A BRAY-BRENDLE-NEVES TYPE SYSTOLIC INEQUALITY FOR COMPACT KÄHLER SURFACES

ZEHAO SHA

ABSTRACT. In this note, we prove a Bray–Brendle–Neves type systolic inequality for compact positive scalar curvature Kähler surfaces admitting a nonconstant holomorphic map to a positive-genus compact Riemann surface.

1. INTRODUCTION

Systolic geometry studies the relationship between the minimal size of non-trivial homological cycles in a Riemannian manifold and its global geometric properties. Given a closed Riemannian manifold (M, h), the k-*systole* is defined as

$$\mathrm{sys}_k(M,h) := \inf \left\{ \mathrm{Vol}_g(Z) \mid Z \subset M \text{ embedded },\ [Z] \neq 0 \in H_k(M;\mathbb{Z}) \right\}.$$

The case $k = 1$ (the minimal length of a non-contractible loop) has been extensively studied. A foundational result of Gromov [Gro83] states that for any essential n-manifold (M, h), the 1-systole satisfies the universal inequality

$$\mathrm{sys}_1(M,h)^n \leq C_n \ \mathrm{Vol}_h(M),$$

where C_n is a constant depending only on the dimension. The study of higher k-systoles, particularly in the case $k = 2$, reveals additional geometric phenomena. Gromov [Gro81] also obtained an optimal inequality for the *stable* 2-systole in complex projective spaces: for any Riemannian metric h on $\mathbb{CP}^n$, we have

$$\mathrm{stsys}_2(\mathbb{CP}^n, h)^n \leq \frac{1}{2^n} \mathrm{Vol}_h(\mathbb{CP}^n),$$

with equality achieved by the Fubini–Study metric.

An attractive rigidity result concerning the π_2-systole and and the minimum of the scalar curvature $\min S_M$ of a positive scalar curvature (PSC for abbreviation) 3-manifold (M, h) was established by Bray, Brendle, and Neves:

Theorem 1.1 (Bray-Brendle-Neves, [BBN10])**.** Let (M^3, h) be a closed, orientable Riemannian 3-manifold with positive scalar curvature. Then the following inequality holds:

$$\mathrm{sys}_{\pi_2}(M,h) \cdot \min_M S_M \leq 8\pi. \tag{1.1}$$

Moreover, equality holds if and only if M^3 is isometrically covered by $S^2 \times S^1$ with the round metric on S^2, product with the flat metric on S^1.

In [Ste22], Stern introduced the following inequality for a non-constant S^1-valued harmonic map u on a 3-manifold (M, h),

$$2\pi \int_{\theta \in S^1} \chi(\Sigma_\theta) \geq \frac{1}{2} \int_{\theta \in S^1} \int_{\Sigma_\theta} \left(|du|^{-2} |\operatorname{Hess}(u)|^2 + S_M \right) \tag{1.2}$$

and generalized the Bray–Brendle–Neves' systolic inequality for the homological 2-systole.

Motivated by Stern's approach, we adapt his method to compact Kähler surfaces fibred over a Riemann surface with positive genus, and obtain a Bray-Brendle-Neves type inequality for the homological 2-systole. In particular, we have the following:

Theorem 1.2. Let (X, ω) be a compact PSC Kähler surface admitting a non-constant holomorphic map $f : X \to C$ to a compact Riemann surface C with genus $g(C) \geq 1$. Then, we have

$$\min_X S_X \cdot \operatorname{sys}_2(X, \omega) \leq 8\pi. \tag{1.3}$$

Moreover, the equality holds if and only if X is isometrically covered by $\mathbb{CP}^1 \times E$ equipped with the product of the standard Fubini–Study metric on $\mathbb{CP}^1$, and a flat metric on E satisfying $\operatorname{Vol}(E) \geq \pi$, where E is an elliptic curve.

