THE 2-SYSTOLE ON COMPACT KÄHLER SURFACES WITH POSITIVE SCALAR CURVATURE

ZEHAO SHA

ABSTRACT. We study systolic inequalities for compact Kähler surfaces of positive scalar curvature. For any such surface (X, ω), we prove the sharp estimate $\min_X S(\omega)\ \mathrm{sys}_2(\omega) \leq 12\pi$, with equality if and only if $X \simeq \mathbb{P}^2$ and ω is the Fubini–Study metric. Using the classification of positive scalar curvature Kähler surfaces by their minimal models, we also determine the optimal constant in each case and describe the corresponding rigid models: 12π when the minimal model is $\mathbb{P}^2$, 8π for rational ruled surfaces, and 4π for non-rational ruled surfaces. In the non-rational ruled case, we give an independent analytic proof, adapting Stern's level set method to the Kähler setting.

1. INTRODUCTION

Systolic geometry studies how the minimal "size" of nontrivial homology classes in a Riemannian manifold controls, and is controlled by, its global geometry. Following Berger's terminology [Ber72a, Ber72b, Ber81] (see also [Ber08]), and Gromov's subsequent development of systolic geometry [Gro83], if (M, g) is a closed Riemannian manifold of dimension $n \geq k$, the k–systole of (M, g) is defined by

$$\mathrm{sys}_k(g) := \inf\big\{\mathrm{Vol}_g(Z)\ \big|\ Z \text{ is a } k\text{-cycle with } [Z] \neq 0 \in H_k(M; \mathbb{Z})\big\}.$$

Here an integral k–cycle may be understood in the sense of geometric measure theory, and $\mathrm{Vol}_g(Z)$ denotes its k-dimensional mass with respect to g.

In dimension three, the interplay between 2–systoles and positive scalar curvature (PSC) is by now well understood. If (M^3, g) has positive scalar curvature $\mathrm{scal}(g)$, Schoen–Yau [SY79] proved that any area–minimizing surface in M is homeomorphic to either S^2 or $\mathbb{RP}^2$, showing that the relevant minimal surfaces are essentially spherical. Building on this, Bray–Brendle–Neves [BBN10] established a sharp π_2–systolic inequality. Denote by $\mathrm{sys}_2^{\pi_2}(g)$ the infimum of the areas of homotopically nontrivial 2–spheres in (M^3, g), then

$$\min_M \mathrm{scal}(g) \cdot \mathrm{sys}_2^{\pi_2}(g)\ \leq 8\pi, \tag{1.1}$$

with equality if and only if the universal cover of (M^3, g) is isometric to the Riemannian product $S^2 \times \mathbb{R}$ endowed with the round metric on S^2 and the flat metric on $\mathbb{R}$. In [Ste22], Stern gave a new proof of (1.1) leveraging the level set method. The bound (1.1) was recently refined by a quantitative gap theorem of Xu [Xu25], and there has been substantial further progress on systolic inequalities under scalar curvature assumptions, see for instance [Zhu20, Ric20, Ori25].

In real dimension four, by contrast, our current understanding of 2–systoles is much more limited: even on $S^2 \times S^2$ with positive scalar curvature, global upper bounds for $\mathrm{sys}_2(g)$ are only known under additional geometric hypotheses. In this paper we initiate a systematic study of 2–systolic inequalities in dimension four under the extra assumption that the metric is Kähler. More precisely, given a compact Kähler surface (X, ω) we denote by g_ω the associated Riemannian metric and we write

$$\mathrm{sys}_2(\omega) := \mathrm{sys}_2(g_\omega),$$

while $S(\omega)$ denotes the (Chern-)scalar curvature of ω (in the Kähler setting, we have $2S(\omega) = \mathrm{scal}(g_\omega)$). With this convention, all our main inequalities are stated purely in terms of the Kähler form ω. Our result shows that:

Theorem 1.1. Let (X, ω) be a compact Kähler surface with positive scalar curvature $S(\omega)$. Then

$$\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \ \leq \ 12\pi.$$

Moreover, equality holds if and only if $X \simeq \mathbb{P}^2$ and ω is the Fubini–Study metric.

It is worth noting that a compact Kähler surface X admits a PSC metric if and only if X is obtained from $\mathbb{P}^2$ or a ruled surface $\mathbb{P}(\mathcal{E})$ by a finite sequence of blow ups. More precisely, Yau showed that the existence of a Kähler metric with positive total scalar curvature forces the Kodaira dimension of X to be $-\infty$ [Yau74], and the Enriques–Kodaira classification then implies that any *minimal* compact Kähler surface with PSC is either $\mathbb{P}^2$ or ruled [BHPVdV04, Ch. V]. Building on this and using Seiberg–Witten theory, LeBrun proved that for minimal complex surfaces of Kähler type, the existence of a Riemannian metric of positive scalar curvature is equivalent to the existence of a Kähler PSC metric, and that this happens if and only if the surface is either $\mathbb{P}^2$ or ruled [LeB95, Thm. 4]. He conjectured that the same characterization remains true after allowing blow ups. The remaining gap, whether blowing up preserves the *sign* of the scalar curvature was settled recently by Brown, who proved that blow-ups with small data preserve the sign of scalar curvature and thereby completed the classification [Bro24, Thm B]. In particular, every PSC Kähler surface has minimal model either $\mathbb{P}^2$ or a ruled surface. It is therefore natural to restate Theorem 1.1 in a form that records how the optimal constant depends on the minimal model. This leads to the following three-way refinement.

Theorem 1.2. Let (X, ω) be a compact Kähler surface with positive scalar curvature. Then:

(1) *(Theorem 3.4)* If the minimal model of X is $\mathbb{P}^2$, then

$$\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \ \leq \ 12\pi,$$

with equality if and only if $(X, \omega) \simeq (\mathbb{P}^2, \omega_{\mathrm{FS}})$.

(2) *(Theorem 4.10)* If the minimal model of X is a Hirzebruch surface $\mathbb{F}_e$ fibred over $\mathbb{P}^1$, then

$$\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \ \leq \ 8\pi,$$

with equality if and only if the minimal model is $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and ω is the product Fubini–Study metric on $\mathbb{P}^1 \times \mathbb{P}^1$ (up to scaling).

(3) *(Theorems 4.5 and 5.4)* If the minimal model of X is a ruled surface fibred over a base curve B of genus $g(B) \geq 1$, then

$$\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \ \leq \ 4\pi,$$

with equality if and only if B is an elliptic curve, the universal cover of X is biholomorphic to $\mathbb{P}^1 \times \mathbb{C}$, and ω is induced by the product of the Fubini–Study metric on $\mathbb{P}^1$ with a flat metric so that $\mathrm{sys}_2(\omega)$ is realized by a $\mathbb{P}^1$-fibre.

Idea of the proof. A key feature of the Kähler setting is that many genuinely Riemannian quantities can be expressed directly in terms of the Kähler class. In real dimension four, complex curves are calibrated by the Kähler form: if (X, ω) is a compact Kähler manifold and $C \subset X$ is a smooth complex curve, then

$$\mathrm{Vol}_\omega(C) = \int_C \omega = [\omega] \cdot [C],$$

so the ω–area of C is entirely encoded by the cohomology class $[\omega] \in H^{1,1}(X)$ and the homology class $[C] \in H_2(X; \mathbb{Z})$. More generally, if C is an effective (possibly singular or reducible) curve, we view $C = \sum m_i C_i$ as an integral 1–cycle and the homology–cohomology pairing can be expressed in the way

$$[\omega] \cdot [C] := \sum_i m_i \int_{C_i^{\mathrm{reg}}} \omega.$$

Motivated by this interplay between geometry and intersection theory, we introduce the following algebraic analogue of the 2–systole.

Definition 1.3 (Holomorphic 2-systole)**.** Let (X, ω) be a compact Kähler manifold. The *holomorphic* 2*-systole* of (X, ω) is defined by:

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) := \inf\big\{ [\omega]\cdot[C] \ \big| \ C \subset X \text{ an effective curve}, \ 0 \neq [C] \in H_2(X; \mathbb{Z}) \big\}. \qquad (\Diamond)$$

In other words, $\mathrm{sys}_2^{\mathrm{hol}}([\omega])$ measures the smallest possible ω–area of a nontrivial effective curve on X, expressed purely in terms of the intersection pairing. We *emphasize* that there exists compact Kähler manifolds that contain no complex curves (for instance, very general K3 surfaces with Picard number 0 or non-algebraic tori), in which case the set of candidates in ($\Diamond$) is empty. Hence, Definition 1.3 is worth using only when X carries at least one effective curve. Fortunately, on PSC Kähler manifolds, ($\Diamond$) is natural (and finite): by [Yan19, Theorem 1.3], a compact complex manifold admits a PSC Hermitian metric if and only if K_X is not pseudo-effective. If moreover X is Kähler, then by Ou's characterization [Ou25, Theorem 1.1] one has that X is uniruled (i.e., covered by rational curves). Summarizing,

$$\boxed{X \text{ is a compact PSC Kähler manifold} \overset{\text{Yang}}{\Longrightarrow} K_X \text{ is not pseudo-effective} \overset{\text{Ou}}{\Longrightarrow} X \text{ is uniruled.}}$$

and hence X carries plenty of effective curves, so the infimum in ($\Diamond$) is taken over a nonempty set.

We then consider the following scale–invariant functional on the Kähler cone $\mathcal{K}$:

$$\mathcal{J}_X([\omega]) := \mathrm{sys}_2^{\mathrm{hol}}([\omega]) \cdot \hat{S}([\omega]) = \Big(\inf_{C \text{ eff.}} [\omega] \cdot [C]\Big) \cdot \frac{2n\pi c_1(X) \cup [\omega]^{n-1}}{[\omega]^n},$$

where $\hat{S}([\omega])$ is the (normalized) average scalar curvature, $n = \dim_{\mathbb{C}} X$ and $c_1(X)$ is the first Chern class. By comparing volume minimizers in a fixed homology class, one always has

$$\mathrm{sys}_2(\omega) \ \leq \ \mathrm{sys}_2^{\mathrm{hol}}([\omega]),$$

so that

$$\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \leq \mathcal{J}_X([\omega]),$$

with equality if and only ω has constant scalar curvature (we will say ω is cscK as tradition in the subsequent sequel), and the minimizer of 2-systole is a holomorphic 1-cycle.

From this point on we restrict to Kähler surfaces, i.e. $n = 2$. In this case the functional

$$\mathcal{J}_X([\omega]) = \Big(\inf_{C \text{ eff.}} [\omega] \cdot [C]\Big) \cdot \frac{4\pi\, c_1(X) \cdot [\omega]}{[\omega]^2}$$

is completely determined by the intersection numbers of the Kähler class $[\omega]$ with curve classes in $H_2(X;\mathbb{Z})$ and $c_1(X)$. In particular, all PSC Kähler surfaces are projective. On a projective surface, by the Nakai–Moishezon–Kleiman criterion a real $(1,1)$–class is Kähler if and only if

$$[\omega]^2 > 0 \quad \text{and} \quad [\omega] \cdot C > 0 \text{ for every irreducible curve } C \subset X.$$

Since we are interested in PSC Kähler metrics, we further restrict to classes with $c_1(X) \cdot [\omega] > 0$. The global inequality in Theorem 1.1 then reduces to obtaining sharp upper bounds on $\sup \mathcal{J}_X$ over the slice

$$\{[\omega] \in \mathcal{K} \mid c_1(X) \cdot [\omega] > 0\},$$

and, as we show in Theorem 2.2, this supremum is always finite. This means that one does not need to worry about sequences of Kähler classes $[\omega_j]$ approaching nef but not big class for which $\mathcal{J}_X([\omega_j]) \to \infty$.

The remaining task is therefore a purely algebraic optimization problem on the Kähler cone. When the Mori cone $\overline{\mathrm{NE}}(X)$ is rational polyhedral and generated by finitely many effective curves (for instance, the del Pezzo surfaces), one can, in principle, write down all generators (even though there are hundreds many!) and obtain precise bounds by solving a finite–dimensional optimization problem. The main difficulty arises for blowups with infinitely generated Mori cone, in which case we can no longer list all effective curve classes, and naive estimates on $\mathcal{J}_X$ along the boundary of the Kähler cone may be too coarse, even allowing the algebraic upper bound to diverge.

To overcome this, we introduce a "mass–shift" method which reduces the degrees of freedom in the algebraic expression. Instead of keeping track of all blowup coefficients t_i of the exceptional divisors individually, we replace them by a small number of aggregate parameters and show that the worst possible configuration for $\mathcal{J}_X$ can always be realized by an extremal arrangement of the t_i. This converts an a priori infinite collection of curve constraints into a manageable finite–dimensional problem. The details of this procedure are worked out first in the case of blow–ups of $\mathbb{P}^2$, and then adapted to Hirzebruch surfaces.

Finally, for non–rational PSC Kähler surfaces (i.e. ruled surfaces over bases of genus $g \geq 1$) we show that the same strategy applies. In addition, we develop an independent analytic approach based on Stern's level–set method for harmonic maps [Ste22] , adapted to the Kähler setting. This provides a second proof of Theorem 1.2 (3) and, as expected, the algebraic and analytic methods lead to exactly the same optimal constant and the same rigid model.

Organization of the paper. In Section 2 we review basic notions that appear in this paper and show that on a projective Kähler surface (X, ω), the functional $\mathcal{J}_X([\omega])$ is well-defined with $\sup \mathcal{J}_X < \infty$ on the Kähler cone.

Section 3 is devoted to the case where the minimal model of X is $\mathbb{P}^2$. We give a detailed analysis of the Kähler cone of blow–ups of $\mathbb{P}^2$, apply the mass–shift technique to obtain the sharp bound $\sup \mathcal{J}_X = 12\pi$ together with the rigidity of the Fubini–Study metric.

