THE CONSTANT SCALAR CURVATURE KÄHLER METRIC ON BOUNDED PSEUDOCONVEX DOMAINS

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Abstract. In this note, we investigate constant scalar curvature Kähler (cscK) metrics on bounded pseudoconvex domains equipped with complete Kähler–Einstein metrics of C^0-bounded geometry. Under a uniform upper bound for the trace of the cscK metric, we establish a quantitative bi-Lipschitz equivalence between the cscK metric and the Kähler-Einstein metric. Moreover, we show that the Kähler-Einstein metric cannot be perturbed to a cscK metric exhibiting similar asymptotic behavior. Building upon the work of Huang and Xiao, we further study an analogue of Cheng's conjecture in the setting of Bergman metrics with constant scalar curvature.

1. Introduction

A classical question in Kähler geometry is on finding an especially "nice" representative of a given Kähler class $[\omega_g]$ for a Kähler manifold (M, g). Generally, the Kähler-Einstein metric is the best candidate for such a representative. A Kähler metric g is called Kähler-Einstein if the Ricci curvature is proportional to the metric, that is

$$\mathrm{Ric}(g) = \lambda g,$$

for some $\lambda \in \mathbb{R}$. Through rescaling, we can assume that $\lambda = 1,\ 0$ or -1.

On compact Kähler manifolds, the existence of Kähler-Einstein metrics depends on the sign of the first Chern class $c_1(M)$:

(1) If $c_1(M) < 0$, M has an ample canonical bundle, a problem solved by Aubin [Aub76] and Yau [Yau78];

(2) If $c_1(M) = 0$, M is Calabi-Yau, as solved by Yau [Yau78];

(3) If $c_1(M) > 0$, M is Fano, there are some obstructions for the existence of Kähler-Einstein metric, introduced by Matsushima [Mat57], Futaki [Fut83] and Tian [Tia97]. It was proven by Chen-Donaldson-Sun [CDS15a,CDS15b,CDS15c] that the K-polystability of a Fano manifold is a sufficient condition for the existence of a Kähler-Einstein metric

For the non-compact case, significant results have been proven by Cheng-Mok-Yau [CY80, MY83] and Guedj-Kolev-Yeganefar [GKY13].

When the canonical bundle K_M is neither trivial, ample, nor anti-ample, the existence of a Kähler–Einstein metric is precluded, as the first Chern class cannot coincide with the Kähler class, making the necessary topological condition unattainable. The constant scalar curvature

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Kähler (often abbreviated as cscK) metric generalizes the concept of the Kähler–Einstein metric. And on compact Kähler manifolds, the average of the scalar curvature $\hat{R}$ is given by

$$\hat{R} = \frac{2n\pi c_1(M) \cup [\omega_g]^{n-1}}{[\omega_g]^n}$$

which is independent of the choice of g.

The Yau-Tian-Donaldson conjecture posits that the existence of a cscK metric in a given Kähler class is intimately related to the K-stability of (M, L) where L is an ample line bundle over M. In [Che18], Chen outlined a program for studying the existence problem for cscK metric: a new continuity path that links the cscK equation to certain second-order elliptic equation, apparently motivated by the classical continuity path for Kähler Einstein metrics and Donaldson's continuity path for conical Kähler Einstein metrics, and showed the openness. Further, Chen and Cheng [CC21b, CC21a] established a priori estimates and proved the closeness.

Despite the progress of the study for the cscK metric on compact Kähler manifold, the case for complete Kähler manifold is less understood. One of the main obstacles is that the complete case lacks a global $\partial\bar{\partial}$-lemma. Consequently, even when a Kähler class $[\omega_g]$ is prescribed, one cannot establish the one-to-one correspondence between Kähler metrics and their potential functions as in the compact case.

Inspired by the outstanding work of Cheng and Yau [CY80] on complete Kähler-Einstein metrics on strictly pseudoconvex domains, we would like to start in a similar setting. In these domains, the defining function naturally produces a complete Kähler metric, and any two such metrics differ by a $\sqrt{-1}\partial\bar{\partial}u$ term for some potential u, as a property inherent to the defining function. Moreover, in several complex variables, there is a growing trend to study invariant metrics on pseudoconvex domains, and since cscK metrics are invariant under biholomorphisms, it is natural to consider them in this context.

In this paper, we focus on bounded pseudoconvex domains $\Omega \subset \mathbb{C}^n$ endowed with a complete Kähler-Einstein metric g. We present the following quantitative bi-Lipschitz equivalence:

Theorem 1.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain equipped with a complete Kähler-Einstein metric g of C^0-bounded geometry. Suppose $\tilde{g}$ is another cscK metric of C^0-bounded geometry on Ω, and assume the ratio of volume forms satisfies $\det(\tilde{g})/\det(g) := e^F \in C^2(\Omega)$. If

$$\sup_{\Omega} \operatorname{tr}_g \tilde{g} < +\infty,$$

then there exist constants $C \geq C' > 0$, depending on the dimension n, the lower bound of the holomorphic bisectional curvature of g, and $\|F\|_{C^2(\Omega)}$, such that

$$C' g \leq \tilde{g} \leq C g \quad \text{in} \quad \Omega.$$

It is well-known that on compact Kähler manifolds, if a cscK metric is cohomologous to the Kähler-Einstein metric, then it is Kähler-Einstein. We expect a similar uniqueness property for the cscK metric which is perturbed from a Kähler-Einstein metric. On bounded strictly pseudoconvex domain, if the cscK metric $\tilde{g}$ is defined via a defining function (as discussed in [CY80]), we show that $\tilde{g}$ must coincide with the Kähler-Einstein metric g.