A direct consequence of the above result is:

Corollary 1.3. Let (X, ω) be a compact PSC Kähler surface admitting a non-constant holomorphic map $f : X \to C$ to a compact Riemann surface C with genus $g(S) \geq 2$. Then, we have

$$\min_X S_X \cdot \operatorname{sys}_2(X, \omega) < 8\pi. \tag{1.4}$$

It is worth-noting that a compact Kähler surface X admits a PSC metric if and only if X is obtained from $\mathbb{P}^2$ or $\mathbb{P}(E)$ by a finite sequence of blow ups, where E is a rank 2 holomorphic vector bundle over a compact Riemann surface: For minimal compact Kähler surfaces (which are not the blow-up of other Kähler surfaces), LeBrun showed that the existence of a *Kähler* PSC metric is equivalent to X is ruled or $\mathbb{P}^2$ [LeB95]. LeBrun also conjectured that this statement is still valid after allowing the blowup. The remaining gap—whether blowing up preserves the *sign* of the scalar curvature was settled recently by Brown, who proved that Kähler blow-ups preserve the sign of scalar curvature and completed the classification [Bro24, Thm B].

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2. The Kähler surface fibred over a Riemann surface with positive genus

Let $f : (X^n, \omega) \to C$ be a non-constant holomorphic map from a compact Kähler manifold to a compact Riemann surface C with genus $g \geq 1$. Thanks to the uniformization theorem, we can always choose a metric ω_0 on C with non-positive Gauss curvature.

Taking $z \in C$ be any point, then we have $f^{-1}(z) := D_z$ is a Cartier divisor in X with complex codimension 1. Indeed, D_z defines a line bundle $\mathcal{O}(D_z)$ with the first Chern class $c_1(\mathcal{O}(D_z))$ represented by $f^*\omega_0$ after normalization. In the subsequent part, we only consider the smooth part of D_z, and denote it by D_z directly.

Recall the adjunction formula for a smooth divisor D, the canonical bundle K_D satisfies

$$K_D = (K_X \otimes \mathcal{O}(D))\,|_D. \tag{2.1}$$

Since D is the fiber of a holomorphic map over C, the normal bundle $\mathcal{N}_D \cong \mathcal{O}(D)|_D$ is trivial. By taking the first Chern class of the adjunction formula, we obtain

$$c_1(D) = c_1(X)|_D,$$

which implies

$$\mathrm{Ric}_D(\omega) = \mathrm{Ric}_X(\omega)|_D. \tag{2.2}$$

Denote ν by the unit normal vector field of D of type $(1,0)$, we obtain the traced Gauss equation

$$S_D(\omega) = \mathrm{tr}_\omega \mathrm{Ric}_D(\omega) = \mathrm{tr}_\omega \mathrm{Ric}_X(\omega)|_D = S_X(\omega) - \mathrm{Ric}_X(\omega)(\nu, \bar{\nu}).$$

Moreover, since $\nu = \nabla^{1,0} f / |\nabla^{1,0} f|$, we obtain

$$\mathrm{Ric}_X(\omega)\left(\nabla^{1,0} f, \nabla^{0,1} f\right) = \left|\nabla^{1,0} f\right|^2 (S_X(\omega) - S_D(\omega)). \tag{2.3}$$

Recall the Bochner formula for holomorphic maps, and the co-area formula:

Lemma 2.1 (Bochner formula)**.** Let $f : (X, \omega) \to (N, \tilde{\omega})$ be holomorphic, then

$$\Delta|\partial f|^2 = |\nabla \partial f|^2 + \langle \mathrm{Ric}(\omega), f^*\tilde{\omega}\rangle - \mathrm{tr}^2_\omega \left(f^* \mathrm{Rm}(\tilde{\omega})\right). \tag{2.4}$$

where $\mathrm{Ric}(\omega)$ is the Ricci form of X, and $\mathrm{Rm}(\tilde{\omega})$ is the curvature form of N.

