THE CONSTANT SCALAR CURVATURE KÄHLER METRIC ON BOUNDED PSEUDOCONVEX DOMAINS

ZEHAO SHA

Abstract. In this paper, we study constant scalar curvature Kähler (cscK) metrics on bounded pseudoconvex domains endowed with the complete Kähler-Einstein metric of C^0-bounded geometry. We prove that the cscK metric is bi-Lipschitz equivalent to the Kähler-Einstein metric, provided the volume form ratio of the two metrics is bounded on the closure of the domain. Moreover, if the cscK metric is asymptotically Einstein, then it is a Kähler-Einstein metric. Based on Huang-Xiao's work, we investigate an analogue of Cheng's conjecture for the Bergman metric with constant scalar curvature.

1. Introduction

A classical question in Kähler geometry is on finding an especially "nice" representative of a given Kähler class $[\omega_g]$ for a Kähler manifold (M, g). Generally, the Kähler-Einstein metric is the best candidate for such a representative. A Kähler metric g is called Kähler-Einstein if the Ricci curvature is proportional to the metric, that is

$$\mathrm{Ric}(g) = \lambda g,$$

for some $\lambda \in \mathbb{R}$. Through rescaling, we can assume that $\lambda = 1,\ 0$ or -1.

On compact Kähler manifolds, the existence of Kähler-Einstein metrics depends on the sign of the first Chern class $c_1(M)$:

(1) If $c_1(M) < 0$, M has an ample canonical bundle, a problem solved by Aubin [Aub76] and Yau [Yau78b];

(2) If $c_1(M) = 0$, M is Calabi-Yau, as solved by Yau [Yau78b];

(3) If $c_1(M) > 0$, M is Fano, there are some obstructions for the existence of Kähler-Einstein metric, introduced by Matsushima [Mat57], Futaki [Fut83] and Tian [Tia97]. It was proven by Chen-Donaldson-Sun [CDS15a,CDS15b,CDS15c] that the K-polystability of a Fano manifold is a sufficient condition for the existence of a Kähler-Einstein metric

For the non-compact case, significant results have been proven by Cheng-Mok-Yau [CY80, MY83] and Guedj-Kolev-Yeganefar [GKY13].

When the canonical bundle K_M is neither trivial, ample, nor anti-ample, the existence of a Kähler–Einstein metric is precluded, as the first Chern class cannot coincide with the Kähler class, making the necessary topological condition unattainable. The constant scalar curvature

Key words and phrases. Strictly pseudoconvex domain; Kähler-Einstein metric; CscK metric.

Kähler (often abbreviated as cscK) metric generalizes the concept of the Kähler–Einstein metric. And on compact Kähler manifolds, the average of the scalar curvature $\hat{R}$ is given by

$$\hat{R} = \frac{2n\pi c_1(M) \cup [\omega_g]^{n-1}}{[\omega_g]^n}$$

which is independent of the choice of g.

The Yau-Tian-Donaldson conjecture posits that the existence of a cscK metric in a given Kähler class is intimately related to the K-stability of (M, L) where L is an ample line bundle over M. In [Che18], Chen outlined a program for studying the existence problem for cscK metric: a new continuity path that links the cscK equation to certain second-order elliptic equation, apparently motivated by the classical continuity path for Kähler Einstein metrics and Donaldson's continuity path for conical Kähler Einstein metrics, and showed the openness. Further, Chen and Cheng [CC21b, CC21a] established a priori estimates and proved the closeness.

Despite the progress of the study for the cscK metric on compact Kähler manifold, the case for complete Kähler manifold is less understood. One of the main obstacles is that the complete case lacks a global $\partial\bar{\partial}$-lemma. Consequently, even when a Kähler class $[\omega_g]$ is prescribed, one cannot establish the one-to-one correspondence between Kähler metrics and their potential functions as in the compact case. Inspired by the outstanding work of Cheng and Yau [CY80] on complete Kähler-Einstein metrics on strictly pseudoconvex domains, we would like to start in a similar setting. In these domains, the defining function naturally produces a complete Kähler metric, and any two such metrics differ by a $\sqrt{-1}\partial\bar{\partial}u$ term for some potential u, as a property inherent to the defining function. Moreover, in several complex variables, there is a growing trend to study invariant metrics on pseudoconvex domains, and since cscK metrics are invariant under biholomorphisms, it is natural to consider them in this context.

In this paper, we focus on the case of bounded strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$ equipped with a complete cscK metric g. We present the following result:

Theorem 1.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain endowed with the complete Kähler-Einstein metric g of C^0-bounded geometry. Suppose $\tilde{g}$ is a complete cscK metric of C^0-bounded geometry such that $\det(\tilde{g})/\det(g) := e^F \in C^2(\bar{\Omega})$. Then there exists constant $C \geq C' > 0$ depending on the dimension n, the lower bound of holomorphic bisectional curvature of g, and $\|F\|_{C^2(\bar{\Omega})}$, such that

$$C'g \leq \tilde{g} \leq Cg \quad \text{in} \quad \Omega. \tag{1.1}$$

Moreover, if $F = 0$ on $\partial\Omega$, then $\tilde{g}$ is Kähler-Einstein.

For the case that if the cscK metric is given by a defining function discussed in [CY80], then we can show that it is Kähler-Einstein.