Proposition 1.2. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^k-boundary for $k \geq 8$. Suppose there exists a complete cscK metric $\tilde{g}$ on Ω of the form

$$\tilde{g}_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log(\rho),$$

where ρ is a defining function for $\Omega = \{\rho > 0\}$. Then $\tilde{g}$ is necessarily Kähler-Einstein.

Considering the Bergman metric on strictly pseudoconvex domains, we provide the following result, which offers a new perspective for extending [HX21, Theorem 1] in the context of the cscK metric.

Theorem 1.3. Let $\Omega \subset \mathbb{C}^n$ be a C^∞ bounded strictly pseudoconvex domain endowed with the Bergman metric g_B. If g_B has constant scalar curvature, then Ω is biholomorphic to the ball.

2. Preliminaries

2.1. **Notations.** Let (M^n, g) denote an n-dimensional complete Kähler manifold. In local coordinates $(z^1, \dots, z^n)$, the Kähler metric g can be expressed in the form

$$g = g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j,$$

where we adopt the Einstein summation convention. We use ∂_i and $\partial_{\bar{j}}$ as shorthand notations for $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \bar{z}^j}$, respectively, and denote by $\left(g^{i\bar{j}}\right)$ the inverse matrix of $\left(g_{i\bar{j}}\right)$.

Given two tensors A and B of type (m, n), their Hermitian inner product with respect to g is defined by

$$\langle A, B\rangle_g := g^{i_1\bar{j}_1} \cdots g^{i_n\bar{j}_n} g_{k_1\bar{l}_1} \cdots g_{k_m\bar{l}_m} A^{k_1\cdots k_m}_{i_1\cdots i_n} \overline{B^{l_1\cdots l_m}_{j_1\cdots j_n}},$$

and the corresponding norm of A is given by $|A|_g^2 := \langle A, A\rangle_g$.

Let ∇ denote the Levi-Civita connection associated with the Kähler metric g. The covariant derivatives are expressed locally as

$$\nabla_i := \nabla_{\partial_i}, \quad \nabla_{\bar{j}} := \nabla_{\partial_{\bar{j}}}.$$

For any vector fields X, Y, Z, W on M, the curvature tensor R is defined by

$$R(X, Y, Z, W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\ W\rangle.$$

In local coordinates, the components of the curvature tensor are calculated as

$$R_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{p\bar{q}} \left(\partial_k g_{i\bar{q}}\right)\left(\partial_{\bar{l}} g_{p\bar{j}}\right).$$

Consider a unitary frame $\{\epsilon_i, \overline{\epsilon_i}\}_{i=1}^n$ constructed by

$$\epsilon_i = \frac{1}{\sqrt{2}}\left(e_i - \sqrt{-1} J e_i\right), \quad \overline{\epsilon_i} = \frac{1}{\sqrt{2}}\left(e_i + \sqrt{-1} J e_i\right),$$

where $\{e_i, Je_i\}_{i=1}^n$ forms an orthonormal frame on TM. The holomorphic bisectional curvature of a complex plane Π spanned by ϵ_i and ϵ_j is defined as

$$\mathrm{HBC}_g(\Pi) = R\left(\epsilon_i, \overline{\epsilon_i}, \epsilon_j, \overline{\epsilon_j}\right).$$

The Ricci curvature tensor is defined as the trace of the curvature tensor:

$$\mathrm{Ric}(g) = R_{i\bar{j}}\, dz^i \otimes d\bar{z}^j, \quad \text{where} \quad R_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log\left(\det(g_{k\bar{l}})\right).$$

The scalar curvature is then obtained by tracing the Ricci curvature with respect to g:

$$R(g) = \mathrm{tr}_g \,\mathrm{Ric} = g^{i\bar{j}} R_{i\bar{j}}.$$

Note that for C^2-functions, the covariant derivative $\nabla_i\nabla_{\bar{j}}$ coincides with the partial derivative $\partial_i\partial_{\bar{j}}$, since the mixed Christoffel symbols $\Gamma_{i\bar{j}}$ vanish identically on Kähler manifolds.

For any C^∞-function $f : M \to \mathbb{R}$, the gradient vector field is defined by

$$\nabla f = g^{i\bar{j}}\left(\partial_i f \cdot \partial_{\bar{j}} + \partial_{\bar{j}} f \cdot \partial_i\right).$$

The complex Hessian of f is given by $\nabla^2 f = \sqrt{-1}\partial\bar{\partial} f$, and the Laplacian operator is defined as the trace of the Hessian:

$$\Delta_g f := \mathrm{tr}_g\left(\nabla^2 f\right) = g^{i\bar{j}}\partial_i\partial_{\bar{j}} f.$$

2.2. **Omori-Yau's maximum principle.** We now introduce Omori-Yau's generalized maximum principle on non-compact manifolds.