Lemma 2.2 (Co-area formula)**.** Let (X^n, ω) be a compact Kähler manifold and let (C, ω_0) be a compact Riemann surface such that

$$\int_C \omega_0 = \frac{1}{n},$$

after normalization. Then for any $g \in C^\infty(X)$ and $z \in E$ regular value of f, we have

$$\int_X g\, \omega^n = \int_C \left(\int_{f^{-1}(z)} \frac{g}{|\partial f|^2}\, \omega^{n-1}\right) \omega_0. \tag{2.5}$$

When $(N, \tilde{\omega}) = (C, \omega_0)$, we have

$$\Delta|\partial f|^2 \geq |\nabla \partial f|^2 + \mathrm{Ric}_X(\omega)\left(\nabla^{1,0} f, \nabla^{0,1} f\right),$$

with equality holds if $\mathrm{Rm}(\omega_0) \equiv 0$. Combining with the traced Gauss equation (2.3), we obtain

$$\Delta|\partial f|^2 \geq |\nabla \partial f|^2 + \left|\nabla^{1,0} f\right|^2 (S_X(\omega) - S_D(\omega)). \tag{2.6}$$

Combining with the co-area formula and (2.6), we can then see the following identity:

Lemma 2.3. Let (X^n, ω) be a compact Kähler manifold and let (C, ω_0) be a compact Riemann surface with genus $g \geq 1$ endowed with a constant curvature metric ω_0. Suppose that $f : X \to C$ is a non-trivial holomorphic map. Then for any $\phi \in C^\infty(X)$, we have

$$\int_C \left[\int_{D_z} \phi^2 \left(\frac{|\nabla \partial f|^2}{|\partial f|^2} + S_X - S_{D_z} \right) \omega^{n-1} \right] \omega_0 \leq -n \int_X \sqrt{-1} \partial(\phi^2) \wedge \bar{\partial} |\partial f|^2 \wedge \omega^{n-1}. \quad (2.7)$$

Moreover, the equality holds if and only if (S, ω_0) is an elliptic curve with a flat metric.

Proof. Let $C = A \cup B$, where A contains the set for all critical values of f. Then, multiplying any smooth function ϕ^2 on the both sides and integrating over $f^{-1}(B)$, we have

$$\int_{f^{-1}(B)} \phi^2 \left[|\nabla \partial f|^2 + \left|\nabla^{1,0} f\right|^2 (S_X(\omega) - S_{D_z}(\omega)) \right] \omega^n \leq \int_{f^{-1}(B)} \phi^2 \left(\Delta |\partial f|^2 \right) \omega^n. \quad (2.8)$$

Indeed, the right-hand side of (2.8) gives

$$\int_{f^{-1}(B)} \phi^2 \left(\Delta |\partial f|^2 \right) \omega^n = n \int_{f^{-1}(B)} \phi^2 \sqrt{-1} \partial \bar{\partial} |\partial f|^2 \wedge \omega^{n-1}$$

We then apply the co-area formula to the left-hand side of (2.8), which yields

$$\begin{aligned} &\int_{f^{-1}(B)} \phi^2 \left[|\nabla \partial f|^2 + \left|\nabla^{1,0} f\right|^2 (S_X - S_{D_z}) \right] \omega^n \\ &\qquad = n \int_B \left[\int_{D_z} \left(\frac{|\nabla \partial f|^2}{|\partial f|^2} + S_X - S_D \right) \omega^{n-1} \right] \omega_0 \end{aligned}$$

By Sard's theorem, we can take the measure of A arbitrarily small, and this gives

$$n \int_X \phi^2 \sqrt{-1} \partial \bar{\partial} |\partial f|^2 \wedge \omega^{n-1} = -n \int_X \sqrt{-1} \partial(\phi^2) \wedge \bar{\partial} |\partial f|^2 \wedge \omega^{n-1}.$$