Corollary 1.2. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^k-boundary, for $k \geq 7$. Assume there exists a complete cscK metric $\tilde{g}_{i\bar{j}} = -\partial_i\partial_{\bar{j}} \log(-\rho)$ for some defining function ρ of Ω. Then $\tilde{g}$ is Kähler-Einstein.

Considering the Bergman metric on strictly pseudoconvex domains, we provide the following result, which offers a new perspective for extending [HX21, Theorem 1] in the context of the cscK metric.

Theorem 1.3. Let $\Omega \subset \mathbb{C}^n$ be a C^∞ bounded strictly pseudoconvex domain equipped with the Bergman metric g_B. If g_B has constant scalar curvature, then Ω is biholomorphic to the ball.

2. Preliminaries

2.1. **Notations.** Let (M^n, g) denote an n-dimensional complete Kähler manifold. In local coordinates $(z^1, \ldots, z^n)$, the Kähler metric g can be expressed in the form

$$g = g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j,$$

where we adopt the Einstein summation convention. We use ∂_i and $\partial_{\bar{j}}$ as shorthand notations for $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \bar{z}^j}$, respectively, and denote by $\left(g^{i\bar{j}}\right)$ the inverse matrix of $\left(g_{i\bar{j}}\right)$.

Given two tensors A and B of type (m, n), their Hermitian inner product with respect to g is defined by

$$\langle A, B\rangle_g := g^{i_1\bar{j}_1} \cdots g^{i_n\bar{j}_n} g_{k_1\bar{l}_1} \cdots g_{k_m\bar{l}_m} A^{k_1\cdots k_m}_{i_1\cdots i_n} \overline{B^{l_1\cdots l_m}_{j_1\cdots j_n}},$$

and the corresponding norm of A is given by $|A|_g^2 := \langle A, A\rangle_g$.

Let ∇ denote the Levi-Civita connection associated with the Kähler metric g. The covariant derivatives are expressed locally as

$$\nabla_i := \nabla_{\partial_i}, \quad \nabla_{\bar{j}} := \nabla_{\partial_{\bar{j}}}.$$

For any vector fields X, Y, Z, W on M, the curvature tensor R is defined by

$$R(X, Y, Z, W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\ W\rangle.$$

In local coordinates, the components of the curvature tensor are calculated as

$$R_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{p\bar{q}} \left(\partial_k g_{i\bar{q}}\right)\left(\partial_{\bar{l}} g_{p\bar{j}}\right).$$

Consider a unitary frame $\{\epsilon_i, \overline{\epsilon_i}\}_{i=1}^n$ constructed by

$$\epsilon_i = \frac{1}{\sqrt{2}}\left(e_i - \sqrt{-1} J e_i\right), \quad \overline{\epsilon_i} = \frac{1}{\sqrt{2}}\left(e_i + \sqrt{-1} J e_i\right),$$

where $\{e_i, Je_i\}_{i=1}^n$ forms an orthonormal frame on TM. The holomorphic bisectional curvature of a complex plane Π spanned by ϵ_i and ϵ_j is defined as

$$\mathrm{HBC}_g(\Pi) = R\left(\epsilon_i, \overline{\epsilon_i}, \epsilon_j, \overline{\epsilon_j}\right).$$

The Ricci curvature tensor is defined as the trace of the curvature tensor:

$$\mathrm{Ric}(g) = R_{i\bar{j}}\, dz^i \otimes d\bar{z}^j, \quad \text{where} \quad R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\left(\det(g_{k\bar{l}})\right).$$

The scalar curvature is then obtained by tracing the Ricci curvature with respect to g:

$$R(g) = \mathrm{tr}_g \mathrm{Ric} = g^{i\bar{j}} R_{i\bar{j}}.$$

Note that for C^2-functions, the covariant derivative $\nabla_i\nabla_{\bar{j}}$ coincides with the partial derivative $\partial_i\partial_{\bar{j}}$, since the mixed Christoffel symbols $\Gamma_{i\bar{j}}$ vanish identically on Kähler manifolds.

For any C^∞-function $f : M \to \mathbb{R}$, the gradient vector field is defined by

$$\nabla f = g^{i\bar{j}}\left(\partial_i f \cdot \partial_{\bar{j}} + \partial_{\bar{j}} f \cdot \partial_i\right).$$

The complex Hessian of f is given by $\nabla^2 f = \sqrt{-1}\partial\bar{\partial} f$, and the Laplacian operator is defined as the trace of the Hessian:

$$\Delta_g f := \operatorname{tr}_g\left(\nabla^2 f\right) = g^{i\bar{j}}\partial_i\partial_{\bar{j}} f.$$

2.2. **Omori-Yau's maximum principle.** We now introduce Omori-Yau's generalized maximum principle on non-compact manifolds.

Proposition 2.1 (Omori, [Omo67],Yau [Yau75])**.** Let (M, g) be a complete Kähler manifold. If g has bounded sectional curvature, then for any function $u \in C^2(M)$ with $\sup_M u < +\infty$, there exists a sequence of points $\{z_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$(1) \lim_{k\to\infty} u(z_k) = \sup_M u,\ (2) \lim_{k\to\infty} |\nabla u(z_k)|_g = 0,\ (3) \lim_{k\to\infty} \nabla^2 u(z_k) \leq 0,$$

where the third inequality holds in the sense of matrices.