In Section 4.1 we study ruled surfaces from an algebro–geometric viewpoint. We first treat non–rational ruled surfaces, which are easier to handle, and show that $\sup \mathcal{J}_X = 4\pi$ together with the corresponding rigid model. After recalling the geometry of Hirzebruch surfaces and their blow–ups, we then adapt the mass–shift argument to the rational ruled case and prove that $\sup \mathcal{J}_X = 8\pi$ for all rational ruled PSC Kähler surfaces, with rigidity characterized by the product Fubini–Study metric on $\mathbb{P}^1 \times \mathbb{P}^1$.

Section 5 provides an alternative proof for non–rational ruled surfaces. Here we introduce the level–set method for holomorphic fibrations $f\colon X \to B$ when B has non-positive Gaussian curvature.

Finally, in Section 6 we present some open problems on systolic inequalities for further study.

Acknowledgment. The author appreciates his advisor, Professor Gérard Besson, for introducing him to the field of PSC manifolds.

2. Preliminaries

Let (X^n, ω) be a compact Kähler manifold of complex dimension n. Locally we write $\omega = \sqrt{-1} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$. We denote by $\mathcal{K} := \{[\omega] \in H^{1,1}(X);\ [\omega] > 0\}$ the Kähler cone of X, which is an open convex cone in the finite dimensional vector space $H^{1,1}(X)$

The (Chern-)Ricci form is defined by $\mathrm{Ric}(\omega) := -\sqrt{-1}\,\partial\bar{\partial}\log\det(g_{i\bar{j}})$, it is a real closed $(1,1)$-form representing the first Chern class

$$c_1(X) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)] \in H^{1,1}(X) \cap H^2(X; \mathbb{R}).$$

The (Chern-)scalar curvature $S(\omega)$ is then determined by

$$S(\omega) := \frac{n\,\mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}.$$

In particular the *normalized average scalar curvature*

$$\hat{S}([\omega]) := \frac{2n\pi\, c_1(X) \cup [\omega]^{n-1}}{[\omega]^n}$$

depends only on the Kähler class $[\omega]$. Clearly $\min_X S(\omega) \le \hat{S}([\omega])$, with equality if and only if ω is cscK.

Throughout the paper we write g_ω for the Riemannian metric associated to ω, and denote by

$$\mathrm{sys}_2(\omega) := \mathrm{sys}_2(g_\omega)$$

the 2-systole in the sense of Berger–Gromov. In contrast, the quantity that we will actually *compute* is the holomorphic 2–systole introduced in Definition 1.3:

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) := \inf\big\{[\omega]\cdot[C] \;\big|\; C \subset X \text{ an effective curve},\ 0 \ne [C] \in H_2(X;\mathbb{Z})\big\}.$$

Here we view an effective curve $C = \sum m_i C_i$ as an integral 2–cycle and set

$$[\omega]\cdot[C] := \sum_i m_i \int_{C_i^{\mathrm{reg}}} \omega.$$

In particular every effective curve is a candidate for the 2–systole, and we always have

$$\mathrm{sys}_2(\omega) \;\le\; \mathrm{sys}_2^{\mathrm{hol}}([\omega]).$$

Note that $\mathrm{sys}_2^{\mathrm{hol}}([\omega])$ depends only on the Kähler class $[\omega]$, whereas $\mathrm{sys}_2(\omega)$ a priori depends on the specific metric in that class.

The basic scale–invariant functional that we will work with is

$$\mathcal{J}_X([\omega]) := \mathrm{sys}_2^{\mathrm{hol}}([\omega]) \cdot \hat{S}([\omega]) = \mathrm{sys}_2^{\mathrm{hol}}([\omega]) \cdot \frac{2n\pi\, c_1(X)\cdot[\omega]^{n-1}}{[\omega]^n}.$$

For Kähler surfaces ($n = 2$) this becomes

$$\mathcal{J}_X([\omega]) = \mathrm{sys}_2^{\mathrm{hol}}([\omega]) \cdot \frac{4\pi\, c_1(X)\cdot[\omega]}{[\omega]^2}.$$

Since $\mathrm{sys}_2^{\mathrm{hol}}(\lambda[\omega]) = \lambda\, \mathrm{sys}_2^{\mathrm{hol}}([\omega])$ and $\hat{S}(\lambda[\omega]) = \lambda^{-1}\hat{S}([\omega])$ for $\lambda > 0$, the value of $\mathcal{J}_X([\omega])$ depends only on the ray $\mathbb{R}_{>0}\,[\omega] \subset \mathcal{K}$.

Lemma 2.1. Let V be a real vector space and let $K \subset V$ be an open cone. Let $f : K \to (0,\infty)$ and $J : K \to \mathbb{R}$ be functions satisfying

$$f(\lambda x) = \lambda f(x), \qquad J(\lambda x) = J(x)$$

for all $x \in K$ and all $\lambda > 0$. Set $K_1 := \{x \in K \mid f(x) = 1\}$. Assume that $K_1 \ne \varnothing$. Then

$$\sup_{x\in K} J(x) \;=\; \sup_{x\in K_1} J(x).$$

Proof. Define a map $\Phi : K \to K$ by

$$\Phi(x) := \frac{x}{f(x)}, \qquad x \in K.$$

This is well-defined because $f(x) > 0$ for every $x \in K$, and the conic property of K implies $\Phi(x) \in K$. Moreover,

$$f(\Phi(x)) = f\left(\frac{x}{f(x)}\right) = \frac{1}{f(x)} f(x) = 1,$$

so that $\Phi(K) \subset K_1$.

Conversely, if $y \in K_1$, then $f(y) = 1$, and we may write $y = \Phi(y)$. Thus every point of K_1 lies in the image of Φ, and hence $\Phi(K) = K_1$. In particular, the restriction $\Phi : K \to K_1$ induces a surjection from the set of rays $\mathbb{R}_{>0}x \subset K$ onto K_1 so that each ray intersects K_1 in at most one point, by the homogeneity of f. Next, for every $x \in K$ we have,

$$J(\Phi(x)) = J\left(\frac{x}{f(x)}\right) = J(x).$$

Therefore

$$\sup_{x\in K} J(x) = \sup_{x\in K} J(\Phi(x)) = \sup_{y\in\Phi(K)} J(y) = \sup_{y\in K_1} J(y),$$

which is the desired equality. □

Theorem 2.2. Let (X, ω) be a projective Kähler surface. Then

(1) $\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \leq \mathcal{J}_X([\omega])$, with equality if and only ω is cscK and $\mathrm{sys}_2(\omega)$ is realized by a holomorphic 1-cycle;

(2) $\sup_{\mathcal{K}} |\mathcal{J}_X| < +\infty$.

Proof. We only prove (2) since (1) is obvious. Define $\mathcal{K}_1 := \left\{[\omega] \in \mathcal{K};\ \mathrm{sys}_2^{\mathrm{hol}}([\omega]) = 1\right\}$, by Lemma 2.1, we have

$$\sup_{\mathcal{K}} \mathcal{J}_X = \sup_{\mathcal{K}_1} \mathcal{J}_X.$$

We first show that $\overline{\mathcal{K}_1} \subset \mathcal{K}$. Assume there is a sequence of Kähler metric $\omega_\varepsilon \to \omega_0 \in \partial\mathcal{K}$. Then ω_0 is nef, and by Nakai-Moishizon criterion, there exists an effective curve C such that $[\omega_0] \cdot [C] = 0$. Hence,

$$1 = \mathrm{sys}_2^{\mathrm{hol}}(\omega_\varepsilon) \leq [\omega_\varepsilon] \cdot [C] \to 0,$$

which is a contradiction. Therefore, $\overline{\mathcal{K}_1} \subset \mathcal{K}$.

Fix a Euclidean norm $\|\cdot\|$ on the finite dimensional vector space $V := H^{1,1}(X)$, then $B = \{\alpha \in V;\ \|\alpha\| = 1\}$ is compact. Set $S_1 = \overline{\mathcal{K}_1} \cap B$. Clearly, $S_1 \subset \mathcal{K}$ is also compact and the continuous function $\omega \mapsto \omega^2 := F(\omega)$ has minimum $m := \min_{S_1} F$ and maximum $M := \max_{S_1} F$ on S_1. For $\omega \in \mathcal{K}_1$, define $u = \omega/\|\omega\| \in S_1$, we then have $\omega^2 = \|\omega\|^2 \cdot u^2$. Thus,

$$m\,\|\omega\|^2 \leq \omega^2 \leq M\,\|\omega\|^2.$$

For any $\alpha, \beta \in V$, $\alpha \mapsto \beta \cdot \alpha$ is a continuous linear functional, so that

$$|\beta \cdot \alpha| \leq C\,\|\alpha\|.$$

Consequently, we have

$$\left|\frac{c_1(X)\cdot\omega}{\omega^2}\right| \le \frac{C\,\|\omega\|}{m\,\|\omega\|^2} = \frac{C'}{\|\omega\|}.$$

It remains to prove $\|\omega\|$ is bounded away from 0 on $\mathcal{K}_1$. In particular, fix any effective curve F, $\omega \mapsto \omega\cdot F$ is a bounded linear functional on $\mathcal{K}_1$. So, we have

$$1 \le \omega\cdot F \le \tilde{C}\,\|\omega\|,$$

where $\tilde{C} > 0$ only depends on F. This completes the proof. □

Blow–ups will play a central role in what follows. Let $\pi\colon \tilde{X} \to X$ be the blow–up of X at a point p, with exceptional divisor E. Given a Kähler class $[\omega]$ on X we consider the family of classes on $\tilde{X}$

$$[\omega_t] := \pi^*[\omega] - t[E], \qquad t \ge 0.$$

The Seshadri constant of $[\omega]$ at p is defined by

$$\varepsilon([\omega];p) := \sup\{\, t \ge 0 \mid \pi^*[\omega] - t[E] \text{ is nef}\,\} = \inf_{C\ni p} \frac{[\omega]\cdot[C]}{\mathrm{mult}_p(C)},$$

where the infimum is taken over irreducible curves $C \subset X$ passing through p and $\mathrm{mult}_p(C)$ denotes the multiplicity of C at p. In particular, for every $0 < t < \varepsilon([\omega];p)$ the class $[\omega_t]$ is Kähler on $\tilde{X}$.

In this paper we will only consider Kähler classes $[\omega]$ with

$$c_1(X)\cdot[\omega] > 0,$$

since these are the classes that can carry PSC Kähler metrics, and we still denote by $\mathcal{K}$ all Kähler classes in the Kähler cone satisfying $c_1(X)\cdot[\omega] > 0$ without furthur mention. Brown's result [Bro24, Theorem A] implies that, if $[\omega]$ contains a PSC Kähler metric, then for all sufficiently small $t > 0$ the classes $[\omega_t] = \pi^*[\omega] - t[E]$ on any blow–up $\tilde{X}$ also contain PSC Kähler metrics. In later sections we will estimate $\mathcal{J}_X$ along such families and, ultimately, prove that $\mathcal{J}_X$ is uniformly bounded from above on all PSC Kähler classes arising from $\mathbb{P}^2$ and ruled surfaces under finitely many blow–ups.

3. The systolic inequality on $\mathbb{P}^2$ and its blow-up

In this section we study the 2-systolic inequality on $\mathbb{P}^2$ and on its blow-up $\mathrm{Bl}_k\mathbb{P}^2$ at k points. On $\mathbb{P}^2$ we write H for the hyperplane class, so that the Néron–Severi group is

$$NS^1(\mathbb{P}^2;\mathbb{R}) = \langle H\rangle, \qquad H^2 = 1.$$

Any Kähler class on $\mathbb{P}^2$ is of the form $[\omega] = aH,\ a > 0$, and every effective curve class is a positive multiple of H. In particular, the holomorphic 2-systole is

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) = \inf_C [\omega]\cdot[C] = aH\cdot H = a,$$

while

$$c_1(\mathbb{P}^2) = 3H, \qquad c_1(\mathbb{P}^2)\cdot[\omega] = 3a, \qquad [\omega]^2 = a^2.$$

Hence

$$\mathcal{J}_{\mathbb{P}^2}([\omega]) = 4\pi \ \mathrm{sys}_2(\omega) \, \frac{c_1(\mathbb{P}^2) \cdot [\omega]}{[\omega]^2} = 4\pi \cdot a \cdot \frac{3a}{a^2} = 12\pi. \tag{3.1}$$

We now pass to the blow-up $X_k := \mathrm{Bl}_k \mathbb{P}^2$ at k points. The Néron–Severi group of X_k is

$$NS^1(X_k; \mathbb{R}) = \langle H, E_1, \dots, E_k \rangle,$$

where H denotes the pullback of the hyperplane class and E_i the exceptional divisors, with

$$H^2 = 1, \qquad E_i^2 = -1, \qquad H \cdot E_i = E_i \cdot E_j = 0 \ (i \neq j).$$

In particular, any Kähler class can be written as

$$[\omega] = aH - \sum_{i=1}^{k} t_i E_i, \qquad a, t_i \in \mathbb{R}_{>0},$$

while a curve class in $H_2(X_k; \mathbb{Z})$ is written

$$[C] = dH - \sum_{i=1}^{k} m_i E_i, \qquad d, m_i \in \mathbb{Z},$$

where we tacitly identify H^2 and H_2 via Poincaré duality.

In view of (3.1), our goal is to show that

$$\mathcal{J}_{X_k}([\omega]) < 12\pi$$

for every Kähler class $[\omega]$ on X_k and every $k \geq 1$. Since $\mathcal{J}_{X_k}$ is defined purely in terms of intersection numbers, this is intrinsically a finite–dimensional optimization problem on the Kähler cone. However, the complexity of the Mori cone $\overline{\mathrm{NE}}(X_k)$ increases rapidly with k: for large k there are many extremal rays, and for $k \geq 9$ (in the non–del Pezzo regime) the cone is not even finitely generated. A direct ray–by–ray analysis of all effective curve classes is therefore hopeless.

