RIGIDITY OF COMPLETE KÄHLER–EINSTEIN METRICS UNDER CSCK PERTURBATIONS

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Abstract. We study constant scalar curvature Kähler (cscK) metrics on a complete Kähler–Einstein manifold. We give a sufficient condition under which any cscK perturbation of the Kähler-Einstein metric remains Kähler–Einstein. As a model case, we extend Huang-Xiao's resolution of Cheng's conjecture if the Bergman metric has constant scalar curvature on bounded strictly pseudoconvex domains with smooth boundary.

1. Introduction

1.1. **Canonical metrics.** A classical question in Kähler geometry is on finding an especially "nice" representative of a given Kähler class $[\omega]$ for a Kähler manifold (M, ω). Generally, the Kähler-Einstein metric is the best candidate for such a representative. A Kähler metric ω is called Kähler-Einstein if the Ricci curvature is proportional to the metric, that is

$$\mathrm{Ric}(\omega) = \lambda\omega,$$

for some $\lambda \in \mathbb{R}$. Through rescaling, we can assume that $\lambda = 1,\ 0$ or -1.

On compact Kähler manifolds, the existence of Kähler-Einstein metrics depends on the sign of the first Chern class $c_1(M)$:

(1) If $c_1(M) < 0$, M has an ample canonical bundle, a problem solved by Aubin [Aub76] and Yau [Yau78];

(2) If $c_1(M) = 0$, M is Calabi-Yau, as solved by Yau [Yau78];

(3) If $c_1(M) > 0$, M is Fano, there are some obstructions for the existence of Kähler-Einstein metric, introduced by Matsushima [Mat57], Futaki [Fut83] and Tian [Tia97]. It was proven by Chen-Donaldson-Sun [CDS15a,CDS15b,CDS15c] that the K-polystability of a Fano manifold is a sufficient condition for the existence of a Kähler-Einstein metric

For the non-compact case, some significant progress for the existence of the Kähler-Einstein metric has been made by Cheng-Mok-Yau [CY80,MY83], Tian-Yau [TY90,TY91], and Guedj-Kolev-Yeganefar [GKY13], etc.

When the canonical bundle K_M is neither trivial, ample, nor anti-ample, the existence of a Kähler–Einstein metric is precluded, as the first Chern class cannot coincide with the Kähler class, making the necessary topological condition unattainable. The constant scalar curvature

Kähler (often abbreviated as cscK) metric generalizes the concept of the Kähler–Einstein metric. And on compact Kähler manifolds, the average of the scalar curvature $\hat{R}$ is given by

$$\hat{R} = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{[\omega]^n}$$

which is independent of the choice of ω.

1.2. **YTD conjecture.** For a polarized manifold (M, L), the Yau-Tian-Donaldson conjecture states that the existence of the cscK metric in $c_1(L)$ is equivalent to K-stability of (M, L), linking the K-energy's analytic behavior to algebraic stability via test configurations [Yau93, Tia97, Don02].

In [Che18], Chen outlined a program for studying the existence problem for cscK metric: a new continuity path that links the cscK equation to a certain second-order elliptic equation, apparently motivated by the classical continuity path for Kähler Einstein metrics and Donaldson's continuity path for conical Kähler Einstein metrics, and showed the openness. Further, Chen and Cheng [CC21b, CC21a] established a priori estimates and proved the existence of the cscK metric under the propness of the K-energy. There has many significant progress made in the resolution of the YTD conjecture; we refer interested readers to see, for instance [Sto09], [BDL20], [BBJ21], [BHJ19, BHJ22], etc.

1.3. **CscK metrics on complete noncompact manifolds.** Despite substantial progress on cscK metrics on compact Kähler manifolds, the complete non-compact case remains far less understood. A principal analytic obstacle is the general failure of a global $\partial\bar{\partial}$-lemma in the non-compact setting, where Hodge decomposition (and related closed-range properties) need not hold.

Fix a background metric ω and consider a cscK perturbation $\omega_\varphi = \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi$ with potential φ. Together with (a priori unknown) volume ratio F, the coupled cscK system reads

$$\begin{cases} \omega_\varphi^n = e^F\,\omega^n, \\ \Delta_\varphi F = -\hat{R} + \operatorname{tr}_\varphi \operatorname{Ric}(\omega), \end{cases} \tag{1.1}$$

whose solvability on certain complete manifolds yields a cscK metric of scalar curvature $\hat{R}$. Note that Kähler–Einstein metrics also solve (1.1). On compact manifolds, if a cscK metric is cohomologous to a Kähler–Einstein metric, then it must be Kähler–Einstein. However, this fact remains less understood on complete non-compact manifolds: there is no general rigidity that *separates* cscK metrics from Kähler–Einstein ones.

Question 1.1. Can we find verifiable hypotheses under which every cscK perturbation of a complete Kähler–Einstein metric is again Kähler–Einstein?

Motivated by these considerations, we expect a rigidity for complete Kähler–Einstein metrics among their cscK perturbations. We therefore seek *sufficient conditions* forcing any complete cscK perturbation to be Kähler–Einstein. More precisely, we have:

Theorem 1.2. Let (M, ω) be a complete Kähler-Einstein manifold with negative scalar curvature. Suppose there exists $\varphi \in C^\infty(M)$ with $\sup_M \varphi < \infty$ defines a complete cscK metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ such that $\mathrm{Ric}(\omega_\varphi)$ is bounded from below and the ratio of the volume form $\omega^n/\omega_\varphi^n$ is bounded from above. If

- (M, ω) is parabolic, or
- (M, ω) is non-parabolic with $\lambda_1(\Delta_\omega) > 0$, and for a fixed point $p \in M$, there exists $C_0, \alpha > 0$, such that for all $r > 1$, we have

$$\int_{B(p,2r)\setminus B(p,r)} |\varphi|^2\, \omega^n \le C_0 r^\alpha, \tag{1.2}$$

then ω_φ is Kähler-Einstein.

Parabolicity will be recalled in Section 4: a complete Riemannian manifold is *nonparabolic* iff it admits a (minimal) positive Green's function, and *parabolic* otherwise. We will see later that the potential φ is subharmonic. It is therefore reasonable to work within the subclass of perturbations $\omega_\varphi = \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi$ with $\sup_M \varphi < +\infty$.

1.4. **Bounded strictly pseudoconvex domains as a model case.** Let $\Omega \subset \mathbb{C}^n$ be a C^k bounded strictly pseudoconvex domain for $k \ge 7$. In [CY80], Cheng and Yau proved that there is a unique complete Kähler-Einstein metric on Ω constructed by a global strictly plurisubharmonic defining function. This leads to the following question:

Question 1.3. Does there exist a complete cscK metric that is not Kähler–Einstein on bounded strictly pseudoconvex domains?