Remark 2.2. If the assumption of bounded sectional curvature is replaced by bounded Ricci curvature, condition (3) can be modified to

$$(3') \lim_{k\to\infty} \Delta_g u(z_k) \leq 0.$$

2.3. **Bounded geometry.** Recall that the injectivity radius at a point $x \in M$ is the maximum radius r of the ball B_r in the tangent space T_xM for which the exponential map $\exp_x : B_r \to \exp_x(B_r) \subset M$ is a diffeomorphism. The injectivity radius of M is the infimum of the injectivity radius at all points in M.

Definition 2.3 (Wu-Yau, [WY20])**.** Let (M, g) be a complete Kähler manifold and let $k \geq 0$ be an integer. We say (M, g) has C^k-quasi-bounded geometry if for each non-negative integer $l \leq k$, there exists a constant $C_l > 0$ such that

$$\sup_M |\nabla^l \operatorname{Rm}|_g \leq C_l, \tag{2.1}$$

where $\operatorname{Rm} = \{R_{i\bar{j}k\bar{l}}\}$ is the Riemann curvature tensor of g and ∇^l is the convariant derivative of order l. Moreover, if (M, g) has a positive injectivity radius, then we say (M, g) has C^k-bounded geometry.

3. CscK metrics on strictly pseudoconvex domains

Let Ω be a bounded pseudoconvex domain, and let g be the Kähler-Einstein metric on Ω of C^0-bounded geometry. The existence of the unique Kähler-Einstein metric g is guaranteed by the result of Mok and Yau [MY83]. Suppose $\tilde{g}$ is a complete cscK metric of C^0-bounded geometry on Ω with scalar curvature $R(\tilde{g}) = -n(n+1)$. Then there exists some F such that

$$\det\left(\tilde{g}_{i\bar{j}}\right) = e^F \det\left(g_{i\bar{j}}\right). \tag{3.1}$$

We first see the following inequality which holds for any Kähler metrics g and $\tilde{g}$ if the holomorphic bisectional curvature of g is bounded from below.

Lemma 3.1. There exists a constant B depending on the holomorphic bisectional curvature of g such that

$$\Delta_{\tilde{g}} \log\left(\operatorname{tr}_g \tilde{g}\right) \geq B \operatorname{tr}_{\tilde{g}} g - \frac{\operatorname{tr}_g \operatorname{Ric}(\tilde{g})}{\operatorname{tr}_g \tilde{g}}. \tag{3.2}$$

Proof. Let $\tilde{R}_{i\bar{j}k\bar{l}}$ and $\tilde{R}_{i\bar{j}}$ denote the local components of Riemann and Ricci curvature tensor of $\tilde{g}$. Taking any $p \in \Omega$, we compute in normal coordinates centered at p where g is identity and $\tilde{g}$ is diagonal. We have

$$\begin{aligned}
\Delta_{\tilde{g}} \operatorname{tr}_g \tilde{g} &= \tilde{g}^{k\bar{l}} \partial_k \partial_{\bar{l}} \left(g^{i\bar{j}} \tilde{g}_{i\bar{j}}\right) \\
&= \tilde{g}^{k\bar{l}} \tilde{g}_{i\bar{j}} \partial_k \partial_{\bar{l}} g^{i\bar{j}} + \tilde{g}^{k\bar{l}} g^{i\bar{j}} \partial_k \partial_{\bar{l}} \tilde{g}_{i\bar{j}} \\
&= \tilde{g}^{k\bar{l}} \tilde{g}_{i\bar{j}} \partial_k \partial_{\bar{l}} g^{i\bar{j}} + \tilde{g}^{k\bar{l}} g^{i\bar{j}} \left(-\tilde{R}_{i\bar{j}k\bar{l}} + \tilde{g}^{p\bar{q}} \partial_k \tilde{g}_{i\bar{q}} \partial_{\bar{l}} \tilde{g}_{p\bar{j}}\right) \\
&= \tilde{g}^{k\bar{l}} \tilde{g}_{i\bar{j}} \partial_k \partial_{\bar{l}} g^{i\bar{j}} - g^{i\bar{j}} \tilde{R}_{i\bar{j}} + \tilde{g}^{k\bar{l}} g^{i\bar{j}} \tilde{g}^{p\bar{q}} \partial_k \tilde{g}_{i\bar{q}} \partial_{\bar{l}} \tilde{g}_{p\bar{j}},
\end{aligned} \tag{3.3}$$

where the third equality follows from the local expression of Riemann curvature tensor in Kähler case. In fact, at p, we have

$$\operatorname{tr}_g \tilde{g} = \sum_i \tilde{g}_{i\bar{i}}, \qquad \operatorname{tr}_{\tilde{g}} g = \sum_i \tilde{g}^{i\bar{i}}.$$

Then for the first term of (3.3), we have

$$\begin{aligned}
\tilde{g}^{k\bar{l}} \tilde{g}_{i\bar{j}} \partial_k \partial_{\bar{l}} g^{i\bar{j}} &= \sum_{k,i} \tilde{g}^{k\bar{k}} \tilde{g}_{i\bar{i}} \partial_k \partial_{\bar{k}} g^{i\bar{i}} \\
&\geq \inf_{\Omega} \mathrm{HBC}_g \sum_{k,i} \tilde{g}^{k\bar{k}} \tilde{g}_{i\bar{i}} \\
&= \inf_{\Omega} \mathrm{HBC}_g \left(\operatorname{tr}_g \tilde{g}\right) \left(\operatorname{tr}_{\tilde{g}} g\right),
\end{aligned}$$

where HBC_g is the holomorphic bisectional curvature of g and we will denote $\inf_{\Omega} \mathrm{HBC}_g$ by B in the remaining part. The second term of (3.3) is just $\operatorname{tr}_g \operatorname{Ric}(\tilde{g})$, and the third term of (3.3) is

$$\tilde{g}^{k\bar{l}} g^{i\bar{j}} \tilde{g}^{p\bar{q}} \partial_k \tilde{g}_{i\bar{q}} \partial_{\bar{l}} \tilde{g}_{p\bar{j}} = \sum_{i,k,p} \tilde{g}^{k\bar{k}} \tilde{g}^{p\bar{p}} |\partial_k \tilde{g}_{i\bar{p}}|^2$$