Instead of keeping track of each coefficient t_i separately, it is convenient to work with the aggregate quantity

$$S := \sum_{i=1}^{k} t_i.$$

Once a Kähler class $[\omega] = aH - \sum t_i E_i$ is fixed, the numbers $t_i > 0$ determine S, and the nef cone imposes a genuine geometric upper bound on S which does not depend on the particular choice of coordinates. Moreover, on X_k one has

$$[\omega]^2 = a^2 - \sum_{i=1}^{k} t_i^2, \qquad c_1(X_k) \cdot [\omega] = \left(3H - \sum_{i=1}^{k} E_i\right) \cdot \left(aH - \sum_{i=1}^{k} t_i E_i\right) = 3a - \sum_{i=1}^{k} t_i,$$

so that $\mathcal{J}_{X_k}([\omega])$ depends on the tuple $(t_1, \dots, t_k)$ only through

$$S = \sum_{i=1}^{k} t_i, \qquad Q := \sum_{i=1}^{k} t_i^2.$$

For a given Kähler class we set $m := \mathrm{sys}_2^{\mathrm{hol}}([\omega])$. In particular, since each exceptional curve E_i is effective and $[\omega] \cdot E_i = t_i$, we always have $m \le t_i \quad$ for all i. Thus, if we fix (a, m, S) and vary the individual t_i subject to

$$t_i \ge m, \qquad \sum_{i=1}^{k} t_i = S,$$

then a and S are held fixed while $\mathcal{J}_{X_k}([\omega])$ changes only through the quantity Q appearing in $[\omega]^2$. In particular, for fixed (a, m, S) the maximal possible value of $\mathcal{J}_{X_k}([\omega])$ is attained when Q is as large as allowed by these constraints. This leads to the following elementary "mass–shift" property, which will play a crucial role not only in the present section but also in our later analysis of Hirzebruch surfaces, i.e., a ruled surface fibred over a rational curve.

Proposition 3.1. Let $k \ge 2$, $m > 0$, and $S \ge km$. Consider

$$\Omega = \Big\{ t = (t_1, \dots, t_k) \in \mathbb{R}^k : \ t_i \ge m \text{ for all } i, \ \sum_{i=1}^{k} t_i = S \Big\}.$$

For $Q(t) := \sum_{i=1}^{k} t_i^2$ one has

$$\sup_{t \in \Omega} Q(t) = (k-1)m^2 + \big(S - (k-1)m\big)^2,$$

and the supremum is attained exactly (up to permutation of coordinates) at

$$t = (\underbrace{m, \dots, m}_{k-1}, \ S - (k-1)m).$$

Proof. Since $S \ge km$, the point $(S/k, \dots, S/k)$ belongs to Ω, so $\Omega \ne \emptyset$. In particular, Ω is compact in $\mathbb{R}^k$, and since Q is continuous, Q attains its maximum on Ω. Let $t = (t_1, \dots, t_k) \in \Omega$ be a maximizer.

We first show that at most one coordinate of t is strictly larger than m. Suppose by contradiction that there exist $i \ne j$ with

$$t_i \ge t_j > m.$$

Choose $\delta \in (0, \ t_j - m]$ and define

$$t'_i = t_i + \delta, \qquad t'_j = t_j - \delta, \qquad t'_\ell = t_\ell \quad (\ell \notin \{i, j\}).$$

Then $t' \in \Omega$, and

$$\begin{aligned} Q(t') - Q(t) &= (t_i + \delta)^2 + (t_j - \delta)^2 - (t_i^2 + t_j^2) \\ &= 2\delta(t_i - t_j) + 2\delta^2 \ \ge \ 2\delta^2 > 0, \end{aligned}$$

which contradicts the maximality of t. Hence at most one coordinate of t exceeds m.

Since all coordinates satisfy $t_i \ge m$, it follows that exactly $k-1$ coordinates are equal to m, and the remaining one equals $S - (k-1)m$. The condition $S \ge km$ ensures $S - (k-1)m \ge m$, so such a point lies in Ω. Evaluating Q there gives

$$Q(t) = (k-1)m^2 + \big(S - (k-1)m\big)^2.$$

Thus

$$\sup_{s\in\Omega} Q(s) = Q(t) = (k-1)m^2 + \big(S-(k-1)m\big)^2.$$

Finally, if $t \in \Omega$ satisfies

$$Q(t) = (k-1)m^2 + \big(S-(k-1)m\big)^2,$$

then t is a maximizer, and the above argument shows that (up to permutation) t must have the stated form. The case $S = km$ corresponds to $S-(k-1)m = m$, i.e. $t_i = m$ for all i. □

Since the blow-up points $p_1, \dots, p_k$ are in very general position, no effective curve is algebraically distinguished. Geometrically, the condition $m = \mathrm{sys}_2^{\mathrm{hol}}([\omega])$ means that every effective curve C has area at least m, and there exists at least one curve for which equality holds. As we vary the vector $(t_1, \dots, t_k)$ while keeping (a, m, S) fixed, the curve that realizes the minimum may change, but the value of the infimum m itself is required to stay the same. In purely numerical terms, this means that whenever we lower one of the t_i to m we simultaneously increase another t_j so that S remains constant, and the new configuration still satisfies $\mathrm{sys}_2^{\mathrm{hol}}([\omega]) = \omega \cdot E_i = t_i = m$. Since $0 < m \le t_i$ for all i, there is always room to transfer mass from those t_j that are strictly larger than m onto coordinates that are closer to m, without ever changing the value of m or of S: what changes is merely *which* curve attains the minimum.

Proposition 3.1 formalizes this idea: for any admissible configuration $(t_1, \dots, t_k)$ with given (a, m, S) there exists another configuration $(t'_1, \dots, t'_k)$ with the same (a, m, S) such that $Q' = \sum_i (t'_i)^2 \ge Q = \sum_i t_i^2$, and equality holds if and only if

$$(t'_1, \dots, t'_k) = (m, \dots, m, S-(k-1)m).$$

In other words, among all Kähler classes with fixed (a, m, S) the extremal vector $(m, \dots, m, S-(k-1)m)$ maximises Q and hence maximises $\mathcal{J}_{X_k}([\omega])$. Thus, to bound $\mathcal{J}_{X_k}$ from above it suffices to study the resulting one–parameter family indexed by S, rather than the full k–tuple $(t_1, \dots, t_k)$.

Lemma 3.2. Suppose $\mathrm{Bl}_k\mathbb{P}^2$ carries a PSC Kähler metric $\omega_t = \pi^*\omega - \sum t_i E_i$, where $\omega = aH$ is a Kähler metric on $\mathbb{P}^2$ for $a > 0$. Then

$$\mathcal{J}_{\mathrm{Bl}_k\mathbb{P}^2}([\omega_t]) < \mathcal{J}_{\mathbb{P}^2}([\omega]) = 12\pi.$$

Proof. We first treat the case $k = 1$. In this situation the Kähler class is $[\omega_t] = aH - tE$, and a direct computation (see Example 3.5) shows that

$$\mathcal{J}_{\mathrm{Bl}_1\mathbb{P}^2}([\omega_t]) = 4\pi\ \min\{a-t, t\}\,\frac{3a-t}{a^2-t^2} \ \le\ \frac{20\pi}{3} < 12\pi.$$

Thus, it remains to consider $k \ge 2$.

For $k \ge 2$, the exceptional curves E_i and the strict transforms of lines through two points, with classes $H - E_i - E_j$ $(1 \le i \ne j \le k)$, are effective. Hence

$$\omega_t \cdot E_i = t_i > 0, \qquad \omega_t \cdot (H - E_i - E_j) = a - t_i - t_j > 0,$$

are candidates for the holomorphic 2-systole. Set $m := \mathrm{sys}^{\mathrm{hol}}(\omega_t)$,$\min\{t_i,\ a-t_i-t_j:\ 1\le i\ne j\le k\}$. Then

$$\mathcal{J}_{\mathrm{Bl}_k\mathbb{P}^2}([\omega_t]) = 4\pi\, m\cdot\frac{3a-S}{a^2-Q} =: 4\pi\,\phi_k(t).$$

We now analyze the supremum of ϕ_k. Observe that $\mathcal{J}$ (and hence ϕ_k) is invariant under rescaling of the Kähler class. Thus, without loss of generality, we may assume $a=1$ in what follows. With this normalization,

$$\phi_k(t) = m\cdot\frac{3-S}{1-Q}.$$

By definition of m we have

$$t_i\ge m,\qquad 1-t_i-t_j\ge m\quad(1\le i\ne j\le k),$$

so in particular $0<m\le 1/3$. Summing $t_i\ge m$ also gives $S\ge km$.

Fix $m>0$ and $S\ge km$, and consider all $t=(t_1,\dots,t_k)$ with

$$t_i\ge m,\qquad \sum_{i=1}^k t_i = S.$$

By Proposition 3.1, we have

$$Q\le (k-1)m^2+\big(S-(k-1)m\big)^2,$$

with equality at

$$t^* = \big(m,\dots,m,\ S-(k-1)m\big)$$

up to permutation of coordinates. Since the denominator $1-Q$ is decreasing in Q, we obtain

$$\phi_k(t) = m\,\frac{3-S}{1-Q}\ \le\ m\,\frac{3-S}{1-(k-1)m^2-\big(S-(k-1)m\big)^2} =: F(m,S), \tag{3.2}$$

whenever $t_i>0$, $3-S>0$ and $1-Q>0$.

We now identify the (m,S) for which the upper bound $F(m,S)$ is sharp, i.e. for which t^* is still feasible and m is still the minimum appearing in the definition of ϕ_k. First, Proposition 3.1 requires $t_i^*\ge m$, i.e.

$$S-(k-1)m\ge m\quad\Longleftrightarrow\quad S\ge km.$$

Next, we need m to remain the minimum in

$$\{t_i^*,\ 1-t_i^*-t_j^*:\ 1\le i\ne j\le k\},$$

so that $\phi_k(t^*)$ is still of the form $m(3-S)/(1-Q(t^*))$. Since $t_i^*\ge m$, it suffices to ensure

$$1-t_i^*-t_j^*\ge m\quad(1\le i\ne j\le k),$$

which gives

$$S\le 1+(k-3)m.$$

Finally, we must have $1-Q(t^*)>0$, i.e.

$$1-(k-1)m^2-\big(S-(k-1)m\big)^2>0.$$

Collecting the constraints, the relevant domain for (m, S) is

$$km \le S \le 1+(k-3)m, \qquad 0 < m \le \tfrac{1}{3}, \qquad 1-(k-1)m^2-\big(S-(k-1)m\big)^2 > 0. \quad (3.3)$$

For (m, S) in this domain we have $\phi_k(t) \le F(m, S)$, and equality holds for $t = t^*$. Now, introduce

$$y := S-(k-1)m, \qquad A := 1-(k-1)m^2, \qquad c := m\big(3-(k-1)m\big),$$

so that $m \le y \le 1-2m$ and $A > 0$ on the domain, and

$$F(m, S) = \frac{c-my}{A-y^2}.$$

Viewing F as a function of y (with m fixed), a direct computation gives

$$\frac{\partial}{\partial y}F(m,(k-1)m+y) = \frac{-mA-my^2+2cy}{(A-y^2)^2} =: \frac{N(y)}{(A-y^2)^2}.$$

The numerator $N(y)$ is a concave quadratic in y with a unique maximum at

$$y_0 = \frac{c}{m} = 3-(k-1)m.$$

One checks that $y_0 \ge 1-2m$ for all admissible m, hence $N(y) \ge 0$ on $[m, 1-2m]$. Therefore, for fixed m, $F(m, S)$ is strictly increasing in S on the domain (3.3), and

$$\max_{km\le S\le 1+(k-3)m} F(m, S) = F\big(m,\, 1+(k-3)m\big).$$

A direct computation yields

$$F\big(m,\, 1+(k-3)m\big) = \frac{m\big(2-(k-3)m\big)}{1-(k-1)m^2-(1-2m)^2} = \frac{2-(k-3)m}{4-(k+3)m} =: G_k(m), \quad (3.4)$$

where the feasibility condition

$$1-(k-1)m^2-(1-2m)^2 > 0$$

is equivalent to

$$0 < m < \frac{4}{k+3}.$$

Differentiating (3.4) gives

$$G_k'(m) = \frac{-2k+18}{\big(4-(k+3)m\big)^2}.$$

We now distinguish three cases.

Case 1: $2 \le k \le 8$. Here $-2k+18 > 0$, so G_k is strictly increasing on $(0, 4/(k+3))$. Since $4/(k+3) \ge 1/3$ for $k \le 8$, the constraint $m \le 1/3$ is active, and

$$\sup_m G_k(m) = G_k\Big(\frac{1}{3}\Big) = 1.$$

This value is attained at $m = \frac{1}{3}$, $S = \frac{k}{3}$, that is, $t_1 = \cdots = t_k = \frac{1}{3}$, and hence

$$\sup_t \phi_k(t) = 1 \quad \text{for } 2 \le k \le 8.$$

Case 2: $k = 9$**.** In this case $G_9'(m) \equiv 0$, so G_9 is constant on $(0, 4/12)$. From (3.4) we obtain

$$G_9(m) = \frac{2-6m}{4-12m} = \frac{1}{2}, \qquad 0 < m < \frac{1}{3}.$$

Since $m < 1/3$ (the endpoint $m = 1/3$ is excluded by the strict inequality in the denominator condition), the value $G_9(m) = 1/2$ is not attained but can be approached as $m \uparrow 1/3$, i.e. along vectors t with $t_i \uparrow \frac{1}{3}$. Thus

$$\sup_t \phi_9(t) = \frac{1}{2}.$$

Case 3: $k \geq 10$**.** Now $-2k + 18 < 0$, so G_k is strictly decreasing on $(0, 4/(k+3))$ and

$$\sup_m G_k(m) = \lim_{m \downarrow 0} G_k(m) = \frac{1}{2}.$$

The corresponding extremal configurations are of the form

$$t = (m, \dots, m, 1-2m), \qquad m \downarrow 0,$$

for which $m \to 0$ and $S \to 1$. Hence

$$\sup_t \phi_k(t) = \frac{1}{2} \quad \text{for } k \geq 10.$$

Collecting the three cases, we arrive at

$$\sup_t \phi_k(t) = \begin{cases} 1, & 2 \leq k \leq 8, \\ \frac{1}{2}, & k \geq 9. \end{cases}$$

In particular,

$$\sup_t \phi_k(t) \leq 1 \quad \text{for all } k \geq 2.$$

Combining this with the case $k = 1$ treated at the beginning, consequently, we obtain

$$\mathcal{J}_{\mathrm{Bl}_k\mathbb{P}^2}([\omega_t]) < \mathcal{J}_{\mathbb{P}^2}([\omega]) = 12\pi$$

for all $k \geq 1$, as claimed. □

Remark 3.3. The dependence of the upper bound of $\mathcal{J}_{\mathrm{Bl}_k\mathbb{P}^2}$ on k is in perfect agreement with the standard classification of the blow-ups $\mathrm{Bl}_k\mathbb{P}^2$ according to the positivity of the canonical class. Indeed,

$$c_1^2(\mathrm{Bl}_k\mathbb{P}^2) = \left(3H - \sum_{i=1}^k E_i\right)^2 = 9 - k,$$

so that $\mathrm{Bl}_k\mathbb{P}^2$ is del Pezzo for $k \leq 8$, satisfies $K_{\mathrm{Bl}_k\mathbb{P}^2}$ nef and $c_1^2 = 0$ for $k = 9$, and has $c_1^2 < 0$ for $k \geq 10$.