On bounded strictly pseudoconvex domains, there are two natural classes of complete Kähler metrics: the Bergman metric ω_B, and metrics ω_ρ arising from strictly plurisubharmonic defining functions: defining functions of Ω naturally produce a class of complete Kähler metrics, and any two such metrics differ by a term of the form $\sqrt{-1}\,\partial\bar{\partial}u$ for some potential u. We will discuss this in Section 5 for details. For these two types of metrics, we obtain:

Theorem 1.4. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with $\partial\Omega \in C^2$.

(1) If ω_ρ is cscK and $\partial\Omega \in C^8$, then ω_ρ coincides with the unique Kähler–Einstein metric constructed by Cheng-Yau.

(2) If the Bergman metric ω_B is cscK, then ω_B is Kähler–Einstein. Moreover, if $\partial\Omega \in C^\infty$, then Ω is biholomorphic to the unit ball.

Item (2) of Theorem 1.4 provides a new perspective on extending Huang–Xiao's resolution of Cheng's conjecture [HX21, Theorem 1] to Bergman metrics with constant scalar curvature. We will elaborate in Section 5.

2. Preliminaries

2.1. **Notations.** Let (M^n, ω) denote an n-dimensional complete Kähler manifold. In local coordinates $(z^1, \ldots, z^n)$, the Kähler metric ω can be expressed in the form

$$\omega = \sqrt{-1} g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j,$$

where we adopt the Einstein summation convention. We use ∂_i and $\partial_{\bar{j}}$ as shorthand notations for $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \bar{z}^j}$, respectively, and denote by $\left(g^{i\bar{j}}\right)$ the inverse matrix of $\left(g_{i\bar{j}}\right)$.

Given two tensors A and B of type (m, n), their Hermitian inner product with respect to g is defined by

$$\langle A, B\rangle_g := g^{i_1\bar{j}_1} \cdots g^{i_n\bar{j}_n} g_{k_1\bar{l}_1} \cdots g_{k_m\bar{l}_m} A^{k_1\cdots k_m}_{i_1\cdots i_n} \overline{B^{l_1\cdots l_m}_{j_1\cdots j_n}},$$

and the corresponding norm of A is given by $|A|_g^2 := \langle A, A\rangle_g$.

Let ∇ denote the Levi-Civita connection associated with the Kähler metric ω. The covariant derivatives are expressed locally as

$$\nabla_i := \nabla_{\partial_i}, \quad \nabla_{\bar{j}} := \nabla_{\partial_{\bar{j}}}.$$

For any vector fields X, Y, Z, W on M, the Riemann curvature is defined by

$$\mathrm{Rm}(\omega)(X, Y, Z, W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\ W\rangle .$$

In local coordinates, the components of the Riemann curvature are calculated as

$$R_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{p\bar{q}} \left(\partial_k g_{i\bar{q}}\right)\left(\partial_{\bar{l}} g_{p\bar{j}}\right).$$

The Ricci curvature is defined as the trace of the Riemann curvature tensor:

$$\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}}\, dz^i \otimes d\bar{z}^j, \quad \text{where} \quad R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\left(\det(g_{k\bar{l}})\right).$$

The scalar curvature is then obtained by tracing the Ricci curvature with respect to g:

$$R(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega) = g^{i\bar{j}} R_{i\bar{j}}.$$

Note that for C^2-functions, the covariant derivative $\nabla_i \nabla_{\bar{j}}$ coincides with the partial derivative $\partial_i \partial_{\bar{j}}$, since the mixed Christoffel symbols $\Gamma_{i\bar{j}}$ vanish identically on Kähler manifolds.

For any C^∞-function $f : M \to \mathbb{R}$, the gradient vector field is defined by

$$\nabla f = g^{i\bar{j}} \left(\partial_i f \cdot \partial_{\bar{j}} + \partial_{\bar{j}} f \cdot \partial_i\right).$$

The complex Hessian of f is given by $\sqrt{-1}\partial\bar{\partial} f$, and the Laplacian operator is defined as the trace of the Hessian:

$$\Delta_\omega f := \mathrm{tr}_\omega \left(\sqrt{-1}\partial\bar{\partial} f\right) = g^{i\bar{j}} \partial_i \partial_{\bar{j}} f.$$

2.2. **Omori-Yau's maximum principle.** We now introduce Omori-Yau's generalized maximum principle on non-compact manifolds.

Proposition 2.1 (Omori [Omo67],Yau [Yau75])**.** Let (M, ω) be a complete Kähler manifold. If the sectional curvature of ω is bounded from below, then for any function $u \in C^2(M)$ with $\sup_M u < +\infty$, there exists a sequence of points $\{z_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$(1) \lim_{k\to\infty} u(z_k) = \sup_M u, \; (2) \lim_{k\to\infty} |\nabla u(z_k)|_\omega = 0, \; (3) \limsup_{k\to\infty} \sqrt{-1}\partial\bar{\partial}u(z_k) \leq 0,$$

where the third inequality holds in the sense of matrices.

Remark 2.2. If the assumption of bounded sectional curvature is replaced by a lower bound of the Ricci curvature, then condition (3) can be modified to

$$(3') \limsup_{k\to\infty} \Delta_\omega u(z_k) \leq 0.$$

2.3. **Bounded geometry.** Recall that the injectivity radius at a point $x \in M$ is the maximum radius r of the ball B_r in the tangent space T_xM for which the exponential map $\exp_x : B_r \to \exp_x(B_r) \subset M$ is a diffeomorphism. The injectivity radius of M is the infimum of the injectivity radius at all points in M.

Definition 2.3. Let (M, ω) be a complete Kähler manifold and let $k \geq 0$ be an integer. We say (M, ω) has C^k-quasi-bounded geometry if for each non-negative integer $l \leq k$, there exists a constant $C_l > 0$ such that

$$\sup_M |\nabla^l \mathrm{Rm}(\omega)|_\omega \leq C_l, \tag{2.1}$$

where ∇^l is the covariant derivative of order l. Moreover, if (M, ω) has a positive injectivity radius, then we say (M, ω) has C^k-bounded geometry.

Remark 2.4. It is clear that if a complete Kähler manifold has C^0-bounded geometry, then the generalized maximum principle is valid.