Proposition 2.1 (Omori, [Omo67],Yau [Yau75])**.** Let (M, g) be a complete Kähler manifold. If g has bounded sectional curvature, then for any function $u \in C^2(M)$ with $\sup_M u < +\infty$, there exists a sequence of points $\{z_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$(1) \lim_{k\to\infty} u(z_k) = \sup_M u, \;\; (2) \lim_{k\to\infty} |\nabla u(z_k)|_g = 0, \;\; (3) \limsup_{k\to\infty} \nabla^2 u(z_k) \le 0,$$

where the third inequality holds in the sense of matrices.

Remark 2.2. If the assumption of bounded sectional curvature is replaced by bounded Ricci curvature, condition (3) can be modified to

$$(3') \limsup_{k\to\infty} \Delta_g u(z_k) \le 0.$$

2.3. **Bounded geometry.** Recall that the injectivity radius at a point $x \in M$ is the maximum radius r of the ball B_r in the tangent space T_xM for which the exponential map $\exp_x : B_r \to \exp_x(B_r) \subset M$ is a diffeomorphism. The injectivity radius of M is the infimum of the injectivity radius at all points in M.

Definition 2.3 (Wu-Yau, [WY20]). Let (M, g) be a complete Kähler manifold and let $k \geq 0$ be an integer. We say (M, g) has C^k-quasi-bounded geometry if for each non-negative integer $l \leq k$, there exists a constant $C_l > 0$ such that

$$\sup_M |\nabla^l \operatorname{Rm}|_g \leq C_l, \tag{2.1}$$

where $\operatorname{Rm} = \{R_{i\bar{j}k\bar{l}}\}$ is the Riemann curvature tensor of g and ∇^l is the convariant derivative of order l. Moreover, if (M, g) has a positive injectivity radius, then we say (M, g) has C^k-bounded geometry.

3. CscK metrics on bounded pseudoconvex domains

Let Ω be a bounded pseudoconvex domain, and let g be the Kähler-Einstein metric on Ω of C^0-bounded geometry. The existence of the unique Kähler-Einstein metric g is guaranteed by the result of Mok and Yau [MY83]. Suppose $\tilde{g}$ is a complete cscK metric of C^0-bounded geometry on Ω with scalar curvature $R(\tilde{g}) = -n(n+1)$. Then there exists some F such that

$$\det\left(\tilde{g}_{i\bar{j}}\right) = e^F \det\left(g_{i\bar{j}}\right). \tag{3.1}$$

We first see the following inequality originally from Yau's famous C^2-estimate [Yau78], which holds for any Kähler metrics g and $\tilde{g}$ if the holomorphic bisectional curvature of g is bounded from below.

Lemma 3.1. There exists a constant B depending on the holomorphic bisectional curvature of g such that

$$\Delta_{\tilde{g}} \log\left(\operatorname{tr}_g \tilde{g}\right) \geq B \operatorname{tr}_{\tilde{g}} g - \frac{\operatorname{tr}_g \operatorname{Ric}(\tilde{g})}{\operatorname{tr}_g \tilde{g}}. \tag{3.2}$$

Proof. Let $\tilde{R}_{i\bar{j}k\bar{l}}$ and $\tilde{R}_{i\bar{j}}$ denote the local components of Riemann and Ricci curvature tensor of $\tilde{g}$. Taking any $p \in \Omega$, we compute in normal coordinates centered at p where g is identity and $\tilde{g}$ is diagonal. We have

$$\begin{aligned}
\Delta_{\tilde{g}} \operatorname{tr}_g \tilde{g} &= \tilde{g}^{k\bar{l}} \partial_k \partial_{\bar{l}} \left(g^{i\bar{j}} \tilde{g}_{i\bar{j}}\right) \\
&= \tilde{g}^{k\bar{l}} \tilde{g}_{i\bar{j}} \partial_k \partial_{\bar{l}} g^{i\bar{j}} + \tilde{g}^{k\bar{l}} g^{i\bar{j}} \partial_k \partial_{\bar{l}} \tilde{g}_{i\bar{j}} \\
&= \tilde{g}^{k\bar{l}} \tilde{g}_{i\bar{j}} \partial_k \partial_{\bar{l}} g^{i\bar{j}} + \tilde{g}^{k\bar{l}} g^{i\bar{j}} \left(-\tilde{R}_{i\bar{j}k\bar{l}} + \tilde{g}^{p\bar{q}} \partial_k \tilde{g}_{i\bar{q}} \partial_{\bar{l}} \tilde{g}_{p\bar{j}}\right) \\
&= \tilde{g}^{k\bar{l}} \tilde{g}_{i\bar{j}} \partial_k \partial_{\bar{l}} g^{i\bar{j}} - g^{i\bar{j}} \tilde{R}_{i\bar{j}} + \tilde{g}^{k\bar{l}} g^{i\bar{j}} \tilde{g}^{p\bar{q}} \partial_k \tilde{g}_{i\bar{q}} \partial_{\bar{l}} \tilde{g}_{p\bar{j}},
\end{aligned} \tag{3.3}$$

where the third equality follows from the local expression of Riemann curvature tensor in Kähler case. In fact, at p, we have

$$\operatorname{tr}_g \tilde{g} = \sum_i \tilde{g}_{i\bar{i}}, \qquad \operatorname{tr}_{\tilde{g}} g = \sum_i \tilde{g}^{i\bar{i}}.$$