On the other hand, the algebraic optimization in the proof above leads to the one-variable function $G_k(m)$. The sign of $G_k'(m)$ is exactly the sign of $c_1^2(X_k)$. Consequently,

- for $2 \leq k \leq 8$ (del Pezzo, $c_1^2 > 0$) the worst-case upper bound for $\mathcal{J}_{X_k}$ is achieved at the largest admissible value of the rescaled systole m;

- for $k = 9$ (K_{X_k} nef, $c_1^2 = 0$) the function G_k is flat, and the corresponding upper bound is approached by Kähler classes, but not attained;
- for $k \geq 10$ ($c_1^2 < 0$) the worst case occurs in the opposite regime $m \to 0$.

We also note that for $k \geq 5$ additional (-1)-curves appear, for instance conics of class $2H - \sum_{i \in I} E_i$ through five points. Their intersection with $[\omega_t]$ gives further inequalities (such as $2a - \sum_{i \in I} t_i > 0$) which can only shrink the feasible region for (m, S) and hence potentially lower the true supremum of $\mathcal{J}_{X_k}$. Thus the upper bounds of $\mathcal{J}_{\mathrm{Bl}_k\mathbb{P}^2}$ above should not be expected to be sharp in general. The fact that their k–dependence mirrors the del Pezzo/nef/negative classification is a genuinely geometric phenomenon rather than an artefact of the estimates.

As a consequence of the previous estimates, we obtain the desired global systolic inequality on $\mathbb{P}^2$ and its blow-ups.

Theorem 3.4. Let $X_0 = \mathbb{P}^2$ and let $X_k = \mathrm{Bl}_k\mathbb{P}^2$ be the blow-up at k points for $k \geq 1$. Suppose ω_k is a PSC Kähler metric on X_k. Then

$$\min_{X_k} S(\omega_k) \cdot \mathrm{sys}_2(\omega_k) \;\leq\; 12\pi, \tag{3.5}$$

with equality if and only if $k = 0$, in which case ω_0 is the Fubini-Study metric and $\mathrm{sys}^{\mathrm{hol}}$ is achieved by $\mathbb{P}^1$. In particular, (3.5) holds strictly for all blow-ups X_k with $k \geq 1$.

Example 3.5. We can compute the precise supremum of $\mathcal{J}_{X_k}([\omega])$ when $k = 1, 2$. For $k = 1$, any Kähler class can be written as $[\omega] = aH - tE$, for $a > 0,\ t > 0,\ a > t$. Also, we have

$$c_1(X_1) = 3H - E, \qquad c_1(X_1) \cdot [\omega] = 3a - t, \qquad [\omega]^2 = a^2 - t^2.$$

The Mori cone of X_1 is generated by E and $H - E$, so

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) = \min\{[\omega] \cdot E,\ [\omega] \cdot (H - E)\} = \min\{t,\ a - t\}.$$

Since $\mathcal{J}_{X_1}([\omega])$ is invariant under overall scaling of $[\omega]$, it only depends on the ratio $x := t/a \in (0, 1)$. We may therefore assume $a = 1$, and obtain

$$\mathcal{J}_{X_1}([\omega]) = 4\pi\, \min\{t, 1 - t\}\, \frac{3 - t}{1 - t^2} =: h_1(t), \qquad 0 < t < 1.$$

A direct computation shows that $h_1(t)$ is strictly increasing on $(0, \frac{1}{2}]$, so its maximum is attained at $t = \frac{1}{2}$, i.e. in the class proportional to $H - \frac{1}{2}E$. Hence

$$\sup_{[\omega]} \mathcal{J}_{X_1}([\omega]) = \mathcal{J}_{X_1}\big([H - \tfrac{1}{2}E]\big) = \frac{20\pi}{3}.$$

For $k = 2$, For $k = 2$, any Kähler class can be written as $[\omega] = aH - t_1E_1 - t_2E_2,\ a > 0,\ t_i > 0$. And,

$$c_1(X_2) = 3H - E_1 - E_2, \qquad c_1(X_2) \cdot [\omega] = 3a - t_1 - t_2, \qquad [\omega]^2 = a^2 - t_1^2 - t_2^2.$$

The Mori cone of X_2 is generated by the (-1)-curves $E_1, E_2, H - E_1 - E_2$, so

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) = \min\big\{t_1,\ t_2,\ a - t_1 - t_2\big\}.$$

Again $\mathcal{J}_{X_2}([\omega])$ is invariant under overall scaling of $[\omega]$, so we may set $a = 1$ and obtain

$$\mathcal{J}_{X_2}([\omega]) = 4\pi \, \min\{t_1, \; t_2, \; 1 - t_1 - t_2\} \frac{3 - t_1 - t_2}{1 - t_1^2 - t_2^2} =: h_2(t_1, t_2), \qquad 0 < t_1 + t_2 < 1.$$

A straightforward analysis of the resulting two-variable function shows that the maximum of h_2 occurs at $t_1 = t_2 = \frac{1}{3}$, so that

$$\sup_{[\omega]} \mathcal{J}_{X_2}([\omega]) = \mathcal{J}_{X_2}\Big(\big[H - \tfrac{1}{3}E_1 - \tfrac{1}{3}E_2\big]\Big) = 4\pi.$$

In both cases $k = 1, 2$, for any PSC Kähler metric ω on X_k we have

$$\min_{X_k} S(\omega) \cdot \mathrm{sys}_2(\omega) < \mathcal{J}_{X_k}([\omega]) \leq \sup_{\mathcal{K}} \mathcal{J}_{X_k},$$

and the first inequality is strict, since there is no cscK metric on X_k for $k = 1, 2$.

4. Systolic inequalities on ruled surfaces

In this section we study PSC Kähler metrics on ruled surfaces and establish a uniform upper bound for the 2-systole. The following theorem summarizes the global picture.

Theorem 4.1. Let $X \to B$ be a ruled surface (not necessarily minimal) fibred over a smooth complex curve B, and let ω be a PSC Kähler metric on X. Then

$$\mathrm{sys}_2(\omega) \cdot \min_X S(\omega) \;\leq\; 8\pi.$$

Moreover, equality holds if and only if $B \simeq \mathbb{P}^1$ and $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and, up to overall scaling, ω is the product Fubini–Study metric.

The proof of Theorem 4.1 will be divided according to the genus of the base curve. In the non-rational case $g(B) \geq 1$ we actually obtain the sharper bound 4π (Theorem 4.5), whereas in the rational case $B \simeq \mathbb{P}^1$ we show that the optimal constant is 8π, with rigidity only on $\mathbb{P}^1 \times \mathbb{P}^1$ (Theorem 4.10).

4.1. Geometry of ruled surfaces and notation.

Let $X_0 \to B$ be a ruled surface over a smooth complex curve B of genus g. Denote by C_0 a minimal section and by F the fibre class in X_0, satisfying

$$F^2 = 0, \quad C_0 \cdot F = 1, \quad C_0^2 = -e, \quad e \in \mathbb{Z}_{\geq 0}.$$

The first Chern class and a Kähler class of X_0 are given by

$$c_1(X_0) = 2C_0 + (2 - 2g + e)F, \qquad [\omega] = aC_0 + bF, \quad a > 0, \; b > ae.$$

A direct computation shows that

$$[\omega]^2 = (aC_0 + bF)^2 = 2ab - ea^2,$$

and

$$c_1(X_0) \cdot [\omega] = \big(2C_0 + (2 - 2g + e)F\big) \cdot (aC_0 + bF) = 2b + (2 - 2g - e)a.$$

Proposition 4.2. Let X_0 be a ruled surface over a smooth complex curve B of genus g, with invariant $e \geq 0$. Then

$$\sup_{\mathcal{K}} \mathcal{J}_{X_0}([\omega]) = \begin{cases} 4\pi \dfrac{e+4}{e+2}, & g = 0, \\ 4\pi, & g \geq 1. \end{cases} \tag{4.1}$$

Moreover, when $g = 0$ the supremum is attained precisely on classes proportional to $C_0 + (e+1)F$, and when $g \geq 1$ the supremum is attained if and only if $g = 1$ and $b/a \geq e+1$.

Proof. It is well known that the Mori cone $\overline{\mathrm{NE}}(X_0)$ is generated by the numerical classes of the fibre and a minimal section, in our notation, this is F and C_0. Every other effective curve is numerically equivalent to $C_0 + nF$ with $n \geq 0$, and has intersection at least as large with $[\omega]$ as one of these generators. Hence

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) = \min\{[\omega] \cdot F,\ [\omega] \cdot C_0\} = \min\{a,\ b - ae\}.$$

Thus

$$\mathcal{J}_{X_0}([\omega]) = \min\{a,\ b - ae\} \cdot 4\pi \, \frac{2b + (2 - 2g - e)a}{2ab - ea^2}.$$

The quantity $\mathcal{J}_{X_0}$ is homogeneous of degree 0 in $[\omega]$, so it depends only on the ratio $t := b/a$. The Kähler condition $b > ae$ becomes $t > e$. Writing

$$\mathcal{J}_{X_0}([\omega]) = 4\pi \, \Phi_{e,g}(t), \quad \text{with} \quad \Phi_{e,g}(t) := \min\{1,\ t - e\} \cdot \frac{2t + 2 - 2g - e}{2t - e}, \qquad t > e,$$

it suffices to compute $\sup_{t>e} \Phi_{e,g}(t)$. We distinguish two cases according to g.

Case 1: $g \geq 1$. For $g \geq 1$ we have $2 - 2g \leq 0$, hence

$$2t + 2 - 2g - e \ \leq\ 2t - e,$$

and therefore

$$\frac{2t + 2 - 2g - e}{2t - e} \leq 1, \qquad t > e.$$

It follows that

$$\Phi_{e,g}(t) = \min\{1,\ t - e\} \cdot \frac{2t + 2 - 2g - e}{2t - e} \leq \min\{1,\ t - e\} \leq 1,$$

so $\Phi_{e,g}(t) \leq 1$ for all $t > e$.

If $g = 1$, then $2 - 2g - e = -e$ and

$$\Phi_{e,1}(t) = \min\{1,\ t - e\} \cdot \frac{2t - e}{2t - e} = \min\{1,\ t - e\}.$$

Hence $\sup_{t>e} \Phi_{e,1}(t) = 1$, and the supremum is attained precisely when $t - e \geq 1$, i.e. $b/a \geq e + 1$.

If $g > 1$, then $2 - 2g < 0$ and the inequality $2t + 2 - 2g - e < 2t - e$ is strict, so

$$\Phi_{e,g}(t) < \min\{1,\ t - e\} \leq 1, \qquad t > e.$$

On the other hand,

$$\lim_{t \to +\infty} \Phi_{e,g}(t) = \lim_{t \to +\infty} \frac{2t + 2 - 2g - e}{2t - e} = 1,$$

so $\sup_{t>e}\Phi_{e,g}(t)=1$, but it is not attained for any finite t. Consequently, in all cases $g\geq 1$ we obtain

$$\sup_{t>e}\Phi_{e,g}(t)=1, \qquad \sup_{\mathcal{K}}\mathcal{J}_{X_0}([\omega])=4\pi.$$

Case 2: $g=0$**.** Now $2-2g-e=2-e$, and for $t>e$ we split

$$\Phi_{e,0}(t)=\begin{cases}(t-e)\dfrac{2t+2-e}{2t-e}, & e<t\leq e+1,\\ \dfrac{2t+2-e}{2t-e}, & t\geq e+1.\end{cases}$$

For $t\geq e+1$, a straightforward derivative computation gives

$$\Phi'_{e,0}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2t+2-e}{2t-e}\right)=-\frac{4}{(2t-e)^2}<0,$$

so on $[e+1,+\infty)$ the function $\Phi_{e,0}$ is strictly decreasing, and

$$\max_{t\geq e+1}\Phi_{e,0}(t)=\Phi_{e,0}(e+1)=\frac{2(e+1)+2-e}{2(e+1)-e}=\frac{e+4}{e+2}.$$

For $e<t\leq e+1$, a direct computation shows that $\Phi'_{e,0}(t)>0$ for all $t>e$, so $\Phi_{e,0}$ is strictly increasing on $(e,e+1]$ and

$$\max_{e<t\leq e+1}\Phi_{e,0}(t)=\Phi_{e,0}(e+1)=\frac{e+4}{e+2}.$$

Combining the two ranges, we see that $\Phi_{e,0}$ attains its global maximum on $(e,+\infty)$ at $t=e+1$, with value

$$\sup_{t>e}\Phi_{e,0}(t)=\Phi_{e,0}(e+1)=\frac{e+4}{e+2}.$$

This corresponds exactly to classes proportional to $C_0+(e+1)F$. Thus, in the case $g=0$,

$$\sup_{\mathcal{K}}\mathcal{J}_{X_0}([\omega])=4\pi\,\frac{e+4}{e+2},$$

and the supremum is attained precisely on Kähler classes with $b/a=e+1$. □

Remark 4.3. It is well known that a ruled surface over the projective line is a Hirzebruch surface $\mathbb{F}_e$ with invariant $e\in\mathbb{Z}_{\geq 0}$. When $e=1$, $\mathbb{F}_1$ is not minimal and is isomorphic to the blowup $\mathrm{Bl}_1\mathbb{P}^2$ of $\mathbb{P}^2$ at one point. As expected, our computations of $\mathcal{J}_{\mathrm{Bl}_1\mathbb{P}^2}$ in Example 3.5 and of $\mathcal{J}_{\mathbb{F}_1}$ in Proposition 4.2 yield the same value.