Thus,

$$\Delta_{\tilde g} \operatorname{tr}_g \tilde g \ge B\left(\operatorname{tr}_g \tilde g\right)\left(\operatorname{tr}_{\tilde g} g\right) - \operatorname{tr}_g \operatorname{Ric}(\tilde g) + \sum_{i,k,p} \tilde g^{k\bar k} \tilde g^{p\bar p} |\partial_k \tilde g_{i\bar p}|^2.$$

Taking logarithm into account, we have

$$\begin{aligned}\Delta_{\tilde g} \log\left(\operatorname{tr}_g \tilde g\right) &= \frac{\Delta_{\tilde g} \operatorname{tr}_g \tilde g}{\operatorname{tr}_g \tilde g} - \frac{\tilde g^{i\bar j}\left(\partial_i \operatorname{tr}_g \tilde g\right)\left(\partial_{\bar j} \operatorname{tr}_g \tilde g\right)}{\left(\operatorname{tr}_g \tilde g\right)^2} \\ &\ge B \operatorname{tr}_{\tilde g} g - \frac{\operatorname{tr}_g \operatorname{Ric}(\tilde g)}{\operatorname{tr}_g \tilde g} + \frac{\sum_{i,k,p} \tilde g^{k\bar k} \tilde g^{p\bar p} |\partial_k \tilde g_{i\bar p}|^2}{\operatorname{tr}_g \tilde g} - \frac{\sum_{i,k,p} \tilde g^{i\bar i}\left(\partial_i \tilde g_{k\bar k}\right)\left(\partial_{\bar i} \tilde g_{p\bar p}\right)}{\left(\operatorname{tr}_g \tilde g\right)^2}. \end{aligned} \tag{3.4}$$

By applying Cauchy–Schwarz to the numerator of the fourth term of (3.4), we have

$$\begin{aligned}\sum_{i,k,p} \tilde g^{i\bar i}\left(\partial_i \tilde g_{k\bar k}\right)\left(\partial_{\bar i} \tilde g_{p\bar p}\right) &= \sum_{i,k,p}\left(\sqrt{\tilde g^{i\bar i}}\partial_i \tilde g_{k\bar k} \cdot \sqrt{\tilde g^{i\bar i}}\partial_{\bar i} \tilde g_{p\bar p}\right) \\ &\le \sum_{k,p}\left(\sum_i \tilde g^{i\bar i}|\partial_i \tilde g_{k\bar k}|^2\right)^{1/2}\left(\sum_i \tilde g^{i\bar i}|\partial_{\bar i} \tilde g_{p\bar p}|^2\right)^{1/2} \\ &= \left(\sum_k\left(\sum_i \tilde g^{i\bar i}|\partial_i \tilde g_{k\bar k}|^2\right)^{1/2}\right)^2 \\ &= \left(\sum_k (\tilde g_{k\bar k})^{1/2}\left(\sum_i \tilde g^{i\bar i}\tilde g^{k\bar k}|\partial_i \tilde g_{k\bar k}|^2\right)^{1/2}\right)^2 \\ &\le \left(\sum_k \tilde g_{k\bar k}\right)\left(\sum_{i,k} \tilde g^{i\bar i}\tilde g^{k\bar k}|\partial_i \tilde g_{k\bar k}|^2\right) \\ &\le \operatorname{tr}_g \tilde g \cdot \sum_{i,k,p} \tilde g^{i\bar i}\tilde g^{k\bar k}|\partial_i \tilde g_{p\bar k}|^2, \end{aligned}$$

where we add some positive terms in the last inequality. This implies the desired inequality. □

Lemma 3.2. Let g and $\tilde g$ be the Kähler-Einstein metric and cscK metric, respectively. Then the following inequality holds for the volume forms:

$$\det(\tilde g_{i\bar j}) \ge \det(g_{i\bar j}). \tag{3.5}$$

Proof. Taking $u = \det(g_{i\bar j})/\det(\tilde g_{i\bar j}) > 0$, we compute

$$\begin{aligned}\Delta_{\tilde g} \log u &= \Delta_{\tilde g} \log\left(\det(g_{i\bar j}) - \det(\tilde g_{i\bar j})\right) \\ &= (n+1) \operatorname{tr}_{\tilde g} g - (n+1)n \\ &\ge (n+1)nu^{1/n} - (n+1)n, \end{aligned}$$

where the last inequality follows from the arithmetic-geometric inequality.