Then for the first term of (3.3), we have

$$\begin{aligned}\tilde{g}^{k\bar{l}}\tilde{g}_{i\bar{j}}\partial_k\partial_{\bar{l}}g^{i\bar{j}} &= \sum_{k,i}\tilde{g}^{k\bar{k}}\tilde{g}_{i\bar{i}}\partial_k\partial_{\bar{k}}g^{i\bar{i}} \\ &\geq \inf_\Omega \mathrm{HBC}_g \sum_{k,i}\tilde{g}^{k\bar{k}}\tilde{g}_{i\bar{i}} \\ &= \inf_\Omega \mathrm{HBC}_g \left(\operatorname{tr}_g \tilde{g}\right)\left(\operatorname{tr}_{\tilde{g}} g\right),\end{aligned}$$

where HBC_g is the holomorphic bisectional curvature of g and we will denote $\inf_\Omega \mathrm{HBC}_g$ by B in the remaining part. The second term of (3.3) is just $\operatorname{tr}_g \mathrm{Ric}(\tilde{g})$, and the third term of (3.3) is

$$\tilde{g}^{k\bar{l}}g^{i\bar{j}}\tilde{g}^{p\bar{q}}\partial_k\tilde{g}_{i\bar{q}}\partial_{\bar{l}}\tilde{g}_{p\bar{j}} = \sum_{i,k,p}\tilde{g}^{k\bar{k}}\tilde{g}^{p\bar{p}}|\partial_k\tilde{g}_{i\bar{p}}|^2$$

Thus,

$$\Delta_{\tilde{g}}\operatorname{tr}_g\tilde{g} \geq B\left(\operatorname{tr}_g\tilde{g}\right)\left(\operatorname{tr}_{\tilde{g}} g\right) - \operatorname{tr}_g \mathrm{Ric}(\tilde{g}) + \sum_{i,k,p}\tilde{g}^{k\bar{k}}\tilde{g}^{p\bar{p}}|\partial_k\tilde{g}_{i\bar{p}}|^2.$$

Taking logarithm into account, we have

$$\begin{aligned}\Delta_{\tilde{g}}\log\left(\operatorname{tr}_g\tilde{g}\right) &= \frac{\Delta_{\tilde{g}}\operatorname{tr}_g\tilde{g}}{\operatorname{tr}_g\tilde{g}} - \frac{\tilde{g}^{i\bar{j}}\left(\partial_i\operatorname{tr}_g\tilde{g}\right)\left(\partial_{\bar{j}}\operatorname{tr}_g\tilde{g}\right)}{\left(\operatorname{tr}_g\tilde{g}\right)^2} \\ &\geq B\operatorname{tr}_{\tilde{g}} g - \frac{\operatorname{tr}_g\mathrm{Ric}(\tilde{g})}{\operatorname{tr}_g\tilde{g}} + \frac{\sum_{i,k,p}\tilde{g}^{k\bar{k}}\tilde{g}^{p\bar{p}}|\partial_k\tilde{g}_{i\bar{p}}|^2}{\operatorname{tr}_g\tilde{g}} - \frac{\sum_{i,k,p}\tilde{g}^{i\bar{i}}\left(\partial_i\tilde{g}_{k\bar{k}}\right)\left(\partial_{\bar{i}}\tilde{g}_{p\bar{p}}\right)}{\left(\operatorname{tr}_g\tilde{g}\right)^2}.\end{aligned} \tag{3.4}$$

By applying Cauchy–Schwarz to the numerator of the fourth term of (3.4), we have

$$\begin{aligned}\sum_{i,k,p}\tilde{g}^{i\bar{i}}\left(\partial_i\tilde{g}_{k\bar{k}}\right)\left(\partial_{\bar{i}}\tilde{g}_{p\bar{p}}\right) &= \sum_{i,k,p}\left(\sqrt{\tilde{g}^{i\bar{i}}}\partial_i\tilde{g}_{k\bar{k}}\cdot\sqrt{\tilde{g}^{i\bar{i}}}\partial_{\bar{i}}\tilde{g}_{p\bar{p}}\right) \\ &\leq \sum_{k,p}\left(\sum_i\tilde{g}^{i\bar{i}}|\partial_i\tilde{g}_{k\bar{k}}|^2\right)^{1/2}\left(\sum_i\tilde{g}^{i\bar{i}}|\partial_{\bar{i}}\tilde{g}_{p\bar{p}}|^2\right)^{1/2} \\ &= \left(\sum_k\left(\sum_i\tilde{g}^{i\bar{i}}|\partial_i\tilde{g}_{k\bar{k}}|^2\right)^{1/2}\right)^2 \\ &= \left(\sum_k\left(\tilde{g}_{k\bar{k}}\right)^{1/2}\left(\sum_i\tilde{g}^{i\bar{i}}\tilde{g}^{k\bar{k}}|\partial_i\tilde{g}_{k\bar{k}}|^2\right)^{1/2}\right)^2 \\ &\leq \left(\sum_k\tilde{g}_{k\bar{k}}\right)\left(\sum_{i,k}\tilde{g}^{i\bar{i}}\tilde{g}^{k\bar{k}}|\partial_i\tilde{g}_{k\bar{k}}|^2\right) \\ &\leq \operatorname{tr}_g\tilde{g}\cdot\sum_{i,k,p}\tilde{g}^{i\bar{i}}\tilde{g}^{k\bar{k}}|\partial_i\tilde{g}_{p\bar{k}}|^2,\end{aligned}$$

where we add some positive terms in the last inequality. This implies the desired inequality. □

Lemma 3.2. Let g and $\tilde{g}$ be the Kähler-Einstein metric and cscK metric, respectively. Then the following inequality holds for the volume forms:

$$\det(\tilde{g}_{i\bar{j}}) \geq \det(g_{i\bar{j}}). \tag{3.5}$$

Moreover, if the equality holds in Ω, then $\tilde{g} = g$.

