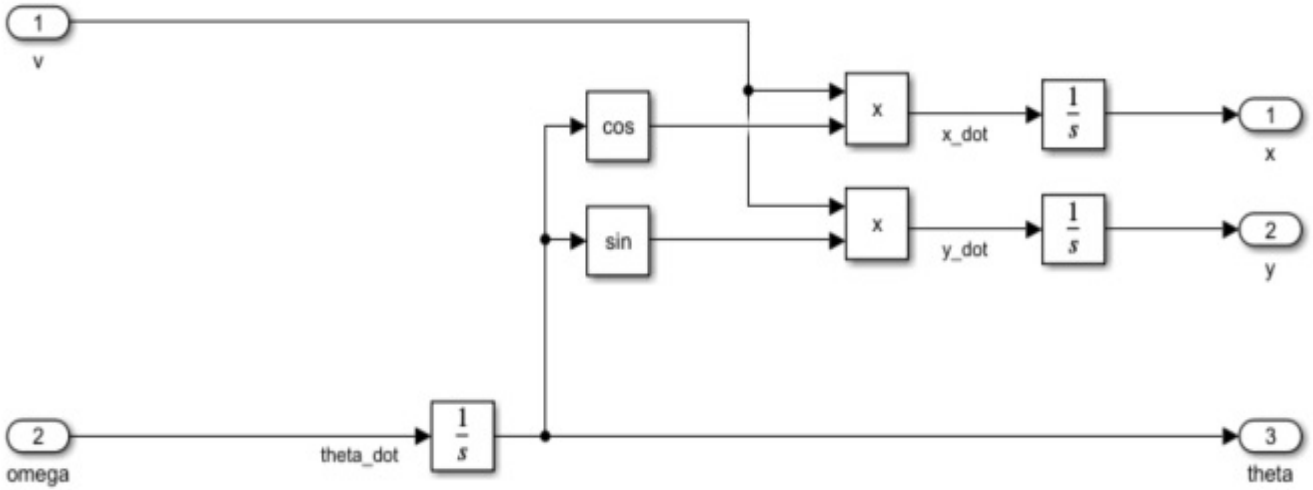
Trajectory tracking via state error feedback - Unicycle model

As presented in class the model is:

$$\dot{x} = v\cos(\theta)$$
$$\dot{y} = v\sin(\theta)$$
$$\dot{\theta} = \omega$$

- x, y: planar coordinates
- θ: orientation of the robot w.r.t the x - axis,
- v: linear velocity (m/s),
- ω: angular velocity (rad/s).

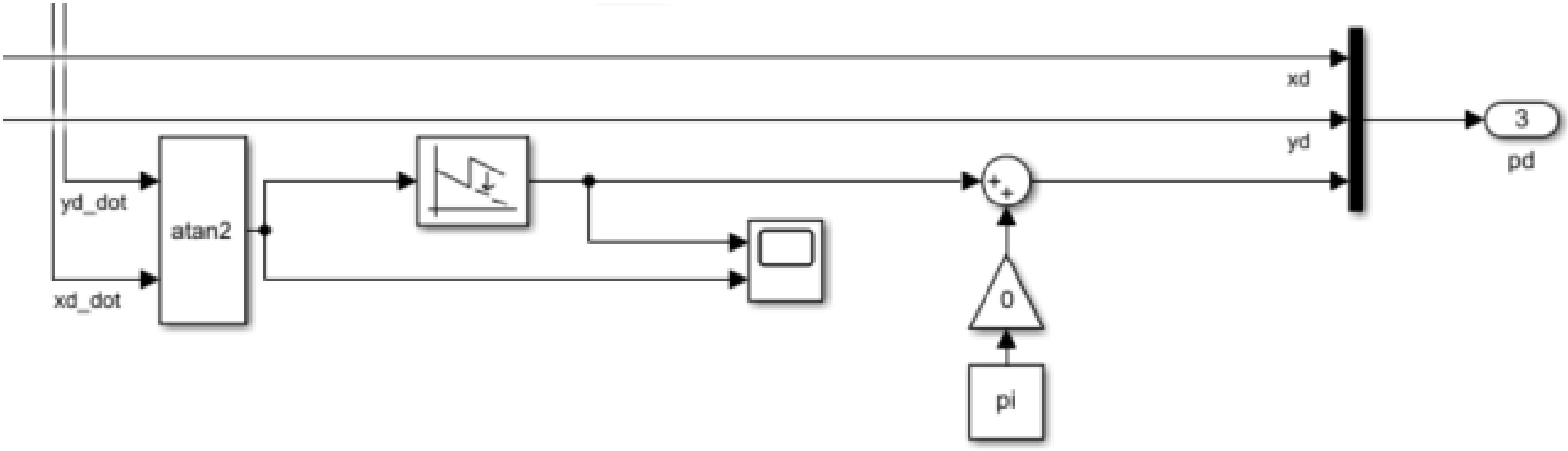


Trajectory tracking via state error feedback - Differential flatness

If we define the output as the desired trajectory in cartesian coordinate then:

$$\theta_d(t) = atan2(\dot{y}_d(t), \dot{x}_d(t)) + k\pi \quad k \in \{0, 1\}$$

- $k = 0 \rightarrow$ forward motion
- $k = 1 \rightarrow$ backward motion



Trajectory tracking via state error feedback - Controller

Idea behind the controller:

- Evaluate the error in body frame
- Write the dynamics of it
- After a change of inputs, linearize the dynamics around the desired trajectory
- Select as input $u = Ke$ where $u \in \mathbb{R}^2, K \in \mathbb{R}^{2\times 3}$, $e \in \mathbb{R}^3$

$$K = \begin{bmatrix} -k_1 & 0 & 0 \\ 0 & -k_2 & -k_3 \end{bmatrix}$$

Which leads to the following error dynamic:

$$\dot{e}(t) = \begin{bmatrix} -k_1 & \omega_d(t) & 0 \\ \omega_d(t) & 0 & v_d(t) \\ 0 & -k_2 & -k_3 \end{bmatrix} e(t)$$

Trajectory tracking via state error feedback - Trajectory

Let $a \in \mathbb{R}^+ \setminus \{0\}$ and $\xi \in (0, 1)$, selecting:

- $k_1 = k_3 = 2\xi a$
- $k_2 = \frac{a^2 - \omega_d(t)^2}{v_d(t)}$

and selecting a circular trajectory with constant $\omega_d(t)$ then the error dynamic becomes time invariant.

Let $R \in \mathbb{R}^+ \setminus \{0\}$ the radius of the circle and ω_d the desired angular velocity, the parametrization is given by:

$$\begin{cases} x_d(t) = R\cos(\omega_d t) \\ y_d(t) = R\sin(\omega_d t) \end{cases}$$

```
% First quarter of circle
if(flag == 1)
    enable_traj = 1;
    enable_park = 0;
    x = r1*cos(omega*t);
    y = r1*sin(omega*t);
end
```

Trajectory tracking via state error feedback - Control law implementation

With the above choice of the coefficients of the controller, namely:

- $k_1 = k_3 = 2\xi a$
- $k_2 = \frac{a^2 - \omega_d(t)^2}{v_d(t)}$

In k_2 we are dividing by $v_d(t) \rightarrow$ particular care must be taken in the implementation of the control law to avoid divisione by zero.

Simulink implementation:

```
function inputs = fcn(vd,wd,eb)
d = 0.4;%csi
a = 8;
k1 = 2*d*a;
if abs(vd) < 1e-4
    if vd >= 0
        vd = 1e-4;
    else
        vd =-1e-4;
    end
end
k2 = (a^2- wd^2)/vd;
k3 = k1;
k = [-k1,0,0;0,-k2,-k3];
inputs = k*eb;
```

Trajectory tracking via state error feedback - eigenvalues

By selecting k_1, k_2, k_3 as in the previous slide we obtain the following eigenvalues of the error dynamics:

- $\lambda_1 = -2\xi a$
- $\lambda_2 = -\xi a + ja\sqrt{1-\xi^2}$
- $\lambda_3 = -\xi a - ja\sqrt{1-\xi^2}$

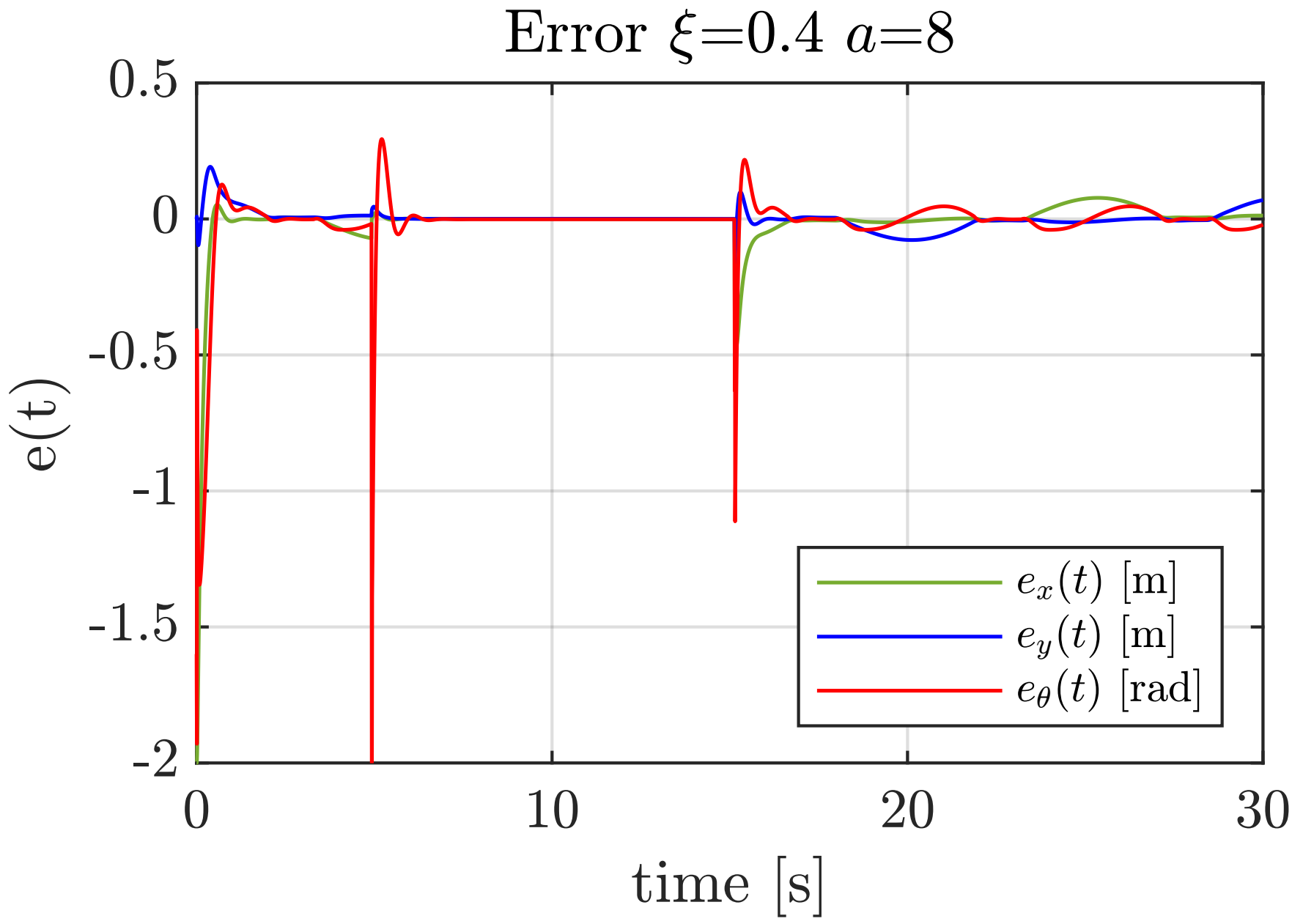
The error at time t will be:

$$e(t) = T \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} T^{-1} e(0)$$

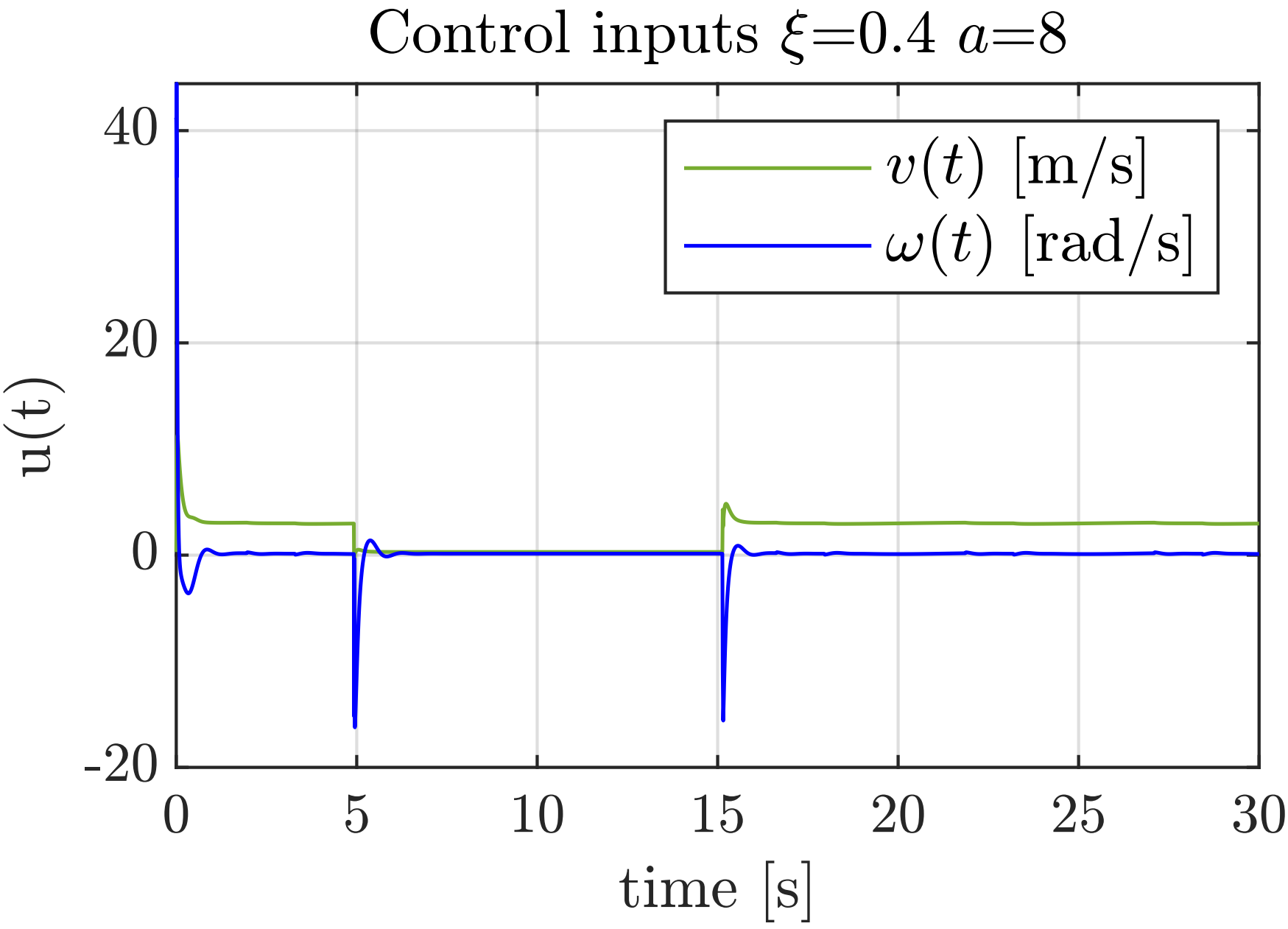
where $T \in \mathbb{R}^{n \times n}$ change of basis.

Note: λ_1 is two times faster than λ_2 and λ_3

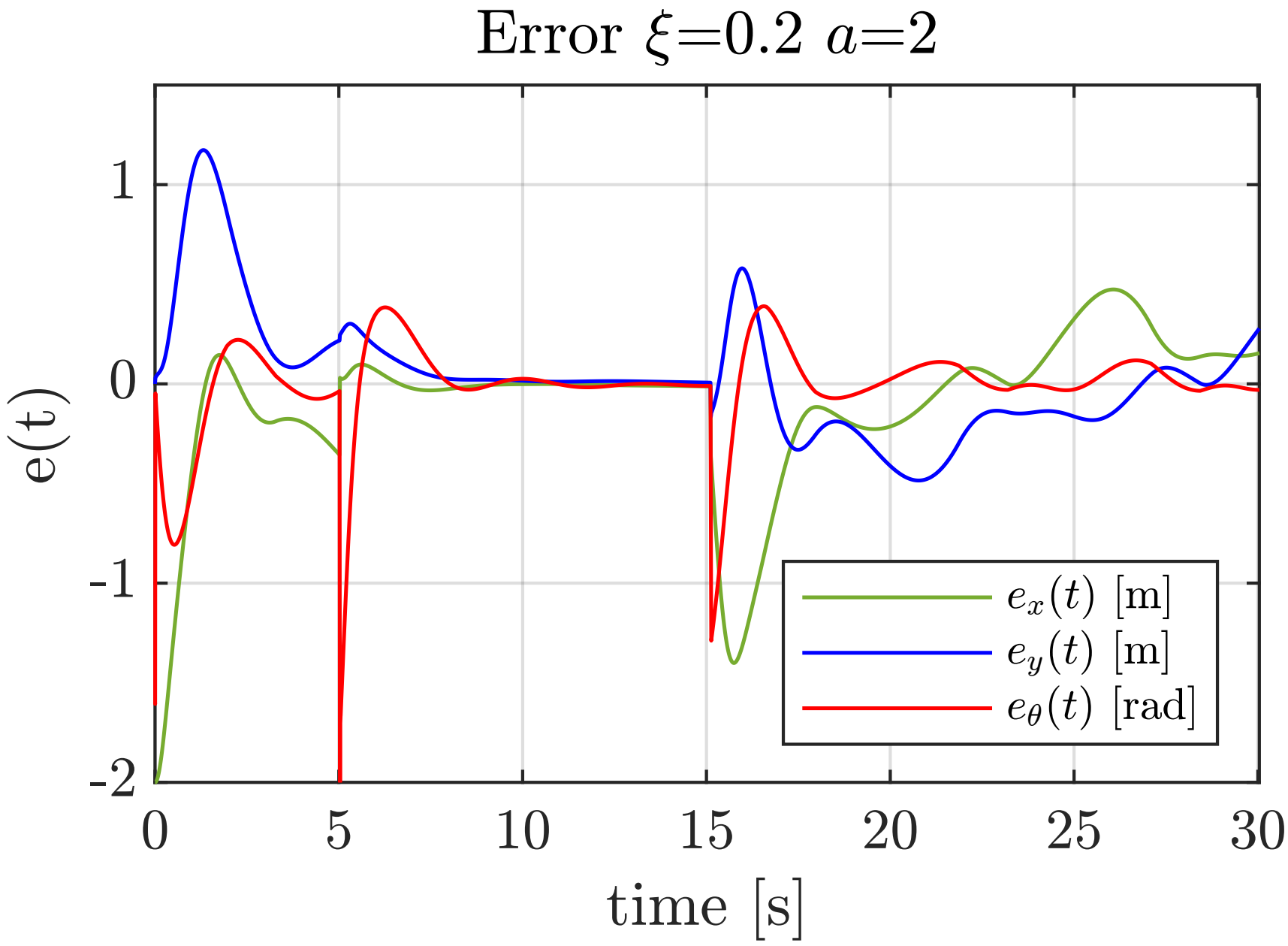
Trajectory tracking via state error feedback - Error



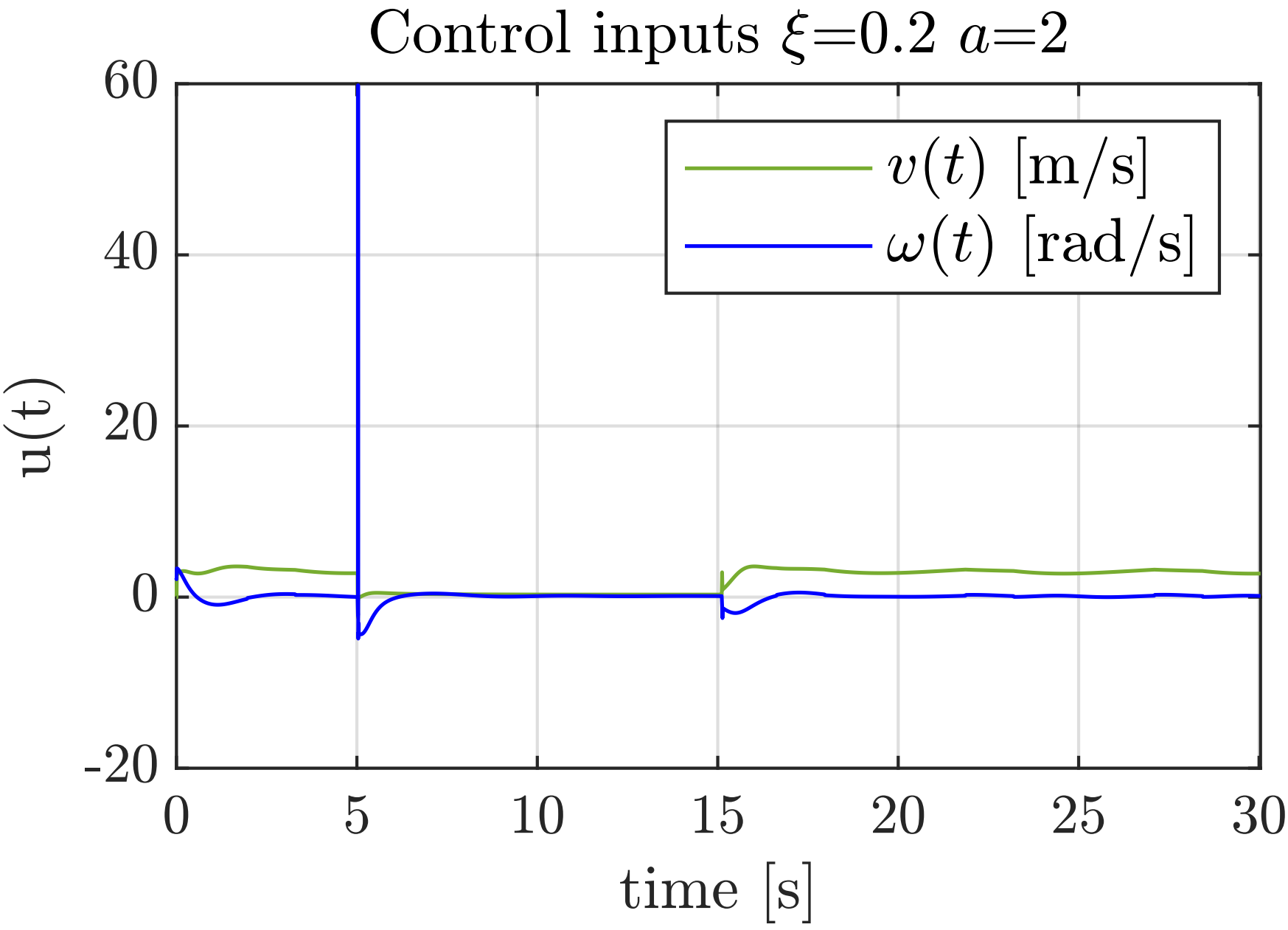
Trajectory tracking via state error feedback - Input



Trajectory tracking via state error feedback - Error



Trajectory tracking via state error feedback - Input



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Obstacle avoidance procedure - Assumptions

In order to address this task we made the following assumptions:

- The position of the obstacle is fixed in time ($x_{obs} = 0, y_{obs} = 10$)
- We suppose to know the cartesian coordinates of the unicycle $\rightarrow$ it is possible to measure the distance between the unicycle and the obstacle

If the distance is below a specified threshold the unicycle enter the obstacle avoidance phase, in the following the simulink implementation:

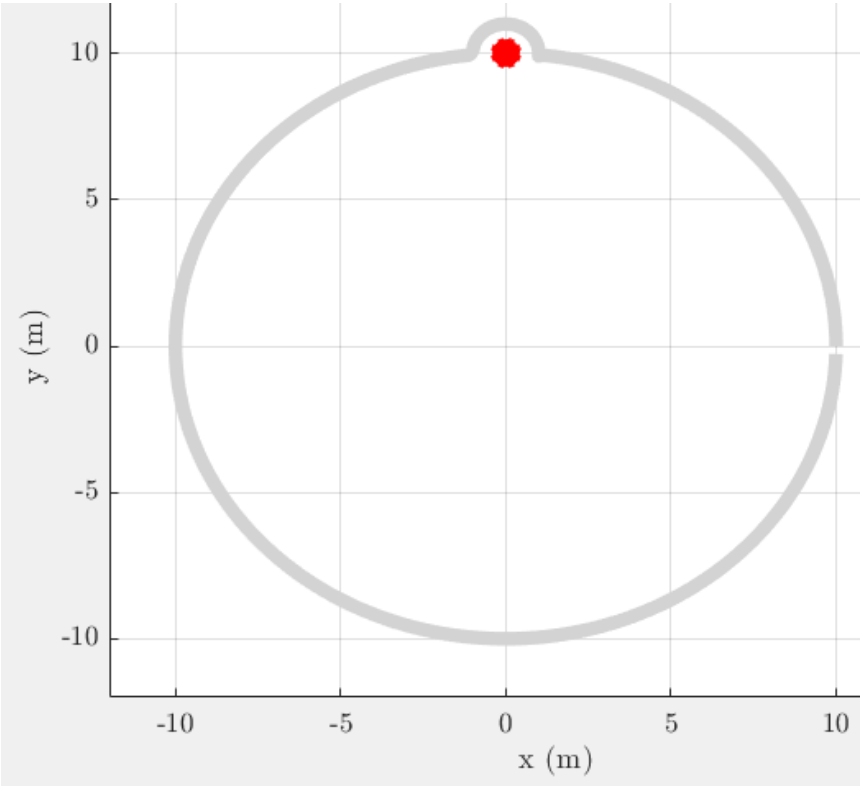
```
if(abs(p(1))<1 && abs(p(2) - r1)<1 && flag==1)
    enable_traj = 1;
    enable_park = 0;
    flag = 2;
end

% Obstacle avoidance
if(flag==2)
    enable_traj = 1;
    enable_park = 0;
    x = r2*cos(omega*(t-pi/2));
    y = r1 + r2*sin(omega*(t-pi/2));
end
if(abs(p(1) + r2)<0.1 && abs(p(2) - r1)<0.1 && flag==2)
    enable_traj = 0;
    enable_park = 1;
    flag = 3;
end
```