Then we obtain

$$\begin{aligned}\Delta_{\tilde{g}} \log u &= \frac{\Delta_{\tilde{g}} u}{u} - \frac{|\nabla u|^2_{\tilde{g}}}{u^2} \\ &\geq (n+1)nu^{1/n} - (n+1)n,\end{aligned}$$

which leads to

$$\Delta_{\tilde{g}} u - \frac{|\nabla u|^2_{\tilde{g}}}{u} \geq (n+1)nu^{1/n+1} - (n+1)nu.$$

Since u is bounded on $\overline{\Omega}$, by the generalized maximum principle, there exists a sequence $\{z_\alpha\} \subset \Omega$, such that

$$0 \geq \lim_{\alpha\to\infty} \Delta_{\tilde{g}} u(z_\alpha) \geq (n+1)n \lim_{\alpha\to\infty} u^{\frac{1}{n}+1}(z_\alpha) - (n+1)n \lim_{\alpha\to\infty} u(z_\alpha).$$

This implies $\sup_\Omega u \leq 1$, yielding the desired inequality. □

Remark 3.3. Note that Lemma 3.2 remains valid provided that the ratio of the volume forms of the Kähler-Einstein metric and the cscK metric is bounded. Furthermore, under this condition, inequality (3.5) implies that the volume form of the Kähler-Einstein metric is the smallest among all cscK metrics.

The following result shows that the Kähler-Einstein metric and the cscK metric are bi-Lipschitz to each other on Ω which is the first part of Theorem 1.1

Proposition 3.4. Assuming the hypotheses of Theorem 1.1 hold, then there exist constant $C \geq C' > 0$ depending on the dimension n, the lower bound of holomorphic bisectional curvature of g, and $\|F\|_{C^2(\bar{\Omega})}$, such that

$$C'g \leq \tilde{g} \leq Cg \quad \text{on} \quad \Omega. \tag{3.6}$$

Proof. It follows from (3.1),

$$\tilde{R}_{i\bar{j}} = -(n+1)g_{i\bar{j}} - F_{i\bar{j}}, \tag{3.7}$$

where $F_{i\bar{j}} := \partial_i \partial_{\bar{j}} F$. Then, (3.2) becomes

$$\begin{aligned}\Delta_{\tilde{g}} \log\left(\operatorname{tr}_g \tilde{g}\right) &\geq B \operatorname{tr}_{\tilde{g}} g - \frac{\operatorname{tr}_g \operatorname{Ric}(\tilde{g})}{\operatorname{tr}_g \tilde{g}} \\ &= B \operatorname{tr}_{\tilde{g}} g + \frac{n(n+1) + \Delta_g F}{\operatorname{tr}_g \tilde{g}}.\end{aligned}$$

Thanks to Cauchy-Schwarz inequality, we have

$$(\operatorname{tr}_g \tilde{g})(\operatorname{tr}_{\tilde{g}} g) = \left(g^{i\bar{j}}\tilde{g}_{i\bar{j}}\right)\left(\tilde{g}^{k\bar{l}} g_{k\bar{l}}\right) \geq n^2.$$

So there exists a constant $C_1 > 0$ depending on n and $\|\Delta_g F\|_{L^\infty(\Omega)}$, such that

$$\Delta_{\tilde{g}} \log\left(\operatorname{tr}_g \tilde{g}\right) \geq B \operatorname{tr}_{\tilde{g}} g - C_1 \operatorname{tr}_{\tilde{g}} g. \tag{3.8}$$

From (3.7),

$$\Delta_{\tilde{g}} F = (n+1)\left(n - \operatorname{tr}_{\tilde{g}} g\right).$$

Taking $C_2 = (-B + C_1 + 1)/(n+1)$, then it follows from (3.8),

$$\Delta_{\tilde{g}}\left(\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2 F\right) \geq \operatorname{tr}_{\tilde{g}} g - C_2(n+1)n.$$

We first claim that under our assumptions

$$\sup_{\Omega} \operatorname{tr}_g \tilde{g} < +\infty.$$

Let us now prove the claim. Suppose that $\sup_{\Omega} \operatorname{tr}_g \tilde{g} = \infty$, then there exists a sequence $\{z_k\}$ such that

$$\operatorname{tr}_g \tilde{g}(z_k) > k.$$

We denote $\tilde{\omega}$ and ω by the Kähler forms of $\tilde{g}$ and g, respectively. A straightforward computation shows,

$$\left(\operatorname{tr}_g \tilde{g}\right) \cdot \omega^n = n\tilde{\omega} \wedge \omega^{n-1}.$$

Thus,

$$k < \operatorname{tr}_g \tilde{g}(z_k) = \frac{n\tilde{\omega} \wedge \omega^{n-1}}{\omega^n}(z_k) \leq \frac{n\tilde{\omega}^n}{\omega^n}(z_k) = ne^{F(z_k)},$$

where the inequality follows from the mixed type Monge-Ampère inequality [BEGZ10, Proposition 1.11]. This contradicts the boundness of F if we let $k \to \infty$.