For later use, we also fix notation for blowups of ruled surfaces. We do assume that all blowup points are in very general positions, that is, each point lies on a distinct fiber of the ruling and that no point lies on the negative section C_0. Indeed, if some points collide on the same fiber or lie on C_0, additional effective curves appear whose strict transforms have smaller intersection with any fixed Kähler class, so they can only decrease $\mathrm{sys}_2^{\mathrm{hol}}([\omega])$ and hence $\mathcal{J}_{X_k}([\omega])$. Thus this configuration is the worst case for the supremum.

Let $X_k = \mathrm{Bl}_k X_0$ be the blowup of X_0 at k points in very general position, and denote by $E_1, \dots, E_k$ the exceptional divisors. Then

$$NS^1(X_k; \mathbb{R}) = \langle C_0, F, E_1, \dots, E_k \rangle,$$

with intersection pairings

$$C_0^2 = -e, \quad F^2 = 0, \quad C_0 \cdot F = 1, \quad E_i^2 = -1, \quad E_i \cdot C_0 = E_i \cdot F = E_i \cdot E_j = 0 \ (i \neq j).$$

A Kähler class on X_k can be written as

$$[\omega] = aC_0 + bF - \sum_{i=1}^{k} t_i E_i, \qquad a > 0, \ b > ae, \ t_i > 0,$$

and the first Chern class is

$$c_1(X_k) = \pi^* c_1(X_0) - \sum_{i=1}^{k} E_i = 2C_0 + (2 - 2g + e)F - \sum_{i=1}^{k} E_i.$$

Consequently,

$$[\omega]^2 = 2ab - ea^2 - \sum_{i=1}^{k} t_i^2, \qquad c_1(X_k) \cdot [\omega] = 2b + (2 - 2g - e)a - \sum_{i=1}^{k} t_i.$$

These expressions will be used repeatedly in the subsequent subsections.

4.2. **Non-rational ruled surfaces.** Suppose $X_0 \to B$ is a compact ruled surface over a smooth complex curve B of genus $g > 0$. It follows from Proposition 4.2 that

$$\sup_{\mathcal{K}} \mathcal{J}_{X_0}([\omega]) = 4\pi.$$

We now study blow-ups of such ruled surfaces.

Lemma 4.4. Let X_0 be a minimal ruled surface over a smooth curve B of genus $g \geq 1$, with invariant $e \geq 0$. For an integer $k \geq 1$, let $X_k = \mathrm{Bl}_k(X_0)$ be the blow-up at k points in very general position, with exceptional divisors $E_1, \dots, E_k$. Then for every $k \geq 1$ one has

$$\sup_{\mathcal{K}} \mathcal{J}_{X_k}([\omega]) = 2\pi.$$

Proof. Since $\mathcal{J}_{X_k}([\omega])$ is homogeneous of degree 0 in $[\omega]$, we may therefore normalize $a = 1$ and write

$$[\omega] = C_0 + bF - \sum_{i=1}^{k} t_i E_i, \qquad b > e, \ t_i > 0,$$

with the Kähler condition implying $2b - e - \sum t_i^2 > 0$. As in the minimal case, $\mathcal{J}_{X_k}([\omega])$ can be written as

$$\mathcal{J}_{X_k}([\omega]) = \mathrm{sys}_2^{\mathrm{hol}}([\omega]) \cdot 4\pi \, \frac{2b + 2 - 2g - e - \sum_{i=1}^{k} t_i}{2b - e - \sum_{i=1}^{k} t_i^2}.$$

Since all blowup points are in very general positions, there is no p_i lies on C_0 and no two p_i lie on the same fibre. For each blownup point $p_i \in X_0$, let F_i be the unique fibre through p_i

and let $\widetilde{F}_i \subset X_k$ be its strict transform. Then $\widetilde{F}_i$ has class $F - E_i$ and is effective. Since each exceptional divisor E_i is also effective, we obtain

$$[\omega] \cdot E_i = t_i, \qquad [\omega] \cdot (F - E_i) = (C_0 + bF - \sum t_j E_j) \cdot (F - E_i) = 1 - t_i.$$

In particular,

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) \leq \min_{1 \leq i \leq k} \big\{ [\omega] \cdot E_i,\ [\omega] \cdot (F - E_i) \big\} = \min_{1 \leq i \leq k} \{t_i, 1 - t_i\}.$$

The Kähler condition $[\omega] \cdot (F - E_i) > 0$ implies $0 < t_i < 1$, hence for each i the function $\min\{t_i, 1 - t_i\}$, viewed as a function of $t_i \in (0, 1)$, attains its maximal value $1/2$ at $t_i = 1/2$. Thus for all Kähler classes

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) \leq \frac{1}{2}.$$

On the other hand, using $g \geq 1$ and $0 < u_i < 1$ we have

$$\big(2b + 2 - 2g - e - \sum t_i\big) - \big(2b - e - \sum t_i^2\big) = (2 - 2g) + \sum (t_i^2 - t_i) < 0.$$

Hence

$$\frac{2b + 2 - 2g - e - \sum t_i}{2b - e - \sum t_i^2} < 1,$$

and therefore

$$\mathcal{J}_{X_k}([\omega]) \leq 4\pi\ \mathrm{sys}_2^{\mathrm{hol}}([\omega]) \leq 4\pi \cdot \frac{1}{2} = 2\pi.$$

It remains to show that the bound 2π is sharp. Consider the family of Kähler classes

$$[\omega_b] = C_0 + bF - \sum_{i=1}^{k} t_i E_i, \qquad b > e.$$

In fact, any effective curve on X_k has the form $C = dC_0 + nF - \sum m_i E_i$ for $d, n, m_i \in \mathbb{Z}$, with the intersection number $\omega \cdot C = (b - e)d + n - m_i t_i$. We take b large enough, so that the curves E_i or $F - E_i$ realize the holomorphic 2-systole, i.e.

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega_b]) = \min_{1 \leq i \leq k} \{t_i, 1 - t_i\}.$$

Observe that, for all $b > 0$ large enough, the classes $[\omega_b]$ remain in the Kähler cone, and

$$\hat{S}([\omega_b]) = 4\pi \frac{2b + 2 - 2g - e - \sum t_i}{2b - e - \sum t_i^2} \nearrow 4\pi, \qquad \text{as} \quad b \nearrow \infty.$$

Therefore, by setting $t_i = 1/2$, we have

$$2\pi \frac{2b + 2 - 2g - e - k/2}{2b - e - k/4} = \mathrm{sys}_2^{\mathrm{hol}}([\omega_b]) \cdot \hat{S}([\omega_b]) = \mathcal{J}_{X_k}([\omega_b]) \leq \sup_{\mathcal{K}} \mathcal{J}_{X_k} \leq 2\pi.$$

Taking $b \nearrow \infty$, we arrive at our conclusion. $\square$

Combining the estimates for the minimal ruled surface X_0 and its blowups X_k we obtain the following systolic inequality in the non-rational case.

Theorem 4.5. Let $X \to B$ be a ruled surface (not necessarily minimal) fibred over a complex curve B of genus $g \geq 1$, and let ω be a PSC Kähler metric on X. Then

$$\mathrm{sys}_2(\omega) \cdot \min_X S(\omega) \leq 4\pi.$$

Moreover, equality holds if and only if B is an elliptic curve, X is minimal and isometrically covered by $\mathbb{P}^1 \times \mathbb{C}$, and ω has constant scalar curvature so that $\omega \cdot F \leq \omega \cdot C_0$, in which case the $\mathrm{sys}_2(\omega)$ is achieved by the $\mathbb{P}^1$-fibre.

Proof. By Proposition 4.2 and Lemma 4.4, we obtain

$$\mathrm{sys}_2(\omega) \cdot \min_X S(\omega) \leq \sup_{\mathcal{K}} \mathcal{J}_X([\omega]) = 4\pi,$$

with equality if and only if B is an elliptic curve, ω has constant scalar curvature, and the holomorphic 2-systole is realised by the $\mathbb{P}^1$-fibre. It therefore suffices to work in this equality case and prove that X is isometrically covered by $\mathbb{P}^1 \times \mathbb{C}$.

Since $X \xrightarrow{\pi} B$ is a ruled surface fibred over an elliptic curve, there exists a rank 2 holomorphic vector bundle $\mathcal{E}$ on B such that $X \simeq \mathbb{P}(\mathcal{E})$ and π is the bundle projection (see [Har77, Chap. V]). By the classification of cscK metrics on ruled surfaces due to Apostolov and Tønnesen-Friedman (see [ATF06, Thm. 2], the existence of a cscK metric on $X \simeq \mathbb{P}(\mathcal{E})$ is equivalent to the slope-polystability of the underlying rank 2 bundle $\mathcal{E}$ over the base curve B. Hence our cscK metric forces $\mathcal{E}$ to be polystable.

Since B is an elliptic curve, the Narasimhan–Seshadri theorem [NS65] (see also [Don83]) combined with the Kobayashi–Hitchin correspondence (see, for instance, [Kob87]) implies that a polystable bundle $\mathcal{E}$ admits a Hermitian–Einstein metric whose Chern connection D is projectively flat; equivalently, its curvature is a scalar multiple of the identity. Since D is projectively flat, the induced $PGL_2(\mathbb{C})$–connection on $\mathbb{P}(\mathcal{E})$ is flat, so X is isomorphic to a quotient $(\widetilde{B} \times \mathbb{P}^1)/\Gamma$ for some projective unitary representation $\Gamma \subset PGL_2(\mathbb{C})$. In Fujiki's terminology [Fuj92, Lemma 2], this means that X is a quasi-stable ruled manifold. By [Fuj92, Lemmas 8 and 10], quasi-stable ruled manifolds admit canonical generalised extremal (in particular, cscK) metrics and, moreover, on a genus 1 base any cscK metric is of this type and is locally symmetric. In particular, the lift of our cscK metric ω to the universal cover of X splits as a Riemannian product

$$(\widetilde{X}, \widetilde{\omega}) \;\simeq\; (\mathbb{C}, \omega_{\text{flat}}) \times (\mathbb{P}^1, \omega_{\text{FS}}).$$

Thus X is isometrically covered by $\mathbb{P}^1 \times \mathbb{C}$ endowed with the product Kähler metric, as claimed. □

4.3. **Rational ruled surfaces.** In this subsection we treat the case where the base curve is rational, so that $B \simeq \mathbb{P}^1$ and $X_0 = \mathbb{F}_e \longrightarrow \mathbb{P}^1$ is a Hirzebruch surface with invariant $e \geq 0$. We denote by $X_k = \mathrm{Bl}_k(\mathbb{F}_e)$ the blow-up of $\mathbb{F}_e$ at k points in very general positions.

Proposition 4.6. Let $\mathbb{F}_e$ be a Hirzebruch surface with invariant $e \geq 0$, and let $X_1 = \mathrm{Bl}_p(\mathbb{F}_e)$ be its blow-up at one point. Then:

(1) If $e=0$ (so $X_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$), one has

$$\sup_{\mathcal{K}} \mathcal{J}_{X_1}([\omega]) \;=\; 4\pi.$$

(2) If $e \geq 1$, one has

$$\sup_{\mathcal{K}} \mathcal{J}_{X_1}([\omega]) \;=\; 4\pi \cdot \frac{2e+5}{4e+3}.$$

Proof. We treat separately the cases $e=0$ and $e \geq 1$. When $e=0$, in which case $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Write the two rulings as $F_1 := [\mathbb{P}^1 \times \{\mathrm{pt}\}],\ F_2 := [\{\mathrm{pt}\} \times \mathbb{P}^1]$, so that $F_1^2 = F_2^2 = 0$, and $F_1 \cdot F_2 = 1$. Any Kähler class on X_1 can be written as $\omega = aF_1 + bF_2 - tE$. Note that the surface X_1 is the degree 7 del Pezzo surface and its Mori cone $\overline{\mathrm{NE}}(X_1)$ is generated by the three (-1)-curves $E,\ F_1 - E$, and $F_2 - E$. Hence,

$$\mathrm{sys}_2^{\mathrm{hol}}([\omega]) = \min\{t,\ a-t,\ b-t\},$$

and

$$\mathcal{J}_{X_1}([\omega]) = \min\{t,\ a-t,\ b-t\} \cdot \frac{4\pi(2a+2b-t)}{2ab-t^2}.$$

Setting $x := a/t,\ y := b/t$, so that $a = xt$, $b = yt$, then the Kähler conditions $t>0$, $a-t>0$, $b-t>0$, and $[\omega]^2 > 0$ translate into

$$x > 1, \quad y > 1, \quad \text{and } \ 2xy > 1.$$

Thus,

$$\mathcal{J}_{X_1}([\omega]) = 4\pi \ \min\{1, x-1, y-1\} \frac{2x+2y-1}{2xy-1},$$

defined on $\{(x,y) \mid x>1,\ y>1\}$. Observe that

$$\sup_{x,y} \left(\min\{1, x-1, y-1\} \frac{2x+2y-1}{2xy-1} \right) = 1$$

on this domain, with equality at $x=y=2$. We then conclude

$$\sup_{[\omega]} \mathcal{J}_{X_1}([\omega]) = 4\pi \sup_{x>1, y>1} \left(\min\{1, x-1, y-1\} \frac{2x+2y-1}{2xy-1} \right) = 4\pi.$$

Moreover, the supremum is achieved by the class $[\omega] = F_1 + F_2 - \frac{1}{2}E$. This proves (1).