Proof. Taking $u = \det(g_{i\bar{j}})/\det(\tilde{g}_{i\bar{j}}) > 0$, we compute

$$\begin{aligned}\Delta_{\tilde{g}} \log u &= \Delta_{\tilde{g}} \log \left(\det(g_{i\bar{j}}) - \det(\tilde{g}_{i\bar{j}})\right) \\ &= (n+1)\operatorname{tr}_{\tilde{g}} g - (n+1)n \\ &\geq (n+1)nu^{1/n} - (n+1)n,\end{aligned}$$

where the last inequality follows from the arithmetic-geometric inequality.

Then we obtain

$$\begin{aligned}\Delta_{\tilde{g}} \log u &= \frac{\Delta_{\tilde{g}} u}{u} - \frac{|\nabla u|^2_{\tilde{g}}}{u^2} \\ &\geq (n+1)nu^{1/n} - (n+1)n,\end{aligned}$$

which leads to

$$\Delta_{\tilde{g}} u - \frac{|\nabla u|^2_{\tilde{g}}}{u} \geq (n+1)nu^{1/n+1} - (n+1)nu.$$

Since u is bounded on $\overline{\Omega}$, by the generalized maximum principle, there exists a sequence $\{z_\alpha\} \subset \Omega$, such that

$$0 \geq \limsup_{\alpha\to\infty} \Delta_{\tilde{g}} u(z_\alpha) \geq (n+1)n \lim_{\alpha\to\infty} u^{\frac{1}{n}+1}(z_\alpha) - (n+1)n \lim_{\alpha\to\infty} u(z_\alpha).$$

This implies $\sup_\Omega u \leq 1$, yielding the desired inequality.

Now, suppose that $\det(\tilde{g}_{i\bar{j}}) = \det(g_{i\bar{j}})$. Then we have $\operatorname{tr}_{\tilde{g}} g = n$. The desired result follows from the equality case of the arithmetic-geometric inequality. □

Remark 3.3. Note that Lemma 3.2 remains valid provided that the ratio of the volume forms of the Kähler-Einstein metric and the cscK metric is bounded. Furthermore, under this condition, inequality (3.5) implies that the volume form of the Kähler-Einstein metric is the smallest among all cscK metrics.

The following result shows that the Kähler-Einstein metric and the cscK metric are bi-Lipschitz to each other on Ω, underlining specifically what quantities the Lipschitz constant depends on.

Proposition 3.4. Assuming the hypotheses of Theorem 1.1 hold, then there exist constant $C \geq C' > 0$ depending on the dimension n, the lower bound of holomorphic bisectional curvature of g, and $\|F\|_{C^2(\bar{\Omega})}$, such that

$$C'g \leq \tilde{g} \leq Cg \quad \text{on} \quad \Omega. \tag{3.6}$$

Proof. It follows from (3.1),

$$\tilde{R}_{i\bar{j}} = -(n+1)g_{i\bar{j}} - F_{i\bar{j}}, \tag{3.7}$$

where $F_{i\bar{j}} := \partial_i \partial_{\bar{j}} F$. Then, (3.2) becomes

$$\begin{aligned} \Delta_{\tilde{g}} \log\left(\operatorname{tr}_g \tilde{g}\right) &\geq B \operatorname{tr}_{\tilde{g}} g - \frac{\operatorname{tr}_g \operatorname{Ric}(\tilde{g})}{\operatorname{tr}_g \tilde{g}} \\ &= B \operatorname{tr}_{\tilde{g}} g + \frac{n(n+1) + \Delta_g F}{\operatorname{tr}_g \tilde{g}}. \end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, we have

$$(\operatorname{tr}_g \tilde{g})(\operatorname{tr}_{\tilde{g}} g) = \left(g^{i\bar{j}} \tilde{g}_{i\bar{j}}\right)\left(\tilde{g}^{k\bar{l}} g_{k\bar{l}}\right) \geq n^2.$$

So there exists a constant $C_1 > 0$ depending on n and $\|\Delta_g F\|_{L^\infty(\Omega)}$, such that

$$\Delta_{\tilde{g}} \log\left(\operatorname{tr}_g \tilde{g}\right) \geq B \operatorname{tr}_{\tilde{g}} g - C_1 \operatorname{tr}_{\tilde{g}} g. \tag{3.8}$$

From (3.7),

$$\Delta_{\tilde{g}} F = (n+1)\left(n - \operatorname{tr}_{\tilde{g}} g\right).$$

Taking $C_2 = (-B + C_1 + 1)/(n+1)$, then it follows from (3.8),

$$\Delta_{\tilde{g}}\left(\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2 F\right) \geq \operatorname{tr}_{\tilde{g}} g - C_2(n+1)n.$$