Since $\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2 F$ is bounded from above on $\overline{\Omega}$. Then by the generalized maximum principle, there exists a sequence $\{z_\alpha\}$, such that

$$0 \geq \lim_{\alpha\to\infty} \Delta_{\tilde{g}}\left(\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2 F\right)(z_\alpha) \geq \lim_{\alpha\to\infty} \operatorname{tr}_{\tilde{g}} g(z_\alpha) - C_2(n+1)n,$$

which implies

$$\lim_{\alpha\to\infty} \operatorname{tr}_{\tilde{g}} g(z_\alpha) \leq C_2(n+1)n.$$

If $z_\alpha \to p \in \Omega$, then taking a normal coordinate at p such that g is identity and $\tilde{g}$ is diagonal, we have

$$\operatorname{tr}_{\tilde{g}} g(p) = \sum_i \tilde{g}^{i\bar{i}}(p) \leq C_2(n+1)n.$$

This yields at point p, for any k,

$$\frac{1}{\tilde{g}_{k\bar{k}}(p)} = \tilde{g}^{k\bar{k}}(p) \leq \sum_i \tilde{g}^{i\bar{i}}(p) \leq C_2(n+1)n. \tag{3.9}$$

On the other hand side, from (3.1), we have

$$\prod_i \tilde{g}_{i\bar{i}}(p) = e^{F(p)} \leq e^{\sup_\Omega F}. \tag{3.10}$$

Combining (3.9) and (3.10), for any k, we obtain

$$\tilde{g}_{k\bar{k}}(p) = \frac{\prod_i \tilde{g}_{i\bar{i}}(p)}{\prod_{i\neq k} \tilde{g}_{i\bar{i}}(p)} \leq (C_2(n+1)n)^{n-1}\, e^{\sup_\Omega F} := \frac{C_3}{n} \tag{3.11}$$

where we recall that C_3 depends on n, the lower bound of holomorphic bisectional curvature of g, $\|F\|_{L^\infty}$ and $\|\Delta F\|_{L^\infty}$. In particular, from (3.11), we have

$$\operatorname{tr}_g \tilde{g}(p) = \sum_k \tilde{g}_{k\bar{k}}(p) \leq C_3.$$

If $z_\alpha \to p \in \overline{\Omega}$, then by the generalized maximum principle, we have

$$\operatorname{tr}_{\tilde{g}} g(z_\alpha) \leq C_2(n+1)n + \frac{1}{\alpha}.$$

Taking a normal coordinate at z_α and following the same approach as above, then we obtain

$$\operatorname{tr}_g \tilde{g}(z_\alpha) \leq C_3 + o\left(\frac{1}{\alpha}\right).$$

Let $\alpha \to \infty$, we have

$$\lim_{\alpha\to\infty} \operatorname{tr}_g \tilde{g}(z_\alpha) \leq C_3.$$

Hence for any $x \in \Omega$, we obtain

$$\begin{aligned} \log\left(\operatorname{tr}_g \tilde{g}\right)(x) - C_2F(x) &\leq \sup_\Omega \left(\log\left(\operatorname{tr}_g \tilde{g}\right) - C_2F\right) \\ &\leq \lim_{\alpha\to\infty} \log \operatorname{tr}_g \tilde{g}(z_\alpha) - C_2 \lim_{\alpha\to\infty} F(z_\alpha) \\ &\leq \log(C_3) - C_2F(p), \end{aligned}$$

which implies

$$\sup_\Omega \operatorname{tr}_g \tilde{g} \leq C_4, \tag{3.12}$$

where C_4 depends on the same factors as C_3. Now, if we choose a normal coordinate at any point $x \in \Omega$ such that g is identity and $\tilde{g}$ is diagonal, it follows from (3.5) and (3.12),

$$\tilde{g}_{k\bar{k}} = \frac{\prod_i \tilde{g}_{i\bar{i}}}{\prod_{i\neq k} \tilde{g}_{i\bar{i}}} \geq C_5,$$

which implies

$$\tilde{g} \geq C_5 g. \tag{3.13}$$

This gives the desired result. □

Before we prove the second part of our main result, we first introduce the following fact which is a consequence of the Schwarz-Yau lemma.

Proposition 3.5. Suppose (M_i, g_i) are complete Kähler manifolds equipped with cscK metrics g_i for $i = 1, 2$ such that their scalar curvatures are equal. If $-C_2 g_i \leq \operatorname{Ric}(g_i) \leq -C_1 g_i$ for some constant $C_1, C_2 > 0$, then any biholomorphism between M_1 and M_2 is an isometry between g_1 and g_2.

Proof. Let ω_i be the Kähler form for g_i, then the volume form of g_i is given by ω_i^n for $i = 1, 2$. It follows from the Schwarz-Yau lemma [Yau78a, Theorem 3] that any biholomorphism ψ between M_1 and M_2 preserve the volume form, that is $\psi^*\omega_2^n = \omega_1^n$. Then we have

$$0 = \psi^*\omega_2^n - \omega_1^n = (\psi^*\omega_2 - \omega_1) \wedge \left((\psi^*\omega_2)^{n-1} + (\psi^*\omega_2)^{n-2} \wedge \omega_1 + \cdots + \omega_1^{n-1}\right),$$

which yields $\psi^*\omega_2 = \omega_1$ since the second term is strictly positive. So we know that ψ also preserves the Kähler form. Let J_i be the complex structure of M_i, we have $d\psi \circ J_1 = J_2 \circ d\psi$ due to ψ is a biholomorphism. Thus for any vector field X, Y on M_1, we have

$$\begin{aligned}
\psi^* g_2(X, Y) &= g_2(d\psi(X), d\psi(Y)) \\
&= \omega_2(d\psi(X), J_2(d\psi(Y))) \\
&= \omega_2(d\psi(X), d\psi(J_1(Y))) \\
&= \psi^*\omega_2(X, J_1(Y)) \\
&= \omega_1(X, J_1(Y)) \\
&= g_1(X, Y),
\end{aligned}$$

which implies that ψ is an isometry. □

In the last part of this section, we study the case when the cscK metric $\tilde{g}$ is asymptotically Kähler-Einstein in the sense that its volume form equals the volume of the Kähler-Einstein metric at infinity, and prove in such case, $\tilde{g}$ is Kähler-Einstein.