Obstacle avoidance procedure - Trajectory

Once the obstacle is detected:

- design a new desired trajectory to avoid the obstacle (a semi circle)
- fed it into the differential flatness block
- once the maneuver is concluded switch back to the original trajectory

Graphically the overall trajectory is the following one:

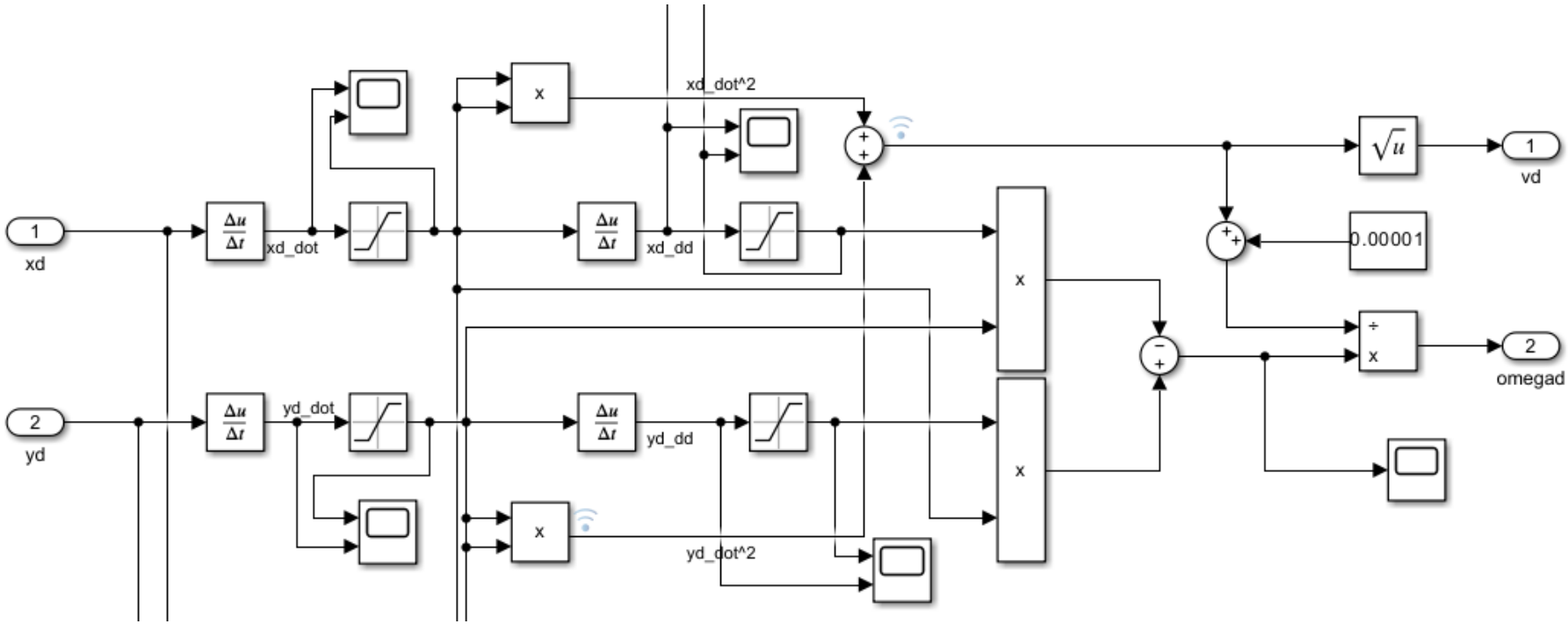


Obstacle avoidance procedure - Switching trajectories

The following problem was encountered:

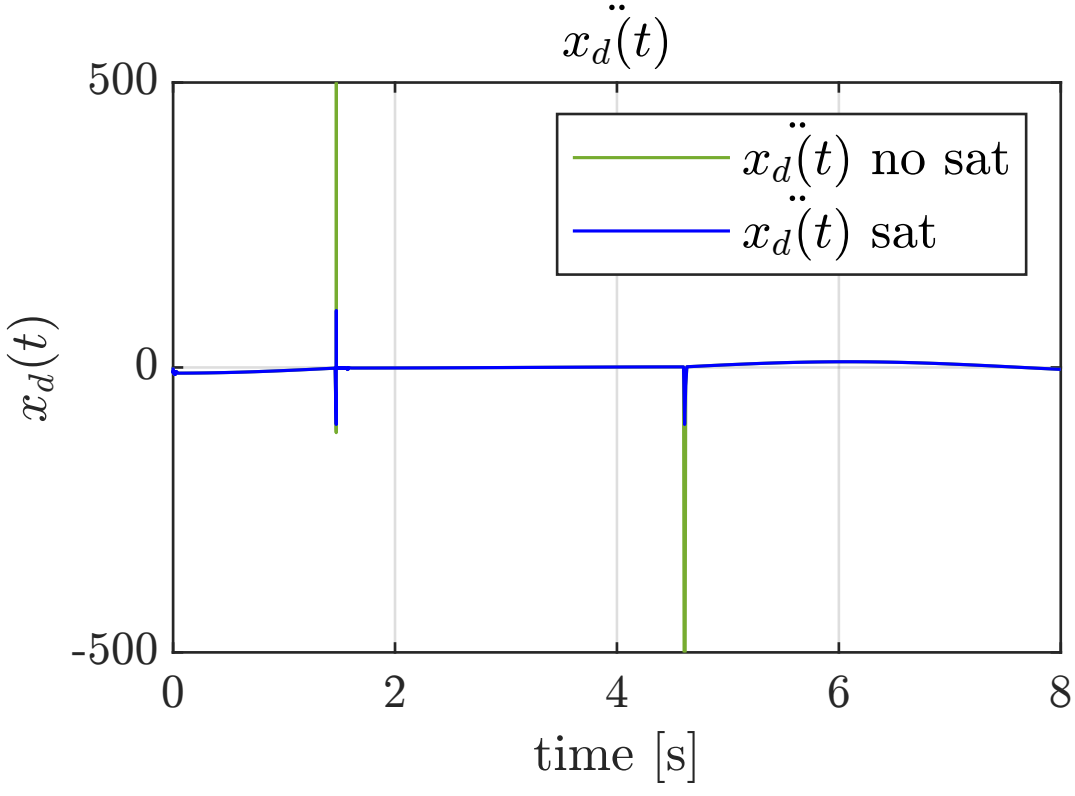
- Switching between different trajectories creates an overall discontinous trajectory fed into the differential flatness block
- This create spikes in $x_d(t), \dot{x}_d(t), y_d(t), \dot{y}_d(t) \rightarrow$ the unicycle will not act properly

Solution $\rightarrow$ add saturation blocks in the differential flatness:



Obstacle avoidance procedure - Switching trajectories solution

In the following a graph reporting the comparision of $\ddot{x}_d(t)$ with and without saturation:



Obstacle avoidance procedure - Results

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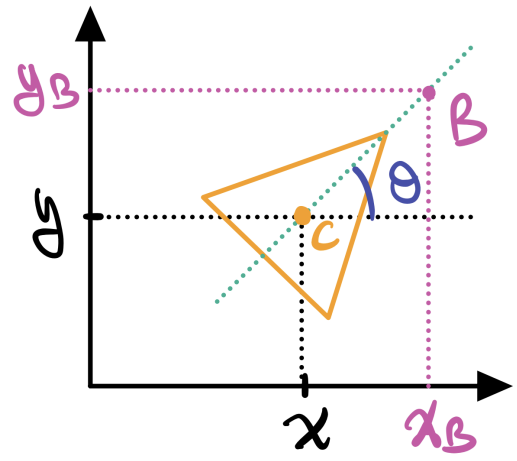
Trajectory Tracking via Output Feedback: B-Point Controller

Idea behind the controller:

- define an invertible map between input and output in order to linearize the system:

$$\begin{cases} x_B = x + b\cos\theta \\ y_B = y + b\sin\theta \end{cases}$$



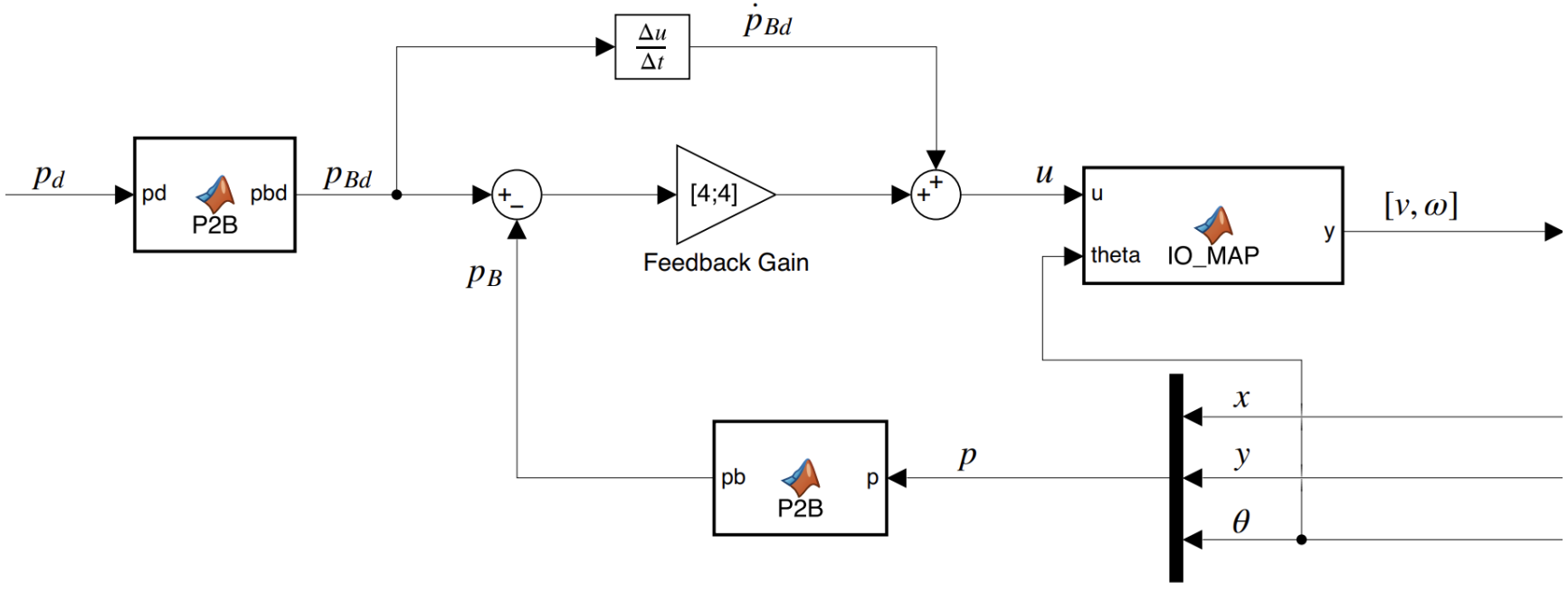
I/O Map: $$\begin{bmatrix} v \\ \omega \end{bmatrix} = \frac{1}{b}\begin{bmatrix} b\cos\theta & b\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad b \neq 0$$

System Dynamics: $$\begin{cases} \dot{x}_B = u_1 \\ \dot{y}_B = u_2 \\ \dot{\theta} = -\frac{\sin\theta}{b}u_1 + \frac{\cos\theta}{b}u_2 \end{cases}$$

Trajectory Tracking via Output Feedback: B-Point Controller

- choose a suitable value for b, *e.g.* $b = 0.1\,m$
- adopt the following linear control law:

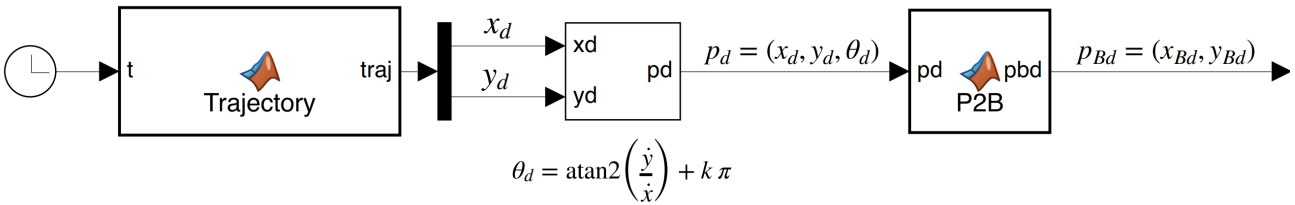
$$\begin{cases} u_1 = K_{P1}\left(x_{Bd} - x_B\right) + \dot{x}_{Bd} \\ u_2 = K_{P2}\left(y_{Bd} - y_B\right) + \dot{y}_{Bd} \end{cases} \qquad K_{P1},\, K_{P2} > 0$$



Trajectory Tracking via Output Feedback: Reference Type

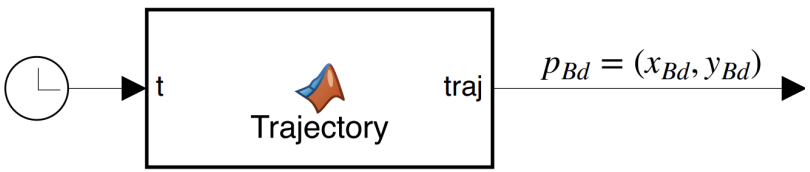
Two different types of reference can be employed for the B-Point controller:

Centroid Reference

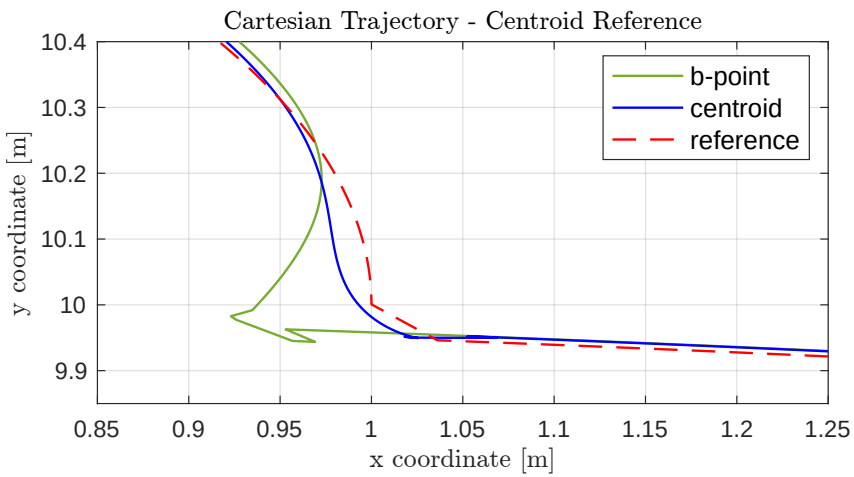


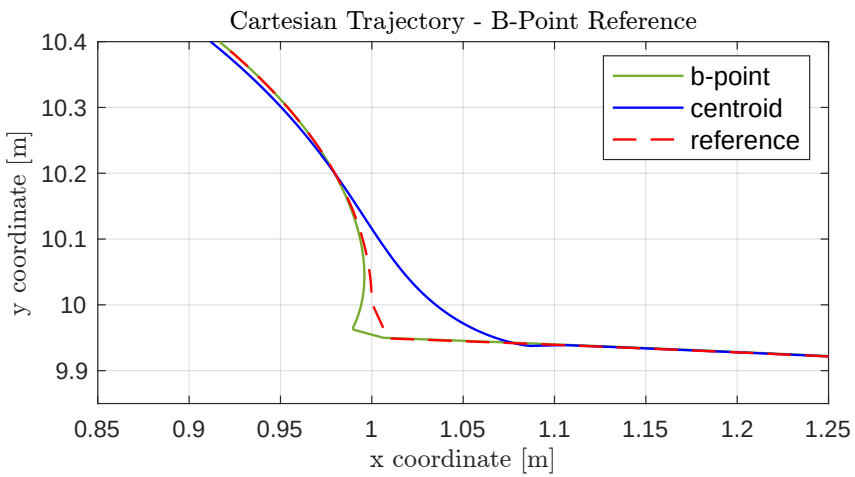
- Allows for more precise tracking
- Subject to numerical errors

B-Point Reference



- Smooth trajectories
- Approximate tracking

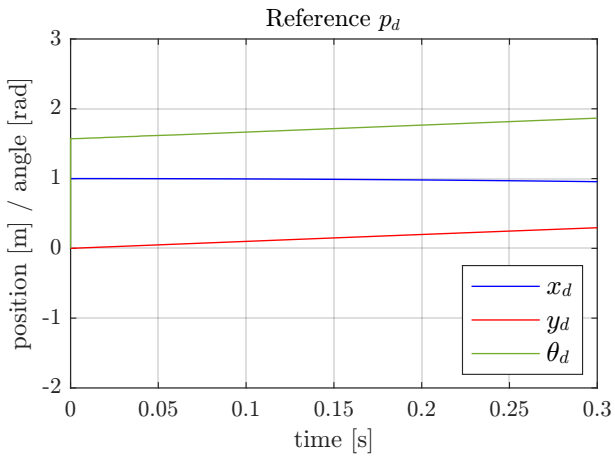


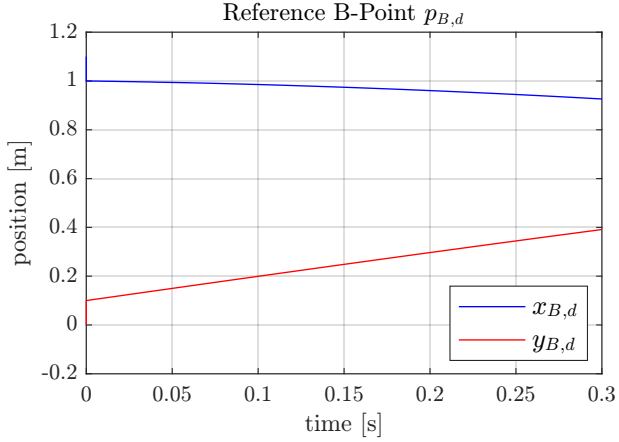


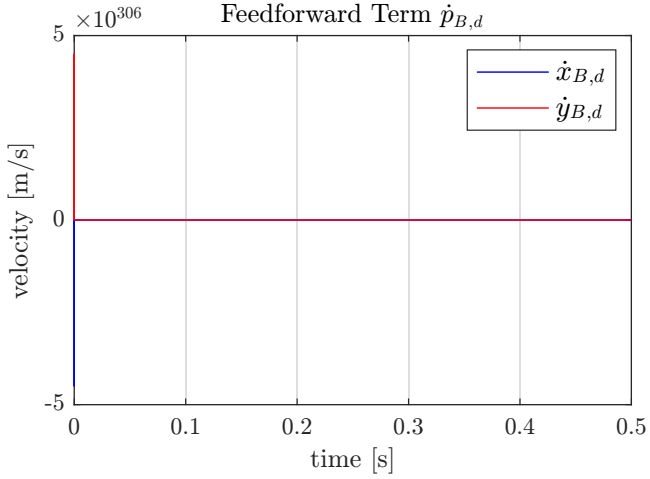
Trajectory Tracking via Output Feedback: Numerical Issues

During the implementation of the B-Point controller, the following numerical error has been encountered:

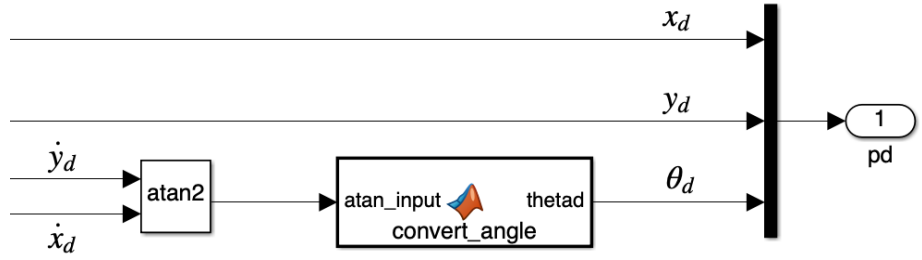
```
An error occurred while running the simulation and the simulation was terminated
Caused by:
  • Derivative of state '1' in block 'unicycle_bpoint_2022b_ISSUE/unicycle_kinematics/Integrator' at time 4.4501477170144E-309
    is not finite. The simulation will be stopped. There may be a singularity in the solution.  If not, try reducing the step
    size (either by reducing the fixed step size or by tightening the error tolerances)
Component: Simulink | Category: Block error
```








```
function thetad = convert_angle(atan_input)
    if(atan_input < 0)
        thetad = atan_input + 2*pi;
    else
        thetad = atan_input;
    end
```



Trajectory Tracking via Output Feedback: Numerical Issues

Possible solutions:

- Saturation block: Ineffective, seems that the issue is not completely related to the magnitude of the derivative term.
- Fixed step-size of the solver: Effective!

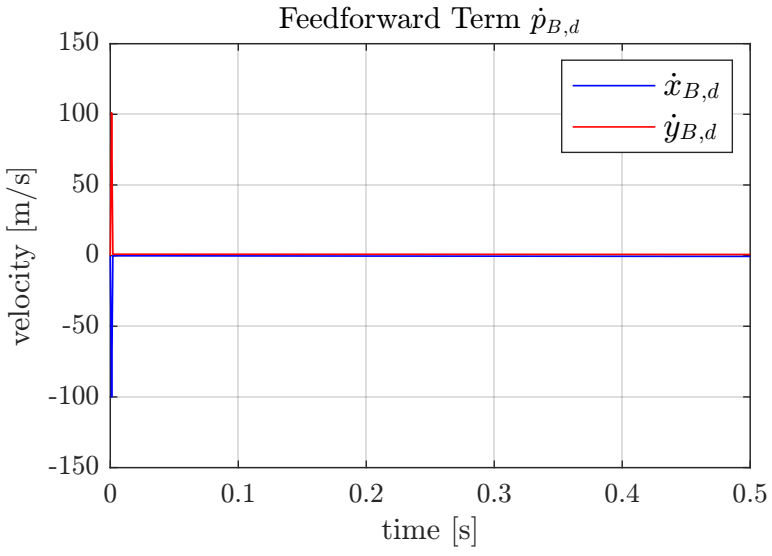


Figure: Step-size $= 10^{-3}$ s

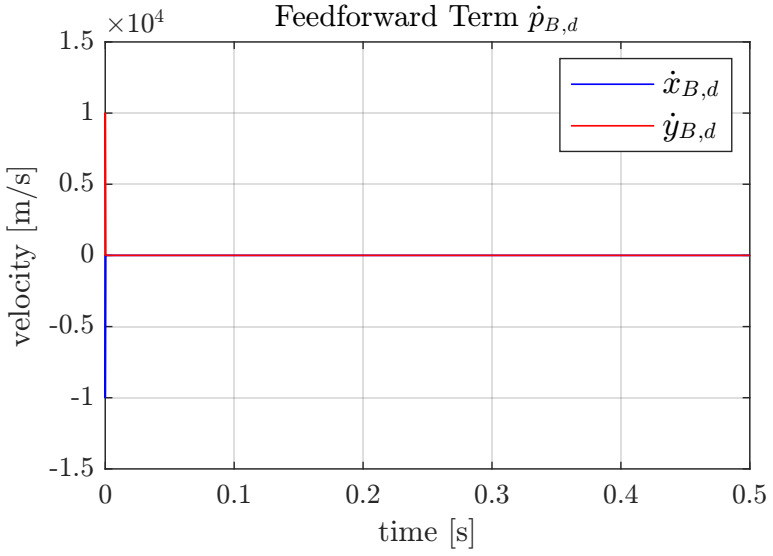
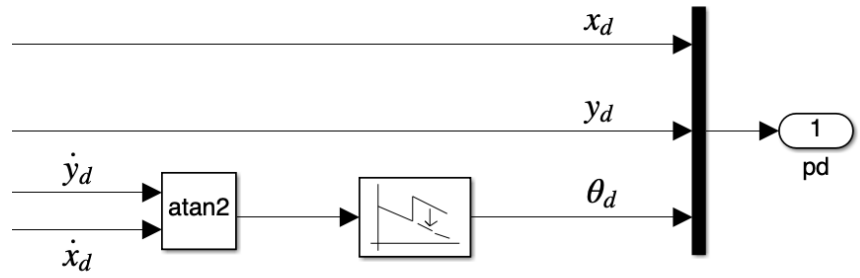
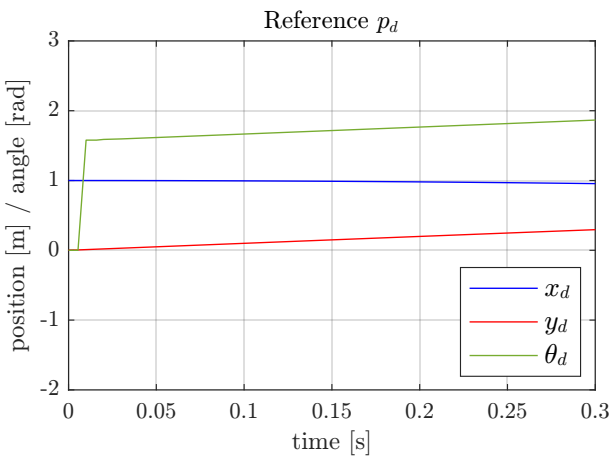