Since $\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2 F$ is bounded from above on $\overline{\Omega}$. Then by the generalized maximum principle, there exists a sequence $\{z_\alpha\}$, such that

$$0 \geq \lim_{\alpha\to\infty} \Delta_{\tilde{g}}\left(\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2 F\right)(z_\alpha) \geq \lim_{\alpha\to\infty} \operatorname{tr}_{\tilde{g}} g(z_\alpha) - C_2(n+1)n,$$

which implies

$$\lim_{\alpha\to\infty} \operatorname{tr}_{\tilde{g}} g(z_\alpha) \leq C_2(n+1)n.$$

If $z_\alpha \to p \in \Omega$, then taking a normal coordinate at p such that g is identity and $\tilde{g}$ is diagonal, we have

$$\operatorname{tr}_{\tilde{g}} g(p) = \sum_i \tilde{g}^{i\bar{i}}(p) \leq C_2(n+1)n.$$

This yields at point p, for any k,

$$\frac{1}{\tilde{g}_{k\bar{k}}(p)} = \tilde{g}^{k\bar{k}}(p) \leq \sum_i \tilde{g}^{i\bar{i}}(p) \leq C_2(n+1)n. \tag{3.9}$$

On the other hand side, from (3.1), we have

$$\prod_i \tilde{g}_{i\bar{i}}(p) = e^{F(p)} \leq e^{\sup_\Omega F}. \tag{3.10}$$

Combining (3.9) and (3.10), for any k, we obtain

$$\tilde{g}_{k\bar{k}}(p) = \frac{\prod_i \tilde{g}_{i\bar{i}}(p)}{\prod_{i\neq k} \tilde{g}_{i\bar{i}}(p)} \leq (C_2(n+1)n)^{n-1} e^{\sup_\Omega F} := \frac{C_3}{n} \tag{3.11}$$

where we recall that C_3 depends on n, the lower bound of holomorphic bisectional curvature of g, $\|F\|_{L^\infty}$ and $\|\Delta F\|_{L^\infty}$. In particular, from (3.11), we have

$$\operatorname{tr}_g \tilde{g}(p) = \sum_k \tilde{g}_{k\bar{k}}(p) \leq C_3.$$

If $z_\alpha \to p \in \overline{\Omega}$, then by the generalized maximum principle, we have

$$\operatorname{tr}_{\tilde{g}} g(z_\alpha) \leq C_2(n+1)n + \frac{1}{\alpha}.$$

Taking a normal coordinate at z_α and following the same approach as above, we obtain

$$\operatorname{tr}_g \tilde{g}(z_\alpha) \leq C_3 + o\left(\frac{1}{\alpha}\right).$$

Let $\alpha \to \infty$, we have

$$\lim_{\alpha\to\infty} \operatorname{tr}_g \tilde{g}(z_\alpha) \leq C_3.$$

Hence for any $x \in \Omega$, we obtain

$$\begin{aligned}\log\left(\operatorname{tr}_g \tilde{g}\right)(x) - C_2F(x) &\leq \sup_\Omega \left(\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2F\right)\\ &\leq \lim_{\alpha\to\infty} \log \operatorname{tr}_g \tilde{g}(z_\alpha) - C_2 \lim_{\alpha\to\infty} F(z_\alpha)\\ &\leq \log(C_3) - C_2F(p),\end{aligned}$$

which implies

$$\sup_\Omega \operatorname{tr}_g \tilde{g} \leq C_4, \tag{3.12}$$

where C_4 depends on the same factors as C_3. Now, if we choose a normal coordinate at any point $x \in \Omega$ such that g is identity and $\tilde{g}$ is diagonal, it follows from (3.5) and (3.12),

$$\tilde{g}_{k\bar{k}} = \frac{\prod_i \tilde{g}_{i\bar{i}}}{\prod_{i\neq k} \tilde{g}_{i\bar{i}}} \geq C_5,$$

which implies

$$\tilde{g} \geq C_5 g. \tag{3.13}$$

This gives the desired result. □

A key observation is that the estimates derived in this section for cscK metrics on bounded pseudoconvex domains extend naturally to complete Kähler manifolds endowed with Kähler-Einstein metrics. This generalization leads to the following relationship, which holds particular significance:

Proposition 3.5. Let (M^n, g) be a complete Kähler manifold equipped with a Kähler-Einstein metric of C^0-bounded geometry. Suppose there exists a cscK metric $\tilde{g}$ of C^0-bounded geometry satisfying $\det(\tilde{g}) = e^F \det(g)$ for some $F \in C^2(M)$ with $\|F\|_{C^2(M)} \leq K$, where $K > 0$. If

$$\sup_\Omega \operatorname{tr}_g \tilde{g} < +\infty,$$

then there exist constants $C, C' > 0$, depending only on n, the holomorphic bisectional curvature of g, and $\|F\|_{C^2(M)}$, such that

$$Cg \leq \tilde{g} \leq C'g.$$

In the last part of this section, we study the case when the cscK metric $\tilde{g}$ is asymptotically Kähler-Einstein in the sense that its volume form equals the volume of the Kähler-Einstein metric at infinity and the eigenvalues of the cscK metric are controlled.

