PLURIPOTENTIAL THEORY ON KÄHLER MANIFOLDS

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Abstract. This document is a self-study note on pluripotential theory. For more details, we refer interested readers to [GZ17], [DDNL23], and [Xia25] for details. It is assumed that the reader has a foundational understanding of Kähler geometry and function theory of several complex variables. Any corrections or suggestions are warmly welcomed.

Contents

1. Canonical metrics and complex Monge-Ampère equation

Let (X, ω) be an n-dimensional compact Kähler manifold. Here X is an n-dimensional compact complex manifold, and ω is a positive closed real $(1,1)$-form, which is called the Kähler form. Let $(z^1, ..., z^n)$ be a local coordinate, then

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $\left(g_{i\bar{j}}\right)$ is a Hermitian matrix.

Example 1.1. (1) The complex projective space $(\mathbb{CP}^{\ltimes}, \omega_{FS})$ endowed with the Fubini-Study metric is a Kähler manifold, where on each $U_k = \{z^k \neq 0\}$

$$\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log(1 + |z|^2).$$

(2) Any hypersurfaces $V^n = \{F = 0\} \subset \mathbb{CP}^{n+1}$ where F is a homogeneous polynomial of degree d is a Kähler manifold with respect to $\omega_{FS}|_V$.
(3) The Euclidean form

$$\omega_0 = \sum_k \sqrt{-1} dz^k \wedge d\bar{z}^k$$

is closed and invariant under translations. This induces Kähler forms on complex tori $X = \mathbb{C}^n/\Lambda$, where $\Lambda \subset \mathbb{R}^{2n}$ is a lattice.

Given a Kähler form ω, we can define the associated volume form ω^n. Let dV_g be the Riemannian volume form, then we have

$$dV_g = \frac{\omega^n}{n!}.$$

The normal coordinate on Kähler manifold will be useful in our calculation.

Proposition 1.2 (Kähler normal coordinates)**.** If ω is a Kähler metric, then for each point p and an open neighbourhood U of p we can choose a local coordinate such that for any i, j, k

$$g_{i\bar{j}}(p) = \delta_{ij} \quad \text{and} \quad \partial_i g_{k\bar{j}} = \partial_{\bar{i}} g_{k\bar{j}} = 0, \tag{1.1}$$

where δ_{ij} is the Kronecker symbol.

The Ricci curvature of ω can be defined by

$$\mathrm{Ric}(\omega) = \sqrt{-1}\partial\bar{\partial}\log\det(\omega).$$

Note that the Ricci curvature is a closed real $(1,1)$-form. And, the scalar curvature $R(\omega) := \mathrm{tr}_\omega \mathrm{Ric}(\omega)$ is the trace of the Ricci curvature.

Proposition 1.3 ($\partial\bar{\partial}$-Lemma)**.** Let (X, ω) be a compact Kähler manifold. Suppose α and β are two closed real $(1,1)$-forms that are cohomologous to each other. Then there exists a smooth function u on X such that

$$\alpha - \beta = \sqrt{-1}\partial\bar{\partial}u.$$

Fixed a Kähler class $[\omega]$, thanks to $\partial\bar{\partial}$-Lemma, the space of Kähler metrics in $[\omega]$ can be formulated by

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(X),\ \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

Question 1.4. Can we find the “best” metric ω_φ in $[\omega]$?

In fact, we expect to find metrics with special curvature properties in $[\omega]$, which often arise as **solutions to geometric PDE** or **critical points of energy functionals**. There are some nice candidates:

Definition 1.5 (Kähler-Einstein metric)**.** A Kähler metric ω is called Kähler-Einstein (KE for short) if there is $\lambda \in \mathbb{R}$, so that

$$\mathrm{Ric}(\omega) = \lambda\omega.$$

Definition 1.6 (CscK metric)**.** A Kähler metric ω is called cscK if ω has constant scalar curvature.

Definition 1.7 (Extremal metric)**.** A Kähler metric ω is called extremal if $\nabla^{1,0}R$ is holomorphic.

Clearly,

$$\{\text{KE metrics}\} \subset \{\text{CscK metrics}\} \subset \{\text{Extremal metrics}\},$$

so the KE metric is the optimal choice for our Question 1.4.

Observe that if we have two Kähler metrics ω and θ, then

$$\mathrm{Ric}(\omega) - \mathrm{Ric}(\theta) = \sqrt{-1}\partial\bar{\partial}\log\left(\frac{\det(\theta)}{\det(\omega)}\right)$$

where the right-hand side is a globally defined exact $(1,1)$-form.

Definition 1.8 (The first Chern class)**.** The first Chern class $c_1(X)$ of X is defined by

$$c_1(X) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)] \in H^{1,1}(X,\mathbb{R}).$$

Once we have a KE metric ω with $\mathrm{Ric}(\omega) = \lambda\omega$, by taking cohomology class on each sides,

$$c_1(X) = \frac{\lambda}{2\pi}[\omega].$$

Since $[\omega] > 0$ is a Kähler class, the existence of a KE metric necessitates a definite sign of $c_1(X)$.

Example 1.9. Let $X = \mathbb{CP}^1 \times S$, where S is a compact Riemann surface of genus 2. Then $c_1(X)$ does not have a definite sign.

Proposition 1.10. Let X be an n-dimensional compact Kähler manifold and $\lambda\omega \in 2\pi c_1(X)$ for some $\lambda \in \mathbb{R}$. Denote by $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. Then the following statements are equivalent:

(1) ω_φ is a Kähler-Einstein metric;

(2) ω_φ is a Kähler metric with constant scalar curvature;

(3) φ solves

$$\left(\omega + \sqrt{-1}\partial\bar{\partial}\varphi\right)^n = e^{-\lambda\varphi+h}\omega^n \tag{MA$_\lambda$}$$

for some smooth $h : X \to \mathbb{R}$ such that $\mathrm{Ric}(\omega) = \lambda\omega + \sqrt{-1}\partial\bar{\partial}h$.

Proof. (2) $\Longleftrightarrow$ (1): If ω_φ is cscK and $\lambda\omega_\varphi \in 2\pi c_1(X)$, then there is $f \in C^\infty(X)$ such that

$$\mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi + \sqrt{-1}\partial\bar{\partial}f.$$

Taking trace w.r.t ω_φ gives $\Delta_\varphi f = 0$. This concludes $f \equiv$ *constant* and ω_φ is KE, since there are no non-constant harmonic functions on compact manifolds.

(3) $\Longleftrightarrow$ (1): Assume ω_φ is KE. Since $\lambda\omega_\varphi \in 2\pi c_1(X)$, there is $h \in C^\infty(X)$ such that $\mathrm{Ric}(\omega) = \lambda\omega + \sqrt{-1}\partial\bar{\partial}h$, and

$$\mathrm{Ric}(\omega_\varphi) = \lambda\omega_\varphi = \mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}h + \sqrt{-1}\partial\bar{\partial}\lambda\varphi.$$

This implies

$$\sqrt{-1}\partial\bar{\partial}\left(\frac{\omega_\varphi^n}{\omega^n}\right) = \sqrt{-1}\partial\bar{\partial}\left(e^{-\lambda\varphi+h}\right),$$

as desired. Conversely, we take $\sqrt{-1}\partial\bar{\partial}$ to both sides of (MA$_\lambda$) which yields ω_φ is KE. □

The solvability of (MA$_\lambda$) is well-known: :

(1) If $c_1(X) < 0$, X is of general type, a problem solved by Aubin [Aub76] and Yau [Yau78];

(2) If $c_1(X) = 0$, X is Calabi-Yau, as solved by Yau [Yau78];

(3) If $c_1(X) > 0$, X is Fano, there are some obstructions for the existence of KE metrics, introduced by Matsushima [Mat57], Futaki [Fut83], and Tian [Tia97]. It was proven by Chen-Donaldson-Sun [CDS15a, CDS15b, CDS15c] that the K-polystability of a Fano manifold is equivalent to the existence of a KE metric.

Question 1.11. What about the "singular" case? Can we solve (MA_λ)?

In practice, "singular" may refer to three different sources:

(i) the *underlying space* X is singular (a Kähler variety: X_{reg} carries a Kähler form);

(ii) the *background class* is only semi-positive / big, so the representative ω may be *degenerate* (not Kähler);

(iii) the *potential* φ is not smooth (e.g. merely psh, hence in L^1).

All three viewpoints lead to the *same* weak complex Monge-Ampère problem once we formulate it in the non-pluripolar sense. We provide a brief explanation here, with a more detailed discussion to follow in a later section.

Fix a closed semi-positive $(1,1)$-form (or current) θ on X and consider

$$\langle(\theta + dd^c\varphi)^n\rangle \;=\; e^{-\lambda\varphi + f}\,\mu, \tag{SMA}$$

where $\varphi \in \mathrm{PSH}(X,\theta)$, $\lambda \in \mathbb{R}$, $f \in L^p$ is given, and μ is a fixed positive measure. Here $\langle\cdot\rangle$ denotes the non-pluripolar Monge–Ampère product, which puts no mass on pluripolar subsets.

From (i) to (ii) + (iii) via resolution. Let Y be a Kähler variety and let $\pi : X \to Y$ be a resolution. A Kähler form ω_Y on Y_{reg} pulls back to a closed semi-positive form $\pi^*\omega_Y$ on X that is *degenerate along the exceptional directions*. Choosing a smooth representative

$$\theta \in \pi^*[\omega_Y] \subset H^{1,1}(X,\mathbb{R}),$$

the equation on Y is equivalent to (SMA) on X with a right-hand side measure

$$\mu \;=\; \pi^*(\omega_Y^n)\cdot e^{f\circ\pi}\cdot(\text{divisorial density coming from the Jacobian}),$$

and the unknown potential replaced by $\varphi\circ\pi$. Equivalence follows from functoriality of the non-pluripolar product and the fact that it charges no mass on analytic sets (hence nothing is lost on Y_{sing}).

Moving singularities between the background current and the potential function. Within a fixed cohomology class, we can always take θ *smooth*. Indeed, if θ is a closed positive current and θ_0 is a smooth form with $[\theta_0] = [\theta]$, then

$$\theta_0 = \theta + dd^c\psi \quad \text{for some (quasi-psh) } \psi,$$

and setting $\tilde\varphi := \varphi - \psi$ gives

$$\theta_0 + dd^c\tilde\varphi \;=\; \theta + dd^c\varphi,$$

so (SMA) is unchanged while the singularity has been moved from the background current to the potential. Thus case (ii) and (iii) are just two faces of the same weak MA equation in a fixed cohomology class.

In summary, whether the singularity sits in the space, the background form, or the potential, the appropriate notion of solution to (SMA) is the same: a θ-psh potential φ solving (SMA) in the non-pluripolar sense, typically sought in the finite energy class $\mathcal{E}^1(X,\theta)$.

2. Quasi-plurisubharmonic functions

2.1. **Basic definition.** Let (X,ω) be a compact Kähler manifold. We abbreviate the adjective "plurisubharmonic" as "psh". From now on, we will use the convention

$$d^c := \frac{\sqrt{-1}}{2\pi}(\bar{\partial}-\partial), \qquad dd^c = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial},$$

so that dd^c maps real-valued functions to real $(1,1)$-forms/currents.

Definition 2.1. Let θ be a closed real $(1,1)$-form.

- We say a function $u : X \to \mathbb{R}\cup\{-\infty\}$ is quasi-psh if u can be locally written as the sum of a smooth function and a psh function.
- We say a function $u : X \to \mathbb{R}\cup\{-\infty\}$ is θ-psh if u is quasi-psh and $\theta_u = \theta + dd^c u \geq 0$ in the sense of current. We denote $\mathrm{PSH}(X,\theta)$ by the space of all θ-psh functions on X.

Example 2.2. Consider $\theta \geq 0$, there are plenty of examples of θ-psh functions:

- If $\theta = \omega$ is Kähler and $u \in C^2(X)$, then there exists $A >> 1$ so that
$$A\omega + dd^c u \geq 0,$$
which gives $u/A \in \mathrm{PSH}(X,\omega)$.
- Let $X = \mathbb{P}^n$ and let $\theta = \omega_{FS}$. If P is a homogeneous polynomial of degree d in $z \in \mathbb{C}^{n+1}$, then
$$\varphi(z) = \frac{1}{d}\log|P|(z) - \log|z| \in \mathrm{PSH}\left(\mathbb{P}^n, \omega_{FS}\right).$$
More generally if $L \to X$ is a positive holomorphic line bundle with metric $h = e^{-\phi}$ of curvature $\omega = \Theta_h = dd^c\phi > 0$ and if $s \in H^0(X,L)$, then
$$\varphi(z) = \log|s|_h = \log|s| - \phi_0 \in \mathrm{PSH}(X,\omega).$$
- Let $B \subset \mathbb{C}^n$ be a ball centered at the origin. Functions $\log|z|$, $-(-\log|z|)^\alpha$ for $0 < \alpha < 1$ and $-\log(-\log|z|)$ on B are all θ-psh functions.

Proposition 2.3. Assume $X = \mathbb{P}^n = \mathbb{C}^n \cup \{z_0 = 0\}$ endowed with $\omega = \omega_{FS}$. Consider

$$\mathcal{L}(\mathbb{C}^n) := \{u \in \mathrm{PSH}(\mathbb{C}^n),\ u(z) \leq \frac{1}{2}\log\left(1 + |z|^2\right) + C \text{ for all } z \in \mathbb{C}^n\}.$$

Then there is a one to one correspondence

$$u \in \mathcal{L}(\mathbb{C}^n) \mapsto u - \frac{1}{2}\log\left(1 + |z'|^2\right) \in \mathrm{PSH}(\mathbb{P}^n, \omega).$$

Proof. Work on the affine chart $U_0 = \{Z_0 \neq 0\} \simeq \mathbb{C}^n$ with coordinates $z_j = Z_j/Z_0$, and set $H_\infty := \mathbb{P}^n \setminus U_0 = \{Z_0 = 0\}$. Let

$$\phi_0(z) := \tfrac{1}{2}\log\left(1 + |z|^2\right), \qquad \omega_{FS} = dd^c\phi_0 \text{ on } U_0.$$

Given $u \in \mathcal{L}(\mathbb{C}^n)$, define on U_0

$$\varphi_0 := u - \phi_0.$$

Since u is psh, $dd^c\varphi_0 + \omega_{FS} = dd^c u \geq 0$ on U_0, so φ_0 is ω_{FS}-psh there. The growth $u \leq \phi_0 + C$ implies a uniform upper bound $\varphi_0 \leq C$ even near H_∞, hence the usc extension

$$\varphi := (\varphi_0)^* \quad \text{on } \mathbb{P}^n$$

is well defined and φ is ω_{FS}-psh. This yields the map $u \mapsto \varphi \in \mathrm{PSH}(\mathbb{P}^n, \omega_{FS})$.

Conversely, let $\varphi \in \mathrm{PSH}(\mathbb{P}^n, \omega_{FS})$. So, $\varphi \leq C$ on $\mathbb{P}^n$. On U_0, define

$$u := \varphi + \phi_0.$$

Then $dd^c u = dd^c\varphi + dd^c\phi_0 \geq -\omega_{FS} + \omega_{FS} = 0$, so u is psh on $\mathbb{C}^n$, and $u \leq \phi_0 + C$, i.e. $u \in \mathcal{L}(\mathbb{C}^n)$.

On any other affine chart U_j one uses the local FS potential ϕ_j with $dd^c\phi_j = \omega_{FS}$; on overlaps $\phi_0 - \phi_j$ is pluriharmonic, so the above constructions glue. Finally, starting from u gives φ with $\varphi|_{U_0} = u - \phi_0$, hence $(\varphi + \phi_0)|_{U_0} = u$; starting from φ gives $u = \varphi + \phi_0$ and then $(u - \phi_0)^* = \varphi$. Thus the correspondence is bijective. □

Remark 2.4. Equivalently, the growth condition defining $\mathcal{L}(\mathbb{C}^n)$ can be phrased as: the "Lelong number at infinity" of u (with respect to the weight ϕ_0) is at most $1/2$. This is precisely the condition ensuring that $u - \phi_0$ does not blow up to $+\infty$ along the hyperplane at infinity H_∞, so it extends as an ω_{FS}-psh function on $\mathbb{P}^n$.

Definition 2.5. A subset $P \subset X$ is pluripolar if for any $x \in X$, there is an open neighborhood U of x and a function $\psi \in \mathrm{PSH}(U)$ such that

$$\psi|_{P\cap U} \equiv -\infty.$$

We say some property about objects on X holds quasi-everywhere (q.e.) if it holds outside a pluripolar set.

Theorem 2.1 (Josefson's theorem)**.** Assume that X is a compact complex manifold and $P \subset X$ is a pluripolar set. Then there is a quasi-psh function ψ on X with $\psi|_P \equiv -\infty$.

We also have:

Proposition 2.6. Let $u, v \in \mathrm{PSH}(X, \theta)$. Assume that there is a dense subset $E \subseteq X$ such that

$$u \leq v \quad \text{on} \quad E,$$

then $u \leq v$ on X.

2.2. **Lelong numbers.** Singularities of quasi-psh functions are at worst logarithmic thanks to the Lelong-Jensen formula and sub-mean value property. An intuitive picture is that psh functions are subharmonic when restricted to each complex line, and the subharmonic functions have only logarithmic level singularities in real dimension 2. We can then use the Lelong number to measure how "singular" a quasi-psh function is, at each point:

Definition 2.7 (Lelong number)**.** Let X be a compact complex manifold and let θ be a closed real $(1,1)$-form.