Consequently, we have

$$\int_C \left[\int_{D_z} \phi^2 \left(\frac{|\nabla \partial f|^2}{|\partial f|^2} + S_X - S_{D_z} \right) \omega^{n-1} \right] \omega_0 \leq -n \int_X \sqrt{-1} \partial(\phi^2) \wedge \bar{\partial} |\partial f|^2 \wedge \omega^{n-1},$$

which implies the desired equality. $\square$

3. The 2-systole in Kähler surface

This section is devoted to the study of the (homological) 2-systole in Kähler surfaces. We begin by recalling the fundamental definition of the k-systole in Riemannian geometry. Let (M, h) be a closed Riemannian manifold of dimension $n \geq k$. The k-systole $\mathrm{sys}_k(M, h)$ is defined as the infimum of the volumes of all integral k-cycles representing nontrivial homology classes:

$$\mathrm{sys}_k(M, h) := \inf \left\{ \mathrm{Vol}_h(Z) \mid Z \subset M \text{ embedded }, [Z] \neq 0 \in H_k(M; \mathbb{Z}) \right\}.$$

In the context of Kähler geometry, additional structure enriches this concept. Let (X, ω) be a compact Kähler surface. The 2-systole is the least area among nonseparating real surfaces in X.

Definition 3.1 (2-systole in Kähler surfaces)**.** For a compact Kähler surface (X, ω), the *2-systole* can be defined by

$$\mathrm{sys}_2(X,\omega) = \inf\{\mathrm{Vol}_\omega(Z) \mid Z \subset X \text{ embedded }, [Z] \neq 0 \in H_2(X;\mathbb{Z})\}.$$

The following result gives a Bray-Brendle-Neves type inequality [BBN10] for 2-systole for compact PSC Kähler surfaces over a Riemann surface with genus $g \geq 1$.

Theorem 3.2. Let (X, ω) be a compact Kähler surface admitting a non-constant holomorphic map $f : X \to C$ to a complex curve C with genus $g(C) \geq 1$. Then, we have

$$\min_X S_X \cdot \mathrm{sys}_2(X,\omega) \leq 8\pi. \tag{3.1}$$

Moreover, the equality holds if and only if X is covered by $\mathbb{CP}^1 \times E$ equipped with the product of the Fubini–Study metric on $\mathbb{CP}^1$ with $\mathrm{Vol}(\mathbb{CP}^1) = \pi$, and a flat metric on E so that the $\mathrm{sys}_2(X,\omega)$ is achieved by the $\mathbb{CP}^1$-fiber, where E is an elliptic curve.

Proof. By taking $\phi = 1$ in (2.7), we obtain

$$\int_C \left[\int_{D_z} \left(\frac{|\nabla\partial f|^2}{|\partial f|^2} + S_X\right)\omega\right]\omega_0 \leq \int_C \left(\int_{D_z} S_{D_z} \cdot \omega\right)\omega_0.$$

It follows from the Gauss-Bonnet formula,

$$\begin{aligned} 4\pi \int_C \chi(D_z)\,\omega_0 &= \int_C \left(\int_{D_z} S_{D_z} \cdot \omega\right)\omega_0 \\ &\geq \int_C \left(\int_{D_z} S_X \cdot \omega\right)\omega_0 \\ &\geq \min_X S_X \cdot \int_C \mathrm{Vol}_\omega(D_z)\,\omega_0. \end{aligned}$$

Denote $N(z)$ by the number of the homological non-zero irreducible components of D_z, we then have

$$\mathrm{Vol}_\omega(D_z) \geq N(z) \cdot \mathrm{sys}_2(X,\omega).$$

Meanwhile, we have

$$\chi(D_z) \leq 2N(z).$$

Thus, we have

$$\begin{aligned} 8\pi \int_C N(z)\,\omega_0 &\geq 4\pi \int_C \chi(D_z)\,\omega_0 \\ &\geq \min_X S_X \cdot \int_C \mathrm{Vol}_\omega(D_z)\,\omega_0 \\ &\geq \min_X S_X \cdot \mathrm{sys}_2(X,\omega) \int_C N(z)\,\omega_0, \end{aligned}$$

which gives the desired result. The equality holds in case C admits a flat metric and ∇f is parallel along D_z, with each irreducible component of D_z is $\mathbb{CP}^1$. $\square$