3. Basic estimates for the cscK metric

Let (M^n, ω) be a complete Kähler-Einstein manifold of C^0-bounded geometry with negative scalar curvature. In our context, it is necessary to assume the background metric has at least C^0-bounded geometry to use the generalized maximum principle. Suppose there exists a complete cscK metric $\tilde{\omega}$ of C^0-bounded geometry on M with $R(\tilde{\omega}) = R(\omega) = -n$, then there is a function F such that

$$\tilde{\omega}^n = e^F \omega^n, \tag{3.1}$$

and

$$\Delta_{\tilde{\omega}} F = n - \mathrm{tr}_{\tilde{\omega}}\, \omega. \tag{3.2}$$

We can verify that

$$\begin{aligned} R(\tilde{\omega}) &= \operatorname{tr}_{\tilde{\omega}} \operatorname{Ric}(\tilde{\omega}) \\ &= -\operatorname{tr}_{\tilde{\omega}} \sqrt{-1}\partial\bar{\partial}\log\left(\frac{\tilde{\omega}^n}{\omega^n}\right) + \operatorname{tr}_{\tilde{\omega}} \operatorname{Ric}(\omega) \\ &= -\Delta_{\tilde{\omega}} F - \operatorname{tr}_{\tilde{\omega}} \omega \\ &= -n. \end{aligned}$$

We will always assume the Kähler-Einstein metric has Einstein constant -1, and the cscK metric has the scalar curvature $-n$ if without making any further mention.

We first see the following property for the volume form of a cscK metric.

Lemma 3.1. Suppose M is a complete Kähler manifold. Let ω and $\tilde{\omega}$ be a complete Kähler-Einstein metric and a complete cscK metric with the same scalar curvature. If $\operatorname{Ric}(\tilde{\omega})$ is bounded from below and the ratio of the volume forms $\omega^n/\tilde{\omega}^n$ is bounded from above, then the following inequality holds:

$$\tilde{\omega}^n \geq \omega^n. \tag{3.3}$$

Moreover, if the equality holds at any interior point $p \in M$, then $\tilde{\omega} = \omega$ on M.

Proof. Assume $R(\omega_\varphi) = R(\omega) = -n$. Taking $u = \omega^n/\tilde{\omega}^n > 0$, we compute

$$\begin{aligned} \Delta_{\tilde{\omega}} \log u &= \Delta_{\tilde{\omega}} \log(\omega^n/\tilde{\omega}^n) \\ &= \operatorname{tr}_{\tilde{\omega}} \omega - n \\ &\geq n u^{\frac{1}{n}} - n, \end{aligned}$$

where the last inequality follows from the arithmetic-geometric inequality.

Then we obtain

$$\Delta_{\tilde{\omega}} \log u = \frac{\Delta_{\tilde{\omega}} u}{u} - \frac{|\nabla u|^2_{\tilde{\omega}}}{u^2} \geq n u^{\frac{1}{n}} - n,$$

which leads to

$$\Delta_{\tilde{\omega}} u - \frac{|\nabla u|^2_{\tilde{\omega}}}{u} \geq n u^{\frac{1}{n}+1} - nu.$$

Since u is bounded from above and $\operatorname{Ric}(\tilde{\omega})$ is bounded from below on Ω, by the generalized maximum principle, there exists a sequence $\{z_\alpha\} \subset \Omega$, such that

$$0 \geq \limsup_{\alpha\to\infty} \Delta_{\tilde{\omega}} u(z_\alpha) \geq n \lim_{\alpha\to\infty} u^{\frac{1}{n}+1}(z_\alpha) - n \lim_{\alpha\to\infty} u(z_\alpha).$$

This implies $\sup_M u \leq 1$, yielding the desired inequality.

Observe that,

$$\Delta_{\tilde{\omega}} \log u \geq n u^{\frac{1}{n}} - n \geq \log u.$$

Suppose there is a point $p \in M$ such that $u(p) = \sup_M u = 1$. By the maximum principle, we have $u \equiv 1$ in M and $\tilde{\omega}^n = \omega^n$. Then (3.2) yields $\mathrm{tr}_{\tilde{\omega}}\, \omega = n$. Consequently, we have $\tilde{\omega} = \omega$ thanks to the equality case of the arithmetic-geometric inequality. □

Remark 3.2. Note that the inequality (3.3) implies that the volume form of the Kähler-Einstein metric is the smallest among all cscK metrics. Therefore, we have $F \geq 0$ in (3.1) and (3.2).

The following trace-type inequality was originally from Yau's famous C^2-estimate [Yau78]. In general, this inequality holds for any two Kähler metrics ω and $\tilde{\omega}$ if the holomorphic bisectional curvature of ω is bounded from below.

Lemma 3.3. There exists a constant B depending on the holomorphic bisectional curvature of ω such that

$$\Delta_{\tilde{\omega}} \log(\mathrm{tr}_\omega \tilde{\omega}) \geq B\, \mathrm{tr}_{\tilde{\omega}}\, \omega - \frac{\mathrm{tr}_\omega \mathrm{Ric}(\tilde{\omega})}{\mathrm{tr}_\omega \tilde{\omega}}. \tag{3.4}$$

We then see the following quantitative estimate. In fact, if $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for some potential φ with $||\varphi||_{C^2} \leq M$ for some $M > 0$, then the cscK metric $\tilde{\omega}$ is bi-Lipschitz to the Kähler-Einstein metric ω.

Proposition 3.4. Let (M^n, ω) be a complete Kähler-Einstein manifold of C^0-bounded geometry. Suppose $\tilde{\omega}$ is a complete cscK metric of C^0-bounded geometry on M such that the volume form ratio $\tilde{\omega}^n/\omega^n := e^F \in C^2(M)$. If

$$\sup_M \mathrm{tr}_\omega \tilde{\omega} < +\infty,$$

then there exist constant $C \geq C' > 0$ depending on the dimension n, the lower bound of holomorphic bisectional curvature of ω and $||F||_{C^2(M)}$, such that

$$C'\omega \leq \tilde{\omega} \leq C\omega \quad \text{on} \quad M. \tag{3.5}$$

Proof. It follows from (3.1),

$$\mathrm{Ric}(\tilde{\omega}) = -(n+1)\omega - \sqrt{-1}\partial\bar{\partial}F. \tag{3.6}$$