(1) The Lelong number of $u \in \mathrm{PSH}(X, \theta)$ at $p \in X$ is defined by

$$\boxed{\nu(u, p) = \sup\{\gamma \geq 0;\ u(z) \leq \gamma \log(d(z, p)) + O(1)\}.}$$

(2) Let T be a positive closed $(1,1)$-current such that $[T] = [\theta]$. The Lelong number of T at $p \in X$ is defined by

$$\boxed{\nu(T, p) \ := \ \nu(u, p),}$$

where u is θ-psh s.t. $T = \theta + dd^c u$.

Remark 2.8. Equivalently, in local coordinates z centered at p,

$$\nu(T, p) \ = \ \lim_{r \downarrow 0} \int_{\{d(z,p)<r\}} T \wedge (dd^c \log(d(z, p)))^{n-1}.$$

Some nice properties for Lelong numbers, thanks to Siu, are:

Theorem 2.2 (Siu, [Siu74])**.** Let X be a compact complex manifold, and let u be quasi-psh. Then

- the function $x \longmapsto \nu(u, x)$ is usc and invariant under local biholomorphism;
- sets $E_c(u) = \{x \in X,\ \nu(u, x) \geq c\}$ are analytic for each $c > 0$.

Theorem 2.3 (Siu's decomposition, [Siu74])**.** Let X be a complex manifold and T a positive closed $(1,1)$-current on X. Then there exist nonnegative numbers λ_i (locally only finitely many nonzero), irreducible divisors $D_i \subset X$, and a positive closed $(1,1)$-current R such that

$$T \ = \ \sum_i \lambda_i [D_i] \ + \ R,$$

where $[D_i]$ denotes integration along D_i, and R has zero *generic* Lelong number along every divisor (equivalently, R has no divisorial component). Moreover, for each i,

$$\lambda_i = \nu(T, x) \quad \text{for a generic point } x \in D_i.$$

Corollary 2.9. Let X be a complex manifold with $\dim X \geq 2$, let $x \in X$, and let $\pi : \widetilde{X} = \mathrm{Bl}_x X \to X$ be the blow-up at x with exceptional divisor $E = \pi^{-1}(x)$. For any positive closed $(1,1)$-current T on X,

$$\boxed{\nu(T, x) = \sup\{\gamma \geq 0;\ \pi^* T - \gamma [E] \text{ is a positive closed } (1,1)\text{-current on } \widetilde{X}\}.}$$

Proposition 2.10 (Properties of Lelong numbers)**.** Let $u, v \in \mathrm{PSH}(X, \theta)$, $\lambda > 0$ and $x \in X$. Then

(1) $\nu(\lambda u, x) = \lambda \nu(u, x)$;

(2) $\nu(u + v, x) = \nu(u, x) + \nu(u, x)$;

(3) $\nu(\max\{u, v\}, x) = \min\{\nu(u, x), \nu(v, x)\}$.

Definition 2.11 (Lelong number of a class)**.** The Lelong number of a class $[\theta]$ is defined by

$$\boxed{\nu([\theta]) := \sup\{\nu(u, x);\ x \in X \text{ and } u \in \mathrm{PSH}(X, \theta)\} < +\infty,}$$

which only depends on the cohomology class $[\theta]$.

2.3. **Skoda's integrability.** It is clearly that $\mathrm{PSH}(X, \theta) \subset L^1(X)$. Moreover, $u \in L^p(X)$ w.r.t. any smooth volume form. This fact can be shown by *Skoda's integrability*.[1]

Theorem 2.4 (Skoda, [Sko72])**.** Let X be a compact complex manifold, and let θ be a closed $(1,1)$-form. Assume $\varphi \in \mathrm{PSH}(X, \theta)$ and $A < 2\left[\sup_{x \in X} \nu(\varphi, x)\right]^{-1}$. Then $\exp(-A\varphi) \in L^1(X)$. Moreover, if $A < 2\nu([\theta])^{-1}$, then

$$\sup\left\{\int_X e^{-A\varphi} dV \mid \varphi \in \mathrm{PSH}(X, \theta) \text{ and } \sup_X \varphi = 0\right\} < +\infty. \tag{2.1}$$

We then decompose $u = u^+ - u^-$, since u is bounded from above, we only need to verify that $u^- \in L^p(X)$. Indeed,

$$\begin{aligned}
\int_X |u^-|^p \, dV &\leq C_{p,\delta} \int_X (1 + e^{\delta(u^-)}) \, dV \\
&\leq C_{p,\delta} \mathrm{Vol}(X) + C_{p,\delta} \int_X e^{-\delta u} \cdot e^{\delta(u^+)} \, dV \\
&< +\infty,
\end{aligned}$$

for some $\delta < 2\left[\sup_{x \in X} \nu(\varphi, x)\right]^{-1}$, thanks to Skoda's integrability and boundness of u^+.

[1] We are cheating here, because the exponential integrability (Skoda's integrability) is stronger than polynomial integrability (L^p property). In fact, the L^p property is a direct consequence of the sub-mean value property for psh functions.

2.4. **Hartogs lemma and Montel property.** We also have the following Hartogs lemma and Montel property for θ-psh functions.

Theorem 2.12 (Hartogs lemma)**.** Let X be a compact complex manifold and let θ be a smooth closed $(1,1)$-form on X. Suppose $(u_j)_{j\geq 1} \subset \mathrm{PSH}(X,\theta)$ satisfies $\sup_X u_j \leq C$ for some $C \in \mathbb{R}$ and all j. Set

$$u := \big(\limsup_{j\to\infty} u_j\big)^*,$$

the usc regularization of the pointwise lim sup. Then either $u \equiv -\infty$, or else

(i) $u \in \mathrm{PSH}(X,\theta)$,

(ii) there is a subsequence $u_{j_k} \to u$ in $L^1(X, dV)$ with respect to any smooth volume form dV.

Corollary 2.13 (Montel property)**.** Let X and θ be as in Theorem 2.12. For all $A \geq 0$, the sets

$$\mathrm{PSH}_A(X,\theta) := \left\{ u \in \mathrm{PSH}(X,\theta,\ -A \leq \sup_X u \leq 0 \right\}$$

are compact in L^1-topology.

Proposition 2.14. We have the following properties for θ-psh functions:

(1) If $\left(u_j\right)_j \subset \mathrm{PSH}(X,\theta)$ and $u_j \searrow u \not\equiv -\infty$, then $u \in \mathrm{PSH}(X,\theta)$.

(2) If $\left(u_j\right)_j \subset \mathrm{PSH}(X,\theta)$ is uniformly bounded from above and $u_j \nearrow u$, then $u^* \in \mathrm{PSH}(X,\theta)$ and $u = u^*$ almost everywhere. Here

$$u^*(z) = \limsup_{w\to z} u(w) = \inf_{r>0} \sup_{B_r(z)} u$$

is the usc regularization.

(3) Let $(u_\alpha)_{\alpha\in A} \subset \mathrm{PSH}(X,\theta)$ be uniformly bounded from above, then

$$U \ := \left(\sup_{\alpha\in A} u_\alpha\right)^* \ \in\ \mathrm{PSH}(X,\theta).$$

(4) If $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, then $\log\left(e^\varphi + e^\psi\right), \max\{\varphi,\psi\} \in \mathrm{PSH}(X,\theta)$.

(5) If $\varphi \in \mathrm{PSH}(X,\theta), \chi \in C^2(X)$ such that $\chi'' \geq 0$ and $0 \leq \chi' \leq 1$, then $\chi \circ \varphi \in \mathrm{PSH}(X,\theta)$.

Proof. See Demailly's book [Dem12]. □

Remark 2.15 (Choquet's lemma)**.** In (3), the index set A can be arbitrary. Moreover, there exists a countable subfamily (u_j) with $u_j \in \{u_\alpha : \alpha \in A\}$ such that

$$\left(\sup_{\alpha\in A} u_\alpha\right)^* = \left(\sup_{j\geq 1} u_j\right)^*.$$

2.5. **Demailly's regularization.** A cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ is called Kähler if there is a Kähler form $\omega \in \alpha$. Besides,

Definition 2.16. Let (X, ω) be a compact Kähler manifold and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a real $(1, 1)$-cohomology class.

(1) We say that α is *pseudo-effective (psef)* if it contains a closed positive $(1, 1)$-current. Equivalently, for any smooth closed $(1, 1)$-form $\theta \in \alpha$, $\mathrm{PSH}(X, \theta) \neq \emptyset$.

(2) We say that α is *big* if it contains a Kähler current, i.e. for any smooth $\theta \in \alpha$, there exists $\varepsilon > 0$, and $u \in \mathrm{PSH}(X, \theta)$ such that
$$\theta + dd^c u \;\geq\; \varepsilon\,\omega_0 \quad \text{(as currents)}.$$
Equivalently, for any some smooth $\theta \in \alpha$ there is $\varepsilon > 0$, one has $\mathrm{PSH}(X, \theta - \varepsilon\omega) \neq \emptyset$.

(3) We say that α is *nef* if for any $\theta \in \alpha$, $\varepsilon > 0$, there exists smooth ψ_ε, s.t. $\theta + dd^c\psi_\varepsilon > -\varepsilon\omega$.

Remark 2.17. The set $\mathrm{Big}(X) \subset H^{1,1}(X, \mathbb{R})$ of big classes is an open convex cone (the *big cone*). The set $\mathrm{Psef}(X)$ of pseudo-effective classes is a closed convex cone, and one has
$$\overline{\mathrm{Big}(X)} \;=\; \mathrm{Psef}(X).$$

Definition 2.18. Given $u, v \in \mathrm{PSH}(X, \theta)$.

- We say $u \preceq_{\mathrm{sing}} v$ (u is more singular than v), if $\exists\, C > 0$, s.t. $u \leq v + C$.
- We say $u \sim v$ (u is as same singular as v), if $u \preceq_{\mathrm{sing}} v$ and $v \preceq_{\mathrm{sing}} u$.

The relation $\preceq_{\mathrm{sing}}$ here is just a partial order, for instance, on $B_1 \subset \mathbb{C}^n$ and θ semi-positive, one can not compare $u_1 = \log|z_1|$ and $u_2 = \log|z_2|$ with $u_1, u_2 \in \mathrm{PSH}(B, \theta)$. The equivalent relation $u \sim v$ gives the class $[u] = \{v \in \mathrm{PSH}(X, \theta),\; v \sim u\}$ which is called the singularity type. As a convention, in this note, we will write $[u] \leq (=)[v]$ if $u \preceq_{\mathrm{sing}} (\sim)\; v$.

Theorem 2.5 (Demailly's regularization [Dem92])**.** Let (X, ω) be a compact Kähler manifold and let T be a closed positive $(1, 1)$-current on X. Set $\alpha := [T] = [\theta]$. Then there exists a sequence of closed $(1, 1)$-currents
$$T_j \;=\; \theta + dd^c\varphi_j \in \alpha,$$
with *analytic singularities* (i.e. locally $\varphi_j = \dfrac{1}{m_j}\log\displaystyle\sum_{k=1}^{N_j}|f_{j,k}|^2 + O(1)$), such that

(1) $T_j \rightharpoonup T$ weakly and the potentials decrease: $\varphi_j \searrow \varphi$;

(2) $T_j \geq -\varepsilon_j\,\omega$ with $\varepsilon_j \downarrow 0$;

(3) the cohomology class is preserved: $[T_j] = [\theta] = \alpha$;

(4) for all $x \in X$, $\nu(T_j, x) \leq \nu(T, x)$ and $\nu(T_j, x) \to \nu(T, x)$.

Moreover, if the class α is big, then there exists a current

$$T_{\mathrm{AS}} = \theta + dd^c\psi \in \alpha$$

with analytic singularities such that $T_{\mathrm{AS}} \geq \varepsilon\,\omega_0$ for some $\varepsilon > 0$ and some Kähler form ω_0 on X; in particular T_{AS} is a Kähler current with analytic singularities.

A direct consequence is:

Corollary 2.19. Given $\varphi \in \mathrm{PSH}(X, \omega)$, then there exists a smooth family $\{\varphi_j\}$ of strictly ω-psh functions, s.t. $\varphi_j \searrow \varphi$. In fact, $\mathrm{PSH}(X, \omega)$ is the closure of the set $\mathcal{K}_\omega$ of all Kähler potentials in L^1.

Theorem 2.20 (Demailly's approximation)**.** Let (X, L) be a polarized compact complex manifold and let h be a smooth Hermitian metric on L with curvature $\omega = \Theta_h > 0$. For any $\varphi \in \mathrm{PSH}(X, \omega)$ there exist a sequence of sections $t_j \in H^0(X, L^{\otimes j})$ such that

$$\frac{1}{j}\log|t_j|_{h^j} \xrightarrow[j\to\infty]{L^1} \varphi.$$

Remark 2.21 (Why this is a big deal)**.** This theorem *algebraizes* quasi-psh weights: every ω-psh potential is an L^1–limit of Bergman weights built from global sections of high tensor powers of L.

Definition 2.22 (Ample locus)**.** Let $[\theta] \in H^{1,1}(X)$, the ample locus of $[\theta]$ is defined by

$$\mathrm{Amp}([\theta]) = \{x \in X,\ \exists\,\psi,\ \text{s.t. } \theta_\psi \text{ is Kähler in a neighborhood of } x\}$$

It follows from our previous discussion,

$$\theta \text{ is big} \iff \mathrm{Amp}(\theta) \neq \emptyset.$$

Moreover, $\mathrm{Amp}([\theta])$ is Zariski dense in X (Boucksom [Bou04]), i.e. $\mathrm{Amp}([\theta]) \cap U \neq \emptyset$ for every non-empty Zariski open set.

3. Non-pluripolar Monge-Ampère measures

Let (X, ω) be a compact Kähler manifold and let θ be a smooth closed real $(1, 1)$-form whose cohomology class $[\theta]$ is *big*. WLOG, we assume $\mathrm{Vol}([\theta]) := \int_X \theta^n = 1$ after normalization. For any $u \in \mathrm{PSH}(X, \theta)$, we want to define the Monge-Ampère measure of u, that is

$$\mathrm{MA}_\theta(u) := \theta_u^n = (\theta + dd^c u)^n.$$

3.1. **The C^∞-case.** If $u \in C^\infty(X) \cap \mathrm{PSH}(X,\theta)$, then θ_u is a smooth form. We can define the MA measure by

$$\mathrm{MA}_\theta(u) \;:=\; \theta_u^n,$$

a smooth positive measure (volume form) on X. The construction is local and multilinear, hence Stokes' theorem yields the cohomological identity

$$\int_X \theta_u^n \;=\; \int_X \theta^n.$$

Moreover, if u is merely C^0, one can still choose a C^∞-approximation $u_k \to u$ uniformly, so that the MA measure $\mathrm{MA}_\theta(u)$ can be defined by the limit of the MA measure $\mathrm{MA}_\theta(u_k)$:

$$\theta_{u_k}^n \overset{*}{\rightharpoonup} \theta_u^n \quad \text{as} \quad k \to \infty \quad \text{in the sense of measure.}$$

We also underline that the limit is independent of the choice of the C^∞-approximation, thanks to Stokes' theorem:

Lemma 3.1. Let $u, v \in \mathrm{PSH}(X,\theta) \cap C^\infty(X)$, and let χ be any test function. Then

$$\int_X \chi\, \mathrm{MA}_\theta(u) - \int_X \chi\, \mathrm{MA}_\theta(v) \leq C(n)\|u-v\|_{L^\infty} \cdot \|\chi\|_{C^2}. \tag{3.1}$$

Proof. In fact,

$$\mathrm{MA}_\theta(u) - \mathrm{MA}_\theta(v) = dd^c(u-v) \wedge \left(\theta_u^{n-1} \wedge \cdots \wedge \theta_v^{n-1}\right).$$

Hence,

$$\begin{aligned}\left|\int_X \chi\,(\mathrm{MA}_\theta(u) - \mathrm{MA}_\theta(v))\right| =& \left|\int_X \chi dd^c(u-v) \wedge \left(\theta_u^{n-1} \wedge \cdots \wedge \theta_v^{n-1}\right)\right| \\ \leq& \left|\int_X (u-v) dd^c\chi \wedge \left(\theta_u^{n-1} \wedge \cdots \wedge \theta_v^{n-1}\right)\right| \\ \leq& C(n)\|u-v\|_{L^\infty} \cdot \|\chi\|_{C^2},\end{aligned}$$

which completes the proof. □

Consequently, for any two different approximations, the resulting MA measures are the same.

Example 3.2. Let $X = \mathbb{P}^n$ with the Fubini–Study metric ω. In homogeneous coordinates $[Z_0 : \cdots : Z_n]$, set

$$u_i([Z]) \;:=\; \log|Z_i| \;-\; \tfrac{1}{2}\log\Big(\sum_{k=0}^n |Z_k|^2\Big), \qquad \varphi \;:=\; \max_{0\leq i\leq n} u_i.$$

Then $\varphi \in \mathrm{PSH}(X,\omega) \cap C^0(X)$. Define

$$\varphi_\varepsilon \;:=\; \varepsilon \log\Big(\sum_{i=0}^n e^{u_i/\varepsilon}\Big), \qquad \varepsilon > 0.$$

Each $\varphi_\varepsilon \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$, and $\varphi_\varepsilon \searrow \varphi$ pointwise as $\varepsilon \downarrow 0$. Since φ is continuous, the MA measure $\mathrm{MA}_\omega(\varphi)$ is well-defined, and

$$\mathrm{MA}_\omega(\varphi) \;:= \lim_{\varepsilon\downarrow 0} \mathrm{MA}_\omega(\varphi) = \lim_{\varepsilon\downarrow 0}\,(\omega + dd^c\varphi_\varepsilon)^n \qquad \text{in the sense of measure.}$$

Thus $\mathrm{MA}_\omega(\varphi)$ is a MA measure in the sense of C^0 ω-psh functions.