Proposition 3.6. Assuming the hypotheses of Theorem 1.1 hold, if $F = 0$ on $\partial\Omega$ and $\operatorname{tr}_{\tilde{g}} g \leq n$ in Ω, then $\tilde{g}$ is Kähler-Einstein.

Proof. It follows from (3.7) that

$$\Delta_{\tilde{g}} F = (n+1)\left(n - \operatorname{tr}_{\tilde{g}} g\right) \geq 0.$$

Since $F = 0$ on $\partial\Omega$, the function F is subharmonic with respect to $\tilde{g}$ and achieves its maximum value of 0 at infinity. By the maximum principle, if $F \not\equiv 0$, then $F < 0$ in Ω. This would imply

$$\det(\tilde{g}_{i\bar{j}}) = e^F \det(g_{i\bar{j}}) < \det(g_{i\bar{j}}),$$

which contradicts (3.5). Therefore, we conclude $F \equiv 0$ and $\det(\tilde{g}_{i\bar{j}}) = \det(g_{i\bar{j}})$ in Ω. The desired result then follows as a consequence of Lemma 3.2. □

In [CY80], Cheng and Yau investigated complete Kähler metrics of the form

$$g_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log \rho,$$

where ρ is a strictly plurisubharmonic defining function for the domain $\Omega = \{\rho > 0\}$. They derived the local expression for the curvature tensor:

$$R_{i\bar{j}k\bar{l}} = -\left(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}\right) + O\left(\frac{1}{|\rho|}\right).$$

Notably, the metric g behaves asymptotically like a Kähler-Einstein metric with Einstein constant $-(n+1)$ near the boundary $\partial\Omega$.

The Kähler-Einstein metric on a strictly pseudoconvex domain is constructed using the Fefferman defining function ρ of class C^k for $k \geq 8$. Let g denote this metric, defined by

$$g = -\sqrt{-1}\,\partial\bar{\partial} \log \rho.$$

If the cscK metric is also defined by a global strictly plurisubharmonic defining function $\tilde{\rho}$, then there exists a positive function $u \in C^{k-1}(\bar{\Omega})$ satisfying $u = 1 + o(1)$ near $\partial\Omega$ such that $\tilde{\rho} = u\rho$. Consequently, the perturbed metric satisfies

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi,$$

where $\varphi = -\log u$. For a detailed discussion of the regularity of u, see [Kra01, Chapter 3.1].

Proposition 3.7. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^k-boundary, for $k \geq 8$. Then there is no complete cscK metric $\tilde{g}$ given by a C^k-defining function $\tilde{\rho}$ unless $\tilde{g}$ is Kähler-Einstein.

Proof. Assume there exists a complete cscK metric defined by a defining function $\tilde{\rho}$. As discussed previously, this implies the metric perturbation

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi,$$

Taking the trace with respect to $\tilde{g}$, we obtain

$$\Delta_{\tilde{g}} \varphi = n - \mathrm{tr}_{\tilde{g}}\, g.$$

Furthermore, the determinant relationship

$$\det\left(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi\right) = \det\left(g_{i\bar{j}}\right) \cdot \det\left(\delta_{i\bar{j}} + g^{i\bar{l}} \partial_l \partial_{\bar{j}} \varphi\right) := e^F \det\left(g_{i\bar{j}}\right)$$

holds. Since g is complete and $\partial_l \partial_{\bar{j}} \varphi \in C^{k-3}(\bar{\Omega})$, the term $g^{i\bar{l}} \partial_l \partial_{\bar{j}} \varphi$ vanishes asymptotically near $\partial\Omega$. Thus,

$$e^F = \det\left(\delta_{i\bar{j}} + g^{i\bar{l}} \partial_l \partial_{\bar{j}} \varphi\right) \to \det\left(\delta_{i\bar{j}}\right) = 1 \quad \text{as} \quad z \to \partial\Omega.$$

This gives:

$$\begin{cases} \Delta_{\tilde{g}} F = (n+1) \Delta_{\tilde{g}} \varphi & \text{in} \quad \Omega, \\ F = \varphi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By the comparison property for the elliptic operator, we have $F = (n+1)\varphi$ in Ω. Substituting this into the determinant equation yields

$$\det\left(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi\right) = e^{(n+1)\varphi} \det\left(g_{i\bar{j}}\right). \tag{3.14}$$

Applying the generalized maximum principle (valid due to the completeness of g and $\sup_\Omega |\varphi| < \infty$), there exists a sequence $\{z_k\} \subset \Omega$ such that

$$1 = \lim_{k\to\infty} \frac{\det(g_{i\bar{j}})}{\det(g_{i\bar{j}})}(z_k) \geq \lim_{k\to\infty} \frac{\det\left(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi\right)}{\det(g_{i\bar{j}})}(z_k) = e^{(n+1)\sup_\Omega \varphi}.$$

This implies $\sup_\Omega \varphi \leq 0$. A symmetric argument gives $\inf_\Omega \varphi \geq 0$, forcing $\varphi \equiv 0$ in Ω. Consequently, $\tilde{g} = g$ in Ω, which complete the proof. □

By a similar argument, we establish the following result on a general complete Kähler–Einstein manifold: if a cscK metric is obtained as a perturbation of a Kähler–Einstein metric with both metrics asymptotically equivalent, then the cscK metric coincides with the Kähler–Einstein metric.