Then, (3.4) becomes

$$\begin{aligned}\Delta_{\tilde{\omega}} \log(\mathrm{tr}_\omega \tilde{\omega}) &\geq B\, \mathrm{tr}_{\tilde{\omega}}\, \omega - \frac{\mathrm{tr}_\omega \mathrm{Ric}(\tilde{\omega})}{\mathrm{tr}_g \tilde{g}} \\ &= B\, \mathrm{tr}_{\tilde{\omega}}\, \omega + \frac{n(n+1) + \Delta_\omega F}{\mathrm{tr}_\omega \tilde{\omega}}.\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, we have

$$(\mathrm{tr}_\omega \tilde{\omega})(\mathrm{tr}_{\tilde{\omega}}\, \omega) = \left(g^{i\bar{j}}\tilde{g}_{i\bar{j}}\right)\left(\tilde{g}^{k\bar{l}}g_{k\bar{l}}\right) \geq n^2.$$

So there exists a constant $C_1 > 0$ depending on n and $||\Delta F||_{L^\infty(M)}$, such that

$$\Delta_{\tilde{\omega}} \log(\mathrm{tr}_\omega \tilde{\omega}) \geq B\, \mathrm{tr}_{\tilde{\omega}}\, \omega - C_1\, \mathrm{tr}_{\tilde{\omega}}\, \omega. \tag{3.7}$$

Taking $C_2 = (-B + C_1 + 1)/(n+1)$, then it follows from (3.7),

$$\Delta_{\tilde{\omega}} (\log (\operatorname{tr}_\omega \tilde{\omega}) - C_2 F) \geq \operatorname{tr}_{\tilde{\omega}} \omega - C_2(n+1)n.$$

Since $\log (\operatorname{tr}_\omega \tilde{\omega}) - C_2 F$ is bounded from above on M and $\tilde{\omega}$ has C^0-bounded geometry. Then by the generalized maximum principle, there exists a sequence $\{z_\alpha\}$, such that

$$0 \geq \limsup_{\alpha\to\infty} \Delta_{\tilde{\omega}} (\log (\operatorname{tr}_\omega \tilde{\omega}) - C_2 F)(z_\alpha) \geq \lim_{k\to\infty} \operatorname{tr}_{\tilde{\omega}} \omega(z_\alpha) - C_2(n+1)n,$$

which implies

$$\lim_{k\to\infty} \operatorname{tr}_{\tilde{\omega}} \omega(z_\alpha) \leq C_2(n+1)n.$$

If $z_\alpha \to p \in M$, then taking a normal coordinate at p such that ω is identity and $\tilde{\omega}$ is diagonal, we have

$$\operatorname{tr}_{\tilde{\omega}} \omega(p) = \sum_i \tilde{g}^{i\bar{i}}(p) \leq C_2(n+1)n.$$

This yields at point p, for any k,

$$\frac{1}{\tilde{g}_{k\bar{k}}(p)} = \tilde{g}^{k\bar{k}}(p) \leq \sum_i \tilde{g}^{i\bar{i}}(p) \leq C_2(n+1)n. \tag{3.8}$$

From (3.1), we have

$$\prod_i \tilde{g}_{i\bar{i}}(p) = e^{F(p)} \leq e^{\sup_\Omega F}. \tag{3.9}$$

Combining (3.8) and (3.9), for any k, we obtain

$$\tilde{g}_{k\bar{k}}(p) = \frac{\prod_i \tilde{g}_{i\bar{i}}(p)}{\prod_{i\neq k} \tilde{g}_{i\bar{i}}(p)} \leq (C_2(n+1)n)^{n-1}\, e^{\sup_\Omega F} := \frac{C_3}{n} \tag{3.10}$$

where we recall that C_3 depends on n, the lower bound of holomorphic bisectional curvature of ω, $\|F\|_{L^\infty}$ and $\|\Delta F\|_{L^\infty}$. In particular, from (3.10), we have

$$\operatorname{tr}_\omega \tilde{\omega}(p) = \sum_k \tilde{g}_{k\bar{k}}(p) \leq C_3.$$

If $\{z_\alpha\}$ does not converge to any interior point, then by the generalized maximum principle, we have

$$\operatorname{tr}_{\tilde{\omega}} \omega(z_\alpha) \leq C_2(n+1)n + \frac{1}{\alpha}.$$

Taking a normal coordinate at z_α and following the same approach as above, we obtain

$$\operatorname{tr}_\omega \tilde{\omega}(z_\alpha) \leq C_3 + o\left(\frac{1}{\alpha}\right).$$

Let $\alpha \to \infty$, we have

$$\lim_{\alpha\to\infty} \operatorname{tr}_\omega \tilde{\omega}(z_\alpha) \leq C_3.$$

Hence for any $x \in M$, we obtain

$$\begin{aligned}\log(\mathrm{tr}_\omega \tilde{\omega})(x) - C_2 F(x) &\le \sup_M (\log(\mathrm{tr}_\omega \tilde{\omega}) - C_2 F)\\ &\le \lim_{\alpha\to\infty} \log \mathrm{tr}_\omega \tilde{\omega}(z_\alpha) - C_2 \lim_{\alpha\to\infty} F(z_\alpha)\\ &\le \log(C_3),\end{aligned}$$

thanks to $F \ge 0$. This implies

$$\sup_M \mathrm{tr}_\omega \tilde{\omega} \le C_4, \tag{3.11}$$

where C_4 depends on the same factors as C_3. Now, if we choose a normal coordinate at any point $x \in M$ such that ω is identity and $\tilde{\omega}$ is diagonal, it follows from (3.3) and (3.11),

$$\tilde{g}_{k\bar{k}} = \frac{\prod_i \tilde{g}_{i\bar{i}}}{\prod_{i\neq k} \tilde{g}_{i\bar{i}}} \ge C_5,$$

which implies

$$\tilde{\omega} \ge C_5 \omega. \tag{3.12}$$

This gives the desired result. □

4. CscK metrics on a complete Kähler-Einstein manifold

On compact manifolds, the $\partial\bar{\partial}$-lemma guarantees that any two Kähler metrics in the same cohomology class differ by a global potential. On a non-compact complete manifold, however, the absence of a global $\partial\bar{\partial}$-lemma prevents one from fixing a Kähler class in the usual sense. To obtain a tackleable framework, we fix a background Kähler metric ω and consider only those perturbations of the form

$$\omega_\varphi := \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi,$$

so that ω_φ remains cohomologically "tethered" to ω via a global potential.

Consider a complete Kähler-Einstein manifold (M, ω). Assume there is $\varphi \in C^\infty(M)$ with $\sup_M \varphi < +\infty$ defines a complete cscK metric $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ which satisfies

$$\omega_\varphi^n = e^F \omega^n, \tag{4.1}$$

and

$$\Delta_{\omega_\varphi} F = n - \mathrm{tr}_{\omega_\varphi} \omega, \tag{4.2}$$

for some $F \in C^\infty(M)$. Without loss of generality, by adding a constant to φ, we always assume $\sup_M \varphi = 0$.