Let $H_i = \{Z_i = 0\}$ be the coordinate hyperplanes. By Poincaré–Lelong, $dd^c u_i = [H_i] - \omega$, hence on the open region where $u_i > \max_{j\neq i} u_j$ one has

$$\omega + dd^c\varphi \;=\; \omega + dd^c u_i \;=\; [H_i],$$

which vanishes on that region (as a current), so $\mathrm{MA}_\omega(\varphi)$ carries no mass there. Consequently, the measure can only live on the ridge where at least two u_i tie. To produce an (n,n)-measure one needs *all* $n{+}1$ terms to tie, i.e.

$$|Z_0| = \cdots = |Z_n|.$$

This locus is the *Clifford torus*

$$T^n \;:=\; \{[Z] \in \mathbb{P}^n : \; |Z_0| = \cdots = |Z_n|\} \;\cong\; (S^1)^n.$$

Since φ is invariant under the torus action $(S^1)^{n+1}/S^1$, the measure is the *Haar measure* on T^n.

3.2. **The L^∞-case: Bedford–Taylor theory.** In this subsection, we explain how the wedge product is constructed when the potentials are locally bounded, as developed by Bedford and Taylor [BT76,BT82] on $\mathbb{C}^n$, and then globalize to θ-psh functions on compact Kähler manifolds.

Let $U \subset \mathbb{C}^n$ be open, let T be a positive closed current of bidegree $(n-p, n-p)$ on U, and let $u_1, \ldots, u_p$ be *bounded* plurisubharmonic functions on U $(1 \le p \le n)$. Define inductively

$$T_0 := T, \qquad T_m := dd^c\!(u_m\, T_{m-1}) \quad (1 \le m \le p),$$

where the product $u_m\, T_{m-1}$ is the current acting on test forms φ by

$$\langle u_m\, T_{m-1}, \varphi\rangle := \langle T_{m-1},\, u_m\, \varphi\rangle.$$

Hence, for every smooth compactly supported test form φ,

$$\langle dd^c(u_m T_{m-1}), \varphi\rangle = \langle T_{m-1},\, u_m\, dd^c\varphi\rangle.$$

The *Bedford–Taylor wedge* of $u_1, \ldots, u_p$ against T is

$$dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T \;:=\; T_p.$$

Equivalently, for every smooth compactly supported *function* χ (i.e. a test $(0,0)$-form) on U,

$$\big\langle dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T,\; \chi\big\rangle = \big\langle dd^c u_2 \wedge \cdots \wedge dd^c u_p \wedge T,\; u_1\, dd^c\chi\big\rangle. \tag{3.2}$$

We can verify that each T_m is closed and positive; it suffices to check T_1. For closedness, for any test form ψ,

$$\langle dT_1, \psi\rangle = \langle dd^c(u_1 T),\, d\psi\rangle = \langle u_1 T,\, dd^c(d\psi)\rangle = 0,$$

since $dd^c d = d\, dd^c$ and $d^2 = 0$; hence T_1 is closed.

For positivity, let $u_1^{(k)} \searrow u_1$ be a sequence of smooth psh functions. By Leibniz' rule and the closedness of T,

$$T_1^{(k)} := dd^c(u_1^{(k)} T) = dd^c u_1^{(k)} \wedge T \;\ge 0.$$

If ϕ is a smooth *strongly positive* test form of bidegree $(p-1, p-1)$, then

$$\langle T_1^{(k)}, \phi\rangle = \langle T,\, dd^c u_1^{(k)} \wedge \phi\rangle \geq 0.$$

Recall that for any $(1,1)$-form, the strong positivity is equivalent to positivity, so $dd^c u_1^{(k)} \wedge \varphi$ is still strongly positive provided φ is a strongly positive test (p,p)-form. We refer interested readers to check Demailly's book [Dem12] for details about the positivity of forms/currents.

Moreover, since the $u_1^{(k)}$ are uniformly bounded and $u_1^{(k)} \to u_1$ pointwise, by dominated convergence with respect to the trace measure of T we have

$$\langle T_1^{(k)}, \phi\rangle = \langle T,\, u_1^{(k)}\, dd^c\phi\rangle \;\longrightarrow\; \langle T,\, u_1\, dd^c\phi\rangle = \langle dd^c(u_1 T), \phi\rangle = \langle T_1, \phi\rangle.$$

Thus $T_1^{(k)} \rightharpoonup T_1$ weakly, and since the weak limit of positive currents is positive, T_1 is positive. The same argument applies inductively to T_m.

Proposition 3.3 (Local Chern–Levine–Nirenberg inequality)**.** Let $U \subset \mathbb{C}^n$ be a bounded domain and $K \subset\subset U$ be a compact set. Fix the standard Kähler form $\omega_0 = \frac{\sqrt{-1}}{\pi}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$. Then there exists $C = C(U, K, n) > 0$ such that for any $u_1, \ldots, u_n \in \mathrm{PSH}(U) \cap L^\infty(U)$,

$$\int_K dd^c u_1 \wedge \cdots \wedge dd^c u_n \;\leq\; C \prod_{j=1}^n \|u_j\|_{L^\infty(U)}.$$

Proof. Choose cutoffs $\chi_0, \chi_1, \ldots, \chi_n \in C_c^\infty(U)$ with

$$0 \leq \chi_0 \leq \cdots \leq \chi_n \leq 1, \quad \chi_n \equiv 1 \text{ on } K, \quad \chi_k \equiv 1 \text{ on } \operatorname{supp} dd^c\chi_{k+1} \;\; (0 \leq k < n).$$

For $0 \leq k \leq n$ set

$$I_k := \int_U \chi_k\, dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \omega_0^{n-k}.$$

We claim that there exists $A = A(U, \{\chi_k\}, n)$ such that for $1 \leq k \leq n$,

$$I_k \;\leq\; A\, \|u_k\|_{L^\infty(U)}\, I_{k-1}. \tag{3.3}$$

Granting this, iterating gives $I_n \leq A^n(\prod_{j=1}^n \|u_j\|_\infty) I_0$, and since $I_n = \int_K dd^c u_1 \wedge \cdots \wedge dd^c u_n$ (because $\chi_n \equiv 1$ on K) and $I_0 = \int_U \chi_0\, \omega_0^n$ is a fixed constant, the proposition follows with $C := A^n I_0$.

It remains to prove (3.3). Let

$$S_k := dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge \omega_0^{n-k},$$

a positive closed $(n-1, n-1)$-current. Hence

$$I_k = \int_U u_k\, (dd^c\chi_k) \wedge S_k \leq \|u_k\|_{L^\infty(U)} \int_U |dd^c\chi_k| \wedge S_k.$$

Because $dd^c\chi_k$ is smooth with compact support in $\{\chi_{k-1} = 1\}$, there exists a constant $B = B(\chi_k, \omega_0)$ such that the smooth $(1,1)$-form inequality $dd^c\chi_k \leq B\chi_{k-1}\,\omega_0$ holds. Wedge with the positive current S_k and integrate to get

$$\int_U |dd^c\chi_k| \wedge S_k \;\leq\; B\int_U \chi_{k-1}\,\omega_0 \wedge S_k = B\int_U \chi_{k-1}\, dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge \omega_0^{n-(k-1)} = B\, I_{k-1}.$$

Combining the above inequalities yields (3.3) with $A := B$. □

Remark 3.4. The CLN inequality is still true if u_k is only L^∞_{loc}.

Example 3.5. In this example, we write $r = |z|$, and consider the MA measure on $\mathbb{C}^n$.

(1) Let $u = \log|z|$, then $(dd^c u)^n = \delta_0$. Consider the smooth regularization

$$u_\varepsilon(z) := \frac{1}{2}\log(r^2+\varepsilon^2) \underset{\varepsilon\downarrow 0}{\longrightarrow} \log r.$$

A direct computation gives

$$dd^c u_\varepsilon = \frac{1}{2}\left(\frac{1}{r^2+\varepsilon^2}\,\omega_0 - \frac{\sqrt{-1}}{\pi}\frac{\partial r^2\wedge\bar\partial r^2}{(r^2+\varepsilon^2)^2}\right), \qquad \omega_0 := \frac{\sqrt{-1}}{\pi}\sum_{j=1}^n dz_j\wedge d\bar z_j.$$

Wedge to the top degree (radial rank 1 computation) yields a smooth radial probability density:

$$(dd^c u_\varepsilon)^n = \frac{n!\,\varepsilon^2}{2^n}\frac{1}{(r^2+\varepsilon^2)^{n+1}}\,\omega_0^n, \qquad \int_{\mathbb{C}^n}(dd^c u_\varepsilon)^n = 1.$$

Hence $(dd^c u_\varepsilon)^n \overset{*}{\rightharpoonup} \delta_0$ as $\varepsilon\downarrow 0$, i.e.

$$\boxed{(dd^c\log|z|)^n = \delta_0\,.}$$

(2) Let $u = \max\{\log|z|, 0\}$, then $(dd^c u)^n = \sigma_{S^{2n-1}}$. Consider

$$u_\varepsilon(r) := \varepsilon\log(e^{\log r/\varepsilon}+e^{0/\varepsilon}) = \varepsilon\log(1+r^{1/\varepsilon}) \searrow \max\{\log r, 0\}.$$

Each u_ε is smooth, radial, psh and bounded on compact sets. By a boring computation,

$$(dd^c u_\varepsilon)^n = f_\varepsilon(r)\,dV_{2n} \quad\text{with}\quad f_\varepsilon(r) = \frac{1}{r}\frac{d}{dr}\Big(r\frac{d}{dr}u_\varepsilon(r)\Big)^n\cdot c_n,$$

for an explicit normalizing constant c_n chosen so that $\int_{\mathbb{C}^n}(dd^c u_\varepsilon)^n = 1$. A direct differentiation shows f_ε concentrates at $r = 1$. More precisely, for any $\phi\in C_c^\infty(\mathbb{C}^n)$,

$$\int_{\mathbb{C}^n}\phi\,(dd^c u_\varepsilon)^n \longrightarrow \frac{1}{\mathrm{Vol}(S^{2n-1})}\int_{S^{2n-1}}\phi\,d\sigma \qquad\text{as}\qquad \varepsilon\downarrow 0.$$

Hence

$$\boxed{(dd^c u)^n = \text{normalized Lebesgue measure on } S^{2n-1}\,.}$$

(3) Consider $u_j = \frac{1}{2j}\log(1+|z_1^j+z_2^j|^2) \to u = \max\{\log|z_1|, \log|z_2|, 0\}$ on $\mathbb{C}^2$, then $(dd^c u)^2 = \sigma_{S^1\times S^1}$, while $(dd^c u_j)^2 \equiv 0$. First, $u_j\in \mathrm{PSH}(\mathbb{C}^2)\cap C^\infty$ and

$$dd^c u_j = \frac{1}{2j}F_j^*\omega_{FS}, \quad F_j = [1, z_1^j + z_2^j] : \mathbb{C}^2\to\mathbb{P}^1.$$

Since $\mathbb{P}^1$ has complex dimension 1,

$$(dd^c u_j)^2 = \frac{1}{4j^2}F_j^*(\omega_{\mathrm{FS}}\wedge\omega_{\mathrm{FS}}) \equiv 0.$$

For the limit, note that for fixed (z_1, z_2),

$$\frac{1}{2j}\log(1+|z_1^j+z_2^j|^2) \longrightarrow \max\{\log|z_1|, \log|z_2|, 0\} \quad\text{in } L^1.$$

As the same as Example 3.2, its MA measure concentrates where

$$|z_1| = |z_2| = 1 \quad \Longleftrightarrow \quad S^1 \times S^1,$$

and by $U(1)^2$–invariance it is the normalized Haar measure on $S^1 \times S^1$:

$$\boxed{(dd^c u)^2 = \text{normalized Lebesgue measure on } S^1 \times S^1.}$$

Remark 3.6. Example 3.5 (3) shows that $u_j \to u$ in L^1, while $0 = (dd^c u_j)^2 \nrightarrow (dd^c u)^2$. This is because $u \mapsto \mathrm{MA}(u)$ is discontinuous in L^1-topology. However, $u \mapsto \mathrm{MA}_\theta(u)$ is continuous for monotonic sequence (for $u_j \searrow u$, almost by definition; for $u_j \nearrow u$, by [BT82]).

Theorem 3.1 (Comparison principle)**.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and let $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$. Assume $\liminf_{z\to\xi}(u - v)(z) \geq 0$ for all $\xi \in \partial\Omega$. Then

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

In particular, if $(dd^c u)^n \leq (dd^c v)^n$ in Ω and $v \leq u$ on $\partial\Omega$, then $v \leq u$ in Ω.

Theorem 3.2 (Dirichlet problem)**.** Let $B := \{z \in \mathbb{C}^n : |z| < 1\}$ and assume $u \in \mathrm{PSH}(\bar{B}) \cap L^\infty(\bar{B})$. Then there exists a unique $v \in \mathrm{PSH}(B) \cap L^\infty(B)$ such that

(1) $\lim\limits_{B \ni z \to \xi} v(z) = u(\xi)$ for every $\xi \in \partial B$;

(2) $(dd^c v)^n = 0$ in B.

Now, let X be a compact complex manifold and let θ be a smooth, closed, real $(1, 1)$-form. Fix an open cover $\{U_\alpha\}$ and choose smooth local potentials ψ_α with $dd^c\psi_\alpha = \theta|_{U_\alpha}$. If $u \in \mathrm{PSH}(X, \theta)$ is bounded, then $u_\alpha := u + \psi_\alpha$ is a bounded psh function on U_α. By Bedford–Taylor theory, we can define on each chart

$$\mathrm{MA}_\theta(u)\big|_{U_\alpha} := (dd^c u_\alpha)^n.$$

On overlaps $U_\alpha \cap U_\beta$, we have $\psi_\beta = \psi_\alpha + h$ with $dd^c h = 0$, hence $dd^c u_\beta = dd^c u_\alpha$ and therefore $(dd^c u_\beta)^n = (dd^c u_\alpha)^n$. Thus these local measures glue to a global positive Radon measure on X, denoted

$$\mathrm{MA}_\theta(u) := (\theta + dd^c u)^n.$$

By multilinearity of the Bedford–Taylor product and the smoothness of θ, the binomial identity holds (in the sense of currents):

$$(\theta + dd^c u)^n = \sum_{k=0}^{n} \binom{n}{k} \theta^{n-k} \wedge (dd^c u)^k,$$

where $(dd^c u)^k$ is the Bedford–Taylor wedge product. Moreover, The BT construction also tells us: for $u_k \in \mathrm{PSH}(X, \theta) \cap L^\infty(X)$, the mixed MA measure

$$\theta_{u_1} \wedge \cdots \wedge \theta_{u_n}$$

is a well-defined probability measure.

Proposition 3.7. Let $u, v \in \mathrm{PSH}(X,\theta) \cap L^\infty(X)$. Then:

(1) (Locality) If $u = v$ on an open set $U \subset X$, then $\mathbf{1}_U\ \mathrm{MA}_\theta(u) = \mathbf{1}_U\ \mathrm{MA}_\theta(v)$.

(2) (Monotone convergence) If $u_j \searrow u$ is a *uniformly bounded* sequence in $\mathrm{PSH}(X,\theta)$, then

$$(\theta + dd^c u_j)^n \ \overset{*}{\rightharpoonup}\ (\theta + dd^c u)^n.$$

(3) (Cohomological mass) If $\theta_u := \theta + dd^c u \geq 0$ globally, then

$$\int_X (\theta + dd^c u)^n = \int_X \theta^n.$$

A direct consequence of Theorem 3.2 is:

Corollary 3.8. Let (X,ω) be a compact Kähler manifold, and let B_r be a ball with radius r small. Suppose $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, then there exists a unique $v_r \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, s.t.

(1) $v_r \equiv u$ in $X \setminus B_r$;

(2) $\omega_{v_r}^n = 0$ in B_r.

Moreover, $v_r \geq u$ and $v_r \searrow u$ as $r \downarrow 0$.