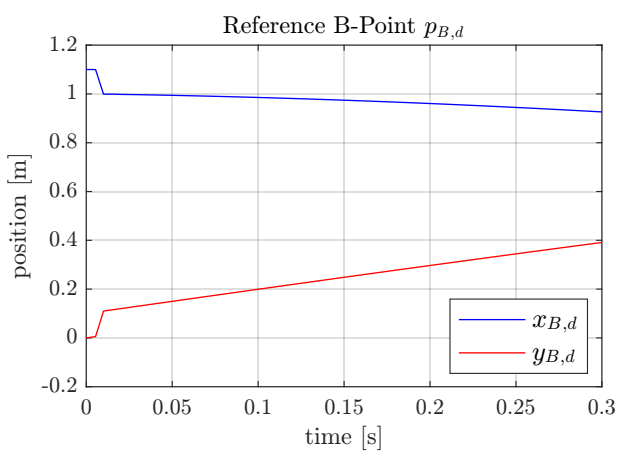
Figure: Step-size $= 10^{-5}$ s

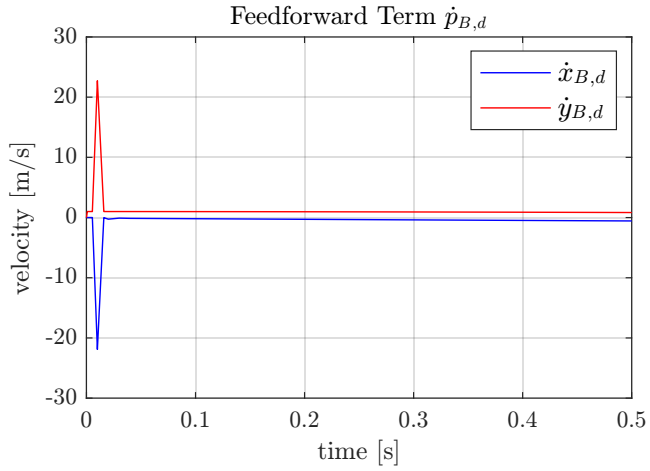
Trajectory Tracking via Output Feedback: Numerical Issues

- Unwrap Block: Effective, since it introduces a delay in the computation of θ_d.



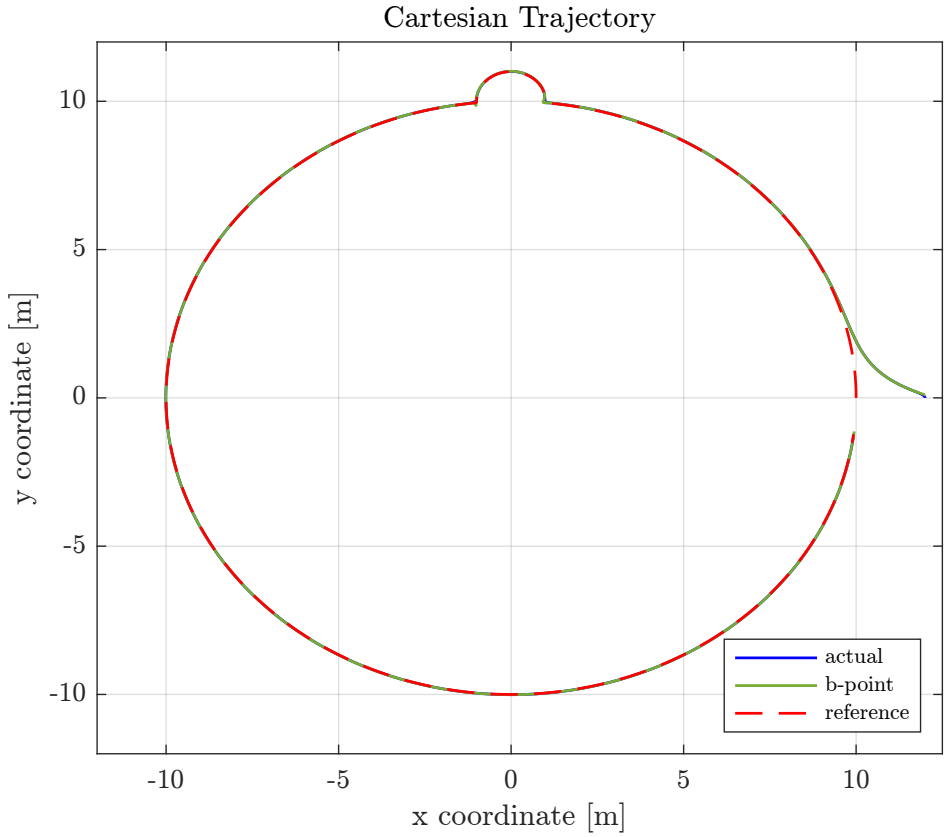


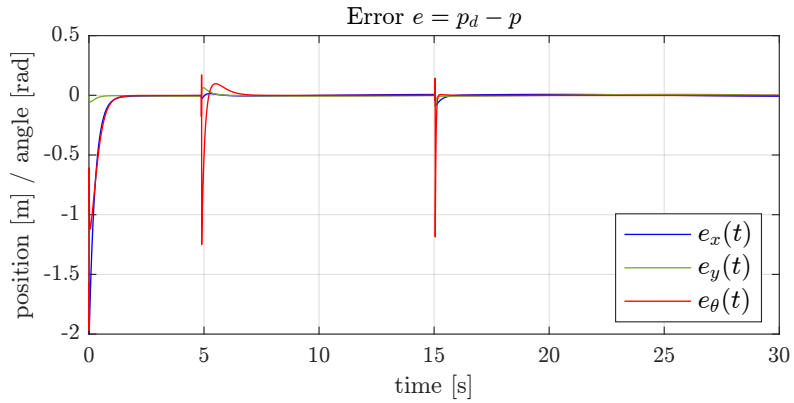


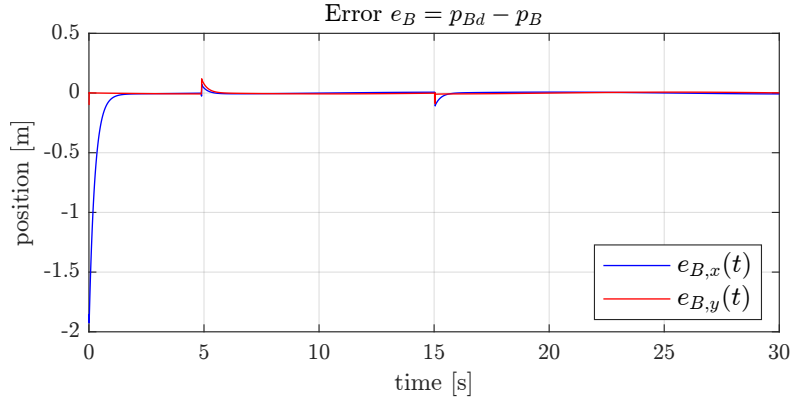


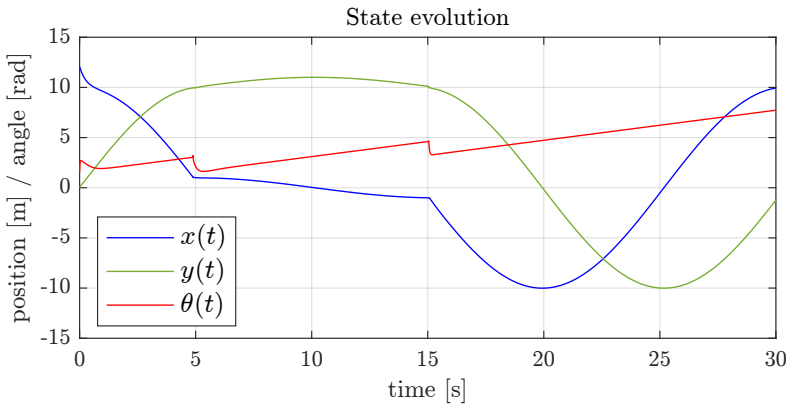
Final observations: try to avoid discontinuities in the reference signal, in particular if it is necessary to compute derivatives of it. If this is not possible, consider adopting pre-computed derivative terms.

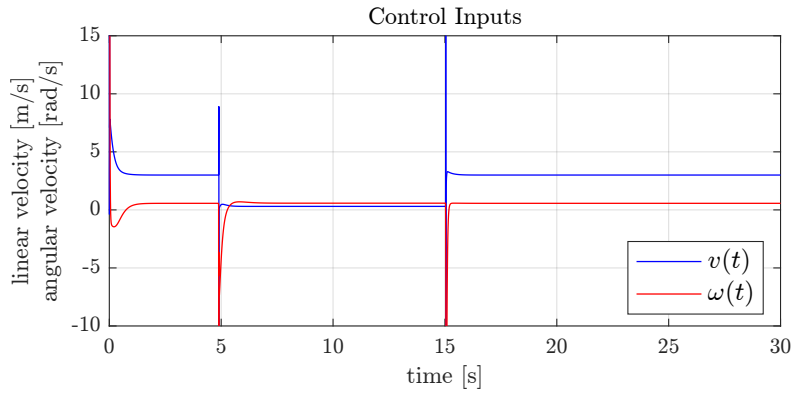
Trajectory Tracking via Output Feedback: Results





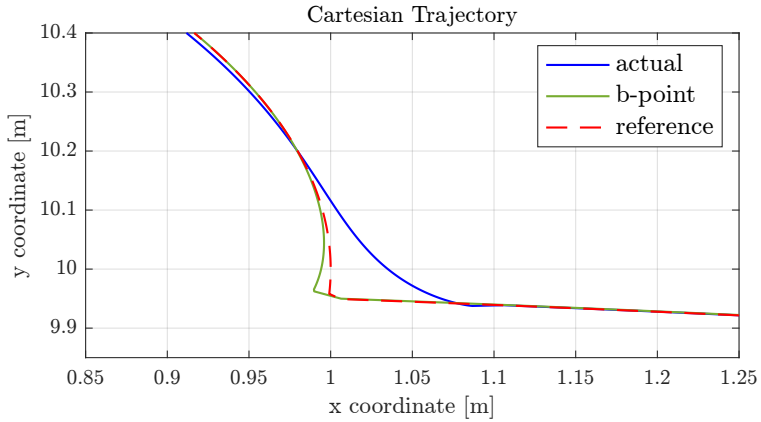


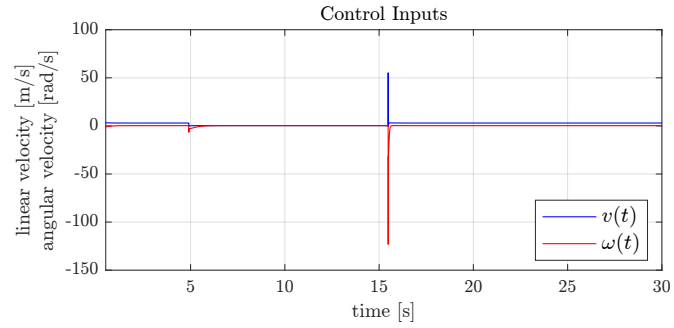




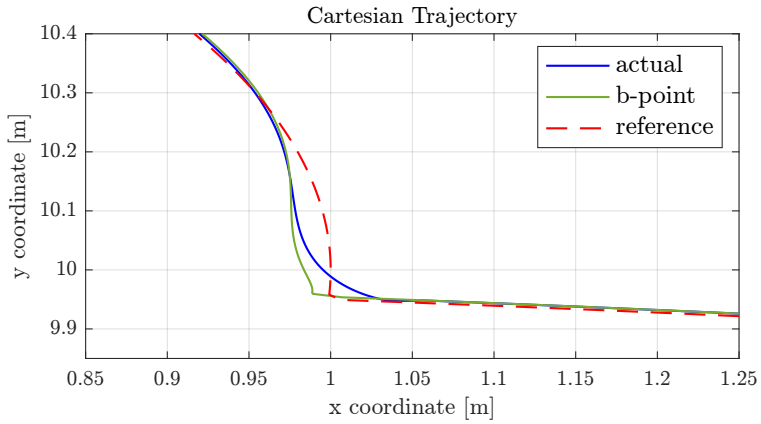
Trajectory Tracking via Output Feedback: Results