We now assume $e \geq 1$ and write the Kähler class on X_1 as $[\omega] = aC_0 + bF - tE,\ a>0,\ b>0,\ t>0$. In this case, he Mori cone $\overline{\mathrm{NE}}(X_1)$ is generated by three extremal rays E, $F-E$ and C_0. Hence,

$$\mathrm{sys}_2([\omega]) = \min\{t,\ a-t,\ b-ea\}.$$

Introduce the scale-invariant variables again $x := a/t$, and $y := b/t$, so that $a = xt$, $b = yt$. The Kähler inequalities $t>0$, $a-t>0$, $b-ea>0$ are equivalent to

$$x > 1, \qquad y > ex.$$

It is convenient to write $y = ex + z$ with $z > 0$, so that

$$\mathcal{J}_{X_1}([\omega]) = 4\pi \ \min\{1,\ x-1,\ z\} \frac{(e+2)x + 2z - 1}{ex^2 + 2xz - 1} := 4\pi \ \Phi_e(x,z),$$

defined on the domain

$$\mathcal{D}_e := \{(x,z) \in \mathbb{R}^2 \mid x > 1,\ z > 0,\ ex^2 + 2xz - 1 > 0\}.$$

A direct computation of the partial derivatives shows that Φ_e has a unique critical point in $\mathcal{D}_e$ at $(x,z) = (2,1)$, so that this point realizes a global maximum. At $(x,z) = (2,1)$ one has

$$\sup_{(x,z)\in\mathcal{D}_e} \Phi_e(x,z) = \Phi_e(2,1) = \frac{2e+5}{4e+3}.$$

The corresponding ray in the Kähler cone is given by $[\omega] = 2t\,C_0 + (2e+1)t\,F - tE$, and we obtain

$$\sup_{\mathcal{K}} \mathcal{J}_{X_1}([\omega]) = 4\pi \cdot \frac{2e+5}{4e+3}.$$

This proves (2) and completes the proof of the lemma. □

Having dealt with the case of a single blow-up of a Hirzebruch surface, we now turn to blowing up several points on $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. As in the $\mathbb{P}^2$-case (see Proposition 3.1 in Section 3), the key input is a simple "mass-shift" optimization for the exceptional parameters t_i, which allows us to reduce to an extremal configuration and then perform a purely algebraic estimate.

Lemma 4.7. Let $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and let $X_k = \mathrm{Bl}_k(X_0)$ be the blow-up of X_0 at $k \geq 2$ points in very general position. Then

$$\sup_{\mathcal{K}} \mathcal{J}_{X_k}([\omega]) \ \leq\ 4\pi.$$

Proof. Let $F_1 = [\mathbb{P}^1 \times \{\mathrm{pt}\}]$ and $F_2 = [\{\mathrm{pt}\} \times \mathbb{P}^1]$ be the two rulings, and let $E_1, \dots, E_k$ be the exceptional curves. Since the two rulings are algebraically equivalent and $\mathcal{J}_{X_k}$ is homogeneous of degree zero, we may assume that the coefficient of the "smaller" ruling is 1. More precisely, any Kähler class can be written (after possibly interchanging F_1, F_2 and rescaling) in the form $[\omega] = F_1 + bF_2 - \sum t_i E_i$, for $b \geq 1$, $t_i > 0$. Set $m = \mathrm{sys}_2^{\mathrm{hol}}([\omega])$. The effective curves $E_i,\ F_1 - E_i,\ F_2 - E_i$ on X_k yield the Kähler condition

$$[\omega]\cdot E_i = t_i \geq m, \quad [\omega]\cdot(F_1 - E_i) = 1 - t_i \geq m, \quad [\omega]\cdot(F_2 - E_i) = b - t_i \geq m.$$

Hence, for all i,

$$m \ \leq\ t_i \ \leq\ 1 - m, \qquad t_i \ \leq\ b - m. \tag{4.2}$$

In particular $0 < m \leq \frac{1}{2}$ and $b \geq 1$.

In this setting, we again apply the mass–shifting reduction introduced in the $\mathbb{P}^2$ case. We will not elaborate further on this point and refer the reader to Section 3 for details. Write $S := \sum_{i=1}^k t_i$ and $Q := \sum_{i=1}^k t_i^2$. Thus

$$\mathcal{J}_{X_k}([\omega]) = 4\pi\, m \cdot \frac{2b + 2 - S}{2b - Q}. \tag{4.3}$$

Fix (b, m, S) in the range $0 < m \leq \frac{1}{2}$, $\quad b \geq 1$, $\quad km \leq S \leq 1 + (k-2)m$. For such parameters any admissible k-tuple $t = (t_1, \dots, t_k)$ satisfies $t_i \geq m$ and $\sum t_i = S$. By Proposition 3.1, under the constraints $t_i \geq m$, $\sum t_i = S$, the quadratic form $Q(t) = \sum t_i^2$ is maximized precisely when

$$t = (\underbrace{m, \dots, m}_{k-1}, S - (k-1)m)$$

up to permutation, and in that case

$$Q_{\max} = (k-1)m^2 + \big(S - (k-1)m\big)^2.$$

Consequently, for each fixed triple (b, m, S), we have

$$\mathcal{J}_{X_k}([\omega]) \ \le\ 4\pi\, m \cdot \frac{2b + 2 - S}{2b - Q_{\max}} =: 4\pi\, F(b, m, S),$$

where

$$F(b, m, S) = \frac{m(2b + 2 - S)}{2b - (k-1)m^2 - \big(S - (k-1)m\big)^2}. \tag{4.4}$$

Set

$$N := m(2b + 2 - S), \qquad D := 2b - (k-1)m^2 - \big(S - (k-1)m\big)^2,$$

so that $F = N/D$ on the domain $\{D > 0\} \cap \{N > 0\}$. We claim that

$$D(b, m, S) \ \ge\ N(b, m, S)$$

for all admissible (b, m, S), which immediately implies $F(b, m, S) \le 1$.

To verify the claim, a direct computation gives

$$\begin{aligned} D - N &= 2b - (k-1)m^2 - \big(S - (k-1)m\big)^2 - m(2b + 2 - S) \\ &= 2b(1-m) - 2m + mS - \big(S - (k-1)m\big)^2. \end{aligned}$$

Introduce $z := S - (k-1)m$ with $m \le z \le 1 - m$. In terms of (m, z) we obtain

$$D - N = 2b(1-m) - 2m + mz - z^2.$$

Observe that $D - N$ is strictly increasing as a function of b. Under the constraint $b \ge 1$ it attains its minimum at $b = 1$. Thus

$$D(b, m, S) - N(b, m, S) \ \ge\ G(m, z),$$

where

$$G(m, z) := D(1, m, S) - N(1, m, S) = 2 - 4m + mz - z^2.$$

On the interval $z \in [m, 1-m]$ we have $\partial_z G(m, z) = m - 2z < 0$, so $G(m, \cdot)$ is strictly decreasing, and hence

$$\min_{z \in [m, 1-m]} G(m, z) = G(m, 1-m).$$

A short calculation yields

$$G(m, 1-m) = -(2m^2 + m - 1) \ge 0.$$

Therefore $G(m, z) \ge 0$ for all $z \in [m, 1-m]$. Combining this with the monotonicity in b gives

$$D(b, m, S) - N(b, m, S) \ \ge\ 0$$

for every admissible triple (b, m, S), and hence $F(b, m, S) \le 1$.

Consequently,

$$\mathcal{J}_{X_k}([\omega]) \;\le\; 4\pi\, F(b,m,S) \;\le\; 4\pi$$

for every Kähler class $[\omega]$ on X_k. This shows that

$$\sup_{\mathcal{K}} \mathcal{J}_{X_k}([\omega]) \le 4\pi,$$

which is the desired estimate. □

We now consider blow-ups of the general Hirzebruch surfaces $\mathbb{F}_e$ with $e \ge 1$. The argument is parallel, with the only new feature being the contribution of the negative section C_0 in the intersection computations. The final bound is again 4π, and when $e = 1$, this recovers the case of $\mathbb{P}^2$ blown up at one point via the identification $\mathbb{F}_1 \simeq \mathbb{P}^2 \# \overline{\mathbb{P}^2}$.

Lemma 4.8. Let $X_0 = \mathbb{F}_e$ be a Hirzebruch surface with $e \ge 1$, and let $X_k = \mathrm{Bl}_k(X_0)$ be the blow-up of X_0 at $k \ge 2$ points in very general position. Then

$$\sup_{\mathcal{K}} \mathcal{J}_{X_k}([\omega]) \;\le\; 4\pi.$$

Proof. Write any Kähler class on X_k in the form $[\omega] \;=\; C_0 + bF - \sum t_i E_i$, for $b > e,\ t_i > 0$. We use the same notations as before, so that

$$\mathcal{J}_{X_k}([\omega]) = 4\pi\, m \cdot \frac{2b + 2 - e - S}{2b - e - Q}. \tag{4.5}$$

Fix parameters (b, m, S) in the range

$$0 < m \le \tfrac{1}{2}, \qquad b \ge e + m, \qquad km \le S \le 1 + (k-2)m.$$

By the mass–shift Proposition 3.1, under the constraints $t_i \ge m,\ \sum t_i = S$, the quadratic form $Q(t) = \sum t_i^2$ is maximized precisely when

$$t = (\underbrace{m, \dots, m}_{k-1},\, S - (k-1)m)$$

up to permutation, and in that case

$$Q_{\max} = (k-1)m^2 + \big(S - (k-1)m\big)^2.$$

Hence, for each fixed triple (b, m, S),

$$\mathcal{J}_{X_k}([\omega]) \;\le\; 4\pi\, m \cdot \frac{2b + 2 - e - S}{2b - e - Q_{\max}} =: 4\pi\, F_{e,k}(b, m, S),$$

where

$$F_{e,k}(b,m,S) := \frac{m\big(2b + 2 - e - S\big)}{2b - e - (k-1)m^2 - \big(S - (k-1)m\big)^2}. \tag{4.6}$$

We use the same trick as in the proof of Lemma 4.7 with just a slight difference when we deal with the extremum. Set

$$N := m\big(2b + 2 - e - S\big), \qquad D := 2b - e - (k-1)m^2 - \big(S - (k-1)m\big)^2,$$

so that $F = N/D$ on the domain $\{D > 0\} \cap \{N > 0\}$. We claim that

$$D(b,m,S) \;\ge\; N(b,m,S)$$

for all admissible (b, m, S), which immediately implies $F(b, m, S) \leq 1$.

A direct computation gives

$$\begin{aligned} D - N &= 2b - e - (k-1)m^2 - \big(S - (k-1)m\big)^2 - m\big(2b + 2 - e - S\big) \\ &= 2b(1-m) - e(1-m) - (k-1)m^2 - \big(S - (k-1)m\big)^2 - 2m + mS. \end{aligned} \tag{4.7}$$

Note that $D - N$ is strictly increasing as a function of b. Under the constraint $b \geq e + m$ it attains its minimum at $b_0 := e + m$. In particular,

$$D(b, m, S) - N(b, m, S) \ \geq \ D(b_0, m, S) - N(b_0, m, S). \tag{4.8}$$

Substituting $b = b_0 = e + m$ into (4.7) yields

$$D(b_0, m, S) - N(b_0, m, S) = e(1-m) - 2m^2 + mS - \big(S - (k-1)m\big)^2.$$

Introduce $z := S - (k-1)m$ with $m \ \leq \ z \ \leq \ 1 - m$. In terms of (m, z) we obtain

$$D(b_0, m, S) - N(b_0, m, S) = \Phi_e(m, z) := e(1-m) - 2m^2 + mz - z^2. \tag{4.9}$$

Combining (4.8) and (4.9) we deduce

$$D(b, m, S) - N(b, m, S) \ \geq \ \Phi_e(m, z)$$

for all admissible (b, m, S). A short computation gives

$$\begin{aligned} \min_{z \in [m, 1-m]} \Phi_e(m, z) = \Phi_e(m, 1-m) &= e(1-m) - 2m^2 + m(1-m) - (1-m)^2 \\ &= e(1-m) - 4m^2 + 3m - 1 \\ &= (e-1) + (3-e)m - 4m^2 \\ &\geq 0, \end{aligned}$$

with the inequality holds strictly if $e \geq 2$. In particular, $\Phi_e(m, t) \geq 0$ for all admissible m, t, and hence $F_{e,k}(b, m, S) \leq 1$. Moreover, if $e > 1$, then $F_{e,k} < 1$.

Consequently,

$$\mathcal{J}_{X_k}([\omega]) \ \leq \ 4\pi \, F_{e,k}(b, m, S) \ \leq \ 4\pi$$

for every Kähler class $[\omega]$ on X_k. Therefore

$$\sup_{\mathcal{K}} \mathcal{J}_{X_k}([\omega]) \leq 4\pi,$$

as claimed. $\square$

Before turning to the global statement for rational ruled surfaces, it is instructive to isolate a concrete situation where the quantity $\mathcal{J}_{X_k}$ can be written down explicitly. In the regime of "small" blow-ups of a Hirzebruch surface (namely $k \leq e$ and the points in very general position), the Mori cone is finitely generated by a short list of curves, so that $\mathrm{sys}_2^{\mathrm{hol}}([\omega])$ and even $\mathcal{J}_{X_k}([\omega])$ reduce to an explicit finite-dimensional optimization problem in the parameters of the Kähler class. The following example makes this reduction precise.