Proposition 3.9. Suppose $f \in C^\infty(X)$. Let

$$P_\omega(f) := \left(\sup\{u \in \mathrm{PSH}(X,\omega),\ u \leq f\}\right)^*.$$

Then $P_\omega(f) \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, and

$$\mathrm{MA}_\omega(P_\omega(f)) = \mathbf{1}_{\{P_\omega(f)=f\}}\,\mathrm{MA}_\omega(P_\omega(f))$$

Proof. We omit the subscript ω for convenience. Let $B_\epsilon \subset \{P(f) < f\}$ be a small ball, such that $\mathrm{MA}(P(f)) \neq 0$ on B_ϵ. Then by Corollary 3.8, there exists a unique v_ϵ, with $\mathrm{MA}(v_\epsilon) = 0$ in B_ϵ and $v_\epsilon = P(f)$ in $X \setminus B_\epsilon$. Since $v_\epsilon \searrow P(f)$ as $\epsilon \downarrow 0$ and $P(f) < f$ on B_ϵ, there exists $A > 0$ small enough, s.t. $v_\epsilon < f$ for $\epsilon < A$ in B_ϵ. Hence, v_ϵ is a candidate in the envelope and $v_\epsilon \leq P(f)$. Consequently, $v_\epsilon = P(f)$ and $\mathrm{MA}(P(f)) = \mathrm{MA}(v_\epsilon) = 0$ which is a contradiction. Our result then follows from Proposition 3.7 (1) directly. □

3.3. **The singular case: BEGZ's construction.** Let $u \in \mathrm{PSH}(X,\theta)$ be arbitrary (possibly singular). Following [BEGZ10], we define the *non-pluripolar Monge–Ampère measure* of u by truncation against the minimal singularity potential V_θ. For each $j \in \mathbb{N}$, set

$$u^{(j)} := \max\{u,\ V_\theta - j\} \in \mathrm{PSH}(X,\theta),$$

which is locally bounded on $\mathrm{Amp}(\theta)$. Then the Bedford–Taylor measure

$$(\theta + dd^c u^{(j)})^n$$

is well-defined on $\mathrm{Amp}(\theta)$. We extend this measure by zero to $X \setminus \mathrm{Amp}(\theta)$ and set

$$\mu_j\ :=\ \mathbf{1}_{\{u > V_\theta - j\}}\,(\theta + dd^c u^{(j)})^n.$$

By positivity and locality, $(\mu_j)_j$ is increasing in j, hence converges in the weak topology of measures. We define

$$\langle(\theta + dd^c u)^n\rangle \;:=\; \lim_{j\to\infty} \mu_j, \qquad \mathrm{MA}_\theta(u) \;:=\; \langle(\theta + dd^c u)^n\rangle. \tag{3.4}$$

Proposition 3.10 (Boucksom-Eyssidieux-Guedj-Zeriahi, [BEGZ10])**.** For every $u \in \mathrm{PSH}(X, \theta)$, the measure $\mathrm{MA}_\theta(u) = \langle(\theta + dd^c u)^n\rangle$ satisfies:

(1) (No mass on pluripolar sets) $\mathrm{MA}_\theta(u)$ does not charge pluripolar sets. In particular, it is supported in $\mathrm{Amp}(\theta)$.

(2) (Consistency) If u is locally bounded on an open $U \subset X$, then

$$\mathbf{1}_U \ \mathrm{MA}_\theta(u) \;=\; \mathbf{1}_U \, (\theta + dd^c u)^n$$

in the sense of Bedford–Taylor.

(3) (Monotone continuity) If $u_j \searrow u$ in $\mathrm{PSH}(X, \theta)$, then $\mathrm{MA}_\theta(u_j) \overset{*}{\rightharpoonup} \mathrm{MA}_\theta(u)$.

(4) (Mass bound and volume) One always has

$$0 \le \int_X \mathrm{MA}_\theta(u) \;\le\; \mathrm{Vol}([\theta]) \;:=\; \sup\left\{\int_X (\theta + dd^c\varphi)^n : \ \varphi \in \mathrm{PSH}(X, \theta),\ \varphi \le 0\right\}.$$

If u has minimal singularities (e.g. $[u] = [V_\theta]$), then equality holds:

$$\int_X \mathrm{MA}_\theta(u) = \mathrm{Vol}([\theta]).$$

The construction (3.4) is independent of the choice of the representative of $[\theta]$ and of the choice of V_θ. If u is smooth with $\theta_u := \theta + dd^c u \ge 0$, then $\mathrm{MA}_\theta(u) = \theta_u^n$. If u is locally bounded on $\mathrm{Amp}(\theta)$, then $\mathrm{MA}_\theta(u)$ coincides there with the Bedford–Taylor measure and vanishes on $X \setminus \mathrm{Amp}(\theta)$. From now on, we will write θ_u^n as the non-pluripolar MA measure of $u \in \mathrm{PSH}(X, \theta)$ directly. Most properties for MA measures in the BT sense are still true for no-pluripolar MA measures. The following mixed MA measure inequality was a generalization of [Din09, Theorem 1.3] from the BT sense to non-pluripolar sense.

Theorem 3.3 ([BEGZ10])**.** Let $T_1, ..., T_n$ be closed positive $(1,1)$-currents, let μ be a positive measure and assume given for each $j = 1, ..., n$ a non-negative measurable function f_j such that

$$T_j^n f_j\,.$$

Then we have

$$T_1 \wedge \cdots \wedge T_n \ge (f_1 \cdots f_n)^{1/n} \mu. \tag{3.5}$$

The non-pluripolar MA measure has an lsc property.

Theorem 3.4 (Darvas-Di Nezza-Lu, [DDNL18a])**.** Let (X, ω) be compact Kähler, θ a big class. Assume $u_j, u \in \mathrm{PSH}(X, \theta)$ with $u_j \searrow u$, $f_j \ge 0$ be quasi-continuous on X, locally uniformly bounded on $\mathrm{Amp}(\theta)$, and $f_j \searrow f$ uniformly on $\mathrm{Amp}(\theta)$. Then

$$\liminf_j \int_X f_j\, \theta^n_{u_j} \geq \int_X f\, \theta^n_u.$$

Moreover, if

$$\limsup_j \int_X \theta^n_{u_j} \leq \int_X \theta^n_u,$$

then $\theta^n_{u_j} \xrightharpoonup{*} \theta^n_u$ in the sense of measure.

Remark 3.11. This result is still valid if we replace $u_j \searrow u$ pointwisely with $u_j \to u$ in $\mathrm{Cap}_{\theta,\Phi}$[2].

An important property for the MA measure w.r.t θ-psh functions with different singularity types is that:

Theorem 3.5 (Witt Nyström, [Nys19])**.** Assume $u, v \in \mathrm{PSH}(X,\theta)$ with $[u] \leq [v]$, then

$$\int_X \theta^n_u \leq \int_X \theta^n_v. \tag{3.6}$$

Moreover, if $[u] = [v]$, then

$$\int_X \theta^n_u = \int_X \theta^n_v. \tag{3.7}$$

Remark 3.12. The comparison property (3.6) is still true if we replace θ^n_u by $\theta_{1,u_1} \wedge \cdots \wedge \theta_{n,u_n}$ for θ_i big and $u_i \in \mathrm{PSH}(X,\theta_i)$, see [DDNL18a].

Usually, the same mass of Monge-Ampère measure generally can not imply the same singularity type of potential θ-psh functions. For instance, consider θ is Kähler and $v = 0$, $u = -(-\log|z|)^\alpha$ for $0 < \alpha < 1$, then

$$\int_X \theta^n_u = \int_X \theta^n_v.$$

However, $u \nsim v$ since u is unbounded. To see that, fix $p \in X$. For $0 < \alpha < 1$ choose local holomorphic coordinates $z = (z_1, \ldots, z_n)$ on a ball $B := \{|z| < r_0\}$ centered at p and define

$$u(z) := -(-\log|z|)^\alpha \quad \text{on } B,$$

while extending u to a globally defined θ-psh function on X that is bounded on $X \setminus B$. After adding a constant, we may assume $u \leq 0$ on X.

Note that the Lelong number of u at p is zero. Since u is radial on B with a single pole at p, by the radial formula for the residual Monge–Ampère mass [Li20, Proposition A.1], we have

$$(dd^c u)^n(\{p\}) = [\nu(u,p)]^n = 0.$$

Put $u_j := \max\{u, -j\} \in \mathrm{PSH}(X,\theta) \cap L^\infty(X)$. By Bedford-Taylor theory and Stokes' theorem, for any j,

$$\int_X \theta^n_{u_j} = \int_X \theta^n.$$

[2] We will introduce the notion of capacity later.

By the BEGZ definition of the non-pluripolar Monge–Ampère measure,

$$\theta_u^n = \lim_{j\to\infty} \mathbf{1}_{\{u>-j\}}\theta_{u_j}^n.$$

Hence

$$\int_X \theta_u^n = \int_X \theta^n - \lim_{j\to\infty}\int_{\{u\le -j\}} \theta_{u_j}^n = \int_X \theta^n.$$

Remark 3.13 (Logarithmic singularity fails)**.** If $u = \lambda \log|z|$ near p with $\lambda > 0$, then $\nu(u,p) = \lambda$ and $(dd^c u)^n(\{p\}) = \lambda^n > 0$ by [Li20, Proposition A.1] again. Hence the above limiting procedure loses λ^n units of mass at p, and $\int_X \theta_u^n = \int_X \theta^n - \lambda^n < \int_X \theta^n$.

Lemma 3.14. Let $u, v \in \mathrm{PSH}(X,\theta)$. Then

$$\theta^n_{\max\{u,v\}} \ge \mathbf{1}_{\{u\ge v\}}\theta_u^n + \mathbf{1}_{\{u<v\}}\theta_v^n. \tag{3.8}$$

In particular, if $u \le v$, then

$$\mathbf{1}_{\{u=v\}}\theta_u^n \le \mathbf{1}_{\{u=v\}}\theta_v^n. \tag{3.9}$$

Proof. The only difficult part is to understanding the contact set $\{u = v\}$. Define

$$u_j = \max\{u, V_\theta - j\}, \qquad v_j = \max\{v, V_\theta - j\}.$$

For any $\varepsilon > 0$, we have

$$\begin{aligned}\theta^n_{\max\{v_j,u_j+\varepsilon\}} &\ge \mathbf{1}_{\{v_j>u_j+\varepsilon\}}\theta^n_{\max\{v_j,u_j+\varepsilon\}} + \mathbf{1}_{\{v_j<u_j+\varepsilon\}}\theta^n_{\max\{v_j,u_j+\varepsilon\}}\\ &= \mathbf{1}_{\{v_j>u_j+\varepsilon\}}\theta^n_{v_j} + \mathbf{1}_{\{v_j<u_j+\varepsilon\}}\theta^n_{u_j}.\end{aligned}$$

Since $\max\{v_j, u_j+\varepsilon\} \searrow \max\{v_j,u_j\} \in \mathrm{PSH}(X,\theta)$, we have $\theta^n_{\max\{v_j,u_j+\varepsilon\}} \xrightarrow{*} \theta^n_{\max\{v_j,u_j\}}$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$, we obtain

$$\theta^n_{\max\{v_j,u_j\}} \ge \mathbf{1}_{\{v_j>u_j\}}\theta^n_{v_j} + \mathbf{1}_{\{v_j\le u_j\}}\theta^n_{u_j}.$$

By multiplying both side with $\mathbf{1}_{\{\min\{u,v\}>V_\theta-j\}}$, we see that

$$\begin{aligned}\mathbf{1}_{\{\min\{u,v\}>V_\theta-j\}}\theta^n_{\max\{v_j,u_j\}} =&\mathbf{1}_{\{\min\{u,v\}>V_\theta-j\}}\theta^n_{\max\{v,u\}}\\ \ge&\mathbf{1}_{\{v_j>u_j\}\cap\{\min\{u,v\}>V_\theta-j\}}\theta^n_{v_j} + \mathbf{1}_{\{v_j\le u_j\}\cap\{\min\{u,v\}>V_\theta-j\}}\theta^n_{u_j}\\ \ge&\mathbf{1}_{\{v>u\}\cap\{\min\{u,v\}>V_\theta-j\}}\theta^n_{v} + \mathbf{1}_{\{v\le u\}\cap\{\min\{u,v\}>V_\theta-j\}}\theta^n_{u}.\end{aligned}$$

Letting $j \to \infty$, we obtain the desired inequality. □

We also have the following domination principle.

Theorem 3.6 (Darvas-Di Nezza-Lu, [DDNL18b])**.** Let $\Phi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_\Phi^n > 0$. Suppose $u, v \in \mathrm{PSH}(X,\theta)$ with $[u] \le [\Phi]$, $[v] \le [\Phi]$, and

$$\int_X \theta_u^n = \int_X \theta_v^n = \int_X \theta_\Phi^n.$$

If $\theta_u^n(\{u < v\}) = 0$, then $u \ge v$ on X.

4. The θ-psh envelope

4.1. **Singularity types of θ-psh functions and P-envelope.** Let $[\theta]$ be big, then there are plenty of θ-psh functions (even many of them have analytic singularities!).

Define the *canonical potential with minimal singularities* by

$$V_\theta := (\sup\{u \in \mathrm{PSH}(X,\theta),\ u \leq 0\})^*$$

When θ is Kähler, we have $0 \in \mathrm{PSH}(X,\theta)$. Hence $V_\theta \equiv 0$. Otherwise, $V_\theta \leq 0$ is usually regard as the "best" θ-psh function.

Proposition 4.1. Let $[\theta]$ be big and let V_θ be defined above, then

(1) V_θ has minimal singularities, i.e. $\forall\, u \in \mathrm{PSH}(X,\theta)$, $[u] \leq [V_\theta]$.

(2) V_θ is locally bounded on $\mathrm{Amp}([\theta])$.

Proof. Since θ-psh is bounded from above on X, for any $u \in \mathrm{PSH}(X,\theta)$, let $\tilde{u} = u - \sup_X u \in \mathrm{PSH}(X,\theta)$. Then $\tilde{u} \leq 0$ is a candidate of V_θ, and $\tilde{u} \leq V_\theta$. Thus, $u \leq V_\theta + \sup_X u$ implies $[u] \leq [V_\theta]$.

For $x \in \mathrm{Amp}([\theta])$, there is $\psi \in \mathrm{PSH}(X,\theta)$, which is smooth in $U \ni x$, and $[\psi] \leq [V_\theta]$. Hence, there exists $C > 0$, s.t.

$$\psi \leq V_\theta + C.$$

This yields V_θ is bounded from below on U. □

Let $f : X \to [-\infty, +\infty]$ which is not identically $\pm\infty$. Define the *rooftop θ-psh envelope of f* by

$$P_\theta(f) := (\sup\{u \in \mathrm{PSH}(X,\theta),\ u \leq f\})^*. \tag{4.1}$$ [3]

Clear, $V_\theta = P_\theta(0)$, $P_\theta(f) \in \mathrm{PSH}(X,\theta)$, and for any $f_1 \leq f_2$ we have $P_\theta(f_1) \leq P_\theta(f_2)$. Let c be any constant, we also have $P_\theta(f + c) = P_\theta(f) + c$.

Proposition 4.2. If f is bounded, then $[P_\theta(f)] = [V_\theta]$.

Proof. Since $-c \leq f \leq c$ for some $c > 0$, note that,

$$\begin{aligned} V_\theta - c = P_\theta(0) - c = P_\theta(-c) \leq P_\theta(f) \\ \leq P_\theta(c) = P_\theta(0) + c = V_\theta + c, \end{aligned}$$

which completes the proof. □

Proposition 4.3 ([DDNL18a]). Let $u, v \in \mathrm{PSH}(X,\theta)$. If $P_\theta(\min\{u,v\}) \not\equiv -\infty$, then

$$\theta^n_{P_\theta(\min\{u,v\})} \leq \mathbf{1}_{\{P_\theta(\min\{u,v\})=u\}}\theta^n_u + \mathbf{1}_{\{P_\theta(\min\{u,v\})=v\}}\theta^n_v. \tag{4.2}$$

[3] If we replace $u \leq f$ in (4.1) by $u \leq f$ q.e., the obtained envelope are same.

Proposition 4.4 ([DDNL18a])**.** Let $u, v \in \mathrm{PSH}(X, \theta)$. If

$$\theta_u^n \le \mu, \qquad \theta_v^n \le \mu,$$

for some Borel measure μ. Then

$$\theta^n_{P_\theta(\min\{u,v\})} \le \mu. \tag{4.3}$$

Proof. By replacing μ with $\mathbf{1}_{X\setminus P}\,\mu$, where $P = \{u = v = -\infty\}$, we can assume that $\mu(P) = 0$. Since $\mu(X) < +\infty$, the function $r \to \mu(\{u \le v + r\})$ is monotone increasing. Monotonic functions have at most countable discontinuous points, and hence for almost every $r \ge 0$ we have $\mu(\{u = v + r\}) = 0$. We set $\varphi_r := P_\theta(\min\{u, v + r\})$, and note that $\varphi_r \searrow P_\theta(\min\{u, v\})$ as $r \to 0$. It then follows from Proposition 4.3 that,

$$\theta^n_{\varphi_r} \le \mathbf{1}_{\{\varphi_r=u\}}\theta^n_u + \mathbf{1}_{\{\varphi_r=v+r\}}\theta^n_v \le \left(\mathbf{1}_{\{\varphi_r=u\}} + \mathbf{1}_{\{\varphi_r=v+r\}}\right)\mu \le \mu,$$

where in the last inequality we used the fact that $\mu(\{u = v + r\}) = 0$. Letting $r \searrow 0$, we arrive at the conclusion. □

For any $u, v \in \mathrm{PSH}(X, \theta)$, we define the *$\theta$-psh envelope with prescribed singularities* (relative to u) by

$$P_\theta[u](v) := \left(\lim_{c\to+\infty} P_\theta\left(\min\{u + c, v\}\right)\right)^*. \tag{4.4}$$

Or equivalently, we have

$$P_\theta[u](v) := \left(\sup\{\psi \in \mathrm{PSH}(X, \theta),\ \psi \le v,\ [\psi] \le [u]\}\right)^*. \tag{4.5}$$

In particular, set

$$\mathcal{F} := \{\psi \in \mathrm{PSH}(X, \theta),\ \psi \le v,\ [\psi] \le [u]\},$$

and for $c \in \mathbb{R}$ write

$$E_c := P_\theta\left(\min\{u + c, v\}\right).$$

Note that $c \mapsto \min\{u + c, v\}$ is non-decreasing, hence $c \mapsto E_c$ is non-decreasing as well. Hence $\lim_{c\to\infty} E_c = \sup_c E_c$ pointwisely.