Proposition 3.6. Assuming the hypotheses of Theorem 1.1 hold, if $F = 0$ on $\partial\Omega$, then $\tilde{g}$ is Kähler-Einstein.

Proof. It follows from Lemma 3.2, $e^F = \det(\tilde{g})/\det(g) \geq 1$. Denote $\tilde{\omega}$ and ω by the Kähler forms of $\tilde{g}$ and g, respectively. Then, we have

$$\begin{aligned}
0 &\leq \left(e^F - 1\right)\omega^n \\
&= \tilde{\omega}^n - \omega^n \\
&= (\tilde{\omega} - \omega) \wedge \left(\tilde{\omega}^{n-1} + \cdots + \omega^{n-1}\right),
\end{aligned}$$

which implies $\tilde{\omega} \geq \omega$. In fact, it shows that we can take $C' = 1$ in (3.6), and thus we have

$$\mathrm{tr}_{\tilde{g}}\, g \leq n.$$

Then,

$$\Delta_{\tilde{g}} F = (n+1)\left(n - \mathrm{tr}_{\tilde{g}}\, g\right) \geq 0.$$

Since we have $F = 0$ on $\partial\Omega$, F is subharmonic with respect to $\tilde{g}$ and has 0 as its maximum at infinity. Then by the strong maximum principle, we have $F < 0$ in Ω if $F \not\equiv 0$. This implies

$$\det(\tilde{g}_{i\bar{j}}) = e^F \det(g_{i\bar{j}}) < \det(g_{i\bar{j}}),$$

which contradicts to (3.5). Thus we have $F \equiv 0$ and $\det(\tilde{g}_{i\bar{j}}) = \det(g_{i\bar{j}})$ in Ω.

By the same trick as in the proof of Proposition 3.5, taking $\psi = \mathrm{Id}_\Omega$, we have $\tilde{g} = g$ in Ω which completes the proof. □

A key observation is that the estimates derived in this section for the cscK metric on bounded pseudoconvex domains extend naturally to general complete Kähler manifolds equipped with a Kähler-Einstein metric. Consequently, we can see the following result, which might be more attractive.

Proposition 3.7. Let (M^n, g) be a complete Kähler manifold equipped with a Kähler-Einstein metric of C^0-bounded geometry. Suppose there exists a cscK metric $\tilde{g}$ of C^0-bounded geometry satisfying $\det(\tilde{g}) = e^F \det(g)$ for some $F \in C^2(M)$ with $\|F\|_{C^2(M)} \le K$, where $K > 0$. Then there exist constants $C, C' > 0$, depending only on n, g, and $\|F\|_{C^2(M)}$, such that

$$Cg \le \tilde{g} \le C'g.$$

Furthermore, if $F(z) \to 1$ uniformly as $d_{\tilde{g}}(z, p) \to \infty$ for any fixed $p \in M$, then $\tilde{g}$ is necessarily Kähler-Einstein.

In [CY80], Cheng and Yau studied the case of complete Kähler metrics in the form of $h_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log(-\rho)$, where ρ is a strictly plurisubharmonic defining function of $\Omega = \{\rho < 0\}$. The local expression for the curvature tensor is calculated as

$$R_{i\bar{j}k\bar{l}} = -\left(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}\right) + O\left(\frac{1}{|\rho|}\right).$$

And, it is notable that h behaves "asymptotically" like a Kähler-Einstein metric with the Einstein constant $-(n+1)$ when approaching the boundary.

The Kähler-Einstein on strictly pseudoconvex domains is given by the Fefferman defining function. Let g be the Kähler-Einstein metric. Suppose the cscK metric is also given by a global strictly plurisubharmonic defining function. In that case, there exists $u \in C^{k-1}(\overline{\Omega})$ such that $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u$ (see [Kra01, Chapter 3.1] for discussion about the regularity of u). And, we have

$$\det\left(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} u\right) = \det\left(g_{i\bar{j}}\right) \cdot \det\left(\delta_{i\bar{j}} + g^{i\bar{l}} \partial_i \partial_{\bar{l}} u\right).$$

Since g is complete and $\partial_l \partial_{\bar{j}} u \in C^{k-3}(\bar{\Omega})$, we immediately see that

$$\det\left(\delta_{i\bar{j}} + g^{i\bar{l}} \partial_l \partial_{\bar{j}} u\right) \to \det\left(\delta_{i\bar{j}}\right) = 1$$

when we approach to $\partial\Omega$. Then Corollary 1.2 follows from Proposition 3.6 directly.

4. Bergman metric on strictly pseudoconvex domains

In this section, we discuss some facts on the Bergman metric on strictly pseudoconvex domains. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ be the space of holomorphic functions in $L^2(\Omega)$. It is clear that $A^2(\Omega)$ is a Hilbert space. The Bergman kernel $K(z)$ on

Ω is a real analytic function defined as

$$K(z) = \sum_{j=1}^{\infty} |\varphi_j(z)|^2, \qquad \forall z \in \Omega,$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $A^2(\Omega)$ with respect to the L^2 inner product. Since the Bergman kernel is positive and independent of the choice of any orthonormal basis [Kra01] on bounded domains, we then can define the Bergman metric

$$g_B := g_{i\bar{j}} dz^i \otimes d\bar{z}^j \qquad \text{where} \quad g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log K.$$

The Bergman metric is a complete real analytic Kähler metric, with the real analytic property inherited from the Bergman kernel. Let $G_B := \det\left(g_{i\bar{j}}\right)$ denote the determinant of the Bergman metric. The Bergman invariant function $B(z) := G(z)/K(z)$, introduced by Bergman in [BS51], is invariant under biholomorphic maps.