The cscK metric forms an important class of canonical metrics in Kähler geometry, generalizing Kähler-Einstein metrics. It is clear that a Kähler-Einstein metric is a cscK metric, but conversely, it is not true. On compact Kähler manifolds, we have:

Proposition 4.1 (A well-known fact)**.** Let (M, ω) be a compact Kähler manifold. Suppose that $2\pi c_1(M) = \lambda[\omega]$ for some constant λ where $c_1(M)$ is the first Chern class of M. If ω is a cscK metric, then ω is Kähler-Einstein.

In this section, we investigate an analogue of the rigidity result above on complete Kähler-Einstein manifolds, in the setting where M admits a complete cscK metric ω_φ that is asymptotically Kähler-Einstein. More precisely, we say a complete cscK metric ω_φ is asymptotically Kähler-Einstein, if

(1) (*Weak version*) the ratio of the volume forms $\omega_\varphi^n/\omega^n \to 1$ asymptotically;

(2) (*Strong version*) the potential function φ satisfies $\sqrt{-1}\partial\bar{\partial}\varphi \to 0$ asymptotically.

The strong version implies the weak version obviously. However, we expect that, on a complete Kähler-Einstein manifold with suitable geometric control, the weak asymptotics already force the rigidity among cscK perturbations: It should suffice to require asymptotic equivalence of the *volume forms* rather than of the metrics themselves, since the volume form determines the Ricci curvature.

It is not difficult to see that if the background metric ω is Calabi-Yau, then the decay of F is enough to show the cscK metric is also Calabi-Yau. Moreover, we do not even need to require that the cscK metric is cohomologous to the Calabi-Yau metric.

Proposition 4.2. Let (M, ω) be a complete Calabi-Yau manifold. Suppose $\tilde{\omega}$ is a complete cscK metric on M with $R(\tilde{\omega}) = 0$, and that $\tilde{\omega}^n = e^F\omega^n$ for some $F \in C^\infty(M)$. If, for any point $p \in M$,

$$\lim_{r\to\infty} \|F\|_{C^0(M\setminus B(p,r))} = 0,$$

then $\tilde{\omega}$ is Calabi–Yau.

Proof. Since $\mathrm{Ric}(\tilde{\omega}) = \mathrm{Ric}(\omega) - \sqrt{-1}\,\partial\bar{\partial}F$ and $\mathrm{Ric}(\omega) = 0$, tracing with respect to $\tilde{\omega}$ gives

$$0 = R(\tilde{\omega}) = \mathrm{tr}_{\tilde{\omega}}\,\mathrm{Ric}(\tilde{\omega}) = -\Delta_{\tilde{\omega}}F,$$

which implies F is harmonic. As $F \to 0$ at infinity and $\tilde{\omega}$ is complete, the maximum principle yields $F \equiv 0$, hence $\tilde{\omega}^n = \omega^n$ and $\mathrm{Ric}(\tilde{\omega}) = \mathrm{Ric}(\omega) = 0$. □

Therefore, only the case of negative scalar curvature needs to be considered more concretely. For the cscK metric ω_φ with negative scalar curvature that is asymptotically Kahler-Einstein in the strong sense, we obtain:

Proposition 4.3. Let (M, ω) be a complete Kähler-Einstein manifold of C^0-bounded geometry with negative scalar curvature. Suppose there is a $\varphi \in C^\infty(M)$ such that $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ defines a complete cscK metric and $R(\omega_\varphi) = R(\omega)$. If for any fix point $p \in M$, we have

$$\lim_{r\to\infty} \|\varphi\|_{C^2(M\setminus B(p,r))} = 0,$$

then $\varphi \equiv 0$ and $\omega_\varphi = \omega$.

Proof. It follows from (4.2),

$$\Delta_{\omega_\varphi}F = \Delta_{\omega_\varphi}\varphi.$$

Since $\lim_{r\to\infty} ||\varphi||_{C^2(M\setminus B(p,r))} = 0$, we also have

$$\lim_{r\to\infty} ||F||_{C^0(M\setminus B(p,r))} = 0,$$

thanks to (3.1). By the maximum principle, we have $F = \varphi$ on M. Substituting this into (3.1) yields

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\varphi}\omega^n. \tag{4.3}$$

Since ω is complete of C^0-bounded geometry and $\sup_M |\varphi| < \infty$, by the generalized maximum principle, there exists a sequence $\{z_k\} \subset \Omega$ such that

$$1 = \lim_{k\to\infty} \frac{\omega^n}{\omega^n}(z_k) \geq \lim_{k\to\infty} \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega^n}(z_k) = e^{\sup_M \varphi}.$$

This implies $\sup_\Omega \varphi \leq 0$. A symmetric argument gives $\inf_M \varphi \geq 0$, forcing $\varphi \equiv 0$ on M. Consequently, ω_φ is Kähler-Einstein, which completes the proof. □

In fact, Proposition 4.3 shows that if a Kähler–Einstein metric and a cscK metric are asymptotically equivalent in the strong sense, then they must coincide. Note that the cscK equations (4.1) and (4.2) depend only on the second derivatives of the potential φ, so that the C^0 control in the associated complex Monge–Ampère equation (4.1) has historically been the most delicate step. In our convention, a C^2-*decay* hypothesis at infinity already entails the C^0-decay of φ. Hence, it already presupposes this delicate part, and is somehow too strong.

Chen-Cheng's breakthrough [CC21b] provides a priori estimates for the coupled cscK system (1.1) in terms of the entropy

$$\mathrm{Ent}(\varphi) = \int_M F\, e^F\, \omega^n.$$

On complete manifolds, this quantity may be infinite. If $\mathrm{Ent}(\varphi) < \infty$ and $\mathrm{Vol}(M, \omega) = \infty$, F must satisfy certain global integrability, which might force some decay of F at ends with infinite volume. A decay assumption for F, together with a growth restriction on the volume, can make $\mathrm{Ent}(\varphi) < \infty$, which could allow us to try to adapt Chen-Cheng's technique on complete manifolds. Moreover, an *optimal* decay rate of F can serve as a sufficient condition to separate cscK metrics from Kähler–Einstein metrics within the solution set of (1.1) if the fixed class is cohomologous to the first Chern class.