For each c, by definition of the envelope, we have

$$E_c \le \min\{u + c,\ v\},$$

hence $E_c \le v$ and $E_c \le u + c$. Therefore $E_c \in \mathcal{F}$, and

$$E_c \le \sup_{\psi\in\mathcal{F}} \psi.$$

Taking the limit in c and then usc regularization yields

$$\left(\lim_{c\to+\infty} P_\theta\left(\min\{u + c, v\}\right)\right)^* = \left(\lim_{c\to\infty} E_c\right)^* \le \left(\sup_{\psi\in\mathcal{F}} \psi\right)^* = P_\theta[u](v).$$

Fix $\psi \in \mathcal{F}$ and choose $C \in \mathbb{R}$ such that $\psi \le u + C$. Since also $\psi \le v$, we have

$$\psi \le \min\{u + C,\ v\}.$$

So ψ is a candidate for the envelope, and

$$\psi \leq P_\theta\left(\min\{u + C, v\}\right) = E_C \;\leq\; \sup_c E_c.$$

Taking the supremum over all $\psi \in \mathcal{F}$ and then usc regularization gives

$$P_\theta[u](v) = \left(\sup_{\psi\in\mathcal{F}} \psi\right)^* \;\leq\; \left(\sup_c E_c\right)^* = \left(\lim_{c\to\infty} E_c\right)^* = \left(\lim_{c\to+\infty} P_\theta\left(\min\{u + c, v\}\right)\right)^*.$$

Proposition 4.5. We have the following properties for $P_\theta[u](v)$:

(1) $P_\theta[u](v) \leq v$;

(2) $P_\theta[u](v) = P_\theta[u](P_\theta(v))$;

(3) If $v_1 \leq v_2$, then $P_\theta[u](v_1) \leq P_\theta[u](v_2)$;

(4) If $[u_1] \leq [u_2]$, then $P_\theta[u_1](v) \leq P_\theta[u_2](v)$;

(5) $P_\theta[u + A](v + A) = P_\theta[u](v) + A$ for $A \in \mathbb{R}$;

(6) If $[v] \leq [u]$, then $P_\theta[u](v) = v$;

Proof. Baby-level exercise. □

Remark 4.6. We are primarily interested in the case where $v = V_\theta$, namely

$$P_\theta[u](V_\theta) = \left(\sup\left\{v \in \mathrm{PSH}(X,\theta) \;\middle|\; v \leq V_\theta \leq 0,\ [v] \leq [u]\right\}\right)^*. \tag{4.6}$$

In this situation, we refer to it simply as the P-envelope and denote it by $P[u]$ for convenience. In particular, for any $c \in \mathbb{R}$, we have

$$P_\theta(\min\{u + c, V_\theta\}) \in \mathrm{PSH}(X,\theta) \quad \text{and} \quad [P_\theta(\min\{u + c, V_\theta\})] = [u].$$

In other words, the condition $[v] \leq [u]$ in (4.6) may be replaced by $[v] = [u]$.

Since $u - \sup_X u \leq 0$ is a candidate of the P-envelope, we have $[u] \leq [P[u]]$. Generally, the equality does not hold.

Definition 4.7. We say $u \in \mathrm{PSH}(X,\theta)$ has model type singularity if $[u] = [P[u]]$.

4.2. **Non-pluripolar MA measures for the envelope.** The following result is a generalization of Proposition 3.9, which will be used many times in the sequel. In particular, the mass of $\theta^n_{P_\theta(f)}$ is concentrated on the contact set $\{P_\theta(f) = f\}$.

Theorem 4.1. Let $f : X \to [-\infty, -\infty]$ be quasi-continuous[4], which is not identically $\pm\infty$. Then $P_\theta(f) \in \mathrm{PSH}(X,\theta)$ and

$$\int_{\{P_\theta(f)<f\}} \theta^n_{P_\theta(f)} = 0. \tag{4.7}$$

[4]Quasi-continuity will be defined in Proposition 5.3 after we introduce the notion of capacity. For now, one may think of such a function as continuous outside sets of arbitrarily small capacity.

Lemma 4.8. Let $u, v \in \mathrm{PSH}(X, \theta)$. Then

$$\theta^n_{P_\theta(\min\{u,v\})} \leq \mathbf{1}_{\{P_\theta(\min\{u,v\}=u\}}\theta^n_u + \mathbf{1}_{\{P_\theta(\min\{u,v\}=v\}}\theta^n_v. \tag{4.8}$$

Proof. It follows from Lemma 3.14 that,

$$\mathbf{1}_{\{P_\theta(\min\{u,v\}=u\}}\theta^n_{P_\theta(\min\{u,v\})} \leq \mathbf{1}_{\{P_\theta(\min\{u,v\}=u\}}\theta^n_u, \quad \mathbf{1}_{\{P_\theta(\min\{u,v\}=v\}}\theta^n_{P_\theta(\min\{u,v\})} \leq \mathbf{1}_{\{P_\theta(\min\{u,v\}=v\}}\theta^n_v.$$

Thus, by Theorem 4.1, we have

$$\begin{aligned}\theta^n_{P_\theta(\min\{u,v\})} \leq & \mathbf{1}_{\{P_\theta(\min\{u,v\}=u\}}\theta^n_{P_\theta(\min\{u,v\})} + \mathbf{1}_{\{P_\theta(\min\{u,v\}=v\}}\theta^n_{P_\theta(\min\{u,v\})} \\ \leq & \mathbf{1}_{\{P_\theta(\min\{u,v\}=u\}}\theta^n_u + \mathbf{1}_{\{P_\theta(\min\{u,v\}=v\}}\theta^n_v,\end{aligned}$$

which concludes (4.8) □

The P-envelope preserves the non-pluripolar masses:

Lemma 4.9. Let θ be big and let $u \in \mathrm{PSH}(X, \theta)$. Then

$$\int_X \theta^n_{P[u]} = \int_X \theta^n_u. \tag{4.9}$$

Proof. Set

$$P[u] := \big(\sup_{c>0} u_c\big)^*, \quad \text{where} \quad u_c := P_\theta(\min\{u + c, 0\}) \in \mathrm{PSH}(X, \theta).$$

Then $(u_c)_{c>0}$ is non-decreasing (since $c \mapsto \min\{u + c, 0\}$ is non-decreasing), and by definition

$$P[u] = \lim_{c\to+\infty} u_c = \big(\sup_{c>0} u_c\big)^*.$$

By Theorem 3.5 we have

$$\limsup_{c\to+\infty} \int_X \theta^n_{u_c} \leq \int_X \theta^n_{P[u]}.$$

On the other hand, by Theorem 3.4 we have the weak convergence

$$\theta^n_{u_c} \xrightarrow{*} \theta^n_{P[u]} \quad \text{as } c \to +\infty,$$

hence

$$\lim_{c\to+\infty} \int_X \theta^n_{u_c} = \int_X \theta^n_{P[u]}.$$

We now show that $\int_X \theta^n_{u_c} = \int_X \theta^n_u$ for every $c > 0$. By construction $u_c \leq \min\{u + c, 0\} \leq u + c$. Choose $A \gg 1$ so that $u - A \leq 0$. Then

$$u - A \leq \min\{u + c, 0\},$$

hence $u - A$ is a candidate in the envelope and thus $u - A \leq u_c$. Consequently,

$$u - A \leq u_c \leq u + c,$$

and $[u] = [u_c]$. Applying Theorem 3.5 yields

$$\int_X \theta^n_{u_c} = \int_X \theta^n_u \qquad \text{for all } c > 0.$$

Passing to the limit $c \to +\infty$ gives (4.9). □

The MA measure of the P-envelope concentrates on the contact set:

Theorem 4.2 ([DNT21]). Let θ be big and let $u \in \mathrm{PSH}(X,\theta)$. Then

$$\theta^n_{P[u]} \leq \mathbf{1}_{\{P[u]=0\}}\theta^n \tag{4.10}$$

Proof. It follows from (4.8) that,

$$\begin{aligned}\theta^n_{u_c} \leq & \mathbf{1}_{\{u_c=u+c\}}\theta^n_u + \mathbf{1}_{\{u_c=V_\theta\}}\theta^n_{V_\theta}\\ \leq & \mathbf{1}_{\{u+c\leq V_\theta\}}\theta^n_u + \mathbf{1}_{\{P[u]=V_\theta\}}\theta^n_{V_\theta}.\end{aligned}$$

We want to let $c \to +\infty$. In particular, we note that the first term of the right-hand side converges to 0[5] as $c \to +\infty$ by the dominant convergence theorem, while the left-hand side converges to $\theta^n_{P[u]}$. Consequently, we have

$$\theta^n_{P[u]} \leq \mathbf{1}_{\{P[u]=V_\theta\}}\theta^n_{V_\theta} \leq \mathbf{1}_{\{P[u]=0\}}\theta^n,$$

where the last step comes from Proposition 3.10 (4) and Theorem 4.1. □

4.3. **The ceiling operator.** It is natural to ask: *for a given mass* $0 < M \leq \mathrm{Vol}([\theta])$*, can we find the least singular* $u \in \mathrm{PSH}(X,\theta)$*, with* $\int_X \theta^n_u = M$*?*

We define the ceiling operator:

$$C(u) := \sup\left\{v \in \mathrm{PSH}(X,\theta),\ v \leq 0,\ [u] \leq [v],\ \int_X \theta^n_v = \int_X \theta^n_u\right\}$$

be the envelope of θ-psh functions has the same (positive) mass as u but no more singular than u. It is clearly that, $C(u) \in \mathrm{PSH}(X,\theta)$, $C(u) \leq 0$ and $[u] \leq [C(u)]$.

Proposition 4.10. Let $u \in \mathrm{PSH}(X,\theta)$. Then

(1) $C(C(u)) = C(u)$;

(2) $C(u) = P[u]$;

(3) $\int_X \theta^n_{C(u)} = \int_X \theta^n_u$.

Proof. If we can prove $P[u] = C(u)$, thanks to (4.9), we are done with point (3). After subtracting a large constant if necessary, we may assume $u \leq 0$. In particular $u_c \leq 0$ for all $c > 0$, hence $P[u] \leq 0$. Thus $P[u]$ is an candidate for $C(u)$ and we have

$$P[u] \leq C(u).$$

For the reverse direction, let v be a candidate of the ceiling operator, then

$$\int_{\{P[u]<v\}} \theta^n_{P[u]} \leq \int_{\{P[u]<v\}\cap\{P[u]=0\}} \theta^n = 0.$$

[5]The set $\{u + c \leq V_\theta\}$ converges to the polar set of u, which has 0 mass w.r.t non-pluripolar MA measure.

Note that we have

$$[P[u]] \leq [P[v]], \quad [v] \leq [P[v]].$$

Moreover, all of these potentials have the same mass, thanks to (4.9). Then by the domination principle (Theorem 3.6), we find

$$P[u] \geq v.$$

Hence, we obtain $P[u] = C(u)$, which completes the proof of (2) and (3).

For (1), we prove $P[P[u]] = P[u]$. First, observe that $P[P[u]] \geq P[u]$ is trivial by definition. For the reverse inequality, denote

$$\mathcal{F}_u = \left\{ v \in \mathrm{PSH}(X, \theta) : v \leq 0,\ [u] \leq [v], \int_X \theta_v^n = \int_X \theta_u^n \right\}$$

. Then we have

$$P[u] = \sup \mathcal{F}_u \geq \sup \mathcal{F}_{P[u]} \geq P[P[u]],$$

since $\mathcal{F}_{P[u]} \subset \mathcal{F}_u$. □

Remark 4.11. In fact, we do not know if $P[P[u]] = P[u]$ if u has null mass. According to the general philosophy, the P-envelope operator is the correct object only when the non-pluripolar mass is positive. We will only consider the positive mass in this note.

Definition 4.12 (Model potential)**.** Let $\Phi \in \mathrm{PSH}(X, \theta)$. If $P[\Phi] = \Phi$ and $\int_X \theta_\Phi^n > 0$, we call Φ a *model potential*.

Example 4.13. There are plenty of model potentials:

(1) The canonical potential V_θ;

(2) $P[u]$ for $u \in \mathrm{PSH}(X, \theta)$ with positive mass;

(3) $\Phi \in \mathrm{PSH}(X, \theta$ with analytic singularities.

4.4. **Relative Full mass classes.** Let θ be big and let Φ be a model potential.

Definition 4.14. We define

$$\mathcal{E}(X, \theta; \Phi) := \left\{ u \in \mathrm{PSH}(X, \theta) : \ [u] \leq [\Phi], \int_X \theta_u^n = \int_X \theta_\Phi^n \right\}$$

$$\mathcal{E}^1(X, \theta; \Phi) := \left\{ \eta \in \mathcal{E}(X, \theta; \Phi) : \int_X |\Phi - u|\, \theta_u^n < \infty \right\}$$

$$\mathcal{E}^\infty(X, \theta; \Phi) := \{ u \in \mathrm{PSH}(X, \theta) : [u] = [\Phi] \}.$$

Potentials in three classes are said to have *full mass*, *finite energy*, and *minimal singularities* relative to Φ, respectively.

Remark 4.15. (1) Note that

$$\mathcal{E}^{\infty}(X,\theta;\Phi) \subset \mathcal{E}^{1}(X,\theta;\Phi) \subset \mathcal{E}(X,\theta;\Phi),$$

where the first inclusion comes from (3.7) in Theorem 3.5.

(2) Since the non-pluripolar MA measure does not charge pluripolar sets, the integral

$$\int_X |\Phi - u|\, \theta_u^n$$

is well-defined, while the difference $|\Phi - u|$ is not defined on polar sets of u and Φ.

When $\Phi = V_\theta$, we simply write

$$\begin{aligned}\mathcal{E}\,(X,\theta;V_\theta) &= \mathcal{E}(X,\theta),\\ \mathcal{E}^1\,(X,\theta;V_\theta) &= \mathcal{E}^1(X,\theta),\\ \mathcal{E}^{\infty}\,(X,\theta;V_\theta) &= \mathcal{E}^{\infty}(X,\theta).\end{aligned}$$

The P-envelope can be used to characterize the relative full mass classes:

Theorem 4.3. [DDNL18a] Let $u \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:

- $u \in \mathcal{E}(X,\theta;\Phi)$;
- $[\Phi] \leq [u]$, and $P_\theta[u](\Phi) = \Phi$.
- $[\Phi] \leq [u]$, and $P[u] = P[\Phi]$

Corollary 4.16. Let $u \in \mathrm{PSH}(X,\theta)$, and let Φ be a model potential. Then the following are equivalent:

- $u \in \mathcal{E}(X,\theta;\Phi)$;
- $P[u] = \Phi$.

Theorem 4.17. Let θ be big, and let $v \in \mathrm{PSH}(X,\theta)$ be a model potential. If $u \in \mathcal{E}(X,\theta;v)$, then for any $x \in X$,

$$\boxed{\nu(u,x) = \nu(v,x).}$$

In particular,

- if $\theta > 0$ is Kähler and $v = 0$, Theorem 4.17 was proven in [GZ05];
- if θ is semi-positive and big, $v = 0$, Theorem 4.17 was proven in [BEGZ10];
- if θ is big and v is a model potential, Theorem 4.17 was proven in [DDNL18b].

Here we give a brief proof when θ is big and $v = V_\theta$. First observe that $[u] \leq [v]$, and this gives $\nu(u,x) \geq \nu(v,x)$ for any $x \in X$. Let $v = V_\theta$, write $\gamma = \nu(u,x)$, we want to show $\nu(V_\theta,x) \geq \gamma$.

Let U be a neighborhood of x, s.t. $\theta = dd^c\rho$ and $\rho << 0$ in U, then we have $u + \rho < 0$ is psh in U. Then

$$V_\theta + \rho = P[u] + \rho \leq \sup\{\varphi \in \mathrm{PSH}(U),\ \varphi \leq 0,\ \varphi \leq \gamma \log|z| + A\},$$

thanks to $[P[u]] = [u]$. Hence,

$$V - \theta + \rho \leq \gamma \log|z| + A,$$

which yields $\nu(V_\theta, x) \geq \gamma$.

Example 4.18. Let $\Phi = V_\theta$. Fix $x_0 \in X$ and a local holomorphic coordinate z centered at x_0. Consider the local models

$$u_\alpha(z) := -(-\log|z|)^\alpha \quad (0 < \alpha < 1), \qquad v(z) := \log|z|.$$

Then the Lelong numbers satisfy

$$\nu(u_\alpha, x_0) = 0, \qquad \nu(v, x_0) = 1.$$

Consequently, for their global θ-psh extensions (obtained by patching with a smooth θ-psh potential outside a small ball), the non-pluripolar Monge–Ampère masses obey

$$\int_X \mathrm{MA}_\theta(u_\alpha) = \int_X \mathrm{MA}_\theta(V_\theta) \quad \text{(full mass)}, \qquad \int_X \mathrm{MA}_\theta(v) < \int_X \mathrm{MA}_\theta(V_\theta) \quad \text{(not full mass)}.$$

5. Relative capacity

5.1. **The notion of capacity.** Let (X, ω) be a compact Kähler manifold, and let $[\theta]$ be a big class. Suppose $E \subset X$ is a Borel set, then the capacity of E is defined by:

$$\mathrm{Cap}_\theta(E) := \sup_u \left\{ \int_E \theta_u^n,\ u \in \mathrm{PSH}(X, \theta),\ V_\theta - 1 \leq u \leq V_\theta \right\}.$$

The capacity is a function that sends Borel subsets of X to a non-negative number. However, the capacity is not a measure due to the lack of additivity.