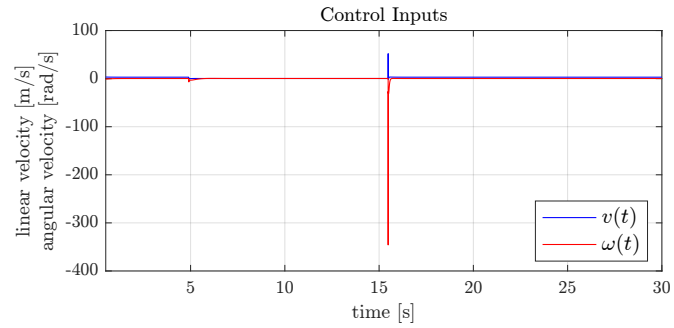
$b = 0.1\,m$:



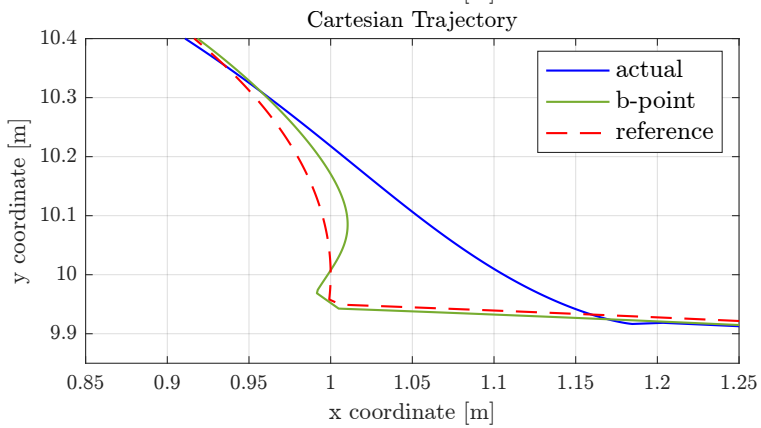


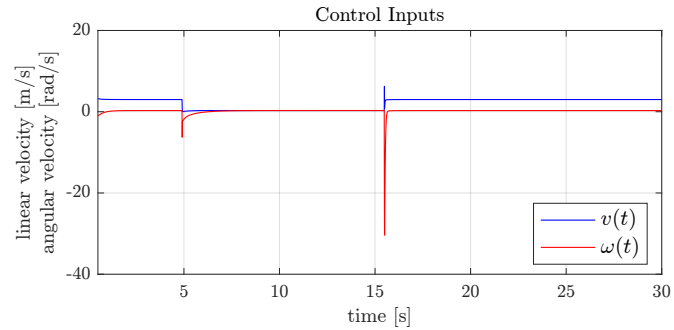
$b = 0.05\,m$:





$b = 0.2\,m$:





Trajectory Tracking: Comparison between State and Output Feedback

	State-Feedback Controller	Output-Feedback Controller
Feedback:	Tracking Error	Cartesian Position Error
	$e_W = p_d - p = \begin{bmatrix} x_d - x \\ y_d - y \\ \theta_d - \theta \end{bmatrix}$	$e_B = p_{Bd} - p_B = \begin{bmatrix} x_{Bd} - x_B \\ y_{Bd} - y_B \end{bmatrix}$
Feedforward:	Differential Flatness	Cartesian Velocities
	$\begin{cases} v = \pm\sqrt{\dot{x}_d^2 + \dot{y}_d^2} \\ \omega = \frac{\dot{x}_d \ddot{y}_d - \dot{y}_d \ddot{x}_d}{\dot{x}_d^2 + \dot{y}_d^2} \end{cases}$	$\dot{p}_{Bd} = \begin{bmatrix} \dot{x}_{Bd} \\ \dot{y}_{Bd} \end{bmatrix}$

Trajectory Tracking: Comparison between State and Output Feedback

	State-Feedback Controller	Output-Feedback Controller
Stability:	A.S. $\iff$ v_d, ω_d constant	A.S. $\iff$ $K_{P1}, K_{P2} > 0$
	$\dot{e} = \begin{bmatrix} -2\xi a & \omega_d & 0 \\ -\omega_d & 0 & v_d \\ 0 & -\frac{a^2-\omega_d^2}{v_d} & -2\xi a \end{bmatrix} e$	$\dot{e}_B = -\begin{bmatrix} K_{P1} & 0 \\ 0 & K_{P2} \end{bmatrix} e_B$
Limitations:	Persistent Trajectory ($v_d \neq 0$) $\implies$ No stopping, No motion inversion	Constraint $b \neq 0$
	$K_2 = \frac{a^2-\omega_d^2}{v_d}$ $\quad a \neq 0$	$\begin{bmatrix} \dot{x}_B \\ \dot{y}_B \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -b\sin(\theta) \\ \sin(\theta) & b\cos(\theta) \end{bmatrix}}_{T(\theta)} \begin{bmatrix} v \\ \omega \end{bmatrix}$
		$\det(T(\theta)) = b$

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Constrained trajectory tracking via NMPC

Idea: Formulate the trajectory tracking task in a MPC fashion, so that obstacle avoidance is embedded in the problem.

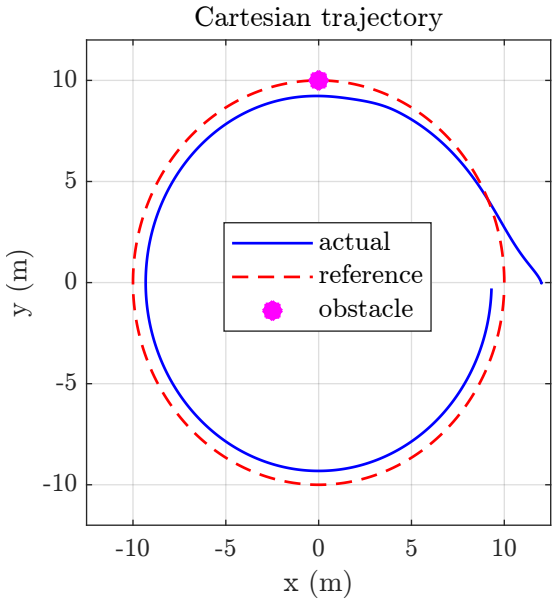
NLP:

$$
\begin{aligned}
\min_{q(\cdot),u(\cdot)} \quad & \int_0^T [\frac{1}{2}||q(t) - ref(t)||^2_{Q_{states}} + \frac{1}{2}||u(t)||^2_{Q_u}]dt + \frac{1}{2}||q(N) - ref(N)||^2_{Q_{N,states}} \\
\text{s.t.} \quad & 0 = q(0) - q_0 \\
& 0 = \dot{q} - f(q(t), u(t)) \quad t \in [0, T] \\
& (x(t) - x_{obs})^2 + (y(t) - y_{obs})^2 > 0.5 \quad t \in [0, T]
\end{aligned} \tag{1}
$$

Tools: Simulations and control design have been carried out using the MATMPC library. In such toolbox, the horizon length is expresses as $T = N \cdot T_{s_{st}}$. Here $T_{s_{st}} = 0.05s$ was used.

Pros & Cons: Obstacle avoidance easy to implement, but the controller is hard to tune and often does not guarantee persistent feasibility.

Constrained trajectory tracking via NMPC - Results



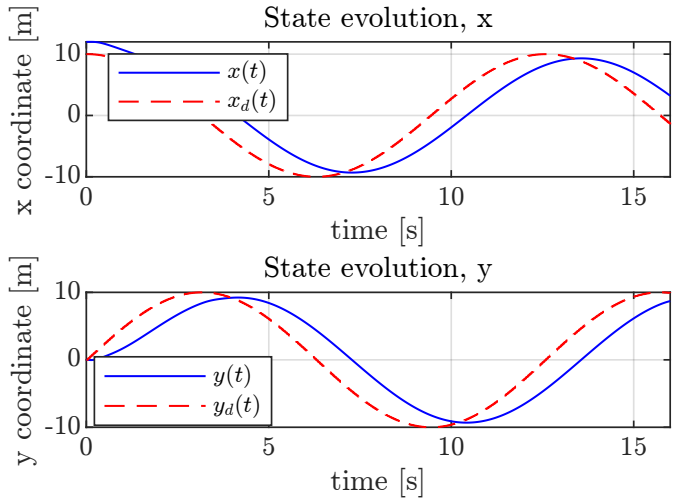
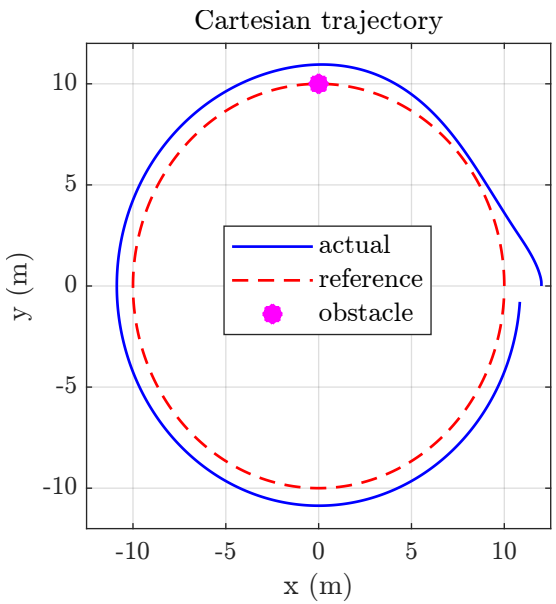


Figure: $N = 100$, $Q = diag\{\underbrace{0.1, 0.1, 0.01}_{Q_{states}}, \underbrace{0.1, 0.1}_{Q_u}\}$



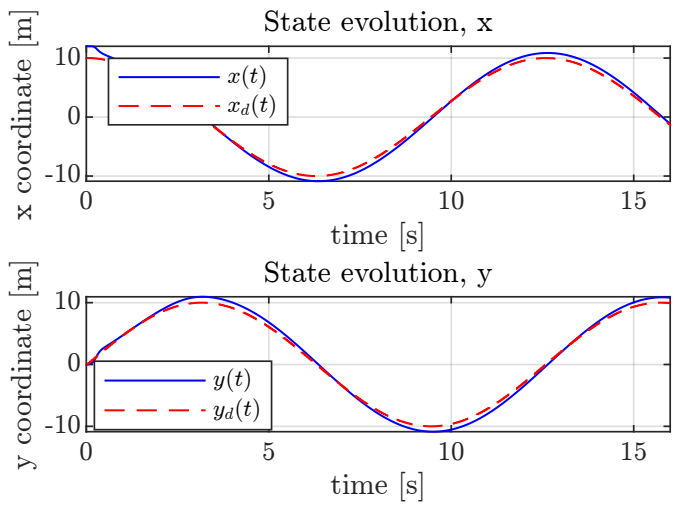


Figure: $N = 5$, $Q = diag\{\underbrace{0.1, 0.1, 0.05}_{Q_{states}}, \underbrace{0.001, 0.01}_{Q_u}\}$

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The parking problem

Goal: Once the unicycle completes one lap, it has to park inside a box.

Approaches:

1. Posture regulation
   - Need to specify only the final posture;
   - There is no complete control over the accomplished trajectory.
2. Trajectory tracking
   - Need to design the whole trajectory;
   - Complete control of the maneuver;

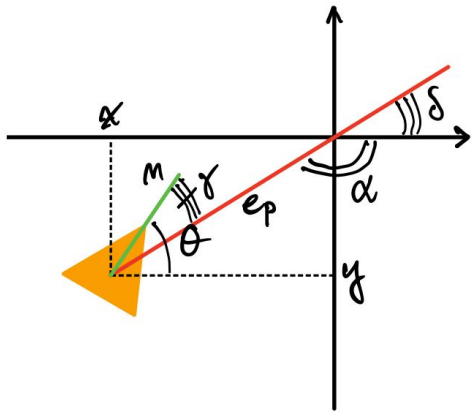
Posture regulation

Goal: From $\bar{q} = [\bar{x}\ \bar{y}\ \bar{\theta}]$ to $q_0 = [0\ 0\ 0]$

Implementation:

1. Convert to polar coordinates

$$\begin{cases} \rho = \sqrt{x^2 + y^2} = |e_\rho| \\ \gamma = atan2(y, x) + \pi - \theta \\ \delta = \gamma + \theta \end{cases}$$