Example 4.9 (Small blow-ups of a Hirzebruch surface)**.** Fix an integer $e \geq 1$ and let $X_0 = \mathbb{F}_e$ be the e-th Hirzebruch surface with section C_0 and fibre class F. Let $X_k := \mathrm{Bl}_k(\mathbb{F}_e)$ be the blow-up of k points $p_1, \ldots, p_k$ with $k \leq e$ in very general position. In this range of parameters the surface X_k is anti-canonical and the Mori cone $\overline{\mathrm{NE}}(X_k)$ is polyhedral. In particular, see for instance [HJNK25, Proposition 2.4 and Lemma 3.4], every extremal ray of $\overline{\mathrm{NE}}(X_k)$ is generated by $C_0,\ E_i,\ F_i - E_i$ for $1 \leq i \leq k$. We now fix a Kähler class on X_k and normalize it as in Lemma 4.8:

$$[\omega] = C_0 + bF - \sum_{i=1}^{k} t_i E_i, \qquad b > e,\ t_i > 0.$$

Using the intersection form on X_k we obtain

$$m := \mathrm{sys}_2(\omega) = \min_{1 \leq i \leq k} \{t_i, 1 - t_i, b - e\}.$$

From $t_i \geq m$ and $1 - t_i \geq m$ we immediately deduce $0 < m \leq \frac{1}{2}$. Therefore the functional $\mathcal{J}_{X_k}([\omega])$ takes the explicit form

$$\mathcal{J}_{X_k}([\omega]) = 4\pi\, m \cdot \frac{2b + 2 - e - S}{2b - e - Q}.$$

and the variables satisfy the Kähler inequalities

$$0 < m \leq \tfrac{1}{2}, \quad 0 < t_i < 1, \quad b > e, \quad 2b - e - Q > 0.$$

In this regime, we can specifically calculate the maximum of the smooth function

$$(b, t_1, \ldots, t_k) \longmapsto m \cdot \frac{2b + 2 - e - S}{2b - e - Q}$$

under the above constraints. In this case, the maximum is achieved at $(e + 1/2, 1/2, ..., 1/2)$, so that

$$\sup_{\mathcal{K}} \mathcal{J}_{X_k}([\omega]) = 4\pi\, \frac{2e + 6 - k}{4e + 4 - k} \leq 4\pi$$

for all such Kähler classes.

Combining the previous lemmas, we obtain the following theorem.

Theorem 4.10. Let $X \to \mathbb{P}^1$ be a rational ruled surface (not necessarily minimal) endowed with a PSC Kähler metric ω. Then

$$\mathrm{sys}_2(\omega) \cdot \min_X S(\omega) \leq 8\pi.$$

Moreover, equality holds if and only if $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (equivalently, the minimal model is $\mathbb{F}_0$), endowed with the product Fubini-Study metric.

5. Level set method on non-rational PSC Kähler surfaces

In [Ste22], Stern introduced the following inequality for a non-constant S^1-valued harmonic map u on a 3-manifold (M, g) through the level set method:

$$2\pi \int_{\theta \in S^1} \chi(\Sigma_\theta)\, d\theta \;\geq\; \frac{1}{2}\int_{\theta\in S^1}\left[\int_{\Sigma_\theta}\left(|du|^{-2}|\operatorname{Hess}(u)|^2 + \mathrm{scal}_M(g)\right) dV_g\right] d\theta, \tag{5.1}$$

and used it to give a new proof of the Bray–Brendle–Neves systolic inequality for the 2-systole. In this section, we adapt the level set method to non-rational PSC Kähler surfaces, and obtain an alternative proof of Theorem 4.5.

5.1. **A Stern-type scalar curvature inequality.** Let X be a compact non-rational PSC Kähler surface. By the classification result recalled in the introduction, X is a ruled surface fibred over a compact Riemann surface B of genus $g(B) \geq 1$. Denote by

$$f\colon (X, \omega) \longrightarrow B$$

the induced non-constant holomorphic fibration. By the uniformization theorem, we may equip B with a constant curvature metric ω_0 of non-positive Gaussian curvature (so ω_0 is hyperbolic if $g(B) \geq 2$ and flat if $g(B) = 1$).

Fix a point $z \in B$. Then $f^{-1}(z)$ is a (possibly singular) Cartier divisor on X, which we denote by D_z. It defines a line bundle $\mathcal{O}(D_z)$ whose first Chern class is represented by $f^*\omega_0$ (after a suitable normalization). In what follows we restrict attention to regular values of f, so that D_z is smooth, and we keep the notation D_z for the smooth fibre. Recall that for a smooth divisor $D \subset X$, the adjunction formula states

$$K_D = \big(K_X \otimes \mathcal{O}(D)\big)|_D.$$

Since $D = D_z$ is a fibre of the holomorphic fibration $f\colon X \to B$, its normal bundle $\mathcal{N}_D \simeq \mathcal{O}(D)|_D$ is holomorphically trivial. Taking the first Chern class in the adjunction formula, we get

$$c_1(D) = c_1(X)|_D,$$

and hence, for the Ricci forms with respect to ω,

$$\mathrm{Ric}_D(\omega) = \mathrm{Ric}_X(\omega)|_D.$$

Let ν be a local unit normal vector field of type $(1,0)$ along D with respect to ω. Tracing the Gauss equation yields

$$S_D(\omega) = \mathrm{tr}_\omega \mathrm{Ric}_D(\omega) = \mathrm{tr}_\omega \mathrm{Ric}_X(\omega)|_D = S_X(\omega) - \mathrm{Ric}_X(\omega)(\nu, \bar\nu),$$

where $S_X(\omega)$ and $S_D(\omega)$ denote the scalar curvatures of (X,ω) and $(D, \omega|_D)$, respectively. In particular, since $\nu = \nabla^{1,0} f/|\nabla^{1,0} f|$, we obtain

$$\mathrm{Ric}_X(\omega)\big(\nabla^{1,0} f, \nabla^{0,1} f\big) = \big|\nabla^{1,0} f\big|^2\big(S_X(\omega) - S_D(\omega)\big). \tag{5.2}$$

We next recall the Bochner formula for holomorphic maps and a co-area formula adapted to the present setting.

Lemma 5.1 (Bochner formula)**.** Let $f\colon (X,\omega) \to (N,\tilde{\omega})$ be a holomorphic map between Kähler manifolds. Then

$$\Delta|\partial f|^2 = |\nabla\partial f|^2 + \big\langle \operatorname{Ric}(\omega), f^*\tilde{\omega}\big\rangle - \operatorname{tr}^2_\omega\big(f^* \operatorname{Rm}(\tilde{\omega})\big),$$

where $\operatorname{Ric}(\omega)$ is the Ricci form of X and $\operatorname{Rm}(\tilde{\omega})$ is the curvature form of N.

Lemma 5.2 (Co-area formula)**.** Let (X^n,ω) be a compact Kähler manifold and let (B,ω_0) be a compact Riemann surface, normalized so that

$$\int_B \omega_0 = \frac{1}{n}.$$

Let $f\colon X \to B$ be a non-constant holomorphic map, and let $g \in C^\infty(X)$. Then for every regular value $z \in B$ of f we have

$$\int_X g\,\omega^n = \int_B \left(\int_{f^{-1}(z)} \frac{g}{|\partial f|^2}\,\omega^{n-1} \right) \omega_0.$$

When $(N,\tilde{\omega}) = (B,\omega_0)$, the curvature term $\operatorname{tr}^2_\omega(f^* \operatorname{Rm}(\omega_0))$ is non-negative and vanishes identically if and only if (B,ω_0) is flat. In particular,

$$\Delta|\partial f|^2 \;\geq\; |\nabla\partial f|^2 + \operatorname{Ric}_X(\omega)\big(\nabla^{1,0} f, \nabla^{0,1} f\big),$$

with equality if and only if (B,ω_0) is an elliptic curve with a flat metric. Combining this with (5.2) we obtain

$$\Delta|\partial f|^2 \;\geq\; |\nabla\partial f|^2 + |\nabla^{1,0} f|^2\big(S_X(\omega) - S_D(\omega)\big). \tag{5.3}$$

We now integrate this inequality in the fibre direction using the co-area formula. The following lemma is the basic Stern-type inequality that we shall use in the non-rational ruled case.

Lemma 5.3. Let (X^n,ω) be a compact Kähler manifold and let (B,ω_0) be a compact Riemann surface of genus $g(B) \geq 1$, endowed with a constant curvature metric ω_0. Suppose that $f\colon X \to B$ is a non-constant holomorphic map, and let $D = f^{-1}(z)$ denote a regular fibre. Then for any $\phi \in C^\infty(X)$ we have

$$\int_B \left[\int_D \phi^2 \left(\frac{|\nabla\partial f|^2}{|\partial f|^2} + S_X(\omega) - S_D(\omega) \right) \omega^{n-1} \right] \omega_0 \;\leq\; -n \int_X \sqrt{-1}\,\partial(\phi^2) \wedge \bar{\partial}|\partial f|^2 \wedge \omega^{n-1}. \tag{5.4}$$

Moreover, equality holds in (5.4) if and only if $g(B) = 1$ and (B,ω_0) is an elliptic curve with a flat metric.

Proof. Write $B = A \cup B_0$, where A is an open neighbourhood of the (finite) set of critical values of f and B_0 consists only of regular values. Multiplying (5.3) by ϕ^2 and integrating over $f^{-1}(B_0)$, we obtain

$$\int_{f^{-1}(B_0)} \phi^2 \Big(\big|\nabla\partial f\big|^2 + \big|\nabla^{1,0} f\big|^2 \big(S_X(\omega) - S_D(\omega)\big) \Big) \omega^n \;\leq\; \int_{f^{-1}(B_0)} \phi^2 \big(\Delta|\partial f|^2\big) \omega^n. \tag{5.5}$$

For the right-hand side of (5.5), integration by parts gives

$$\int_{f^{-1}(B_0)} \phi^2 \left(\Delta|\partial f|^2\right) \omega^n = n \int_{f^{-1}(B_0)} \phi^2 \sqrt{-1}\, \partial\bar{\partial}|\partial f|^2 \wedge \omega^{n-1}.$$

On the other hand, applying the co-area formula to the left-hand side of (5.5) yields

$$\begin{aligned}\int_{f^{-1}(B_0)} & \phi^2 \Big(\left|\nabla\partial f\right|^2 + \left|\nabla^{1,0} f\right|^2 (S_X - S_D)\Big)\omega^n \\ &= n \int_{B_0} \left[\int_D \phi^2 \left(\frac{|\nabla\partial f|^2}{|\partial f|^2} + S_X(\omega) - S_D(\omega)\right) \omega^{n-1}\right] \omega_0.\end{aligned}$$

By Sard's theorem, the set of critical values has measure zero, and by choosing A with arbitrarily small measure we can pass to the limit and replace $f^{-1}(B_0)$ and B_0 by X and B, respectively. Finally, another integration by parts shows that

$$n \int_X \phi^2 \sqrt{-1}\, \partial\bar{\partial}|\partial f|^2 \wedge \omega^{n-1} = -\,n \int_X \sqrt{-1}\, \partial(\phi^2) \wedge \bar{\partial}|\partial f|^2 \wedge \omega^{n-1},$$

and combining the above identities with (5.5) gives (5.4). The equality statement follows from the discussion before (5.3): equality in the Bochner formula holds if and only if $\mathrm{Rm}(\omega_0) \equiv 0$, i.e. $g(B) = 1$ and (B, ω_0) is a flat elliptic curve. $\square$

5.2. **The 2-systole on non-rational PSC Kähler surfaces.** In this subsection, we study the homological 2-systole on non-rational PSC Kähler surfaces. Recall that, by the classification of PSC Kähler surfaces, a non-rational PSC Kähler surface is precisely a (possibly blown-up) ruled surface fibred over a curve of genus $g \geq 1$.

By leveraging (5.4), we provide an alternative proof of Theorem 4.5 with an analytic method. It is worth noting that for a Kähler metric the Chern scalar curvature $S(\omega)$ differs from the Riemannian scalar curvature $\mathrm{scal}(g_\omega)$ by a factor 2.

Theorem 5.4. Let (X, ω) be a non-rational PSC Kähler surface admitting a holomorphic fibration $X \to B$ to a compact Riemann surface B with genus $g(B) \geq 1$. Then

$$\min_X S_X(\omega) \cdot \mathrm{sys}_2(\omega) \ \leq\ 4\pi. \tag{5.6}$$

Moreover, equality holds if and only if $g(B) = 1$, X is covered by $\mathbb{P}^1 \times \mathbb{C}$ equipped with the product of the Fubini–Study metric on $\mathbb{P}^1$ and a flat metric on $\mathbb{C}$, in such a way that $\mathrm{sys}_2(\omega)$ is achieved by the $\mathbb{P}^1$-fibre.