Proposition 5.1. We have the following properties for the capacity:

(1) If $E_1 \subset E_2$ are both Borel, then $\mathrm{Cap}_\theta(E_1) \leq \mathrm{Cap}_\theta(E_2)$;

(2) $\mathrm{Cap}_\theta(X) = \mathrm{Vol}([\theta])$;

(3) For $E \subset X$ Borel, $\int_E \theta_{V_\theta}^n \leq \mathrm{Cap}_\theta(E)$;

(4) ([Lu21]) Let $[\theta_1], [\theta_2]$ be big, and $E \subset X$ Borel. Then there exists $f, g : [0, \infty) \to [0, \infty)$ continuous with $f(0) = g(0) = 0$, such that

$$\mathrm{Cap}_{\theta_1}(E) \leq f\left(\mathrm{Cap}_{\theta_2}(E)\right), \quad \text{and} \quad \mathrm{Cap}_{\theta_2}(E) \leq g\left(\mathrm{Cap}_{\theta_1}(E)\right).$$

Proposition 5.2 ([BEGZ10]). Fix $p > 1$ and $f \in L^p(X, \omega^n)$, $f \geq 0$. Then there exists a constant $C = C(X, \omega, \theta, p) > 0$ such that for all Borel sets $E \subset X$,

$$\int_E f\,\omega^n \ \leq\ C\,\|f\|_{L^p(X,\omega^n)}\ \mathrm{Cap}_\theta(E)^2. \tag{5.1}$$

In particular, the measure $f\,\omega^n$ is dominated by the square of the θ-capacity:

$$f\,\omega^n \;\le\; C\,\|f\|_{L^p}\;\mathrm{Cap}_\theta^2\,.$$

Proposition 5.3 (Quasi-continuous)**.** Suppose $u \in \mathrm{PSH}(X,\theta)$, then for any $\varepsilon > 0$, there is $U \subset X$ open with $\mathrm{Cap}_\theta(U) < \varepsilon$, s.t. u is continuous in $X \setminus U$.

Proposition 5.4. Let $P \subset X$. Then P is pluripolar if and only if $\mathrm{Cap}_\theta(P) = 0$.

The notion of capacity is the right one when working with the MA measure. We need to know that the capacity does not distinguish between "big" sets.

Let θ be Kähler. Assume D is a Cartier divisor such that $[D] = k[\theta]$ for some $k \in \mathbb{Z}_{>0}$. Choose a smooth Hermitian metric h_0 on $\mathcal{O}(D)$ whose curvature form is positive and represents $c_1(\mathcal{O}(D)) = k[\theta]$. Let s_D be the canonical section with $\mathrm{div}(s_D) = D$, and set

$$\varphi \;:=\; \frac{1}{k}\log\|s_D\|_{h_0}.$$

After multiplying h_0 by a positive constant, we may assume $\varphi \le 0$ on X. By the Lelong–Poincaré formula,

$$dd^c\log\|s_D\|_{h_0} \;=\; [D] - k\theta, \qquad \text{hence} \qquad k(\theta + dd^c\varphi) \;=\; [D] > 0.$$

In particular, $\varphi \in \mathrm{PSH}(X,\theta)$, $\varphi \in C^\infty(X\setminus D)$, and $\{\varphi = -\infty\} = D$.

Define $\varphi_j := \max\{\varphi, -j\} \in \mathrm{PSH}(X,\theta)$ and $V_j := \{\varphi < -j\}$. Then $\varphi_1 \in \mathrm{PSH}(X,\theta)\cap L^\infty(X)$ with $-1 \le \varphi_1 \le 0$, hence (by the bounded case)

$$\int_X \theta_{\varphi_1}^n \;=\; \int_X \theta^n \;=\; \mathrm{Cap}_\theta(X).$$

Moreover, we have

$$\mathbf{1}_{V_1}\,\theta_{\varphi_1}^n \;=\; 0,$$

since $\theta_{\varphi_1} = \theta_\varphi = \frac{1}{k}[D] = 0$ in $X \setminus V_1$. Therefore,

$$\mathrm{Cap}_\theta(X) \;=\; \int_X \theta_{\varphi_1}^n \;=\; \int_{X\setminus V_1} \theta_{\varphi_1}^n \;\le\; \mathrm{Cap}_\theta(X\setminus V_1) \;\le\; \mathrm{Cap}_\theta(X),$$

and consequently

$$\mathrm{Cap}_\theta(X\setminus V_1) \;=\; \mathrm{Cap}_\theta(X).$$

Thus, removing a small tubular neighborhood of a divisor does not change the capacity. In particular, the capacity does not distinguish between certain large (proper) subsets of X.

Let Φ be a model potential, then the relative capacity of a Borel subset E is defined by:

$$\mathrm{Cap}_{\theta,\Phi}(E) := \sup\left\{\int_E \theta_u^n\,;\; u \in \mathrm{PSH}(X,\theta),\ \Phi - 1 \le u \le \Phi\right\}.$$

In the case, $\Phi = V_\theta$, the relative capacity recover the classical capacity.

The relative capacity has the same properties of the classical one. More preciously,

(1) If $E_1 \subset E_2 \subset X$, then

$$\mathrm{Cap}_{\theta,\Phi}(E_1) \leq \mathrm{Cap}_{\theta,\Phi}(E_2);$$

(2) A subset $P \subset X$ has 0 relative capacity if and only lf P is pluripolar;

(3) The non-pluripolar MA mass of any $E \subset X$ is dominated by the relative capacity, that is

$$\int_E \theta_\Phi^n \leq \mathrm{Cap}_{\theta,\Phi}(E).$$

In particular, the relative capacity is also inner regular, that is:

$$\mathrm{Cap}_{\theta,\Phi}(E) = \sup\{\mathrm{Cap}_{\theta,\Phi}(K) : \ K \subset E, \ K \text{ is compact}\}. \tag{5.2}$$

To see this, fix $\varepsilon > 0$, there is $u \in \mathrm{PSH}(X,\theta)$, s.t. $\Phi - 1 \leq u \leq \Phi$ and

$$\int_E \theta_\Phi^n \geq \mathrm{Cap}_{\theta,\Phi}(E) - \varepsilon.$$

Since θ_u^n is an inner regular Borel measure, there exists $K \subset E$ compact, such that

$$\int_K \theta_u^n \geq \int_E \theta_u^n - \varepsilon \geq \mathrm{Cap}_{\theta,\Phi}(E) - 2\varepsilon.$$

Hence, we have

$$\mathrm{Cap}_{\theta,\Phi}(K) \geq \mathrm{Cap}_{\theta,\Phi}(E) - 2\varepsilon.$$

Taking the supremum over all compact subsets $K \subset E$, we arrive at the conclusion.

Theorem 5.1 (Comparison principle: I)**.** Let $u, v \in \mathrm{PSH}(X,\theta)$, s.t. $[v] \leq [P[u]]$. Then

$$\int_{\{u<v\}} \theta_v^n \leq \int_{\{u<v\}} \theta_u^n. \tag{5.3}$$

Proof. **Step 1**: Let $u, v \in \mathcal{E}(X,\theta;\Phi)$. Note that $\max\{u,v\} \in \mathcal{E}(X,\theta;\Phi)$, since $\max\{u,v\} \in \mathrm{PSH}(X,\theta)$, $[\max\{u,v\}] \leq [\Phi]$, and

$$\int_X \theta_\Phi^n = \int_X \theta_u^n \leq \int_X \theta_{\max\{u,v\}}^n \leq \int_X \theta_\Phi^n.$$

Therefore, we have

$$\begin{aligned}\int_X \theta_u^n = \int_X \theta_{\max\{u,v\}}^n &\geq \int_{\{u<v\}} \theta_{\max\{u,v\}}^n + \int_{\{u>v\}} \theta_{\max\{u,v\}}^n \\ &= \int_{\{u<v\}} \theta_v^n + \int_{\{u>v\}} \theta_u^n \\ &= \int_{\{u<v\}} \theta_v^n + \int_X \theta_u^n - \int_{\{u\leq v\}} \theta_u^n,\end{aligned}$$

which gives

$$\int_{\{u<v\}} \theta_v^n \leq \int_{\{u\leq v\}} \theta_u^n.$$

Replacing u by $u + \varepsilon$ and letting $\varepsilon \to 0$, we arrive at (5.3) within the class $\mathcal{E}(X,\theta;\Phi)$.

Step 2: Writing $\varphi = \max\{u, v\}$. We observe that $u, \varphi \in \mathcal{E}(X, \theta; P[u])$: Clearly, $u \in \mathcal{E}(X, \theta; P[u])$. Since $[u], [v] \le [P[u]]$, we have $[\varphi] \le [P[u]]$. Moreover,

$$\int_X \theta_u^n \le \int_X \theta_\varphi^n \le \int_X \theta_{P[u]}^n = \int_X \theta_u^n,$$

which implies $\varphi \in \mathcal{E}(X, \theta; P[u])$. We then can apply comparison principle within the class $\mathcal{E}(X, \theta; P[u])$ to u and φ,

$$\begin{aligned}\int_{\{u<v\}} \theta_v^n = \int_{\{u<v\}} \theta_\varphi^n &= \int_{\{u<\varphi\}} \theta_\varphi^n \\ &\le \int_{\{u<\varphi\}} \theta_u^n \\ &= \int_{\{u<v\}} \theta_u^n,\end{aligned}$$

which arrives at (5.3). □

We also have the comparison principle within the full mass class, which is slightly different from the previous one (see [DDNL18a, Proposition 3.5] for the proof).

Theorem 5.2 (Comparison principle: II)**.** Let $\Phi \in \mathrm{PSH}(X, \theta)$, and let $u, v \in \mathcal{E}(X, \theta; \Phi)$. Then

$$\int_{\{u<v\}} \theta_v^n \le \int_{\{u<v\}} \theta_u^n. \tag{5.4}$$

5.2. **An oscillation estimate.** In this subsection, we introduce a generalization of Kołodziej's oscillation estimate, due to DDL [DDNL21].

Theorem 5.3 (Kołodziej, [Koł98])**.** Let (X, ω) be a compact Kähler manifold. Suppose $u \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ satisfies

$$\omega_u^n = f\omega^n \tag{5.5}$$

for some $f \in L^p(X)$ with $p > 1$. Then there exists $C > 0$ depending on ω, n, and $\|f\|_{L^p}$ s.t.

$$\mathrm{osc}_X\, u \le C.$$

We first need the following lemm, where the proof can be found in [DDNL18a, Proposition 4.30]

Lemma 5.5. Let Φ be a model potential and let $f \in L^p(X, \omega^n)$ for $p > 1$. Then for any $E \subset X$, there exists $A > 0$, s.t.

$$\int_E f\,\omega^n \le A\,\mathrm{Cap}_{\theta,\Phi}(E)^2. \tag{5.6}$$

Theorem 5.4 (DDL, [DDNL21])**.** Fix $a \in [0, 1)$. Let $0 \ge \Phi \in \mathrm{PSH}(X, \theta)$ be a model potential and $0 < f \in L^p(X, \omega^n)$ for $p > 1$. Assume $u \in \mathrm{PSH}(X, \theta)$ with $\sup_X u \le 0$, s.t.

$$\theta_u^n \le f\omega^n + a\theta_\Phi^n,$$

and $[\Phi] \le [P[u]]$. Then there exists $C > 0$ depending on p, n, ω, a, and $\|f\|_{L^p}$, s.t.

$$\Phi - C \le u \le 0.^{6} \tag{5.7}$$

Proof. **Step 1**: Denote

$$g(t) = \left(\mathrm{Cap}_{\theta,\Phi}(\{u < \Phi - t\})\right)^{\frac{1}{n}}.$$

Our goal is to prove for any $s \in [0,1]$, we have

$$sg(t+s) \le bg(t)^2. \tag{5.8}$$

Observe that $g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is decreasing, since for $t_2 < t_1$, we have $\{u < \Phi - t_1\} \subset \{u < \Phi - t_2\}$, and $g(+\infty) = 0$ since if $t = \infty$, we have

$$\bigcap_{t>0}^{\infty} \{u < \Phi - t\} = \{u = -\infty\}.$$

Let $s \in [0,1]$, taking $v \in \mathrm{PSH}(X,\theta)$ with $\Phi - 1 \le v \le \Phi$, we have

$$s^n \int_{\{u<\Phi-t-s\}} \theta_v^n \le s^n \int_{\{u<(1-s)\Phi+sv-t\}} \theta_v^n,$$

since $\{u < \Phi - t - s\} = \{u < (1-s)\Phi + s(\Phi - 1) - t\} \subset \{u < (1-s)\Phi + sv - t\}$. Meanwhile,

$$\theta^n_{(1-s)\Phi+sv} = ((1-s)\theta_\Phi + s\theta_v)^n \ge s^n\theta_v^n,$$

which gives

$$s^n \int_{\{u<(1-s)\Phi+sv-t\}} \theta_v^n \le \int_{\{u<(1-s)\Phi+sv-t\}} \theta^n_{(1-s)\Phi+sv}.$$

Note that $[(1-s)\Phi + sv - t] = [\Phi] \le [P[u]]$, then by the comparison principle (5.3),

$$\int_{\{u<(1-s)\Phi+sv-t\}} \theta^n_{(1-s)\Phi+sv} \le \int_{\{u<(1-s)\Phi+sv-t\}} \theta_u^n \le \int_{\{u<\Phi-t\}} \theta_u^n,$$

where the second inequality is due to $\{u < (1-s)\Phi + sv - t\} \subset \{u < \Phi - t\}$. Hence, we have

$$s^n \int_{\{u<\Phi-t-s\}} \theta_v^n \le \int_{\{u<\Phi-t\}} \theta_u^n.$$

Taking supremum among all such v, we arrive at

$$s^n g(t+s)^n = s^n\,\mathrm{Cap}_{\theta,\Phi}(\{u < \Phi - t - s\}) \le \int_{\{u<\Phi-t\}} \theta_u^n.$$

On the other hand, by our assumption,

$$\begin{aligned}\int_{\{u<\Phi-t\}} \theta_u^n \le& \int_{\{u<\Phi-t\}} f\,\omega^n + a\int_{\{u<\Phi-t\}} \theta_\Phi^n \\ \le& A\,\mathrm{Cap}_{\theta,\Phi}(\{u < \Phi - t\})^2 + a\int_{\{u<\Phi-t\}} \theta_u^n,\end{aligned}$$

where the first last inequality is due to (**??**) and (5.3). Therefore,

$$\int_{\{u<\Phi-t\}} \theta_u^n \le \frac{A}{1-a}\mathrm{Cap}_{\theta,\Phi}(\{u < \Phi - t\})^2 = \frac{A}{1-a} g(t)^{2n},$$

which yields (5.8).

[6] In this situation, we have $[u] = [\Phi]$.

Step 2: We want to prove that for any $t > 0$ and $0 \leq s \leq 1$, if $g(t)$ is a decreasing, right-continuous function satisfying (5.8), with $g(+\infty) = 0$. Then there is $T > 0$ big enough, s.t. $g(t) = 0$, for $t > T$. The original proof can be founded in [EGZ09, Lemma 2.4].

Taking $t_0 > 0$ large enough s.t. $g(t_0) \leq 1/2b$. We define a sequence $\{t_j\}$ by induction in the following way: If $f(t_0) = 0$, we stop here, otherwise, set

$$t_1 := \sup_t \{t > t_0 : \ g(t) > g(t_0)/2\} = \sup_t A_1.$$

Note that $t_1 \leq 1 + t_0$. If not, assume $t_1 > 1 + t_0$, then $1 + t_0 \in A_1$, and $g(t_0 + 1) > g(t_0)/2$. Taking $s = 1$ and combing with (5.8), we immediately obtain that

$$g(t_0)/2 < g(t_0 + 1) \leq b g(t_0)^2,$$

which gives $1/2b < g(t_0)$. But this contradicts our choice of t_0, hence we must have $t_1 \leq 1 + t_0$.

Because g is right continuous, we have $g(t_1) \leq g(t_0)/2$. If not there is $\varepsilon > 0$ small enough, s.t. $g(t_1 + \varepsilon) > g(t_0)/2$ thanks to right continuous. This gives $t_i < t_1 + \varepsilon \in A_1$, which contradicts the definition of t_1. If $s_1 = 0$, we stop here. Otherwise, define

$$t_{j+1} := \sup_t \{t > t_j : \ g(t) > g(t_j)/2\}.$$

We have $t_{j+1} \leq 1 + t_j$ and $g(t_{j+1}) \leq g(t_j)/2$, and this sequence does not grow too fast. Taking $t \in (t_j, t_{j+1})$, by (5.8), we have

$$(t - t_j) g(t) \leq b g(t_j)^2 \leq 2b g(t) \cdot g(t_j).$$

Therefore,

$$t_{j+1} - t_j \leq 2b g(t_j) \leq 2b 2^{-j} g(t_0) \leq 2^{-j}.$$

Consequently, the sequence $\{t_j\}$ is bounded from above with the limit

$$t_\infty = t_0 + 2.$$

Step 3: We now can choose T large enough, so that

$$g(T) = \left(\mathrm{Cap}_{\theta,\Phi}(\{u < \Phi - T\})\right)^{\frac{1}{n}} \leq \frac{1}{2b}.$$

Then by Step 2, for any $t > T + 2$, we have $g(t) = 0$. Write $T_\infty = T + 3$, we then obtain

$$\mathrm{Cap}_{\theta,\Phi}(\{u < \Phi - T_\infty\}) = 0,$$

which implies $\{u < \Phi - T_\infty\}$ is pluripolar. Therefore

$$u \geq \Phi - T_\infty \qquad \text{q.e. (thus a.e.).}$$

Thanks to u is usc, we then arrive at our conclusion. □

Remark 5.6. This result is still true if Φ is not a model potential. In this case, we need to put (**??**) as an assumption.