2. Control law

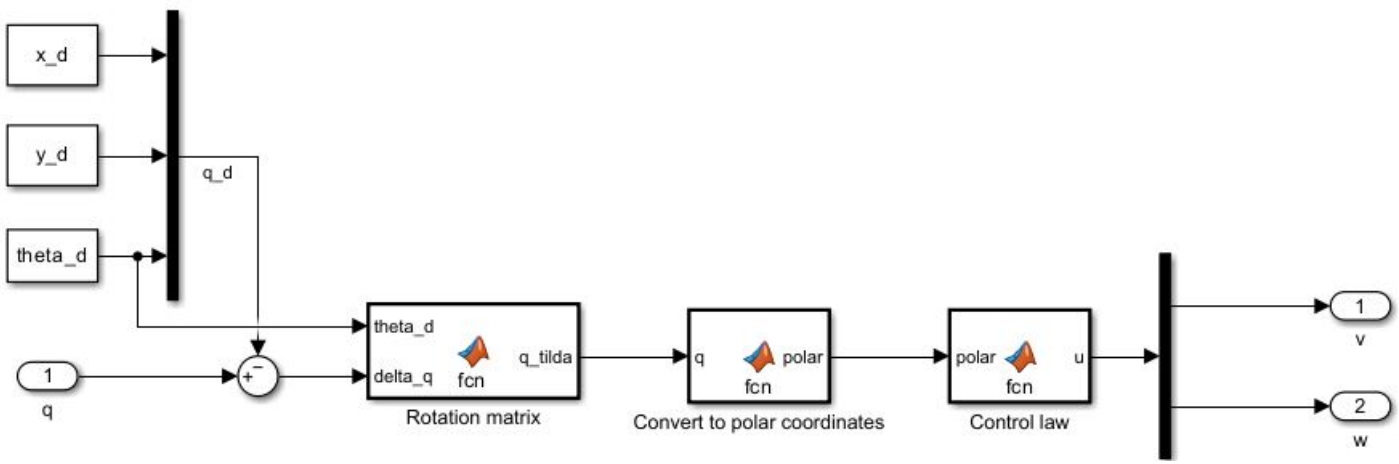
$$\begin{cases} v = K_1\ \rho\ cos(\theta) \\ \omega = K_2\ \gamma + [K_1\ \frac{sin(\gamma)\ cos(\gamma)}{\gamma}\ (\gamma + K_3\ \delta)] \end{cases}$$

Posture regulation - Practical considerations 1

Problem: In practice we want to reach any posture $q_d = [x_d \ y_d \ \theta_d]^T$.
Solution: Apply a change of coordinates before converting to polar ones.

$$\begin{cases} \Delta_q = q - q_d \\ \tilde{q} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} cos(\theta_d) & sin(\theta_d) & 0 \\ -sin(\theta_d) & cos(\theta_d) & 0 \\ 0 & 0 & 1 \end{bmatrix} \Delta_q \end{cases}$$

So, the controller's scheme is



In the next simulations, the target posture is set to $q_d = [12 \quad -2 \quad \frac{\pi}{2}]^T$.

Posture regulation - Practical considerations 2

Problem: Singularity/inputs cause θ to change its steady state after converging to the desired posture.

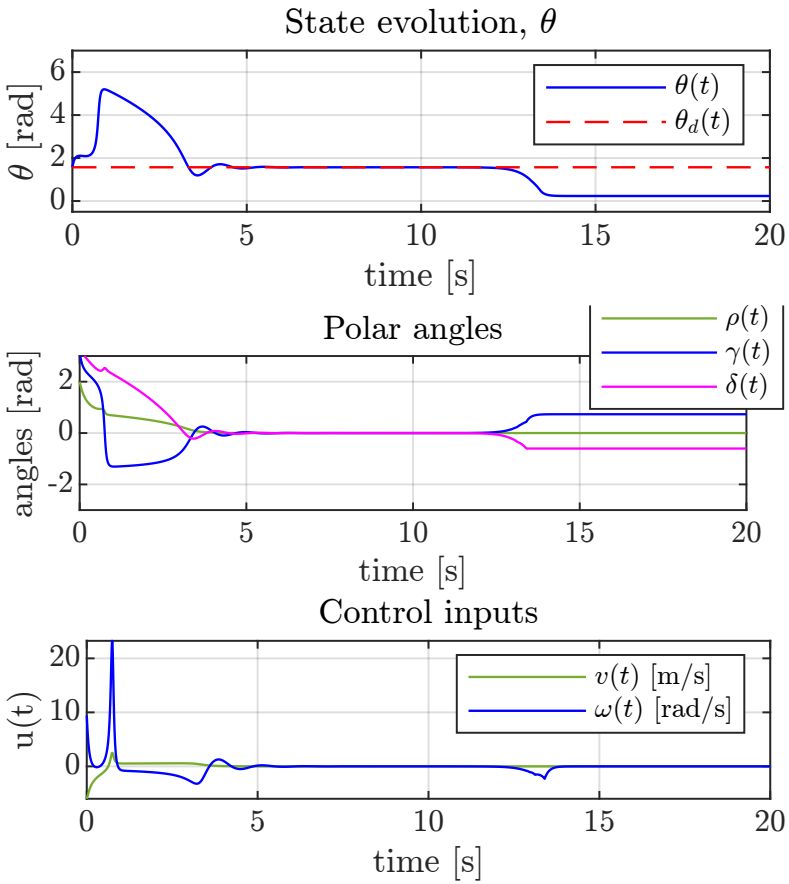


Figure: Controller staying on

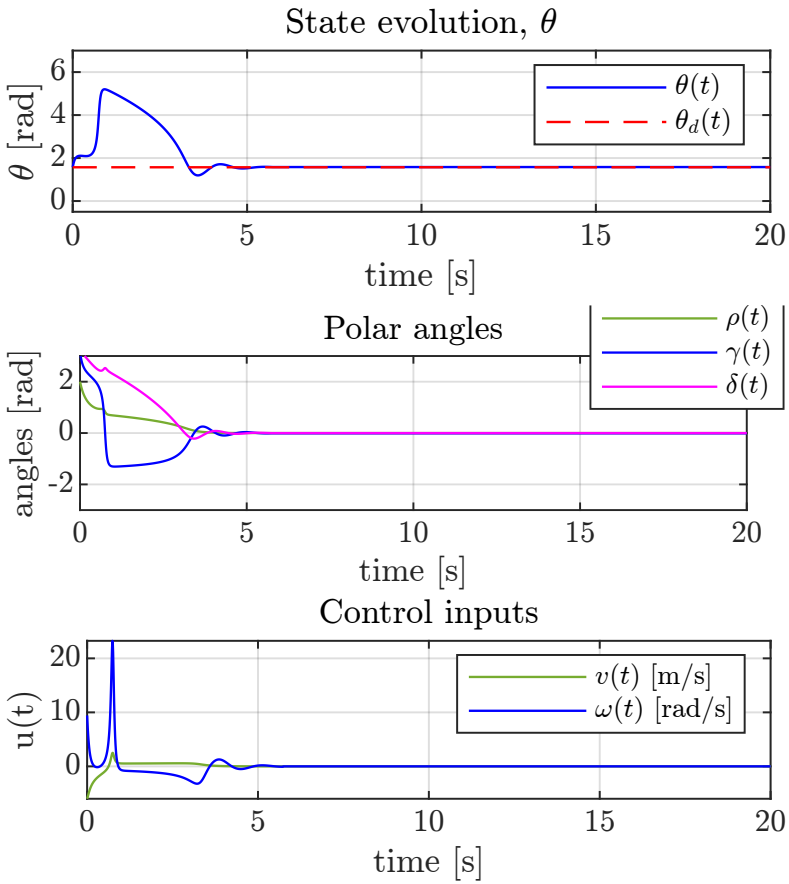
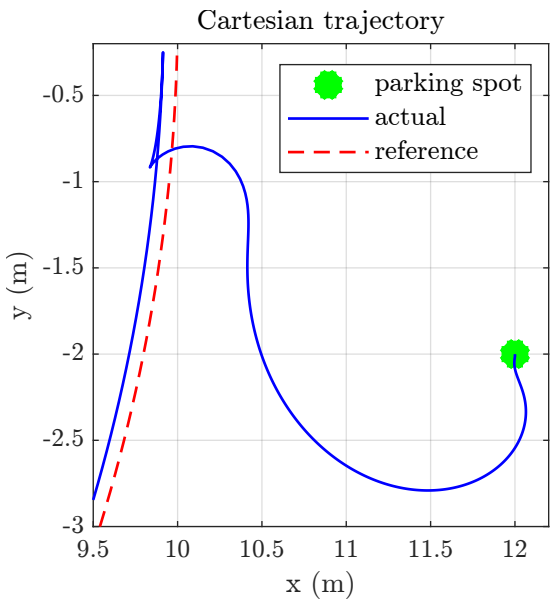


Figure: With our solution

Solution: Stop the controller when ρ, γ and θ are all close to zero.

Posture regulation - Results 1



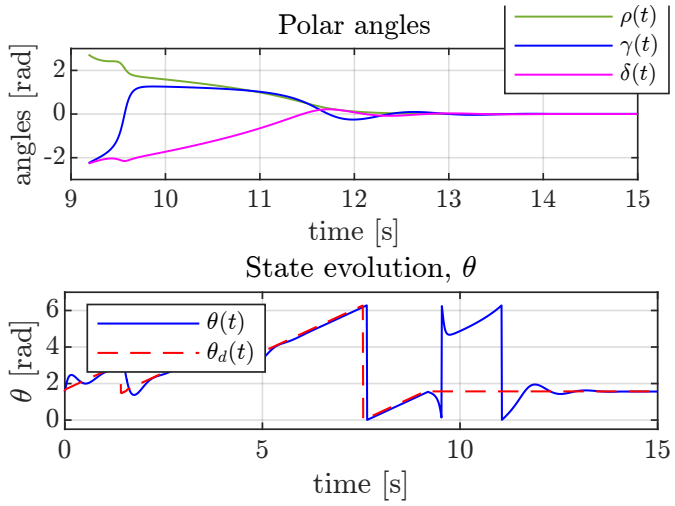
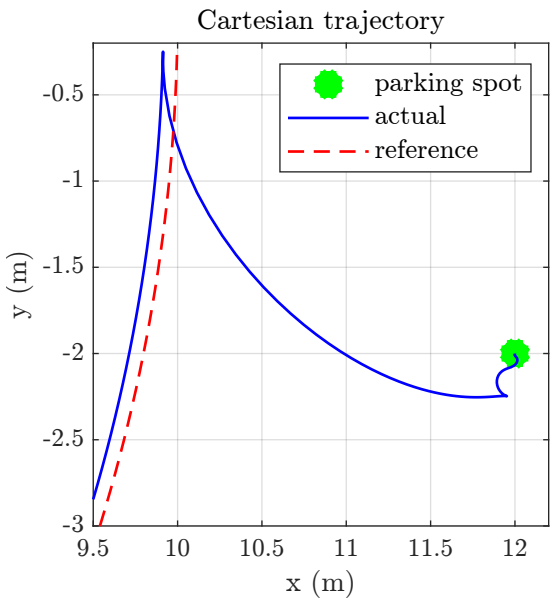


Figure: $K_1 = K_2 = K_3 = 3$



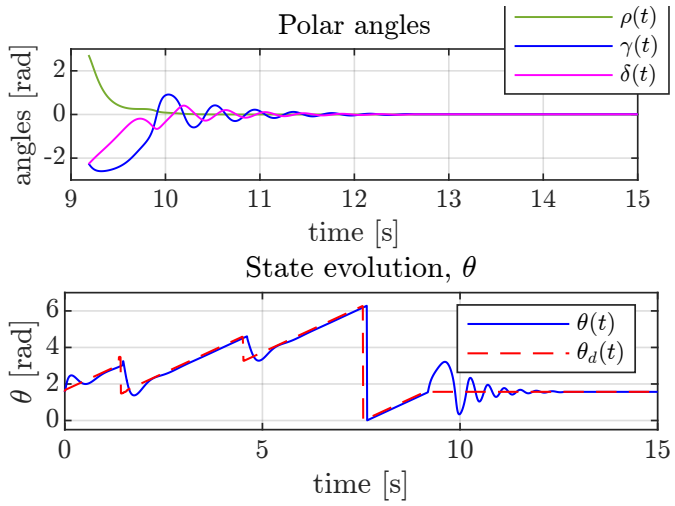
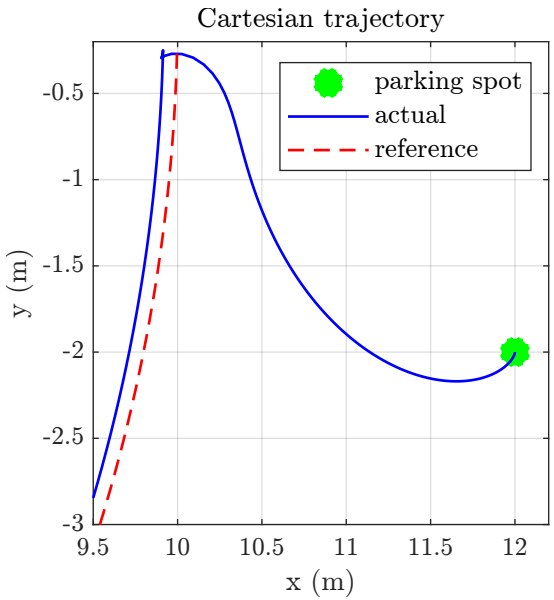