4.1. **Parabolicity and rigidity of complete Kähler-Einstein metrics.** In this subsection, we study the case by replacing the C^2 decay of φ in Proposition 4.3 by an L^2 condition. To realize this purpose, we need first to introduce the *parabolicity*.

The Green's function plays a significant role in the analysis of elliptic PDEs on complete Riemannian manifolds. Malgrange [Mal56] first proved that the Laplace operator admits a symmetric Green's function. Afterward, Li and Tam [LT87] gave a constructive proof for the existence of the Green's function on any complete Riemannian manifold. In particular, the existence of a positive minimal Green's function allows one to classify complete manifolds into two categories: *parabolic* manifolds and *non-parabolic* manifolds.

Definition 4.4 (Paracolicity)**.** We say a complete Riemannian manifold (M, g) is non-parabolic if it admits a positive minimal Green's function. Otherwise, (M, g) is said to be parabolic.

It is well known that, on a parabolic manifold, all subharmonic functions bounded from above are constant (see, for instance, Grigor'yan [Gri99] for more discussion on parabolicity). It follows from Lemma 3.1,

$$\Delta_\omega \varphi = \mathrm{tr}_\omega\, \omega_\varphi - n \geq n \left(\omega_\varphi^n / \omega^n\right)^{1/n} - n \geq 0. \tag{4.4}$$

Therefore, we immediately obtain:

Proposition 4.5. Let (M, ω) be a complete Kähler-Einstein manifold. Suppose there exists $\varphi \in C^\infty(M)$ with $\sup_M \varphi < \infty$ such that ω_φ defines a complete cscK metric such that $\mathrm{Ric}(\omega_\varphi)$ is bounded from below, and the ratio of the volume form $\omega^n / \omega_\varphi^n$ is bounded from above. If (M, ω) is parabolic, then ω_φ is Kähler-Einstein.

Remark 4.6 (Quadratic volume growth implies parabolicity)**.** A standard volume growth criterion asserts that if

$$\int^\infty \frac{r}{\mathrm{Vol}\, B(p, r)}\, dr = \infty,$$

then M is parabolic. In particular, if $\mathrm{Vol}\, B(p, r) \leq C\, r^2$ for all sufficiently large r, the integral diverges and M is parabolic (see also [CY75]).

We now turn to the non-parabolic case. In particular, a positive bottom of the spectrum implies non-parabolicity: if

$$\lambda_1(\Delta_\omega) := \inf_{0 \neq f \in C_c^\infty(M)} \frac{\int_M |\nabla f|_\omega^2\, \omega^n}{\int_M f^2\, \omega^n} > 0,$$

then (M, ω) admits a minimal positive Green's function $G(x, \cdot)$ with $G(x, y) \to 0$ as $d(x, y) \to \infty$ for each fixed x. Moreover, Cheng and Yau gave a necessary condition for $\lambda_1 > 0$: if the manifold has polynomial volume growth, then $\lambda_1 = 0$. Recall that non-parabolicity and the positivity of the bottom of the spectrum are stable under bi-Lipschitz changes of the metric.

On a complete Riemannian manifold (M, g) with positive spectrum, we have some nice decay estimates for the minimal positive Green's function on the annular area:

Lemma 4.7 (Li-Wang [LW01])**.** Let M^n be a complete manifold with $\lambda_1(\Delta) > 0$. Then the minimal positive Green's function $G(p, \cdot)$ with pole at $p \in M$ must satisfy the decay estimate

$$\int_{B_{r+1}(p) \setminus B_r(p)]} G^2(p, y) dy \leq C e^{-2\sqrt{\lambda_1(\Delta)} r} \int_{B_2(p) \setminus B_1(p)} G^2(p, y) dy \tag{4.5}$$

for $r > 1$, where $C > 0$ only depends on $\lambda_1(\Delta) > 0$.

In what follows, we therefore assume $\lambda_1(\Delta_\omega) > 0$, which allows us to exploit quantitative estimates for the Green's function G.

Theorem 4.8. Let (M^n, ω) be a complete Kähler-Einstein manifold with $\lambda_1(\Delta_\omega) > 0$. Suppose there exists $\varphi \in C^\infty(M)$ with $\sup_M \varphi < +\infty$ such that ω_φ defines a complete cscK metric such that $\mathrm{Ric}(\omega_\varphi)$ is bounded from below, and the ratio of the volume form $\omega^n/\omega_\varphi^n$ is bounded from above. Fix a base point $p \in M$, denote $A_r(p) := B_{2r}(p) \setminus B_r(p)$ by the annulur area. If there exists constants $C_0, \alpha > 0$, such that for all $r > 0$,

$$\int_{A_r(p)} |\varphi|^2\, \omega^n \;\leq\; C_0 r^\alpha,$$

then ω_φ is Kähler-Einstein.

Proof. Since we only use the Laplacian Δ_ω in this proof, we omit the subscript ω for simplicity. Let G be the minimal positive Green's function on M, and let $r >> 1$. Consider a cut-off function η with $\eta \equiv 1$ in $B_r(p)$, $\eta \equiv 0$ outside $B_{2r}(p)$, with $|\nabla\eta| < C/r$ and $|\Delta\eta| < C/r^2$ for some constant $C > 0$. Define $\psi = \eta(\varphi - \sup_M \varphi)$, then by the Green's identity, we have

$$\begin{aligned}
\psi(p) &= -\int_{B_{2r}} G(p,x)\Delta\psi(x)\,\omega^n(x)\\
&= -\frac{\sqrt{-1}}{n}\int_{B_{2r}} G\left(\partial\varphi\wedge\bar\partial\eta + \psi\partial\bar\partial\eta + \partial\eta\wedge\bar\partial\varphi + \eta\partial\bar\partial\varphi\right)\wedge\omega^{n-1}\\
&= -\int_{B_{2r}} G\psi\Delta\eta\,\omega^n - \int_{B_{2r}} G\eta\Delta\varphi\,\omega^n - \frac{\sqrt{-1}}{n}\int_{B_{2r}} G\left(\partial\varphi\wedge\bar\partial\eta + \partial\eta\wedge\bar\partial\varphi\right)\wedge\omega^{n-1}\\
&= \int_{B_{2r}} G\psi\Delta\eta\,\omega^n - \int_{B_{2r}} G\eta\Delta\varphi\,\omega^n + \frac{\sqrt{-1}}{n}\int_{B_{2r}} \varphi\left(\partial G\wedge\bar\partial\eta + \partial\eta\wedge\bar\partial G\right)\wedge\omega^{n-1}\\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where we use integration by parts in the fourth equality.