6. Resolution of degenerate Monge-Ampère equations with prescribed singularities

Let (X, ω) be a compact Kähler manifold and let $[\theta]$ be a big class. Given a *model potential* $\Phi \in \mathrm{PSH}(X, \theta)$ (i.e. $P[\Phi] = \Phi \le 0$ and $\int_X \theta_\Phi^n > 0$), and a Borel measure μ, we study

$$\begin{cases} \theta_u^n = \mu, \\ [u] = [\Phi], \end{cases} \qquad u \in \mathrm{PSH}(X, \theta), \tag{6.1}$$

in two basic regimes:

(i) $\mu = f\,\omega^n$ with $f \in L^p(X, \omega^n)$, $p > 1$;

(ii) μ is a non-pluripolar measure with total mass $\mu(X) = \displaystyle\int_X \theta_\Phi^n$.

6.1. Overview.

Case 1: If $\theta = \omega$ is Kähler, $f > 0$ is smooth, then (6.1) recovers to celebrated Calabi conjecture, which was solved by Yau.

Theorem 6.1 ([Yau78]). If ω is Kähler and $f > 0$ is smooth with $\int_X f\,\omega^n = \int_X \omega^n$, then there exists a unique (up to additive constant) smooth ω-psh u solving

$$(\omega + dd^c u)^n = f\,\omega^n.$$

Case 2: If $\theta = \omega$ is Kähler, $0 < f \in L^p(X, \omega)$ for $p > 1$, then (6.1) was solved by Kołodziej.

Theorem 6.2 ([Koł98]). If ω is Kähler and $f \in L^p(X, \omega^n)$ with $p > 1$ and $\int_X f\,\omega^n = \int_X \omega^n$, then there exists a unique bounded ω-psh solution u (normalized e.g. by $\sup_X u = 0$) of

$$(\omega + dd^c u)^n = f\,\omega^n.$$

Moreover, an a priori L^∞ estimate depends only on (X, ω) and $\|f\|_{L^p}$ (Theorem 5.3).

Case 3: If $\theta \ge 0$ is big, $0 < f \in L^p(X, \omega)$ for $p > 1$ and $\Phi = V_\theta$, then (6.1) was solved by Eyssidieux-Guedj-Zeriahi.

Theorem 6.3 ([EGZ09]). If θ is smooth, semi-positive and big, and $\mu = f\,\omega^n$ with $f \in L^p$, $p > 1$, then

$$\theta_u^n = \mu$$

admits a unique bounded solution u with *minimal singularities* (hence $[u] = [V_\theta]$).

Case 4: If θ is big, μ is any non-pluripolar measure, and $\Phi = V_\theta$, then (6.1) was solved by Boucksom-Eyssidieux-Guedj-Zeriahi.

Theorem 6.4 ([BEGZ10]). Let $[\theta]$ be big and let μ be a non-pluripolar measure with $\mu(X) = \displaystyle\int_X \theta_{V_\theta}^n$. Then there exists a unique $u \in \mathcal{E}(X, \theta)$ with *minimal singularities* solving the non-pluripolar Monge–Ampère equation

$$\langle(\theta + dd^c u)^n\rangle = \mu.$$

6.2. **Approximation by supersolution.** We first construct the supersolutions.

Theorem 6.5. Given $0 \leq f \in L^p(X, \omega^n)$ s.t.

$$\int_X f\omega^n = \int_X \theta_\Phi^n.$$

Then for any $b > 1$, there exists a *unique* $v \in \mathrm{PSH}(X, \theta)$ with $[\Phi] \leq [v]$ satisfying

$$\theta_v^n \leq bf\,\omega^n. \tag{6.2}$$

Proof. Fix $a \in (0, 1)$ and k positive integer. Thanks to Theorem 6.4, we can solve

$$\theta_{\varphi_k}^n = c_k f\,\omega^n + a\mathbf{1}_{\{\Phi \leq V_\theta - k\}}\,\theta^n_{\max\{\Phi, V_\theta - k\}}, \tag{6.3}$$

where c_k is a constant, s.t.

$$\int_X \theta_{\varphi_k}^n = \int_X \theta_{V_\theta}^n,$$

and $\varphi_k \in \mathcal{E}(X, \theta)$. In particular, integrating on both sides of (6.3) over X, we have

$$\begin{aligned}\int_X \theta_{\varphi_k}^n &= c_k \int_X f\,\omega^n + a\int_{\{\Phi \leq V_\theta - k\}} \theta^n_{\max\{\Phi, V_\theta - k\}}\\ &= c_k \int_X f\,\omega^n + a\left(\int_X \theta^n_{\max\{\Phi, V_\theta - k\}} - \int_{\{\Phi > V_\theta - k\}} \theta^n_{\max\{\Phi, V_\theta - k\}}\right).\end{aligned}$$

Note that $[\max\{\Phi, V_\theta - k\}] = [V_\theta]$, and $\varphi_k \in \mathcal{E}(X, \theta)$, we obtain

$$\int_X \theta_{V_\theta}^n = c_k \int_X \theta_\Phi^n + a\int_X \theta_{V_\theta}^n - a\int_{\{\Phi > V_\theta - k\}} \theta^n_{\max\{\Phi, V_\theta - k\}},$$

which implies

$$c_k = \frac{(1-a)\int_X \theta_{V_\theta}^n + a\int_{\{\Phi > V_\theta - k\}} \theta^n_{\max\{\Phi, V_\theta - k\}}}{\int_X \theta_\Phi^n}.$$

Hence, $c_k \nearrow C(a)$ as $k \to +\infty$, where

$$C(a) = a + (1-a)\frac{\int_X \theta_{V_\theta}^n}{\int_X \theta_\Phi^n}.$$

For a fix $1 > \varepsilon > 0$, define

$$\psi_k = (1-\varepsilon)\max\{\Phi, V_\theta - k\} + \varepsilon V_\theta.$$

Obviously, $\psi_k \in \mathrm{PSH}(X, \theta)$ and $[\psi_k] = [V_\theta]$, so that

$$\theta_{\psi_k}^n \geq (1-\varepsilon)^n\,\theta^n_{\max\{\Phi, V_\theta - k\}}.$$

Then,

$$\theta_{\varphi_k}^n \leq C(a)f\,\omega^n + a(1-\varepsilon)^{-n}\theta_{\psi_k}^n.$$

Since $f \in L^p(X, \omega^n)$ for $p > 1$, it follows from Proposition 5.2,

$$\int_E f\,\omega^n \leq C\ \mathrm{Cap}_\theta(E)^2, \tag{6.4}$$

for every Borel subsets, where C depends on $\theta, n, p, \|f\|_{L^p}$.

We claim
$$\operatorname{Cap}_\theta \leq \varepsilon^{-n} \operatorname{Cap}_{\theta,\psi_k}. \tag{6.5}$$
To see this, taking $\chi \in \mathrm{PSH}(X,\theta)$ with $V_\theta - 1 \leq \chi \leq V_\theta$. Then $w = (1-\varepsilon)\max\{\Phi, V_\theta - k\} + \varepsilon\chi \in \mathrm{PSH}(X,\theta)$ and satisfies $\psi_k - 1 \leq w \leq \psi_k$. Therefore,
$$\varepsilon^n \int_E \theta_\chi^n \leq \int_E \left((1-\varepsilon)\theta_{\max\{\Phi,V_\theta-k\}} + \varepsilon\theta_\chi\right)^n \leq \int_E \theta_w^n \leq \operatorname{Cap}_{\theta,\psi_k}.$$
Taking the supremum over all such χ gives our claim.

It follows from (6.4) and (6.5),
$$\int_E f\,\omega^n \leq C\varepsilon^{2n} \operatorname{Cap}_{\theta,\psi_k}(E).$$
We then can apply Theorem 5.4 to obtain
$$\varphi_k \geq \psi_k - C_1 \geq V_\theta - C_2 \geq \Phi - C_3.$$
Define
$$v_{k,j} = P(\min\{\varphi_k, \ldots, \varphi_{k+j}\}),$$
then $v_{k,j} \geq \Phi - C$ and $v_{k,j} \searrow v_k$ as $j \to +\infty$. Moreover, $v_k \nearrow v$ as $k \to +\infty$, which implies
$$\theta_v^n \leq C(a) f\,\omega^n \leq bf\,\omega^n,$$
as desired after choosing a s.t. $C(a) \leq b$. □

Remark 6.1. When we are working with a model potential $\Phi \neq V_\theta$, we must have
$$\int_X \theta_\Phi^n < \int_X \theta_{V_\theta}^n.$$
Otherwise, if $\int_X \theta_\Phi^n = \int_X \theta_{V_\theta}^n$, then $\Phi \in \mathcal{E}(X,\theta)$, and $P[\Phi] = V_\theta = \Phi$ which is a contradiction.

Theorem 6.6. Given μ a non-pluripolar measure with
$$\mu(X) = \int_X \theta_\Phi^n.$$
If
$$\mu \leq B \operatorname{Cap}_{\theta,\Phi},$$
then there exists a unique $u \in \mathcal{E}(X,\theta;\Phi)$ with $\sup_X u = 0$ solving $\theta_u^n = \mu$.

Proof. Theorem 5.4 + Theorem 6.5. I will fill in this when I am happy :). □

Theorem 6.7. Given μ a non-pluripolar measure with
$$\mu(X) = \int_X \theta_\Phi^n.$$
Then there exists a unique $u \in \mathcal{E}(X,\theta;\Phi)$ with $\sup_X u = 0$ solving $\theta_u^n = \mu$. Moreover, if $\mu = f\,\omega^n$, for $0 \leq f \in L^p(X,\omega^n)$ for $p > 1$, then $[u] = [\Phi]$.

Sketch of the proof. Indeed, we have $\mu = f\nu$ (see [DDNL18a]) for $f \in L^1(X, \nu)$ and

$$\nu \leq \mathrm{Cap}_{\theta,\Phi}.$$

Define $\mu_j = c_j \min\{f, j\}\nu$ where $c_j \geq 1$ is chosen so that $\mu_j(X) = \mu(X)$. Observe that

$$\mu_j \leq c_j \cdot j \cdot \nu \leq C \cdot \mathrm{Cap}_{\theta,\Phi}.$$

Then by Theorem 6.6, there exists a unique $u_j \in \mathcal{E}(X, \theta; \Phi)$, s.t.

$$\theta^n_{u_j} = \mu_j.$$

Consequently, we can extract a subsequence and we still denote it by $\{u_j\} \subset \mathcal{E}(X, \theta; \Phi)$, s.t. $u_j \to u \in \mathrm{PSH}(X, \theta)$ in L^1, $u \in \mathcal{E}(X, \theta; \Phi)$ and $u \leq \Phi \leq 0$. Then, by the below Lemma 6.2, we have

$$\theta^n_u \geq \mu.$$

On the other hand, it follows from Theorem 3.5,

$$\mu(X) = \int_X \theta^n_\Phi \geq \int_X \theta^n_u \geq \mu(X).$$

Hence, we have $\theta^n_u = \mu$. The fact $[u] = [\Phi]$ can be obtained by Theorem 5.4. □

Lemma 6.2. Let $\left\{u_j\right\}_j \subset \mathrm{PSH}(X, \theta)$ such that $\theta^n_{u_j} \geq f_j\, \mu$, where $f_j \in L^1(X, \mu)$ and μ is a positive non-pluripolar Borel measure on X. Assume that $f_j \to f \in L^1(X, \mu)$ in $L^1(X, \mu)$, and $u_j \to u \in \mathrm{PSH}(X, \theta)$ in $L^1(X, \omega^n)$. Then

$$\theta^n_u \geq f\mu.$$

6.3. **The Aubin-Yau equation.** Let (X, ω) be a compact Kähler manifold and let θ be a big class. Fix $\Phi \in \mathrm{PSH}(X, \theta)$ a model potential, and μ a non-pluripolar measure, we consider the following equation:

$$\theta^n_u = e^u \mu. \tag{6.6}$$

We recall

Theorem 6.8 (Schauder fixed-point theorem)**.** Let X be a Banach space, and let $K \subset X$ be a non-empty, compact and convex set. Then given any continuous mapping $F : K \to K$ there exists $x \in K$ such that $F(x) = x$.

We also need the following tool:

Lemma 6.3. Assume μ is a non-pluripolar measure on X. Let $u_j, u \in \mathrm{PSH}(X, A\omega)$ for some $A > 0$. Assume $u_j \to u$ in $L^1(X, \omega^n)$ and $\sup_j \int_X |u_j|^2\, d\mu < +\infty$. Then

$$\int_X \left|u_j - u\right|\, d\mu \to 0.$$

Proof. It follows from [GZ17, Lemma 11.5] that

$$\int_X \left(u_j - u\right)\, d\mu \to 0. \tag{6.7}$$

For each $j > 0$ we set $\tilde{u}_j := \left(\sup_{k\geq j} u_k\right)^*$. Then $\tilde{u}_j \in \mathrm{PSH}(X,\theta)$ and $\tilde{u}_j$ decrease to u pointwise. Since $\tilde{u}_j \geq \max\left\{u_j, u\right\}$, we can have

$$\left|u_j - u\right| = 2\max\left\{u_j, u\right\} - u_j - u \leq 2\left(\tilde{u}_j - u\right) + \left(u - u_j\right).$$

It thus follows from the monotone convergence theorem and (6.7) that

$$\int_X \left|u_j - u\right| d\mu \leq 2\int_X \left(\tilde{u}_j - u\right) d\mu + \int_X \left(u - u_j\right) d\mu \to 0.$$

□

Theorem 6.9. Given μ a non-pluripolar measure of finite mass. Then there exists a unique $u \in \mathcal{E}(X,\theta;\Phi)$ that solves (6.6). Moreover, if $\mu = f\,\omega^n$ with $f \in L^p(X,\omega^n)$ for $p > 1$, then $[u] = [\Phi]$.

Proof. Consider

$$K = \left\{u \in \mathrm{PSH}(X,\theta),\ [u] \leq [\Phi],\ \int_X u\,\omega^n = 0\right\}.$$

In fact, K is convex and compact in $L^1(X,\omega^n)$. Define

$$F : K \to K, \qquad \varphi \mapsto u \in \mathcal{E}(X,\theta;\Phi),$$

satisfying

$$\begin{cases} \theta_u^n = C(\varphi)\, e^{\varphi}\mu; \\ \int_X u\,\omega^n = 0, \end{cases}$$

where $C(\varphi)$ is chosen s.t.

$$\int_X \theta_u^n = C(\varphi)\int_X e^{\varphi} d\mu = \int_X \theta_\Phi^n$$

Thanks to Theorem 6.7, u is unique, so that F is well-defined. We now show F is continuous in L^1-topology. Let $\varphi_k \to \varphi$ in $L^1(X,\omega^n)$, and denote $F(\varphi_k) := u_k$. Since K is compact, after extracting a subsequence, we assume $u_k \to u$ in $L^1(X,\omega^n)$. The goal is to prove that

$$\theta_u^n = C(\varphi)\, e^{\varphi}\mu.$$

From the normalization, we obtain a uniform bound for $\sup_k \varphi_k$, and thus a uniform for e^{φ_k}. It follows from [GZ17, Lemma 11.5] that

$$\int_X e^{\varphi_k}\, d\mu \to \int_X e^{\varphi}\, d\mu.$$

Then,

$$C(\varphi_k) = \frac{\int_X \theta_\Phi^n}{\int_X e^{\varphi_k} d\mu} \longrightarrow \frac{\int_X \theta_\Phi^n}{\int_X e^{\varphi} d\mu} := C(\varphi).$$

Hence $C(\varphi_k)e^{\varphi_k} \to C(\varphi)e^{\varphi}$ in $L^1(X,\mu)$. By Lemma 6.2, we obtain

$$\theta_u \geq C(\varphi)\, e^{\varphi}\mu.$$

Since $[u] \le [\Phi]$, thanks to Theorem 3.5, we have

$$C(\varphi)\int_X e^{\varphi} d\mu = \int_X \theta_\Phi^n \ge \int_X \theta_u^n \ge C(\varphi)\int_X e^{\varphi} d\mu,$$

which forces $\theta_u = C(\varphi)\, e^{\varphi}\mu$. Consequently, F is continuous in L^1-topology.

Applying Schauder fixed-point theorem, there exists φ s.t. $F(\varphi) = \varphi$. Then we have

$$\begin{cases} \theta_\varphi^n = C(\varphi) e^{\varphi}\mu \\ \int_X \varphi\, \omega^n = 0. \end{cases}$$

Taking $\tilde{\varphi} = \varphi + \log C(\varphi)$, we arrive at our solution $\tilde{\varphi} \in \mathcal{E}(X, \theta; \Phi)$. This gives the existence of (6.6).