Proposition 3.8. Let (M, g_{KE}) be a complete Kähler manifold endowed with a Kähler-Einstein metric g_{KE} of C^0-bounded geometry. Suppose there is a $\varphi \in C^2(M)$ such that $g_{\mathrm{cscK}} = g_{KE} + \partial\bar{\partial}\varphi$ defines a complete cscK metric. If for any fix point $p \in M$, we have

$$\lim_{r\to\infty} \|\varphi\|_{C^2(M \setminus B(p,r))} = 0,$$

then $g_{\mathrm{cscK}} = g_{KE}$.

Following the above proposition, it can be shown that if the two metrics are asymptotically equivalent, then there are no non-trivial cscK metrics obtained as perturbations of the Kähler–Einstein metric. However, we conjecture that the asymptotic condition can be weakened: it suffices to require only the asymptotic equivalence of the volume forms, rather than the metrics themselves. Specifically, suppose the volume form of the cscK metric is given by e^F times the volume form of the Kähler–Einstein metric, with F vanishing at infinity. We then propose the following ambitious conjecture, which may be more attractive:

Question 3.9. Let (M, g_{KE}) be a complete Kähler manifold endowed with a Kähler–Einstein metric g_{KE} of C^0-bounded geometry. Suppose there is a $\varphi \in C^2(M)$ such that $g_{\mathrm{cscK}} = g_{KE} + \partial\bar{\partial}\varphi$ defines a complete cscK metric. Assume that the volume forms satisfy $\det(g_{\mathrm{cscK}}) = e^F \det(g_{KE})$ for some function $F \in C^0(M)$, and for any fix point $p \in M$, we have

$$\lim_{r\to\infty} \|F\|_{C^0(M\setminus B(p,r))} = 0.$$

Then, $g_{\mathrm{cscK}} = g_{KE}$.

A standard method to construct a cscK metric is to consider the product of two Kähler-Einstein manifolds with different Einstein constants. Let (Ω_1, g_1) and (Ω_2, g_2) be two C^∞ bounded strictly pseudoconvex domains, each endowed with its unique Kähler–Einstein metric with different Einstein constants. So the product domain $\Omega = \Omega_1 \times \Omega_2$ is only pseudoconvex, and the product metric $g = g_1 \oplus g_2$ is a cscK metric but not Kähler-Einstein. Moreover, it follows from [MY83], there exists a unique complete Kähler–Einstein metric g_0 on Ω, and the volume forms of g_0 and g exhibit different asymptotic behaviors near the boundary.

4. Bergman metric on strictly pseudoconvex domains

In this section, we discuss some facts on the Bergman metric on strictly pseudoconvex domains. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ be the space of holomorphic functions in $L^2(\Omega)$. It is clear that $A^2(\Omega)$ is a Hilbert space. The Bergman kernel $K(z)$ on Ω is a real analytic function defined as

$$K(z) = \sum_{j=1}^{\infty} |\varphi_j(z)|^2, \qquad \forall z \in \Omega,$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $A^2(\Omega)$ with respect to the L^2 inner product. Since the Bergman kernel is positive and independent of the choice of any orthonormal basis [Kra01] on bounded domains, we then can define the Bergman metric

$$g_B := g_{i\bar{j}} dz^i \otimes d\bar{z}^j \qquad \text{where} \quad g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log K.$$

The Bergman metric is a complete real analytic Kähler metric, with the real analytic property inherited from the Bergman kernel. Let $G := \det(g_B)$ denote the determinant of the Bergman metric. The Bergman invariant function $B(z) := G(z)/K(z)$, introduced by Bergman in [BS51], is invariant under biholomorphic maps.

A significant result regarding the Bergman invariant function, established by Diederich in [Die70], states that as one approaches the boundary of a strictly pseudoconvex domain with C^∞-boundary, B tends to constant when we approach the boundary.

Proposition 4.1 (Diederich, [Die70])**.** Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $p \in \partial\Omega$ be a C^2 strictly pseudoconvex point. Then

$$\lim_{z \to p} B(z) = \frac{(n+1)^n \pi^n}{n!}.$$

Then we have the following result, as an extending for Huang and Xiao's resolution of Cheng's conjecture.

Theorem 4.2. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary and let g_B be the Bergman metric. If g_B has constant scalar curvature, then Ω is biholomorphic to the ball.

Proof. Expressed in local coordinates, g_B satisfies the equation

$$R_{i\bar{j}} + \partial_i \partial_{\bar{j}} \log B = -g_{i\bar{j}}, \tag{4.1}$$

where $R_{i\bar{j}}$ and $g_{i\bar{j}}$ are the local components of the Ricci curvature $\mathrm{Ric}(g_B)$ and the metric g_B, respectively, and B represents the Bergman invariant function. Taking the trace of equation (4.1) with respect to g_B, we derive

$$\Delta_{g_B} \log B = 0 \quad \text{in} \quad \Omega.$$

This implies that $\log B$ is a harmonic function with constant "boundary value" (Proposition 4.1). By the maximum principle, it follows that $\log B$ must be constant throughout Ω, specifically

$$\log B = \log\left(\frac{(n+1)^n \pi^n}{n!}\right).$$

Combining this result with the characterization in [FW97, Proposition 1.1], we conclude that g_B is Kähler-Einstein. The final statement then follows immediately from [HX21, Theorem 1]. $\square$

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