A significant result regarding the Bergman invariant function, established by Diederich in [Die70], states that as one approaches the boundary of a strictly pseudoconvex domain with C^∞-boundary, B tends to constant when we approach the boundary.

Proposition 4.1 (Diederich, [Die70])**.** Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $p \in \partial\Omega$ be a C^2 strictly pseudoconvex point. Then

$$\lim_{z \to p} B(z) = \frac{(n+1)^n \pi^n}{n!}.$$

Then we have the following result, as an extending for Huang and Xiao's resolution of Cheng's conjecture.

Theorem 4.2. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary and let g_B be the Bergman metric. If g_B has constant scalar curvature, then Ω is biholomorphic to the ball.

Proof. Expressed in local coordinates, g_B satisfies the equation

$$R_{i\bar{j}} + \partial_i \partial_{\bar{j}} \log B = -g_{i\bar{j}}, \tag{4.1}$$

where $R_{i\bar{j}}$ and $g_{i\bar{j}}$ are the local components of the Ricci curvature $\mathrm{Ric}(g_B)$ and the metric g_B, respectively, and B represents the Bergman invariant function. Taking the trace of equation (4.1) with respect to g_B, we derive

$$\Delta_{g_B} \log B = 0 \quad \text{in} \quad \Omega.$$

This implies that $\log B$ is a harmonic function with constant "boundary value" (Proposition 4.1). By the strong maximum principle, it follows that $\log B$ must be constant throughout Ω, specifically

$$\log B = \log\left(\frac{(n+1)^n \pi^n}{n!}\right).$$

Combining this result with the characterization in [FW97, Proposition 1.1], we conclude that g_B is Kähler-Einstein. The final statement then follows immediately from [HX21, Theorem 1]. □

References

[Aub76] T. Aubin, *Équations du type Monge-Ampère sur les varietés Kählériennes compactes*, CR Acad. Sci. Paris Sér. AB **283** (1976), 119–121.

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Monge–ampère equations in big cohomology classes*, Acta Math. **205** (2010), 199–262.

[BS51] S. Bergman and M. Schiffer, *Kernel functions and conformal mapping*, Compositio Mathematica **8** (1951), 205–249.

[CC21a] X.-X. Chen and J.-R. Cheng, *On the constant scalar curvature Kähler metrics (II)—Existence results*, J. Amer. Math. Soc. **34** (2021), no. 4, 937–1009.

[CC21b] ———, *On the constant scalar curvature Kähler metrics (I)—A priori estimates*, J. Amer. Math. Soc. **34** (2021), no. 4, 909–936.

[CDS15a] X.-X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197.

[CDS15b] ———, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π*, J. Amer. Math. Soc. **28** (2015), no. 1, 199–234.

[CDS15c] ———, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278.

[Che18] X.-X. Chen, *On the existence of constant scalar curvature Kähler metric: a new perspective*, Ann. Math. Québec **42** (2018), 169–189.

[CY80] S.-Y. Cheng and S.-T. Yau, *On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math **33** (1980), 507–544.

[Die70] K. Diederich, *Das randverhalten der Bergmanschen kernfunktion und metrik in streng pseudo-konvexen gebieten*, Math. Ann. **187** (1970), no. 1, 9–36.

[Fut83] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), no. 3, 437–443.

[FW97] S. Fu and B. Wong, *On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics*, Math. Res. Lett. **4** (1997), no. 5, 697–703.

[GKY13] V. Guedj, B. Kolev, and N. Yeganefar, *Kähler-Einstein fillings*, J. Lond. Math. Soc. **88** (2013), no. 3, 737–760.

[HX21] X. Huang and M. Xiao, *Bergman-Einstein metrics, a generalization of Kerner's theorem and Stein spaces with spherical boundaries*, J. Reine Angew. Math. **2021** (2021), no. 770, 183–203.

[Kra01] S. Krantz, *Function theory of several complex variables*, vol. 340, American Mathematical Soc., 2001.

[Mat57] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété Käehlérinne*, Nagoya Math. J. **11** (1957), 145–150.

[MY83] N. Mok and S.-T. Yau, *Completeness of the Kähler-Einstein metric on bound domains and the characterization of domain of holomorphy by curvature condition*, Proceddings of Symposia in Pure mathematics, V.39 **Part 1** (1983), 41—-59.

[Omo67] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), no. 2, 205–214.

[Tia97] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37.

[WY20] D.-M. Wu and S.-T. Yau, *Invariant metrics on negatively pinched complete Kähler manifolds*, J. Amer. Math. Soc. **33** (2020), no. 1, 103–133.

[Yau75] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), no. 2, 201–228.

[Yau78a] ———, *A general schwarz lemma for kahler manifolds*, Amer. J. Math. **100** (1978), no. 1, 197–203.

[Yau78b] ———, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

Institut Fourier, UMR 5582, Laboratoire de Mathématiques, Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France

Email address: `zehao.sha@univ-grenoble-alpes.fr`