For I_3, we apply Cauchy-Schwartz inequality and Hölder's inequality,

$$\begin{aligned}
I_3 &= -\frac{\sqrt{-1}}{n}\int_{A_r} \varphi\left(\partial G\wedge\bar\partial\eta + \partial\eta\wedge\bar\partial G\right)\wedge\omega^{n-1}\\
&\leq C\int_{A_r} |\varphi|\,|\nabla G|\,|\nabla\eta|\,\omega^n\\
&\leq \frac{C}{r}\left(\int_{A_r} |\varphi|^2\,\omega^n\right)^{1/2}\left(\int_{A_r} |\nabla G|^2\,\omega^n\right)^{1/2}.
\end{aligned}$$

For any $x \in A_r$, take $\rho = r/4$ such that $B(x,\rho) \subset M \setminus \{p\}$. By Cheng-Yau's gradient estimate [CY75],

$$\frac{|\nabla G(p,x)|}{|G(p,x)|} \leq \sup_{B(x,\rho)} \frac{|\nabla G(p,\cdot)|}{|G(p,\cdot)|} \leq \sqrt{2n-1} + \frac{C'}{r}.$$

We then decompose A_r into annular area $B(p, r+k+1) \setminus B(p, r+k)$ of thickness 1 and apply Li-Wang's decay estimate (4.5), which gives

$$\begin{aligned} I_3 &\le C\sqrt{C_0}\left(\frac{\sqrt{2n-1}}{r} + \frac{C'}{r^2}\right) r^{\frac{\alpha}{2}} \left(\int_{A_r} |\nabla G|^2\, \omega^n\right)^{1/2} \\ &\le C r^{\frac{\alpha}{2}-2}(r+1)\cdot\left(\sum_{k=0}^{\lfloor r\rfloor}\int_{B_{r+k+1}\setminus B_{r+k}} |G|^2\,\omega^n\right)^{1/2} \\ &\le C r^{\frac{\alpha}{2}-2}(r+1)\, e^{-r\sqrt{\lambda_1(\Delta)}}\left(\sum_{k=0}^{\lfloor r\rfloor} e^{-2\sqrt{\lambda_1(\Delta)}k}\right)^{1/2} \\ &\le C_3 r^{\frac{\alpha}{2}-2}(r+1)\, e^{-r\sqrt{\lambda_1(\Delta)}}. \end{aligned}$$

For I_2, since ω and ω_φ are bi-Lipschitz, we have $0 \le \Delta\varphi$, which yields $I_2 \le 0$.

For I_1, we adapt a similar approach with I_3, we then obtain

$$I_1 = \int_{A_r} G\psi\Delta\eta\,\omega^n \le \frac{C}{r^2}\left(\int_{A_r}|\varphi|^2\,\omega^n\right)^{1/2}\left(\int_{A_r}|G|^2\,\omega^n\right)^{1/2} \le C_1 r^{\frac{\alpha}{2}-2} e^{-r\sqrt{\lambda_1(\Delta)}}$$

Combining with the above estimates, we immediately have,

$$|\psi(p)| \le C_1 r^{\frac{\alpha}{2}-2} e^{-r\sqrt{\lambda_1(\Delta)}} + C_2 r^{\frac{\alpha}{2}-2}(r+1)\, e^{-r\sqrt{\lambda_1(\Delta)}} \to 0,$$

as $r \to \infty$. This gives $\varphi(p) = \sup_M \varphi$. Together with (4.4), we have $\varphi \equiv C$ on M for some constant C, thanks to the strong maximum principle. Consequently, $\omega_\varphi = \omega$ is Kähler-Einstein. □

Remark 4.9. In fact, the polynomial growth assumption for the L^2-norm of φ on A_r can be improved to exponential growth with rate $e^{2\delta r}$, for $0 < \delta \le \lambda_1(\Delta)$.

5. The cscK metric on bounded strictly pseudoconvex domains

In this section, we discuss the cscK metric on bounded pseudoconvex domains in $\mathbb{C}^n$. Note that such domains always admit a complete Kähler-Einstein metric of negative scalar curvature, thanks to [MY83].

5.1. **The Cheng-Yau metric.** Let $\Omega \subset \mathbb{C}^n$ be a C^k bounded strictly pseudoconvex domain. In [CY80], Cheng and Yau investigated complete Kähler metrics of the form

$$\omega_\rho = -\sqrt{-1}\partial\bar\partial\log\rho,$$

where ρ is a strictly plurisubharmonic defining function for the domain $\Omega = \{\rho > 0\}$. They derived the local expression for the curvature tensor:

$$R_{i\bar{j}k\bar{l}} = -\left(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}\right) + O\left(|\rho|^{-1}\right).$$

Notably, the metric ω behaves asymptotically like a Kähler-Einstein metric with Einstein constant $-(n+1)$ near the boundary $\partial\Omega$.

The Kähler-Einstein metric on a strictly pseudoconvex domain is constructed using the Fefferman defining function ρ of class C^k for $k \geq 8$. Let ω denote this metric, defined by

$$\omega = -\sqrt{-1}\,\partial\bar{\partial}\log\rho.$$

If the cscK metric is also defined by a global strictly plurisubharmonic defining function $\tilde{\rho}$, then there exists a positive function $u \in C^{k-1}(\bar{\Omega})$ satisfying $u = 1 + o(1)$ near $\partial\Omega$ such that $\tilde{\rho} = u\rho$. Consequently, the perturbed metric satisfies

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\varphi = -\log u$. For a detailed discussion of the regularity of u, see [Kra01, Chapter 3.1].

Proposition 5.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^k-boundary, for $k \geq 8$. Then there is no complete cscK metric $\tilde{\omega}$ given by a C^k-defining function $\tilde{\rho}$ unless $\tilde{\omega}$ is Kähler-Einstein.

Proof. Assume there exists a complete cscK metric $\tilde{\omega}$ defined by a defining function $\tilde{\rho}$. Let ω be the Kähler-Einstein metric on Ω given by the defining function ρ. As discussed previously, this implies the metric perturbation

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\varphi = -\log u$ for some $u \in C^{k-1}(\bar{\Omega})$ such that $\tilde{\rho} = u\rho$. The method is similar to the proof of Proposition 4.3, and the only difference is that $\partial\bar{\partial}\varphi \nrightarrow 0$.