Figure: $K_1 = 9,\ K_2 = K_3 = 3$

Posture regulation - Results 2



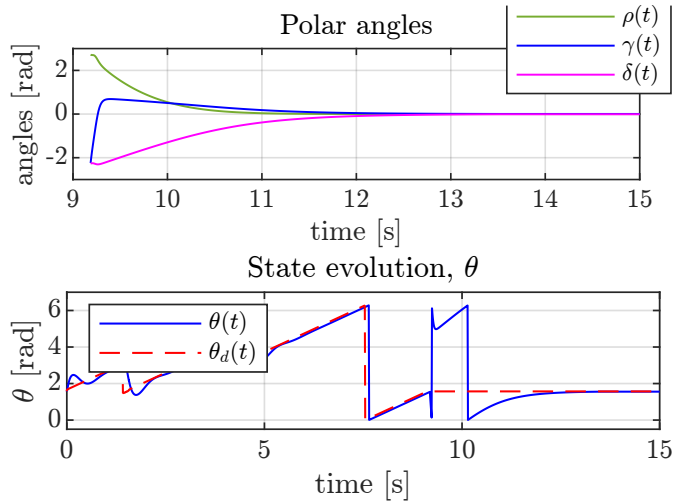
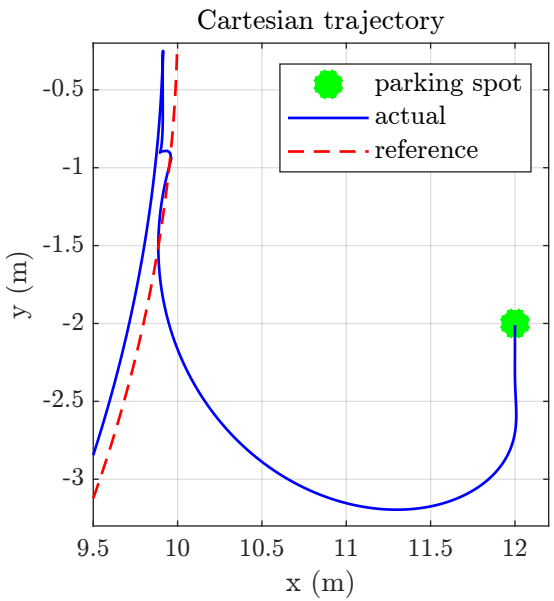


Figure: $K_1 = 3,\ K_2 = 20,\ K_3 = 3$



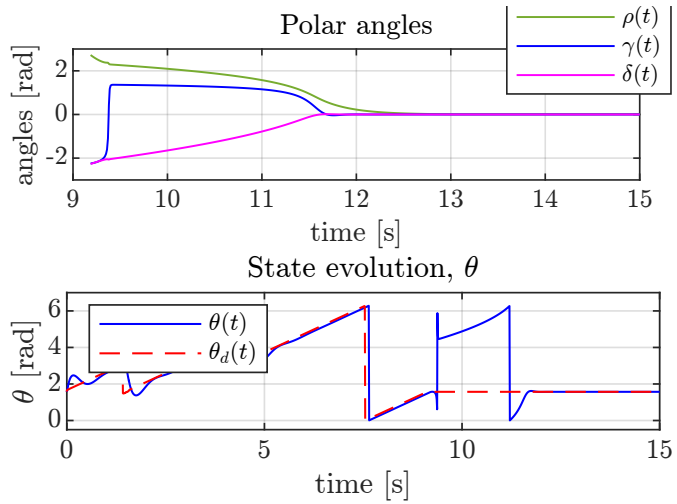


Figure: $K_1 = 3,\ K_2 = 20,\ K_3 = 30$

Full task

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Parking maneuver via trajectory tracking

Goal: Design a path that finishes in the target posture $q_d = [14 \quad 0 \quad -\frac{\pi}{2}]^T$ and track it using the previous controllers.

We opted for a half circle trajectory and we compared it with the posture regulator on the same goal.

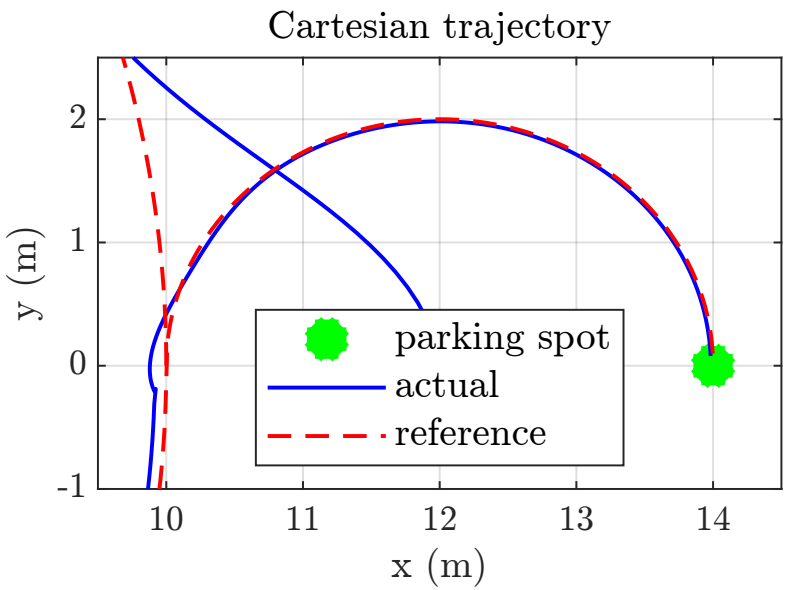

Figure: Full task using trajectory controller

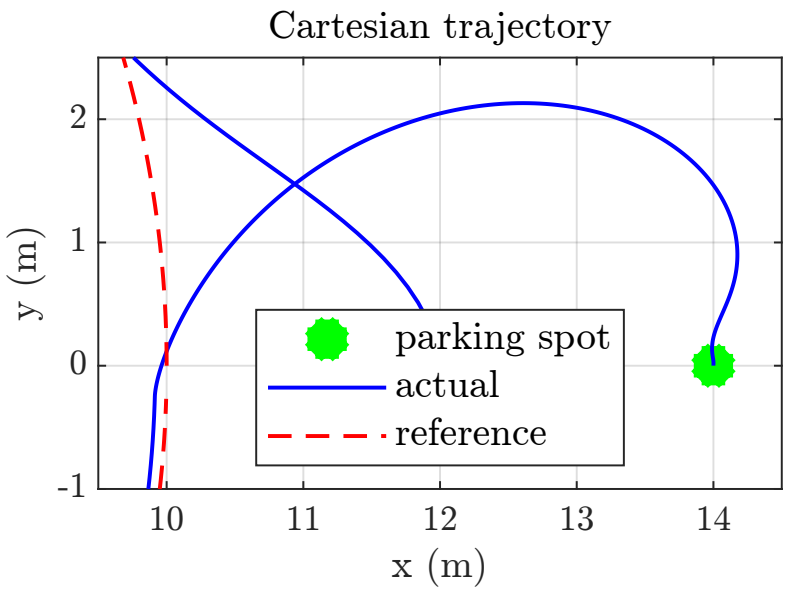

Figure: Full task using posture regulator

Main difficulty is correctly parametrizing the parking trajectory.

Parking maneuver via trajectory tracking - Comparison

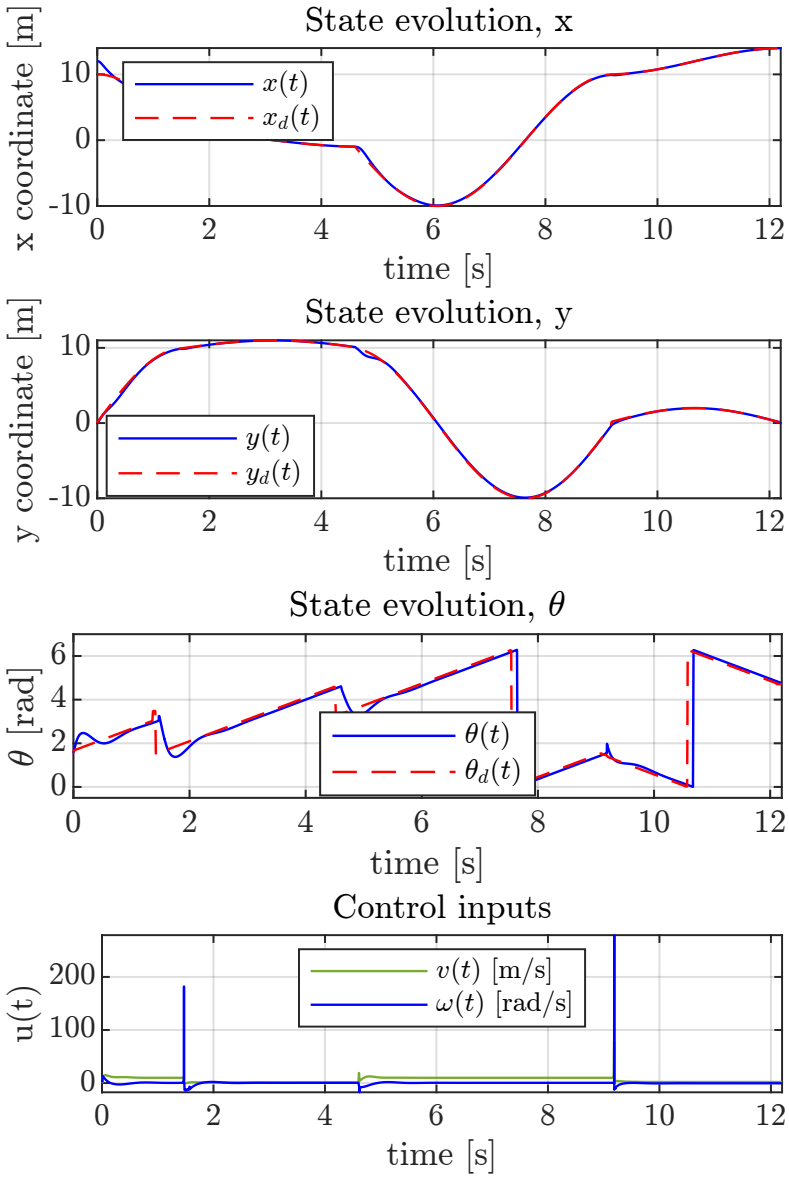

Figure: Trajectory controller

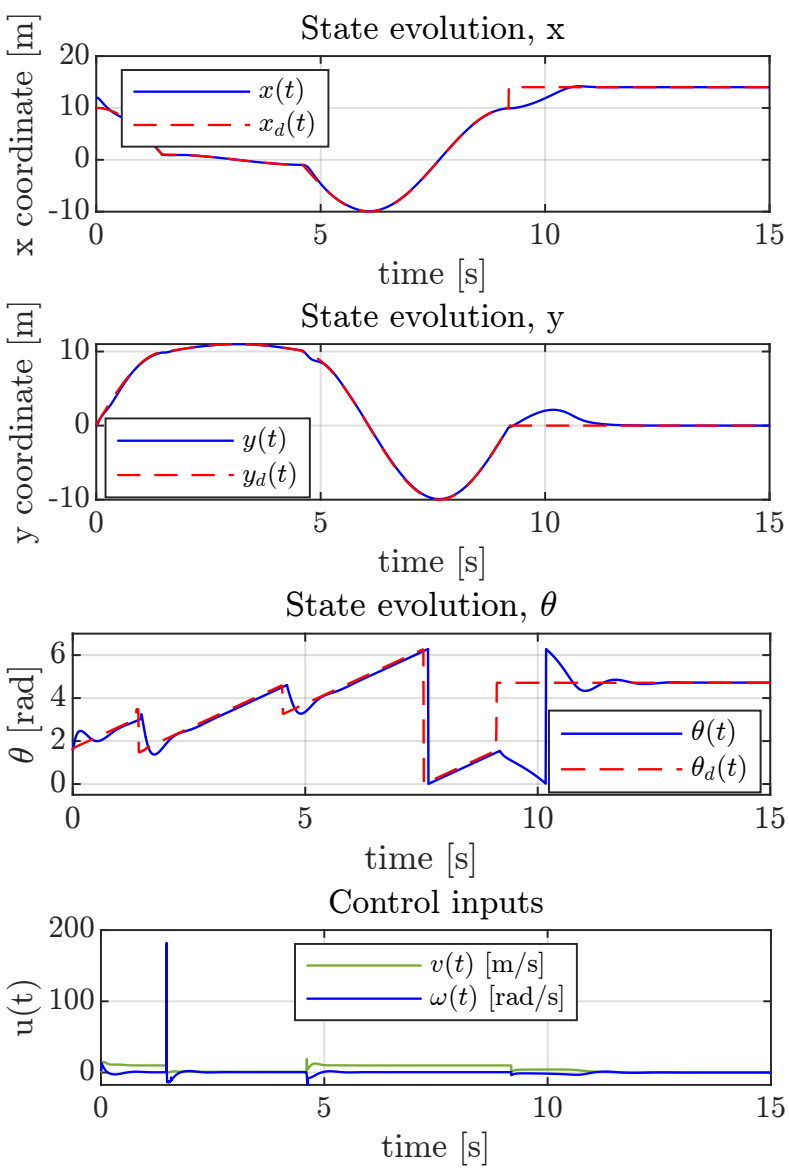

Figure: Posture regulator