Proof. Let $f\colon X \to B$ be the holomorphic fibration, and let ω_0 be a constant curvature metric on B of non-positive Gaussian curvature, so that (5.4) holds. Taking $\phi \equiv 1$ in (5.4), we have

$$\int_B \left[\int_D \left(\frac{|\nabla\partial f|^2}{|\partial f|^2} + S_X(\omega)\right) \omega\right] \omega_0 \leq \int_B \left(\int_D S_D(\omega)\, \omega\right) \omega_0.$$

For every regular value z, the fibre D is a smooth rational curve, hence $D \simeq \mathbb{P}^1$ and $\chi(D) = 2$. By Gauss–Bonnet formula,

$$\int_D S_D(\omega)\,\omega = 2\pi\,\chi(D) = 4\pi.$$

Integrating over B, we obtain

$$\begin{aligned} 4\pi \int_B \omega_0 = 2\pi \int_B \chi(D)\,\omega_0 &= \int_B \Big(\int_D S_D(\omega)\,\omega\Big)\,\omega_0 \\ &\geq \int_B \Big(\int_D S_X(\omega)\,\omega\Big)\,\omega_0 \\ &\geq \min_X S_X(\omega) \cdot \int_B \mathrm{Vol}_\omega(D)\,\omega_0. \end{aligned}$$

By definition of the homological 2-systole we have

$$\mathrm{Vol}_\omega(D) \;\geq\; \mathrm{sys}_2(\omega)$$

for every regular fibre D. Hence

$$\begin{aligned} 4\pi \int_B \omega_0 &\geq \min_X S_X(\omega) \cdot \int_B \mathrm{Vol}_\omega(D)\,\omega_0 \\ &\geq \min_X S_X(\omega) \cdot \mathrm{sys}_2(\omega) \int_B \omega_0, \end{aligned}$$

and since $\int_B \omega_0 > 0$ this yields

$$\min_X S_X(\omega) \cdot \mathrm{sys}_2(\omega) \;\leq\; 4\pi,$$

as claimed. The equality case happens if B is an elliptic curve endowed with a flat metric, ∇f is parallel, which implies X is isometrically covered by $\mathbb{P}^1 \times \mathbb{C}$, and (X, ω) is cscK so that $\mathrm{sys}_2(\omega)$ is realized by $\mathbb{P}^1$-fibre. □

Corollary 5.5. Let (X, ω) be a non-rational PSC Kähler surface admitting a non-constant holomorphic map $f\colon X \to B$ to a compact Riemann surface B with genus $g(B) \geq 2$. Then

$$\min_X S_X(\omega) \cdot \mathrm{sys}_2(\omega) \;<\; 4\pi.$$

Proof. If $g(B) \geq 2$, then B is hyperbolic and cannot carry a flat metric. Hence equality in (5.6) cannot occur in Lemma 5.3, nor in Theorem 5.4, and the inequality in Theorem 5.4 is strict. □

Example 5.6. Let $X = \mathbb{P}^1 \times B$ be a compact complex surface, where B is a compact Riemann surface of genus $g(B) \geq 2$. Equip X with the product Kähler metric

$$\omega = \omega_{\mathrm{FS}} \oplus \omega_B,$$

where on $\mathbb{P}^1$ we take the Fubini–Study metric normalized by

$$\mathrm{Vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^1) = \pi, \qquad S_{\mathbb{P}^1}(\omega_{\mathrm{FS}}) \equiv 4,$$

and on B we choose a constant Chern scalar curvature metric with

$$S_B(\omega_B) = -4 + \varepsilon \quad \text{for some } \varepsilon \in (0, 4).$$

Then the Chern scalar curvature of the product metric is constant and given by

$$S_X(\omega) = S_{\mathbb{P}^1}(\omega_{\mathrm{FS}}) + S_B(\omega_B) = 4 + (-4 + \varepsilon) = \varepsilon,$$

so $\min_X S_X(\omega) = \varepsilon > 0$ and X has positive scalar curvature in our convention.

Next we compare the areas of the two basic complex curves:

- For the $\mathbb{P}^1$-fibre $F = \mathbb{P}^1 \times \{p\}$, calibration by ω gives

$$\mathrm{Vol}_\omega(F) = \mathrm{Vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^1) = \pi.$$

- For the B-fibre $B_p = \{q\} \times B$, Gauss–Bonnet for the Chern scalar curvature gives

$$\int_B S_B(\omega_B)\, \omega_B = 2\pi\chi(B) = 2\pi(2 - 2g(B)) = 4\pi(1 - g(B)).$$

  Since $S_B(\omega_B) \equiv -4 + \varepsilon < 0$, we obtain

$$\mathrm{Vol}_\omega(B_p) = \mathrm{Vol}_{\omega_B}(B) = \frac{4\pi(g(B) - 1)}{4 - \varepsilon}.$$

For $g(B) \geq 2$ and $\varepsilon \in (0, 4)$ one has $\mathrm{Vol}_\omega(B_p) > \pi$, so the 2-systole is realized by the $\mathbb{P}^1$-fibre:

$$\mathrm{sys}_2(\omega) = \min\big\{\mathrm{Vol}_\omega(F), \mathrm{Vol}_\omega(B_p)\big\} = \pi.$$

Consequently,

$$\min_X S_X(\omega) \cdot \mathrm{sys}_2(\omega) = \varepsilon \cdot \pi \; < \; 4\pi.$$

In particular, this product is independent of the genus $g(B)$, and it can be made arbitrarily close to 4π by letting $\varepsilon \uparrow 4$.

6. Open problems

The results of this paper show that on a PSC Kähler surface (X, ω) one has a sharp inequality of the form

$$\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \; \leq \; 12\pi,$$

with equality precisely for the Fubini–Study metric on $\mathbb{P}^2$ (up to scaling). In terms of the Riemannian scalar curvature $\mathrm{scal}(g_\omega) = 2S(\omega)$, this gives

$$\min_X \mathrm{scal}(g_\omega) \cdot \mathrm{sys}_2(g_\omega) \; \leq \; 24\pi$$

for every PSC Kähler surface, and this inequality is sharp in the Kähler category, again with equality on $(\mathbb{P}^2, \omega_{\mathrm{FS}})$. We raise the following question: to our knowledge, it is open.

Higher-dimensional PSC Kähler manifolds. The functional $\mathcal{J}_X$ and the holomorphic 2–systole are defined for Kähler manifolds of any complex dimension $n \geq 2$:

$$\mathcal{J}_X([\omega]) := \mathrm{sys}_2^{\mathrm{hol}}(\omega) \cdot \hat{S}([\omega]),$$

It is therefore natural to ask whether the surface result has a genuine higher-dimensional analogue.

Question 6.1. Let $n \geq 2$ and let (X^n, ω) be a compact PSC Kähler manifold of complex dimension n. Do we have

$$\min_X S(\omega) \cdot \mathrm{sys}_2(\omega) \leq \sup_{\mathcal{K}} \mathcal{J}_X([\omega]) = 2\pi n(n+1)$$

for every such (X, ω)? Is it true that one can take with equality if and only if (X, ω) is biholomorphic to $(\mathbb{P}^n, \omega_{\mathrm{FS}})$?

Even in the projective case, this problem seems widely open for $n \geq 3$. One obstruction to directly extending the surface argument is that, in higher dimensions, the Kähler cone is characterized by positivity on all analytic subvarieties, not just curves. By Demailly–Păun theorem [DP04], a real $(1,1)$–class α is Kähler if and only if

$$\int_V \alpha^{\dim V} > 0$$

for every positive-dimensional irreducible analytic subvariety $V \subset X$. Thus a class can remain uniformly positive on all curves while degenerating along some higher-dimensional subvariety, a phenomenon which our holomorphic 2-systolic does not directly detect.

A more speculative extension is to consider higher even-dimensional systoles. For $1 \leq k \leq n-1$ one can define the $2k$–systole $\mathrm{sys}_{2k}(\omega)$ of g_ω in the usual way, and a holomorphic counterpart by

$$\mathrm{sys}_{2k}^{\mathrm{hol}}(\omega) := \inf\bigl\{[\omega]^k \cdot [V] \bigm| V \subset X \text{ effctive analytic } k\text{-cycle},\ 0 \neq [V] \in H_{2k}(X;\mathbb{Z})\bigr\}.$$

On $\mathbb{P}^n$ with $[\omega] = aH$ and a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$ one easily computes $[\omega]^k \cdot [\mathbb{P}^k] = a^k$, so the model value of $\mathrm{sys}_{2k}^{\mathrm{hol}}(\omega)$ is explicit.

Question 6.2. Let $n \geq 2$ and $1 \leq k \leq n-1$. For PSC Kähler manifolds (X^n, ω), is there a natural scale–invariant functional $\mathcal{J}_{X,k}([\omega])$ built from $\mathrm{sys}_{2k}^{\mathrm{hol}}(\omega)$ and the average scalar curvature, and a universal constant $C_{n,k} > 0$, such that

$$\Bigl(\min_X S(\omega)\Bigr)^k \cdot \mathrm{sys}_{2k}(\omega) \ \leq\ \mathcal{J}_{X,k}([\omega]) \ \leq\ C_{n,k}$$

for all such (X, ω)? Is it true that one can take $C_{n,k} = \bigl(2\pi n(n+1)\bigr)^k$, with equality realized precisely by $(\mathbb{P}^n, \omega_{\mathrm{FS}})$.

Here the main difficulty is precisely that, in complex dimension $n \geq 3$, the Kähler condition and the positivity of scalar curvature interact with the whole hierarchy of analytic subvarieties of dimensions $1, \ldots, n-1$, rather than being controlled solely by curves as in the surface case.

It is therefore unclear whether the algebro–geometric optimization techniques developed in this paper can be adapted to higher-dimensional PSC Kähler manifolds.

Universal upper bound for the 2-systole in PSC Riemannian 4-manifolds. On the other hand, in the purely Riemannian setting, the situation is far from understood. For instance, even on $S^2 \times S^2$ it is currently unknown whether the 2-systole is bounded from above among all metrics with scalar curvature bounded below by a fixed positive constant. In particular, any universal upper bound of the type considered below would already imply a positive answer to this difficult problem. Motivated by the Kähler case, we nevertheless propose the following questions.

Question 6.3. Let (M^4, g) be a closed Riemannian 4-manifold with $\mathrm{scal}(g) > 0$ and $H_2(M; \mathbb{Z}) \neq 0$. Does there exist a universal constant $C > 0$ such that

$$\min_M \mathrm{scal}(g) \cdot \mathrm{sys}_2(g) \;\leq\; C$$

for all such (M, g)?

In view of the Kähler realm, any such universal constant must satisfy $C \geq 24\pi$, and equality $C = 24\pi$ is already realized by the Fubini–Study metric on $\mathbb{P}^2$. Is it true that one can take $C = 24\pi$ in Question 6.3? If so, is the constant 24π sharp, with equality occurring precisely for the Fubini–Study metric on $\mathbb{P}^2$ (up to scaling), or perhaps for a larger but still explicitly describable family of rigid models?

References

[ATF06] V. Apostolov and C. W. Tønnesen-Friedman, *A remark on Kähler metrics of constant scalar curvature on ruled complex surfaces*, Bull. Lond. Math. Soc. **38** (2006), no. 3, 494–500.

[BBN10] H. Bray, S. Brendle, and A. Neves, *Rigidity of area–minimizing two–spheres in three–manifolds*, Comm. Anal. Geom. **18** (2010), no. 4, 821–830.

[Ber72a] M. Berger, *Du côté de chez Pu*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 1–44 (French).

[Ber72b] ———, *A l'ombre de Loewner*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 241–260 (French).

[Ber81] ———, *Isosystolic and isembolic inequalities*, Preprint, Université Paris 7, 1981.

[Ber08] ———, *What is... a systole?*, Notices Amer. Math. Soc. **55** (2008), no. 3, 374–376.

[BHPVdV04] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2 ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 4, Springer-Verlag, Berlin, 2004.

[Bro24] Garrett M. Brown, *The sign of scalar curvature on Kähler blowups*, arXiv e-prints (2024), arXiv:2405.12189 [math.DG].

[Don83] S. K. Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Differential Geom. **18** (1983), no. 2, 269–277.

[DP04] J.-P. Demailly and M. Păun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. **159** (2004), no. 3, 1247–1274.

[Fuj92] A. Fujiki, *Remarks on extremal Kähler metrics on ruled manifolds*, Nagoya Math. J. **126** (1992), 89–101.

[Gro83] Mikhail Gromov, *Filling riemannian manifolds*, J. Differential Geom. **18** (1983), 1–147.

[Har77] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52, Springer-Verlag, New York–Heidelberg, 1977.

[HJNK25] Krishna H., Cyril J. J., Suhas B. N., and Singh A. K., *Seshadri constants on blow-ups of hirzebruch surfaces*, Math. Nachr. **298** (2025), no. 2, 437–455.

[Kob87] S. Kobayashi, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan, vol. 15, Princeton Univ. Press, Princeton, NJ, 1987, Kan Memorial Lectures, 5.

[LeB95] C. LeBrun, *On the scalar curvature of complex surfaces*, Geom. Funct. Anal. **5** (1995), no. 3, 619–628.

[NS65] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540–567.

[Ori25] Shunichiro Orikasa, *Systolic inequality and scalar curvature*, arXiv:2509.17376 (2025).

[Ou25] W. Ou, *A characterization of uniruled compact $k\backslash$" ahler manifolds*, arXiv preprint arXiv:2501.18088 (2025).

[Ric20] T. Richard, *On the 2-systole of stretched enough positive scalar curvature metrics on $S^2 \times S^2$*, SIGMA. Symmetry Integrability Geom. Methods Appl. **16** (2020).

[Ste22] D. Stern, *Scalar curvature and harmonic maps to S^1*, J. Differential Geom. **122** (2022), no. 2, 259–269.

[SY79] R. Schoen and S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with non-negative scalar curvature*, Ann. Math. **110** (1979), no. 1, 127–142.

[Xu25] K. Xu, *A topological gap theorem for the π_2-systole of positive scalar curvature 3-manifolds*, Duke Math. J. **174** (2025), no. 8, 1647–1664.

[Yan19] X. Yang, *Scalar curvature on compact complex manifolds*, Trans. Amer. Math. Soc. **371** (2019), no. 3, 2073–2087.

[Yau74] S.-T. Yau, *On the curvature of compact Hermitian manifolds*, Invent. Math. **25** (1974), no. 3, 213–239.

[Zhu20] Jintian Zhu, *Rigidity of area-minimizing 2-spheres in n-manifolds with positive scalar curvature*, Proc. Amer. Math. Soc. **148** (2020), no. 8, 3479–3489.

INSTITUTE FOR MATHEMATICS AND FUNDAMENTAL PHYSICS, 1899 SHILIAN SOUTH ROAD, 230088 HEFEI, CHINA

Email address: zhsha@imfp.org.cn