For the uniqueness, let $u, v \in \mathcal{E}(X, \theta; \Phi)$ be two solutions. Assume $\{v < u\} \neq \emptyset$, then

$$\begin{aligned} \int_{\{v<u\}} e^u d\mu &= \int_{\{v<u\}} \theta_u^n \\ &\le \int_{\{v<u\}} \theta_v^n && \text{(by Theorem 5.2)} \\ &= \int_{\{v<u\}} e^v d\mu \\ &< \int_{\{v<u\}} e^u d\mu, \end{aligned}$$

which implies $\mu(\{v < u\})$. Note that, $\mu = e^{-v}\theta_v^n$, so that $\theta_v^n(\{v < u\})$. Applying domination principle (Theorem 3.6), we have $v \ge u$ on X and $\{v < u\} = \emptyset$. Conversely, we also have $\{u < v\} = \emptyset$, and therefore $u = v$ on X. □

6.4. **Singular KE metrics with prescribed singularities.** Solutions of complex Monge-Ampère equations are linked to existence of canonical Kähler metrics. In particular, we can think of the solution to

$$\theta_u^n = e^f \omega^n$$

as a potential with prescribed singularity type and prescribed Ricci curvature in the philosophy of the Calabi-Yau theorem. As an immediate application of the resolution of the Monge-Ampère equation

$$\theta_u^n = e^{u+f}\omega^n$$

with prescribed singularities $[u] = [\Phi]$, we obtain existence of singular KE metrics with prescribed singularity type on Kähler manifolds of general type.

Corollary 6.4. Let X be a smooth projective manifold with ample canonical bundle K_X and let h be a smooth Hermitian metric on K_X with $\theta := \Theta(h) > 0$. Suppose also that $\Phi \in \mathrm{PSH}(X, \theta)$ is a model potential, has small unbounded locus. Then there exists a unique singular KE metric he^{-u} on K_X with $u \in \mathrm{PSH}(X, \theta)$ and $[u] = [\Phi]$.

Remark 6.5. An analogous result also holds on Calabi-Yau manifolds.

We want to underline that the assumption of the model potential is necessary.

Theorem 6.10. Let $\phi \in \mathrm{PSH}(X, \theta)$ with positive mass. Suppose for any $0 \le f \in L^\infty(X)$, there exists u that satisfying

$$\theta_u^n = f\omega^n$$

and $[u] = [\phi]$. Then ϕ has model type singularity, i.e. $[P[\phi]] = [\phi]$.

Proof. Assume $[\phi] \neq [P[\phi]]$, and $[\phi] \le [P[\phi]]$ strictly, then the inclusion

$$\mathcal{E}(X, \theta; \phi) \subset \mathcal{E}(X, \theta; P[\phi])$$

holds also strictly. Note that $[u] = [\phi] \le [P[u]]$, and

$$\int_X \theta_u^n = \int_X \theta_\phi^n = \int_X \theta_{P[\phi]}^n,$$

Therefore,

$$\theta_{P[\phi]}^n = \mathbf{1}_{\{P[\phi]=0\}}\theta^n = f\,\omega^n,$$

where $f \in L^\infty(X)$ that satisfying $f \equiv 0$ on $X \setminus \{P[\phi] = 0\}$ and $f = \theta^n/\omega^n$ on $\{P[\phi] = 0\}$. Therefore, $P[\phi]$ is the unique solution solves $\theta_{P[\phi]}^n = f\omega^n$ in the class $\mathcal{E}(X, \theta; P[\phi])$. Since $P[\phi] \notin \mathcal{E}(X, \theta; \phi)$, there is no solution $u \in \mathcal{E}(X, \theta; \phi)$ that solving $\theta_u^n = f\omega^n$ s.t. $[u] = [\phi]$. □

Remark 6.6. The above result is still true if we replace $f \in L^\infty(X)$ by $f \in L^p(X, \omega^n)$ for $p > 1$.

6.5. **Log concavity of volume.** In this subsection, we introduce the log concavity of MA measure, which is a direct consequence of the solvability of complex Monge-Ampère equations with prescribed singularity type:

Theorem 6.11. Let $T_1, ..., T_n$ be closed positive $(1, 1)$-currents. Then

$$\int_X T_1 \wedge \cdots \wedge T_n \ge \left(\int_X T_1^n\right)^{\frac{1}{n}} \cdots \left(\int_X T_n^n\right)^{\frac{1}{n}}. \tag{6.8}$$

In particular, the map

$$T \longmapsto \left(\int_X T^n\right)^{\frac{1}{n}}$$

is concave on the sets of positive currents.

Proof. Assume $\int_X \omega^n = \int_X T_j^n = 1$, for $1 \le j \le n$. Assume all T_j are big, otherwise there is nothing need to prove.

Consider a smooth closed $(1, 1)$ form θ_j cohomologeous to T_j, s.t. $T_j = \theta_j + dd^c u_j$ for some $u_j \in \mathrm{PSH}(X, \theta_j)$. Then we know $P_{\theta_j}[u_j]$ is a model potential. It follows from Theorem 6.7, there exists $\varphi_j \in \mathcal{E}(X, \theta; P_{\theta_j}[u_j])$ s.t.

$$(\theta_j + dd^c\varphi_j)^n = \omega^n.$$

Hence,

$$\int_X T_1 \wedge \cdots \wedge T_n = \int_X (\theta_1 + dd^c\varphi_1) \wedge \cdots \wedge (\theta_n + dd^c\varphi_n) \geq \int_X \omega^n = 1,$$

thanks to [BEGZ10, Proposition 1.11], and we are done. □

Remark 6.7. The log concavity of volume was first proven by Boucksom–Favre–Jonsson [BFJ09] when the class is big and nef. After, Boucksom-Eyssidieux-Guedj-Zeriahi [BEGZ10] showed the case when the current has analytic singularities. The full version was affirmatively proven by Darvase-Di Nezza-Lu [DDNL21].

7. Uniform C^0 estimate for cscK equations

In this section, we introduce a uniform C^0 estimate for cscK equations given by Deruelle-Di Nezza [DDN22].[7]

7.1. **Overview.** The constant scalar curvature Kähler (often abbreviated as cscK) metric generalizes the concept of the Kähler–Einstein metric. And on compact Kähler manifolds, the average of the scalar curvature $\hat{R}$ is given by

$$\hat{R} = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{[\omega]^n}$$

which is independent of the choice of ω.

For a polarized manifold (M, L), the Yau-Tian-Donaldson conjecture states that the existence of the cscK metric in $c_1(L)$ is equivalent to K-stability of (M, L), linking the K-energy's analytic behavior to algebraic stability via test configurations [Yau93, Tia97, Don02].

In [Che18], Chen outlined a program for studying the existence problem for cscK metric: a new continuity path that links the cscK equation to a certain second-order elliptic equation, apparently motivated by the classical continuity path for Kähler Einstein metrics and Donaldson's continuity path for conical Kähler Einstein metrics, and showed the openness. Further, Chen and Cheng [CC21b, CC21a] established a priori estimates and proved the existence of the cscK metric under the propness of the K-energy. There has many significant progress made in the resolution of the YTD conjecture; we refer interested readers to see, for instance [Sto09], [BDL20], [BBJ21], [BHJ19, BHJ22], etc.

Fix ω, consider $\omega_\varphi = \omega + dd^c\varphi$. Set

$$\omega_\varphi^n = e^F \omega^n,$$

then applying $dd^c \log$ to this equality gives

$$\mathrm{Ric}(\omega_\varphi) = \mathrm{Ric}(\omega) - dd^c \log \frac{\omega_\varphi^n}{\omega^n}.$$

Tracing both sides w.r.t ω_φ leads to

$$\hat{R} = R(\omega_\varphi) = \mathrm{tr}_\varphi \mathrm{Ric}(\omega) - \Delta_\varphi F.$$

[7] I will also add the C^2 estimate after.

Therefore, the cscK equation can be written as a system of coupled equations:

$$\begin{cases} \omega_\varphi^n & = e^F \omega^n; \\ \Delta_\varphi F & = \operatorname{tr}_\varphi \operatorname{Ric}(\omega) - \hat{R}. \end{cases} \tag{7.1}$$

The (classical) idea is then to deform the above system using a continuity path in such a way that the initial system (at time $t = 0$) has an obvious solution while the system of equations at $t = 1$ is the one for which we want to prove existence of solutions. The goal is to show that the set S of parameters $t \in [0, 1]$ such that a smooth solution exists is open, closed and non-empty. This would imply in turn that $t = 1$ is in S, meaning that the desired solution exists.

The closedness part is historically the most difficult. In the framework of the continuity method (specific to this setting) it suffices to prove uniform estimates for cscK potentials. Indeed, such estimates generalize easily to potentials which are solutions of the intermediate equations we have to deal with in the continuity method. The key result that Chen-Cheng [CC21b] are able to obtain states as follows:

Theorem 7.1. Let (X, ω) be a compact Kähler manifold. Assume ω_φ is a cscK metric for some smooth function φ on X normalized such that $\sup_X \varphi = 0$. Then all the derivatives of φ can be estimated in terms of $\operatorname{Ent}(\varphi)$, i.e. for each $k \geq 0$, there exists a positive constant $C_k = C(k, \operatorname{Ent}(\varphi))$ such that

$$||\varphi||_{C^k} \leq C_k$$

Here $\operatorname{Ent}(\varphi)$ denotes the *entropy* of the measure ω_φ^n and it is defined as

$$\operatorname{Ent}(\varphi) = \int_X \log \frac{\omega_\varphi^n}{\omega^n}\, \omega_\varphi^n = \int_X F e^F \omega^n.$$

Our goal is to get a priori estimate within the realm of pluripotential theory.

7.2. **A priori C^0 estimate.** We normalized $V = \operatorname{Vol}_\omega(X) = 1$. Let φ and F be the solution of (7.1). Let u be the solution of

$$\omega_u^n = b^{-1} e^F \sqrt{F^2 + 1}\, \omega^n, \qquad \text{with} \qquad \sup_X u = 0, \tag{7.2}$$

where b is set so that $\int_X \omega_u^n = 1$. Thanks to Yau's theorem 6.1, the existence of u is guaranteed. Note that

$$\begin{aligned} 0 < b &= \int_X e^F \sqrt{F^2 + 1}\, \omega^n \\ &= \int_{\{F<1\}} e^F \sqrt{F^2 + 1}\, \omega^n + \int_{\{F \geq 1\}} e^F \sqrt{F^2 + 1}\, \omega^n \\ &\leq \sqrt{2}\, e + \sqrt{2} \operatorname{Ent}(\varphi) < +\infty, \end{aligned}$$

if $\operatorname{Ent}(\varphi)$ is bounded.

Theorem 7.2. Given $\varepsilon \in (0, 1)$, there exists $C = C(\varepsilon, \omega, b)$, s.t.

$$F + \varepsilon u - A\varphi \leq C,$$

where $A > 0$ depends only on the lower bound of $\mathrm{Ric}(\omega)$.

Proof. Suppose $\mathrm{Ric}(\omega) \geq -K\omega$, we take $A = K + 1$ and define

$$v := F + \varepsilon u - A\varphi.$$

We calculate that

$$\begin{aligned}
\Delta_\varphi v &= \Delta_\varphi F + \varepsilon \Delta_\varphi u - A\Delta_\varphi \varphi \\
&= \mathrm{tr}_\varphi \mathrm{Ric}(\omega) - \hat{R} + n\varepsilon \frac{\omega_u \wedge \omega_\varphi^n}{\omega_\varphi^n} - \varepsilon \,\mathrm{tr}_\varphi \omega + A \,\mathrm{tr}_\varphi \omega - An \\
&\geq -(\hat{R} + An) + (A - K - \varphi)\, \mathrm{tr}_\varphi \omega + n\varepsilon \frac{\omega_u \wedge \omega_\varphi^n}{\omega_\varphi^n} \\
&\geq -(\hat{R} + An) + n\varepsilon b^{-\frac{1}{n}} (F^2 + 1)^{\frac{1}{2n}},
\end{aligned}$$

where the last inequality is given by Theorem 3.3. By the maximum principle, at the maximum point p of v, we have

$$n\varepsilon b^{-\frac{1}{n}} (F^2 + 1)^{\frac{1}{2n}}(p) \leq \hat{R} + An,$$

which implie $F(p) \leq C(\varepsilon, K, b)$.

Let $a, \delta \in (0, 1)$, we have either

$$\sqrt{F^2 + 1} \geq \frac{b}{a\delta^n}, \qquad \text{or} \qquad F \leq \sqrt{F^2 + 1} \leq \frac{b}{a\delta^n}.$$

Hence, we have

$$\begin{aligned}
\omega_\varphi^n = e^F \omega^n &\leq \mathbf{1}_{\{\sqrt{F^2+1} \geq \frac{b}{a\delta^n}\}} e^F \omega^n + \mathbf{1}_{\{\sqrt{F^2+1} \leq \frac{b}{a\delta^n}\}} e^F \omega^n \\
&\leq \frac{a\delta^n}{b} \sqrt{F^2 + 1}\, e^F \omega^n + e^{\frac{b}{a\delta^n}} \omega^n \\
&= a\delta^n \omega_u^n + e^{\frac{b}{a\delta^n}} \omega^n \\
&\leq a\omega_{\delta u}^n + e^{\frac{b}{a\delta^n}} \omega^n.
\end{aligned}$$

Applying Theorem 5.4, we then obtain

$$\varphi \geq \delta u - C_0.$$

Taking δ small enough so that $A\delta = \varepsilon$, consequently, we have

$$F + \varepsilon u - A\varphi = v \leq v(p) \leq C(\varepsilon, b, \omega),$$

which complete the proof. □

Remark 7.1. We verify the assumption in Theorem 5.4 here. Setting $\theta = \omega > 0$, $u = \varphi$, $\Phi = \delta u$ and $f = e^{\frac{b}{a\delta^n}} \in L^p(X, \omega^n)$ for $p \geq 1$. Note that δu is a candidate of $P[\varphi] = 0$, then $[\delta u] \leq [P[\varphi]]$. Consider $v \in \mathrm{PSH}(X, \omega)$, s.t. $\delta u - 1 \leq v \leq \delta u$. Define $\chi = v - \delta u$, then we have $-1 \leq \chi \leq 0$, and

$$\mathrm{Cap}_{\omega, \delta \mathrm{u}} = \mathrm{Cap}_{\omega_{\delta \mathrm{u}}}.$$

Let $g = f\omega^n/\omega^n_{\delta u} \in L^p(X)$ for $p \geq 1$, it follows from Lemma 5.5,

$$\int_E f\,\omega^n = \int_E g\,\omega^n_{\delta u} \leq A\,\mathrm{Cap}_{\omega_{\delta u}}(E)^2 = A\,\mathrm{Cap}_{\omega,\delta u}(E)^2,$$

for any Borel subsets E, which completes the verification.

Corollary 7.2. The functions φ, F, u are uniformly bounded by contant only depends on ω and $\mathrm{Ent}(\varphi)$.

Proof. Thanks to Theorem 7.2 and $\sup_X \varphi = 0$, we now have

$$F \leq C - \varepsilon u.$$

Thus,

$$\int_X e^{2F}\omega^n \leq C\int_X e^{-2\varepsilon u}\omega^n$$

Choosing $\varepsilon < \nu([\omega])$, then by Skoda's integrability (Theorem 2.4), we obtain a uniform upper bound for $\|e^F\|_{L^2}$. Applying Kołodziej's estimate (Theorem 5.3) to $\omega^n_\varphi = e^F\omega^n$, we have

$$\varphi \geq -C(\|e^F\|_{L^2}, \omega),$$

which give a uniform control on $\|\varphi\|_{L^\infty}$.

Repeating this trick again,

$$\int_X e^{2F}\left(F^2+1\right)\,\omega^n \leq \int_X e^{4F}\omega^n \leq C\int_X e^{-4\varepsilon u}\omega^n.$$

We choose $\varepsilon < (2\nu([\omega]))^{-1}$ this time, then by Skoda's integrability, we obtain a uniform upper bound for $\|e^F\sqrt{F^2+1}\|_{L^2}$. Combing with Kołodziej's estimate, we have a control on $\|u\|_{L^\infty}$ with the constant depending on $\|e^F\sqrt{F^2+1}\|_{L^2}$ and ω.

By the uniform bounds for $\|\varphi\|_{L^\infty}$ and $\|u\|_{L^\infty}$, together with Theorem 7.2, we obtain an upper bound for F.

It remains to prove the lower bound for F. To finish this, assume first $\mathrm{Ric}(\omega) \leq B\omega$. Taking $s = B + 1$, then

$$\begin{aligned}\Delta_\varphi(F + s\varphi) &= -\hat{R} + \mathrm{tr}_\varphi \mathrm{Ric}(\omega) + sn - s\,\mathrm{tr}_\varphi\,\omega\\ &\leq sn - \hat{R} - \mathrm{tr}_\varphi\,\omega\\ &\leq sn - \hat{R} - ne^{-\frac{F}{n}},\end{aligned}$$

where the last inequality comes from the arithmetic–geometric mean inequality. Let p be the minimum point of $F + s\varphi$, we than have

$$sn - \hat{R} - ne^{-\frac{F(p)}{n}} \geq 0,$$

or equivalently

$$F(p) \geq -C.$$

Therefore,

$$F + s\varphi \geq F(p) - s\|\varphi\|_{L^\infty} \geq -\tilde{C},$$

which gives the desired estimate. □

To be continue...

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