However, note that the determinant relationship

$$\det\left(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi\right) = \det\left(g_{i\bar{j}}\right)\cdot\det\left(\delta_{i\bar{j}} + g^{i\bar{l}}\partial_l\partial_{\bar{j}}\varphi\right) := e^F\det\left(g_{i\bar{j}}\right)$$

holds. Moreover, we have $g^{i\bar{l}} = O(|\rho|)$ and $\partial_l\partial_{\bar{j}}\varphi \in C^{k-3}(\bar{\Omega})$, which implies that $g^{i\bar{l}}\partial_l\partial_{\bar{j}}\varphi$ vanishes asymptotically near $\partial\Omega$. Thus,

$$e^F = \det\left(\delta_{i\bar{j}} + g^{i\bar{l}}\partial_l\partial_{\bar{j}}\varphi\right) \to \det\left(\delta_{i\bar{j}}\right) = 1 \quad \text{as} \quad z \to \partial\Omega.$$

Combing with $\varphi = 0$ on $\partial\Omega$, we complete the proof. □

5.2. **The Bergman metric.** Besides ω_ρ, there is another important complete Kähler metric on C^2 bounded pseudoconvex domain, which is called the Bergman metric. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ be the space of holomorphic functions in $L^2(\Omega)$. It is clear that $A^2(\Omega)$ is a Hilbert space. The Bergman kernel $K(z)$ on Ω is a real analytic function defined as

$$K(z) = \sum_{j=1}^{\infty}|\varphi_j(z)|^2, \qquad \forall z \in \Omega,$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $A^2(\Omega)$ with respect to the L^2 inner product. Since the Bergman kernel is positive and independent of the choice of any orthonormal basis [Kra01] on bounded domains, we then can define the Bergman metric by

$$\omega_B := \sqrt{-1}\partial\bar{\partial}\log K.$$

The Bergman metric is a complete real analytic Kähler metric, with the real analytic property inherited from the Bergman kernel.

A key observation is that, on a bounded pseudoconvex domain Ω with C^2 boundary, the Bergman metric ω_B behaves asymptotically like a Kähler–Einstein metric near the boundary.

Proposition 5.2 (Krantz-Yu [KY96]). Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $p \in \partial\Omega$ be a C^2 strictly pseudoconvex point. Then

$$\lim_{z \to p} |\operatorname{Ric}(\omega_B) + \omega_B|(z) = 0, \quad \text{and} \quad \lim_{z \to p} R(\omega_B) = -n.$$

This leads to a natural and fundamental question: *under what conditions does this identity hold on a general strictly pseudoconvex domain?*

The simplest and most striking example is the unit ball $B^n \subset \mathbb{C}^n$, where the Bergman metric is exactly Kähler–Einstein everywhere. But more generally, when can we say that the Bergman metric ω_B is Kähler–Einstein? Indeed, both the KE metric and the Bergman metric are invariant under biholomorphisms, and the boundary behavior of ω_B can be understood throughout the geometry of the boundary, which is determined by the defining function. However, the interior curvature behavior of the Bergman metric remains far less known.

To access this interior behavior, one promising idea is to study the automorphism group of Ω. If we can find a biholomorphism that maps a point near the boundary, where ω_B behaves like a Kähler–Einstein metric, into the central region of the domain, the invariance properties of both metrics allow us to carry the asymptotic Kähler–Einstein behavior into the interior. In this way, the understanding of the automorphism group becomes a crucial key in exploring the interior geometry of ω_B. This problem was explicitly posed by Yau in the 1980s, who conjectured the following:

Conjecture 5.3 (Yau [SY94]). A bounded domain Ω, which is not a product domain admits a complete Kähler–Einstein Bergman metric if and only if Ω is homogeneous.

Recall that a bounded domain Ω is said to be *homogeneous* if its automorphism group $\operatorname{Aut}(\Omega)$ acts transitively; that is, for any $x, y \in \Omega$, there exists a $g \in \operatorname{Aut}(\Omega)$ such that $g(x) = y$. Notably, the celebrated ball characterization theorem by Rosay [Ros79] and Wong [Won77] implies that any C^2 bounded homogeneous domain must be a ball. The product domain is automatically excluded in this case since the corner of the product boundary has only C^0 regularity.

A more tractable case of Yau's conjecture is when Ω is strictly pseudoconvex with C^∞-boundary. In particular, Cheng asked:

Conjecture 5.4 (Cheng [Che79]). Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^∞ boundary. If ω_B is Kähler-Einstein, then Ω is biholomorphic to the ball.

In 2021, Huang and Xiao answered Cheng's conjecture affirmatively.

Theorem 5.5 (Huang-Xiao [HX21])**.** The Bergman metric of a bounded strictly pseudoconvex domain Ω with C^∞-boundary is Kähler-Einstein if and only if the domain is biholomorphic to the ball.

Given the rigidity of this result, it is natural to ask whether similar characterizations can be obtained under weaker curvature conditions.

Question 5.6. Can we characterize bounded domains whose Bergman metric has constant scalar curvature?

Let $G := \det(\omega_B)$ denote the determinant of the Bergman metric. The Bergman invariant function defined by $B(z) := G(z)/K(z)$ is invariant under biholomorphic maps. A significant result regarding the Bergman invariant function, established by Diederich, states that if a bounded strictly pseudoconvex domain has C^2-boundary, then B tends to constant when we approach the boundary.

Proposition 5.7 (Diederich [Die70])**.** Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $p \in \partial\Omega$ be a C^2 strictly pseudoconvex point. Then

$$\lim_{z\to p} B(z) = \frac{(n+1)^n \pi^n}{n!}.$$

We then have the following statement, as an extending for Huang and Xiao's resolution of Cheng's conjecture.

Proposition 5.8. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary and let ω_B be the Bergman metric. If ω_B has constant scalar curvature, then ω_B is Kähler-Einstein. Moreover, if $\partial\Omega \in C^\infty$, then Ω is biholomorphic to the ball.

Proof. Note that, for the Bergman metric ω_B,

$$\mathrm{Ric}(\omega_B) + \sqrt{-1}\partial\bar{\partial}\log B = -\omega_B \tag{5.1}$$

Taking trace with respect to ω_B, we have

$$\Delta_{\omega_B} \log B = 0 \quad \text{in} \quad \Omega,$$

which implies $\log B$ is harmonic with a constant boundary value (Proposition 5.7). By the maximum principle, it follows that $\log B$ must be constant throughout Ω, specifically

$$\log B = \log\left(\frac{(n+1)^n \pi^n}{n!}\right).$$

Combining this with the characterization in [FW97, Proposition 1.1], we conclude that ω_B is Kähler-Einstein. The final statement then follows immediately from Theorem 5.5. □

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