We finally see a simple but interesting example:

Example 3.3. Let $X = \mathbb{P}^1 \times C$ be a compact complex surface, where C is a compact Riemann surface of genus $g \geq 2$. Equip X with the product Kähler metric

$$\omega = \omega_{\mathrm{FS}} \oplus \omega_C,$$

where on $\mathbb{P}^1$ we take the Fubini–Study metric normalized by

$$\mathrm{Vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^1) = \pi, \qquad S_{\mathbb{P}^1} = 8,$$

and on C we choose a constant scalar curvature metric with

$$S_C = -8 + \varepsilon \quad \text{for some } \varepsilon \in (0, 8).$$

Then the product scalar curvature $S_X = S_{\mathbb{P}^1} + S_C = 8 + (-8 + \varepsilon) = \varepsilon$ is constant, hence $\min_X S_X = \varepsilon$ and X has positive scalar curvature.

Next, compare the areas of the two basic complex curves:

- For the $\mathbb{P}^1$-fiber $F = \mathbb{P}^1 \times \{p\}$, calibration by ω gives $\mathrm{Vol}_\omega(F) = \pi$.
- For the C-fiber $C_p = \{q\} \times C$, Gauss–Bonnet formula yields
$$\int_C K_C \, dA = 2\pi\chi(C) = 2\pi(2 - 2g) = -4\pi(g - 1).$$
Then we obtain
$$\mathrm{Vol}_\omega(C_p) = \frac{8\pi(g-1)}{8 - \varepsilon}.$$

For $g \geq 2$ and $\varepsilon \in (0, 8)$ one has $\mathrm{Area}_\omega(C_p) > \pi$, so the 2-systole is realized by the $\mathbb{P}^1$-fiber:

$$\mathrm{sys}_2(X, \omega) = \min\left\{ \mathrm{Area}_\omega(F), \mathrm{Area}_\omega(C_p) \right\} = \pi.$$

Consequently,

$$\min_X S_X \cdot \mathrm{sys}_2(X, \omega) = \varepsilon \cdot \pi < 8\pi.$$

In particular, this product is independent of the genus g, and it can be made arbitrarily close to 8π by letting $\varepsilon \uparrow 8$.

4. Discussion

We have introduced the BBN-type 2-systolic inequality for PSC Kähler surfaces admitting a non-constant holomorphic map to a base curve with positive genus. The remaining natural case is the *rational base* $C \simeq \mathbb{P}^1$.

Question 4.1 (Rational base)**.** Let (X, ω) be a compact PSC Kähler surface admitting a non-constant holomorphic map $f : X \to \mathbb{P}^1$. Then

$$\min_X S_X \cdot \mathrm{sys}_2(X, \omega) \;\leq\; 16\pi,$$

with equality realized by the product $(\mathbb{P}^1 \times \mathbb{P}^1,\ \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}})$ normalized by $\mathrm{Vol}_{\omega_{\mathrm{FS}}}(\mathbb{P}^1) = \pi$ on each factor.

More ambitiously, we expect the same universal bound without assuming a fibration:

Question 4.2 (Universal bound)**.** Every compact PSC Kähler surface satisfies

$$\min_X S_X \cdot \mathrm{sys}_2(X,\omega) \ \leq \ 16\pi.$$

This broader form is motivated by surfaces such as $\mathbb{P}^2$, which do not admit nonconstant holomorphic maps onto $\mathbb{P}^1$ (nor onto positive-genus curves). Explicit computations in standard models remain $\leq 16\pi$. At present, we are not aware of any example exceeding 16π.

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INSTITUT FOURIER, UMR 5582, LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ GRENOBLE ALPES, CS 40700, 38058 GRENOBLE CEDEX 9, FRANCE

Email address: zhsha0251@gmail.com